Strict Positivity of the Density for a Poisson Driven S.D.E.

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Abstract

We consider a one-dimensional stochastic differential equation driven by a compensated Poisson measure. We assume that this equation admits a unique solution X_t. We prove that under a strong non-degeneracy condition, for each $t > 0$, the law of X_t is bounded below by a measure that admits a strictly positive density with respect to the Lebesgue measure on $I\!R$. To this aim, we develop Bismut's approach of the Malliavin calculus for Poisson functionals.

Key words : Stochastic differential equations, Jump processes, Stochastic calculus of variations.

MSC 91 : 60H10, 60J75, 60H07.

1 Introduction.

Consider the following stochastic differential equation :

$$X_t = x_0 + \int_0^t \int_O h(X_{s-}, z)\tilde{N}(ds, dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-}, u)\tilde{N}_1(ds, du) \tag{1.1}$$

where

Assumption (M) : N and N_1 are two independant Poisson measures on $I\!R^+ \times O$ and $I\!R^+ \times E$, where O is an open subset of $I\!R$ and E is a Blackwell space (see Jacod, Shiryaev, [11], p 65). The intensity measures of N and N_1 are respectively

$$\nu(ds, dz) = \varphi(z)dsdz \quad ; \quad \nu_1(ds, du) = dsq(du) \tag{1.2}$$

where φ is a strictly positive C^1 function on O, and q is a positive σ-finite measure on E. We denote by $\tilde{N}$ and $\tilde{N}_1$ the associated compensated measures.

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We assume that equation (1.1) admits a unique solution X_t, and we fix $T > 0$. Our problem is to prove that under some conditions, the law of X_T is bounded below by a measure that admits a strictly positive density. This mean that we are interested in the existence of a strictly positive function $\theta : I\!R \mapsto I\!R^+$ such that for all $f \in C_b^+(I\!R)$,

$$E\left(f(X_T)\right) \geq \int_{I\!R} f(y)\theta(y)dy \tag{1.3}$$

In particular, if the law of X_T admits a continuous density p_T, this will yield that $p_T(x) > 0$ for all $x \in I\!R$. In order to study this problem, we transpose to our context a method based on the Malliavin Calculus for Gaussian functionals, investigated by Ben Arous and Léandre in [3], and later by Bally and Pardoux in [2]. Bismut's approach of the Malliavin Calculus is used in [2]. Following this approach and the work of Bichteler and Jacod, [5], we will build a sequence of perturbations, then we will differentiate our perturbed process and study the obtained derivatives.

Comparing our work with that of Bally, Pardoux, we see that the big difficulty (and also the limit) of our work is that when we differentiate, we obtain integrals against the Poisson measure instead of the Lebesgue measure. We thus have to choose non deterministic perturbations and to deal with stopping times, which makes everything hard and drives to stringent conditions.

Of course, the result we obtain here in a quite general context is not completely satisfying : as for the existence of a smooth density, the assumptions we need are not very explicit. However, we have applied in [7] the present method for the case of a particular nonlinear S.D.E. in order to study the strict positivity of the solution of a generalized Kac equation, and we have obtained quite a good result.

The present work is organized as follows. In the second section, we recall the assumptions given by Bichteler, Gravereaux, and Jacod in [4] under which the law of X_T admits a continuous density (in the case where $\varphi \equiv 1$), then we give our assumptions and we state our main result. The third section is devoted to the exposition of our notations and to the proof of the criterion of strict positivity we will use. Finally, we prove our main theorem in the last sections.

Let us now mention alternative methods that could be used to study our problem. First, one might use the Markov property of equation (1.1), in order to obtain minorations of the density. This method looks natural, but in fact, it seems difficult to apply, and probably necessitates more regularity of the density. Another idea could consist in applying the results of Simon, [15], who characterizes the support (in $I\!D([0,T], I\!R)$) of the law of X, and in using a method as that of Millet, Sanz, [13].

In the case of equation (1.1), all these methods might work and give different results. Anyway, our method, based on the stochastic calculus of variations, seems to be the only (probabilistic) way to prove that the solution of the Kac equation is strictly positive. Indeed, the S.D.E. associated with the Kac equation is nonlinear, thus it may be very difficult to use a Markov property or to prove a support theorem.

Let us finally mention Léandre, [12], Ishikawa, [9], and Picard, [14], who proved lowerbounds of the density in small time.

2 Statement of the main result.

This section is divided in three parts. In the first part, we recall the result of Bichteler et al. in [4]. We state our assumptions and results in the second one. At last, the third part deals with remarks and examples of applications.

2.1 Existence of a continuous density in the case where $\varphi \equiv 1$.

When $\varphi \equiv 1$, Bichteler, Gravereaux, and Jacod give in [4] a sufficient condition under which the law of X_T admits a continuous density. In fact, they do not prove a minimal assumption for this problem, since they are interested in the (at least) continuous differentiability of the density. The existence of a density for equation (1.1) has been studied in [8], see also Denis, [6], but no continuity result seems to be known, even if the method of Bichteler et al. in [4] could probably be easily extended. Let us recall the assumptions in [4].

Assumption $(A-4)$: the function g is four times differentiable on $I\!R$, and its derivatives of order 1 to 4 are bounded. The function h is four times differentiable on $I\!R \times O$, the partial derivatives $h^{(n+q)}_{x^n z^q}$ are bounded as soon as $q \geq 1$ (with $n + q \leq 4$), and there exists a function $\eta \in \cap_{2 \leq p < \infty} L^p(O, dz)$ such that

$$|h(0,z)| + |h'_x(x,z)| + ... + |h^{(4)}_{x^4}(x,z)| \leq \eta(z) \tag{2.1}$$

For any $u \in E$, the function $f(.,u)$ is four times differentiable on $I\!R$, and there exists a function $\sigma \in \cap_{2 \leq p < \infty} L^p(E, q)$ such that

$$|f(0,u)| + |f'_x(x,u)| + ... + |f^{(4)}_{x^4}(x,u)| \leq \sigma(u) \tag{2.2}$$

Assumption (SC) : there exists $c_0 > 0$ such that identically,

$$1 + h'_x(x,z) \geq c_0 \quad ; \quad 1 + f'_x(x,u) \geq c_0 \tag{2.3}$$

In [4], a positive function δ on O is called (ζ,θ)-broad (for some fixed $\zeta \geq 0$, $\theta > 0$) if

$$\int_0^\infty \lambda^{\zeta-1} \exp\left\{-\theta \int_O \left(1 - e^{-\lambda\delta(z)}\right) dz\right\} d\lambda < \infty \tag{2.4}$$

and one also considers functions α on O satisfying

$$\alpha \geq 0 \quad ; \quad \alpha \text{ is } C_b^\infty \text{ on } O \quad ; \quad \alpha(z) \longrightarrow_{z \to \partial O} 0 \quad ; \quad \forall\, r \in I\!N,\ \frac{\partial^r \alpha}{\partial z^r} \in L^1(O, dz) \tag{2.5}$$

Their last assumption is the following

Assumption $(SB)(\zeta,\theta)$: there exists $\epsilon > 0$, $q \geq 0$, a (ζ,θ)-broad function δ, and a function α satisfying (2.5) such that

$$\frac{{h'_z}^2(x,z)\alpha(z)}{(1 + h'_x(x,z))^2} \geq \frac{\epsilon}{1 + |x|^q}\delta(z) \tag{2.6}$$

We now can state the result in [4]:

Theorem 2.1 *Let $T > 0$ be fixed. Assume (M) with $\varphi \equiv 1$, $(A-4)$, (SC), and $(SB)(\zeta,\theta)$ for some $\theta \leq T$, and some $\zeta > \dfrac{4}{\left[\frac{T}{\theta}\right]}$. Then the law of X_T admits a C^1 density $p_T(x)$ with respect to the Lebesgue measure on $I\!R$.*

2.2 Strict positivity of the density.

Let us now turn back to our problem. A classical way to write S.D.E.s consists in assuming that $h(x,z)$ splits into $\psi(x)\eta(z)$. Although we state a more general formulation, all the hypotheses below are especially adapted to this case.
The two first assumptions are quite similar to $(A-4)$ and (SC).

<u>Assumption (H)</u> : the function g is C^3 on $I\!R$, and its derivatives g', g'', and g''' are bounded. The function h admits the continuous partial derivatives $h^{(n+q)}_{x^n z^q}$ for $n,q \in \{0,1,2,3\}$ on $I\!R \times O$, the derivatives $h^{(n+q)}_{x^n z^q}$ are bounded as soon as $q \geq 1$, and there exists a function $\eta \in L^2(O, \varphi(z)dz) \cap L^\infty(O, \varphi(z)dz)$ such that

$$|h(0,z)| + |h'_x(x,z)| + |h''_{xx}(x,z)| + |h'''_{xxx}(x,z)| \leq \eta(z) \tag{2.7}$$

For any $u \in E$, the function $f(.,u)$ is C^3 on $I\!R$, and there exists a function $\sigma \in L^2(E,q) \cap L^\infty(E,q)$ such that

$$|f(0,u)| + |f'_x(x,u)| + |f''_{xx}(x,u)| + |f'''_{xxx}(x,u)| \leq \sigma(u) \tag{2.8}$$

<u>Assumption (P)</u> : there exists $c_0 > 0$ such that

$$\forall\, x \in I\!R,\ \forall\, z \in O,\ \ 1 + h'_x(x,z) \geq c_0 \quad ; \quad \forall\, x \ \in\ I\!R,\ \int_E 1_{\{1+f'_x(x,u)=0\}} q(du) = 0 \tag{2.9}$$

In order to state our non-degeneracy condition, we introduce some notation.

Notation 2.2 *We set*

$$\beta(z) = \sup\left\{ \left| h^{(2+q)}_{x^q z^2}(x,z) \right| \ ;\ x \in I\!R,\ q \in \{0,1,2,3\} \right\} \tag{2.10}$$

Consider a C^1 positive function α on O such that $\| \alpha' \|_\infty < 1$ and such that $\alpha(z)$ goes to 0 when z goes to the boundary of O (this implies that for all $z \in O$, $[z-\alpha(z), z+\alpha(z)]$ is contained in O). Then we denote

$$\phi_\alpha(z) = \frac{\sup\{|\varphi'(w)|\ ;\ |w-z| \leq \alpha(z)\}}{\varphi(z)} \quad ; \quad \xi_\alpha(z) = |\alpha'(z)| + 3\alpha(z)\phi_\alpha(z) \tag{2.11}$$

$$\zeta_\alpha(z) = \sup\left\{ \left| h^{(3+q)}_{x^q z^3}(x,w) \right| \ ;\ x \in I\!R\ ,\ q \in \{0,1,2,3\}\ ,\ |w-z| \leq \alpha(z) \right\} \tag{2.12}$$

<u>Assumption (SP)</u> :

1. There exists continuous functions $\psi > 0$ on $I\!R$ and $\delta \geq 0$ on O such that :

$$\psi(x)\delta(z) \leq |h'_z(x,z)| \leq \delta(z) \tag{2.13}$$

$$\forall n \in \{1,2,3\}, \quad |h^{(n+1)}_{x^n z}(x,z)| \leq \delta(z) \tag{2.14}$$

Thanks to (H), ψ and δ are bounded.

2. There exists a sequence of C^1 positive functions α_n on O, a sequence of real numbers a_n decreasing to 0, a constant $d_0 \in]0,1[$ such that, if $\xi_n = \xi_{\alpha_n}$, $\zeta_n = \zeta_{\alpha_n}$, (see Notation 2.2)

$$\alpha_n + \xi_n \in L^1(O, \varphi(z)dz) \quad ; \quad \| \alpha_n \|_\infty + \| \xi_n \|_\infty \leq d_0 \quad ; \quad \alpha_n(z) \longrightarrow_{z \to \partial O} 0 \tag{2.15}$$

$$\forall\, n \;,\; \{\delta = 0\} \subset \{\alpha_n = 0\} \tag{2.16}$$

and

$$a_n \int_O \alpha_n(z)\delta(z)\varphi(z)dz \longrightarrow +\infty \tag{2.17}$$

$$a_n \int_O \left[\alpha_n^2(z)\delta^2(z) + \alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\right] \varphi(z)dz \longrightarrow 0 \tag{2.18}$$

Now we can state our main result :

Theorem 2.3 *Assume (M), (H), (P), (SP), and let $T > 0$ be fixed. Then the law of X_T is bounded below by a measure that admits a strictly positive density. This mean that there exists a strictly positive function $\theta : I\!R \mapsto I\!R^+$ such that for all $f \in C_b^+(I\!R)$,*

$$E\left(f(X_T)\right) \geq \int_{I\!R} f(y)\theta(y)dy \tag{2.19}$$

In particular, if the law of X_T admits a continuous density, then this density does never vanish.

