Existence and regularity study for a 2-dimensional Kac equation without cutoff by a probabilistic approach.

Nicolas FOURNIER

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Abstract : We consider a 2-dimensional Kac equation without cutoff, which we relate to a stochastic differential equation. We prove the existence of a solution for this S.D.E., and we use the Malliavin calculus (or stochastic calculus of variations) to prove that the law of this solution admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}$. This density satisfies the Kac equation.

1 Introduction.

The Boltzmann equation describes the density $f(t, r, v)$ of particles which have the position r and the velocity v at the instant $t > 0$, in a sufficiently dilute gas. The 2-dimensional Kac equation deals with a simplified model. Indeed, the particles take place in the plane, and the density f is supposed to be spatially homogeneous : the interaction is meanfield. In this paper, we will take in account the difficulty generated by the possible explosion of the mass of the collision kernel.
The Kac equation can be written as follows :

$$(B) \qquad \frac{\partial f}{\partial t}(t, v) = K_\beta(f, f)(t, v)$$

The collision kernel K_β is given by :

$$K_\beta(f,f)(t,v) = \int_{v^* \in \mathbb{R}^2} \int_{-\pi}^{\pi} \left[f(t, c(v, v^*, \theta)) f(t, c^*(v, v^*, \theta)) - f(t, v) f(t, v^*) \right] \beta(\theta, |v - v^*|) d\theta dv^*$$

where, if R_θ is the θ-rotation centered at 0,

$$c(v, v^*, \theta) = \frac{v + v^*}{2} + R_\theta \left(\frac{v - v^*}{2} \right) \quad \text{and} \quad c^*(v, v^*, \theta) = \frac{v + v^*}{2} - R_\theta \left(\frac{v - v^*}{2} \right)$$

We will need the following computation of $c(v, v^*, \theta)$:

$$c(v, v^*, \theta) = \begin{pmatrix} c_x(v, v^*, \theta) \\ c_y(v, v^*, \theta) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (v_x + v_x^*) + (v_x - v_x^*) \cos\theta - (v_y - v_y^*) \sin\theta \\ (v_y + v_y^*) + (v_y - v_y^*) \cos\theta + (v_x - v_x^*) \sin\theta \end{pmatrix}$$

In fact, $c(v, v^*, \theta)$ and $c^*(v, v^*, \theta)$ represent the velocities of two particles after their collision, if these particles had the velocities v and v^* before the collision, and if the angle due to the collision is θ.
We will assume that we are in a case of Maxwellian particles, i.e. that the cross section β depends only on θ, and is even : $\beta(\theta, |v - v^*|) = \beta(|\theta|)$. We will also suppose the physically reasonnable condition :

$$\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty \tag{1.1}$$

The Kac equation "with cutoff", namely when $\int_0^\pi \beta(\theta)d\theta < \infty$, has been much investigated by the analysts. It is really more difficult to assume only (1.1), and the only analytical existence and regularity result under (1.1) is due to Desvillettes in [3].
A probabilistic approach using the underlying evolution Markov process allows to work under (1.1) thanks to the L^2-calculus. We obtain a slightly better existence result than Desvillettes, and our regularity result is much better. Desvilletes builds a solution $g(t,v)$ of (B), and he proves that for each $t>0$, $f(t,.)$ is in $H^{1-\epsilon}(\mathbb{R}^2)$ for all $\epsilon > 0$. The solution $f(t,v)$ we build is continuous on $]0,T]\times \mathbb{R}^2$, and for each $t>0$, $f(t,.)$ is in $C^\infty(\mathbb{R}^2)$.
Another advantage of a probabilistic approach is that we can assume that the initial data is a probability, and not necessarily a density of probability. Finally, we give a (probabilistic) notion of uniqueness

In order to define the weak solutions, we consider the following kernel, which depends on the test function $\phi \in C_b^2(\mathbb{R}^2)$ (the set of C^2 functions on $\mathbb{R}^2$ of which the derivatives of order 0 to 2 are bounded) :

$$\begin{aligned} K_\beta^\phi(v,v^*) &= \int_{-\pi}^{\pi} \Big[\phi(c(v,v^*,\theta)) - \phi(v) - \phi'_x(v)\,(c_x(v,v^*,\theta)-v_x) - \phi'_y(v)\,(c_y(v,v^*,\theta)-v_y)\Big]\beta(\theta)d\theta \\ &- \frac{b}{2}\Big[\phi'_x(v)(v_x-v_x^*) + \phi'_y(v)(v_y-v_y^*)\Big] \end{aligned} \tag{1.2}$$

where $b = \displaystyle\int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)d\theta$. This expression is well defined for every test function thanks to the assumption (1.1). Now we can define the weak solutions of (B).

Definition 1.1 *Let β be a cross section (even and positive on $[-\pi,\pi]\backslash\{0\}$) satisfying (1.1). Let P_0 be a probability on $\mathbb{R}^2$ that admits a moment of order 2. A positive function f on $\mathbb{R}^+\times\mathbb{R}^2$ is a weak solution of (B) with initial data P_0 if for every test function $\phi \in C_b^2(\mathbb{R}^2)$,*

$$\int_{v\in\mathbb{R}^2} f(t,v)\phi(v)dv = \int_{v\in\mathbb{R}^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in\mathbb{R}^2}\int_{v^*\in\mathbb{R}^2} K_\beta^\phi(v,v^*)f(s,v)f(s,v^*)dvdv^*ds \tag{1.3}$$

Let us explain this definition : a priori, we should look for weak solutions satisfying, for every test function,

$$\int_{v\in\mathbb{R}^2} f(t,v)\phi(v)dv = \int_{v\in\mathbb{R}^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in\mathbb{R}^2} K_\beta(f,f)(s,v)\phi(v)dvds$$

Let us subsitute $v' = c(v,v^*,\theta)$, $v'^* = c^*(v,v^*,\theta)$, and $\theta' = -\theta$ in the first part of $K_\beta(f,f)$. The Jacobian of this substitution is equal to 1, and an easy drawing shows that $v = c(v',v'^*,\theta')$, $v^* = c^*(v',v'^*,\theta')$ and $\theta = -\theta'$. We obtain :

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi} f(t,c(v,v^*,\theta))f(t,c^*(v,v^*,\theta))\phi(v)\beta(\theta)d\theta dvdv^* \\ &\quad= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi} f(t,v)f(t,v^*)\phi(c(v,v^*,\theta))\beta(\theta)d\theta dvdv^* \end{aligned}$$

and hence

$$\int_{v\in\mathbb{R}^2} f(t,v)\phi(v)dv = \int_{v\in\mathbb{R}^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in\mathbb{R}^2}\int_{v^*\in\mathbb{R}^2} k_\beta^\phi(v,v^*)f(s,v)f(s,v^*)dvdv^*ds$$

where

$$k_\beta^\phi(v,v^*) = \int_{-\pi}^{\pi}\Big[\phi(c(v,v^*,\theta)) - \phi(v)\Big]\beta(\theta)d\theta$$

But this kernel does not make sense for every test function $\phi \in C_b^2(\mathbb{R}^2)$, except if we suppose that $\int_0^\pi \theta\beta(\theta)d\theta < \infty$. Consequently, we replace k_β^ϕ by K_β^ϕ, in which there is a compensated term. Notice that if $\int_0^\pi \theta\beta(\theta)d\theta < \infty$, then $\int_{-\pi}^{\pi} \sin\theta\beta(\theta)d\theta = 0$, and the two kernels are identical.

The method is partially adapted from the papers of L. Desvillettes, C. Graham and S. Méléard in [4] and [5], who solved a simpler problem in dimension 1. We first show that there exists a stochastic differential equation associated with the equation (B). This means that if V_t is a solution of this S.D.E., then its law is a measure solution and if V_t admits a density for of (B). If furthermore for each $t > 0$, the law of V_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^2$, then f will be a solution of (B) in the sense of Definition 1.1.
The first section is devoted to the statement of the S.D.E., to the existence and the uniqueness in law of a solution of this S.D.E., and to the study of some moment conservations for this solution, which can be related to physical conservations. The aim of the second section is to use the Malliavin Calculus in order to show the existence of a weak solution of (B), and to study the smoothness of this solution. We will use Bismut's approach of the Malliavin Calculus, by following the methods of Bichteler, Gravereaux, and Jacod in [1] and [2]. However, we can not apply trheir results, because our model does not satisfy their assumptions, for several reasons.
The most difficult and original part of this paper is the proof of the regularity (see Lemmas 3.22, 3.23, and Theorem 3.24), for which we need to use the particular form of our S.D.E.

In the sequel, β is a fixed cross section satisfying (1.1).

The uniqueness for the equation (B) is an open problem. But it is possible to prove that if all the moments of P_0 are finite, if f and g are two weak solutions of (B) on $[0,T]$, and if for every $p \geq 0$,

$$\sup_{[0,T]} \int_{\mathbb{R}^2} \| v \|^p f(t,v)dv + \sup_{[0,T]} \int_{\mathbb{R}^2} \| v \|^p g(t,v)dv < \infty$$

then f and g have the same moments : for every $p,q \geq 0$, for all $t \in [0,T]$,

$$\int_{\mathbb{R}^2} v_x^p v_y^q f(t,v)dv = \int_{\mathbb{R}^2} v_x^p v_y^q g(t,v)dv$$

This can be shown recursively (on $p+q$) by using Newton's formula. (We will compute explicitely the moments of order 1 in Proposition 2.10).

