Communication classes, irreductibility

Communication classes are exactly that of the corresponding jump chain. In particular the chain is irreducible iff the associated jump chain is irreducible.
Note however in the continuous-time setting that if x leads to y, then $P_{xy}(t) > 0 \; \forall t > 0$.
In particular if the chain is irreducible then $P_{xy}(t) > 0 \; \forall t > 0, \; \forall (x, y) \in E^2$.

Transience/recurrence

Let $T_x^+ = \inf\{t \geq J_1 : X_t = x\}$.

Theorem (4.10)

(i) *If $q_x = 0$ or if $\mathbb{P}_x(T_x^+ < \infty) = 1$ then x is said recurrent, moreover $\int_0^\infty P_{xx}(t)dt = +\infty$ and finally x is also recurrent for the jump chain.*

(ii) *If $\mathbb{P}_x(T_x^+ < \infty) < 1$ then x is said transient, moreover $\int_0^\infty P_{xx}(t)dt < \infty$ and finally x is also transient for the jump chain.*

Transience/recurrence : proof

Denote $\mathbb{P}^X$ for the law of the continuous-time chain and $\mathbb{P}^Y$ for the law of the corresponding jump chain.
Proof of Theorem (4.10) : If $q_x = 0$ then $P_{xx}(t) = 1$ and the jump chain never leaves x so everything in (i) is obvious.
Otherwise $\mathbb{P}^X_x(T_x^+ < \infty) = \mathbb{P}^Y(T_x^+ < \infty)$, and

$$\int_0^\infty P_{xx}(t)dt = \mathbb{E}_x\left[\sum_{n\geq 0} S_{n+1}\mathbb{1}_{\{Y_n = x\}}\right] = \frac{1}{q_x}\sum_{n\geq 0}\Pi_{xx}^{(n)}.$$

Now use the recurrence criterion for the jump chain to conclude.

Transience/recurrence : proof

Theorem (4.11)

For $h > 0$ fixed set $Z^h_n = X_{nh}$. Then x is recurrent for X iff x is recurrent for Z.

Proof : Assume x is recurrent for Z. Then a.s. the number of returns at x for Z is infinite, this implies the number of returns at x for X is infinite a.s., at each return X spends an exponential random time with fixed parameter q_x, so the mean total holding time $\int_0^\infty P_{xx}(t)dt = +\infty$.
Reciprocally, assume $\int_0^\infty P_{xx}(t)dt = +\infty$. However $P_{xx}(nh + h) \geq \exp(-q_x h)P_{xx}(t)$ for any $nh \leq t \leq (n+1)h$, and thus

$$+\infty = \int_0^\infty P_{xx}(t)dt \leq \exp(q_x h)h \sum_{n\geq 1} P_{xx}(nh),$$

so the expected number of returns at x for Z is infinite, which leads to the desired conclusion.

Invariant measures and probabilities

Definition

The measure λ is invariant iff $\lambda Q = 0$.

Theorem (4.12)

The measure λ is invariant iff $\mu\Pi = \mu$ where $\mu(x) = q_x\lambda(x), x \in E$.

Proof : For any $y \in E$,

$$\begin{aligned}
\mu\Pi(y) &= \mu(y)\mathbb{1}_{q_y=0} + \sum_{x\in E, x\neq y, q_x\neq 0} \mu(x)\frac{q_{xy}}{q_x} \\
&= \sum_{x\in E, x\neq y, q_x\neq 0} \lambda(x)q_{xy} \\
&= (\lambda Q)(y) + q_y\lambda(y) = (\lambda Q)(y) + \mu(y)
\end{aligned}$$

Thus $\mu\Pi = \lambda Q + \mu$, yielding the desired result.

Corollary

If X is irreducible recurrent then any two nondegenerate invariant measures of X are proportional.

Proof : Fix $x_0 \in E$. By the preceding results the jump chain Y is irreducible and recurrent since X is, thus all invariant measures for Y are proportional to the unique one which attributes mass one at x_0. But because of irreducibility $q_x > 0$ for all $x \in E$, so by Theorem 4.12 there is a one-to-one correspondence between invariant measures for X and invariant measures of the jump chain.

