HITTING PROBABILITY OF A DISTANT POINT FOR THE VOTER MODEL STARTED WITH A SINGLE ONE

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Abstract. The goal of this work is to find the asymptotics of the hitting probability of a distant point for the voter model on the integer lattice started from a single 1 at the origin. In dimensions $d = 2$ or 3, we obtain the precise asymptotic behaviour of this probability. We use the scaling limit of the voter model started from a single 1 at the origin in terms of super-Brownian motion under its excursion measure. This invariance principle was stated by Bramson, Cox and Le Gall, as a consequence of a theorem of Cox, Durrett and Perkins. Less precise estimates are derived in dimension $d \geq 4$.

1. Introduction, Notation and Statement of result

The voter model is one of the most classical interacting particle systems. This model is of great interest because it exhibits a range of interesting phenomena and also because it is dual to a system of coalescing random walks. The voter model was first introduced in [5], [10], and some of its basic properties were investigated by Liggett [16], Sawyer [20], Arratia [1], Bramson and Griffeath [3].

More recently, Cox, Durrett and Perkins [4] showed an important invariance principle, establishing that, after a suitable renormalization, voter models in dimension $d \geq 2$ converge to super-Brownian motion. Super-Brownian motion is a continuous measure-valued process which arises as the weak limit of branching particle systems (see Watanabe [22]). It was discussed by Dawson [6], and studied extensively in the nineties (see in particular [7], [18], [13]). In the recent years, it was shown that super-Brownian motion also appears in scaling limits of a wide range of lattice systems such as lattice trees, contact processes or oriented percolation. The main idea of this work is to exploit known properties of super-Brownian motion to get asymptotic results for the voter model.

Let us now describe the voter model and state our main result. Let $d \geq 2$. At each site of the integer lattice $\mathbb{Z}^d$ there is a voter holding an opinion. We will study here a two-type model, where there are only two possible opinions, say 0 or 1. At rate 1 exponential times, the voter at $x \in \mathbb{Z}^d$ chooses a neighbor y according to a given jump kernel p and adopts the opinion of y. The voting times and neighbor selections are supposed independent. The jump kernel $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ will be supposed symmetric, translation invariant, irreducible, centered, isotropic, and having exponential moments :

- $p(x, y) = p(0, y - x), \qquad p(x, y) = p(y, x), \qquad p(0, 0) = 0,$
- $\sum_{y\in\mathbb{Z}^d} yp(0, y) = 0,$
- $\sum_{y\in\mathbb{Z}^d} p(0, y)y^i y^j = \sigma^2\delta_{ij}$ for some $0 < \sigma^2 < \infty$,
- there exists a constant $C > 0$ such that $\sum_{y\in\mathbb{Z}^d} p(0, y) \exp(C|y|) < \infty$.

If $t \geq 0$, we denote by ξ_t the set of sites where voters hold opinion 1 at time t; $(\xi_t)_{t\geq 0}$ is the two-type voter model. If $A \subset \mathbb{Z}^d$, we write P_A for the probability measure under which $\xi_0 = A$. Throughout this paper, we will consider the particular case when $\xi_0 = \{0\}$. In this case, $(\xi^0_t)_{t\geq 0}$ will denote the two-type voter model started from a single opinion 1 at the origin, and for simplicity, we will write P for $P_{\{0\}}$.

It is often convenient to work with the associated measure-valued processes

$$X_t := \sum_{y\in\xi_t} \delta_y, \quad X_t^0 := \sum_{y\in\xi_t^0} \delta_y.$$

For $\alpha > 0$ we define the conditional probability

$$P_\alpha^*(.) := P(.|\xi_\alpha^0 \neq \emptyset).$$

We are interested in estimating the probability that a voter located at a distance of order c from the origin ever holds opinion 1. If $x \in \mathbb{R}^d$, we denote by $[x]_c$ the point in $c^{-1}\mathbb{Z}^d$ closest to x. If there is more than one such point, we choose the point closest to the origin. Our goal is to find the asymptotic order as $c \to \infty$ of

$$P(\exists t \geq 0 : c[x]_c \in \xi_t^0).$$

We introduce the notation $T_{c[x]_c} = \inf\{t \geq 0 : c[x]_c \in \xi_t^0\}$ so that the previous quantity can also be written $P(T_{c[x]_c} < \infty)$. Set $\beta_2 = 2\pi$, and for $d \geq 3$, let β_d be the probability that a rate 1 continuous time random walk with jump kernel p started from the origin never returns to it.

Theorem 1. *Let $x \in \mathbb{R}^d \setminus 0$ be fixed. Let us define*

$$\phi_d(c) = \begin{cases} \frac{c^2}{2\ln(c)} & \text{if } d = 2, \\ c^2 & \text{if } d = 3, \\ c^{d-2} & \text{if } d \geq 5. \end{cases}$$

Then, if $d = 2$ or $d = 3$,

$$\lim_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) = \frac{2\sigma^2}{\beta_d}\left(2 - \frac{d}{2}\right)|x|^{-2}.$$

If $d \geq 5$, there exist positive constants a_d, b_d depending on x such that

$$a_d \leq \liminf_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq \limsup_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq b_d.$$

In dimension 4 we obtain less precise results. We will prove the existence of a positive constant a_4 and we conjecture the existence of a positive b_4 such that a statement similar to the one in $d \geq 5$ holds for $d = 4$ with the function $\phi_4(c) := c^2\ln(c)$. The upper bound in dimension 4 seems more difficult than the corresponding results in other dimensions. Adapting the proof of the upper bound for $2 \leq d \leq 3$ to the case $d = 4$ only gives $\limsup_{c\to\infty} c^2 P(T_{c[x]} < \infty) = 0$.

Theorem 1 immediately extends to the multitype voter model $\overline{\xi}_t$, which is described as follows. We assume that the initial opinions are all distinct. The dynamics of the multitype voter model are the same as those of the two-type voter model. In this multitype setting, Theorem 1 gives the asymptotics of the probability that the voter at x ever adopts the initial opinion of y, as $|x - y|$ tends to infinity.

In dimensions 2 and 3, we will let $T > 0$ and argue under the measure $P^*_{c^2T}$. Motivated by the results of [4], Bramson, Cox and Le Gall [2] proved that for $T > 0$, the voter model ξ^0 under $P^*_{c^2T}$ converges as $c \to \infty$ modulo a suitable rescaling to a nondegenerate limit that can be expressed in terms of the excursion measure $\mathbb{N}_0$ of super-Brownian motion (see Theorem 2 below). This invariance principle of [2] will be our main tool in the proof of Theorem 1 for small dimensions. We will also need properties of super-Brownian motion under its excursion measure $\mathbb{N}_0$.

The Brownian snake approach of Le Gall [13] gives a good understanding of the measure $\mathbb{N}_0$, and will be used to prove an intermediate result.

As mentioned earlier, the voter model and coalescing random walks are dual processes. In a coalescing random walk system, particles are assumed to execute rate 1 random walks with jump kernel p. Particles move independently until they meet, then coalesce and move together afterwards. The duality property also serves as a major tool for our results.

In Section 2.1, we introduce super-Brownian motion and its excursion measure $\mathbb{N}_0$. Scaling limits of the voter model (invariance principles) are discussed in Section 2.2. The duality property is explained in Section 2.3, and preliminary results on rate 1 random walks and system of coalescing random walks are discussed in Section 2.4 and 2.5.

We establish the asymptotic upper bounds on $P(T_{c[x]} < \infty)$ in Section 3. This requires interesting intermediate results. Lemma 4 expresses that the probability for the voter model under P^*_α to escape $B(0, A)$ before time 2α decays exponentially with A. Lemma 3 informally expresses that for any fixed $\epsilon > 0$, then, $\cup_{t\geq\epsilon\alpha}\xi^0_t$ does not contain any "isolated" point, with arbitrarily high probability under P^*_α, when α is taken large enough.

We prove the asymptotic lower bounds in Section 4. Sections 4.1 is devoted to the case $d \geq 4$, and Sections 4.2 and 4.3 to the case $d = 2$ or 3. Finally, we prove the results of Sections 2.4 and 2.5 in Section 5.

2. Further notation and preliminary results

Let f and g be two functions from $\mathbb{R}$ into $(0,\infty)$. We will write $f(x) = o(g(x))$ as $x \to \infty$, respectively $f(x) \sim g(x)$ as $x \to \infty$ whenever $\lim_{x\to\infty} f(x)(g(x))^{-1}$ is equal to 0, respectively 1.

For $x \in \mathbb{R}^d, r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^d$ centered at x with radius r, and $B(x, r)^c$ its complement.

For real numbers $x \leq y$, the set $\{n \in \mathbb{Z} : x \leq n \leq y\}$ of integers between x and y will be denoted by $[\![x, y]\!]$; also, the integer part of x : $\max\{n \in \mathbb{Z} : n \leq x\}$ will be denoted by $\lfloor x\rfloor$, while $\lfloor x\rfloor + 1 = \min\{n \in \mathbb{Z} : n > x\}$ will be denoted by $\lceil x\rceil$.

2.1. **Super-Brownian motion.**

Let $M_F(\mathbb{R}^d)$ be the space of all finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. For $\mu \in M_F(\mathbb{R}^d)$, f a function on $\mathbb{R}^d$, the notation $\mu(f)$ will stand for $\int_{\mathbb{R}^d} f(x)\mu(dx)$ whenever this integral is well-defined. We let $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ be the space of continuous paths from $\mathbb{R}_+$ into $M_F(\mathbb{R}^d)$, and we let $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$ be the Skorohod space of cadlag functions from $\mathbb{R}^d$ into $M_F(\mathbb{R}^d)$. We denote by $(Y_t, t \geq 0)$ the canonical process on either $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ or $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$.

The law of super-Brownian motion with branching rate γ and diffusion coefficient σ^2, starting from $\mu \in M_F(\mathbb{R}^d)$, is the probability measure $\mathbf{Q}^{\gamma,\sigma^2}_\mu$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ that solves the following well-posed martingale problem (see [18], Theorem II.5.1) :

(MP) For any $\phi \in C^\infty_b(\mathbb{R}^d)$,

$$Y_t(\phi) \;=\; \mu(\phi) + M_t(\phi) + \frac{1}{2}\int_0^t Y_s(\sigma^2\Delta\phi)ds,$$

where $M_t(\phi)$ is a $\mathbf{Q}_\mu^{\gamma,\sigma^2}$ continuous square integrable martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$< M(\phi) >_t = \int_0^t Y_s(\gamma\phi^2)ds.$$

One can show (see for example Section II.7 of [18]) that there exists a family $\{R_t^{\gamma,\sigma^2}(y,.), y \in \mathbb{R}^d, t > 0\}$ of finite measures on $M_F(\mathbb{R}^d)$, called the canonical measures of super-Brownian motion, which assign zero mass to the 0 measure and are such that the following holds. The law of Y_t under $\mathbf{Q}_\mu^{\gamma,\sigma^2}$ is the same as the law of $\sum_{i\in I} Y_t^i$, where $\sum_{i\in I}\delta_{Y_t^i}$ is a Poisson measure with intensity $\int_{M_F(\mathbb{R}^d)} R_t^{\gamma,\sigma^2}(y,.)\mu(dy)$. It follows that for any Borel subset $\mathcal{Y}$ of $M_F(\mathbb{R}^d)$ with $0 \notin \mathcal{Y}$,

$$\lim_{\epsilon\to 0}\epsilon^{-1}\mathbf{Q}_{\epsilon\delta_y}^{\gamma,\sigma^2}(Y_t\in\mathcal{Y}) = R_t^{\gamma,\sigma^2}(y,\mathcal{Y}). \tag{1}$$

It is also well-known (see [18], Theorem II.7.2) that for any $y \in \mathbb{R}^d$,

$$R_t^{\gamma,\sigma^2}(y, M_F(\mathbb{R}^d)) = \frac{2}{\gamma t}. \tag{2}$$

From [18], Theorem II.7.3 (see also formula (3.10) in [2]), for each $y \in \mathbb{R}^d$ there is a σ-finite measure $\mathbb{N}_y$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ called the excursion measure of super-Brownian motion with branching rate γ and diffusion coefficient σ^2 such that the following holds. For any $\alpha > 0$ fixed, then for any bounded continuous function F on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$, such that $F(\omega) = 0$ for any ω with $\omega(t) = 0$ for all $t \geq \alpha$,

$$\lim_{\epsilon\to 0}\epsilon^{-1}\mathbf{Q}_{\epsilon\delta_y}^{\gamma,\sigma^2}(F((Y_t, t\geq 0))) = \mathbb{N}_y(F). \tag{3}$$

The convergence (1) is a particular case of (3). Thus, for any Borel subset $\mathcal{Y}$ of $M_F(\mathbb{R}^d)$ with $0 \notin \mathcal{Y}$, we have

$$\mathbb{N}_y(Y_t \in \mathcal{Y}) = R_t^{\gamma,\sigma^2}(y,\mathcal{Y}).$$

Also, for any $T > 0, y \in \mathbb{R}^d$, we get from (2)

$$\mathbb{N}_y(Y_T \neq 0) = \frac{2}{\gamma T}, \tag{4}$$

and we can define the probability measure $\mathbb{N}_y^{(T)} := \mathbb{N}_y(.|Y_T \neq 0)$.

A better understanding of the measures $\mathbb{N}_y$ is given by the Brownian snake approach of Le Gall (see [13], and Section 4.4 below). The Brownian snake approach corresponds to $\gamma = 4$, but scaling properties of super-Brownian motion can then be used to deal with a general value of γ.

Finally, we will use the following result about hitting probabilities of a single point. Let $\mathcal{R}_t$ denote the topological support of the measure Y_t, and $\mathcal{R} = \bigcup_{t>0}\mathcal{R}_t$. It follows from [13], Section 6.1 that

$$\mathbb{N}_0(x\in\mathcal{R}) = \frac{4\sigma^2}{\gamma}\left(2-\frac{d}{2}\right)^+ |x|^{-2}. \tag{5}$$

In particular, in the case $d \geq 4$, $\mathbb{N}_0(x \in \mathcal{R}) = 0$, which explains why our results are less precise. Also, as (5) suggests, the case of dimension 4 is critical, and thus harder.

Since $\mathbb{N}_0(Y_T = 0 \big| x \in \mathcal{R}) \to 0$ as T goes to 0, we deduce from (4) and (5) that

$$\mathbb{N}_0^{(T)}(x \in \mathcal{R}) \underset{T\to 0}{\sim} 2\sigma^2 T \left(2 - \frac{d}{2}\right)^+ |x|^{-2}. \tag{6}$$

2.2. **Extinction probability, invariance principle.** Set $p_t := P(\xi_t^0 \neq \emptyset)$. The asymptotic rate at which p_t converges to 0 was found in [3]. As $t \to \infty$,

$$p_t \sim \begin{cases} \log(t)/(\beta_2 t) \text{ if } d = 2 \\ 1/(\beta_d t) \text{ if } d \geq 3, \end{cases} \tag{7}$$

where $\beta_d, d \geq 2$ was defined before Theorem 1. Hence, for any $d \geq 2$ there exist a positive κ_0 depending only on d such that for any $1/4 < t' \leq t$,

$$\frac{p_{t'}}{p_t} \leq \kappa_0 \frac{t}{t'} \tag{8}$$

If $|C|$ denote the cardinality of a finite set C, Bramson and Griffeath ([3]) established that the law of $p_t|\xi_t^0|$ under P_t^* converges as $t \to \infty$ to an exponential distribution with parameter 1.

Bramson and Griffeath [3] also conjectured that ξ_t^0 would obey a certain asymptotic shape theorem. Such a result was derived in 2001 by Bramson, Cox and Le Gall [2] using the invariance principle relating the voter model and super-Brownian motion, which was proved by Cox, Durrett and Perkins in [4]. We rescale the voter model as follows. For $N > 0$, the lattice is now $S_N := \mathbb{Z}^d/\sqrt{N}$. Individuals change opinion at rate N instead of 1, and the jump kernel becomes $p_N : S_N \times S_N \to \mathbb{R}_+$ such that $p_N(x, y) = p(\sqrt{N}x, \sqrt{N}y)$. We denote by $(\xi_t^{N,0})_{t\geq 0}$ the corresponding process ($\xi_t^{N,0}$ represents the set of sites having opinion 1 at time t). If we let

$$m_N := \frac{N}{\log(N)} \text{ if } d = 2, \qquad m_N := N \text{ if } d \geq 3,$$

we can define an associated measure-valued processes :

$$X_t^{N,0} := \frac{1}{m_N} \sum_{y \in \xi_t^{N,0}} \delta_y.$$

Similarly, when at time 0, opinion 1 is started from a given set ξ_0, we may define for $N > 0$ a rescaled voter model ξ_t^N and the corresponding measure valued process X_t^N. Theorem 1.2 of [4] states that whenever X_0^N converges to a non-degenerate measure $X_0 \in M_F(\mathbb{R}^d)$, then $(X_t^N)_{t\geq 0}$ converges to a super-Brownian motion on $\mathbb{R}^d$ with branching rate $2\beta_d$ and diffusion coefficient σ^2, started from X_0.

Theorem 2 below states the convergence in law of the process $(X_t^{N,0})_{t\geq 0}$ under the conditional distribution $P(.\big|X_\alpha^{N,0} \neq 0)$ towards super-Brownian motion under $\mathbb{N}_0^{(\alpha)}$. This result, which is taken from [2] (Theorem 4) will be a key ingredient of the proof of Theorem 1 in dimensions 2 and 3.

Theorem 2. *Assume $d \geq 2$, and let $\mathbb{N}_0$ be the excursion measure of super-Brownian motion on $\mathbb{R}^d$ with branching rate $2\beta_d$ and diffusion coefficient σ^2. Let $\alpha > 0$, and let F be a bounded continuous function on $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$. Then*

$$\lim_{N\to\infty} E\left[F\left((X_t^{N,0})_{t\geq 0}\right) \big| X_\alpha^{N,0} \neq 0\right] = \mathbb{N}_0^{(\alpha)}[F]. \tag{9}$$

Let us now turn to the well-known relation between the voter model and coalescing random walks.

2.3. **Dual process to the voter model.** Let us introduce further notation in order to describe the dual process to the voter model. The times at which the voter at x adopts the opinion of the voter at y are the jump times of a standard Poisson process with rate $p(x,y)$. We denote by $\Lambda(x,y)$ this set of times. Then, $\{\Lambda(x,y), x,y \in \mathbb{Z}^d\}$ forms a family of independent Poisson point processes on $[0,\infty)$.

We now describe the useful graphical representation of the voter model. Horizontal axis represents $\mathbb{Z}^d$, vertical axis represents time. For $x, y \in \mathbb{Z}^d$ we draw a horizontal arrow from y to x at each time $s \in \Lambda(x,y)$.

For $s < t$ we say there is a path up from (y,s) to (x,t) or equivalently a path down from (x,t) to (y,s) and we will write

$$(y,s) \nearrow (x,t) \Leftrightarrow (x,t) \searrow (y,s)$$

if there exist times $s = s_0 < s_1 < ... < s_n \leq s_{n+1} = t$ and sites $y = x_0, x_1, ..., x_n = x$ such that

- for $1 \leq i \leq n$ there is an arrow pointing from x_{i-1} towards x_i at time s_i,
- for $0 \leq i \leq n$, there is no arrow pointing towards x_i in the time interval (s_i, s_{i+1}).

Clearly for every $x \in \mathbb{Z}^d$ and every choice of $0 \leq s \leq t$, there is a unique $y \in \mathbb{Z}^d$ such that $(y,s) \nearrow (x,t)$. In such a case, the opinion of (x,t) is "descended" from that at (y,s). We will say that x at time t is a "descendant" of y at time s, or equivalently that y at time s is an "ancestor" of x at time t.

We are now in a position to describe the dual process to the voter model. For $t > 0$ and $x \in \mathbb{Z}^d$ we define $(Z_s^{x,t})_{0\leq s\leq t}$ by setting $Z_0^{x,t} = x$ and for $0 < s \leq t$, $Z_s^{x,t} = y$ if and only if $(x,t) \searrow (y, t-s)$. Clearly, $(Z_s^{x,t})_{0\leq s\leq t}$ is a rate 1 random walk with jump kernel p starting from x. Moreover, for $x \in \mathbb{Z}^d$, $y \in \mathbb{Z}^d$, the two walks $(Z_s^{x,t})_{0\leq s\leq t}$, $(Z_s^{y,t})_{0\leq s\leq t}$ start respectively from x and y, move independently until they meet, and move together afterwards. That is, $(Z_s^{x,t})_{0\leq s\leq t, x\in\mathbb{Z}^d}$ forms a coalescing random walk system with jump kernel p. Furthermore

$$\xi_t^0 = \{y \in \mathbb{Z}^d : Z_t^{y,t} = 0\}. \tag{10}$$

For $t \geq 0$, we denote by $\hat{\xi}_s^{y,t}$ the set of descendants at time $t+s$ of y at time t, that is

$$\hat{\xi}_s^{y,t} := \{z \in \mathbb{Z}^d : (y,t) \nearrow (z, t+s)\} = \{z \in \mathbb{Z}^d : Z_s^{z,t+s} = y\}$$

Notice that $(\hat{\xi}_s^{y,t})_{s\geq 0}$ has the same law as $(\xi_s^0 + y)_{s\geq 0}$. For $u \leq t$ we will denote by Ω_u^t the set of points having opinion 1 at time u and having descendants at time t, that is

$$\Omega_u^t := \{y \in \xi_u^0 : \hat{\xi}_{t-u}^{y,u} \neq \emptyset\} = \xi_u^0 \cap \{Z_{t-u}^{z,t}, z \in \mathbb{Z}^d\}.$$

The coalescing random walk perspective, combined with the Bramson and Griffeath results and Theorem 2, gives us a heuristic explanation of our main result Theorem 1. If $c[x]_c$ has opinion 1 at time t, then $Z_t^{c[x]_c,t} = 0$ so that from well-known properties of random walks, t should be of order c^2. The probability for the voter model to survive a time of order c^2 is of order p_{c^2}, and conditionally on that event, the rescaled voter model converges to super-Brownian motion under its excursion measure. Informally, formula (5) is then exactly what we need to conclude in the case $2 \leq d \leq 3$. Also, not rigourously, one should expect that for $d \geq 4$, the probability of hitting $c[x]_c$ should be of order $p_{c^2} \times \mathbb{N}_0(Y \text{ hits } B(x, 1/c)) \approx A_d(x) \times \phi_d(c)^{-1}$, where $A_d(x)$ is a constant depending only on d and $|x|$ (see [9]).