2 The probabilistic approach.

The whole section is an easy adaptation of the paper of Desvillettes, Graham and Méléard, [4], although there is a quite important difference between the S.D.E. in dimension 1 and 2.
Since we are looking for a solution $f(t,v)$ which is a density of particles at each instant t, it is quite natural to relate $f(t,v)$ to the flow of marginals of a stochastic process. We restrict our study to the time interval $[0,T]$, where $T > 0$ is fixed.

Definition 2.1 *We will say that a flow $\{P_t\}_{t\in[0,T]}$ of probability measures on $\mathbb{R}^2$ such that P_0 admits a moment of order 2 is a weak solution of (B) with initial data P_0 if for every test function $\phi \in C_b^2(\mathbb{R}^2)$,*

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_0^t \left\langle K_\beta^\phi(v,v^*), P_s(dv)P_s(dv^*) \right\rangle ds \tag{2.1}$$

Remark 2.2 *If a flow $\{P_t\}_{t\in[0,T]}$ of probability measures on $\mathbb{R}^2$ is a weak solution of (B), and if for every $t \in]0,T]$, P_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^2$, then f is a solution of (B) with initial data P_0 in the sense of Definition 1.1.*

In order to state a S.D.E. associated with our problem, we introduce some notations. Following Tanaka, [9], we will consider two probability spaces : the first one is an abstract space $(\Omega, \mathcal{F}, P)$, and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be some α-processes, the expectation under $d\alpha$ will be denoted E_α, and the laws $\mathcal{L}_\alpha$.
On $(\Omega, \mathcal{F}, P)$, we consider a Poisson measure $N(d\theta d\alpha dt)$ on $[-\pi, \pi] \times [0,1] \times [0,T]$ with intensity measure $\nu(d\theta d\alpha dt) = \beta(\theta) d\theta d\alpha dt$ and with compensated measure $\tilde{N}(d\theta d\alpha dt)$.
If Q is a probability on $\mathbb{D}_T$, and if $p \geq 1$, we will say that $Q \in \mathcal{P}_p(\mathbb{D}_T)$ if $\int_{x \in \mathbb{D}_T} \sup_{[0,T]} \parallel x(t) \parallel^p Q(dx) < \infty$. A càdlàg adapted process Y_s on $[0,T]$ will be a $\mathbb{L}^p_T$-process if its law is in $\mathcal{P}_p(\mathbb{D}_T)$.

Definition 2.3 *Let $V_0(\omega) \in L^2(\Omega)$, let $Y_s(\omega)$ be a $\mathbb{L}^2_T$-process, and let $Z_s(\alpha)$ be a $\mathbb{L}^2_T$-α-process, every of these elements with values in $\mathbb{R}^2$. Then we denote by $V = \Phi(Y, Z, V_0, N)$ the process defined (and well defined) by*

$$V_t(\omega) = V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} [c(Y_{s-}(\omega), Z_{s-}(\alpha), \theta) - Y_{s-}(\omega)]\, \tilde{N}(d\theta d\alpha ds) - \frac{b}{2} \int_0^t \int_0^1 (Y_s(\omega) - Z_s(\alpha))\, d\alpha ds \tag{2.2}$$

This can also be written, by using the matrix $A(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}$:

$$V_t = V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\, (Y_{s-} - Z_{s-}(\alpha))\, \tilde{N}(d\theta d\alpha ds) - \frac{b}{2} \int_0^t \int_0^1 (Y_s - Z_s(\alpha))\, d\alpha ds \tag{2.3}$$

Definition 2.4 *Let $\{V_t\}_{t \in [0,T]}$ be a $\mathbb{L}^2_T$-process and let $\{W_t\}_{t \in [0,T]}$ be a $\mathbb{L}^2_T$-α-process, with values in $\mathbb{R}^2$. We will say that (V, W) is a solution of (SB) with initial data V_0 if*

$$\mathcal{L}(V) = \mathcal{L}_\alpha(W) \qquad \textit{and} \qquad V = \Phi(V, W, V_0, N)$$

We notice here that this S.D.E. is symmetric in V and W, which is not the case in dimension 1. This yields that the solution of this S.D.E. does not behaves in the same way when the dimension is 1 or 2. In particular the conservation of the momentum (i.e. $E(V_t) = E(V_0)$ for $t > 0$) will hold. The next remark follows from the Itô formula.

Remark 2.5 *If (V, W) is a solution of (SB) with initial data V_0, then the probability flow $\{\mathcal{L}(V_t)\}_{t \in [0,T]} = \{\mathcal{L}_\alpha(W_t)\}_{t \in [0,T]}$ is a weak solution of (B) with initial data $\mathcal{L}(V_0)$.*

In order to prove the existence and the uniqueness in law for the non classical S.D.E. (SB), we first solve the associated classical S.D.E.

Proposition 2.6 *Let $V_0 \in L^2(\Omega)$, and let Z be a $\mathbb{L}^2_T$-α-process. Then the classical S.D.E. $V = \Phi(V, Z, V_0, N)$ admits a unique solution, that belongs to $\mathbb{L}^2_T$. Furthermore, the law of the solution depends only on $\mathcal{L}(V_0)$ and on the flow $\{\mathcal{L}_\alpha(Z_t)\}_{t \in [0,T]}$.*

Proof : the existence and the uniqueness for this kind of S.D.E. is standard. In order to study the law of the solution, let us write the Poisson measure as $N = \sum_{s \in [0,T]} \mathbb{1}_D(s) \delta_{(\theta_s, \alpha_s, s)}$, and let us set $N^* = \sum_{s \in [0,T]} \mathbb{1}_D(s) \delta_{(\theta_s, Z_s(\alpha_s), s)}$. Then N^* is a Poisson measure on $[0,T] \times [-\pi, \pi] \times \mathbb{R}^2$ with intensity $\beta(\theta) d\theta \mathcal{L}_\alpha(Z_s)(dz) ds$. (Recall that Z_t is "ω-deterministic"). Then

$$V_t = V_0 + \int_0^t \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} (c(V_{s-}, z, \theta) - V_{s-})\, \tilde{N}^*(d\theta dz ds) - \frac{b}{2} \int_0^t V_s ds + \frac{b}{2} \int_0^t E_\alpha(Z_s) ds$$

and the law of V_t is entirely determined by $\mathcal{L}(V_0)$, by the intensity of N^*, and by $\{E_\alpha(Z_s)\}_{s \leq T}$. The result follows.

We now define recursively the Picard iterations that will converge to a solution of (SB).

Definition 2.7 *Let $V_0 \in L^2$. Let V^0 be the process identically equal to V_0. Assuming that we have defined the $\mathbb{L}^2_T$-processes V^0,...,V^k, and the $\mathbb{L}^2_T$-α-processes Z^0,...,Z^{k-1}, we choose a $\mathbb{L}^2_T$-α-process Z^k satisfying*

$$\mathcal{L}_\alpha(Z^k|Z^{k-1},...,Z^0) = \mathcal{L}(V^k|V^{k-1},...,V^0)$$

then we set

$$V^{k+1} = \Phi(V^k, Z^k, V_0, N)$$

Notice here that we build the pathwises of the V^k, and only the laws of the Z^k. The following theorem shows the existence of a solution for S.D.E. (SB).

Theorem 2.8 *The sequences V^k and Z^k converge a.s. and in $\mathbb{L}^2_T$ to some processes V and W. The process V is in $\mathbb{L}^2_T$, and W is a $\mathbb{L}^2_T$-α-process. Furthermore,*

$$\mathcal{L}(V) = \mathcal{L}_\alpha(W) = P^\beta \qquad \text{and} \qquad V = \phi(V, W, V_0, N)$$

Hence (V, W) is a solution of (SB) with initial data V_0. The law P^β does not depend on the possible choices for Ω, for N, for V_0, and for the Picard approximations, but only on $\mathcal{L}(V_0)$.
If furthermore $E(|V_0|^p) < \infty$ for all $p < \infty$, then V is a $\mathbb{L}^p_T$-process for all $p < \infty$.

Proof : we show that these sequences are Cauchy by using a simple computation and the fact that for every k, $\mathcal{L}_\alpha(Z^k - Z^{k-1}) = \mathcal{L}(V^k - V^{k-1})$. Letting k go to infinity in the equality $V^{k+1} = \Phi(V^k, Z^k, V_0, N)$, we see that $V = \Phi(V, W, V_0, N)$. Finally, $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ because the sequences $\{V^k\}$ and $\{Z^k\}$ have the same law, and because the processes V^k and Z^k converge uniformly in L^2.
As in Proposition 2.6, we can check that the law of the sequence $\{V^k\}$ does not depend on the choices for Ω, N, V_0, and $\{Z^k\}$, but only on the laws of these elements.

We now prove the uniqueness in law for (SB). : it suffices to consider a fixed "space" (Ω, V_0, N), and to check that any solution of (SB) on this space have the law P^β.

Theorem 2.9 *Let Ω, $V_0 \in L^2(\Omega)$, and N be fixed. We consider the solution (V, W) (with $P^\beta = \mathcal{L}(V) = \mathcal{L}_\alpha(W)$) of (SB) that we have built in Theorem 2.8. We also assume that there exists another solution (U, Y), and we set $Q = \mathcal{L}(U) = \mathcal{L}_\alpha(Y)$. Then $Q = P^\beta$.*

This theorem can be shown by following the methods of Desvillettes et al. in [4] Theorem 3.7 p 12.