The main supposition in (SP) is the following : $h'_z(x,.) \notin L^1(O, \varphi(z)dz)$ (obtained by (2.13), (2.15), and (2.17)). This assumption looks like (SB), but is much more stringent.
Somewhere in the proof of Theorem 2.3, we will need a function $v(s,z)$ of class C^1 on O such that $h'_z \times v \geq 0$. The first idea consists in choosing $v = h'_z \times w$, with w nonnegative, but this would drive us to the assumption $h'_z(x,.) \notin L^2(O, \varphi(z)dz)$, which is more stringent than (SP). This is why we state the following proposition.

Proposition 2.4 *1. Thanks to (2.16), we can assume that the sign of h'_z is constant on $I\!R \times O$.*

2. In (SP), we can assume that $d_0 \in]0,1[$ is as small as we want.

<u>Proof</u> :
1. We consider the solution X of equation (1.1). We assume only (SP), and we prove that X is the solution of another S.D.E. satisfying (SP), with $h'_z \geq 0$ (or $h'_z \leq 0$) identically. We first set

$$\mathcal{H} = \{z \in O \;/\; \delta(z) > 0\}$$
$$\mathcal{H}^+ = \{z \in \mathcal{H} \;/\; \forall x \in I\!R \;,\; h'_z(x,z) > 0\} \quad ; \quad \mathcal{H}^- = \{z \in \mathcal{H} \;/\; \forall x \in I\!R \;,\; h'_z(x,z) < 0\} \tag{2.20}$$

Since h'_z, and ψ are continuous, since ψ does never vanish, and since (2.13) is satisfied, we see that $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. We furthermore deduce that

$$\mathcal{H}^+ = \{z \in O \;/\; h'_z(0,z) > 0\} \tag{2.21}$$

is an open set. Of course, so is $\mathcal{H}^-$. From the definition of $\mathcal{H}$, and since $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, equation (2.17) yields the existence of a subsequence n_k such that either

$$a_{n_k} \int_{\mathcal{H}^+} \alpha_{n_k}(z)\delta(z)\varphi(z)dz \longrightarrow +\infty \tag{2.22}$$

or

$$a_{n_k} \int_{\mathcal{H}^-} \alpha_{n_k}(z)\delta(z)\varphi(z)dz \longrightarrow +\infty \tag{2.23}$$

Let us for example assume (2.22). We rewrite equation (1.1) as

$$X_t = x_0 + \int_0^t \int_{\mathcal{H}^+} h(X_{s-}, z)\tilde{N}^*(ds, dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-}, u)\tilde{N}_1(ds, du)$$
$$+ \int_0^t \int_{O\backslash\mathcal{H}^+} h(X_{s-}, z)\tilde{N}_2(ds, dz) \qquad (2.24)$$

where $N^* = N|_{I\!R^+\times\mathcal{H}^+}$ and $N_2 = N|_{I\!R^+\times(O\backslash\mathcal{H}^+)}$. It is clear that our method would not fail with two independent Poisson measures (independent of N^*) instead of one. Furthermore, (M), (H), and (P) are clearly satisfied with the new coefficients. At last, the only problem to check that (SP) is still satisfied (with of course h'_z nonnegative on $\mathcal{H}^+$) is to see that for all n, $\alpha_n(z)$ goes to 0 when z goes to $\partial\,\mathcal{H}^+\cap O$. Let $z_0 \in \partial\,\mathcal{H}^+ \cap O$. Thanks to (2.21), $\lim_{z_0} h'_z(0, z) = 0$, and thus $\lim_{z_0} \delta(z) = 0$. Since $\{\delta = 0\} \subset \{\alpha_n = 0\}$, we immediately conclude.

2. It suffices to notice that we can replace α_n by $C \times \alpha_n$ in (SP), for any $C \in]0, 1]$ fixed.

Let us say a word about N_1. We add an independent Poisson measure, in order to generalize our Theorem. Assume for example that the parameter h of equation (1.1) satisfies (SP) only on an open subset $A \subset O$. Then, replacing N by $N|_A$ and N_1 by $N_1 + N|_{O/A}$, the conditions might hold. The first part of Proposition 2.4 gives an example for this kind of method.

In the whole work, we will assume that $T > 0$ is fixed and that (M), (H), (P) and (SP) are satisfied. We will also suppose that h'_z is nonnegative on $I\!R \times O$.

2.3 Examples.

First, we give one "simple" way to obtain (SP) when $\varphi \equiv 1$. Similar methods may be used for any other particular φ.

Remark 2.5 *Assume that $\varphi \equiv 1$, that (SP)-1 and (SPI) below hold. Then (SP) is satisfied.*

Assumption (SPI) : there exists a positive C_b^1 function γ on O such that, for some $d_0 \in]0, 1[$,

$$\| \gamma \|_\infty + \| \gamma' \|_\infty \leq d_0 \quad ; \quad \gamma(z) \longrightarrow_{z\to\partial O} 0 \quad ; \quad \{\delta = 0\} \subset \{\gamma = 0\} \qquad (2.25)$$

$$\delta\gamma \notin L^1(O, dz) \quad ; \quad \gamma^2\delta^2 + \gamma^2\beta + \gamma^3\zeta_\gamma \in L^1(O, dz) \qquad (2.26)$$

where ζ_γ is defined in Notation 2.2.

Proof : in order to check this claim, choose a smooth version of

$$\alpha_n(z) = \begin{cases} \gamma(z) & \text{if} \quad z \in O\ ,\ |z| \leq n \\ 0 & \text{if} \quad z \in O\ ,\ |z| \geq n + 1 \end{cases} \qquad (2.27)$$

and set $a_n = \left(\int_O \alpha_n(z)\delta(z)dz \right)^{-\frac{1}{2}}$.

At last, we give two examples of function $h(x, z)$ satisfying (H), (P), and (SP).

Example 1 : we assume that $O =]1, \infty[$, that $\varphi \equiv 1$, and that $h(x, z) = c(x)\eta(z)$, where $\eta(z) = \sin z/z$, and where c is a strictly positive C^3 function on $I\!R$, bounded with all its derivatives. A simple computation shows that there exists $C < \infty$ such that $|\eta(z)| + ... + |\eta'''(z)| \leq C/z$. Thus (H) is satisfied. If $\parallel c' \parallel_\infty < 1$, it is clear that (P) is satisfied. Else, we replace O by $]2 \parallel c' \parallel_\infty, \infty[$, and we use the presence of N_1 as in the proof of Proposition 2.4-1. Condition (SP)-1 is satisfied with $\psi(x) = c(x)/A$, where $A = \parallel c \parallel_\infty$ +...+ $\parallel c''' \parallel_\infty$, and with

$$\delta(z) = A|\eta'(z)| = A\left|\frac{z\cos z - \sin z}{z^2}\right| \tag{2.28}$$

Thus $\delta(z)$ behaves as $1/z$ on a “large” subset of O. More precisely, there exists a constant $K < \infty$ such that

$$\text{for all} \quad z \in \cup_{k\geq 1}]2k\pi - \pi/4, 2k\pi + \pi/4[\,, \quad \delta(z) \geq A\frac{\sqrt{2}}{2} \times \frac{z-1}{z^2} \geq \frac{K}{z} \tag{2.29}$$

Hence we can say that δ, β, and ζ_γ, defined in (SP), will behave as $1/z$. We now search a function γ such that h satisfies (SPI) : we need $\gamma\delta \notin L^1(O, dz)$, but $\gamma^2\beta + \gamma^3\zeta_\gamma \in L^1(O, dz)$. We thus will choose γ behaving as $1/\ln z$. We also need that $\{\delta = 0\} \subset \{\gamma = 0\}$. This will hold if the support of γ is contained in $\cup_{k\geq 1}]2k\pi - \pi/4, 2k\pi + \pi/4[$. At last, γ has to be C^1, and bounded with its derivative. The function we search for is given by $\gamma(z) = b_0 \sum_{k\geq 1} \gamma_k(z)$, where for each k, the function γ_k is C^1 on O, satisfies

$$\gamma_k(z) = \begin{cases} \dfrac{1}{\ln(k+1)} & \text{if} \quad z \in [2k\pi - \pi/8, 2k\pi + \pi/8] \\ \\ 0 & \text{if} \quad z \notin [2k\pi - \pi/4, 2k\pi + \pi/4] \end{cases} \tag{2.30}$$

and $0 \leq \gamma_k \leq \dfrac{1}{\ln(k+1)}$, $|\gamma_k'| \leq \dfrac{16}{\pi}\dfrac{1}{\ln(k+1)}$. For b_0 small enough, $\parallel \gamma \parallel_\infty + \parallel \gamma' \parallel_\infty \leq d_0$, for some $d_0 < 1$. At last, it is clear that (2.26) is satisfied.

Example 2 : we now set $O =]0, \infty[$, and we consider the standard Levy measure on O, $\varphi(z)dz = z^{-2}dz$. We assume that $h(x, z) = c(x)\eta(z)$, where c is as in example 1, and where $\eta(z) = z/(z+1)$. Assumptions (H) and (P) are obviously satisfied, at least if $\parallel c' \parallel_\infty < 1$, and (SP)-1 is met with $\psi(x) = c(x)/A$, where $A = \parallel c \parallel_\infty$ +...+ $\parallel c''' \parallel_\infty$, and with

$$\delta(z) = A|\eta'(z)| = A(1+z)^{-2} \tag{2.31}$$

We now consider a sequence of C^1 nonnegative functions α_n on O satisfying, for some $0 < k < 1/2$, $\alpha_n(z) \leq k(z \wedge 2)$, and

$$\alpha_n(z) = \begin{cases} 0 & \text{if} \quad z \in]0, 1/n] \\ kz & \text{if} \quad z \in]2/n, 1] \\ 0 & \text{if} \quad z \geq 2 \end{cases} \quad \text{and} \quad |\alpha_n'(z)| \leq \begin{cases} 0 & \text{if} \quad z \in]0, 1/n] \\ 4k & \text{if} \quad z \in]1/n, 2] \\ 0 & \text{if} \quad z \geq 2 \end{cases} \tag{2.32}$$

One can easily check that for some constants B, C, $\phi_{\alpha_n}(z) \leq C/z$, and that $\xi_{\alpha_n}(z) \leq |\alpha_n'(z)| + C\alpha_n(z)/z$. Choosing k small enough, we see that $\parallel \alpha_n \parallel_\infty + \parallel \xi_{\alpha_n} \parallel_\infty \leq d_0$, for some $d_0 < 1$. Since α_n and ξ_{α_n} are bounded and vanish near 0, they belong to $L^1(]0, \infty[, z^{-2}dz)$. At last, the existence of a sequence a_n yielding (2.17) and (2.18) is immediate, since

$$\int_O \alpha_n(z)\delta(z)\varphi(z)dz \geq C\int_{2/n}^1 \frac{kz}{(1+z)^2} \times \frac{dz}{z^2} \geq C\int_{2/n}^1 \frac{dz}{z} \longrightarrow_{n\to\infty} \infty \tag{2.33}$$

$$\int_O \left[\alpha_n^2(z)\delta^2(z) + \alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\right]\varphi(z)dz \leq C\int_0^\infty \alpha_n^2(z)\frac{dz}{z^2} \leq C\int_0^\infty z^2 \wedge 1\frac{dz}{z^2} \leq C \tag{2.34}$$

3 A criterion of strict positivity.

In the whole work, $\Omega = \Omega^N \times \Omega^{N_1}$ is the canonical product space associated with the independent random elements N and N_1. We will in fact be interested only in N.

This section contains two parts. We first introduce some general notations and definitions about Bismut's approach of the Malliavin Calculus on the Poisson space associated with N. We extend here the work of Bichteler, Jacod, [5], who work with $\varphi \equiv 1$. Then we adapt the criterion of strict positivity of Bally, Pardoux, [2] (which deals with the Wiener functionals) to the Poisson functionals.