Invariant measures and probabilities

In fact the above corollary can be established directly by an argument similar to the discrete-time setting when the chain is recurrent, by looking at excursion away from x and averaging the time spent at each vertex.
Beware, however, that things can get counter-intuitive in case the chain is explosive : we may then be able to build in a similar fashion an invariant probability for the chain even though because it is explosive, it must be transient.

Invariant measures and probabilities

Theorem (4.13)

Assume X is irreducible. The chain X is positive recurrent iff X is non explosive and *has an invariant probability. In that case, this invariant probability is unique, and can be written*

$$\lambda(y) = \frac{1}{\mathbb{E}_x[T_x^+]}\mathbb{E}_x\left[\int_0^{T_x^+} \mathbb{1}_{\{X_s=y\}} ds\right], \quad y \in E$$

Invariant measures and probabilities

Proof of Theorem (4.13) : Since X is irreducible $q_y > 0$ for any $y \in E$, and at each visit y, it spends an average of $1/q_y$ there before it jumps. Thus for any $y \in E$

$$\begin{aligned} \Lambda(y) &:= \mathbb{E}_x\left[\int_0^{\zeta\wedge T_x^+} \mathbb{1}_{\{X_s=y\}} ds\right] \\ &= \frac{1}{q_y}\mathbb{E}_x^Y\left[\sum_{k=0}^{T_x^+-1} \mathbb{1}_{\{X_k=y\}}\right] = \frac{1}{q_y}\nu_x(y), \end{aligned}$$

so by theorem (4.12), and the corresponding result for discrete-time chains (the jump chain is recurrent since the continuous-time chain is), we get that Λ is invariant, and that it is the unique invariant measure giving mass $1/q_x$ to state x, and total mass $\mathbb{E}_x[T_x^+]$. Of course when the chain is positive recurrent, i.e. $\mathbb{E}_x[T_x^+] < \infty$, we deduce a unique invariant probability $\lambda = \Lambda/\mathbb{E}_x[T_x^+]$.

Invariant measures and probabilities

Conversely when X is non explosive and has an invariant probability λ_0, then we may set $\nu(y) = \frac{\lambda_0(y) q_y}{\lambda_0(x) q_x}, y \in E$, for some x such that $\lambda_0(x) > 0$.
By Theorem (4.12) ν is an invariant measure for the jump chain, and gives mass 1 to x, so we must have, following the proof of the discrete-time result, $\nu(y) \geq \nu_x(y), y \in E$. But then, because the chain is non-explosive,

$$
\begin{aligned}
\mathbb{E}_x^X[T_x^+] = \mathbb{E}_x[T_x^+ \wedge \zeta] = \Lambda(E) &= \sum_{y \in E} \frac{\nu_x(y)}{q_y} \\
&\leq \sum_{y \in E} \frac{\nu(y)}{q_y} \\
&= \sum_{y \in E} \frac{\lambda_0(y)}{\lambda_0(x) q_x} = \frac{1}{\lambda_0(x) q_x} < \infty
\end{aligned}
$$

Theorem (4.14)

Assume X irreducible, recurrent. TFAAE

(i) $\lambda Q = 0$

(ii) $\lambda P(t) = \lambda$ *for any* $t \geq 0$

(iii) $\exists h > 0 \; : \; \lambda P(h) = \lambda$.

Invariant measures and probabilities

Assume (i). We may as well work with Λ of the previous proof (other invariant measures are proportional). By Markov at T_x^+,

$$\mathbb{E}_x\left[\int_0^t \mathbb{1}_{\{X_s=y\}}\,ds\right] = \mathbb{E}_x\left[\int_{T_x^+}^{T_x^+ + t} \mathbb{1}_{\{X_s=y\}}\,ds\right],$$

so that

$$\begin{aligned}
\Lambda(y) &= \mathbb{E}_x\left[\int_t^{t+T_x^+} \mathbb{1}_{\{X_s=y\}}\,ds\right] \\
&= \mathbb{E}_x\left[\int_t^{t+T_x^+} \sum_{z\in E} \mathbb{1}_{\{X_{s-t}=z}\mathbb{1}_{\{X_s=y\}}\,ds\right] \\
&= \mathbb{E}_x\left[\int_0^{T_x^+} \sum_{z\in E} \mathbb{1}_{\{X_u=z}\mathbb{1}_{\{X_{u+t}=y\}}\,du\right] \\
&= \sum_{z\in E} \Lambda(z)P_{zy}(t) = (\Lambda P(t))(y)
\end{aligned}$$

where we used Markov at time u to get the before to last equality.