In the following paragraph, we present a few well-known properties of random walks, then some estimates for coalescing random walks. These will prove useful when using the duality property in the course of the proof of Theorem 1.

2.4. **Random walks with jump kernel** p. We denote by $(Z_t, t \geq 0)$ a continuous-time random walk on $\mathbb{Z}^d$ with jump kernel p and exponential holding times with parameter 1. For $x \in \mathbb{Z}^d$, Z starts from x under the probability measure $\mathbb{P}_x$. For $x, y \in \mathbb{Z}^d, t \geq 0$ we let

$$q_t(x,y) = q_t(y-x) := \mathbb{P}_x(Z_t = y)$$

be the transition kernel of our random walk. For $x, y \in \mathbb{R}^d$ and $t > 0$ let

$$p_t(x,y) = p_t(x-y) := (2\pi\sigma^2 t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2\sigma^2 t}\right)$$

be the transition density of d-dimensional Brownian motion. We denote by P_t the associated semigroup. For $d \geq 3$ and $x \in \mathbb{R}^d \setminus 0$, we also denote by $G(x)$ the Green function associated with p :

$$G(x) = \int_0^\infty p_s(x) ds = c_d |x|^{2-d}.$$

The asymptotic behaviour of $q_t(y)$ as $t \to \infty$ is given by standard local limit theorems (see [21], and [11] for an equivalent statement for discrete random walks).

Theorem 3. *If q and p are defined as above,*

$$\lim_{t\to\infty} \sup_{y \in \mathbb{Z}^d} \left| t^{d/2} q_t(y) - p_1(y t^{-1/2}) \right| = 0.$$

We will also need an upper bound on the transition kernel q that is valid for any $t \geq 1/2$:

Lemma 1. *There exist two positive constants κ_1, κ_2 such that for every $t \geq 1/2$, $y \in \mathbb{Z}^d$,*

$$q_t(y) \leq \frac{\kappa_1}{t^{d/2}} \exp\left(-\frac{\kappa_2 |y|}{\sqrt{t}}\right).$$

For the reader's convenience, we provide a short proof of Lemma 1 in Section 5. For $t > 0$ and $y \in \mathbb{R}^d$ let us define

$$f_t(y) := \frac{\kappa_1}{t^{d/2}} \exp\left(-\frac{\kappa_2 |y|}{\sqrt{t}}\right).$$

We also set for $t > 0$ and $y \in \mathbb{R}^d$

$$\tilde{\kappa}_1 := 2\kappa_1;\ \tilde{\kappa}_2 := \kappa_2/4; \qquad \tilde{f}_t(y) := \frac{\tilde{\kappa}_1}{t^{d/2}} \exp\left(-\frac{\tilde{\kappa}_2 |y|}{\sqrt{t}}\right);$$

$$\hat{\kappa}_1 := 4\kappa_1;\ \hat{\kappa}_2 := \kappa_2/8; \qquad \hat{f}_t(y) := \frac{\hat{\kappa}_1}{t^{d/2}} \exp\left(-\frac{\hat{\kappa}_2 |y|}{\sqrt{t}}\right),$$

so that $f_t(x) \leq \tilde{f}_t(x) \leq \hat{f}_t(x)$. We need to control integrals of these functions. Note that, for $x \neq 0$, the supremum of the function $t \to \hat{f}_t(x)$ is reached at $t_0 = \frac{|x|^2 \hat{\kappa}_2^2}{d^2}$. Let us introduce for $r > 0$

$$\psi_2(r) = 2\ln(r \vee e),$$
$$\psi_d(r) = r^{d-2} \text{ if } d \geq 3.$$

We then observe that for $T>0$, there exists a constant L_0 depending only on d and T such that for any $x\in\mathbb{R}^d\setminus 0$,

$$\int_0^T f_t(x)dt \le \int_0^T \tilde{f}_t(x)dt \le \int_0^T \hat{f}_t(x)dt \le L_0\psi_d\left(|x|^{-1}\right). \tag{11}$$

Furthermore, whenever $|x|\ge \frac{d\sqrt{T}}{\hat\kappa_2}$ we have

$$\int_0^T \hat{f}_t(x)dt \le \hat\kappa_1 T^{1-d/2}\exp(-\frac{\hat\kappa_2|x|}{\sqrt{T}}). \tag{12}$$

Finally, when $d=2$, the integral $\int_0^T f_t(x)dt$ diverges when $T\to\infty$, but, when $d\ge 3$, there exist a constant L_1 depending only on d such that

$$\int_0^\infty \hat{f}_t(x)dt \le L_1\psi_d\left(|x|^{-1}\right) = L_1|x|^{2-d}. \tag{13}$$

We also need an exponential bound on the probability for a random walk with jump kernel p to escape $B(0,A\sqrt{t})$ before time t. As a consequence of Lemma 1 and Doob's maximal inequality applied to a suitable exponential martingale of the random walk, there exist positive constants κ_3,κ_4 such that for any $t\ge 1/2$, for any $A>0$,

$$\mathbb{P}_0(\sup_{s\in[0,t]}|Z_s|\ge A\sqrt{t})\le \kappa_3\exp(-\kappa_4 A). \tag{14}$$

We may and will assume that the constant κ_2 in Lemma 1 is such that $\kappa_4\ge 4\kappa_2$.

We then deduce easy consequences of Theorem 3 and Lemma 1. From Theorem 3, we obtain, for $x\ne 0$ and $s>0$,

$$c^d q_{c^2 s}(c[x]_c) = s^{-d/2}(c^2 s)^{d/2} q_{c^2 s}(c[x]_c) \underset{c\to\infty}{\longrightarrow} s^{-d/2}p_1\left(\frac{x}{\sqrt{s}}\right) = p_s(x). \tag{15}$$

On the other hand, using (14), we get

$$c^d\int_0^{c^{-2}} q_{c^2 s}(c[x]_c)ds \le \kappa_3 c^{d-2}\exp\left(-\kappa_4 c[x]_c\right)\underset{c\to\infty}{\longrightarrow} 0,$$

whereas, from Lemma 1, for any $s\ge c^{-2}$ we have

$$c^d q_{c^2 s}(c[x]_c)\le f_s([x]_c). \tag{16}$$

We can use (15) and dominated convergence to deduce that for $x\ne 0$ and $T>0$ we have

$$c^d\int_0^T q_{c^2 s}(c[x]_c)ds \underset{c\to\infty}{\longrightarrow} \int_0^T p_s(x)ds. \tag{17}$$

By a similar argument, we obtain, for any $T>0$, $y\in\mathbb{Z}^d$,

$$\int_0^T q_{c^2 s}(y)ds \le c^{-2}\kappa_3\exp(-\kappa_4|y|) + c^{-d}\int_{c^{-2}}^T f_s(y/c)ds.$$

Using (11), (12), it is then easy to establish that there exist constants L_2, L_2', depending only on T and d, such that for any $c\ge 1$, we have

$$\int_0^T q_{c^2 s}(y)ds \le \begin{cases} L_2 c^{-d}\psi_d(c) & \text{if } y=0,\\ L_2 c^{-d}\psi_d\left(c|y|^{-1}\right) & \text{if } y\in\mathbb{Z}^d\setminus 0,\\ L_2 c^{-d}\exp\left(-L_2'\frac{|y|}{c}\right) & \text{if } y\in\mathbb{Z}^d, |y|>c.\end{cases} \tag{18}$$

We now discuss some preliminary results on coalescing random walks.

2.5. **Preliminary results on coalescing random walks.** Consider two independent copies Z^1, Z^2 of the random walk Z with transition kernel q, starting respectively at points $y_1, y_2 \in \mathbb{Z}^d$ under the probability measure $\mathbb{P}_{y_1,y_2}$. The time at which Z^1 and Z^2 first meet is the stopping time $T_1 = \inf\{t \geq 0 : Z^1_t = Z^2_t\}$. We will need the following result. The first bound below holds in the case $d \geq 3$, for which we recall that $\psi_d(r) = r^{d-2}$. The second bound holds in the case $d = 2$, for which we recall $\psi_2(r) = 2\ln(r \vee e)$.

Lemma 2. *Let $d \geq 2$ and $T > 0$. There exists a positive constant L_4 depending only on T and d such that for any $x \in \mathbb{R}^d \setminus 0$, for any $c \geq 1 \vee |x|^{-2}$ and for any $y \in \mathbb{Z}^d \setminus 0$,*

$$\begin{cases} \textit{if } d \geq 3, & c^d \psi_d(|y|) \int_0^T dt \mathbb{P}_{0,y}\left[T_1 \leq c^2 t, Z^1_{c^2 t} = c[x]_c\right] \leq L_4 \ \psi_d(|x|^{-1}), \\ \textit{if } d = 2, & c^2 \frac{\psi_2(|y|)}{\psi_2(c/|y|)} \int_0^T dt \mathbb{P}_{0,y}\left[T_1 \leq c^2 t, Z^1_{c^2 t} = c[x]_c\right] \leq L_4 \psi_2(|x|^{-1}). \end{cases}$$

We postpone the proof of this result to Section 5.

3. Upper bound

In the case $d \geq 5$, the upper bound of Theorem 1 follows from the next proposition.

Proposition 1. *Let $d \geq 5, x \in \mathbb{R}^d$. For c large enough*

$$P(T_{c[x]_c} < \infty) \leq 2ec^{2-d} G(x).$$

In the case $d \leq 3$, we will argue under $P^*_{c^2 T}$ and use Theorem 2 to establish the following sharp asymptotic upper bound. This bound also holds when $d \geq 4$ but is not sharp in that case.

Proposition 2. *Let $d \geq 2$, $T > 0, x \in \mathbb{R}^d \setminus \{0\}$,*

$$\limsup_{c \to \infty} P^*_{c^2 T}(T_{c[x]_c} < \infty) \leq \mathbb{N}_0^{(T)} \ (x \in \mathcal{R}). \tag{19}$$

In the cases $d = 2$ or $d = 3$, we will see in Section 3.3 that Proposition 2 implies the asymptotic upper bound in Theorem 1. Notice that the right-hand side of (19) is 0 if $d \geq 4$. We begin with the proof of Proposition 1, which only requires very simple arguments.

3.1. **The case** $d \geq 5$**.** Fix $x \in \mathbb{R}^d \setminus \{0\}$. Proving Proposition 1 reduces to establishing the following two results :

$$E\left[\int_0^\infty ds \mathbf{1}_{\{c[x]_c \in \xi^0_s\}}\right] \underset{c \to \infty}{\sim} c^{2-d} \int_0^\infty ds p_s(x), \tag{20}$$

$$P\left(T_{c[x]_c} < \infty\right) \leq eE\left[\int_0^\infty ds \mathbf{1}_{\{c[x]_c \in \xi^0_s\}}\right]. \tag{21}$$

Let us fix $T>0$, and observe that

$$E\left[\int_0^{c^2T} ds\mathbf{1}_{\{c[x]_c\in\xi_s^0\}}\right] = c^2E\left[\int_0^T ds\mathbf{1}_{\{c[x]_c\in\xi^0_{c^2s}\}}\right]$$
$$= c^2\int_0^T dsP\left(Z^{c[x]_c,c^2s}_{c^2s}=0\right) = c^2\int_0^T dsq_{c^2s}(c[x]_c)$$
$$\text{(22)} \qquad \underset{c\to\infty}{\sim} c^{2-d}\int_0^T p_s(x)ds,$$

where the asymptotics at the last line come from (17). Furthermore, using (16), we have similarly

$$E\left[\int_{c^2T}^\infty \mathbf{1}_{\{c[x]_c\in\xi_s^0\}}ds\right] = c^2\int_T^\infty q_{c^2s}(c[x]_c)ds$$
$$\leq c^{2-d}\int_T^\infty f_s([x]_c)ds,$$

and since $d\geq 3$, $\int_T^\infty f_s([x]_c)ds$ goes to 0 as $c\to\infty$. Thus from (22), we obtain (20). Let us now prove (21).

When $T_{c[x]_c}<\infty$, denote by N the numbers of arrows pointing towards $c[x]_c$ in the time interval $(T_{c[x]_c},T_{c[x]_c}+1]$. Under $P\left(.\middle|T_{c[x]_c}<\infty\right)$, N is a Poisson variable with parameter 1. It follows that

$$P(T_{c[x]_c}<\infty)=eP(T_{c[x]_c}<\infty,N=0).$$

Furthermore, on the event $\{N=0\}$ we have $c[x]_c\in\xi_s^0$ for every $s\in[T_{c[x]_c},T_{c[x]_c}+1]$. Hence,

$$E\left[\int_0^\infty ds\mathbf{1}_{\{c[x]_c\in\xi_s^0\}}\right]\geq P(T_{c[x]_c}<\infty,N=0).$$

This completes the proof of (21), and of Proposition 1. $\square$

3.2. **Proof of Proposition 2.** Let $d\geq 2$ and fix $T>0$, $x\in\mathbb{R}^d\setminus 0$, and $\eta\in(0,|x|/2)$. Recall the notation m_N from Section 2.2. We have for any $\delta>0$, $\varepsilon>0$:

$$P^*_{c^2T}(T_{c[x]_c}<\infty)\leq P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^0(\overline{B}(cx,\eta c))\geq\delta m_{c^2}\}}\geq\varepsilon c^2\right]$$
$$\text{(23)} \qquad +P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^0(\overline{B}(cx,\eta c))\geq\delta m_{c^2}\}}<\varepsilon c^2, T_{c[x]_c}<\infty\right].$$

Intuitively, when c tends to infinity, the second term of the sum above should remain small when ε and δ are small enough, while the first term, using the invariance principle, should be bounded by a corresponding rescaled quantity under $\mathbb{N}_0^{(T)}$. Let us be more precise. Using rescaling, the first term of the sum in the right-hand side of (23) is equal to

$$P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right].$$

It is easy to see that for any $A>0$, the set

$$\left\{\omega\in D(\mathbb{R}_+,M_F(\mathbb{R}^d)):\int_0^A ds\mathbf{1}_{\{\omega_s(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right\}$$

is closed for the Skorohod J_1 topology. Then, Theorem 2 implies that

$$\limsup_{c\to\infty} P^*_{c^2T}\left[\int_0^A ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\leq \mathbb{N}_0^{(T)}\left[\int_0^A ds\mathbf{1}_{\{Y_s(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]$$
$$\leq \mathbb{N}_0^{(T)}\left[Y \text{ hits } \overline{B}(x,\eta)\right].$$

Furthermore, we have, for $A\geq T$,

$$P^*_{c^2T}\left[\int_A^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\leq P^*_{c^2T}(X_A^{c^2,0}\neq 0)=\frac{p_{c^2A}}{p_{c^2T}},$$

which goes to 0 as $A\to\infty$. Hence, we obtain for every $\delta>0,\epsilon>0$,

$$\limsup_{c\to\infty} P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\leq \mathbb{N}_0^{(T)}\left[Y \text{ hits } \overline{B}(x,\eta)\right]. \tag{24}$$

To control the second term of the sum in the right-hand side of (23), we will use the following argument. When c is large and point $c[x]_c$ is hit by opinion 1, then with arbitrarily high probability, a sufficient number (of order m_{c^2}) of its neighbors (at distance less than ηc) should also be hit by opinion 1 during a certain time interval (with length of order c^2).

We will prove a somewhat more general result, which will be valid uniformly over all points in ξ_t^0, with the restriction that t should be at least of order c^2.

Lemma 3. *Let $T>0,\rho>0$, $\eta>0$ be fixed. We can find $\varepsilon_0>0$ so that for any $\varepsilon\in(0,\varepsilon_0]$, there exists $\delta>0$ such that for c sufficiently large,*

$$P^*_{c^2T}\left(\exists t\geq 4\varepsilon c^2\ \exists x\in\xi_t^0:\inf_{s\in[t-3\varepsilon c^2,t-2\varepsilon c^2]}|\xi_s^0\cap\overline{B}(x,\eta c)|<\delta m_{c^2}\right)\leq\rho. \tag{25}$$

We will also need a useful exponential bound on the probability for the voter model to escape a ball of radius $A\sqrt{\alpha}$ before time 2α :

Lemma 4. *There exists constants $K_1>0,K_2>0$ such that for any $\alpha>1$, for any $A>0$,*

$$P^*_\alpha\left(\sup_{t\leq 2\alpha}\sup_{x\in\xi_t^0}|x|>A\sqrt{\alpha}\right)\leq K_1\exp(-K_2A). \tag{26}$$

Let us postpone the proofs of Lemma 3 and Lemma 4, and finish the proof of Proposition 2. Recall $x,T,\eta\in(0,|x|/2)$ have been fixed. Notice that, when c is large enough, $\overline{B}(c[x]_c,\eta c/2)\subset\overline{B}(cx,\eta c)$. Thus,

$$P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^0(\overline{B}(cx,\eta c))\geq\delta m_{c^2}\}}<\varepsilon c^2,4\varepsilon c^2\leq T_{c[x]_c}<\infty\right]$$
$$\leq P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^0(\overline{B}(c[x]_c,\eta c/2))\geq\delta m_{c^2}\}}<\varepsilon c^2,4\varepsilon c^2\leq T_{c[x]_c}<\infty\right].$$

Hence, using Lemma 3, for any $\rho>0$, we can choose $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$, there exists $\delta>0$ such that for c large enough,

$$P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^0\left(\overline{B}(c[x]_c,\eta c/2)\right)\geq\delta m_{c^2}\}}<\varepsilon c^2,4\varepsilon c^2\leq T_{c[x]_c}<\infty\right]\leq\rho. \tag{27}$$

Furthermore, provided $2\varepsilon\leq T$, we have

$$P^*_{c^2T}\left[T_{c[x]_c}<4\varepsilon c^2\right]\leq\frac{p_{2\varepsilon c^2}}{p_{c^2T}}P^*_{2\varepsilon c^2}\left[T_{c[x]_c}<4\varepsilon c^2\right].$$

If c is sufficiently large, we can thus use (8) and the fact that $c|[x]_c| \geq c|x|/\sqrt{2}$, then Lemma 4 with $\alpha = 2\varepsilon c^2$ and $A = \frac{|x|}{2\sqrt{\varepsilon}}$ to get

$$\begin{aligned} P^*_{c^2T}\left[T_{c[x]_c} < 4\varepsilon c^2\right] &\leq \kappa_0 \frac{T}{2\varepsilon} P^*_{2\varepsilon c^2}\left(\sup_{t\leq 4\varepsilon c^2} \sup_{y\in\xi^0_t} |y| > \frac{c|x|}{\sqrt{2}} \right) \\ &\leq \kappa_0 K_1 \frac{T}{2\varepsilon} \exp\left(-\frac{K_2|x|}{2\sqrt{\varepsilon}}\right). \end{aligned}$$

Combining (23), (24), (27) and the last inequality now yields

$$\limsup_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \leq \mathbb{N}_0^{(T)}\left[Y \text{ hits } \overline{B}(x,\eta)\right] + \rho + \kappa_0 K_1 \frac{T}{2\varepsilon} \exp\left(-\frac{K_2|x|}{2\sqrt{\varepsilon}}\right),$$

for any $\rho > 0$ and $\varepsilon \in (0, \varepsilon_0(\eta,\rho)]$. By letting ε and then ρ go to 0, we get

$$\limsup_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \leq \mathbb{N}_0^{(T)}\left[Y \text{ hits } \overline{B}(x,\eta)\right]. \tag{28}$$

Our reasonning is valid for any $\eta \in (0, |x|/2)$. Thus, letting η go to 0 in (28) finishes the proof of Proposition 2. □

It remains to prove Lemma 4 and Lemma 3. We start with the proof of Lemma 4, since it will appear to be a key tool in the proof of Lemma 3.

3.2.1. *Proof of Lemma 4.* Let us first outline the proof and summarize the intermediate results. We need to discretize the time scale. Introduce the integer

$$N := \min\{n \in \mathbb{N} : \alpha 2^{-n} < 1\} = \left\lfloor \frac{\ln(\alpha)}{\ln(2)} \right\rfloor,$$

and the time intervals

$$B_n := [(n-1)2^{-N-1}\alpha, n2^{-N-1}\alpha], \quad n \in [\![1, 2^{N+2}]\!].$$

Let us introduce the set of points having, for some odd $n \in [\![1, 2^{N+2}]\!]$, opinion 1 at a time belonging to B_n, and descendants at time $(n+1)2^{-N-1}\alpha$:

$$\Xi_{N+1} := \bigcup_{\substack{n=1 \\ n \text{ odd}}}^{2^{N+1}} \bigcup_{u\in B_n} \Omega_u^{(n+1)2^{-N-1}\alpha}.$$

Informally, our interest in this set Ξ_{N+1} comes from the fact that if $x \in \bigcup_{t\leq 2\alpha} \xi^0_t$, a "close" ancestor of x belongs to Ξ_{N+1}, and hence, Ξ_{N+1} should not be too far from $\bigcup_{t\leq 2\alpha} \xi^0_t$. More precisely, for $t < 1$, set $u_t = 0$, and for $t \in [1, 2\alpha]$, let us choose

$$u_t \in \left[t - \frac{\alpha}{2^N}, t - \frac{\alpha}{2^{N+1}}\right] \bigcap \bigcup_{\substack{n=1 \\ n \text{ odd}}}^{2^{N+1}} B_n.$$

We have $t - u_t \leq \alpha 2^{-N} < 1$, and, if $x \in \xi^0_t$ for some $t \in [0, 2\alpha]$, the ancestor of x at time u_t indeed belongs to Ξ_{N+1}.