Definition 3.1 *A predictable function $v(\omega, s, z)$ on $\Omega \times [0,T] \times O$ is said to be a* **perturbation** *if for all fixed ω, s, $v(\omega, s, .)$ is C^1 on O, if there exists two positive functions α and ρ on O such that,*

$$|v(\omega, s, z)| \leq \alpha(z) \quad ; \quad |v'(\omega, s, z)| \leq \rho(z) \tag{3.1}$$

and such that, if

$$\phi_\alpha(z) = \frac{\sup\{|\varphi'(w)| \; ; \; |w - z| \leq \alpha(z)\}}{\varphi(z)} \quad ; \quad \xi(z) = \rho(z) + 3\alpha(z)\phi_\alpha(z) \tag{3.2}$$

then, for some constant $c < 1$,

$$\alpha + \xi \in L^1 \cap L^\infty(O, \varphi(z)dz) \quad ; \quad \alpha(z) \longrightarrow_{z \to \partial O} 0 \quad ; \quad \xi(z) \leq c \tag{3.3}$$

We now consider a fixed perturbation v. For each $\omega \in \Omega$, $\lambda \in [-1, 1]$ and $s \in [0,T]$, the map

$$z \mapsto \gamma^\lambda(\omega, s, z) = z + \lambda v(\omega, s, z) \tag{3.4}$$

is an increasing bijection from O to O, thanks to (3.1) and (3.3). We now set

$$Y^\lambda(\omega, s, z) = (1 + \lambda v'(\omega, s, z)) \times \frac{\varphi(\gamma^\lambda(\omega, s, z))}{\varphi(z)} \tag{3.5}$$

Then, a simple substitution shows that

$$\gamma^\lambda(Y^\lambda.\nu) = \nu \tag{3.6}$$

i.e. that for all Borel set $A \subset [0,T] \times O$,

$$\int_0^T \int_O 1_A(s, \gamma^\lambda(s,z)) Y^\lambda(s,z)\varphi(z) dz ds = \int_0^T \int_O 1_A(s, z')\varphi(z')dz'ds \tag{3.7}$$

We also denote by $N^\lambda = \gamma^\lambda(N)$ the image measure of N by γ^λ

$$N^\lambda(A) = \int_0^T \int_O 1_A(s, \gamma^\lambda(s,z)) N(ds, dz) \tag{3.8}$$

and by S^λ the shift on Ω defined by

$$N \circ S^\lambda(\omega) = N^\lambda(\omega) \quad ; \quad N_1 \circ S^\lambda(\omega) = N_1(\omega) \tag{3.9}$$

Then we consider the following martingale

$$M_t^\lambda = \int_0^t \int_O (Y^\lambda(s,z) - 1)\tilde{N}(ds, dz) \tag{3.10}$$

and its Doléans-Dade exponential (see Jacod, Shiryaev, [11], p 59)

$$G_t^\lambda = 1 + \int_0^t G_{s-}^\lambda dM_s^\lambda = e^{M_t^\lambda} \prod_{0\le s\le t} \left(1 + \Delta M_s^\lambda\right) e^{-\Delta M_s^\lambda} \tag{3.11}$$

which clearly is a square integrable martingale. Using the fact that Y^λ is always strictly positive, we see that G^λ is strictly positive a.s. We now set $P^\lambda = G_T^\lambda.P$. Thanks to (3.6), the Girsanov Theorem for random measures (see Jacod, Shiryaev, [11], p 157) shows that $P^\lambda \circ (S^\lambda)^{-1} = P$, i.e. that the law of (N^λ, N_1) under P^λ is the same as that of (N, N_1) under P.

It is easy to check that :

$$|Y^\lambda(s,z) - Y^\mu(s,z)| \le |\lambda - \mu|\xi(z) \tag{3.12}$$

We at last check the following lemma :

Lemma 3.2 *If v is a perturbation, and if G^λ is the associated exponential martingale, then a.s., the map $\lambda \mapsto G_T^\lambda$ is continuous.*

Proof : using (3.11), we obtain

$$G_T^\lambda = \exp\left[-\int_0^T \int_O \left(Y^\lambda(s,z) - 1\right)\varphi(z)dzds\right] \times \exp\left[\int_0^T \int_O \ln Y^\lambda(s,z) N(ds,dz)\right] \tag{3.13}$$

Using (3.12) and the fact that $\xi \in L^1(O, \varphi(z)dz)$, it is obvious that the first term in the product is a.s. continuous. Furthermore,

$$|Y^\lambda(s,z) - 1| = |Y^\lambda(s,z) - Y^0(s,z)| \le \xi(z) \le c < 1 \tag{3.14}$$

Thus for all $\lambda, \mu \in [-1,1]$,

$$\left|\ln\left(Y^\lambda(s,z)\right) - \ln\left(Y^\mu(s,z)\right)\right| \le \frac{1}{1-c}\left|Y^\lambda(s,z) - Y^\mu(s,z)\right| \le \frac{1}{1-c}|\lambda-\mu|\xi(z) \tag{3.15}$$

and the second term is also continuous.

We now give a criterion of strict positivity.

Theorem 3.3 *Let X be a real valued random variable on Ω and let $y_0 \in I\!R$. Assume that there exists a sequence v_n of perturbations such that, if $X^n(\lambda) = X \circ S_n^\lambda$, then for each n, the map*

$$\lambda \mapsto X^n(\lambda) \tag{3.16}$$

is a.s. twice differentiable on $[-1,1]$. Assume that there exists $c > 0$, $\delta > 0$, and $k < \infty$, such that for all $r \in]0,1]$,

$$\lim_{n\to\infty} P\left(\Lambda^n(r)\right) > 0 \tag{3.17}$$

where

$$\Lambda^n(r) = \left\{|X - y_0| < r\ ,\ \left|\frac{\partial}{\partial\lambda}X^n(0)\right| \ge c\ ,\ \sup_{|\lambda|\le\delta}\left[\left|\frac{\partial}{\partial\lambda}X^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X^n(\lambda)\right|\right] \le k\right\} \tag{3.18}$$

Then there exists a continuous function $\theta_{y_0}(.) : I\!R \mapsto I\!R^+$ such that $\theta_{y_0}(y_0) > 0$ and such that for all $f \in C_b^+(I\!R)$,

$$E(f(X)) \ge \int_{I\!R} f(y)\theta_{y_0}(y)dy \tag{3.19}$$

In order to prove this criterion, we will use the following uniform local inverse Theorem, that can be found in Aida, Kusuoka, Stroock, [1] :

Lemma 3.4 *Let $c > 0$, $\delta > 0$, and $k < \infty$ be fixed. Consider the following set :*

$$\mathcal{G} = \left\{ g : I\!R \mapsto I\!R \Big/ |g'(0)| \geq c\,,\ \sup_{|x|\leq\delta} \left[|g(x)| + |g'(x)| + |g''(x)|\right] \leq k \right\} \tag{3.20}$$

Then there exists $\alpha > 0$ and $R > 0$ such that for every $g \in \mathcal{G}$, there exists a neigbourhood $\mathcal{V}_g$ of 0 contained in $]-R, R[$ such that g is a diffeomorphism from $\mathcal{V}_g$ to $]g(0) - \alpha, g(0) + \alpha[$.

Since this Lemma deals with the behaviour of functions near 0, it can be obviously adapted to functions on $[-1, 1]$.

Proof of Theorem 3.3 :
Step 1 : first notice that for all $r \leq 1$, all n, and all $\omega \in \Lambda_n(r)$,

$$\sup_{|\lambda|\leq\delta} |X^n(\omega,\lambda)| \leq |X^n(\omega, 0)| + \delta k = |X(\omega)| + \delta k \leq |y_0| + 1 + \delta k = k' \tag{3.21}$$

Thus, using Lemma 3.4, there exists $\alpha > 0$ and $R \in]0, 1]$, depending only on δ, c, k, and k', such that for all $r \leq 1$, all n, and all $\omega \in \Lambda_n(r)$, there exists $V_n(\omega)$ a neighbourhood of 0 contained in $]-R, R[$ such that the map $\lambda \mapsto X^n(\omega, \lambda)$ is a diffeomorphism from $V_n(\omega)$ to $]X^n(\omega, 0) - \alpha, X^n(\omega, 0) + \alpha[=]X(\omega) - \alpha, X(\omega) + \alpha[$. Choosing α small enough, we can assume that $R \leq c/2k$. Thus, for all $\omega \in \Lambda_n(r)$ and $\lambda \in V_n(\omega)$, we have

$$\left|\frac{\partial}{\partial\lambda} X^n(\lambda)\right| \geq c/2 \tag{3.22}$$

We now fix $r < \alpha$, and we choose n large enough such that $P(\Lambda_n(r)) > 0$.

Step 2 : the perturbations have been built in order to obtain, for all λ and all $f \in C_b^+(I\!R)$,

$$E(f(X)) = E(f(X^n(\lambda))G_T^n(\lambda)) \tag{3.23}$$

Thus

$$\begin{aligned} E(f(X)) &= \frac{1}{2}\int_{-1}^{1} E(f(X^n(\lambda))G_T^n(\lambda))d\lambda \\ &\geq \frac{1}{2} E\left[\int_{V_n} f(X^n(\lambda))G_T^n(\lambda)d\lambda \times 1_{\Lambda_n(r)}\right] \end{aligned} \tag{3.24}$$

Using the first step, we substitute $y = X^n(\lambda)$, and we obtain :

$$\begin{aligned} E(f(X)) &\geq \frac{1}{2} E\left[\int_{]X-\alpha, X+\alpha[} f(y) \frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda} X^n(\{X^n\}^{-1}(y))\right|} dy \times 1_{\Lambda_n(r)}\right] \\ &\geq \int_{I\!R} f(y) E\left[\frac{1}{2}\psi(|X-y|)\left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda} X^n(\{X^n\}^{-1}(y))\right|}\right) \times 1_{\Lambda_n(r)}\right] dy \end{aligned} \tag{3.25}$$

where ψ is a continuous function on $I\!R^+$ such that $1_{[0,r]} \leq \psi \leq 1_{[0,\alpha]}$. Let

$$\theta_n(y) = E\left[\frac{1}{2}\psi(|X-y|) \times \left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda} X^n(\{X^n\}^{-1}(y))\right|}\right) \times 1_{\Lambda_n(r)}\right] \tag{3.26}$$

Step 3 : on one hand, it is clear that $\theta_n(y_0) > 0$ (recall the definition of $\Lambda_n(r)$). On the other hand, one can show by using the Lebesgue Theorem and Lemma 3.2 that θ_n is continuous. This concludes the proof.

We at last state a usefull remark.

Remark 3.5 *Let X be a real-valued random variable on Ω. Suppose that for each $y_0 \in \mathrm{supp}\, P \circ X^{-1}$, the assumptions of Theorem 3.3 hold. Then the law of X is bounded below by a measure with a strictly positive density.*

Proof : for each y_0 in $\mathrm{supp}\, P \circ X^{-1}$, we consider the continuous function θ_{y_0} built in Theorem 3.3. Since $\theta_{y_0}(y_0) > 0$, there exists a neighbourhood $\mathcal{W}_{y_0}$ of y_0 on which θ_{y_0} does not vanish.
We easily deduce from (3.19) that for each y_0 in $\mathrm{supp}\, P \circ X^{-1}$,

$$\mathcal{W}_{y_0} \subset \mathrm{supp}\, P \circ X^{-1} \tag{3.27}$$

Thus $\mathrm{supp}\, P \circ X^{-1}$ is an open set, and therefore is the whole real line.
For each $n \in \mathbf{Z}$, we thus can build a strictly positive function θ^n on the compact set $[n, n+1]$ such that for all $f \in C_b^+(I\!R)$ of which the support is contained in $[n, n+1]$,

$$E(f(X)) \geq \int_{I\!R} f(y)\theta^n(y)dy \tag{3.28}$$

Using the function $\theta(y) = \sum_{n\in\mathbf{Z}} \theta^n(y) 1_{[n,n+1[}(y)$, one concludes easily.

In order to prove Theorem 2.3, we will of course apply the previous criterion. In the next section, we will consider a fixed perturbation v_n, and we will compute $X_t^n(\lambda) = X_t \circ S_n^\lambda$ and its derivatives for any $t \in [0, T]$. Section 5 is devoted to the explicit choice of the sequence v_n of perturbations. In Section 6, we will prove that for each y_0 in $I\!R$, there exists a constant $C(y_0) > 0$ such that for any $r > 0$,

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| \geq C(y_0) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \tag{3.29}$$

At last, we will check in Section 7 that for some constant K,

$$P\left(\sup_{|\lambda|\leq 1}\left\{\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X_T^n(\lambda)\right|\right\} \leq K\right) \longrightarrow 1 \tag{3.30}$$

Since for all $y_0 \in \mathrm{supp}\, P \circ X_T^{-1}$, for all $r > 0$, $P(X_T \in]y_0 - r, y_0 + r[) > 0$, we will easily conclude at the end of Section 7.

4 Computation of the derivatives of X.

Recall that

$$X_t = x_0 + \int_0^t \int_O h(X_{s-}, z)\tilde{N}(ds, dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-}, u)\tilde{N}_1(ds, du) \tag{4.1}$$

We consider in this section a fixed perturbation v_n, and compute $X_t^n(\lambda) = X_t \circ S_n^\lambda$. Then we prove that for each t, $X_t^n(\lambda)$ is a.s. differentiable on $[-1, 1]$ and obtain $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$. At last, we study the second derivative $\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)$.