We now assume that Ω, N, and $V_0 \in L^2(\Omega)$ are fixed. We consider a solution (V, W) of (SB) with initial data V_0.

Proposition 2.10 *The conservations of the momentum and of the kinetic energy hold : for every $t \in [0, T]$,*

$$E(V_t) = E(V_0) \qquad \text{and} \qquad E\left(\| V_t \|^2\right) = E\left(\| V_0 \|^2\right)$$

Notice that the conservation of the momentum does not hold in dimension 1.

Proof : in order to prove these equalities, it suffices to use the fact that the flow $P_t = \mathcal{L}(V_t)$ is a weak solution of (B) in the sense of Definition 2.1. Let us first consider the test function $\phi(v) = v_x$: it is easy to check that $K^\phi_\beta(v, v^*) = 0 - \frac{b}{2}(v_x - v^*_x)$. Hence for every $s > 0$, $\left\langle K^\phi_\beta(v, v^*), P_s(dv)P_s(dv^*)\right\rangle = 0$, and we obtain $\int_{R^2} v_x P_t(dv) = \int_{R^2} v_x P_0(dv)$. In the same way, $\int_{R^2} v_y P_t(dv) = \int_{R^2} v_y P_0(dv)$, and the conservation of the momentum is proved.
Then we set $\phi(v) = v_x^2 + v_y^2$: since $K^\phi_\beta(v, v^*) = \frac{b}{2}\left(v_x^{*2} - v_x^2 + v_y^{*2} - v_y^2\right)$, it is clear that for every $s > 0$, $\left\langle K^\phi_\beta(v, v^*), P_s(dv)P_s(dv^*)\right\rangle = 0$, and we can conclude as above that the conservation of the kinetic energy holds.

We now deduce a useful corollary :

Corollary 2.11 *If $\mathcal{L}(V_0)$ is not a Dirac mass, then for every $t \in [0,T]$, $\mathcal{L}(V_t)$ is not a Dirac mass either.*

Proof : let us assume that there exists $t > 0$ and $X \in I\!R^2$ such that $\mathcal{L}(V_t) = \delta_X$. Then from Proposition 2.10, $E\left(\| V_0 - X \|^2\right) = E\left(\| V_t - X \|^2\right) = 0$, which implies that $V_0 = X$ a.s.

3 Existence and smoothness of a weak solution by using the stochastic calculus of variations.

We now want to study the existence and the smoothness of a density with respect to the Lebesgue measure on $I\!R^2$ for the law of a solution of (SB). Indeed, if this density exists, it will satisfy (B) in the sense of Definition 1.1. We thus will use the stochastic calculus of variations (namely the Malliavin calculus). Bismut's methods are here easier than Malliavin's original approach. The papers of Bichteler, Jacod [2] and of Bichteler, Gravereaux, Jacod [1] explain the Malliavin calculus for diffusion processes with jumps when the intensity of the Poisson measure is the Lebesgue measure ; and although we cannot apply directly their results, we will follow their methods. In [2], Bichteler and Jacod study the existence of a density for these processes in dimension 1, and Bichteler, Gravereaux and Jacod extend in [1] the methods to the existence and the smoothness of this density in any finite dimension. This second paper is very complete, but the assumptions that yield the existence of a density are too much stringent, so that we have to use a mixed method to show the existence of a weak solution of (B).
First, let us state our assumptions.
Assumption (H) :

1. The initial data P_0 admits a moment of order 2, and is not a Dirac mass.
2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi,\pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0,\pi[$, and $r \in]1,3[$ such that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r} 1\!\!1_{[-\theta_0,\theta_0]}(\theta)$. We still assume $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$.

Assumption (S) :

1. All the moments of P_0 are finite.
2. The cross section β satisfies : $\left|\dfrac{\sin\theta}{1+\cos\theta}\right| 1\!\!1_{|\theta|\in[\pi/2,\pi]} \in \cap_{p\geq 1} L^p(\beta(\theta)d\theta)$

Then we state our main theorems.

Theorem 3.1 *Under the assumption (H), the equation (B) admits a solution with initial data P_0 in the sense of Definition 1.1.*

Theorem 3.2 *We assume (H) and (S), and we consider the solution $f(t,v)$ of the equation (B) with initial data P_0 built in Theorem 3.1. Then for each $t \in]0,T]$ fixed, $f(t,.)$ is of class C^∞ on $I\!R^2$.*

Theorem 3.3 *Assume (H) and (S). Let $f(t,v)$ be the solution of (B) on $[0,T]$ with initial data P_0 built in Theorem 3.1. The map $(t,v) \longrightarrow f(t,v)$ is continuous on $]0,T] \times I\!R^2$.*

Let us notice that Assumption (H)-1 is natural. Indeed, if P_0 is a Dirac mass at $v_0 \in I\!R^2$, then all the particles have the initial velocity v_0, and there cannot be any collision. Hence $P_t = P_0$ for all t is a solution of (B) in the sense of Definition 2.1, and it is clear that in this case, P_t does not admit any density.
It seems also natural to suppose (S)-2, which means that β is small near $\theta = \pi$. If the angle of a collision between two particles is π, then these particles exchange their velocities, and this has no effect on the density $f(t,.)$. Thus if P_0 does not admit any density, and if $\beta(\theta)$ is large near π, there cannot be any

regularization property.

In [3], the analyst Desvillettes states a comparable theorem under the following assumption (here the initial data is a density of probability) :
Assumption (h) : There exists $\beta_0 > 0$, $\beta_1 > 0$, and $\gamma \in]1,3[$ such that :

$$\beta_0|\theta|^{-\gamma} \leq \beta(\theta) \leq \beta_1|\theta|^{-\gamma}$$

and the initial data $f_0 \ : \ I\!R^2 \to I\!R^+$ satisfies :

$$\int_{I\!R^2} f_0(v) \left(1 + |v|^2 + |\ln f_0(v)|\right) dv < \infty$$

Theorem : *Under (h), the Kac equation (B) admits a weak solution f satisfying, for every $t_0 > 0$, $\epsilon > 0$:*

$$f \in L^1_{loc}\left([t_0, \infty[, H^{1-\epsilon}(I\!R^2_v)\right) \cap L^\infty_{loc}\left([t_0, \infty[, H^{\frac{3-\gamma}{2}-\epsilon}(I\!R^2_v)\right)$$

Comparing this theorem and Theorems 3.2 and 3.3, we see how the probabilistic approach is efficient. Let us come back to our method.

Notations : *In the whole section, Ω and N are fixed as in Section 2, and we assume at least (H). We also consider on Ω a random variable V_0 such that $\mathcal{L}(V_0) = P_0$, and a solution (V, W) of the S.D.E. (SB) with initial data V_0 in the sense of Definition 2.4.*

3.1 The techniques.

The Malliavin Calculus is based on the *integration by parts settings* (IBPS). Of course, the IBPS needed for the existence of a density (which we will name *weak IBPS*) are less stringent than the ones used for the smoothness of the density.
In the next definition, we follow [1] p 27, and we introduce the weak IBPSs. Recall that $C^2_p(I\!R^d)$ is the set of C^2 functions on $I\!R^d$ of which all derivatives of order 0 to 2 have at most a polynomial growth.

Definition 3.4 *Let ϕ be a random variable with values in $I\!R^2$. We will say that $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS (resp. a weak IBPS) for ϕ if*

1. *σ is a random variable with values in $\mathcal{M}_2(I\!R)$ (the set of the 2×2-matrices on $I\!R$).*
2. *γ is a random variable with values in $I\!R^2$ such that $\gamma \in \cap_{p<\infty} L^p$ (resp. $\gamma \in L^2$).*
3. *$\mathcal{D}$ is a linear space of random variables contained in $\cap_{p<\infty} L^p$ (resp. L^2), and is stable under C^2_p (resp. C^2_b).*
4. *$\delta = (\delta_1, \delta_2)$, where δ_i is a linear map on $\mathcal{D}$ such that if $n \geq 1$, if $F \in C^2_p(I\!R^n)$ (resp. $C^2_b(I\!R^n)$), and if $\psi = (\psi_1, ..., \psi_n) \in \mathcal{D}^n$, then*
$$\delta_i(F \circ \psi) = \sum_{j=1}^n \frac{\partial F}{\partial x_j}(\psi)\delta_i(\psi_j)$$
5. *For every $g \in C^2_p(I\!R^2)$ (resp. $C^2_b(I\!R^2)$), for every $\psi \in \mathcal{D}$, for $j = 1, 2$ the following equality holds :*

$$E\left(\psi \sum_{i=1}^2 d_i g(\phi)\sigma^{ij}\right) = E\left(g(\phi)[\psi\gamma^j + \delta_j(\psi)]\right) \qquad (3.1)$$

We will use the following criteria :

Theorem 3.5 *Let ϕ be a random variable with values in $I\!\!R^2$. Assume that $(\sigma, \gamma, \mathcal{D}, \delta)$ is a weak IBPS for ϕ. If for each $i, j \in \{1, 2\}$, σ^{ij} is in $\mathcal{D}$, and if $\det \sigma \neq 0$ a.s., then the law of ϕ admits a density with respect to the Lebesgue measure on $I\!\!R^2$.*

Theorem 3.6 *Let ϕ be a random variable with values in $I\!\!R^2$. We assume that $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS for ϕ, and we consider the following sets :*

$$C_0 = \left\{\sigma^{ij}, \gamma^i \mid i, j \in \{1, 2\}\right\} \quad \text{and} \quad C_{n+1} = C_n \cup \{\delta_j(\psi) \mid j \in \{1, 2\},\ \psi \in C_n\}$$

Then ϕ admits a density of class C^∞ with respect to the Lebesgue measure on $I\!\!R^2$ provided for all $n \geq 0$, $C_n \subset \mathcal{D}$, and $(\det \sigma)^{-1} \in \cap_{p<\infty} L^p$.