We will show that, under P_{α^*}, Ξ_{N+1} intersects $B\left(0, \frac{A}{2}\sqrt{\alpha}\right)$ with a probability which decays exponentially with A.

Lemma 5. *There exist positive constants K_3, K_4 such that for any $A > 0$, for any $\alpha > 1$,*

$$P^*_\alpha\left(\Xi_{N+1} \not\subseteq B(0, \frac{A}{2}\sqrt{\alpha})\right) \leq K_3 \exp(-K_4 A). \tag{29}$$

Then, we will argue that the probability under P^*_α for $\bigcup_{t\le 2\alpha}\xi^0_t$ to escape the ball $B(0,A\sqrt{\alpha})$ and simultaneously to have $\Xi_{N+1}\subset B(0,\frac{A}{2}\sqrt{\alpha})$ also decays exponentially with A. This is seen below as a consequence of the following result.

Lemma 6. *There exist positive constants K_5, K_6 such that for any $A>0$,*

$$P\left(\exists t\in[0,1]\ \exists x\in\mathbb{Z}^d\setminus B(0,A)\ \exists y\in B(0,\frac{A}{2}):(t,x)\searrow(0,y)\right)\le K_5\exp(-K_6A).$$

Let us postpone the proofs of Lemmas 5 and 6 and show how Lemma 4 is deduced from these two results. Introduce the event

$$\mathcal{A}:=\left\{\exists t\le 2\alpha\ \exists x\in\xi^0_t:|x|>A\sqrt{\alpha}\right\}\bigcap\left\{\Xi_{N+1}\subset B(0,\frac{A}{2}\sqrt{\alpha})\right\}.$$

Clearly, $\mathcal{A}=\bigcup_{n=1}^{2^{N+2}}\mathcal{A}_n$, where

$$\mathcal{A}_n:=\left\{\exists t\in B_n\ \exists x\in\xi^0_t\ :\ |x|>A\sqrt{\alpha}\right\}\bigcap\left\{\Xi_{N+1}\subset B(0,\frac{A}{2}\sqrt{\alpha})\right\}.$$

As we noticed earlier, when $x\in\xi^0_t$, the ancestor of x at time u_t belongs to Ξ_{N+1}. Hence, using the Markov property at time u_t, we get, for every $n\in[\![1,2^{N+2}]\!]$,

$$P(\mathcal{A}_n)\le P\bigg(\exists s\in\left[0,\frac{\alpha}{2^N}\right]\ \exists x\in\mathbb{Z}^d\setminus B(0,A\sqrt{\alpha})\ \exists y\in B(0,\frac{A}{2}\sqrt{\alpha})\ :\ (s,x)\searrow(0,y)\bigg),$$

where we used that $t-u_t\le\alpha 2^{-N}$. Using the fact that $\alpha 2^{-N}<1$, it then follows from Lemma 6 that

$$P(\mathcal{A})\le 2^{N+2}K_5\exp(-K_6A\sqrt{\alpha}).$$

Since $2^{N+2}\le 8\alpha$ from the definition of N, it follows from the above that $P^*_\alpha(\mathcal{A})\le 8K_5\alpha^2\exp(-K_6A\sqrt{\alpha})$. Hence, there exists positive constants K'_1,K'_2 such that $P^*_\alpha(\mathcal{A})\le K'_1\exp(-K'_2A)$. This fact and Lemma 5 imply Lemma 4. □

It now remains to prove Lemmas 5 and 6. We first establish Lemma 6.

Proof of Lemma 6 : Fix $x\in\mathbb{Z}^d\setminus B(0,A)$. There is a Poisson number n_x with parameter 1 of arrows pointing towards x during the time interval $[0,1]$. Denote by $1\ge T_1>T_2>...>T_{n_x}\ge 0$ the times at which these arrows occur and by $z_1,z_2,...,z_{n_x}$ the respective origins of these arrows. We also set $T_i=0$ when $i>n_x$. For $t\in[0,1]$ and $y\in B(0,A/2)$, a path up $(0,y)\nearrow(x,t)$ has to "follow" one of the n_x arrows pointing towards x in the time interval $[0,1]$, say the ith one at time T_i, in this case we then have $(z_i,T_i)\searrow(y,0)$.

For $t\in[0,1]$, let us define $\mathcal{G}_t$ the σ-field which is generated by the random sets $(\Lambda(x,y)\cap[1-t,1])$ for all $x,y\in\mathbb{Z}^d$.

The times $1-T_i, i\in\mathbb{N}$ are stopping times for the filtration $(\mathcal{G}_t)_{t\in[0,1]}$, and conditionally on $\{n_x=k\}$, the points $z_i, 1\le i\le k$ are located independently according to $p(x,.)$. In particular, using the exponential moments assumption on p, there exist positive $\tilde\kappa_3,\tilde\kappa_4$ such that for any $1\le i\le k$,

$$P\left(z_i\in B\left(0,\frac{3|x|}{4}\right)\Bigg|n_x=k\right)\le\tilde\kappa_3\exp(-\tilde\kappa_4|x|). \tag{30}$$

For $i\in[\![1,k]\!]$, let us define $(\tilde Z^{x,T_i}_s)_{0\le s\le T_i}$ as follows

- for $0<s\le T_i$, $\tilde Z^{x,T_i}_s=Z^{x,T_i}_s$
- $\tilde Z^{x,T_i}_0=z_i$.

For $t > 0$, conditionally on $\{T_i = t\}$, $(\tilde{Z}_s^{x,T_i})_{0\le s\le t}$ is a rate 1 random walk with jump kernel p started from z_i, and is thus distributed as $(Z_s)_{0\le s\le t}$ under $\mathbb{P}_{z_i}$. Furthermore, using (14), we have for any $z \in \mathbb{Z}^d \setminus B\left(0, \frac{3|x|}{4}\right)$

$$\mathbb{P}_z\left(\exists s \in [0,1] : Z_s \in B\left(0, \frac{|x|}{2}\right)\right) \le \kappa_3 \exp\left(-\kappa_4 \frac{|x|}{4}\right). \tag{31}$$

Combining (30) and (31), we see that there exist positive constants K_5', K_6' such that for any $x \in \mathbb{Z}^d \setminus B(0,A)$, for any $k \in \mathbb{N}$

$$P\left(\exists i \in [\![1,k]\!] : Z_{T_i}^{x,T_i} \in B\left(0, \frac{A}{2}\right) \;\middle|\; n_x = k\right) \le K_5' k \exp\left(-K_6'|x|\right).$$

We thus get

$$\begin{aligned}
&P\left(\exists t \in [0,1]\ \exists x \in \mathbb{Z}^d \setminus B(0,A)\ \exists y \in B\left(0,\frac{A}{2}\right)\ :\ (t,x) \searrow (0,y)\right)\\
&\le \sum_{x\in\mathbb{Z}^d\setminus B(0,A)} \sum_{k=0}^{\infty} P\left(n_x = k\ ,\ \exists i \in [\![1,k]\!]\ :\ Z_{T_i}^{x,T_i} \in B(0,\frac{A}{2})\right)\\
&\le \sum_{x\in\mathbb{Z}^d\setminus B(0,A)} \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{(k-1)!} K_5' \exp\left(-K_6'|x|\right).
\end{aligned}$$

Lemma 6 follows. □

To prove Lemma 5, we need the following key result.

Lemma 7. *Let $t \ge 0, s > r > 0$ and $A > A' \ge 0$.*

$$P\left(\bigcup_{u\in[0,t]} \Omega_u^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right) \le p_{s-r}\kappa_3 \exp\left(-\kappa_4 \frac{A-A'}{\sqrt{r}}\right).$$

Proof of Lemma 7: The event $\left\{\Omega_t^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]}\Omega_u^{t+s} \not\subseteq B(0,A)\right\}$ considered in Lemma 7 is contained in the event that there exists a point $z \in \xi_{t+r}^0$ having descendants at time $t+s$, such that the ancestor of z at time t belongs to $B(0,A')$, and moreover, z has an ancestor in $B(0,A)^c$ at a time belonging to $[t,t+r]$. More precisely, using duality over the time interval $[0,t+r]$, and then decomposing over all possible values of the point z,

$$\begin{aligned}
&P\left(\Omega_t^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right)\\
&\le P\left(\exists z \in \xi_{t+r}^0 : \hat{\xi}_{s-r}^{z,t+r} \ne \emptyset, \sup_{u\in[0,r]} |Z_u^{z,t+r}| > A, |Z_r^{z,t+r}| \le A', Z_{t+r}^{z,t+r} = 0\right)\\
&\le \sum_{z\in\mathbb{Z}^d} P\left(\hat{\xi}_{s-r}^{z,t+r} \ne \emptyset, \sup_{u\in[0,r]} |Z_u^{z,t+r}| > A, |Z_r^{z,t+r}| \le A', Z_{t+r}^{z,t+r} = 0\right).
\end{aligned} \tag{32}$$

Using the Markov property at time $(t+r)$, we obtain that the quantity in the right-hand side of (32) is equal to

$$p_{s-r}\sum_{z\in\mathbb{Z}^d}\mathbb{P}_z\left(\sup_{u\in[0,r]}|Z_u|>A,|Z_r|\le A',Z_{t+r}=0\right)$$
$$=p_{s-r}\sum_{z\in\mathbb{Z}^d}\mathbb{P}_0\left(|Z_r|\le A',\sup_{u\in[t,t+r]}|Z_u|>A,Z_{t+r}=z\right)$$
$$\le p_{s-r}\mathbb{P}_0\left(|Z_r|\le A',\sup_{u\in[t,t+r]}|Z_u|>A\right),\tag{33}$$

where, at the second line above, we used a time-reversal argument together with the symmetry assumption we made on the jump kernel p. From (32), (33) and the Markov property for the random walk at time t, we now obtain

$$P\left(\Omega_t^{t+s}\subset B(0,A'),\bigcup_{u\in[t,t+r]}\Omega_u^{t+s}\not\subseteq B(0,A)\right)$$
$$\le p_{s-r}\mathbb{P}_0(\sup_{u\in[0,r]}|Z_u|>A-A'),$$

and we conclude using (14). $\square$

Proof of Lemma 5: Let us first note that we only need to establish the existence of positive K_3',K_4' such that (29) holds for any $A\ge 1$ and $\alpha>1$. Indeed, Lemma 5 will follow from taking $K_3=K_3'\vee\exp(K_4'),K_4:=K_4'$. For $p\in[\![0,N+1]\!]$, let us introduce the sets

$$\Xi_p:=\bigcup_{\substack{n=1\\ n\text{ odd}}}^{2^{p+1}}\bigcup_{u\in[(n-1)2^{-p}\alpha,n2^{-p}\alpha]}\Omega_u^{(n+1)2^{-p}\alpha}.$$

For convenience, we also set $\Xi_{-1}:=\emptyset$. In the case $p=N+1$, this is of course consistent with our definition of Ξ_{N+1}. For $p\ge 0$, let $A_p:=\frac{A}{14}\sum_{i=0}^{p}2^{-i/4}$ and set $A_{-1}=0$. so that for any $k\in\mathbb{N}$, $A_k\le\frac{A}{2}$. For $p\in[\![0,N+1]\!]$, let

$$\mathcal{E}_p:=\left\{\Xi_{p-1}\subset B(0,A_{p-1}\sqrt{\alpha}),\Xi_p\not\subseteq B(0,A_p\sqrt{\alpha})\right\}.$$

Note that $\mathcal{E}_p$ is a subset of

$$\mathcal{F}_p:=\Bigg\{\exists n\in[\![1,2^{p+1}]\!],n\text{ odd }:\Omega_{(n-1)2^{-p}\alpha}^{(n+1)2^{-p}\alpha}\subset B(0,A_{p-1}\sqrt{\alpha}),$$
$$\bigcup_{u\in[(n-1)2^{-p}\alpha,n2^{-p}\alpha]}\Omega_u^{(n+1)2^{-p}\alpha}\not\subseteq B(0,A_p\sqrt{\alpha})\Bigg\}.$$

Hence,

$$P_\alpha^*\left(\Xi_{N+1}\not\subseteq B(0,\frac{A}{2}\sqrt{\alpha})\right)\le\sum_{p=0}^{N+1}P_\alpha^*\left(\mathcal{E}_p\right)\le\sum_{p=0}^{N+1}P_\alpha^*\left(\mathcal{F}_p\right)\tag{34}$$

From Lemma 7, we obtain

$$P(\mathcal{F}_p) \leq \sum_{\substack{n=1 \\ n \text{ odd}}}^{2^{p+1}} p_{2^{-p}\alpha} \exp\Big(-\kappa_4 2^{p/2}(A_p - A_{p-1})\Big).$$

Hence, using our definition of the numbers $A_p, p \geq -1$, then (8), we get

$$P^*_\alpha(\mathcal{F}_p) \leq p_\alpha^{-1} p_{2^{-p}\alpha} \exp\left(-\kappa_4 2^{p/4}\frac{A}{14}\right) \leq \kappa_0 2^{2p} \exp\left(-\kappa_4 2^{p/4}\frac{A}{14}\right).$$

From (34) and the last inequality, elementary arguments then give Lemma 5. □

This completes the proof of Lemma 4. To finish the one of Proposition 2, it remains to establish Lemma 3.

3.2.2. *Proof of Lemma 3.* Fix $T > 0, \rho > 0, \eta > 0$. Recall K_1, K_2 are the constants appearing in the statement of Lemma 4. We can choose $\varepsilon_0 \in (0,1)$ so that for any $\varepsilon \in (0, \varepsilon_0]$,

$$K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right) \leq \frac{\rho^2\varepsilon^2}{8T^2}.$$

Let us now fix $\varepsilon \in (0, \varepsilon_0]$. We can then choose $\delta > 0$ small enough so that

$$\mathbb{N}_0^{(4\varepsilon)}\left[\inf_{t\in[\varepsilon,3\varepsilon]} Y_t(1) \leq \delta\right] \leq \frac{\rho^2\varepsilon^2}{8T^2}.$$

The reasons for our choices of ϵ_0 and δ will become clear in the following.

We first need to reduce the problem to a finite time interval. Notice that $P^*_{c^2T}(\xi_{c^2\frac{4T}{\rho}} \neq \emptyset) \leq p^{-1}_{c^2T} p_{c^2\frac{4T}{\rho}}$ which, using (7), is bounded by $\rho/2$ for c large enough. Thus, to establish Lemma 3 we only need to prove that provided c is sufficiently large,

(35)

$$P^*_{c^2T}\left(\exists t \in [4\varepsilon c^2, \frac{4T}{\rho}c^2]\ \exists x \in \xi^0_t : \inf_{s\in[t-3\varepsilon c^2, t-2\varepsilon c^2]} |\xi^0_s \cap \overline{B}(x, \eta c)| < \delta m_{c^2}\right) \leq \frac{\rho}{2},$$

Set $M := \left\lceil \frac{4T}{\rho\varepsilon} \right\rceil$. Let us discretize the time scale via introducing the levels $\mathcal{L}_k := k\varepsilon c^2, k \in [\![0, M-4]\!]$.

We are going to establish, using Lemma 4, that with arbitrarily high probability, when c is large enough, each point holding opinion 1 at such a level $\mathcal{L}_k$ and having descendants at time $\mathcal{L}_k + 4\varepsilon c^2$ is close (at a distance less than $\eta c/2$) to all its descendants during the time interval $[\mathcal{L}_k, \mathcal{L}_k + 5\varepsilon c^2]$. Then, using Theorem 2, we will prove that such a point has more than δm_{c^2} descendants in the time interval $[\mathcal{L}_k + \varepsilon c^2, \mathcal{L}_k + 3\varepsilon c^2]$.

Let us be more precise. We shall prove that if c is large enough,

$$(36)\quad P^*_{c^2T}\left(\exists k \in [\![0, M-4]\!]\quad \exists y \in \Omega^{(k+4)\varepsilon c^2}_{k\varepsilon c^2} : \bigcup_{s\in[0,5\varepsilon c^2]} \hat{\xi}^{y,k\varepsilon c^2}_s \not\subset \overline{B}(y, \eta c/2)\right) \leq \frac{\rho}{4},$$

$$(37)\quad P^*_{c^2T}\left(\exists k \in [\![0, M-4]\!]\quad \exists y \in \Omega^{(k+4)\varepsilon c^2}_{k\varepsilon c^2} : \inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}^{y,k\varepsilon c^2}_s| < \delta m_{c^2}\right) \leq \frac{\rho}{4}.$$

Let us postpone the proof of these two results and show how (35) follows from (36) and (37). Consider $t \in [4\varepsilon c^2, M\varepsilon c^2]$ and $x \in \xi_t^0$. Introduce

$$k_t := \sup\{n \in \mathbb{N} : n\varepsilon c^2 \leq t - 4\varepsilon c^2\}; \quad z_x := Z^{x,t}_{t-k_t\varepsilon c^2},$$

so that $z_x \in \Omega^t_{k_t\varepsilon c^2} \subset \Omega^{(k_t+4)\varepsilon c^2}_{k_t\varepsilon c^2}$ (indeed z_x is the ancestor of $x \in \xi_t^0$ at time $k_t\varepsilon c^2$). Thus,

- using (36), with probability at least $1-\rho/4$, for any $x \in \xi_t^0$, $t \in [4\varepsilon c^2, M\varepsilon c^2]$, all descendants of z_x until time $(k_t+5)\varepsilon c^2$ belong to $\overline{B}(x, \eta c)$.
- using (37), with probability at least $1-\rho/4$, for any $x \in \xi_t^0$, $t \in [4\varepsilon c^2, M\varepsilon c^2]$, there are more than δm_c^2 descendants of z_x at every time $s \in [(k_t+1)\varepsilon c^2, (k_t+3)\varepsilon c^2]$.

Since $[t-3\varepsilon c^2, t-2\varepsilon c^2] \subset [(k_t+1)\varepsilon c^2, (k_t+3)\varepsilon c^2]$, we now deduce from the above that with probability at least $1-\rho/2$, for any $x \in \xi_t^0$, z_x has at least δm_c^2 descendants in $\overline{B}(x, \eta c)$ at every time $s \in [t-3\varepsilon c^2, t-2\varepsilon c^2]$. Assertion (35) follows.

Let us now prove (36). Let us consider $k \in [\![0, M-4]\!]$, and c large enough so that $4\varepsilon c^2 > 1$. Using the Markov property at time $k\varepsilon c^2$ and the fact that $(\hat{\xi}_s^{y,k\varepsilon c^2})_{s\geq 0}$ has the same law as $(y + \xi_s^0)_{s\geq 0}$ we get :

$$\begin{aligned}
&P\left(\exists y \in \Omega^{(k+4)\varepsilon c^2}_{k\varepsilon c^2} : \bigcup_{s\in[0,5\varepsilon c^2]} \hat{\xi}_s^{y,k\varepsilon c^2} \not\subset \overline{B}(y, \eta c/2)\right) \\
&\leq \sum_{y\in\mathbb{Z}^d} P(y \in \xi^0_{k\varepsilon c^2}) P\left(\xi^0_{4\varepsilon c^2} \neq \emptyset, \bigcup_{s\in[0,5\varepsilon c^2]} \xi_s^0 \not\subset \overline{B}(0, \eta c/2)\right) \\
&= \sum_{y\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) p_{4\varepsilon c^2} P^*_{4\varepsilon c^2}\left(\sup_{s\leq 5\varepsilon c^2} \sup_{x\in\xi_s^0} |x| > \frac{\eta c}{2}\right). \\
&\leq p_{4\varepsilon c^2} K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right), \qquad (38)
\end{aligned}$$

where at the last line we used Lemma 4 with $\alpha = 4\varepsilon c^2 > 1$ and $A = \eta(4\sqrt{\varepsilon})^{-1}$, and the fact that $\sum_{z\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) = 1$ from the symmetry assumption on p. Since $M - 3 \leq 4T(\varepsilon\rho)^{-1}$, we deduce from the above that

$$\begin{aligned}
&P^*_{c^2T}\left(\exists k \in [\![0, M-4]\!]\ \exists y \in \Omega^{(k+4)\varepsilon c^2}_{k\varepsilon c^2} : \bigcup_{s\in[0,5\varepsilon c^2]} \hat{\xi}_s^{y,k\varepsilon c^2} \not\subset \overline{B}(y, \eta c/2)\right) \\
&\leq p^{-1}_{c^2T} \frac{4T}{\varepsilon\rho} p_{4\varepsilon c^2} K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right). \qquad (39)
\end{aligned}$$

Provided c is sufficiently large, we then deduce (36) from (39), (7), and our choice of ε_0.