4.1 The perturbed process.

The direct expression of $X_t^n(\lambda)$ is given by

$$\begin{aligned} X_t^n(\lambda) &= x_0 + \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \nu)(ds, dz) + \int_0^t g(X_{s-}^n(\lambda))ds \\ &+ \int_0^t \int_E f(X_{s-}^n(\lambda), u)\tilde{N}_1(ds, du) \end{aligned} \tag{4.2}$$

But

$$\begin{aligned} \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \nu)(ds, dz) &= \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \gamma_n^\lambda(\nu))(ds, dz) \\ &+ \int_0^t \int_O h(X_{s-}^n(\lambda), z)(\gamma_n^\lambda(\nu) - \nu)(ds, dz) \\ &= \int_0^t \int_O h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z))\tilde{N}(ds, dz) \\ &+ \int_0^t \int_O \Big[h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z) - h(X_{s-}^n(\lambda), z)\Big] \varphi(z)dzds \end{aligned} \tag{4.3}$$

Finally,

Proposition 4.1 *For each $\lambda \in [-1, 1]$, the perturbed process $X_t^n(\lambda) = X_t \circ S_n^\lambda$ is solution of the following equation :*

$$\begin{aligned} X_t^n(\lambda) &= x_0 + \int_0^t \int_O h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z))\tilde{N}(ds, dz) \\ &+ \int_0^t \int_O \Big[h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z)) - h(X_{s-}^n(\lambda), z)\Big] \varphi(z)dzds \\ &+ \int_0^t g(X_{s-}^n(\lambda))ds + \int_0^t \int_E f(X_{s-}^n(\lambda), u)\tilde{N}_1(ds, du) \end{aligned} \tag{4.4}$$

4.2 The first derivative.

We now would like to differentiate the paths of the map $\lambda \mapsto X^n(\lambda)$. Consider the following linear equation, that is obtained by differentiating formally (4.4).

$$\begin{aligned} \frac{\partial}{\partial \lambda} X_t^n(\lambda) &= \int_0^t \int_O h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z)) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda)\tilde{N}(ds, dz) \\ &+ \int_0^t \int_O h'_z(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z))v_n(s, z)\tilde{N}(ds, dz) \\ &+ \int_0^t \int_O \Big[h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z) - h'_x(X_{s-}^n(\lambda), z)\Big] \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda)\varphi(z)dzds \\ &+ \int_0^t \int_O h'_z(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z))v_n(s, z)\varphi(z)dzds \\ &+ \int_0^t g'(X_{s-}^n(\lambda)) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda)ds + \int_0^t \int_E f'_x(X_{s-}^n(\lambda), u) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda)\tilde{N}_1(ds, du) \end{aligned} \tag{4.5}$$

Proposition 4.2 *Fix an integer n. The map $\lambda \mapsto X_T^n(\lambda)$ is a.s. differentiable on $[-1,1]$, and its derivative $\frac{\partial}{\partial \lambda} X_T^n(\lambda)$ is the terminal value of the solution of equation (4.5).*

Proof : for simplicity, we drop the superscript n, since it is fixed. We break the proof in several steps.

Step 1 : we consider an increasing sequence O_k (resp. E_k) of subsets of O (resp. of E) such that O_k goes to O (resp. E_k goes to E), and for all k,

$$\int_{O_k} \varphi(z)dz < \infty \qquad ; \qquad \int_{E_k} q(du) < \infty \tag{4.6}$$

Then we denote by $\bar{X}_t^k(\lambda)$ and $\frac{\partial}{\partial \lambda}\bar{X}_t^k(\lambda)$ the solutions of (4.4) and (4.5) where we have replaced O and E by O_k and E_k. Using classical estimates, as Burkholder's inequality and Gronwall's Lemma, one can easily check that $\bar{X}_T^k(\lambda) - X_T(\lambda)$ and $\frac{\partial}{\partial \lambda}\bar{X}_T^k(\lambda) - \frac{\partial}{\partial \lambda}\bar{X}_T(\lambda)$ satisfy the assumptions of Lemma 8.1 of the Appendix. Thus there exists a subsequence such that a.s., when l goes to infinity,

$$\sup_{|\lambda|\leq 1}\left|\bar{X}_T^{k_l}(\lambda) - X_T(\lambda)\right| + \sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial \lambda}\bar{X}_T^{k_l}(\lambda) - \frac{\partial}{\partial \lambda}X_T(\lambda)\right| \longrightarrow 0 \tag{4.7}$$

Step 2 : We now fix k, and we prove that there exists a random variable $Z^k < \infty$ a.s. such that for all λ, μ, and all $t \in [0,T]$,

$$\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| \leq |\mu| Z^k \tag{4.8}$$

Indeed, it is possible, using strongly the fact that $N([0,T]\times O_k) + N_1([O,T]\times E_k) < \infty$ a.s., that there exists a constant $C < \infty$ and a random variable $A^k < \infty$ a.s. such that for all $t \in [0,T]$ and all λ, μ,

$$\begin{aligned}\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| &\leq |\mu|A^k + C\int_0^t \left|\bar{X}_{s-}^k(\lambda+\mu) - \bar{X}_{s-}^k(\lambda)\right| ds \\ &+C\int_0^t\int_{O_k}\left|\bar{X}_{s-}^k(\lambda+\mu) - \bar{X}_{s-}^k(\lambda)\right| N(ds,dz) + C\int_0^t\int_{E_k}\left|\bar{X}_{s-}^k(\lambda+\mu) - \bar{X}_{s-}^k(\lambda)\right| N_1(ds,du)\end{aligned} \tag{4.9}$$

We denote by $0 < S_1 < ... < S_\nu < T$ the times of jump of the process $N([0,t]\times O_k) + N_1([0,t]\times E_k)$. This way, we obtain

$$\begin{aligned}\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| &\leq |\mu|A^k + C\int_0^t \left|\bar{X}_{s-}^k(\lambda+\mu) - \bar{X}_{s-}^k(\lambda)\right| ds \\ &\quad +C\sum_{i=1}^{\nu}\left|\bar{X}_{S_i-}^k(\lambda+\mu) - \bar{X}_{S_i-}^k(\lambda)\right| 1_{\{t\geq S_i\}}\end{aligned} \tag{4.10}$$

Using Gronwall's Lemma on $[0,S_1[$, we obtain for all $t \in [0,S_1[$,

$$\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| \leq |\mu|A^k e^{CS_1} \leq |\mu|A^k e^{CT} \tag{4.11}$$

This way, we can write for t in $[S_1,S_2[$

$$\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| \leq |\mu|A^k + CS_1|\mu|A^k e^{CT} + C\int_{S_1}^t \left|\bar{X}_{s-}^k(\lambda+\mu) - \bar{X}_{s-}^k(\lambda)\right| ds + C|\mu|A^k e^{CT} \tag{4.12}$$

Using again Gronwall's Lemma, we obtain for all $t \in [S_1,S_2[$,

$$\left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda)\right| \leq |\mu|\left[A^k + A^kCTe^{CT} + CA^ke^{CT}\right]e^{CT} \tag{4.13}$$

Iterating this argument, we obtain the existence of the announced random variable Z^k, and the second step is finished.

Step 3 : we again fix k, and we set

$$\Delta_t^k(\lambda,\mu) = \left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda) - \mu\frac{\partial}{\partial\lambda}\bar{X}_t^k(\lambda)\right| \tag{4.14}$$

Then one can check, by using Step 2 and the same arguments as in this previous step, that there exists a random variable $U^k < \infty$ a.s. such that for all $t \in [0,T]$, and all λ, μ,

$$\Delta_t^k(\lambda,\mu) \le \mu^2 U^k \tag{4.15}$$

This of course implies that a.s., the map $\lambda \mapsto \bar{X}_T^k(\lambda)$ is differentiable on $[-1,1]$, and its derivative is $\frac{\partial}{\partial\lambda}\bar{X}_T^k(\lambda)$.

This and the convergence (4.7) yield that the Proposition is proved.

Rewriting equation (4.5), we obtain

Proposition 4.3 *for each $\lambda \in [-1,1]$, the process $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$ is solution of the following S.D.E.*

$$\begin{aligned}
\frac{\partial}{\partial\lambda}X_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}(ds,dz) \\
&+ \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\varphi(z)dzds \\
&+ \int_0^t g'(X_{s-}^n(\lambda))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}_1(ds,du) \\
&+ \int_0^t\int_O h'_z(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))v_n(s,z)N(ds,dz)
\end{aligned} \tag{4.16}$$

This can also be written

$$\frac{\partial}{\partial\lambda}X_t^n(\lambda) = \int_0^t \frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)dK_s^n(\lambda) + H_t^n(\lambda) \tag{4.17}$$

where

$$\begin{aligned}
K_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\tilde{N}(ds,dz) \\
&+ \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]\varphi(z)dzds \\
&+ \int_0^t g'(X_{s-}^n(\lambda))ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\tilde{N}_1(ds,du)
\end{aligned} \tag{4.18}$$

and

$$H_t^n(\lambda) = \int_0^t\int_O h'_z(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))v_n(s,z)N(ds,dz) \tag{4.19}$$

Thanks to assumption (P), $1+\Delta K_{s-}^n(\lambda)$ does never vanish, thus the Doléans-Dade exponential martingale $\mathcal{E}(K^n(\lambda))$ is a.s. invertible, and we can write (see Jacod, [10]) :

$$\frac{\partial}{\partial\lambda}X_t^n(\lambda) = \mathcal{E}(K^n(\lambda))_t \times \int_0^t \mathcal{E}(K^n(\lambda))_{s-}^{-1}\left(1+\Delta K_s^n(\lambda)\right)^{-1}dH_s^n(\lambda) \tag{4.20}$$

Finally, since N and N_1 are independent, they never jump together a.s., and we obtain :

Remark 4.4 *The process* $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$ *is given by*

$$\frac{\partial}{\partial\lambda}X_t^n(\lambda) = \mathcal{E}(K^n(\lambda))_t \times \int_0^t\int_O \mathcal{E}(K^n(\lambda))_{s-}^{-1}\frac{h_z'(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))}{1+h_x'(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))}v_n(s,z)N(ds,dz) \tag{4.21}$$

4.3 The second derivative.

Exactly as in the previous subsection, we obtain :

Proposition 4.5 *The second derivative* $\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)$ *is solution of the following equation :*

$$\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda) = \int_0^t \frac{\partial^2}{\partial\lambda^2}X_{s-}^n(\lambda)dK_s^n(\lambda) + L_t^n(\lambda) \tag{4.22}$$

where

$$\begin{aligned}
L_t^n(\lambda) &= 2\int_0^t\int_O h_{xz}''(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)v_n(s,z)N(ds,dz)\\
&+\int_0^t\int_O h_{zz}''(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))v_n^2(s,z)N(ds,dz)\\
&+\int_0^t\int_O h_{xx}''(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\left(\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right)^2\tilde N(ds,dz)\\
&+\int_0^t\int_O \left[h_{xx}''(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h_{xx}''(X_{s-}^n(\lambda),z)\right]\left(\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right)^2\varphi(z)dzds \qquad (4.23)\\
&+\int_0^t g''(X_{s-}^n(\lambda))\left(\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right)^2ds + \int_0^t\int_E f_{xx}''(X_{s-}^n(\lambda),u)\left(\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right)^2\tilde N_1(ds,du)
\end{aligned}$$

We end this section with a classical remark.