Theorem 3.6 is proved in Bichteler, Gravereaux, Jacod, [1] p 33, and Theorem 3.5 is also proved in [1] p 28 in the case where $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS for ϕ. But it is easy to see that they use only the fact $(\sigma, \gamma, \mathcal{D}, \delta)$ is a weak IBPS.

3.2 An I.B.P.S. for V_t.

The existence of the density for the law of a jump process is based on an accumulation of small jumps. Recalling that $\beta = \beta_0 + \beta_1$ and that β_0 explodes near 0, we will in fact be interested only in β_0. Hence, we suppose that the Poisson measure N splits into $N_0 + N_1$, where N_0 and N_1 are independent Poisson measures on $[0, T] \times [0, 1] \times [-\pi, \pi]$ with intensities $\nu_0(d\theta d\alpha ds) = \beta_0(\theta) d\theta d\alpha ds$ and $\nu_1(d\theta d\alpha ds) = \beta_1(\theta) d\theta d\alpha ds$. We will denote by $\tilde{N}_0$ and $\tilde{N}_1$ the associated compensated measures. We also assume that our probability space is the canonical one associated with the independent random elements V_0, N_0, and N_1 :

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P) = (\Omega', \mathcal{F}', \{\mathcal{F}'\}, P') \otimes (\Omega^0, \mathcal{F}^0, \{\mathcal{F}^0_t\}, P^0) \otimes (\Omega^1, \mathcal{F}^1, \{\mathcal{F}^1_t\}, P^1) \tag{3.2}$$

An element $\omega \in \Omega$ can be written $\omega = (\omega', \omega^0, \omega^1)$, where ω' is a real number, and ω^0 and ω^1 are integer valued measures on $[0, T] \times [0, 1] \times [-\pi, \pi]$.

Notations : *Although N_0 has its support in $[0, T] \times [0, 1] \times [-\theta_0, \theta_0]$, we will still integrate against N_0 and $\tilde{N}_0$ on $[0, T] \times [0, 1] \times [-\pi, \pi]$, even if the functions in the integrals are defined only on $[0, T] \times [0, 1] \times [-\theta_0, \theta_0]$.*

Let us briefly present the method we will use to build an I.B.P.S. for V_t. We will first build a *perturbation*, in order to obtain a new family of integer valued random measures N_0^λ (for $\lambda \in \Lambda$, where Λ is a neighbourhood of 0 in $I\!\!R^2$). Of course, N_0^0 must equal N_0. Then we will build a family of probability measures $P^\lambda = G_t^\lambda.P$ on Ω, such that $\mathcal{L}(V_0, N_0^\lambda, N_1 | P^\lambda) = \mathcal{L}(V_0, N_0, N_1 | P)$. By this way, we will obtain a perturbed process V_t^λ satisfying $\mathcal{L}(V_t^\lambda | P^\lambda) = \mathcal{L}(V_t | P)$, and thus $E(\phi(V_t^\lambda) G_t^\lambda) = E(\phi(V_t))$ for any borel bounded function ϕ on $I\!\!R^2$. Then we will differentiate this equality at $\lambda = 0$ (if ϕ is regular enough), by using a L^2-*derivative* of V_t^λ and G_t^λ. We will obtain something like

$$E\left(\phi'(V_t).DV_t\right) = -E(\phi(V_t) DG_t)$$

which looks like (3.1).

We now build the perturbation. Let ρ be a positive $C_b([-\theta_0, \theta_0])$ function satisfying :

$$\rho(\theta) \leq \left(ce^{-|\theta|^{-r'}}\right) \wedge \frac{|\theta|}{2} \wedge M \ ; \quad \rho(\theta) \overset{0}{\sim} ce^{-|\theta|^{-r'}} \ ; \quad \{\rho = 0\} = \{-\theta_0, 0, \theta_0\} \tag{3.3}$$

where $r' = \frac{1}{8}(r - 1) > 0$, and where c and M are positive constants that we will choose soon. In particular, this yields that $\rho \in \cap_{p \geq 1} L^p(\beta_0(\theta) d\theta)$.

We also need a predictable function $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ from $\Omega \times [0,T] \times [-\theta_0, \theta_0] \times [0,1]$ to $I\!R^2$, such that for every ω, t, α, the map $\theta \longrightarrow v(\omega,t,\theta,\alpha)$ is of class C^1, and

$$\| v(\omega,t,\theta,\alpha) \| \vee \| v'(\omega,t,\theta,\alpha) \| \leq \rho(\theta) \tag{3.4}$$

where $v' \in I\!R^2$ is the derivative of v with respect to θ. This function will be chosen at the end of the section.
We consider a neighbourhood $\Lambda \subset B(0,1)$ of 0 in $I\!R^2$. For $\lambda \in \Lambda$, we define the following *perturbation* :

$$\gamma^\lambda(\omega,t,\theta,\alpha) = \theta + \langle \lambda, v(\omega,t,\theta,\alpha) \rangle = \theta + \lambda_x v_x(\omega,t,\theta,\alpha) + \lambda_y v_y(\omega,t,\theta,\alpha) \tag{3.5}$$

If Λ is small enough (which we assume), we can check that for every $\lambda \in \Lambda$, for every ω, t, α, the map $\theta \longrightarrow \gamma^\lambda(\omega,t,\theta,\alpha)$ is an increasing bijection from $[-\theta_0,\theta_0]$ into itself (by using (3.3) and (3.4)). For $\lambda \in \Lambda$, we set $N_0^\lambda = \gamma^\lambda(N_0)$: if $A \subset [0,T] \times [0,1] \times [-\pi,\pi]$ is a Borel set,

$$N_0^\lambda(\omega,A) = \int_0^T \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_A(s, \gamma^\lambda(\omega,s,\theta,\alpha), \alpha) N_0(\omega, d\theta d\alpha ds)$$

We consider the shift S^λ defined (and entirely defined) by

$$V_0 \circ S^\lambda(\omega) = V_0(\omega) \ , \quad N_0 \circ S^\lambda(\omega) = N_0^\lambda(\omega) \ , \quad \text{and} \quad N_1 \circ S^\lambda(\omega) = N_1(\omega) \tag{3.6}$$

We now look for a family of probability measures P^λ on Ω satisfying $P^\lambda \circ (S^\lambda)^{-1} = P$. To this end, we consider the following predictable real valued function on $\Omega \times [0,T] \times [-\theta_0,\theta_0] \times [0,1]$:

$$Y^\lambda(\omega,t,\theta,\alpha) = \Big(1 + \lambda_x v_x'(\omega,t,\theta,\alpha) + \lambda_y v_y'(\omega,t,\theta,\alpha)\Big) \times \frac{\beta_0(\gamma^\lambda(\omega,t,\theta,\alpha))}{\beta_0(\theta)} \tag{3.7}$$

If $\tilde\rho(\theta) = \rho(\theta) + r2^{r+1}\dfrac{\rho(\theta)}{|\theta|} + r2^{r+1}\rho(\theta)\dfrac{\rho(\theta)}{|\theta|}$, then

$$|Y^\lambda(t,\theta,\alpha) - 1| \leq \| \lambda \| \tilde\rho(\theta) \tag{3.8}$$

Let us notice that $\tilde\rho \in \cap_{p\geq 1} L^p(\beta_0(\theta)d\theta)$. We choose c and M such that $\tilde\rho \leq \frac{1}{2}$.
Then we consider the following square integrable Doléans-Dade martingale :

$$G_t^\lambda = 1 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} G_{s-}^\lambda (Y^\lambda(s,\theta,\alpha) - 1) \tilde N_0(d\theta d\alpha ds) \tag{3.9}$$

Proposition 3.7 *G_t^λ is strictly positive for every $t \in [0,T]$. If P^λ is the probability measure defined by $P^\lambda = G_T^\lambda.P$, then $P^\lambda \circ (S^\lambda)^{-1} = P$.*

The proof of this proposition follows from the Girsanov theorem for random measures (see Jacod, Shiryaev [7]), as Lemme 3.8 in [2] (except that the initial data V_0 is not deterministic). This proof is based on the choice of Y^λ : one can check that $\gamma^\lambda(Y^\lambda.\nu_0) = \nu_0$.

We now introduce the following derivatives :

Definition 3.8 *Recall that Λ is a neighbourhood of* 0 *in $I\!R^2$. Let $p \geq 2$.*

1. *Let $\{X^\lambda\}_{\lambda\in\Lambda}$ be a family of real valued L^p random variables. We will say that X^λ is L^p-differentiable at $\lambda = 0$ if there exists a* **derivative** $DX = \begin{pmatrix} D^x X \\ D^y X \end{pmatrix} \in L^p$ *such that, when λ goes to* 0*,*

$$E\left(\left|X^\lambda - X^0 - \langle \lambda, DX \rangle\right|^p\right) = o\left(\| \lambda \|^p\right)$$

2. Let $\{X^\lambda\}_{\lambda\in\Lambda}$ be a family of $\mathbb{R}^2$ valued L^p random variables. We will say that X^λ is L^p-differentiable at $\lambda = 0$ if there exists a derivative $DX = \begin{pmatrix} D^xX^x & D^yX^x \\ D^xX^y & D^yX^y \end{pmatrix} \in L^p$ such that, when λ goes to 0,

$$E\left(\| X^\lambda - X^0 - DX.\lambda \|^p\right) = o(\| \lambda \|^p)$$

3. We denote by $\mathcal{D}$ (resp. $\mathcal{D}^\infty$) the set of the real valued random variables X such that $X^\lambda = X \circ S^\lambda$ is L^2-differentiable (resp. L^q-differentiable for every $q < \infty$) at 0, and by $\mathcal{D}_t$ (resp. $\mathcal{D}_t^\infty$) its restriction to the set of the $\mathcal{F}_t$-measurable random variables.