Let us now prove (37). Fix $k \in [\![0, M-4]\!]$. Using the same arguments as in the proof of (36), we obtain

$$\begin{aligned} & P\left(\exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}_s^{y,k\varepsilon c^2}| < \delta m_{c^2}\right) \\ & \leq \sum_{y\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) P\left(\inf_{s\in[\varepsilon c^2,3\varepsilon c^2]} |\xi_s^0| < \delta m_{c^2}, \xi^0_{4\varepsilon c^2} \neq \emptyset\right) \\ & = p_{4\varepsilon c^2} P^*_{4\varepsilon c^2}\left(\inf_{s\in[\varepsilon c^2,3\varepsilon c^2]} |\xi_s^0| < \delta m_{c^2}\right). \end{aligned} \tag{40}$$

Furthermore, by rescaling, when c is large enough so that $(m_{c^2})^{-1} m_{\varepsilon c^2} \leq 1$ (recall $\varepsilon < 1$ from our choice of ε_0), we get

$$\begin{aligned} P^*_{4\varepsilon c^2}\left(\inf_{s\in[\varepsilon c^2,3\varepsilon c^2]} |\xi_s^0| < \delta m_{\varepsilon c^2}\right) &= P^*_{4\varepsilon c^2}\left(\inf_{s\in[\varepsilon,3\varepsilon]} X_s^{c^2,0}(1) < \delta \frac{m_{\varepsilon c^2}}{m_{c^2}}\right) \\ &\leq P^*_{4\varepsilon c^2}\left(\inf_{s\in[\varepsilon,3\varepsilon]} X_s^{c^2,0}(1) \leq \delta\right). \end{aligned} \tag{41}$$

Since the set $\{\omega \in D(\mathbb{R}_+, M_F(\mathbb{R}^d)) : \inf_{s\in[\varepsilon,3\varepsilon]} \omega_s(1) \leq \delta\}$ is closed for the Skorohod J_1 topology, Theorem 2 implies

$$\limsup_{c\to\infty} P^*_{4\varepsilon c^2}\left(\inf_{s\in[\varepsilon,3\varepsilon]} X_s^{c^2,0}(1) \leq \delta\right) \leq \mathbb{N}_0^{(4\varepsilon)}\left[\inf_{t\in[\varepsilon,3\varepsilon]} Y_t(1) \leq \delta\right] \leq \frac{\varepsilon^2\rho^2}{8T^2},$$

by our choice of δ. Assertions (40), (41), and the above now imply

$$\begin{aligned} & \limsup_{c\to\infty} P^*_{c^2T}\left(\exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \inf_{s\in[\varepsilon c^2,3\varepsilon c^2]} |\hat{\xi}_s^{y,k\varepsilon c^2}| < \delta m_{c^2}\right) \\ & \leq \limsup_{c\to\infty} p^{-1}_{c^2T} p_{4\varepsilon c^2} \frac{\varepsilon^2\rho^2}{8T^2} = \frac{\varepsilon\rho^2}{32T}, \end{aligned}$$

where we used (7) at the last line. Hence, using the fact that $M-3 \leq 4T(\rho\varepsilon)^{-1}$, we get (37), provided c is sufficiently large. This ends the proof of Lemma 3. □

We have thus finished the proof of the asymptotic upper bound on $P^*_{c^2T}(T_{c[x]_c} < \infty)$ (Proposition 2). However, to complete the proof of the asymptotic upper bound for $d = 2$ or 3 in Theorem 1, we need to establish a corresponding result under the measure P. Let us briefly explain how Lemma 4 allows us to do so.

3.3. **Back to non-conditioned results.** First, we shall prove a result corresponding to Lemma 4 without conditioning upon survival.

Claim 1. - There exists a positive K_0 such that for any $\alpha > 1$, for any $A \geq 1$,

$$P\left(\sup_{t\leq 2\alpha} \sup_{x\in\xi_t^0} |x| > A\sqrt{\alpha}\right) \leq K_0 p_\alpha \exp(-K_2 A).$$

Proof of Claim 1: For any $i \in [\![0, N-1]\!]$ we have

$$P\left(\sup_{t\le 2^{1-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)$$
$$= p_{2^{-i}\alpha}P^*_{2^{-i}\alpha}\left(\sup_{t\le 2^{1-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)+P\left(\sup_{t\le 2^{-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha},\xi^0_{2^{-i}\alpha}=\emptyset\right)$$
$$\le p_{2^{-i}\alpha}K_1\exp\left(-K_2A2^{i/2}\right)+P\left(\sup_{t\le 2^{-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha},\xi^0_{2^{-i}\alpha}=\emptyset\right),$$

where we used Lemma 4 at the last line. It easily follows that

$$P\left(\sup_{t\le 2\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)$$
$$\le \sum_{i=0}^{N-1} p_{2^{-i}\alpha}K_1\exp\left(-K_2A2^{i/2}\right)+P\left(\sup_{t\le 2^{1-N}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right). \tag{42}$$

Furthermore, by an easy application of Lemma 6,

$$P\left(\sup_{t\le 2^{1-N}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\le 2K_5\exp\left(-K_6A\sqrt{\alpha}/2\right).$$

Thus, from (42) and (8), we obtain

$$P\left(\sup_{t\le 2\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\le\kappa_0K_1\sum_{i=0}^{N-1}2^ip_\alpha\exp\left(-K_2A2^{i/2}\right)+2K_5\exp(-K_6A\sqrt{\alpha}/2),$$

and Claim 1 follows. □

Let us now finish the proof of the upper bound in Theorem 1 in dimensions 2 and 3. As before, $x\in\mathbb{R}^d\setminus 0$ is fixed. Simply observe that, for every $T>0$,

$$P(T_{c[x]_c}<\infty)=P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty)+p_{c^2T}P^*_{c^2T}(T_{c[x]_c}<\infty). \tag{43}$$

On the one hand

$$\begin{aligned}\phi_d(c)P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty) &\le \phi_d(c)P\left(\sup_{t\le c^2T}\sup_{y\in\xi^0_t}|y|\ge c|[x]_c|\right)\\ &\le \phi_d(c)K_0p_{c^2T}\exp\left(-K_2\frac{|[x]_c|}{\sqrt{T}}\right),\end{aligned}$$

where we used Claim 1 at the last line. We can now use (7) to obtain

$$\limsup_{c\to\infty}\phi_d(c)P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty)\le\frac{K_0}{\beta_dT}\exp\left(-K_2\frac{|x|}{2\sqrt{T}}\right),$$

which goes to 0 as $T\to 0$.

On the other hand, using Proposition 2 and (7), we get, for every $T>0$,

$$\limsup_{c\to\infty}\phi_d(c)p_{c^2T}P^*_{c^2T}(T_{c[x]_c}<\infty)\le\frac{1}{T\beta_d}\mathbb{N}^{(T)}_0(x\in\mathcal{R}),$$

and by (6), the right-hand side converges, as $T\to 0$, to

$$\frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}.$$

From (43) and the preceeding observations we get

$$\limsup_{c\to\infty} \phi_d(c)P(T_{c[x]_c} < \infty) \le \frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}, \tag{44}$$

which completes the proof of the upper bound in Theorem 1, in the case $d = 2$ or 3. We already noticed that the case $d \ge 5$ follows from Proposition 1. Finally, note that in the case $d = 4$, Proposition 2 and a similar proof imply

$$\limsup_{c\to\infty} c^2 P(T_{c[x]_c} < \infty) = 0,$$

as we already mentioned in the introduction.

4. Lower bound.

In this section we finish the proof of Theorem 1 by establishing the required asymptotic lower bounds on $P(T_{c[x]_c} < \infty)$. We also prove a similar result in dimension 4.

Proposition 3. *Fix $x \in \mathbb{R}^d$, $x \ne 0$.*

Rough lower bound : *Let $d \ge 4$. There exists a positive constant a_d depending on $|x|$ and d such that*

$$\liminf_{c\to\infty} \phi_d(c)P(\exists t \ge 0 : c[x]_c \in \xi^0_t) \ge a_d,$$

where we recall that for $d \ge 5$, $\phi_d(c) = c^{d-2}$, and $\phi_4(c) = c^2 \ln(c)$.

Sharp lower bound : *Let $d = 2$ or 3. Recall $\phi_2(c) = c^2(\ln(c))^{-1}$ and $\phi_3(c) = c^2$. Then*

$$\liminf_{c\to\infty} \phi_d(c)P(\exists t \ge 0 : c[x]_c \in \xi^0_t) \ge \frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}.$$

4.1. **Proof of the rough lower bound,** $d \ge 4$**.** For $T > 0$ let us introduce the random variable

$$U_T := \int_0^T \mathbf{1}_{\{c[x]_c \in \xi^0_{c^2 s}\}} ds,$$

so that $c^2 U_T$ is the occupation time of opinion 1 for the voter at $c[x]_c$ in the time interval $[0, c^2 T]$.

We clearly have for any $T > 0$, $P(\exists t \ge 0 : c[x]_c \in \xi^0_t) \ge P(U_T > 0)$. Using the Cauchy-Schwarz inequality, we thus obtain

$$P(\exists t \ge 0 : c[x]_c \in \xi^0_t) \ge \frac{(E[U_T])^2}{E[(U_T)^2]}. \tag{45}$$

Hence, proving the lower bound reduces to establishing the following two estimates

$$c^d E[U_T] \underset{c\to\infty}{\longrightarrow} \int_0^T ds p_s(x), \tag{46}$$

$$\limsup_{c\to\infty} c^{2d}\phi_d(c)^{-1}E[(U_T)^2] \le L, \tag{47}$$

where L is a constant depending only on $|x|, d$ and T.

The first moment of U_T is

$$E[U_T] = \int_0^T ds P\left(Z^{c[x]_c, c^2 s}_{c^2 s} = 0\right) = \int_0^T ds q_{c^2 s}(c[x]_c),$$

so that (46) is a consequence of (17). Let us now estimate the second moment of U_T. We have

$$\frac{1}{2}E[(U_T)^2] = \int_0^T dt \int_t^T dr P\left[c[x]_c \in \xi^0_{c^2t}, c[x]_c \in \xi^0_{c^2r}\right].$$

Let us fix r and t with $0 < t < r \leq T$. Using duality over the time interval $[0, c^2r]$ and setting $s := r - t$, we see that $P\left[c[x]_c \in \xi^0_{c^2t}, c[x]_c \in \xi^0_{c^2r}\right]$ is the probability for two coalescing random walks starting at point $c[x]_c$ respectively at times 0 and c^2s, to be both located at point 0 at time c^2r. Using the symmetry properties of p and the Markov property for the first walk at time c^2s, we get

$$\frac{1}{2}E[(U_T)^2] = \int_0^T dt \int_0^{T-t} ds \sum_{y\in\mathbb{Z}^d} q_{c^2s}(y)\mathbb{P}_{0,y}\left[T_1 \leq c^2t, Z^1_{c^2t} = c[x]_c\right], \tag{48}$$

recalling that the notation $\mathbb{P}_{0,y}$ was introduced in Section 2.5.

With a slight abuse, in the remaining part of the section we use L to denote a positive constant that only depends on T, d and $|x|$ and may change from line to line. We suppose that $c \geq 1 \vee |x|^{-2}$ in order to use Lemma 2. Let us set

$$I(y) := \int_0^T dt \left(\int_0^{T-t} ds\ q_{c^2s}(y)\right)\mathbb{P}_{0,y}\left[T_1 \leq c^2t, Z^1_{c^2t} = c[x]_c\right].$$

Note that $I(y)$ also depends on d, T and x, although this does not appear in our notation. From (48), we have $E[(U_T)^2] = 2\sum_{y\in\mathbb{Z}^d} I(y)$. Hence, we need to bound $I(y)$ over different regions of $\mathbb{Z}^d$, in order to control $c^{2d}(\phi_d(c))^{-1}E[(U_T)^2]$. Recall from Section 2.4 that $\psi_d(c) = c^{d-2}$ for $d \geq 4$. Using (18) twice in the case $y = 0$, and using (18) together with Lemma 2 in the case $y \neq 0$, we get

- for $y = 0$,
$$I(0) \leq Lc^{-2d}\psi_d(c),$$
- for $y \in \mathbb{Z}^d, y \neq 0$,
$$I(y) \leq Lc^{-2d}\psi_d(c/|y|)\psi_d(|y|)^{-1},$$
- for $y \in \mathbb{Z}^d, |y| > c^2$,
$$I(y) \leq Lc^{-d-2}\exp(-L_2'\sqrt{|y|})\psi_d(|y|)^{-1}.$$

Since $\phi_d(c) \geq \psi_d(c)$ when $d \geq 4$, we then obtain

$$c^{2d}(\phi_d(c))^{-1}I(0) \leq L. \tag{49}$$

Furthermore, if $y \in \mathbb{Z}^d \setminus 0, d \geq 4$, we have $\psi_d(c/|y|) = \psi_d(c)(\psi_d(|y|))^{-1}$. Hence,

$$\begin{aligned} c^{2d}(\phi_d(c))^{-1} \sum_{y\in\mathbb{Z}^d, 0<|y|\leq c^2} I(y) &\leq L(\phi_d(c))^{-1}\psi_d(c) \sum_{y\in\mathbb{Z}^d,0<|y|\leq c^2} |y|^{4-2d} \\ &\leq L(\phi_d(c))^{-1}\psi_d(c)\sum_{k=1}^{c^2} k^{d-1}k^{4-2d} \leq L, \end{aligned} \tag{50}$$

Finally,

$$c^{2d}(\phi_d(c))^{-1}\sum_{y\in\mathbb{Z}^d,|y|>c^2} I(y) \leq Lc^{d-2}(\phi_d(c))^{-1}\sum_{|y|>c^2} |y|^{-d+2}\exp(-L_2'\sqrt{|y|}) \underset{c\to\infty}{\longrightarrow} 0. \tag{51}$$

Since $E[(U_T)^2] = 2\sum_{y\in\mathbb{Z}^d} I(y)$, the desired result (47) follows from (49), (50) and (51). This finishes the proof of the lower bound for $d \geq 4$. □

A similar proof in the case $d = 2$ or 3 would give us a rough lower bound, but we need to get sharper estimates.

4.2. **Outline of the proof of the sharp lower bound,** $d = 2$ **or** 3.
Fix $x \in \mathbb{R}^d \setminus \{0\}$. For any $T > 0$, $c > 0$, we have

$$\phi_d(c)P(T_{c[x]_c} < \infty) \geq \phi_d(c)p_{c^2T}P^*_{c^2T}(T_{c[x]_c} < \infty).$$

We deduce from (7) that for any $T > 0$, $\lim_{c\to\infty} \phi_d(c)p_{c^2T} = (\beta_d T)^{-1}$ so that

$$\liminf_{c\to\infty} \phi_d(c)P(T_{c[x]_c} < \infty) \geq (\beta_d T)^{-1} \liminf_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty). \tag{52}$$

Claim 2. - For any $\rho > 0$, if $T > 0$ is sufficiently small,

$$\liminf_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \geq (1-\rho)\mathbb{N}_0^{(T)}(x \in \mathcal{R}).$$

The desired lower bound follows from (52), the above claim and (6) by letting T go to 0. Let us now outline the proof of Claim 2.

For $z \in \mathbb{R}^d$, $\varepsilon' > \varepsilon > 0$, let

$$\mathcal{C}(z,\varepsilon,\varepsilon') = \left\{y \in \mathbb{R}^d : \varepsilon < |y - z| < \varepsilon'\right\}, \qquad h(\varepsilon) := \varepsilon^2 \ln(\ln(\varepsilon^{-1})).$$

We also set for $r \in (0,1)$

$$g_d(r) = \begin{cases} 2^{-\left(\frac{\ln(r)}{\ln(2)}\right)^4} & \text{if } d = 2, \\ r^{16} & \text{if } d = 3. \end{cases}$$

For $\alpha > 0, T > 0$ and $\varepsilon_0 \in (0,1)$ we consider the events

$$\mathcal{E}^{(c)}_{\varepsilon_0} = \left\{\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : X^0_s\left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \geq \alpha h(\varepsilon)\phi_d(c)\right\},$$

$$\mathcal{F}^{(c)}_{\varepsilon_0} = \left\{\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : X^0_s\left(\mathcal{C}\left(cx, \frac{c\varepsilon}{2}, c\varepsilon\right)\right) > \alpha h(\varepsilon)\phi_d(c)\right\}.$$

For c large enough, we have $\mathcal{F}^{(c)}_{\varepsilon_0} \subset \mathcal{E}^{(c)}_{\varepsilon_0}$, hence

$$P^*_{c^2T}(T_{c[x]_c} < \infty) \geq P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) \times P^*_{c^2T}\left(T_{c[x]_c} < \infty \middle| \mathcal{E}^{(c)}_{\varepsilon_0}\right). \tag{53}$$

The idea of the proof of Claim 2 is the following. Rescaling and using Theorem 2, we will show that for c large, the first term of the product in (53), namely $P^*_{c^2T}(\mathcal{F}^{(c)}_{\varepsilon_0})$, is bounded below by a corresponding rescaled quantity under $\mathbb{N}_0^{(T)}$. For α small enough, this quantity will then be bounded from below by a quantity arbitrarily close to $\mathbb{N}_0^{(T)}(x \in \mathcal{R})$ (see assertions (55) and (56) below). To finish the proof of Claim 2 we shall then establish that if we take T, ε_0 small enough, the second term of the product in (53), namely $P^*_{c^2T}\left(T_{c[x]_c} < \infty \middle| \mathcal{E}^{(c)}_{\varepsilon_0}\right)$, is, for c large, arbitrarily close to 1 (see Lemma 9 below).

Let us reformulate the preceeding discussion in more precise terms. Using rescaling we have

$$P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) = P^*_{c^2T}\left(\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : X^{c^2,0}_s\left(\mathcal{C}(x, \frac{\varepsilon}{2}, \varepsilon)\right) > \alpha h(\varepsilon)\right). \tag{54}$$

It is easy to see that the set

$$\left\{\omega \in D(\mathbb{R}_+, M_F(\mathbb{R}^d)) : \exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0),\ \omega_s\left(\mathcal{C}(x, \frac{\varepsilon}{2}, \varepsilon)\right) > \alpha h(\varepsilon)\right\}$$

is open for the Skorohod J_1 topology. Theorem 2 thus implies that

$$(55)\quad \liminf_{c\to\infty} P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) \geq \mathbb{N}_0^{(T)}(\exists s\geq 0\ \exists\varepsilon\in(g_d(\varepsilon_0),\varepsilon_0): Y_s(C(x,\frac{\varepsilon}{2},\varepsilon))>\alpha h(\varepsilon)).$$

Lemma 8. *Let $d=2$ or 3. We can choose $\alpha>0$ so that, for any $\delta>0$, there exists $\varepsilon_1\in\left(0,1\wedge\frac{|x|}{2}\right)$ such that for any $\varepsilon_0\in(0,\varepsilon_1)$,*

$$\mathbb{N}_0\left[\exists s\geq 0\ \exists\ \varepsilon\in(g_d(\varepsilon_0),\varepsilon_0): Y_s\left(\mathcal{C}\left(x,\frac{\varepsilon}{2},\varepsilon\right)\right)>\alpha h(\varepsilon)\ \middle|\ x\in\mathcal{R}\right]\geq 1-\delta.$$

In the following, we fix α as in Lemma 8. The following lemma estimates the second term of the product in the right-hand side of (53).

Lemma 9. *For any fixed $\gamma>0$, there exists $\varepsilon_2\in(0,1)$ such that for any $\varepsilon_0\in(0,\varepsilon_2)$, we have*

(a) $\displaystyle\liminf_{c\to\infty} P\left(T_{c[x]_c}<\infty\ \middle|\ \mathcal{E}^{(c)}_{\varepsilon_0}\right)\geq 1-\gamma,$

(b) $\displaystyle\liminf_{T\to 0}\left(\liminf_{c\to\infty} P^*_{c^2T}\left(T_{c[x]_c}<\infty\middle|\mathcal{E}^{(c)}_{\varepsilon_0}\right)\right)\geq 1-\gamma.$

Let us now fix $\delta\in(0,1)$, $\gamma>0$ and let ε_1 and ε_2 be as in Lemma 8 and 9 respectively. Since

$$\mathbb{N}_0\left(\left\{\sup\{|y|:y\in\mathcal{R}\}>\frac{|x|}{2}\right\}\cap\left\{Y_T=0\right\}\right)\underset{T\to 0}{\longrightarrow}0,$$

we deduce from Lemma 8 that for any $\varepsilon_0\in(0,\varepsilon_1)$, for $T>0$ sufficiently small,

$$(56)\quad \mathbb{N}_0^{(T)}\left[\exists s\geq 0\ \exists\varepsilon\in(g_d(\varepsilon_0),\varepsilon_0): Y_s\left(C\left(x,\frac{\varepsilon}{2},\varepsilon\right)\right)>\alpha h(\varepsilon)\right]\geq(1-2\delta)\mathbb{N}_0^{(T)}(x\in\mathcal{R}).$$

From (55) and (56), we have for T sufficiently small

$$(57)\qquad \liminf_{c\to\infty} P^*_{c^2T}(\mathcal{F}^{(c)}_{\varepsilon_0})\geq(1-2\delta)\mathbb{N}_0^{(T)}(x\in\mathcal{R}).$$

Now use (53) and Lemma 9 (b) to get for T small,

$$\liminf_{c\to\infty} P^*_{c^2T}(T_{c[x]_c}<\infty)\geq(1-2\delta)(1-2\gamma)\mathbb{N}_0^{(T)}(x\in\mathcal{R}),$$

which gives Claim 2, hence Proposition 3.

