Theorem (4.9)

X is Markov (λ, Q) iff $X_0 \sim \lambda$ and for any $n \in \mathbb{N}$, any $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n+1}$,

$$\mathbb{P}(X_{t_{n+1}} = x_{n+1} \mid X_{t_0} = x_0, \ldots X_{t_n} = x_n) = P_{x_n x_{n+1}}(t_{n+1} - t_n),$$

where $P = (P(t), t \geq 0) = (P(t) = (P_{xy}(t))_{(x,y)\in E^2}, t \geq 0)$ is the minimal nonnegative solution of Kolomgorov's "backward equation"

$$P'(t) = QP(t), t \geq 0 \quad P(0) = Id.$$

Remark By the above it must be that for any $t \geq 0, (x, y) \in E^2$,

$$P_{xy}(t) = \mathbb{P}_x(X_t = y).$$

Forward equation

Theorem (4.9.2)

The "semigroup" P is also the minimal nonnegative solution of Kolmogorov's "forward equation"

$$P'(t) = P(t)Q, t \geq 0, \quad P(0) = Id.$$

Proof of Theorem (4.9)

For $t \geq 0, (x, y) \in E^2$, let $p_{xy}(t) = \mathbb{P}_x(X_t = y)$. Obviously $p_{xy}(0) = \mathbb{1}_{\{x=y\}}$. Then, according to wether the chain has already jumped at time t or not, then using Markov property at J_1, we can write

$$\begin{aligned} p_{xy}(t) &= \mathbb{1}_{\{x=y\}} \exp(-q_x t) + \sum_{z \neq x} \mathbb{P}_x(J_1 \leq t, X_{J_1} = z, X_t = y) \\ &= \mathbb{1}_{\{x=y\}} \exp(-q_x t) + \sum_{z \neq x} \int_0^t q_{xz} \exp(-q_x s) \mathbb{P}_z(X_{t-s} = y) ds \end{aligned}$$

We deduce, by Fubini-Tonelli,

$$\exp(q_x t) p_{xy}(t) = \mathbb{1}_{\{x=y\}} + \int_0^t \sum_{z \neq x} q_{xz} \exp(q_x u) p_{zy}(u) du.$$

From the above, we first deduce that $t \to p_{xy}(t)$ is continuous, then differentiable with respect to t (the last term in the righthand side is the antiderivative of a normally converging sum of continuous functions).

Proof of Theorem (4.9)

Differentiating the above gives

$$q_x \exp(q_x t) p_{xy}(t) + \exp(q_x t) p'_{xy}(t) = \sum_{z \neq x} q_{xz} \exp(q_x t) p_{zy}(t),$$

and simplifying by $\exp(q_x t)$ yields $p'(t) = Qp(t)$, thus p solves backward equation.

Proof of Theorem (4.9)

Now if $\tilde{P}$ solves the backward equation, going back through the above steps yields

$$\tilde{P}_{xy}(t) = \mathbb{1}_{\{x=y\}} + \int_0^t \sum_{z \neq x} q_{xz} \exp(-q_x s) \tilde{P}_{zy}(t-s) ds.$$

By the same reasoning as in the finite state case we deduce by induction on n that for any $n \in \mathbb{N}$, $(x, y) \in E^2$, $\mathbb{P}_x(X_t = y, t < J_n) \leq \tilde{P}_{xy}(t)$, and so letting $n \to \infty$ yields $p_{xy}(t) \leq \tilde{P}_{xy}(t)$.

The proof of Theorem 4.9.2 is harder, it requires decomposing according to the last value the process takes before it reaches y. But the corresponding time is not a stopping time for the chain, and one needs to work instead with the time-reversed chain. See paragraph 2.8.6 of [Norris] for more details.