4. Let now $\{Y_t^\lambda\}_{\lambda\in\Lambda}$ be a family of real valued $\mathbb{L}_T^p$-processes. We will say that Y^λ is L^p-differentiable at $\lambda = 0$ if there exists a $\mathbb{L}_T^p$-process $DY_t = \begin{pmatrix} D^xY_t \\ D^yY_t \end{pmatrix}$ such that :

$$E\left(\sup_{[0,T]} |Y_t^\lambda - Y_t^0 - \langle \lambda, DY_t\rangle |^p\right) = o(\| \lambda \|^p)$$

Let us describe the process $V_t^\lambda = V_t \circ S^\lambda$. The α-process W behaves here as a parameter.

Proposition 3.9 *The perturbed process V^λ satisfies the following equation under P :*

$$E(\lambda)\left\{\begin{array}{rcl} V_t^\lambda &=& V_0 - \dfrac{b}{2}\displaystyle\int_0^t\int_0^1 (V_s^\lambda - W_s(\alpha))d\alpha ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_1(d\theta d\alpha ds) \\ &+& \displaystyle\int_0^t\int_0^1\int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_0(d\theta d\alpha ds) \\ &-& \displaystyle\int_0^t\int_0^1\int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha) - 1)A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\beta_0(\theta)d\theta d\alpha ds \end{array}\right.$$

Proof : we work here under P. The direct expression of V^λ is given by

$$\begin{array}{rcl} V_t^\lambda &=& V_0 - \frac{b}{2}\displaystyle\int_0^t\int_0^1 (V_s^\lambda - W_s(\alpha))d\alpha ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_1(d\theta d\alpha ds) \\ &+& \displaystyle\int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))(N_0^\lambda - \nu_0)(d\theta d\alpha ds) \end{array}$$

But the last term is equal to

$$\int_0^t\int_0^1\int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_0(d\theta d\alpha ds)$$

$$- \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))(\nu_0 - \gamma^\lambda(\nu_0))(d\theta d\alpha ds)$$

Since $\nu_0 - \gamma^\lambda(\nu_0) = \gamma^\lambda(Y.\nu_0) - \gamma^\lambda(\nu_0) = \gamma^\lambda((Y-1).\nu_0)$ (see Proposition 3.7), the proof is finished.

As we will study V^λ as a solution of $E(\lambda)$, (we have no other information), we may need the following proposition of which the proof is standard :

Proposition 3.10 *For every $\lambda \in \Lambda$, the equation $(E(\lambda))$ admits one and only one solution $V^\lambda \in \mathbb{L}_T^2$. If furthermore $P_0 = \mathcal{L}(V_0)$ admits moments of all orders, then $V^\lambda \in \mathbb{L}_T^p$ for every $p < \infty$.*

Let us differentiate G^λ (see Definition 3.8).

Proposition 3.11 *The family $\{G^\lambda\}$ is L^p differentiable for every $p<\infty$, and has the following derivative*

$$DG_t = \begin{pmatrix} D^x G_t \\ D^y G_t \end{pmatrix} = \begin{pmatrix} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \frac{\partial}{\partial \lambda_x} Y^\lambda(s,\theta,\alpha)\Big|_{\lambda=0} \tilde{N}_0(d\theta d\alpha ds) \\ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \frac{\partial}{\partial \lambda_y} Y^\lambda(s,\theta,\alpha)\Big|_{\lambda=0} \tilde{N}_0(d\theta d\alpha ds) \end{pmatrix} \tag{3.10}$$

We omit this proof and the following one, because they are very simple in their principle, but the computations are fastidious. The method can be found in [2] Lemma 3.7 p 138 and Lemma 3.11 p 140, or [1] Subsection 5-b.

Notations : *We will denote in the sequel* $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$.

Theorem 3.12 *The family $\{V^\lambda\}$ is L^2-differentiable at $\lambda=0$, and its derivative $DV \in \mathcal{M}_2(I\!R)$ satisfies the equation :*

$$(ED) \left\{ \begin{array}{lcl} DV_t & = & -\frac{b}{2}\int_0^t DV_s ds + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta) DV_{s-} \tilde{N}(d\theta d\alpha ds) \\ & + & \int_0^t \int_0^1 \int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha)) v^T(s,\theta,\alpha) \tilde{N}_0(d\theta d\alpha ds) \\ & - & \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha)) \left((v(s,.,\theta)\beta_0(.))'(\theta) \right)^T d\theta d\alpha ds \end{array} \right.$$

If furthermore P_0 has moments of all orders, then V is L^p-differentiable for every $p<\infty$.

We can now state an IBPS for V_t.

Proposition 3.13 *Let $t \geq 0$. If $X \in \mathcal{D}_t$ (or if $X \in \mathcal{D}_t^\infty$, cf Definition 3.8), we set $\delta_t(X) = -DX$. Under (H), $(DV_t, -DG_t, \mathcal{D}_t, \delta_t)$ is a weak IBPS for V_t. Under (H) and (S), $(DV_t, -DG_t, \mathcal{D}_t^\infty, \delta_t)$ is an IBPS for V_t.*

Proof : let us for example assume (H) and (S) and prove the second claim. DV_t is of course a $\mathcal{M}_2(I\!R)$ valued random variable. By Proposition 3.11, $-DG_t$ is a $I\!R^2$ valued random variable which is in $\cap_p L^p$. $\mathcal{D}_t^\infty$ is a linear space, and it is classical to show that if $X_1,...,X_n$ are in $\mathcal{D}_t^\infty$, and if $F \in C_p^2(I\!R^n)$, then $F(X_1,...,X_n) \in \mathcal{D}_t^\infty$, and has the following derivative :

$$DF(X_1,...,X_n) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(X_1,...,X_n) DX_i$$

It remains to prove that if $f \in C_p^2(I\!R^2)$, and if $X \in \mathcal{D}_t^\infty$, then $E(D_t)=0$, where

$$D_t = DXf(V_t) + X\left(f'_x(V_t) \quad f'_y(V_t)\right) DV_t + Xf(V_t)DG_t$$

By using the facts that $V_t \in \cap L^p$ and $f \in C_p^2(I\!R^2)$, it is standard and natural to show that

$$E\left(\left|X^\lambda f(V_t^\lambda)G_t^\lambda - Xf(V_t) - \langle \lambda, D_t\rangle\right|\right) = o(\parallel \lambda \parallel)$$

Hence,

$$\left|E\left(X^\lambda f(V_t^\lambda)G_t^\lambda\right) - E\left(Xf(V_t)\right) - \langle \lambda, E(D_t)\rangle\right| = o(\parallel \lambda \parallel)$$

But, since $X^\lambda f(V_t^\lambda) = Xf(V_t)\circ S^\lambda$ and since $P^\lambda \circ (S^\lambda)^{-1} = P$, we deduce that

$$E\left(X^\lambda f(V_t^\lambda)G_t^\lambda\right) = E\left(Xf(V_t)\right)$$

Hence $|\langle \lambda, E(D_t)\rangle| = o(\parallel \lambda \parallel)$, and $E(D_t)=0$, which was our aim.

3.3 The choice of v.

In order to apply Theorems 3.5 and 3.6, we have to study the inversibility of DV_t. We will use the Doléans-Dade martingales, in order to obtain a suitable expression of DV_t. Then we will choose v, which is really more difficult in dimension 2 than in dimension 1. Only a good choice of v will allow DV_t to admit moments of all orders (see Theorem 3.24) : v must be "large" (this way, DV_t will be invertible) but also "small" (in particular, we need $\| v \| \leq \rho$).
We denote by I the unit matrix on $\mathbb{R}^2$.

Lemma 3.14 *One can rewrite the S.D.E. (ED) in the following way :* $DV_t = \int_0^t dK_s.DV_{s-} + L_t$ *where* $K_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\tilde{N}(d\theta d\alpha ds) - \frac{b}{2} tI$
and $L_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)N_0(d\theta d\alpha ds)$.

Proof : it suffices to prove that

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha)) \left([v(s,.,\alpha)\beta_0]'(\theta) \right)^T d\theta d\alpha ds$$
$$= \int_0^t \int_0^1 \int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)\beta_0(\theta) d\theta d\alpha ds$$

This can be shown by using a (standard) integration by parts formula in the variable θ, and by noticing that

$$\forall \omega, s, \alpha \qquad v(\omega,s,-\theta_0,\alpha) = v(\omega,s,0,\alpha) = v(\omega,s,\theta_0,\alpha) = 0$$

Proposition 3.15 *Let M (with values in $\mathcal{M}_2(\mathbb{R})$) be the following Doléans-Dade martingale :*

$$M_t = \int_0^t dK_s.M_{s-} + I \tag{3.11}$$

For all t, $(I + \Delta K_t)$ is a.s. invertible. We thus know (see Jacod, [6]) that for all s, M_s and M_{s-} are also a.s. invertible, and $DV_t = M_t H_t$ where

$$H_t = \int_0^t M_{s-}^{-1}(I + \Delta K_{s-})^{-1} dL_s = \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}^{-1}(I + A(\theta))^{-1}A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)N_0(d\theta d\alpha ds) \tag{3.12}$$

The only claim we need to show here is that for every t, $(I + \Delta K_t)$ is a.s. invertible. To this end, let us write $N = \sum_{s \in [0,T]} \mathbb{1}_D(s)\delta_{(s,\theta_s,\alpha_s)}$. Then, when N jumps at s, $I + \Delta K_s = I + A(\theta_s)$ is invertible except if $\theta_s \in \{-\pi, \pi\}$, which never happens a.s.