To complete our proof of the lower bound Proposition 3, we still need to establish Lemma 8 and Lemma 9. Establishing that part (b) of Lemma 9 follows from part (a) requires a result which is a consequence of Lemma 8. However, we first give the proof of Lemma 9 (Section 4.3 below), because it is more closely related to our results. We then provide a proof of Lemma 8 in Section 4.4. In these two sections, we will assume for simplicity that $\sigma=1$. Adapting the proofs to a general σ is easy.

4.3. **Proof of Lemma 9.** We assume in this section that Lemma 8 has been proved, and in particular that (57) holds. Let us first explain how to derive part

(b) from part (a). We have

$$P^*_{c^2T}\left(T_{c[x]_c}<\infty \;\middle|\; \mathcal{E}^{(c)}_{\varepsilon_0}\right)$$
$$= \frac{P\left(\{T_{c[x]_c}<\infty\}\cap\mathcal{E}^{(c)}_{\varepsilon_0}\right)}{P\left(\mathcal{E}^{(c)}_{\varepsilon_0}\cap\{\xi^0_{c^2T}\neq\emptyset\}\right)} - \frac{P\left(\{T_{c[x]_c}<\infty\}\cap\mathcal{E}^{(c)}_{\varepsilon_0}\cap\{\xi_{c^2T}=\emptyset\}\right)}{P\left(\mathcal{E}^{(c)}_{\varepsilon_0}\cap\{\xi^0_{c^2T}\neq\emptyset\}\right)}$$

$$(58)\qquad \geq P\left(T_{c[x]_c}<\infty \;\middle|\; \mathcal{E}^{(c)}_{\varepsilon_0}\right) - \frac{P\left(\{T_{c[x]_c}<\infty\}\cap\{\xi_{c^2T}=\emptyset\}\right)}{p_{c^2T}P^*_{c^2T}(\mathcal{E}^{(c)}_{\varepsilon_0})}.$$

Take $\delta = 1/2$ in Lemma 8, and choose ε_1 so that the conclusion of this lemma holds. From the fact that $\mathcal{F}^{(c)}_{\varepsilon_0}\subset\mathcal{E}^{(c)}_{\varepsilon_0}$, and then from (57), we get that for $\varepsilon_0\in(0,\varepsilon_1)$, for $T>0$ small,

$$(59)\qquad \liminf_{c\to\infty} T^{-1}P^*_{c^2T}(\mathcal{E}^{(c)}_{\varepsilon_0}) \geq \frac{1}{2T}\mathbb{N}^{(T)}_0(x\in\mathcal{R}) \geq \frac{1}{2\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2},$$

using (6). Since $|[x]_c|>|x|/2$ for c large enough, we have

$$P\left(\{T_{c[x]_c}<\infty\}\cap\{\xi^0_{c^2T}=\emptyset\}\right) \leq P\left(\sup_{t\leq c^2T}\sup_{y\in\xi^0_t}|y|>c|x|/2\right).$$

For T small enough so that $\frac{|x|}{\sqrt{2T}}\geq 1$, and c large enough so that $\frac{c^2T}{2}>1$, we can use Claim 1 to deduce that

$$P\left(\{T_{c[x]_c}<\infty\}\cap\{\xi_{c^2T}=\emptyset\}\right)\leq p_{c^2T}K_0\exp\left(-K_2\frac{|x|}{\sqrt{2T}}\right).$$

Combining (59) and this last inequality, we obtain that for any $\varepsilon_0\in(0,\varepsilon_1)$

$$\limsup_{T\to 0}\limsup_{c\to\infty}\frac{P\left(\{T_{c[x]_c}<\infty\}\cap\{\xi_{c^2T}=\emptyset\}\right)}{p_{c^2T}P^*_{c^2T}\left(\mathcal{E}^{(c)}_{\varepsilon_0}\right)}=0.$$

It is now clear from (58) and the above that part (b) of Lemma 9 follows from part (a). $\square$

Proving part (a) of Lemma 9 requires the following intermediate result. Recall that for $A\subset\mathbb{Z}^d$ the voter model ξ starts from $\xi_0=A$ under P_A.

Lemma 10. *For any $\gamma>0$, there exists $M>0$ and $U>0$ such that for every $u\geq U$ and every subset A of $\mathbb{Z}^d\cap\mathcal{C}(0,\frac{u}{4},2u)$ with $|A|\geq M\phi_d(u)$, one has*

$$P_A(\exists t\geq 0: 0\in\xi_t)\geq 1-\gamma.$$

Let us postpone the proof of Lemma 10 and proceed to the proof of Lemma 9 (a).

Let us fix $\gamma>0$. Let us choose $M>0$ and $U>0$ such that the conclusion of Lemma 10 holds. We can then choose $\varepsilon_2>0$ small enough so that

$$\frac{\alpha}{2}\ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right)\geq M.$$

Let us fix $\varepsilon_0\in(0,\varepsilon_2)$. Let $c>0$ be large enough so that

$$cg_d(\varepsilon_0)\geq U,\qquad\text{and}\qquad 2\ln(cg_d(\varepsilon_0))\geq\ln(c).$$

We then set

$$T^{(c)}_{\varepsilon_0} := \inf\Big\{t \geq 0 : X^0_t \left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \geq \alpha h(\varepsilon)\phi_d(c) \text{ for some } \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0)\Big\}.$$

Clearly $T^{(c)}_{\varepsilon_0}$ is a stopping time of the filtration generated by the voter model, and $\mathcal{E}^{(c)}_{\varepsilon_0} = \{T^{(c)}_{\varepsilon_0} < \infty\}$.

From the definition of $T^{(c)}_{\varepsilon_0}$, on the event $\mathcal{E}^{(c)}_{\varepsilon_0}$ we can choose a random $\varepsilon \in (g_d(\varepsilon_0), \varepsilon_0)$ such that

$$X^0_{T^{(c)}_{\varepsilon_0}} \left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \geq \alpha h(\varepsilon)\phi_d(c).$$

On the event $\mathcal{E}^{(c)}_{\varepsilon_0}$, we can consider the set

$$\mathcal{A}^{(c)}_\varepsilon := \left(\xi^0_{T^{(c)}_{\varepsilon_0}} \cap \mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) - c[x]_c,$$

where, for $A \subset \mathbb{Z}^d$, $z \in \mathbb{Z}^d$, $A - z = \{y \in \mathbb{Z}^d : y + z \in A\}$. The random set $\mathcal{A}^{(c)}_\varepsilon$ is a subset of $\mathbb{Z}^d \cap \mathcal{C}(0, \frac{c\varepsilon}{4}, 2c\varepsilon)$, and has cardinality $|\mathcal{A}^{(c)}_\varepsilon| = \lceil \alpha h(\varepsilon)\phi_d(c)\rceil$.

Let us argue on $\mathcal{E}^{(c)}_{\varepsilon_0}$ and set $u(\varepsilon, c) = c\varepsilon$. Note that $u(\varepsilon, c) \geq c g_d(\varepsilon_0) \geq U$. When $d = 3$, we have

$$|\mathcal{A}^{(c)}_\varepsilon| \geq \alpha h(\varepsilon)c^2 \geq \alpha \ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right)\varepsilon^2 c^2 \geq M\phi_d(u(\varepsilon, c)).$$

When $d = 2$, noticing that $2\ln(c\varepsilon) \geq 2\ln(c g_d(\varepsilon_0)) \geq \ln(c)$, we also have

$$|\mathcal{A}^{(c)}_\varepsilon| \geq \alpha h(\varepsilon)\frac{c^2}{\ln(c)} \geq \alpha \ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right)\frac{c^2\varepsilon^2}{2\ln(c\varepsilon)} \geq M\phi_d(u(\varepsilon, c)).$$

From Lemma 10, we deduce that, on the event $\mathcal{E}^{(c)}_{\varepsilon_0}$,

$$P_{\mathcal{A}^{(c)}_\varepsilon}(\exists t \geq 0 : 0 \in \xi_t) \geq 1 - \gamma.$$

Using the strong Markov property for ξ^0 at time $T^{(c)}_{\varepsilon_0}$, then the fact that $\mathcal{A}^{(c)}_\varepsilon + c[x]_c \subset \xi^0_{T^{(c)}_{\varepsilon_0}}$, we obtain

$$\begin{aligned} P\left(\mathcal{E}^{(c)}_{\varepsilon_0} \cap \{T_{c[x]_c} < \infty\}\right) &\geq E\left(\mathbf{1}_{\{T^{(c)}_{\varepsilon_0} < \infty\}} P_{\xi^0_{T^{(c)}_{\varepsilon_0}}}(\exists t \geq 0 : c[x]_c \in \xi_t)\right) \\ &\geq E\left(\mathbf{1}_{\{T^{(c)}_{\varepsilon_0} < \infty\}} P_{\mathcal{A}^{(c)}_\varepsilon}(\exists t \geq 0 : 0 \in \xi_t)\right) \\ &\geq (1-\gamma)P(\mathcal{E}^{(c)}_{\varepsilon_0}), \end{aligned}$$

which gives part (a) of Lemma 9. □

Let us now fix $\gamma > 0$ and establish Lemma 10. First, notice that the function $A \to P_A(\exists t \geq 0 : 0 \in \xi_t)$ is increasing. It thus suffices to find $M > 0$ and $U > 0$ such that for $u \geq U$,

$$\inf\Big\{P_A(\exists t \geq 0 : 0 \in \xi_t) : A \subset \mathbb{Z}^d \cap \mathcal{C}\left(0, \frac{u}{4}, 2u\right), |A| = \lceil M\phi_d(u)\rceil\Big\} \geq 1 - \gamma. \tag{60}$$

For $M > 0, u > 0$ let us introduce

$$\mathfrak{A}^{(M)}_u := \Big\{A \subset \mathbb{Z}^d \cap \mathcal{C}\left(0, \frac{u}{4}, 2u\right) \ : \ |A| = \lceil M\phi_d(u)\rceil\Big\}.$$

We then use a similar method as for establishing the rough lower bound. Let us set $V_T = \int_0^T \mathbf{1}_{\{0\in\xi_{u^2t}\}}dt$. As in Section 4.1, we use the Cauchy-Schwarz inequality to get for any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$

$$P_A(\exists t \geq 0 : 0 \in \xi_t) \geq \frac{(E_A[V_T])^2}{E_A[(V_T)^2]}. \tag{61}$$

We will verify that for any fixed $M > 0$, there exists a constant $U(M)$ such that if $u \geq U$, then for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[V_T] \geq \frac{M}{2}(\psi_d(u))^{-1}\int_{T/2}^T \frac{1}{(2\pi s)^{d/2}}\exp\left(-\frac{2}{s}\right)ds, \tag{62}$$

$$E_A[(V_T)^2] \leq (E_A[V_T])^2 + L'M\,(\psi_d(u))^{-2}, \tag{63}$$

where L' is a constant depending only on d and T. Let us postpone the proof of these two assertions and finish the proof of Lemma 10. We can choose $M > 0$ sufficiently large so that

$$L'M \leq \gamma\frac{M^2}{4}\left(\int_{T/2}^T ds\frac{1}{(2\pi s)^{d/2}}\exp\left(-\frac{2}{s}\right)\right)^2.$$

From (62) and (63), we then deduce that for $u \geq U(M)$, for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[(V_T)^2] \leq (1+\gamma)\,(E_A[V_T])^2.$$

Lemma 10 now follows from (61).

Proof of (62): Let us now fix $M > 0$ and establish that (62) is valid for u sufficiently large, and for any $A \in \mathfrak{A}_u^{(M)}$. Note that for any $A \in \mathfrak{A}_u^{(M)}$, $u^{-1}A$ is a subset of $u^{-1}\mathbb{Z}^d \cap \mathcal{C}(0,1/4,2)$ and has cardinality $\lceil M\phi_d(u)\rceil$. For any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$, duality gives

$$E_A[V_T] = \int_0^T \mathbb{P}_0(Z_{u^2t}\in A)dt = \sum_{y\in u^{-1}A}\int_0^T q_{u^2t}(uy)dt \geq \sum_{y\in u^{-1}A}\int_{T/2}^T q_{u^2t}(uy)dt. \tag{64}$$

It is easy to deduce from Theorem 3 that uniformly in $t \in [T/2, T]$,

$$\lim_{u\to\infty}\sup_{y\in u^{-1}\mathbb{Z}^d}|u^d q_{u^2t}(uy) - p_t(y)| = 0.$$

Thus, if u is sufficiently large, for any $y \in u^{-1}\mathbb{Z}^d$ with $|y| \leq 2$,

$$u^d q_{u^2t}(uy) \geq \frac{1}{2}p_t(y) \geq \frac{1}{2}(2\pi t)^{-d/2}\exp\left(-\frac{2}{t}\right).$$

We deduce from the above and (64) that for u large enough, and for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[V_T] \geq \frac{1}{2}u^{-d}|A|\int_{T/2}^T(2\pi s)^{-d/2}\exp\left(-\frac{2}{s}\right)dt.$$

(62) now follows from the fact that $u^{-d}\phi_d(u) = (\psi_d(u))^{-1}$.

Proof of (63): Let us now estimate the second moment of V_T and prove (63). Using the same arguments as in the proof of the rough estimate, we obtain for any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$,

$$E_A[(V_T)^2] = 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')\mathbb{E}_{0,z'}[Z^1_{u^2 t} \in A, Z^2_{u^2 t} \in A].$$

It follows that

$$E_A[(V_T)^2] = 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')(H_{1,u}(t,z') + H_{2,u}(t,z')), \tag{65}$$

where

$$H_{1,u}(t,z') := \sum_{y\in A} \mathbb{P}_{0,z'}[T_1 \le u^2 t, Z^1_{u^2 t} = y],$$
$$H_{2,u}(t,z') := \sum_{y\in A}\sum_{y'\in A} \mathbb{P}_{0,z'}[T_1 > u^2 t, Z^1_{u^2 t} = y, Z^2_{u^2 t} = y'].$$

Since two coalescing walks behave independently before they meet, we can bound $\mathbb{P}_{0,z'}[T_1 > u^2 t, Z^1_{u^2 t} = y, Z^2_{u^2 t} = y']$ by $q_{u^2 t}(y)q_{u^2 t}(z'-y')$ so we obtain

$$\begin{aligned}
& 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')H_{2,u}(t,z') \\
\le\ & 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z') \sum_{y\in A}\sum_{y'\in A} q_{u^2 t}(y)q_{u^2 t}(z'-y') \\
=\ & 2\int_0^T dt \int_0^{T-t} ds \sum_{y\in A}\sum_{y'\in A} q_{u^2 t}(y)q_{u^2(t+s)}(y') \\
=\ & \left(\int_0^T dt \sum_{y\in A} q_{u^2 t}(y)\right)^2 = (E_A[V_T])^2
\end{aligned} \tag{66}$$

With a slight abuse of notation, in the remaining part of the section we use L to denote a constant depending only on d and T and which may change from line to line.

Using (18), we obtain

$$\begin{aligned}
&\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')H_{1,u}(t,z') \\
&\le L_2 u^{-d}\Bigg[L_2\psi_d(u)u^{-d}\sum_{y\in A}\psi_d(|y|^{-1}u) + \sum_{z'\in\mathbb{Z}^d, 0<|z'|\le u} \psi_d\left(\frac{u}{|z'|}\right)\int_0^T H_{1,u}(t,z') \\
&\qquad + \sum_{z'\in\mathbb{Z}^d, |z'|>u} \exp\left(-L_2'\frac{|z'|}{u}\right)\int_0^T H_{1,u}(t,z')\Bigg].
\end{aligned} \tag{67}$$

Note that we used (18) a second time to bound $\int_0^T dt H_{1,u}(t,0)$ and get the first term in the sum above. Then, Lemma 2 is exactly what we need to bound $\int_0^T H_{1,u}(t,z')$ when $z' \neq 0$. However, the cases $d = 2$ and $d = 3$ are slightly different.

When $d = 3$, for any $u > 0$, for any $A \in \mathfrak{A}_u^{(M)}$, and any $z' \neq 0$, we obtain from Lemma 2 that

$$\int_0^T dt H_{1,u}(t,z') \quad \leq \quad L_4 u^{-3} \left(\psi_3(|z'| \vee 1)\right)^{-1} \sum_{y \in u^{-1}A} \psi_3\left(\frac{1}{|y|}\right) \tag{68}$$

From the fact that $\min_{y \in u^{-1}A} |y| > 1/4$ and $|A| = \lceil M\phi_3(u) \rceil = \lceil Mu^2 \rceil$, we obtain that

$$\sum_{y \in u^{-1}A} \psi_3\left(\frac{1}{|y|}\right) \leq 4\lceil Mu^2 \rceil.$$

Hence, from (67) and (68), we deduce that, when $d = 3$, for u sufficiently large, and for any $A \in \mathfrak{A}_u^{(M)}$,

$$\begin{aligned} &\int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t,z') \\ \leq \quad & Lu^{-6} Mu^2 \left[M + \sum_{z' \in \mathbb{Z}^3, 0 < |z'| \leq u} |z'|^{-2} + \sum_{z' \in \mathbb{Z}^3, |z'| \geq u} |z'|^{-1} \exp\left(-L_2' \frac{|z'|}{u}\right) \right] \\ \leq \quad & Lu^{-6} Mu^4 = LM(\psi_3(u))^{-2}, \end{aligned}$$

where we used that $\sum_{z' \in \mathbb{Z}^3, |z'| \geq u} |z'|^{-1} \exp\left(-L_2' \frac{|z'|}{u}\right) \leq L \int_u^\infty \rho \exp(-L_2' \rho/u) d\rho$.

Similarly, when $d = 2$, for any $u > 1$, for any $A \in \mathfrak{A}_u^{(M)}$, and any $z' \neq 0$, we get from Lemma 2 that

$$\begin{aligned} \int_0^T dt H_{1,u}(t,z') \quad & \leq \quad L_4 u^{-2} \frac{\psi_2(u/|z'|)}{\psi_2(|z'|)} \sum_{y \in u^{-1}A} \ln(|y|^{-1}) \\ & \leq \quad \ln(4) L_4 u^{-2} \lceil M \frac{u^2}{\ln(u)} \rceil \frac{\psi_2(u/|z'|)}{\psi_2(|z'|)}, \end{aligned} \tag{69}$$

Furthermore, we have $\min_{y \in u^{-1}A} |y| > 1/4$ and $|A| = \lceil M\phi_2(u) \rceil = \lceil Mu^2/(2\ln(u)) \rceil$, so that

$$\sum_{y \in u^{-1}A} \psi_2\left(\frac{1}{|y|}\right) \leq 4\lceil Mu^2/2(\ln(u)) \rceil.$$

Hence, from (67) and (69), we deduce that for any u sufficiently large, for any $A \in \mathfrak{A}_u^{(M)}$,

$$\begin{aligned}
&\int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t,z') \\
\leq\ & Lu^{-4} M \frac{u^2}{2\ln(u)} \Bigg[\ln(u) + \sum_{z' \in \mathbb{Z}^3, 0<|z'|\leq u} \ln\left(\frac{u}{|z'|} \vee e\right)^2 \times \frac{1}{\ln(|z'| \vee e)} \\
&\qquad\qquad + \sum_{z' \in \mathbb{Z}^3, |z'| \geq u} \frac{1}{\ln(|z'|)} \exp\left(-L_2' \frac{|z'|}{u}\right) \Bigg] \\
\leq\ & Lu^{-4} M \frac{u^2}{2\ln(u)} \left[\ln(u) + \int_{\sqrt{2}}^u \rho \frac{(\ln(u/\rho))^2}{\ln(\rho)} d\rho + \int_u^\infty \frac{\rho}{\ln(\rho)} \exp(-L_2'\rho/u) d\rho \right] \\
\leq\ & Lu^{-2} (\ln(u))^{-1} M \frac{u^2}{2\ln(u)} = 2LM(\psi_2(u))^{-2}.
\end{aligned}$$

To get to the last line above, we have used elementary computations to check that for u large, one has

$$\int_{\sqrt{2}}^u \rho \frac{(\ln(u/\rho))^2}{\ln(\rho)} d\rho \leq \frac{1}{2} \frac{u^2}{\ln(u)}, \qquad \int_u^\infty \frac{\rho}{\ln(\rho)} \exp(-L_2'\rho/u) d\rho \leq L \frac{u^2}{\ln(u)}.$$

In both $d = 2$ and $d = 3$, we have thus obtained that for any u sufficiently large, for any $A \in \mathfrak{A}_u^{(M)}$,

$$\int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t,z') \leq L' M (\psi_d(u))^{-2},$$

where L' is a constant depending on d and T. From (65), (66) and the above, we deduce (63). As explained earlier, this completes the proof of Lemma 10. $\square$

4.4. **Proof of Lemma 8.** The proof of Lemma 8 is somewhat lengthy. It is inspired by the first part of [15], where an upper bound for the Hausdorff measure of the support of two-dimensional super-Brownian motion is established. In particular, we use the Brownian snake as a main tool. The Brownian snake gives an alternative construction of super-Brownian motion under its excursion measure. Moreover, this object introduces time dynamics in the analysis of super-Brownian motion which prove to be critical for our arguments to work. We briefly introduce the Brownian snake and related notation in paragraph 4.4.1, then discuss the link between Brownian snake and super-Brownian motion.

For convenience, we work in this section with super-Brownian motion with branching rate 4 and diffusion coefficient 1 under its excursion measure. Simple scaling arguments then give the general case.

We only give a detailed proof of Lemma 8 in the three-dimensional case (paragraph 4.4.4), after having summarized the basic idea (paragraph 4.4.2), and presented three intermediate lemmas (paragraph 4.4.3). Using the results of the first part of [15], the case $d = 2$ easily adapts. In fact, we even establish a stronger result in the plane (see Lemma 14 below), which we discuss in paragraph 4.4.5.