Remark 4.6 *For each* $n\in I\!N$, *and each* $\lambda\in[-1,1]$, *the processes* $X_t^n(\lambda)$, $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$, *and* $\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)$ *are the solution of standard S.D.E.s. Thanks to* (H), *it is well-known that for each* λ, *each* n, *and each* $p<\infty$, *there exists a constant* $C(n,p,\lambda)$ *such that*

$$E\left(\sup_{[0,T]}|X_t^n(\lambda)|^p\right) + E\left(\sup_{[0,T]}\left|\frac{\partial}{\partial\lambda}X_t^n(\lambda)\right|^p\right) + E\left(\sup_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)\right|^p\right) \le C(n,p,\lambda) \tag{4.24}$$

5 Choice of the perturbation.

Let us first recall that, thanks to (4.18) and (4.21) :

$$\frac{\partial}{\partial\lambda}X_T^n(0) = \mathcal{E}(K)_T \times \int_0^T\int_O \mathcal{E}(K)_{s-}^{-1}\frac{h_z'(X_{s-},z)}{1+h_x'(X_{s-},z)}v_n(s,z)N(ds,dz) \tag{5.1}$$

where

$$K_t = \int_0^t\int_O h_x'(X_{s-},z)\tilde N(ds,dz) + \int_0^t g'(X_{s-})ds + \int_0^t\int_E f_x'(X_{s-},u)\tilde N_1(ds,du) \tag{5.2}$$

The main idea consists in choosing v_n in such a way that for some $0<\epsilon<K<\infty$, the probability

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right|\in[\epsilon,K] \Big/ X_T\in[y_0-r,y_0+r]\right)$$

goes to 1 when n goes to infinity. In order to get rid of the random terms $\mathcal{E}(K)_T$ and $\mathcal{E}(K)_{s-}^{-1}$, we will choose $v_n(s,z)$ equal to 0 for $0 \leq s < T - a_n$, for some sequence a_n decreasing to 0, and we will use the a.s. continuity of $\mathcal{E}(K)$ at T. The term $(1 + h'_x(X_{s-}, z))^{-1}$ is not a problem, since it always belongs to $[1/(1+ \parallel \eta \parallel_\infty), 1/c_0]$, thanks to (P) and (H). Thanks to (SP), and proposition 2.4, $h'_z(X_{s-}, z) \sim \psi(y_0)\delta(z)$ near T on the set $\{X_T \in [y_0 - r, y_0 + r]\}$. Then, choosing $v_n \sim \alpha_n$, we will use the fact that thanks to (2.17) in (SP),

$$\int_{T-a_n}^{T} \int_O \delta(z)\alpha_n(z)N(ds, dz)$$

goes a.s. to infinity when n goes to infinity. By this way, we will get a good lowerbound. The upperbound will be obtained by using a well-chosen stopping time.

Let us now describe the sequence v_n of perturbations. We will develop the arguments above in the next sections.

Definition 5.1 *Consider the sequences of increasing processes*

$$Z_t^n = \int_0^t \int_O \alpha_n(z)\delta(z)N(ds, dz) \quad ; \quad U_t^n = \int_0^t \int_O \left[\alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\right] N(ds, dz) \tag{5.3}$$

where a_n, α_n, δ, β, and ζ_n are defined in (SP). Define the stopping times

$$T_n = \inf\{t > T - a_n \, / Z_t^n - Z_{T-a_n}^n \geq l\} \quad ; \quad R_n = \inf\{t > T - a_n \, / U_t^n - U_{T-a_n}^n \geq l\} \tag{5.4}$$

where $l > 0$ is a strictly positive real number that we will choose (very small) in Section 7. We now set

$$v_n(s, z) = 1_{[T-a_n, T\wedge T_n \wedge R_n]}(s)\alpha_n(z) sg(\mathcal{E}(K)_{s-}^{-1}) \tag{5.5}$$

where $sg(x)$ denotes the sign of x.

It is clear from (SP) that for each n, v_n is a perturbation in the sense of Definition 3.1. We will need the following key Lemma :

Lemma 5.2 *When n goes to infinity,*

$$P(T_n \leq T < R_n) \longrightarrow 1 \tag{5.6}$$

<u>Proof</u> : first,

$$\begin{aligned}
P(T_n \leq T) &= P(Z_T^n - Z_{T-a_n}^n \geq l) \\
&\geq 1 - e^l E\left[\exp\{-(Z_T^n - Z_{T-a_n}^n)\}\right] \\
&\geq 1 - e^l \exp\left[-\int_{T-a_n}^{T} \int_O \left(1 - e^{-\delta(z)\alpha_n(z)}\right) \varphi(z) dz ds\right] \\
&\geq 1 - e^l \exp\left\{-K a_n \int_O \alpha_n(z)\delta(z)\varphi(z) dz\right\} \longrightarrow 1
\end{aligned} \tag{5.7}$$

We have used condition (2.17) of (SP), and the fact that, since $\parallel \delta\alpha_n \parallel_\infty \leq d_0 \parallel \delta \parallel_\infty$, there exists a constant K such that for all z,

$$1 - e^{-\delta(z)\alpha_n(z)} \geq K\delta(z)\alpha_n(z) \tag{5.8}$$

Furthermore, thanks to (2.18) in (SP),

$$\begin{aligned} P(R_n \leq T) &= P(U^n_T - U^n_{T-a_n} \geq l) \\ &\leq \frac{1}{l} E\left(U^n_T - U^n_{T-a_n}\right) \\ &= \frac{1}{l} a_n \int_O \left[\alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\right] \varphi(z)dz \longrightarrow 0 \end{aligned} \quad (5.9)$$

The lemma is proved.

At last, we notice that since U^n and Z^n are càdlàg and increasing, since $\alpha_n(z)\delta(z) \leq d_0 \parallel \delta \parallel_\infty$, and since $\alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z) \leq d_0 \left(\parallel \beta \parallel_\infty + \parallel \zeta_n \parallel_\infty\right)$, for all $t \geq T - a_n$,

$$Z^n_{t\wedge T_n} - Z^n_{T-a_n} \leq l + d_0 \parallel \delta \parallel_\infty \quad ; \quad Z^n_{T_n} - Z^n_{T-a_n} \geq l \quad (5.10)$$

$$U^n_{t\wedge T_n} - U^n_{T-a_n} \leq l + d_0 \left(\parallel \beta \parallel_\infty + \sup_n \parallel \zeta_n \parallel_\infty\right) \quad ; \quad U^n_{T_n} - U^n_{T-a_n} \geq l \quad (5.11)$$

6 Lowerbound for the derivative at 0.

Thanks to our choice for v_n, since $1 + h'_x \geq 0$ thanks to (P) and since $h'_z \geq 0$ thanks to Proposition 2.4, we can write

$$\left|\frac{\partial}{\partial\lambda} X^n_T(0)\right| = |\mathcal{E}(K)_T| \times \int_{T-a_n}^{R_n\wedge T_n\wedge T} \int_O \left|\mathcal{E}(K)^{-1}_{s-}\right| \frac{h'_z(X_{s-},z)}{1 + h'_x(X_{s-},z)} \alpha_n(z) N(ds,dz) \quad (6.1)$$

This section is devoted to the proof of the following proposition :

Proposition 6.1 *For each $y_0 \in I\!R$ and each $l > 0$, there exists a constant $C(y_0,l) > 0$ such that for any $r \in]0,1]$,*

$$P\left(\left|\frac{\partial}{\partial\lambda} X^n_T(0)\right| \geq C(y_0,l) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \quad (6.2)$$

Proof : thanks to (H), $1 + h'_x \leq 1 + \parallel \eta \parallel_\infty$, and thanks to (SP), $h'_z(x,z) \geq \psi(x)\delta(z)$. Hence,

$$\begin{aligned} \left|\frac{\partial}{\partial\lambda} X^n_T(0)\right| &\geq \frac{1}{1+\parallel \eta \parallel_\infty} |\mathcal{E}(K)_T| \times \int_{T-a_n}^{R_n\wedge T_n\wedge T} \int_O \left|\mathcal{E}(K)^{-1}_{s-}\right| \psi(X_{s-})\delta(z)\alpha_n(z)N(ds,dz) \\ &\geq \frac{1}{1+\parallel \eta \parallel_\infty} \inf_{[T-a_n,T]} \left|\mathcal{E}(K)_T\mathcal{E}(K)^{-1}_{s-}\right| \times \inf_{[T-a_n,T]} \psi(X_{s-}) \times \int_{T-a_n}^{R_n\wedge T_n\wedge T} \int_O \delta(z)\alpha_n(z)N(ds,dz) \\ &= \frac{1}{1+\parallel \eta \parallel_\infty} \times A_n \times B_n \times (Z^n_{T\wedge T_n\wedge R_n} - Z^n_{T-a_n}) \end{aligned} \quad (6.3)$$

where Z^n is defined by (5.3).

Since $\mathcal{E}(K)$ is a.s. continuous at T, since, thanks to (P), $\mathcal{E}(K)$ does never vanish a.s., it is clear that A_n goes a.s. to 1 when n goes to infinity.

Since, from (SP)-1, ψ is continuous, since X is a.s. continuous at T, it is clear that on the set $X_T \in]y_0 - r, y_0 + r[$, a.s.,

$$\underline{\lim}\, B_n \geq \inf_{]y_0-2r,y_0+2r[} \psi(x) \geq \inf_{]y_0-2,y_0+2[} \psi(x) = \bar{\psi}(y_0) > 0 \quad (6.4)$$

At last, since $P(T_n < T < R_n)$ goes to 1 (see Lemma 5.2), and since $Z^n_{T_n} - Z^n_{T-a_n} \geq l$ a.s.,

$$P(Z^n_{T\wedge T_n} - Z^n_{T-a_n} \geq l) \longrightarrow 1 \tag{6.5}$$

Thus

$$P\left(\left|\frac{\partial}{\partial\lambda}X^n_T(0)\right| \geq \frac{1}{2}\left(\frac{1}{1+\parallel \eta \parallel_\infty} \times 1 \times \bar{\psi}(y_0) \times l\right) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \tag{6.6}$$

and we have checked the first part of Criterion 3.3 for X_T.

7 Upperbound for the derivatives.

We have to prove now that there exists a constant $K < \infty$ such that, when n goes to infinity,

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X^n_T(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.1}$$

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial^2}{\partial\lambda^2}X^n_T(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.2}$$

7.1 Upperbound for the first derivative.

Let us observe equation (4.16). Thanks to our choice for the perturbation v_n, it is clear that $\frac{\partial}{\partial\lambda}X^n_t(\lambda)$ does vanish for $s \leq T - a_n$. Thus only the second and fifth terms in (4.16) seem not to go to 0. Hence, we consider the following S.D.E. : it looks like (4.16), but we have kept only the second and fifth terms.

$$\begin{aligned} I^n_t(\lambda) \quad &= \quad \int_0^t \int_O \left[h'_x(X^n_{s-}(\lambda), \gamma^\lambda_n(s,z)) - h'_x(X^n_{s-}(\lambda), z)\right] I^n_{s-}(\lambda) N(ds, dz) \\ &\quad + \int_0^t \int_O h'_z(X^n_{s-}(\lambda), \gamma^\lambda_n(s,z)) v_n(s,z) N(ds, dz) \end{aligned} \tag{7.3}$$

Notice that the first term is an integral against $N(ds, dz)$ instead of $\varphi(z)dzds$. This comes from the fact that we will have to control the paths of $I^n_t(\lambda)$, and because our perturbation contains a stopping time which gives information about N. Furthermore, the difference between the integral against N and the one against $dzds$ will go to 0.

Recall now that we can choose $d_0 \in]0, 1[$ (see (SP), and Proposition 2.4) and $l > 0$ (see Definition 5.1) as small as we want. Notice that it is obvious from (H) that $\sup_n \parallel \zeta_n \parallel_\infty < \infty$. We will prove in this section the following estimate.

Proposition 7.1 *If we choose*

$$0 < l \leq \frac{1}{64} \qquad ; \qquad 0 < d_0 \leq \frac{1}{32\,(\parallel \delta \parallel_\infty + \parallel \beta \parallel_\infty + \sup_n \parallel \zeta_n \parallel_\infty)} \wedge 1 \tag{7.4}$$

then, when n goes to infinity,

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X^n_T(\lambda)\right| \leq 1\right) \longrightarrow 1 \tag{7.5}$$

In order to prove this, we will use the following lemmas.

Lemma 7.2 *Assume that l and d_0 are as in Proposition 7.1. Then for all n,*

$$\text{a.s.}\,, \qquad \sup_{|\lambda|\leq 1}\sup_{[0,T]}|I^n_t(\lambda)| \leq 1/2 \tag{7.6}$$

Lemma 7.3 *Assume that l and d_0 are as in Proposition 7.1. Let $p \in [1,\infty[$ be fixed. There exists a constant $C_p < \infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup_{[0,T]} |X_t^n(\lambda)|^p\right) \le C_p \tag{7.7}$$

$$E\left(\sup_{[0,T]} |X_t^n(\lambda) - X_t^n(\mu)|^p\right) \le C_p|\lambda-\mu|^p \tag{7.8}$$

$$E\left(\sup_{[0,T]} \left|\frac{\partial}{\partial\lambda}X_t^n(\lambda)\right|^p\right) \le C_p \tag{7.9}$$

$$E\left(\sup_{[0,T]} \left|\frac{\partial}{\partial\lambda}X_t^n(\lambda) - \frac{\partial}{\partial\lambda}X_t^n(\mu)\right|^p\right) \le C_p|\lambda-\mu|^p \tag{7.10}$$

Lemma 7.4 *Assume that l and d_0 are as in Proposition 7.1. Let $p \in [1,\infty[$ be fixed. There exists a constant $C_p < \infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup_{[0,T]} |I_t^n(\lambda) - I_t^n(\mu)|^p\right) \le C_p|\lambda-\mu|^p \tag{7.11}$$

Furthermore, for all $\lambda \in [-1,1]$,

$$E\left(\sup_{[0,T]} \left|I_t^n(\lambda) - \frac{\partial}{\partial\lambda}X_t^n(\lambda)\right|^p\right) \longrightarrow_{n\to\infty} 0 \tag{7.12}$$

We now assume for a moment that these Lemmas hold.