We now choose v. First we need a positive C_b^∞ function δ on $[-\theta_0, \theta_0]$ such that ($C > 0$ is a constant) :

$$|\delta(\theta)| + |\delta'(\theta)| \leq \rho(\theta) \quad ; \qquad \{\delta = 0\} = \{-\theta_0, 0, \theta_0\} \quad ; \quad \delta(\theta) \overset{0}{\sim} Ce^{-|\theta|^{-2r'}} \tag{3.13}$$

We will also use a function on $\mathbb{R}^2 \times (\mathcal{M}_2(\mathbb{R})) \times [-\theta_0, \theta_0]$ with values in $\mathbb{R}^2$:

$$\bar{g}(x,y,\theta) = (A'(\theta)x)^T((I + A(\theta))^{-1})^T(y^{-1})^T$$

We consider the C^∞ function $h(x) = \frac{1}{1 + \| x \|^2}$ from $\mathbb{R}^2$ to $]0,1]$. Finally, we will use a function k from $\mathcal{M}_2(\mathbb{R})$ to $[0,1]$, such that $k(y) = 0$ if and only if $\det y = 0$, and such that the map

$$y \longrightarrow \begin{cases} (y^{-1})^T k(y) & \text{if} \quad \det y \neq 0 \\ 0 & \text{if} \quad \det y = 0 \end{cases}$$

is C_b^∞ from $\mathcal{M}_2(\mathbb{R})$ to itself.
Then, the function on $\mathbb{R}^2 \times \mathcal{M}_2(\mathbb{R}) \times [-\theta_0, \theta_0]$ with values in $\mathbb{R}^2$ defined by

$$g(x,y,\theta) = \bar{g}(x,y,\theta) h\left(A'(\theta)x\right) k\left(I + A(\theta)\right) k(y)$$

is of class C_b^∞.
We now set $\Delta(x,y,\theta) = g(x,y,\theta)\delta(\theta)$. This function is of class C_b^∞.

Definition 3.16 *We set $v(s,\theta,\alpha) = \Delta\left(V_{s-} - W_{s-}(\alpha), M_{s-}, \theta\right)$. (This function satisfies the assumptions of the subsection 3.2).*

The last preliminary consists in talking about the higher derivatives of V_t and G_t : in order to apply Theorems 3.5 and 3.6, we have either to differentiate DV (under (H)) or to differentiate infinitely DV and DG (under (H) and (S)). To this end, we first notice that M_t satisfies a quite similar (but easier) equation than V_t. Hence, since the initial condition $M_0 = I$ is deterministic, $M^\lambda = M \circ S^\lambda$ is L^p-differentiable at 0 for every $p < \infty$. Let us compute $v^\lambda(\omega, s, \theta, \alpha) = v(S^\lambda(\omega), s, \theta, \alpha)$: with the notations of the Definition 3.16,

$$v^\lambda(s,\theta,\alpha) = \Delta(V_{s-}^\lambda - W_{s-}(\alpha), M_{s-}^\lambda, \theta)$$

By using the expression of DV in Lemma 3.14, we can write $DV^\lambda = DV \circ S^\lambda$ as

$$\begin{aligned}
DV_t^\lambda &= -\tfrac{b}{2}\int_0^t DV_s^\lambda ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta) DV_{s-}^\lambda \tilde{N}_1(d\theta d\alpha ds) \\
&+ \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha)) DV_{s-}^\lambda \tilde{N}_0(d\theta d\alpha ds) \\
&- \int_0^t\int_0^1\int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha) - 1) A(\gamma^\lambda(s,\theta,\alpha)) DV_{s-}^\lambda \beta_0(\theta) d\theta d\alpha ds \\
&+ \int_0^t\int_0^1\int_{-\pi}^{\pi} A'(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha)) \left(v^\lambda(s,\gamma^\lambda(s,\theta,\alpha),\alpha)\right)^T N_0(d\theta d\alpha ds)
\end{aligned}$$

One can show that under (H), the family DV^λ is L^2-differentiable at 0, by using the properties of v.

Assume now (H) and (S), and set $X_t = (DV_t, M_t, DG_t, V_t)$. Then X_t satisfies a S.D.E. with initial condition $X_0 = (0, I, 0, V_0)$. Using the properties of v, one can show that $X^\lambda = X \circ S^\lambda$ is L^p differentiable at 0 for every $p < \infty$, with $DX_t = (D^x X_t, D^y X_t)$. Hence, $DV_t \circ S^\lambda$, $M_t \circ S^\lambda$ and $DG_t \circ S^\lambda$ are L^p differentiable at 0 for every $p < \infty$.
Finally, we can iterate this method for $Y_t = (DX_t, X_t)$, and so on. We may state the following theorem :

Theorem 3.17 *Under (H), the derivative DV_t is in $\mathcal{D}_t$ for every $t \in [0,T]$. Under (H) and (S), V and G are infinitely L^p differentiable for every $p < \infty$.*

The first conditions of Theorems 3.5 and 3.6 are thus satisfied, and we still have to study the inversibility of DV_t.

3.4 Existence of a weak solution.

The following remark shows the way to prove that $DV_t = M_t H_t$ is invertible.

Remark 3.18 *We set $\Gamma(x,\theta) = (I + A(\theta))^{-1}(A'(\theta)x)(A'(\theta)x)^T\left((I+A(\theta))^{-1}\right)^T$, which is a symmetric nonnegative matrix. Then we set*
$$R_t = \int_0^t\int_0^1\int_{-\pi}^{\pi} \Gamma(V_{s-} - W_{s-}(\alpha), \theta) \times h\left(A'(\theta)(V_{s-} - W_{s-}(\alpha))\right) \times k(I + A(\theta)) \times k(M_{s-}) \times \delta(\theta) N_0(d\theta d\alpha ds).$$
This matrix is also symmetric, nonnegative, and is increasing for the strong order (on the set of symmetric nonnegative matrices : for every $s \le t$, $R_t - R_s$ is a.s. symmetric and nonnegative). We can write

$H_t = \int_0^t M_{s-}^{-1} dR_s \left(M_{s-}^{-1}\right)^T$. *Hence, in order to show that* H_t *(and hence* DV_t*) is a.s. invertible, it suffices to prove that a.s.,* $R_t - R_s$ *is invertible for every* $0 \le s < t \le T$*. Finally, since the real valued expression in* R_t *is always in* $]0,1]$*, it suffices in fact to show that a.s.,* $\bar{R}_t - \bar{R}_s$ *is invertible for all* $0 \le s < t \le T$*, where*

$$\bar{R}_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Gamma(V_{s-} - W_{s-}(\alpha), \theta)\delta(\theta) N_0(d\theta d\alpha ds)$$

Theorem 3.19 *Let* $t \in]0,T]$*. Under (H),* DV_t *is a.s. invertible.*

Proof : we break the proof in several steps.
Step 1 : If Y is a (random) vector of $\mathbb{R}^2$ not equal to 0 an easy computation shows that for $\theta \in]-\pi, \pi[$,

$$Y^T \Gamma(V_{s-} - W_{s-}(\alpha), \theta) Y = \Big(\frac{\sin\theta}{1+\cos\theta} \left[Y_x(V_{s-}^x - W_{s-}^x(\alpha)) + Y_y(V_{s-}^y - W_{s-}^y(\alpha)) \right] + \left[-Y_y(V_{s-}^x - W_{s-}^x(\alpha)) + Y_x(V_{s-}^y - W_{s-}^y(\alpha)) \right] \Big)^2 \quad (3.14)$$

Let us fix ω, s, and α. It is easy to see that if $V_{s-}(\omega) \ne W_{s-}(\alpha)$, then

$$d\theta \left\{ \theta \in]-\pi, \pi[\ / \ Y^T(\omega) \Gamma(V_{s-}(\omega) - W_{s-}(\alpha), \theta) Y(\omega) = 0 \right\} = 0$$

Step 2 : Let $s > 0$ be fixed, and let Y be a (random) unit vector in $\mathbb{R}^2$ that is $\mathcal{F}_s$-measurable. The aim of this step is to show that a.s. $\forall t > s$, $Y^T(\bar{R}_t - \bar{R}_s)Y > 0$. To this end, we consider the following stopping time :

$$\tau(Y) = \inf \left\{ t > s \Big/ Y^T(\bar{R}_t - \bar{R}_s) Y > 0 \right\} = \inf \left\{ t > s \Big/ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y)}(r, \theta, \alpha) N_0(d\theta d\alpha ds) > 0 \right\}$$

where $B(Y) = \left\{ (r, \theta, \alpha) \Big/ r > s \quad \text{and} \quad Y^T \Gamma(V_{r-} - W_{r-}(\alpha), \theta) Y > 0 \right\}$ (recall that $\bar{R}_u$ is "increasing"). It thus suffices to check that $\tau(Y) = s$ a.s. By assumption, $\mathcal{L}(V_0)$ is not a Dirac mass. By Lemma 2.11, for every $t > 0$, $\mathcal{L}(V_t) = \mathcal{L}_\alpha(W_t)$ is not a Dirac mass either. This implies that for every $r \ge 0$, for every ω,

$$\int_0^1 \mathbb{I}_{\{W_{r-}(\alpha) \ne V_{r-}(\omega)\}} d\alpha = P_\alpha(W_{r-} \ne V_{r-}(\omega)) > 0$$