4.4.1. *Brownian snake.* For a precise definition of the Brownian snake, we refer to [13], Chapter IV. Let $\mathcal{W}$ be the set of continuous finite paths from $\mathbb{R}_+$ into $\mathbb{R}^d$. For $w \in \mathcal{W}$, we denote by ζ_w the lifetime of w, and by $\hat{w}$ the terminal point of the path w, that is $w(\zeta_w)$. The trivial path $\overline{y}$ is the path with initial point $y \in \mathbb{R}^d$ and lifetime 0. The space $\mathcal{W}$ is Polish when equipped with the distance

$$d(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{r \geq 0} |w(r \wedge \zeta_w) - w'(r \wedge \zeta_{w'})|.$$

We then consider $\Omega = \mathcal{C}(\mathbb{R}_+, \mathcal{W})$, the space of continuous paths from $\mathbb{R}_+$ into $\mathcal{W}$ with the topology of uniform convergence on compact sets, and $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-field on Ω. The canonical process on this space is denoted $(W_s, s \geq 0)$, and we define for $s \geq 0$, $\zeta_s := \zeta(W_s)$, and $\sigma(\zeta) := \inf\{s > 0 : \zeta_s = 0\}$. We also let $(\mathcal{F}_t)_{t \geq 0}$ be the canonical filtration on Ω.

For $w \in \mathcal{W}$, we let Π_w be the law on $(\Omega, \mathcal{F})$ of the Brownian snake starting from the path w. Under Π_w, $(W_s, s \geq 0)$ is a $\mathcal{W}$-valued diffusion and $(\zeta_s, s \geq 0)$ is a one-dimensional reflecting Brownian motion. Informally, when ζ_s "increases", the path W_s grows like a d-dimensional Brownian motion, whereas it is erased when ζ_s "decreases" (see [13], Chapter IV for more precisions).

For $y \in \mathbb{R}^d$, the measure $\mathbb{N}_y$ is the excursion measure of W away from the trivial path $\overline{y}$. We abuse the notation by using the same notation $\mathbb{N}_y$ for the excursion measure of the Brownian snake away from $\overline{y}$ and for the excursion measure of super-Brownian motion (cf Section 2.1). This abuse will be justified below when we construct the excursion measure of super-Brownian motion from the Brownian snake under $\mathbb{N}_y$. Under $\mathbb{N}_y$, the law of ζ is the Itô measure of positive Brownian excursions and $\sigma(\zeta)$ is the length of this excursion.

Denote by Π^*_w the law under Π_w of $(W_{s \wedge \sigma_{\zeta_s}}, s \geq 0)$, that is the law of the Brownian snake stopped when its lifetime process hits 0. The strong Markov property of W under $\mathbb{N}_y$ can be expressed in the following way. Let θ_t denotes the usual shift operator on Ω. If T is a $(\mathcal{F}_t)_{t \geq 0}$-stopping time such that $T > 0$ $\mathbb{N}_y$-a.e., then, for any nonnegative $\mathcal{F}_T$-measurable F, for any nonnegative $\mathcal{F}$-measurable G,

$$\mathbb{N}_y \left(\mathbf{1}_{\{T < \infty\}} F \times G \circ \theta_T \right) = \mathbb{N}_y \left(\mathbf{1}_{\{T < \infty\}} F \times \Pi^*_{W_T}(G) \right). \tag{70}$$

The link between Brownian snake and super-Brownian motion can be expressed as follows. Let L^t_s denote the local time of ζ at time s and level t. Since the law of $(\zeta_s, s \geq 0)$ under $\mathbb{N}_y$ is the Itô measure of positive Brownian excursions, $(L^t_s, s \geq 0)$ is, for any $t \geq 0$, well-defined, increasing and continuous, $\mathbb{N}_y$-a.s. We denote by $d_s L^t_s$ the measure associated with the function $u \to L^t_u$ and we let $(Y_t(W), t \geq 0)$ be the measure-valued process defined by the formula

$$Y_t(W)(.) = \int_0^{\sigma(\zeta)} d_s L^t_s \mathbf{1}_{\{\hat{W}_s \in .\}}.$$

Then, the law of $(Y_t(W), t \geq 0)$ under $\mathbb{N}_y$ is the excursion measure of super-Brownian motion with branching rate 4 and diffusion coefficient 1*.

*Moreover, if we let $\mu \in M_F(\mathbb{R}^d)$ and $\sum_{i \in I} \delta_{y_i, W_i}$ be a Poisson measure with intensity $\mu(dy)\mathbb{N}_y(dW)$, then a super-Brownian motion $(\overline{Y}_t, t \geq 0)$ starting from μ can be obtained by setting

$$\overline{Y}_t = \sum_{i \in I} Y_t(W_i).$$

4.4.2. *Outline of the proof of Lemma 8.* Using a symmetry argument, we can interchange the roles of 0 and x, and we will thus work under the probability measure $\mathbb{N}_x(.\big|0 \in \mathcal{R}) = \mathbb{N}_x(.\big|T_0 < \infty)$, where $T_0 = \inf\{t \geq 0 : \hat{W}_t = 0\}$. It is possible to precise the law of $(|W_{T_0}|)_{t \leq \zeta_{T_0}}$ under $\mathbb{N}_x(.|0 \in \mathcal{R})$ (see Lemma 11 below).

For $j \in \mathbb{N}$, let us introduce $r_j = \exp(-j^2)$. To $n_1 \in \mathbb{N}$ we associate $\varepsilon_1 := r_{2^{n_1}}$, and for $\varepsilon_0 > 0$, we set $n_0 := \min\{p \in \mathbb{N}^* : r_{2^p} \leq \varepsilon_0\}$. Note that we have

$$r_j \in (g_d(\varepsilon_0), \varepsilon_0) \quad \forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!].$$

Claim 3. - One can choose $\alpha > 0$ such that, for any $\delta > 0$, there exists $n_1 \in \mathbb{N}$ such that for any $n_0 \geq n_1$, one has

$$\mathbb{N}_x \left(\forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!] : Y_{\zeta_{T_0}}(C(0, r_j/2, r_j)) < \alpha h(r_j) \;\Big|\; T_0 < \infty\right) \leq \delta.$$

From our preceeding remarks, Lemma 8 follows from Claim 3 (even if it means changing α to loosen the inequality).

The idea of the proof of Claim 3 is the following. For given $w \in \mathcal{W}$, $n_0 \in \mathbb{N}$ and $j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!]$, we will express further the contribution $\mathfrak{Y}_w(r_j)$ to $Y_{\zeta_w}(\mathcal{C}(0, r_j/2, r_j))$ of particules which split off the path w in the time interval $[\zeta_w - r_j^2 \ln(1/r_j), \zeta_w - r_j^2]$ (see (76) below). We will observe that for large enough n, the contributions $\mathfrak{Y}_w(r_j), j \in [\![2^n, 2^{n+1}]\!]$ are independent. Using estimates on these contributions (see Lemma 13 below), this independence will lead us to a bound on the probability that for any $j \in [\![2^n, 2^{n+1}-1]\!]$, $\mathfrak{Y}_w(r_j)$ remains smaller than $\alpha h(r_j)$ (see (82) below).

For a well-choosen $\alpha > 0$, we will deduce from this bound and the knowledge of the law of the path $|W_{T_0}|$ the existence of integers N_0, N, and of a family of sets of "good paths" $(\mathbb{W}_n, n \geq N_0)$ such that, with a probability arbitrarily close to 1 when N is large enough,

- $(|W_{T_0}|)_{t \leq \zeta_{T_0}}$ belongs to $\mathbb{W}_n$ for any $n \geq N$.
- for any $w \in \mathbb{W}_n$, $n \geq N$, there exists $j \in [\![2^n, 2^{n+1} - 1]\!]$ such that $r_j^2 Z_w(r_j) > \alpha h(r_j)$.

The desired claim will follow (see assertions (77), (80) and (81) below).

We now present three intermediate lemmas.

4.4.3. *Preliminary results.* Let us start by investigating the law of the path $|W_{T_0}|$ under $\mathbb{N}_x[.|0 \in \mathcal{R}]$.

Lemma 11. *Under* $\mathbb{N}_x[.|0 \in \mathcal{R}]$, $|W_{T_0}(t)|_{t \in [0, \zeta_{T_0}]}$ *has the law of a Bessel process with index* $-5/2$ *started from* $|x|$ *and stopped when it first hits* 0.

Proof of Lemma 11. Introduce a d-dimensional Brownian motion $(B_t)_{t \geq 0}$ and the function $u(y) := \mathbb{N}_y(T_0 < \infty) = (2 - d/2)\,|y|^{-2}$. Let us denote by Q the law of the solution of the stochastic differential equation

$$dY_t = dB_t + \frac{\nabla u}{u}(Y_t), Y_0 = x,$$

stopped when it hits 0.

We know from [8], Proposition 1.4, that for any nonnegative continuous function F on $\mathcal{W}$,

$$\mathbb{N}_x \left[\mathbf{1}_{\{T_0 < \infty\}} F(W_{T_0})\right] = \int \pi_{0,x}(dW) F(W), \tag{71}$$

where $\pi_{0,x} := \left(2 - \frac{d}{2}\right) |x|^{-2} Q$.

For $t \in [0, \zeta_{T_0}]$ let us set $R_t := |W_{T_0}(t)|$. From (71) we deduce that under the probability measure $\mathbb{N}_x[.|T_0 < \infty]$, $(R_t)_{0 \le t \le \zeta_{T_0}}$ solves the stochastic differential equation

$$dR_t = d\beta_t - \frac{2}{R_t} dt,$$

where $(\beta_t)_{t \ge 0}$ is a linear Brownian motion. Thus, $(|W_{T_0}|(t))_{0 \le t \le \zeta_{T_0}}$ has under $\mathbb{N}_x(.\big|\{T_0 < \infty\})$ the law of a Bessel process with index $-5/2$ started from $|x|$ and stopped when it first hits 0. We have completed the proof of Lemma 11. □

For $x \in \mathbb{R}^d$, $p \in \mathbb{N}^*$, $\varepsilon > 0$ and $t > 0$, we will need a lower bound on

$$\psi(t, x, \varepsilon, p) := \mathbb{N}_x \left[Y_t \left(\mathcal{C}(0, \varepsilon/2, \varepsilon) \right)^p \right].$$

Recall P_t denotes the semigroup of d-dimensional Brownian motion. We know (see [15], Proposition 3.2) that we have

$$\psi(t, x, \varepsilon, 1) = P_t \mathbf{1}_{\mathcal{C}(0,\varepsilon/2,\varepsilon)}(x),$$

and the following recursion relation for $p \ge 2$

$$\psi(t, x, \varepsilon, p) = 2 \sum_{j=1}^{p-1} \binom{p}{j} \int_0^t P_{t-s} \left(\psi(s, ., \varepsilon, j) \psi(s, ., \varepsilon, p-j) \right)(x) ds. \tag{72}$$

Fix $c_1 > 1$ and let $c_2 = 1 - \frac{1}{2c_1}$. Note that $1/2 < c_2 < 1$. First observe that there exists a positive c_3 such that

$$\psi(t, x, \varepsilon, 1) = P_t \mathbf{1}_{\mathcal{C}(0,\varepsilon/2,\varepsilon)}(x) \ge c_3 \varepsilon^3 \exp\left(-\frac{|x|^2}{2t} \right) t^{-3/2} \mathbf{1}_{\{t \ge \varepsilon^2\}}. \tag{73}$$

Lemma 12. *For $d = 3$, there exists a positive constant c_4 so that for any $p \in \mathbb{N}^*$, $t > 0$, $x \in \mathbb{R}^3$ and $\varepsilon \ge 0$,*

$$(\mathcal{H}_p) \qquad \psi(t, x, \varepsilon, p) \ge c_4^p p! \frac{\varepsilon^{2p+1}}{t^{3/2}} \exp\left(-\frac{c_1 |x|^2}{t} \right) \mathbf{1}_{\{t \ge c_2^{-2} \varepsilon^2\}}.$$

Corollary 3.3 of [15] is the corresponding result for the two-dimensional case.
Proof of Lemma 12. Note that there exists a constant $c_5 \ge 1$ such that for any $p \ge 2$,

$$\sum_{j=1}^{p-1} (j(p-j))^{3/2} \ge \frac{1}{c_5} p^4.$$

Let us set $c_6 := c_3 c_5^{-1} (\pi c_1)^{-3/2}$. We first verify that for any $p \in \mathbb{N}^*$,

$$(\tilde{\mathcal{H}}_p) \qquad \psi(t, x, \varepsilon, p) \ge \pi^{3/2} c_5 c_6^p p! \varepsilon^{3p} p_{t/2pc_1}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{t}} \right)^{p-1} \mathbf{1}_{\{t \ge \varepsilon^2\}}.$$

We use induction on p to establish $(\tilde{\mathcal{H}}_p)$. If $p = 1$, using (73) and our definition of c_6, we obtain

$$\psi(t, x, \varepsilon, 1) \ge (\pi c_1)^{3/2} c_5 c_6 \varepsilon^3 t^{-3/2} \exp\left(-\frac{|x|^2}{2t} \right) \mathbf{1}_{\{t \ge \varepsilon^2\}}.$$

Since $c_1 > 1$, $(\tilde{\mathcal{H}}_1)$ follows.

Let $p \geq 2$ and assume that the result holds for all $p' < p$. Using (72) and the induction assumption we get for $t \geq \varepsilon^2$

$$\psi(t,x,\varepsilon,p) \geq 2\pi^3 c_5^2 c_6^p p! \varepsilon^{3p} \times \sum_{j=1}^{p-1} \int_{\varepsilon^2}^t ds \int_{\mathbb{R}^3} dy \; p_{t-s}(x-y) p_{s/2jc_1}(y) p_{s/2(p-j)c_1}(y) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2}.$$

For any $j \in [\![1, p-1]\!]$, $s > 0$ and $y \in \mathbb{R}^3$ we have

$$p_{s/2jc_1}(y) p_{s/2(p-j)c_1}(y) = \left(\frac{c_1 j(p-j)}{\pi p s}\right)^{3/2} p_{s/2pc_1}(y)$$

From the last two displays, the choice of c_5 and the fact that $p_{t-s} * p_{s/2pc_1} = p_{t-s+(s/2pc_1)}$ we obtain

$$\psi(t,x,\varepsilon,p) \geq 2\pi^{3/2} c_5 c_6^p p! \varepsilon^{3p} p^{5/2} c_1^{3/2} \int_{\varepsilon^2}^t \frac{ds}{s^{3/2}} p_{t-s+s/2pc_1}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2}. \tag{74}$$

Since $c_1 > 1$ the function $s \to t - s + s/2pc_1$ is decreasing, so that for any $s \in [\varepsilon^2, t]$, we have $t \geq t - s + \frac{s}{2pc_1} \geq \frac{t}{2pc_1}$. Thus,

$$(2pc_1)^{3/2} p_{t-s+\frac{s}{2pc_1}}(x) \geq p_{\frac{t}{2pc_1}}(x).$$

It follows that

$$\begin{aligned}\psi(t,x,\varepsilon,p) &\geq 2^{-1/2} \pi^{3/2} c_5 c_6^p p.p! \varepsilon^{3p} p_{\frac{t}{2pc_1}}(x) \int_{\varepsilon^2}^t \frac{ds}{s^{3/2}} \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2} \\ &= 2^{1/2} \pi^{3/2} c_5 c_6^p p! \varepsilon^{3p} p_{\frac{t}{2pc_1}}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{t}}\right)^{p-1},\end{aligned}$$

which finishes the proof of $(\tilde{\mathcal{H}}_p)$. We have established $(\tilde{\mathcal{H}}_p)$ for any $p \in \mathbb{N}^*$. Note in particular that (74) holds for any $p \in \mathbb{N}^*$.

Let us now complete the proof of Lemma 12. We set $c_4 := (1 - c_2^{1/2}) c_6$. The case $p = 1$ follows from (73) since $c_3 \geq c_4$.

Let us now suppose $p \geq 2$ and $t \geq c_2^{-2} \varepsilon^2$. Since $c_2 = 1 - \frac{1}{2c_1} \leq \frac{2c_1 - 1}{2c_1 - 1/p}$, we get, for any $s \leq c_2 t$, $t - s + s/2pc_1 \geq t/2c_1$, so that

$$p_{t-s+\frac{s}{2pc_1}}(x) \geq (2\pi t)^{-3/2} \exp\left(-\frac{c_1 |x|^2}{t}\right).$$

Hence, it follows from (74) that for any $p \geq 2$,

$$\begin{aligned}\psi(t,x,\varepsilon,p) &\geq 2^{-1/2} c_5 c_6^p p^{5/2} p! \varepsilon^{3p} t^{-3/2} \exp\left(-\frac{c_1 |x|^2}{t}\right) \int_{\varepsilon^2}^{c_2 t} \frac{ds}{s^{3/2}} \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2} \\ &\geq 2^{1/2} c_5 c_6^p p^{3/2} p! \varepsilon^{3p} t^{-3/2} \exp\left(-\frac{c_1 |x|^2}{t}\right) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{c_2 t}}\right)^{p-1}.\end{aligned}$$

Since $t \geq c_2^{-2} \varepsilon^2$, we have $\left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{c_2 t}}\right)^{p-1} \geq \varepsilon^{-p+1} (1 - c_2)^{p-1}$. Moreover, $c_5 \geq 1$, hence, $(\mathcal{H}_p)$ follows for $p \geq 2$, which completes the proof of Lemma 12. □

As explained briefly in paragraph 4.4.2, we will need to estimate, for a fixed $w \in \mathcal{W}$, the contribution under Π_w to $Y_{\zeta_w}(\mathcal{C}(0, \varepsilon/2, \varepsilon))$ of particules which split off the path w shortly before ζ_w. For a given $t > 0$, Lemma V.5 of [13] allows one to decompose Y_t under Π^*_w as the sum of independent contributions corresponding to the decomposition of the path ζ into its excursions above its minimum-to-date. Let us state this more precisely.

Fix $w \in \mathcal{W}$ with $\zeta_w > 0$. Under Π^*_w we can construct a Poisson point measure Λ on $[0, \zeta_w] \times \Omega$ with intensity $2dt\mathbb{N}_{w(t)}(dW)$ such that

$$Y_{\zeta_w} = \int_{[0,\zeta_w]\times\Omega} Y_{\zeta_w - t}(W)\Lambda(dt, dW), \qquad \Pi^*_w - \text{a.s.} \tag{75}$$

Hence, when $w \in \mathcal{W}$ satisfies $\zeta_w \geq \varepsilon^2 \ln(1/\varepsilon)$, the contribution to $Y_{\zeta_w}(\mathcal{C}(0, \varepsilon/2, \varepsilon))$ of particules which split off the path w in the time interval $[\zeta_w - \varepsilon^2 \ln(1/\varepsilon), \zeta_w - \varepsilon^2]$ can be written

$$\mathfrak{Y}_w(\varepsilon) := \int_{\zeta_w - \varepsilon^2 \ln(1/\varepsilon)}^{\zeta_w - \varepsilon^2} \int_\Omega Y_{\zeta_w - t}(\mathcal{C}(0, \varepsilon/2, \varepsilon))\Lambda(dt, dW). \tag{76}$$

For $\varepsilon > 0, w \in \mathcal{W}$, we also introduce $Z_w(\varepsilon) := \varepsilon^{-2}\mathfrak{Y}_w(\varepsilon)$ and we then estimate the moments of $Z_w(\varepsilon)$ under Π^*_w.

Lemma 13. *For $w \in \mathcal{W}$ and $\varepsilon > 0$ such that $\zeta_w \geq \varepsilon \ln(1/\varepsilon)$, let us set*

$$I_w(\varepsilon) := \varepsilon \int_{\varepsilon^2}^{\varepsilon^2 \ln(1/\varepsilon)} \exp\left(-c_1 \frac{|w(\zeta_w - s)|^2}{s}\right) s^{-3/2} \mathbf{1}_{\{s \geq \varepsilon^2 c_2^{-2}\}} ds.$$

There exist positive constants c_7, c_8, c_9 such that for any $w \in \mathcal{W}$, $\varepsilon_2 \in (0, 1/e)$ such that $\zeta_w \geq \varepsilon_2^2 \ln(1/\varepsilon_2)$, the following holds.

(a): *For any $\varepsilon \in (0, \varepsilon_2)$ and for any $p \in \mathbb{N}$*

$$c_7^p p^p \geq \Pi^*_w(Z_w(\varepsilon)^p) \geq c_8^p p^p I_w(\varepsilon).$$

(b): *For any $A > 0$, let $p = \lceil 2A/c_6 \rceil$. Then, for any $\varepsilon \in (0, \varepsilon_2)$*

$$\Pi^*_w(Z_w(\varepsilon) \geq A) \geq \exp(-c_9 A) \left((2^p I_w(\varepsilon) - 1)^+\right)^2.$$

The lower bounds on $\Pi^*_w(Z_w(\varepsilon)^p), p \in \mathbb{N}$ in Lemma 13 (a) are a direct consequence of Lemma 12. Furthermore, the proof of the upper bound in Lemma 13 (a) easily adapts from the one of Lemma 3.4 in [15]. Then, part (b) of Lemma 13 is deduced from part (a) in the exact same manner as, in [15], Lemma 3.5 is deduced from Lemma 3.4. We leave details to the reader. □

4.4.4. Let us now complete the proof of Lemma 8 by establishing Claim 3. We let $\alpha = 1/(4c_9)$ and fix $\delta > 0$. Note that T_0 is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$. Using the strong Markov property (70) at time T_0, we have, for $n_0 > 0$,

$$\begin{aligned} &\mathbb{N}_x \left[\forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!] : Y_{\zeta_{T_0}}(C(0, r_j/2, r_j)) < \alpha h(r_j),\ T_0 < \infty\right] \\ &\leq \mathbb{N}_x \left[\Pi^*_{W_{T_0}}\left(\forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!] : Y_r(C(0, r_j/2, r_j)) < \alpha h(r_j)\right)_{r=\zeta_{T_0}}, T_0 < \infty\right]. \end{aligned} \tag{77}$$

Notice that (77) is an inequality and not an equality, because we used that

$$Y_r\left(\mathcal{C}(0, r_j/2, r_j)\right) = \int_0^{\sigma(\zeta)} d_s L^r_s \mathbf{1}_{\mathcal{C}(0,r_j/2,r_j)}(\hat{W}_s) \geq \int_{T_0}^{\sigma(\zeta)} d_s L^r_s \mathbf{1}_{\mathcal{C}(0,r_j/2,r_j)}(\hat{W}_s),$$

on the event $\{T_0 < \infty\}$.