Proof of Proposition 7.1 : from estimates (7.10), (7.12), (7.11), and Lemma 8.1 of the Appendix, we deduce that, when n goes to infinity,

$$P\left(\sup_{|\lambda|\le 1} \left|I_T^n(\lambda) - \frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \ge 1/2\right) \longrightarrow 0 \tag{7.13}$$

Thus, using (7.6),

$$\begin{aligned} P\left(\sup_{|\lambda|\le 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \le 1\right) &\ge P\left(\sup_{|\lambda|\le 1}|I_T^n(\lambda)| \le 1/2\ ;\ \sup_{|\lambda|\le 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda) - I_T^n(\lambda)\right| \le 1/2\right) \\ &\ge P\left(\sup_{|\lambda|\le 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda) - I_T^n(\lambda)\right| \le 1/2\right) \longrightarrow 1 \end{aligned} \tag{7.14}$$

and the proposition is proved.

Proof of Lemma 7.2 : we work here for a fixed $\omega \in \Omega$. Using the functions δ, β and ζ_n defined in (SP) and Notation 2.2, the fact that $|\gamma_n^\lambda(s,z) - z| \le |v_n(s,z)|$ and $|v_n(s,z)| \le \alpha_n(z)$, we obtain, for all $x \in I\!R$,

$$\begin{aligned} \left|h'_z(x,\gamma_n^\lambda(s,z))\right| &\le |h'_z(x,z)| + |\gamma_n^\lambda(s,z) - z| \times \sup_{|z-w|\le|v_n(s,z)|}|h''_{zz}(x,w)| \\ &\le \delta(z) + |v_n(s,z)| \times \left[|h''_{zz}(x,z)| + |v_n(s,z)| \times \sup_{|z-w|\le|v_n(s,z)|}|h'''_{zzz}(x,w)|\right] \\ &\le \delta(z) + |v_n(s,z)|\beta(z) + v_n^2(s,z)\zeta_n(z) \end{aligned} \tag{7.15}$$

and in the same way,

$$\left|h'_x(x,\gamma_n^\lambda(s,z)) - h'_x(x,z)\right| \le |v_n(s,z)|\delta(z) + v_n^2(s,z)\beta(z) + |v_n(s,z)|^3\zeta_n(z) \tag{7.16}$$

Thus, since $|v_n(s,z)| \le \alpha_n(z)1_{[T-a_n,T_n\wedge R_n\wedge T]}(s)$, we see that

$$\begin{aligned}
\sup_{[0,t]}|I_s^n(\lambda)| &\le \int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O \left[\delta(z)+\alpha_n(z)\beta(z)+\alpha_n^2(z)\zeta_n(z)\right]\times\left[|I_{s-}^n(\lambda)|+1\right]\alpha_n(z)N(ds,dz)\\
&\le \left[1+\sup_{[0,t]}|I_s^n(\lambda)|\right]\times\Big[\int_{T-a_n}^{T_n}\int_O \delta(z)\alpha_n(z)N(ds,dz)\\
&\qquad\qquad + \int_{T-a_n}^{R_n}\int_O\left[\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]N(ds,dz)\Big]\\
&\le \left[1+\sup_{[0,t]}|I_s^n(\lambda)|\right]\times\left[2l+d_0(\|\delta\|_\infty+\|\beta\|_\infty+\|\zeta_n\|_\infty)\right]
\end{aligned} \tag{7.17}$$

We have used here only the definitions of T_n and R_n. But we have chosen d_0 and l satisfying :

$$[2l+d_0(\|\delta\|_\infty+\|\beta\|_\infty+\|\zeta_n\|_\infty)] \le 1/16 \tag{7.18}$$

we thus obtain

$$\sup_{[0,t]}|I_s^n(\lambda)| \le \frac{1}{16}+\frac{1}{16}\sup_{[0,t]}|I_s^n(\lambda)| \tag{7.19}$$

and the proof of (7.6) is finished.

<u>Proof of Lemma 7.3</u> : first of all, we omit the proofs of (7.7), (7.8), and (7.9), because they are similar but easier than the one of (7.10). In order to prove (7.10), we set :

$$\Gamma_t^n(\lambda,\mu) = X_t^n(\lambda)-X_t^n(\mu) \quad ; \quad \theta_t^n(\lambda,\mu) = \frac{\partial}{\partial\lambda}X_t^n(\lambda)-\frac{\partial}{\partial\lambda}X_t^n(\mu) \tag{7.20}$$

Then, using the expression (4.16), we see that (the constant K depends only on p)

$$\begin{aligned}
&E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)\\
&\le KE\left[\sup_{[0,t]}\left|\int_0^u\int_O \Big(h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))-h'_x(X_{s-}^n(\mu),\gamma_n^\lambda(s,z))\Big)\times\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde N(ds,dz)\right|^p\right]\\
&+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O \Big(h'_x(X_{s-}^n(\mu),\gamma_n^\mu(s,z))-h'_x(X_{s-}^n(\mu),\gamma_n^\lambda(s,z))\Big)\times\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde N(ds,dz)\right|^p\right]\\
&+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O h'_x(X_{s-}^n(\mu),\gamma_n^\mu(s,z))\times\theta_{s-}^n(\lambda,\mu)\tilde N(ds,dz)\right|^p\right]\\
&+2^{p-1}E\left[\left|\int_0^t\int_O \left|h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))-h'_x(X_{s-}^n(\lambda),z)\right||\theta_{s-}^n(\lambda,\mu)|\varphi(z)dzds\right|^p\right]\\
&+KE\left[\left|\int_0^t\int_O \left|h'_x(X_{s-}^n(\lambda),\gamma_n^\mu(s,z))-h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\right|\left|\frac{\partial}{\partial\lambda}X_{s-}^n(\mu)\right|\varphi(z)dzds\right|^p\right]
\end{aligned}$$

$$+KE\Big[\Big|\int_0^t\int_O\Big|h'_x(X^n_{s-}(\lambda),\gamma^\mu_n(s,z))-h'_x(X^n_{s-}(\lambda),z)-h'_x(X^n_{s-}(\mu),\gamma^\mu_n(s,z))$$

$$+h'_x(X^n_{s-}(\mu),z)\Big|\,\Big|\frac{\partial}{\partial\lambda}X^n_{s-}(\mu)\Big|\,\varphi(z)dzds\Big|^p\Big]$$

$$+KE\left[\left|\int_0^t|g'(X^n_{s-}(\lambda))-g'(X^n_{s-}(\mu))|\times\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|ds\right|^p\right]$$

$$+KE\left[\left|\int_0^t|g'(X^n_{s-}(\mu))|\times|\theta^n_{s-}(\lambda,\mu)|\,ds\right|^p\right]$$

$$+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O(f'_x(X^n_{s-}(\lambda),u)-f'_x(X^n_{s-}(\mu),u))\times\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\tilde N_1(ds,du)\right|^p\right]$$

$$+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O f'_x(X^n_{s-}(\mu),u)\times\theta^n_{s-}(\lambda,\mu)\tilde N_1(ds,du)\right|^p\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^t\int_O\left|h'_z(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))-h'_z(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\right|\times\alpha_n(z)N(ds,dz)\right|^p\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^t\int_O\left|h'_z(X^n_{s-}(\mu),\gamma^\mu_n(s,z))-h'_z(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\right|\times\alpha_n(z)N(ds,dz)\right|^p\right]$$

$$=J^{n,1}_t(\lambda,\mu)+\ldots+J^{n,12}_t(\lambda,\mu)\tag{7.21}$$

First, we compute $J^{n,1}$. Using Burkholder's inequality, we obtain

$$J^{n,1}_t(\lambda,\mu)\le KE\left[\left|\int_0^t\int_O\left(h'_x(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))-h'_x(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\right)^2\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2N(ds,dz)\right|^{p/2}\right]$$

Using the functions η, δ, β, ζ_n defined in (SP) and (H), using the facts that $|v_n(s,z)|\le\alpha_n(z)$ and $|\gamma^\lambda_n(s,z)-z|\le|v_n(s,z)|$, we see that for all x,y in $I\!R$,

$$\left|h'_x(x,\gamma^\lambda_n(s,z))-h'_x(y,\gamma^\lambda_n(s,z))\right|\le|x-y|\left[\eta(z)+|v_n(s,z)|\delta(z)+|v_n(s,z)|^2\beta(z)+|v_n(s,z)|^3\zeta_n(z)\right]\tag{7.22}$$

Using Definition 5.1 of v_n, we see that $J^{n,1}_t(\lambda,\mu)$ is smaller than

$$KE\left[\left|\int_0^t\int_O(\Gamma^n_{s-}(\lambda,\mu))^2\times\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\times\eta^2(z)N(ds,dz)\right|^{p/2}\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^t\int_O(\Gamma^n_{s-}(\lambda,\mu))^2\times\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\times\left[\delta\alpha_n+\beta\alpha_n^2+\zeta_n\alpha_n^3\right]^2(z)N(ds,dz)\right|^{p/2}\right]$$

We now apply Lemma 8.2 for the first term. For the second term, we use Lemma 8.4-1, of which the conditions are satisfied thanks to (SP). This gives :

$$\begin{aligned}J^{n,1}_t(\lambda,\mu)\;\;\le\;\;& K\int_0^tE\left[|\Gamma^n_s(\lambda,\mu)|^p\times\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right]ds\\&+\frac{K}{a_n}\int_{(T-a_n)\wedge t}^tE\left[|\Gamma^n_s(\lambda,\mu)|^p\times\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right]ds\end{aligned}\tag{7.23}$$

At last, we deduce from Cauchy-Schwarz's inequality, (7.8) and (7.9) that

$$J_t^{n,1}(\lambda,\mu) \leq K|\lambda-\mu|^p \tag{7.24}$$

In order to estimate $J^{n,2}$, we will use Burkholder's inequality, then the fact that, since $|\gamma_n^\lambda(s,z)-\gamma_n^\mu(s,z)| \leq |\lambda-\mu| \times |v_n(s,z)|$, for all $x \in I\!R$,

$$|h'_x(x,\gamma_n^\lambda(s,z))-h'_x(x,\gamma_n^\mu(s,z))| \leq |\lambda-\mu|\times|v_n(s,z)|\times\Big[\delta(z)+|v_n(s,z)|\times\beta(z)+|v_n(s,z)|^2\times\zeta_n(z)\Big] \tag{7.25}$$

This way, $J_t^{n,2}(\lambda,\mu)$ is smaller than

$$\begin{aligned}
&K|\lambda-\mu|^p \times E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \Big[\delta(z)\alpha_n(z)+\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\Big]^2 \times \left|\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right|^2 N(ds,dz)\right|^{\frac{p}{2}}\right] \\
&\leq K|\lambda-\mu|^p \times \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right)ds \\
&\leq K|\lambda-\mu|^p
\end{aligned} \tag{7.26}$$

for the same reasons as in the computation of $J^{n,1}$.