Since $\int_{-\pi}^{\pi} \beta_0(\theta) d\theta = \infty$, and thanks to the first step, for all ω, for all $r > s$,

$$\int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y(\omega))}(r, \theta, \alpha) \beta_0(\theta) d\theta d\alpha \ge \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{\{W_{r-}(\alpha) \ne V_{r-}(\omega)\}} \mathbb{I}_{B(Y(\omega))}(r, \theta, \alpha) \beta_0(\theta) d\theta d\alpha = \infty$$

Consequently, except if $\tau(Y(\omega)) = s$,

$$\int_0^{\tau(Y(\omega))} \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y(\omega))}(r, \theta, \alpha) \beta_0(\theta) d\theta d\alpha dr = \infty$$

But a.s., $\int_0^{\tau(Y)} \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y)}(r, \theta, \alpha) N_0(d\theta d\alpha dr) \le 1$, which yields

$$E\left(\int_0^{\tau(Y)} \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y)}(r, \theta, \alpha) \beta_0(\theta) d\theta d\alpha dr \right) = E\left(\int_0^{\tau(Y)} \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y)}(r, \theta, \alpha) N_0(d\theta d\alpha dr) \right) \le 1$$

and thus $\int_0^{\tau(Y)}\int_0^1\int_{-\pi}^{\pi} 1\!\!1_{B(Y)}(r,\theta,\alpha)\beta_0(\theta)d\theta d\alpha ds < \infty$ a.s. Hence $\tau(Y) = s$ a.s., which was our aim.

Step 3 : We now show that if $s > 0$ is fixed, then a.s., $\forall\ t > s$, $\bar{R}_t - \bar{R}_s$ is invertible. We set $Ker_t = Ker(\bar{R}_t - \bar{R}_s)$. For each random unit vector Y in $I\!R^2$, that is $\mathcal{F}_s$-measurable, we know that a.s., $\forall t > s$, $Y \notin Ker_t$. Hence, as Ker_t is increasing when t decreases, a.s., $Y \notin Ker_{s+} = \cup_{t>s} Ker_t$. Since Ker_{s+} is $\mathcal{F}_s$-measurable, and since this is true for every unit vector $\mathcal{F}_s$-measurable, we deduce that $Ker_{s+} = \{0\}$, and the step 3 is finished.

Step 4 : We just have to change the "a.s.". First,

$$\text{a.s.}\,, \quad \forall\, s < t \quad \text{with} \quad s,t \in [0,T] \cap \mathbb{Q}\,, \quad \bar{R}_t - \bar{R}_s \quad \text{is} \quad \text{invertible}$$

Since $\bar{R}_t$ is increasing, it is easy to drop the "$\cap \mathbb{Q}$", and the theorem follows.

Proof of Theorem 3.1 : it is immediate, thanks to Theorems 3.19 and 3.17, Proposition 3.13, Theorem 3.5, and Remarks 2.5, and 2.2.

3.5 Smoothness of the weak solution.

We now have to study the inverse moments of $\det DV_t$. We use the notations of the previous subsection. Recall that $DV_t = M_t H_t$, where M_t is the Doléans-Dade martingale given in Proposition 3.15, and where

$$H_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}^{-1}\Gamma(V_{s-} - W_{s-}(\alpha),\theta)(M_{s-}^{-1})^T \zeta(V_{s-} - W_{s-}(\alpha), M_{s-},\theta)\delta(\theta) N_0(d\theta d\alpha ds)$$

where, for $x \in I\!R^2$ and $y \in \mathcal{M}_2(I\!R)$,

$$\Gamma(x,\theta) = (I + A(\theta))^{-1} \times (A'(\theta)x) \times (A'(\theta)x)^T \times ((I + A(\theta))^{-1})^T$$

and

$$\zeta(x,y,\theta) = h(A'(\theta)x) \times k(I + A(\theta)) \times k(y)$$

where h and k are defined in Subsection 3.3.

We first study the **inverse moments of** M_t.

Theorem 3.20 *Assume (H) and (S). For every $t \geq 0$, $(\det M_t)^{-1}$ admits moments of all orders.*

Proof : we notice that under (S)-2,

$$\begin{aligned} M_t^{-1} &= I + \tfrac{b}{2}\int_0^t M_s^{-1} ds - \int_0^t\int_0^1\int_{-\pi}^{\pi} M_{s-}^{-1}(I + A(\theta))^{-1}A(\theta)\tilde{N}(d\theta d\alpha ds) \\ &+ \int_0^t\int_0^1\int_{-\pi}^{\pi} M_{s-}^{-1}A(\theta)(I + A(\theta))^{-1}A(\theta)\beta(\theta)d\theta d\alpha ds \end{aligned} \tag{3.15}$$

In order to check this equality, it suffices to apply the Itô formula to the product $M_t.M_t^{-1}$, (where M_t^{-1} is defined by 3.15) : one obtains that $M_t.M_t^{-1}$ is a solution of a classical S.D.E. of which I is also a solution. Then a simple computation shows that :

$$(I + A(\theta))^{-1}A(\theta) = \frac{\sin\theta}{\cos\theta + 1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$A(\theta)(I + A(\theta))^{-1}A(\theta) = \frac{1}{2}\frac{\sin\theta}{\cos\theta + 1}\begin{pmatrix} -\sin\theta & 1 - \cos\theta \\ \cos\theta - 1 & -\sin\theta \end{pmatrix}$$

Thanks to Assumption (S)-2, and since $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$, one can check that

$$\frac{|\sin\theta|}{1+\cos\theta} \in \cap_{p\geq 2} L^p(\beta(\theta)d\theta) \quad ; \qquad \frac{\sin^2\theta + |\sin\theta(1-\cos\theta)|}{1+\cos\theta} \in \cap_{p\geq 1} L^p(\beta(\theta)d\theta)$$

Hence it is clear that M_t^{-1} (and thus its determinant) is well defined and admits moments of all orders (this S.D.E. is classical, and the initial data I is deterministic).

It is more difficult to prove that H_t **admits moments of all orders**. In fact, we will only study **the case where** $E(V_0) = 0$ by using the Malliavin Calculus. The generalization (see the final proof of this section) will then follow from the uniqueness in law for (SB). We begin with three lemmas.

Lemma 3.21 *The map* $(t, Y) \longrightarrow \mathcal{L}(\langle V_t, Y\rangle)$ *is weakly continuous on* $[0, T] \times \{Y \in I\!R^2 \mid \parallel Y \parallel = 1\}$.

Proof : it suffices to show that for every $\phi \in C_b^2(I\!R)$, the map $(t, Y) \longrightarrow E(\phi(\langle V_t, Y\rangle))$ is continuous, which can be checked by using the fact that the flow $\mathcal{L}(V_t)$ is a solution of (B) in the sense of Definition 2.1.

Lemma 3.22 *Assume* (H), (S), *and* $E(V_0) = 0$. *Let* $t_0 > 0$ *be fixed. There exists* $\eta > 0$, $q > 0$, *and* $\xi > 0$ *(depending on* t_0*) such that for every* $t \in [t_0, T]$, *for every* $X \in I\!R^2$, *for every unit vector* $Y \in I\!R^2$,

$$P_\alpha\left(\langle W_t - X, Y\rangle^2 > \eta \ , \ \parallel W_t \parallel^2 < \xi\right) > q \tag{3.16}$$

Proof : since $\sup_{[0,T]} \parallel W_t \parallel$ is in $\cap_p L^p$, it suffices to show that there exists $\eta > 0$, $q > 0$ such that for every $t \in [t_0, T]$, for every $X \in I\!R^2$, for every $Y \in I\!R^2$ such that $\parallel Y \parallel = 1$,

$$P_\alpha\left(\langle W_t - X, Y\rangle^2 > \eta\right) > 2q$$

In order to check this claim, notice (by using Bienaymé Tchebichev's inequality) that there exists $\xi > 0$ such that for every t, $P_\alpha(\parallel W_t \parallel^2 \leq \xi) > 1 - q$. We now break the proof in several steps :
Step 1 : Let $t \geq t_0$ and $\parallel Y \parallel = 1$ be fixed. Thanks to the previous section, the law of W_t admits a density on $I\!R^2$, and hence the law of $\langle W_t, Y\rangle$ admits a density with respect to the Lebesgue measure on $I\!R$. By Proposition 2.10 and since $E(V_0) = 0$, we also know that $E_\alpha(W_t) = E_\alpha(W_0) = 0$, and hence $E_\alpha(\langle W_t, Y\rangle) = 0$. It is then easy to show that there exists $\eta(t, Y) > 0$ and $q(t, Y) > 0$ such that

$$P_\alpha\left(\langle W_t, Y\rangle > \sqrt{\eta(t, Y)}\right) > 2q(t, Y) \quad \text{and} \quad P_\alpha\left(\langle W_t, Y\rangle < -\sqrt{\eta(t, Y)}\right) > 2q(t, Y)$$