Now $(ii) \Rightarrow (iii)$ is obvious. Finally assume (iii), this means λ is an invariant measure for Z^h, which is recurrent by Theorem (4.11), and since $(i) \Rightarrow (ii)$, any invariant measure for X has to be invariant for Z^h, so Λ is. Now λ and Λ have to be proportional because of the result for discrete-time chains, and we conclude that λ has to be invariant for X.

Convergence theorem

Theorem (4.15)

Assume X is irreducible, positive recurrent, λ its unique invariant probability. Then

$$P_{xy}(t) \underset{t\to\infty}{\longrightarrow} \lambda(y), \quad \forall (x, y) \in E^2$$

Convergence theorem

For any $h > 0$, Z^h is irreducible and recurrent positive since λ is an invariant probability for Z^h by Theorem (4.14). It is of course aperiodic since $P_{xx}(h) > 0$ for any $x \in E$. By the convergence theorem for discrete-time chains, $P_{xy}(nh) \to \lambda(y)$ as $n \to \infty$. Now, fix $\varepsilon > 0$. We have

$$\begin{aligned} & |P_{xy}(t) - \lambda(y)| \\ \leq \; & \left| P_{xy}(t) - P_{xy}\left(\lfloor \frac{n}{h} \rfloor h\right) \right| + \left| P_{xy}\left(\lfloor \frac{n}{h} \rfloor h\right) - \lambda(y) \right| \end{aligned}$$

The first term of the sum is bounded above by the probability that there is a jump for X in the time interval $[0, \lfloor \frac{n}{h} \rfloor h - t]$, because if there isn't, one can always couple X with a version which may only start to move after that time, and hence will have the law of Z_n^h at time t. Thus this first term is bounded above by $1 - \exp(-q_x h)$. Now take h small enough that this is less than $\varepsilon/2$. Then take $t \geq t_0(h)$ so that the second term also is bounded by $\varepsilon/2$, and we are done.

Ergodic theorem

Theorem (4.16)

Assume X is irreducible. Then $\frac{1}{t}\int_0^t \mathbb{1}_{\{X_s=x\}}ds \xrightarrow[t\to\infty]{\text{a.s}} \frac{1}{q_x\mathbb{E}_x[T_x^+]}$.
Moreover if X is assumed recurrent, and μ is an invariant measure for X, $f : E \to \mathbb{R}, g : E \to \mathbb{R}_+$ such that $\sum_{x\in E}|f(x)|\mu(x) < \infty, 0 < \sum_{x\in E} g(x)\mu(x) < \infty$,

$$\frac{\int_0^t f(X_s)ds}{\int_0^t g(X_s)ds} \xrightarrow[t\to\infty]{\text{a.s}} \frac{\sum_{x\in E} f(x)\mu(x)}{\sum_{x\in E} g(x)\mu(x)}.$$

In particular if X is positive recurrent with invariant probability λ, $f : E \to \mathbb{R}$ such that $\sum_{x\in E}|f(x)|\mu(x) < \infty$,

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{\text{a.s}} \sum_{x\in E} f(x)\mu(x).$$