We now notice that for any real number x,

$$\Big|h'_x(x,\gamma_n^\lambda(s,z))\Big| \leq \eta(z)+|v_n(s,z)|\times\delta(z)+|v_n(s,z)|^2\times\beta(z)+|v_n(s,z)|^3\times\zeta_n(z) \tag{7.27}$$

Thus, exactly as for $J^{n,1}$, we obtain :

$$J_t^{n,3}(\lambda,\mu) \leq K\int_0^t E\left[|\theta_s^n(\lambda,\mu)|^p\right]ds + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left[|\theta_s^n(\lambda,\mu)|^p\right]ds \tag{7.28}$$

We now are interested in $J^{n,4}$. First notice that

$$|h'_x(x,\gamma_n^\lambda(s,z))-h'_x(x,z)| \leq |v_n(s,z)|\delta(z)+v_n^2(s,z)\beta(z)+|v_n(s,z)|^3\zeta_n(z) \tag{7.29}$$

Hence

$$\begin{aligned}
J_t^{n,4}(\lambda,\mu) &\leq 2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)|\times\Big[\delta(z)\alpha_n(z)+\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\Big]\varphi(z)dzds\right|^p\right] \\
&\leq 2^{p-1}2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)|\times\delta(z)\alpha_n(z)\varphi(z)dzds\right|^p\right] \\
&\quad +2^{p-1}2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{R_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)|\times\Big[\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\Big]\varphi(z)dzds\right|^p\right]
\end{aligned} \tag{7.30}$$

Using Assumption (SP), Lemma 8.5, and the definitions of T_n and R_n (see Definition 5.1), we deduce that

$$\begin{aligned}
J_t^{n,4}(\lambda,\mu) &\leq 2^{p-1}2^{p-1}\left[2^{p-1}(l+d_0 \parallel \delta \parallel_\infty)^p \times E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)\right] \\
&\quad +2^{p-1}2^{p-1}\left[2^{p-1}(l+d_0(\parallel \beta \parallel_\infty + \parallel \zeta_n \parallel_\infty))^p \times E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)\right] \\
&\quad +\frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\right)ds
\end{aligned} \tag{7.31}$$

But we have chosen l and d_0 satisfying (7.18). Thus

$$J_t^{n,4}(\lambda,\mu) \leq \frac{1}{2^p} E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\right) ds \tag{7.32}$$

A similar computation allows us to write

$$J_t^{n,5}(\lambda,\mu) \leq K|\lambda-\mu|^p \times \left[KE\left(\sup_{[0,t]}\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right) + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right) ds\right] \tag{7.33}$$

Thanks to (7.9), we obtain

$$J_t^{n,5}(\lambda,\mu) \leq K|\lambda-\mu|^p \tag{7.34}$$

Since

$$\left|h'_x(x,\gamma_n^\lambda(s,z)) - h'_x(x,z) - h'_x(y,\gamma_n^\lambda(s,z)) + h'_x(y,z)\right|$$

$$\leq |x-y| \left[|v_n(s,z)|\delta(z) + v_n^2(s,z)\beta(z) + |v_n(s,z)|^3\zeta_n(z)\right] \tag{7.35}$$

one can check, as in the computation of $J^{n,4}$, that

$$\begin{aligned} J_t^{n,6}(\lambda,\mu) &\leq KE\left(\sup_{[0,t]}|\Gamma_s^n(\lambda,\mu)|^p\left|\frac{\partial}{\partial\lambda}X_s^n(\mu)\right|^p\right) \\ &\quad + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\left|\frac{\partial}{\partial\lambda}X_s^n(\mu)\right|^p\right) ds \\ &\leq K|\lambda-\mu|^p \end{aligned} \tag{7.36}$$

thanks to the Cauchy-Schwarz inequality, and estimates (7.8) and (7.9).
Using (H), Lemma 8.2 for the Poissonian terms, one easily checks that

$$J_t^{n,7}(\lambda,\mu) + ... + J_t^{n,10}(\lambda,\mu) \leq K\int_0^t E\left(|\Gamma_s^n(\lambda,\mu)|^p \times \left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right) ds + K\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right) ds \tag{7.37}$$

Using the Cauchy-Schwarz inequality, and estimations (7.8) and (7.9), we can conclude that

$$J_t^{n,7}(\lambda,\mu) + ... + J_t^{n,10}(\lambda,\mu) \leq K|\lambda-\mu|^p + K\int_0^t E\left(|\theta_{s-}^n(\lambda)|^p\right) ds \tag{7.38}$$

It is easy to check that

$$|h'_z(x,\gamma_n^\lambda(s,z)) - h'_z(y,\gamma_n^\lambda(s,z))| \leq |x-y| \times \left[\delta(z) + |v_n(s,z)| \times \beta(z) + |v_n(s,z)|^2 \times \zeta_n(z)\right] \tag{7.39}$$

Thus $J_t^{n,11}(\lambda,\mu)$ is smaller than

$$\begin{aligned} & KE\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O |\Gamma_{s-}^n(\lambda,\mu)| \times \left[\delta(z)\alpha_n(z) + \beta(z)\alpha_n^2(z) + \zeta_n(z)\alpha_n^3(z)\right] N(ds,dz)\right|^p\right] \\ & \leq KE\left[\left|\int_{T-a_n}^{T_n}\int_O \delta(z)\alpha_n(z)N(ds,dz)\right|^p \times \sup_{[0,T]}|\Gamma_s^n(\lambda,\mu)|^p\right] \\ & + KE\left[\left|\int_{T-a_n}^{R_n}\int_O \left[\beta(z)\alpha_n^2(z) + \zeta_n(z)\alpha_n^3(z)\right] N(ds,dz)\right|^p \times \sup_{[0,T]}|\Gamma_s^n(\lambda,\mu)|^p\right] \\ & \leq K|\lambda-\mu|^p \end{aligned} \tag{7.40}$$

thanks to the definitions of T_n, R_n, and equation (7.8).

At last, since

$$|h'_z(x,\gamma_n^\lambda(s,z)) - h'_z(x,\gamma_n^\mu(s,z))| \le |\lambda-\mu| \times |v_n(s,z)| \times [\beta(z) + |v_n(s,z)|\zeta_n(z)] \tag{7.41}$$

we see that (thanks to the definition of R_n) :

$$J_t^{n,12}(\lambda,\mu) \le KE\left[\left|\int_{T-a_n}^{R_n}\int_O |\lambda-\mu| \times \left[\beta(z)\alpha_n^2(z) + \zeta_n(z)\alpha_n^3(z)\right] N(ds,dz)\right|^p\right] \le K|\lambda-\mu|^p \tag{7.42}$$

Finally, we obtain, for some constants K_1, K_2, K_3 depending only on p,

$$\begin{aligned} E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) &\le K_1|\lambda-\mu|^p + K_2\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right)ds + \frac{K_3}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda,\mu)|^p\right)ds \\ &\quad + \frac{1}{2^p}E\left[\sup_{[0,t]}|\theta_s^n(\lambda)|^p\right] \end{aligned} \tag{7.43}$$

Thus, since $p \ge 1$,

$$E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) \le 2K_1|\lambda-\mu|^p + 2K_2\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right)ds + \frac{2K_3}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda,\mu)|^p\right)ds \tag{7.44}$$

Furthermore, it is clear, from Remark 4.6, that for each λ, μ, n, the function $f(t) = E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)$ is bounded on $[0,T]$. We thus can apply the extended Gronwall Lemma 8.6 proved in the Appendix, and conclude that there exists a constant C_p, independent of λ, μ, and n, such that

$$E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) \le C_p|\lambda-\mu|^p \tag{7.45}$$

<u>Proof of Lemma 7.4</u> : we first check (7.12). We set $A_t^n(\lambda) = \frac{\partial}{\partial\lambda}X_t^n(\lambda) - I_t^n(\lambda)$. Writing $\varphi(z)dzds$ as $N(ds,dz) - \tilde N(ds,dz)$ in the expression (4.16) of $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$, we see that

$$\begin{aligned} A_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde N(ds,dz) \\ &\quad + \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]A_{s-}^n(\lambda)N(ds,dz) \\ &\quad - \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde N(ds,dz) \\ &\quad + \int_0^t g'(X_{s-}^n(\lambda))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde N_1(ds,du) \end{aligned} \tag{7.46}$$

But it is easy to check that $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$ vanishes as soon as $t \le T - a_n$. Hence, using Burkholder's inequality for the first, third, and fifth terms, using inequalities (7.27) and (7.29), the expression of v_n (see Definition

5.1), and (H), we see that

$$
\begin{aligned}
&E\left(\sup_{[0,t]}|A^n_s(\lambda)|^p\right)\\
&\le KE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_O\left[\eta(z)+\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]^2\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|^2N(ds,dz)\right|^{\frac{p}{2}}\right]\\
&+2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O\left[\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]\times|A^n_s(\lambda)|\,N(ds,dz)\right|^p\right]\\
&+KE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_O\left[\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]^2\times\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|^2N(ds,dz)\right|^{\frac{p}{2}}\right]\\
&+K\int_{(T-a_n)\wedge t}^{t}E\left[\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|^p\right]ds+KE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_E\sigma^2(u)\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|^2N_1(ds,du)\right|^{\frac{p}{2}}\right]\\
&\le D^{n,1}_t(\lambda)+...+D^{n,5}_t(\lambda)
\end{aligned}
\tag{7.47}
$$

First,

$$
\begin{aligned}
D^{n,1}_t(\lambda) &\le KE\left[\sup_{[0,T]}\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right]^{\frac12}\\
&\times E\left[\left|\int_{T-a_n}^{T}\int_O\left[\eta^2(z)+\alpha_n^2(z)\delta^2(z)+\alpha_n^4(z)\beta^2(z)+\alpha_n^6(z)\zeta_n^2(z)\right]N(ds,dz)\right|^p\right]^{\frac12}
\end{aligned}
\tag{7.48}
$$

Using Lemma 8.4, since from (SP),

$$
a_n\int_O\left[\eta^2(z)+\alpha_n^2(z)\delta^2(z)+\alpha_n^4(z)\beta^2(z)+\alpha_n^6(z)\zeta_n^2(z)\right]\varphi(z)dz\longrightarrow_{n\to\infty}0 \tag{7.49}
$$

and using (7.9), it is clear that $D^{n,1}_t(\lambda)\le K_n$, where the sequence K_n goes to 0. One can check in the same way that $D^{n,3}_t(\lambda)\le L_n$, where L_n goes to 0. It is clear from (7.9) that $D^{n,4}_t(\lambda)\le Ka_n$. Thanks to Lemma 8.2 and estimation (7.9), since $\sigma\in L^2(E,q)$, we see that $D^{n,5}_t(\lambda)\le Ka_n$. At last, the definitions of T_n and R_n yield that

$$
\begin{aligned}
D^{n,2}_t(\lambda) &\le 4^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge t}\alpha_n(z)\delta(z)|A^n_{s-}(\lambda)|N(ds,dz)\right|^p\right]\\
&\quad+4^{p-1}E\left(\left|\int_{(T-a_n)\wedge t}^{R_n\wedge t}[\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)]|A^n_{s-}(\lambda)|N(ds,dz)\right|^p\right]\\
&\le 4^pE\left(\sup_{[0,t]}|A^n_s(\lambda)|^p\right)\times\left[2l+d_0(\parallel\delta\parallel_\infty+\parallel\beta\parallel_\infty+\sup_n\parallel\zeta_n\parallel_\infty)\right]^p\\
&\le \frac{1}{4^p}E\left(\sup_{[0,t]}|A^n_s(\lambda)|^p\right)
\end{aligned}
\tag{7.50}
$$

since d_0 and l satisfy (7.18). We finally can write

$$
E\left(\sup_{[0,t]}|A^n_s(\lambda)|^p\right)\le C_n+\frac{1}{4^p}E\left[\sup_{[0,t]}|A^n_s(\lambda)|^p\right] \tag{7.51}
$$

where the sequence C_n goes to 0. Thus

$$E\left(\sup_{[0,t]}|A_s^n(\lambda)|^p\right)\leq 2C_n\longrightarrow 0 \tag{7.52}$$

when n goes to infinity.

The proof of (7.11) is quite similar to the one of (7.10).

7.2 Upperbound for the second derivative.

The method is exactly the same as in the previous subsection : we set

$$\begin{aligned} J_t^n(\lambda) &= \int_0^t\int_O\left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))-h'_x(X_{s-}^n(\lambda),z)\right]J_{s-}^n(\lambda)N(ds,dz)\\ &+2\int_0^t\int_O h''_{xz}(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))I_{s-}^n(\lambda)v_n(s,z)N(ds,dz)\\ &+\int_0^t\int_O\left[h''_{xx}(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))-h''_{xx}(X_{s-}^n(\lambda),z)\right]\left(I_{s-}^n(\lambda)\right)^2N(ds,dz) \end{aligned}\tag{7.53}$$

and we state the following lemma, which can be proved as Lemmas 7.2, 7.3 and 7.4 :

Lemma 7.5 *Let d_0 and l be as in Proposition 7.1. Let $p\in[1,\infty[$ be fixed. There exists a constant $C_p<\infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup\nolimits_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)\right|^p\right)\leq C\quad ;\quad E\left(\sup\nolimits_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)-\frac{\partial^2}{\partial\lambda^2}X_t^n(\mu)\right|^p\right)\leq C|\lambda-\mu|^p \tag{7.54}$$

$$\text{a.s.,}\qquad \sup_{|\lambda|\leq 1}\sup_{[0,T]}|J_t^n(\lambda)|\leq\frac{1}{2} \tag{7.55}$$

$$E\left(\sup\nolimits_{[0,T]}\left|J_t^n(\lambda)-\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)\right|^p\right)\longrightarrow_{n\to\infty}0\quad ;\quad E\left(\sup\nolimits_{[0,T]}|J_t^n(\lambda)-J_t^n(\mu)|^p\right)\leq C|\lambda-\mu|^p \tag{7.56}$$

Comparing equations (4.17) and (4.22), we see that in order to prove (7.54), we just have to check that

$$E\left(\sup_{[0,T]}|L_t^n(\lambda)|^p\right)\leq C_p\quad ;\quad E\left(\sup_{[0,T]}|L_t^n(\lambda)-L_t^n(\mu)|^p\right)\leq C_p|\lambda-\mu|^p \tag{7.57}$$

Similar simplifications may be used to prove (7.56). Let us just prove (7.55). Using Lemma 7.2, inequality (7.16), the same inequality for h''_{xx}, the expression of v_n, and inequality (7.18), we see that

$$\begin{aligned} |J_t^n(\lambda)| &\leq \sup_{[0,t]}|J_s^n(\lambda)|\times\int_{T-a_n}^{T_n\wedge R_n}\left[\delta\alpha_n+\beta\alpha_n^2+\zeta_n\alpha_n^3\right](z)N(ds,dz)\\ &+2\sup_{[0,t]}|I_s^n(\lambda)|\times\int_{T-a_n}^{T_n\wedge R_n}\left[\delta\alpha_n+\beta\alpha_n^2+\zeta_n\alpha_n^3\right](z)N(ds,dz)\\ &+\sup_{[0,t]}|I_s^n(\lambda)|^2\times\int_{T-a_n}^{T_n\wedge R_n}\left[\delta\alpha_n+\beta\alpha_n^2+\zeta_n\alpha_n^3\right](z)N(ds,dz) \end{aligned}$$

$$\leq \quad [2l + d_0(\| \delta \|_\infty + \| \beta \|_\infty + \| \zeta \|_\infty)] \times \left(\sup_{[0,t]} |J_s^n(\lambda)| + 2 \times 1/2 + 1/4 \right)$$

$$\leq \quad \frac{1}{16} \sup_{[0,t]} |J_s^n(\lambda)| + \frac{5}{64} \tag{7.58}$$

This yields (7.55).