Introduce the sequence $u_j := r_{2^j}^2 \ln(1/r_{2^j})$, and choose n_2 large enough so that

$$r_{2^{n_2}} \leq e^{-4} \qquad \text{and} \qquad \mathbb{N}_x\left[\zeta_{T_0} \geq s_{n_2} \middle| T_0 < \infty\right] \leq \delta/4.$$

Let us set

$$\mathfrak{W} := \{w \in \mathcal{W} \ : \ w(0) = x, \hat{w} = 0, \zeta_w \geq s_{n_2}\}.$$

Our choice of n_2 ensures that

$$\mathbb{N}_x[W_{T_0} \in \mathfrak{W} | T_0 < \infty] \geq 1 - \delta/4. \tag{78}$$

Since $c_2 > 1/2$, it also guarantees that for any $w \in \mathfrak{W}$, $j \geq 2^{n_2}$, $2r_j^2 c_2^{-2} \leq \zeta_w$. We can then define, for $B > 0$, $j \geq 2^{n_2}$,

$$\mathscr{W}_{B,j} := \left\{ w \in \mathfrak{W} \ : \ \sup_{s \leq 2r_j^2 c_2^{-2}} |w(\zeta(w) - s)| > B r_j c_2^{-1} \right\}.$$

For $w \in \mathfrak{W}$, $n \geq n_2$, we then introduce

$$F_{n,B}(w) := 2^{-n} \sum_{j=2^n}^{2^{n+1}-1} \mathbf{1}_{\{w \in \mathscr{W}_{B,j}\}},$$

and for $n \geq n_2$, we finally let

$$\mathbb{W}_n := \{w \in \mathfrak{W} \ : \ \forall p \geq n \ \ F_{p,B}(w) < 1/2\}.$$

From (78), it follows that

$$\mathbb{N}_x \left[W_{T_0} \notin \mathbb{W}_n \middle| T_0 < \infty \right] \leq \frac{\delta}{4} + \sum_{p \geq n} \mathbb{N}_x \left[W_{T_0} \in \mathfrak{W}, F_{p,B}(W_{T_0}) \geq 1/2 \middle| T_0 < \infty \right].$$

Using Lemma 11 and following the arguments of the proof of Lemma 1 in [14], one can easily establish that there exist constants $B > 0, C > 0$ such that

$$\mathbb{N}_x \left[W_{T_0} \in \mathfrak{W}, F_{p,B}(W_{T_0}) \geq 1/2 | T_0 < \infty \right] \leq C e^{-n}.$$

Hence, there exists $n_3 \geq n_2$ large enough so that for any $n \geq n_3$,

$$\mathbb{N}_x \left[W_{T_0} \notin \mathbb{W}_n \middle| T_0 < \infty \right] \leq \delta/2, \tag{79}$$

which yields

$$\mathbb{N}_x \left[\Pi^*_{W_{T_0}} \Big(\forall j \in [\![2^n, 2^{n+1} - 1]\!] : Y_r(C(0, r_j/2, r_j)) < \alpha h(r_j) \Big)_{r = \zeta_{T_0}} \ \middle| \ T_0 < \infty \right]$$

$$\leq \delta/2 + \sup_{w \in \mathbb{W}_n} \left\{ \Pi^*_w \left[\forall j \in [\![2^n, 2^{n+1} - 1]\!] : Y_{\zeta_w}(C(0, r_j/2, r_j)) < \alpha h(r_j) \right] \right\}. \tag{80}$$

Let $n \geq n_3$ and $w \in \mathbb{W}_n$. Since $r_j^2 \geq r_{j+1}^2 \ln(1/r_{j+1})$ for $2^n \leq j \leq 2^{n+1} - 1$, the independence properties of Poisson measures imply that for any $n \in \mathbb{N}$, the variables $Z_w(r_j), 2^n \leq j \leq 2^{n+1} - 1$ are independent under Π^*_w. Using (75) and the definition of $Z_w(\varepsilon)$, we then get

$$\Pi^*_w \left[\forall j \in [\![2^n, 2^{n+1} - 1]\!] : Y_{\zeta_{T_0}}(C(0, r_j/2, r_j)) < \alpha h(r_j) \right] \tag{81}$$

$$\leq \Pi^*_w \left[Z_w(r_j) < \alpha \theta(r_j) \ \forall j \in [\![2^n, 2^{n+1} - 1]\!] \right] = \prod_{j=2^{n_0}}^{2^{n_0+1}-1} \Pi^*_w \Big(Z_w(r_j) < \alpha \theta(r_j) \Big),$$

where we set, for $r > 0$, $\theta(r) := r^{-2}h(r)$. Furthermore, Lemma 13 (b) leads to

$$\prod_{j=2^{n_0}}^{2^{n_0+1}-1} \Pi_w^*\Big(Z_w(r_j) < \alpha\theta(r_j)\Big) \leq \prod_{j=2^{n_0}}^{2^{n_0+1}-1} \left(1 - e^{-c_9\alpha\theta(r_j)}\left((2^{p_j} I_w(r_j) - 1)^+\right)^2\right)$$

$$\leq \exp\left\{ - \sum_{j=2^n}^{2^{n+1}-1} j^{-2c_9\alpha}\left((2^{p_j} I_w(r_j) - 1)^+\right)^2 \right\}. \tag{82}$$

We then note that, if $j \geq 2^{n_2}$ and $w \in \mathfrak{W} \setminus \mathscr{W}_{B,j}$, an easy computation provides

$$I_w(r_j) \geq c_2 \exp\left(-c_1 B^2\right) =: K(B).$$

Hence, from our choice of α, there exists $n_4 \geq n_3$ so that for any $w \in \mathbb{W}_n$, $n \geq n_4$, one has

$$\sum_{j=2^n}^{2^{n+1}-1} j^{-2c_9\alpha}\left((2^{p_j} I_w(r_j) - 1)^+\right)^2 \geq 2^{n+1/2}.$$

We finally choose $n_0 > n_1 \geq n_4 \geq n_3 \geq n_2$ large enough so that $\exp(-2^{n+1/2}) \leq \delta/2$, and combine (77), (80), (81) and (82) with the above inequality to obtain Claim 3. As explained in paragraph 4.4.2, Lemma 8 follows. □

4.4.5. *The case $d = 2$.* We know from [18], Section III.3 that for any $t > 0$, h is for $d = 3$ the correct Hausdorff measure function of $\mathcal{R}_t$. On the other hand, when $d = 2$, the correct Hausdorff measure function of $\mathcal{R}_t$, is, as it is proven in [15], the function

$$h_2(\varepsilon) = \varepsilon^2 \ln(\varepsilon^{-1}) \ln(\ln(\ln(\varepsilon^{-1}))).$$

Not surprinsingly, when $d = 2$, one can in fact establish a stronger result than Lemma 8.

Lemma 14. *We can choose $\alpha > 0$ so that, for any $\delta > 0$, there exists $\varepsilon_1 \in \left(0, 1 \wedge \frac{|x|}{2}\right)$ such that for any $\varepsilon_0 \in (0, \varepsilon_1)$,*

$$\mathbb{N}_0\left[\exists s \geq 0\ \exists\ \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s\left(\mathcal{C}\left(x, \frac{\varepsilon}{2}, \varepsilon\right)\right) > \alpha h_2(\varepsilon)\ \middle|\ x \in \mathcal{R}\right] \geq 1 - \delta.$$

Lemma 14 clearly implies the two-dimensional case of Lemma 8. The proof is similar to that of Lemma 8 in the three-dimensional case. Let us only point out the main differences, and leave details to the reader.

Obviously, one should work with h_2 instead of h, g_2 instead of g and the function θ_2 such that $\theta_2(r) := \ln\ln\ln(1/r)$ instead of θ. Moreover, the sequence r_j is to be replaced with $r_j^{(2)} = 2^{-2^j}$, so that $r^{(2)}_{2^{n+1}-1} \geq g_2(r^{(2)}_{2^n})$. We already noted that Lemma 12 for the three-dimensional case corresponds to Corollary 3.3 of [15] in the plane. In particular, note that c_1, c_2, c_4 should be replaced with $c_1^{(2)} > 1/2, c_2 = (4c_1^{(2)} - 2)/(4c_1^{(2)} - 1), c_4^{(2)} = c_3^{(2)}(2c_1^{(2)})^{-1}$. We also already remarked that Lemma 13 (a) and (b) are to be respectively related with Lemma 3.4, respectively Lemma 3.5 of [15]. Lemma 13 remains valid in the plane when one replaces $Z_w(\varepsilon)$ with

$$Z_w^{(2)}(\varepsilon) := \left(\varepsilon^2 \ln(\varepsilon^{-1})\right)^{-1} \int_{\zeta_w - \varepsilon^2\ln(1/\varepsilon)}^{\zeta_w - \varepsilon^2} Y_{\zeta_w - t}(\mathcal{C}(0, \varepsilon/2, \varepsilon))\Lambda(dt, dW),$$

then $I_w(\varepsilon)$ with

$$I_w^{(2)}(\varepsilon,p) := p\left(\log(\varepsilon^{-1})\right)^{-p} \times \int_{\varepsilon^2}^{\varepsilon^2\ln(1/\varepsilon)} \exp\left(-c_1^{(2)}\frac{|w(\zeta_w - s)|^2}{s}\right)\left(\log^+\left(s(c_2^{(2)})^p\varepsilon^{-2}\right)\right)^{p-1} s^{-1}ds.$$

Is is then straightforward to check that, once all these changes have been made, the exact same proof as in paragraph 4.4.4 leads to assertions similar to (77), (78), (79), (80), (81), (82) and Claim 3.

5. Results on coalescing random walks

In this section, we prove† Lemmas 1 and 2, which we used in the proof of Theorem 1. First, let us introduce further notation. We write $\mathbb{P}_x^{(2)}$ for a probability measure under which $(Z_t, t \geq 0)$ is a continuous time random walk with rate 2 (instead of rate 1 for $\mathbb{P}_x$) and jump kernel p, starting from x. Let us also denote

$$H_t(x) := \mathbb{P}_x(Z \text{ hits } 0 \text{ before } t); \qquad G_t(x) := \mathbb{E}_x\left[\int_0^t \mathbf{1}_{\{Z_s=0\}}ds\right];$$

and $H_t^{(2)}(x), G_t^{(2)}(x)$ the corresponding quantities under $\mathbb{P}_x^{(2)}$. It is well-known that when $d = 2$,

$$G_{c^2t}^{(2)}(0) \underset{c\to\infty}{\sim} G_{c^2\varepsilon}^{(2)}(0) \underset{c\to\infty}{\sim} \frac{1}{2\pi}\ln(c). \tag{83}$$

When $d = 3$, from the definition of k_d, we have

$$G_{c^2t}^{(2)}(0) \underset{c\to\infty}{\longrightarrow} k_d^{-1}. \tag{84}$$

In dimension $d = 2$, an easy adaptation of [11], Theorem 1.6.1, to the continuous time setting, ensures the existence of

$$a(x) := \lim_{t\to\infty}[G_t(0) - G_t(x)], \quad a^{(2)}(x) := \lim_{t\to\infty}[G_t^{(2)}(0) - G_t^{(2)}(x)].$$

An easy consequence of the proof of Theorem 1.6.2 in [11] is the existence of a constant κ_5 depending only on d such that for any $t \geq 1$,

$$G_t(0) - G_t(x) \leq \kappa_5 a(x), \quad G_t^{(2)}(0) - G_t^{(2)}(x) \leq \kappa_5 a^{(2)}(x). \tag{85}$$

Furthermore, from Theorem 1.6.2 of [11], both $a(x)$ and $a^{(2)}(x)$ are $O(\ln|x|)$.

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†Some of the ideas involved in the results below are borrowed from [12], such as in particular, the case $d \geq 3$ of Lemma 15. The proof of Lemma 1 is also borrowed from this unpublished manuscript. Moreover, it is interesting to note that it would be possible to get precise asymptotics on the quantities we are bounding below. In particular, for $\alpha > 0$ and $\phi \in C_b(\mathbb{R}^d)$, it is possible to get the exact asymptotics of quantities such as $\mathbb{E}_{0,y}\left[T_1 \leq |y|^\alpha t, \phi(X_{T_1^1}/|y|\right]$, as $|y| \to \infty$. Such asymptotics were computed in [12] in the case $d \geq 3$, $\alpha = 2$, and it is possible to extend the results to a general $\alpha > 0$ and to the case $d = 2$. For $d \geq 4$, these exact estimates would allow one to get a more precise upper bound on $E[(U_T)^2]$ than the one obtained in Section 4.1, and therefore improve the constants $a_d, d \geq 4$ appearing in the statement of Proposition 3 (however, this would not be enough to get precise asymptotics on the hitting probability of a far point for $d \geq 5$). We chose not to present these asymptotics here, as they only had this minor impact on our main result.

5.1. **Proof of Lemma 1.** This proof was taken from [12]. In the following, we denote by $C_i, i \geq 0$ positive constants depending only on d. Let us denote by $(S_n, n \in \mathbb{N})$ a discrete time random walk with jump kernel p, starting from x under the probability measure $\mathbb{Q}_x$. By combining the well-known bound $\mathbb{Q}_x[S_n = y] \leq Kn^{-d/2}$ and the martingale inequality of Ledoux and Talagrand ([17], Lemma 1.5), we get for any $n \geq 1, y \in \mathbb{Z}^d$,

$$\mathbb{Q}_x[S_n = y] \leq \frac{C_1}{n^{d/2}} \exp\left(-\frac{C_2|y-x|^2}{n}\right).$$

Let $(N_t, t \geq 0)$ be a standard Poisson process. Then

$$\begin{aligned} \mathbb{Q}_0[Z_t = y] &= \sum_{n=0}^{\infty} P[N_t = n]\mathbb{Q}_0[S_n = y] \\ &\leq \exp(-t)\mathbf{1}_{\{y=0\}} + \sum_{n=1}^{\infty} P[N_t = n]\frac{C_1}{n^{d/2}} \exp\left(-\frac{C_2|y|^2}{n}\right). \end{aligned} \tag{86}$$

We also have for any $t \geq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{d/2}} P[N_t = n] \leq C_3 t^{-d/2}.$$

It follows from the above that

$$\sum_{n=1}^{\lfloor 2t \rfloor} P[N_t = n]\frac{C_1}{n^{d/2}} \exp\left(-\frac{C_2|y|^2}{n}\right) \leq \frac{C_1C_3}{t^{d/2}} \exp\left(-\frac{C_2|y|^2}{2t}\right). \tag{87}$$

For values of n greater then $2t$, a simple large deviation estimate gives for every $t > 0, m \geq 1$,

$$P[N_t \geq 2^m t] \leq C_4 \exp(-C_5 2^m t).$$

Hence,

$$\begin{aligned} &\sum_{n>2t} P[N_t = n]n^{-d/2} \exp\left(-\frac{C_2|y|^2}{n}\right) \\ \leq\ &\sum_{m=1}^{\infty} C_4 \exp(-C_5 2^m t)(2^m t)^{-d/2} \exp\left(-\frac{C_2|y|^2}{2^{m+1}t}\right) \\ =\ &C_4 t^{-d/2} \sum_{m=1}^{\infty} 2^{-md/2} \exp\left(-C_5 2^m t - \frac{C_2|y|^2}{2^{m+1}t}\right). \end{aligned}$$

Setting $C_6 = \sqrt{2C_2C_5}$ we now obtain from the above

$$\sum_{n>2t} P[N_t = n]n^{-d/2} \exp\left(-\frac{C_2|y|^2}{n}\right) \leq C_7 t^{-d/2} \exp(-C_6|y|).$$

Combining (86), (87) and the above now gives

$$q_t(y) \leq \exp(-t)\mathbf{1}_{\{y=0\}} + \frac{C_1C_3}{t^{d/2}} \exp\left(-\frac{C_2|y|^2}{2t}\right) + \frac{C_7}{t^{d/2}} \exp(-C_6|y|),$$

which clearly implies Lemma 1. $\square$

5.2. **Proof of Lemma 2.** In the following, we use K, K' to denote positive constants depending only on d, T and which may change from line to line. We consider only the case when c and x are such that $cx \in \mathbb{Z}^d$. The general case immediately follows.

We are first going to rule out small values of t. We deal with the integral over the interval $[0, c^{-2}]$. Note that

$$\mathbb{P}_{0,y}\left[T_1 \le c^2 t, Z^1_{c^2 t} = cx\right] \le \min\left\{q_{c^2 t}(0, cx), q_{c^2 t}(y, cx)\right\}.$$

Considering separately the cases $|y| \le 2c|x|$ and $|y| > 2c|x|$ and using (14), we easily obtain

$$\begin{aligned} & c^d \psi_d(|y|) \int_0^{c^{-2}} dt \mathbb{P}_{0,y}\left[T_1 \le c^2 t, Z^1_{c^2 t} = cx\right] \\ \le \;& \kappa_3 c^{d-2} \exp(-\kappa_4 c|x|/2) \times \psi_d(|y|) \exp(-\kappa_4(|y|/4)) \\ \le \;& K|x|^{2-d} \le K\psi_d(|x|^{-1}). \end{aligned}$$

Let us now deal with small values of T_1. In a similar way as in the previous computation, one gets

$$\begin{aligned} & c^d \psi_d(|y|) \int_0^{c^{-2}} dt \mathbb{P}_{0,y}\left[T_1 \le 1, Z^1_{c^2 t} = cx\right] \\ \le \;& \kappa_3 c^{d} \exp(-\kappa_4 c|x|/2) \times \psi_d(|y|) \exp(-\kappa_4(|y|/4\sqrt{2})) \\ \le \;& K|x|^{2-d} \le K\psi_d(|x|^{-1}). \end{aligned}$$

Note that to obtain the last line above, we used the assumption $c \ge |x|^{-2}$.

Let us then deal with large values of T_1. For $t \ge c^{-2}$, we have

$$\begin{aligned} & c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[c^2 t - 1 \le T_1 \le c^2 t, Z^1_{c^2 t} = cx\right] \\ \le \;& e c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[c^2 t - 1 \le T_1 \le c^2 t, Z^1_{c^2 t} = Z^1_{T_1} = cx\right] \\ \le \;& e c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[Z^1_{c^2 t} = Z^2_{c^2 t} = cx\right] \\ \le \;& e c^{-d} \psi_d(|y|) f_t(x) f_t(x - y/c), \end{aligned}$$

where we used Lemma 1 at the last line above. Hence, from (11), we get that

$$\begin{aligned} & c^d \psi_d(|y|) \int_{c^{-2}}^{T} dt \mathbb{P}_{0,y}\left[c^2 t - 1 \le T_1 \le c^2 t, Z^1_{c^2 t} = cx\right] \\ \le \;& K c^{-d} \psi_d(|y|) \left(|x|^{2-2d} \wedge \left(\frac{|y|}{c}\right)^{2-2d} \right) \le K|x|^{2-d} \le K\psi_d(|x|^{-1}). \end{aligned}$$

Note that, at the last line above, we used the assumption $c \ge |x|^{-2}$.

We can now suppose $1 \le c^2 t - 1$, that is $t \ge 2c^{-2}$, and restrict our attention to estimating

$$\int_{2c^{-2}}^{T} dt \mathbb{P}_{0,y}\left[1 \le T_1 \le c^2 t - 1, Z^1_{c^2 t} = cx\right].$$

Using the Markov property at time T_1, then Lemma 1, we obtain

$$\int_{2c^{-2}}^{T} dt\ \mathbb{P}_{0,y}\left[1 \le T_1 \le c^2t-1, Z^1_{c^2t}=cx\right]$$
$$= \int_{2c^{-2}}^{T} dt\ \mathbb{E}_{0,y}\left[\mathbf{1}_{\{1\le T_1\le c^2t-1\}}\sum_{z\in\mathbb{Z}^d}\mathbf{1}_{\{Z^1_{T_1}=z\}}q_{c^2t-T_1}(z-cx)\right]$$
$$\le Kc^{-d}\int_{2c^{-2}}^{T} dt\sum_{z\in\mathbb{Z}^d}\mathbb{E}_{0,y}\left[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}f_{t-\frac{T_1}{c^2}}\left(\frac{z}{c}-x\right)\right]. \tag{88}$$

In order to bound the above quantity, we need the following intermediate result.