Ergodic theorem

The claim is obvious in the transient case, so we assume the chain to be recurrent and prove the second assertion when it is started at x. Then the variables $\left(Z_r(f) := \int_{T_x^{(r)}}^{T_x^{(r+1)}} f(X_s)ds, r \geq 0\right)$ are i.i.d, where of course $T_x^{(0)} = 0$, and $T_x^{(r)}$ denotes the rth return time at x. Obviously $r_t := \max\{r \geq 0 : T_x^{(r)} \leq t\}$ almost surely goes to $+\infty$ if the chain is recurrent. But then

$$\frac{1}{r_t}\sum_{k=0}^{r_t-1} Z_k(f) \leq \frac{1}{r_t}\int_0^t f(X_s)ds \leq \frac{1}{r_t}\sum_{k=0}^{r_t} Z_k(f),$$

and by SLLN, both left and right hand sides converge a.s. to $\mathbb{E}_x[Z_1(f)] = \sum_{y\in E} \Lambda(y)f(y) = \frac{\sum_{y\in E}\mu(y)f(y)}{\mu(x)q_x}$. By the same reasoning for g we conclude.

Reversibility

Definition

We say λ and Q satisfy detailed balance iff

$$\lambda(x)q_{xy} = \lambda(y)q_{yx}, \ \forall (x,y) \in E^2.$$

If λ and Q satisfy detailed balance *and* X is non explosive we say that X is reversible.

Of course, if λ and Q satisfy detailed balance then $\lambda Q = 0$. Also, the corresponding jump chain remains reversible, since

$$\lambda(x)q_x\Pi_{xy} = \lambda(y)q_y\Pi_{yx},$$

and $\nu(x) = \lambda(x)q_x$ indeed is invariant measure for Π.

Reversibility

Of course, the jump chain corresponds to a conductance model (with, e.g., $c(x, y) = \nu(x)\Pi_{xy} = \lambda(x)q_{xy}$), however, the conductance function, if it allows to recover the jump kernel Π, is not enough to recover the generator Q (one is missing the information on the parameters of holding times at each site). In other words, to one reversible jump chain correspond many different reversible continuous-time chains.

Reversibility : mean commute time

Assume X is a continuous-time irreducible, reversible, positive recurrent chain. Then, for $T = \inf\{t\ geT_y : X_t = x\}$, the measure

$$\mu(z) := \mathbb{E}_x \left[\int_0^T \mathbb{1}_{\{X_s = z\}} ds \right]$$

is invariant, it has total mass the mean commute-time $\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x]$, and its mass at x is given by the average holding time at x times the mean number of returns at x before it reaches y, so

$$\mu(x) = \frac{1}{q_x} \mathbb{P}_x(T_y < T_x^+)^{-1}.$$

Now the quantity $\mathbb{P}_x(T_y < T_x^+)$ is exactly the same for the (also reversible) jump chain, and we have seen it equals $c(x)\mathcal{R}(x \leftrightarrow y)$ for the corresponding conductance model.

On the other hand, if λ is the invariant probability of X, it has total mass 1, and its mass at x is $\lambda(x) = \frac{1}{q_x \mathbb{E}_x[T_x^+]}$ by ergodic theorem. By matching the ratios for total mass and mass at x we obtain

$$\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x] = \frac{c(x)}{\lambda(x) q_x} \mathcal{R}(x \leftrightarrow y).$$

Note that if we choose the conductance function to be $c(x, y) = \pi(x) P(x, y)$, and $c(x) = \lambda(x) q_x, x \in E$, we find $\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x] = \mathcal{R}(x \leftrightarrow y)$.

SRW on a graph

A simple reversible, continuous-time chain on a graph is the continuous-time SRW. When at a given site, start an exponential(1) clock along each neighbouring edge, and jump along the edge whose clock rings first, at the time it rings. In other words $q_x = \deg(x)$, and the jump kernel is that of the discrete-time SRW on the same graph.

The uniform measure is clearly invariant, so that continuous-time SRW on a graph is positive recurrent iff the graph is finite. In that case, and if $\#E = n$, we get $\lambda(x) = \frac{1}{n}$ for any $x \in E$. In particular the mean commute-time becomes

$$\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x] = n\mathcal{R}(x \leftrightarrow y),$$

where here, $\mathcal{R}(x \leftrightarrow y)$ is computed for the conductance function $c(x,y) = 1\ \forall (x,y) \in E^2$ (so $c(x) = q_x = \deg(x)$).