This Lemma allows to conclude, as in the previous subsection, that the proposition below hold.

Proposition 7.6 *Let d_0 and l be as in Proposition 7.1. Then, when n goes to infinity,*

$$P \left(\sup_{|\lambda| \leq 1} \left| \frac{\partial^2}{\partial \lambda^2} X_T^n(\lambda) \right| \leq 1 \right) \longrightarrow 1 \tag{7.59}$$

7.3 Conclusion.

Fix y_0 in the support of $P \circ X_T^{-1}$. We have proved that there exists $C_1 > 0$ and $C_2 < \infty$ such that for all $r > 0$,

$$P \left(|X_T - y_0| < r \quad ; \quad \left| \frac{\partial}{\partial \lambda} X_T^n(0) \right| \geq C_1 \quad ; \quad \sup_\lambda \left| \frac{\partial}{\partial \lambda} X_T^n(\lambda) \right| + \left| \frac{\partial^2}{\partial \lambda^2} X_T^n(\lambda) \right| \leq C_2 \right) \tag{7.60}$$

goes to $P(|X_T - y_0| < r)$, that is strictly positive.

Thus each y_0 in the support of $P \circ X_T^{-1}$ satisfies the assumptions of Theorem 3.3. Remark 3.5 allows us to conclude that the proof of Theorem 2.3 is proved.

8 Appendix.

We begin this annex with a consequence of a standard limit theorem for continuous processes.

Lemma 8.1 *Let $\{Y_n(\lambda)\}_{\lambda \in [-1,1]}$ be a sequence of real valued processes. Assume that there exists a constant $C < \infty$ such that for all $\lambda, \mu \in [-1, 1]$, all $n \in I\!N$,*

$$E \left(|Y_n(\lambda) - Y_n(\mu)|^2 \right) \leq C |\lambda - \mu|^2 \tag{8.1}$$

Suppose also that for all $\lambda \in [-1, 1]$, when n goes to infinity,

$$E \left(|Y_n(\lambda)|^2 \right) \longrightarrow 0 \tag{8.2}$$

Then the following convergence holds in probability :

$$\sup_{|\lambda| \leq 1} |Y_n(\lambda)| \longrightarrow 0 \tag{8.3}$$

Proof : first notice that thanks to (8.1) and to the Kolmogorov criterion of continuity, see e.g. Revuz, Yor, [17], p 25, there exists a constant $K < \infty$ such that for all n,

$$E \left(\sup_{\lambda \neq \mu} \frac{|Y_n(\lambda) - Y_n(\mu)|}{|\lambda - \mu|^{\frac{1}{4}}} \right) \leq K \tag{8.4}$$

Thus for all $\eta > 0$,

$$\lim_{\delta\to 0}\sup_n P\left(\sup_{|\lambda-\mu|\leq\delta}|Y_n(\lambda)-Y_n(\mu)|\geq\eta\right)\leq\lim_{\delta\to 0}K\eta^{-1}\delta^{\frac{1}{4}}=0 \tag{8.5}$$

Furthermore, the family of the laws of $Y_n(0)$ is clearly tight, thanks to (8.2). Thus, we deduce from Theorem 1.3.1 p 31 in [16] that the family of the laws of Y_n is tight.
On the other hand, it is obvious from (8.2) that for all λ_1, ..., λ_k in $[-1,1]$, $(Y_n(\lambda_1),...,Y_n(\lambda_k))$ goes to 0 in law when n goes to infinity. The standard limit Theorems for continuous processes yield that Y_n goes to 0 in law for the supremum norm, i.e. that

$$\sup_{[-1,1]}|Y_n(\lambda)|\longrightarrow 0 \tag{8.6}$$

in law. But the convergence in law to a constant implies the convergence in probability, and the Lemma is proved.

We now give five technical lemmas. We omit the proof of the first one, because it is similar to the one of Lemma 8.4 below.

Lemma 8.2 *Let $\eta\in L^2(O,\varphi(z)dz)\cap L^\infty(O,\varphi(z)dz)$, and let $p\geq 1$. There exists some constants C_p and $K_p(\eta)$ such that for every predictable process Y on $[0,T]$,*

$$\begin{aligned} E\left(\sup_{[0,t]}\left|\int_0^u\int_O\eta(z)Y_s\tilde{N}(ds,dz)\right|^p\right) &\leq C_pE\left(\left|\int_0^t\int_O\eta^2(z)Y_s^2N(ds,dz)\right|^{\frac{p}{2}}\right)\\ &\leq K_p(\eta)\int_0^t E\left(|Y_s|^p\right)ds \end{aligned} \tag{8.7}$$

Lemma 8.3 *Let γ_n be a sequence of positive functions on O and let a_n be a real valued sequence decreasing to 0, such that for some constant $C<\infty$,*

$$a_n\int_O\gamma_n(z)\varphi(z)dz\leq C \tag{8.8}$$

Let $p\geq 1$. Then there exists a constant K_p such that for all predictable process Y on $[0,T]$,

$$E\left(\left|\int_{(T-a_n)\wedge t}^t\int_O\gamma_n(z)|Y_s|\varphi(z)dzds\right|^p\right)\leq\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right)ds \tag{8.9}$$

<u>Proof</u> : the left member of equation (8.9) is smaller than

$$E\left(\left|\int_{(T-a_n)\wedge t}^t|Y_s|ds\times\frac{C}{a_n}\right|^p\right)\leq\frac{K}{a_n^p}E\left[\left|\int_{(T-a_n)\wedge t}^t|Y_s|ds\right|^p\right] \tag{8.10}$$

Using Holder's inequality (for the measure ds), we obtain

$$\leq\frac{K}{a_n^p}\times\left|\int_{(T-a_n)\wedge t}^t ds\right|^{p-1}\times E\left[\int_{(T-a_n)\wedge t}^t|Y_s|^pds\right]\leq\frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right)ds \tag{8.11}$$

which was our aim.

Lemma 8.4 *Consider a sequence of positive functions $\gamma_n\in L^1(O,\varphi(z)dz)$, such that $\|\gamma_n\|_\infty\leq k_0$, let a_n be a real valued sequence decreasing to 0, and let $p\geq 1$.*

1. *if* $a_n \int_O \gamma_n(z)\varphi(z)dz \leq C$, *then there exists* K_p *such that for all predictable process* Y *on* $[0,T]$,

$$E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|^p\right] \leq \frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right)ds \tag{8.12}$$

2. *if* $a_n \int_O \gamma_n(z)\varphi(z)dz \to 0$, *then*

$$E\left[\left|\int_{T-a_n}^T \int_O \gamma_n(z)N(ds,dz)\right|^p\right] \to 0 \tag{8.13}$$

Proof : let us for example check 1. : we will prove (8.12) for every $p = 2^q$, recursively (on q). First, if $p = 1$,

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|\right] &= \int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)E\left(|Y_s|\right)\varphi(z)dzds \\ &\leq \int_{(T-a_n)\wedge t}^t E\left(|Y_s|\right)\times\frac{C}{a_n}ds \end{aligned} \tag{8.14}$$

Assume now that (8.12) holds for $p = 2^q$. Then

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|^{2p}\right] &\leq KE\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|\tilde{N}(ds,dz)\right|^{2p}\right] \\ &+ KE\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|\varphi(z)dzds\right|^{2p}\right] \end{aligned} \tag{8.15}$$

Thanks to Burkholder's inequality and the inductive assumption, the first term is smaller than

$$K\times C_pE\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n^2(z)|Y_s|^2N(ds,dz)\right|^p\right] \leq K\times k_0\times C_p\times\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^{2p}\right)ds \tag{8.16}$$

By using Lemma 8.3, the second term is smaller than

$$\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^{2p}\right)ds \tag{8.17}$$

which was our aim.

Lemma 8.5 *Let* Y *be a predictable process on* $[0,T]$, *let* a_n *be a sequence decreasing to* 0, *and let* $l > 0$ *be fixed. Consider a sequence of positive functions* γ_n *on* O *such that*

$$\parallel \gamma_n \parallel_\infty \leq k_0 \quad ; \quad a_n\int_O \gamma_n^2(z)\varphi(z)dz \leq K \tag{8.18}$$

Consider also the following stopping time :

$$\tau_n = \inf\left\{t > T-a_n \;\Big/\; \int_{T-a_n}^t \int_O \gamma_n(z)N(ds,dz) \geq l\right\} \tag{8.19}$$

Then for all $p \in [1,\infty[$,

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t} \int_O |Y_s|\gamma_n(z)\varphi(z)dzds\right|^p\right] &\leq 2^{p-1}\times|l+k_0|^p\times E\left(\sup_{[0,t]}|Y_s|^p\right) \\ &+ \frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right)ds \end{aligned} \tag{8.20}$$

Proof : we first write $\varphi(z)dzds$ as $N(ds,dz) - \tilde{N}(ds,dz)$, to upperbound the left member of (8.20) with :

$$2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t}\int_O |Y_s|\gamma_n(z)N(ds,dz)\right|^p\right]$$
$$+2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t}\int_O |Y_s|\gamma_n(z)\tilde{N}(ds,dz)\right|^p\right] \tag{8.21}$$

Thanks to Burkholder's inequality, this is smaller than

$$2^{p-1}E\left[\sup_{[0,t]}|Y_s|^p \times \left|\int_{(T-a_n)}^{\tau_n}\int_O \gamma_n(z)N(ds,dz)\right|^p\right]$$
$$+C_pE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_O Y_s^2\gamma_n^2(z)N(ds,dz)\right|^{\frac{p}{2}}\right] \tag{8.22}$$

At last thanks to the definition of τ_n, Lemma 8.4, and (8.18) this is smaller than

$$2^{p-1}\times|l+k_0|^p\times E\left(\sup_{[0,t]}|Y_s|^p\right)+\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left(|Y_s|^p\right)ds \tag{8.23}$$

which was our aim.

At last, the next lemma is a consequence of Gronwall's Lemma.

Lemma 8.6 *If f is a positive and bounded function on $[0,T]$ satisfying, for some $a\in]0,T]$ and some positive constants C_1, C_2, and C_3 :*

$$f(t)\leq C_1+C_2\int_0^t f(s)ds+\frac{C_3}{a}\int_{(T-a)\wedge t}^{t}f(s)ds \tag{8.24}$$

then

$$\sup_{[0,T]}f(t)\leq C_1\left[1+C_2Te^{C_2T}\right]e^{C_2T+C_3} \tag{8.25}$$

Proof : first, it is clear from Gronwall's lemma that

$$\sup_{[0,T-a]}f(t)\leq C_1e^{C_2T} \tag{8.26}$$

We now set $g(t)=f((T-a)+t)$, which is well defined on $[0,a]$. We obtain

$$g(t)\leq C_1+C_2C_1e^{C_2T}(T-a)+\left(C_2+\frac{C_3}{a}\right)\int_0^t g(s)ds \tag{8.27}$$

Thus Gronwall's Lemma yields that

$$\sup_{[T-a,T]}f(t)=\sup_{[0,a]}g(t)\leq C_1\left[1+C_2Te^{C_2T}\right]e^{C_2a+C_3} \tag{8.28}$$

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