Lemma 15. *Let $d\ge 2$. There exists a positive constant L_5 depending only on d such that for any $c\ge 1$, $z\in\mathbb{Z}^d$, $t\ge 2c^{-2}$, $y\in\mathbb{Z}^d\setminus 0$ and every measurable function $\phi:\mathbb{R}_+\times\mathbb{Z}^d\to\mathbb{R}_+$,*

$$|y|^d\psi_d(|y|)\mathbb{E}_{0,y}\left[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}\phi\left(\frac{T_1}{|y|^2}, Z^1_{T_1}\right)\right]$$
$$\le L_5\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\ \Phi(u,z)\,\tilde f_u\left(\frac{z}{|y|}\right)\tilde f_u\left(\frac{z-y}{|y|}\right),$$

where $\Phi(u,z)=\sup_{\left(u-\frac{1}{2|y|^2}\right)^+\le r\le u}\phi(r,z)$, and $\tilde f_u$ was defined in Section 2.4.

Proof of lemma 15. In this proof, we use L to denote a constant depending only on d and which may change from line to line. Obviously,

$$\phi(r,z)\le 2|y|^2\int_r^{r+\frac{1}{2|y|^2}}\Phi(u,z)du.$$

It follows that

$$|y|^d\psi_d(|y|)\mathbb{E}_{0,y}\left[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}\phi\left(\frac{T_1}{|y|^2}, Z^1_{T_1}\right)\right]$$
$$\le 2|y|^{d+2}\psi_d(|y|)\mathbb{E}_{0,y}\Bigg[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}$$
$$\times\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\Phi(u,z)\mathbf{1}_{\{\frac{T_1}{|y|^2}\le u\le\frac{T_1}{|y|^2}+\frac{1}{2|y|^2}\}}\Bigg]$$
$$= 2|y|^{d+2}\psi_d(|y|)\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\Phi(u,z) \tag{89}$$
$$\times\mathbb{P}_{0,y}\left[1\le T_1\le u|y|^2\le T_1+\frac{1}{2}\le c^2t-\frac{1}{2}, Z^1_{T_1}=z\right].$$

where we use the Fubini theorem at the last line. Hence, proving Lemma 15 reduces to establishing the following claim

Claim 4. - If $u>|y|^{-2}$,

$$2|y|^{d+2}\psi_d(|y|)\mathbb{P}_{0,y}\left[1\le T_1\le u|y|^2\le T_1+\frac{1}{2}\le c^2t-\frac{1}{2}, Z^1_{T_1}=z\right]$$
$$\le L_5\tilde f_u\left(\frac{z}{|y|}\right)\tilde f_u\left(\frac{z-y}{|y|}\right).$$

Let us first rule out the easy cases of Claim 4.

First note that the case $d \geq 3$ is simple, because

$$\mathbb{P}_{0,y}\left[T_1 \leq |y|^2 u \leq T_1 + \frac{1}{2}, Z^1_{T_1} = z\right] \leq e q_{|y|^2 u}(z) q_{|y|^2 u}(y-z),$$

and we can use Lemma 1 to conclude.

In the case $d = 2$, using the same argument as in the case $d \geq 3$ only gives

$$\begin{aligned}&|y|^4 \ln(|y| \vee e)\mathbb{P}_{0,y}\left[1 \leq T_1 \leq |y|^2 u \leq T_1 + \frac{1}{2}, Z^1_{T_1} = z\right]\\ &\leq e \ln(|y| \vee e) f_u\left(\frac{z}{|y|}\right) f_u\left(\frac{z-y}{|y|}\right).\end{aligned}$$

However, in the particular cases when $|y| \leq A$ for some fixed constant $A \geq 1$, or when $|y|^{-2} \leq u \leq |y|^{-1}$, we have

$$\ln(|y| \vee e) \exp\left(-\frac{\kappa_2 |z|}{2|y|\sqrt{u}}\right) \exp\left(-\frac{\kappa_2 |z-y|}{2|y|\sqrt{u}}\right) \leq L.$$

This easily leads to the desired claim in these particular cases.

We now suppose $d = 2$, $|y| \geq A := 6^8$, and $u \geq |y|^{-1}$ and outline of the proof of Claim 4. We have

$$\begin{aligned}&\mathbb{P}_{0,y}\left[1 \leq T_1 \leq |y|^2 u \leq T_1 + \frac{1}{2}, Z^1_{T_1} = z\right]\\ &\leq e\mathbb{P}_{0,y}\left[1 \leq T_1 \leq |y|^2 u \leq T_1 + \frac{1}{2}, Z^1_{|y|^2 u} = Z^2_{|y|^2 u} = z\right]\\ &\leq e\mathbb{P}_{z,z}\left[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^2 u], Z^1_{|y|^2 u} = 0, Z^2_{|y|^2 u} = y\right], \end{aligned} \tag{90}$$

where we used a time-reversal argument in the last line.

We are going to use (90) and argue under $\mathbb{P}_{z,z}$. On the one hand, with high probability, both $Z^1_{|y|^{3/2}(u\wedge 1)}$ and $Z^2_{|y|^{3/2}(u\wedge 1)}$ should remain close to z. More precisely, if we set

$$B_{z,y} := B\left(z, |y|^{7/8}\left(\frac{|z|}{3|y|} \vee 1\right)\right), \qquad t(u,y) := |y|^2 u - |y|^{3/2}(u \wedge 1),$$

we will establish that

$$\begin{aligned}&\ln(|y|)|y|^4 \mathbb{P}_{z,z}\left[Z^1_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y} \text{ or } Z^2_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y}, Z^1_{|y|^2 u} = 0, Z^2_{|y|^2 u} = y\right]\\ &\leq L \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right).\end{aligned} \tag{91}$$

On the other hand, when both $Z^1_{|y|^{3/2}(u\wedge 1)}, Z^2_{|y|^{3/2}(u\wedge 1)}$ are close to z, we obtain from the Markov property for the walks Z^1, Z^2 at time $|y|^{3/2}(u \wedge 1)$ that

$$\begin{aligned}&\mathbb{P}_{z,z}\Big[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u \wedge 1)],\ Z^1_{|y|^2 u} = 0,\ Z^2_{|y|^2 u} = y,\\ &\qquad\qquad Z^1_{|y|^{3/2}(u\wedge 1)} \in B_{z,y},\ Z^2_{|y|^{3/2}(u\wedge 1)} \in B_{z,y}\Big]\\ &\leq \mathbb{P}_{z,z}\left[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u \wedge 1)]\right] \sup_{(x_1,x_2)\in B^2_{z,y}} q_{t(u,y)}(x_1) q_{t(u,y)}(x_2 - y).\end{aligned} \tag{92}$$

We will then establish, using Lemma 1, that

$$\sup_{(x_1,x_2)\in B^2_{z,y}} |y|^4 q_{t(u,y)}(x_1) q_{t(u,y)}(x_2-y) \le \tilde f_u\left(\frac{z}{|y|}\right)\tilde f_u\left(\frac{z-y}{|y|}\right). \tag{93}$$

Moreover, we will finally prove that the probability for (Z^1,Z^2) to avoid each other in the time interval $[1,|y|^{3/2}(u\wedge 1)]$ is of order $\ln(|y|\vee e)^{-1}$:

$$\ln(|y|\vee e)\mathbb{P}_{z,z}\Big[Z^1_s\neq Z^2_s\ \forall s\in[1,|y|^{3/2}(u\wedge 1)]\Big]\le L. \tag{94}$$

Combining (91) (92), (93) and (94), we obtain

$$\begin{aligned}&\ln(|y|)|y|^4\mathbb{P}_{z,z}\left[Z^1_s\neq Z^2_s\ \forall s\in[1,|y|^2u], Z^1_{|y|^2u}=0, Z^2_{|y|^2u}=y\right]\\&\le L\tilde f_u\left(\frac{z}{|y|}\right)\tilde f_u\left(\frac{z-y}{|y|}\right).\end{aligned}$$

Claim 4 then follows from (90) and the above. To complete the proof of Claim 4, hence the one of Lemma 15, it remains to establish (91), (93) and (94).

Proof of (91) : As a consequence of (14),

$$\begin{aligned}&\ln(|y|)\mathbb{P}_{z,z}\Big[Z^1_{|y|^{3/2}(u\wedge 1)}\notin B_{z,y}\text{ or } Z^2_{|y|^{3/2}(u\wedge 1)}\notin B_{z,y}\Big]\\&\le 2\kappa_3\ln(|y|)\exp\left(-\kappa_4\frac{|y|^{1/8}}{\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right)\\&\le L\exp\left(-\kappa_4\frac{|y|^{1/8}}{2\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right),\end{aligned} \tag{95}$$

where at the last line, we used the bound $\ln(|y|)\exp\left(-\kappa_4|y|^{1/8}/2\right)\le L$. By studying separately the cases $|z|\ge 3|y|$, $|z|\le 3|y|$, and using the fact that $\kappa_4/4\ge\kappa_2$, we get

$$\exp\left(-\kappa_4\frac{|y|^{1/8}}{2\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right)\le\exp\left(-\kappa_2\frac{|z|}{4|y|\sqrt{u}}\right)\exp\left(-\kappa_2\frac{|z-y|}{4|y|\sqrt{u}}\right),$$

Furthermore, since $t(u,y)\ge|y|^2u/2$, we get from Lemma 1 that

$$\sup\{|y|^2q_{t(u,y)}(x,x'), x\in\mathbb{Z}^d, x'\in\mathbb{Z}^d\}\le Lu^{-1}$$

Hence, (91) follows from using the Markov property for the walks Z^1,Z^2 at time $|y|^{3/2}(u\wedge 1)$ and combining the above remarks.

Proof of (93) : Lemma 1 implies that

$$\sup_{(x_1,x_2)\in B_{z,y}} q_{t(u,y)}(x_1)q_{t(u,y)}(x_2-y)\le 4|y|^{-4}\sup_{(x_1,x_2)\in B^2_{z,y}} f_u\left(\frac{x_1}{|y|}\right)f_u\left(\frac{x_2-y}{|y|}\right). \tag{96}$$

The bound (93) in the case $|z|\ge 3|y|$ easily follows from (96) and the fact that for x_1,x_2 both in $B_{z,y}$, we have $|x_1|\ge 2|z|/3$, $|x_2-y|\ge|z|/3\ge|z-y|/4$.

Let us now suppose $|z|\le 3|y|$, and recall that we assumed $|y|\ge 6^8$, so that the balls $B(0,3|y|^{7/8})$ and $B(y,3|y|^{7/8})$ are disjoint. Now, if $z\in B(0,2|y|^{7/8})$ we easily see that for any $x_2\in B(z,|y|^{7/8})$, we have $|x_2-y|\ge|z-y|/2\ge|z|/2$. It follows that

$$\exp\left(-\frac{\kappa_2|x_2-y|}{|y|\sqrt{u}}\right)\le\exp\left(-\frac{\kappa_2|z-y|}{4|y|\sqrt{u}}\right)\exp\left(-\frac{\kappa_2|z|}{4|y|\sqrt{u}}\right).$$

Assertion (96) and the above imply (93) in the case $z \in B(0, 2|y|^{7/8})$. We can use a similar argument to conclude in the case $z \in B(y, 2|y|^{7/8})$. At last, if z satisfies $|z| \le 3|y|$ but is not in any of the two aforementioned balls, then, for any $x_1, x_2 \in B(z, |y|^{7/8})$ we have $|x_1| \ge |z|/2$, $|x_2 - y| \ge |z-y|/2$, and (93) easily follows from (96).

Proof of (94) : First note that under $\mathbb{P}_{z,z}$, $Z^1 - Z^2$ has law $\mathbb{P}_0^{(2)}$. Recall that the notation $a^{(2)}(x)$, $G_x^{(2)}$, $H_t^{(2)}(x)$ have been introduced at the beginning of the section. From the simple bound $H_t^{(2)}(x) \ge G_t^{(2)}(0)^{-1} G_t^{(2)}(x)$, then (85), we get

$$1 - H_{|y|}^{(2)}(x) \le \frac{G_{|y|}^{(2)}(0) - G_{|y|}^{(2)}(x)}{G_{|y|}^{(2)}(0)} \le \frac{\kappa_5 a^{(2)}(x)}{\ln(|y|)}. \tag{97}$$

From the Markov property for $Z^1 - Z^2$ at time 1, we have

$$\begin{aligned} & \mathbb{P}_{z,z}\left[Z_s^1 \neq Z_s^2 \ \forall s \in [1, |y|^{3/2}(u \wedge 1)]\right] \\ = \ & \mathbb{E}_{z,z}\left[\sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\{Z_1^1 - Z_1^2 = x\}} \mathbf{1}_{\{Z_s^1 - Z_s^2 \neq 0 \ \forall s\in[1,|y|^{3/2}(u\wedge 1)]\}}\right] \\ = \ & \sum_{x\in\mathbb{Z}^d} q_2(x)\left(1 - \mathbb{P}_x^{(2)}\left(Z \text{ hits } 0 \text{ before time } |y|^{3/2}(u\wedge 1)\right)\right) \\ \le \ & \sum_{x\neq 0} q_2(x) \frac{\kappa_5 a^{(2)}(x)}{\ln\left(|y|^{3/2}(u\wedge 1)\right)} \end{aligned}$$

where we used (97) at the last line. Since $u \ge |y|^{-1}$, we have $\ln\left(|y|^{3/2}(u\wedge 1)\right) \ge \ln(|y|)/2$. Hence, using Lemma 1 and the fact that $a^{(2)}(x) = O(\ln(|x|))$, we get (94).

As explained earlier on, this completes the proof of Claim 4, hence the one of Lemma 15. □

Let us now complete the proof of Lemma 2. We will apply Lemma 15 to bound the right-hand side of (88). Fix $t \in [2c^{-2}, T]$. Let us consider the nonnegative functions $\Phi_c(u,z) = f_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right)$. For $u \in [\frac{1}{|y|^2}, \frac{2tc^2-1}{2|y|^2}]$ we have

$$\Phi_c(u,z) := \sup_{u-(2|y|^2)^{-1}\le r\le u} \phi_c(r,z) \le \tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right).$$

Thus, from (88) and Lemma 15, it follows that

$$\begin{aligned} & c^d \psi_d(|y|) \int_{2c^{-2}}^T dt \mathbb{P}_{0,y}\left[1 \le T_1 \le c^2 t - 1, Z^1_{c^2 t} = cx\right] \\ & \le K|y|^{-d} \int_{2c^{-2}}^T dt \sum_{z\in\mathbb{Z}^d} \int_{|y|^{-2}}^{\frac{tc^2}{|y|^2} - \frac{1}{2|y|^2}} du \tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right) \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right) \\ & \le K|y|^{-d} \int_{2c^{-2}}^T dt \sum_{z\in\mathbb{Z}^d} \int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du \hat{f}_t\left(\frac{z}{c} - x\right) \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right) \\ & + K|y|^{-d} \int_{2c^{-2}}^T dt \sum_{z\in\mathbb{Z}^d} \int_{\frac{tc^2}{2|y|^2}}^{\frac{tc^2}{|y|^2} - \frac{1}{2|y|^2}} du \tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right) \hat{f}_{\frac{tc^2}{|y|^2}}\left(\frac{z}{|y|}\right) \hat{f}_{\frac{tc^2}{|y|^2}}\left(\frac{z-y}{|y|}\right). \end{aligned} \tag{98}$$

For convenience, let us define, for $z \in \mathbb{Z}^d$, $y \in \mathbb{Z}^d \setminus 0$, $x \in c^{-1}\mathbb{Z}^d \setminus 0$ and $c \geq |x|^{-1} \vee 1$,

$$F_1(c,x,y,z) := \int_{2c^{-2}}^{T} dt \int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du \hat{f}_t\left(\frac{z}{c} - x\right) \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right)$$

$$F_2(c,x,y,z) := \int_{2|y|^{-2}}^{\frac{Tc^2}{|y|^2}} dt' \int_{\frac{1}{2|c|^2}}^{\frac{t}{2}} du' \tilde{f}_{u'}\left(\frac{z}{c} - x\right) \hat{f}_{t'}\left(\frac{z}{|y|}\right) \hat{f}_{t'}\left(\frac{z-y}{|y|}\right).$$

so that (98) can be rewritten

$$\begin{aligned} (99) \qquad & c^d \psi_d(|y|) \int_{2c^{-2}}^{T} dt \mathbb{P}_{0,y}\left[1 \leq T_1 \leq c^2 t - 1, Z^1_{c^2 t} = cx\right] \\ & \leq K|y|^{-d} \sum_{z \in \mathbb{Z}^d} F_1(c,x,y,z) + K|y|^{-d} \sum_{z \in \mathbb{Z}^d} F_2(c,x,y,z). \end{aligned}$$

Thus, completing the proof of Lemma 2, in the case $d = 2$, reduces to verify the bounds

$$(100) \qquad |y|^{-2} \ln\left(\frac{c}{|y|} \vee e\right)^{-1} \sum_{z \in \mathbb{Z}^2} F_1(c,x,y,z) \leq K \ln(|x|^{-1} \vee e),$$

$$(101) \qquad |y|^{-2} \ln\left(\frac{c}{|y|} \vee e\right)^{-1} \sum_{z \in \mathbb{Z}^2} F_2(c,x,y,z) \leq K \ln(|x|^{-1} \vee e).$$

Similarly, in the case $d = 3$, in order to complete the proof of Lemma 2, we need to establish that

$$(102) \qquad |y|^{-3} \sum_{z \in \mathbb{Z}^3} F_1(c,x,y,z) \leq K(|x|^{-1} \vee 1),$$

$$(103) \qquad |y|^{-3} \sum_{z \in \mathbb{Z}^3} F_2(c,x,y,z) \leq K(|x|^{-1} \vee 1).$$

We first deal with the first term of the sum in the right-hand side of (99).
Proof of (100), (102): From (13), we obtain

$$(104) \qquad \int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right) \leq K\left(\frac{|z|}{|y|} \vee 1\right)^{2-2d}.$$

Then, from (11) and (12), we easily get

$$(105) \qquad \int_{2c^{-2}}^{T} dt \hat{f}_t\left(\frac{z}{c} - x\right) \leq \begin{cases} K\psi_d(c) & \text{if } z = cx, \\ K\psi_d\left(\left|\frac{z}{c} - x\right|^{-1}\right) & \text{if } z \neq cx, \\ K\exp\left(-K'\left|\frac{z}{c} - x\right|\right) & \text{if } \left|\frac{z}{c} - x\right| \geq \sqrt{T}. \end{cases}$$

We then split $\mathbb{Z}^d$ into the following subsets

$D_0 := \{cx\}$, $D_1 := \left(\mathbb{Z}^d \cap B(cx, c|x|/2)\right) \setminus D_0$, $D_2 := \left(\mathbb{Z}^d \cap B(0, |y| \wedge 2c^2T)\right) \setminus D_1$, $D_3 := \left(\mathbb{Z}^d \cap B(0, |y| \vee 2c^2T)\right) \setminus (D_2 \cup D_1)$, $D_4 := \mathbb{Z}^d \setminus (D_1 \cup D_3)$.

We now combine the displays (104), (105), in order to obtain bounds on $F_1(c,x,y,z)$ over the regions D_i, $0 \le i \le 4$. We also use that, for $z \in D_1$, $|z| \ge Kc|x|$, while, for $z \notin D_1$, $\psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) \le K\psi_d(|x|^{-1})$. We have

$$F_1(c,x,y,z) \le K \times \begin{cases} \psi_d(c)\left(\frac{c|x|}{|y|}\vee 1\right)^{2-2d} & \text{if } z = cx, \\ \psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right)\left(\frac{c|x|}{|y|}\vee 1\right)^{2-2d} & \text{if } z \in D_1, \\ \psi_d(|x|^{-1}) & \text{if } z \in D_2, \\ \psi_d(|x|^{-1})\left(\frac{|z|}{|y|}\vee 1\right)^{2-2d} & \text{if } z \in D_3 \cup D_4, \\ \exp\left(-K'\frac{|z|}{c}\right)|z|^{2-2d}|y|^{2d-2} & \text{if } z \in D_4. \end{cases} \tag{106}$$

Then, observe that

$$\begin{aligned} &\sum_{z\in D_1} \psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) \le Kc^{d-2}(c|x|)^2, \\ &|D_2| \le K(|y|\wedge c^2)^d, \\ &|D_3| \le K(|y|\vee c^2)^d, \\ &\text{if } d=3, \quad \sum_{z\in D_3\cup D_4} |z|^{-4} \le K(|y|\wedge c^2)^{-1} \\ &\text{if } d=2, \quad \begin{cases} \sum_{z\in D_3}|z|^{-2} \le K\ln\left(\frac{c}{|y|}\vee e\right), \\ \sum_{z\in D_4}|z|^{-2}\exp\left(-K'\frac{|z|}{c}\right) \le K. \end{cases} \end{aligned} \tag{107}$$

Combining the bounds (106) and (107), and doing some elementary computations then leads to (100), (102).

Proof of (101), (103): From (11) and (12), we obtain

$$\int_{\frac{1}{2c^2}}^{\frac{t}{2}} du' \tilde{f}_{u'}\left(\frac{z}{c}-x\right) \le \begin{cases} K\psi_d(c) & \text{if } z = cx, \\ K\psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) & \text{if } z \ne cx, \\ K\exp\left(-K'\left|\frac{z}{c}-x\right|\right) & \text{if } \left|\frac{z}{c}-x\right| \ge \sqrt{T}. \end{cases} \tag{108}$$

Also, from (13),

$$\int_{2|y|^{-2}}^{\frac{Tc^2}{|y|^2}} dt' \hat{f}_{t'}\left(\frac{z}{|y|}\right)\hat{f}_{t'}\left(\frac{z-y}{|y|}\right) \le K\left(\frac{|z|}{|y|}\vee 1\right)^{2-2d}. \tag{109}$$

Thus, the bounds in (106) remain true when replacing F_1 with F_2, and (101), (103) follow. This completes the proof of Lemma 2. □

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