Step 2 : Using Lemma 3.21, Portemanteau's Theorem, and the step 1, it is classical to show that for every t in $[t_0, T]$, for every $\parallel Y \parallel = 1$, there exists a neighbourhood $\mathcal{V}(t, Y)$ of (t, Y) such that for every $(t', Y') \in \mathcal{V}(t, Y)$,

$$P_\alpha\left(\langle W_{t'}, Y'\rangle > \sqrt{\eta(t, Y)}\right) > 2q(t, Y)$$

Let us consider a finite covering $\cup_{i=1}^N \mathcal{V}(t_i, Y_i)$ of the compact set $[t_0, T] \times \{Y \in I\!R^2 \big/ \parallel Y \parallel = 1\}$. Then, if $\eta = \inf_{i\leq N} \eta(t_i, Y_i)$ and if $q = \inf_{i\leq N} q(t_i, Y_i)$, then for all $t \geq t_0$ and $\parallel Y \parallel = 1$,

$$P_\alpha(\langle W_t, Y\rangle > \sqrt{\eta}) > 2q$$

In the same way, we get $P_\alpha(\langle W_t, Y\rangle < -\sqrt{\eta}) > 2q$ for all $t \geq t_0$ and $\parallel Y \parallel = 1$.
Step 3 : Finally, let X be in $I\!R^2$, $t \geq t_0$, and $\parallel Y \parallel = 1$ be fixed. If $\langle X, Y\rangle \leq 0$,

$$P_\alpha(\langle W_t - X, Y\rangle^2 > \eta) \geq P_\alpha(\langle W_t - X, Y\rangle > \sqrt{\eta}) \geq P_\alpha(\langle W_t, Y\rangle > \sqrt{\eta} + \langle X, Y\rangle) \geq P_\alpha(\langle W_t, Y\rangle > \sqrt{\eta}) > 2q$$

If $\langle X, Y\rangle \geq 0$, the same kind of argument does work, and the proof is finished.

Lemma 3.23 *Assume (H), (S), and $E(V_0)=0$. Let $t_0>0$ be fixed, and let η, q, and ξ be the strictly positive numbers associated with t_0 introduced in the previous lemma. If $X\in I\!R^2$, $\| Y \|=1$, and $s\geq t_0$, we consider the set :*

$$\mathcal{H}_s(X,Y)=\Big\{(\theta,\alpha)\in[-\theta_0,\theta_0]\times[0,1]\Big/\| W_s(\alpha)\|^2\leq\xi \quad and \quad Y^T\Gamma(X-W_s(\alpha),\theta)Y\geq\eta\Big\} \tag{3.17}$$

Then for every even positive function z on $[-\theta_0,\theta_0]$,

$$\iint_{\mathcal{H}_s(X,Y)} z(\theta)\beta_0(\theta)d\theta d\alpha\geq q\int_0^{\theta_0} z(\theta)\beta_0(\theta)d\theta \tag{3.18}$$

Proof : let $X\in I\!R^2$, let $\| Y \|=1$, and let $s\geq t_0$ be fixed. Recall (see equation (3.14) in the proof of Theorem 3.19) that :

$$Y^T\Gamma(X-W_s(\alpha),\theta)Y=\langle f(\theta)Y+PY,X-W_s(\alpha)\rangle^2$$

where $P=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $f(\theta)=\dfrac{\sin\theta}{\cos\theta+1}$ is an increasing bijection from $]-\pi,\pi[$ to $I\!R$ satisfying $f(0)=0$. We denote

$$h_s(X,PY)=\Big\{\alpha\in[0,1]\Big/\langle W_s(\alpha)-X,PY\rangle^2>\eta\ ,\ \| W_s(\alpha)\|^2<\xi\Big\}$$

Thanks to Lemma 3.22, we know that $P_\alpha(h_s(X,PY))>q$. We will show that if $\alpha\in h_s(X,PY)$, then $Y^T\Gamma(X-W_s(\alpha),\theta)Y\geq\eta$ either for all $\theta\in]0,\pi[$ or for all $\theta\in]-\pi,0[$ (and the lemma will be proved). Let $\alpha\in h_s(X,PY)$. If $\langle Y,X-W_s(\alpha)\rangle=0$, then

$$Y^T\Gamma(X-W_s(\alpha),\theta)Y=\langle PY,X-W_s(\alpha)\rangle^2>\eta$$

for every θ. Else, $Y^T\Gamma(X-W_s(\alpha),\theta)Y\geq\eta$ for every θ such that $f(\theta)\in I\!R\backslash[x_1,x_2]$, where $x_1\leq x_2$ are the solutions of

$$x^2\times\langle Y,X-W_s(\alpha)\rangle^2+2x\times\langle Y,X-W_s(\alpha)\rangle\langle PY,X-W_s(\alpha)\rangle+\langle PY,X-W_s(\alpha)\rangle^2-\eta=0$$

Hence, it suffices to show that the signs of x_1 and x_2 are equal. But

$$x_1,x_2=\frac{-\langle PY,X-W_s(\alpha)\rangle\pm\sqrt{\eta}}{\langle Y,X-W_s(\alpha)\rangle}$$

Since $\langle PY,X-W_s(\alpha)\rangle^2\geq\eta$, the lemma follows.

Theorem 3.24 *Assume (H), (S), and $E(V_0)=0$. For every $t>0$, $(\det H_t)^{-1}$ admits moments of all orders (and thus so does $(\det DV_t)^{-1}$).*

Proof : we fix $t_0>0$, and we prove the theorem for every $t>t_0$, which of course suffices. Since $\theta_0<\pi$, there exists $d_0>0$ such that, for every $|\theta|\leq\theta_0$, $|\det(I+A(\theta))|=\frac{1}{2}(1+\cos\theta)\geq d_0$. We choose k such that $k(y)=1$ as soon as $|\det y|\geq d_0$.
For every X in $I\!R^2$, one has $\| A'(\theta)X\|^2=\frac{1}{4}\| X\|^2$. Hence, if α is in any set $\mathcal{H}_s(X,Y)$, then

$$h(A'(\theta)(V_s-W_s(\alpha)))\geq\left(1+\frac{1}{4}(\| V_s\|^2+\xi)\right)^{-1}$$

Hence, for every $\| Y \|>0$, a simple computation (using the Lemma 3.23) shows that for every $t\geq t_0$, Y^TH_tY is greater or equal than

$$\int_{t_0}^t\iint_{\mathcal{H}_s\left(V_{s-},\frac{M_{s-}^{-1T}Y}{\| M_{s-}^{-1T}Y\|}\right)}\| M_{s-}^{-1T}Y\|^2\times\eta\times\left(1+\frac{1}{4}(\| V_{s-}\|^2+\xi)\right)^{-1}\times k(M_{s-})\times\delta(\theta)N_0(d\theta d\alpha ds)$$

Let us notice that the function on $\Omega \times [0,T] \times [-\pi,\pi] \times [0,1]$ defined by

$$\omega, s, \theta, \alpha \longrightarrow 1\!\!1_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1\,T} Y}{\|M_{s-}^{-1\,T} Y\|}\right)}(\theta,\alpha) = 1\!\!1_{\left\{|\theta|\le\theta_0,\ \|W_{s-}(\alpha)\|^2\le\xi,\ Y^T \frac{M_{s-}^{-1}}{\|M_{s-}^{-1\,T} Y\|}\Gamma(V_{s-}(\omega)-W_{s-}(\alpha),\theta)\frac{M_{s-}^{-1\,T}}{\|M_{s-}^{-1\,T} Y\|}Y\ge\eta\right\}}$$

is predictable, because V_{s-} and M_{s-}^{-1} are predictable, and because W is a measurable α-process.
Let us define the following random variable :

$$F = \sup_{[0,T]} \left\{ \left(1 + \frac{1}{4}(\| V_s \|^2 + \xi)\right) \times \left(k(M_{s-}) \| M_{s-}^{-1\,T} \|_{op}^2\right)^{-1} \right\}$$

where $\| M_{s-}^{-1\,T} \|_{op}$ is the operator norm of $M_{s-}^{-1\,T}$. Thus, for every $\| Y \| = 1$, $t \ge t_0$,

$$F \times Y^T H_t Y \ge \eta \int_{t_0}^t \iint_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1\,T} Y}{\|M_{s-}^{-1\,T} Y\|}\right)} \delta(\theta) N_0(d\theta d\alpha ds)$$

In order to use the Appendix (4.1), we have to compute $E\left(e^{-\zeta F \times Y^T H_t Y}\right)$ for $\zeta > 0$, $t \ge t_0$. To this end, we set

$$n_\zeta(s) = \frac{q \int_0^{\theta_0} \left(1 - e^{-\zeta\delta(\theta)}\right) \beta_0(\theta) d\theta}{\iint_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1\,T} Y}{\|M_{s-}^{-1\,T} Y\|}\right)} \left(1 - e^{-\zeta\delta(\theta)}\right) \beta_0(\theta) d\theta d\alpha}$$

Choosing δ even, and using Lemma 3.23, we see that $n_\zeta(s) \in]0,1[$ a.s. for every $s \ge t_0$, $\zeta > 0$. Furthermore for every $\zeta > 0$, the following function on $\Omega \times [t_0,T] \times [-\pi,\pi] \times [0,1]$ is predictable and takes its values in $[0,1]$:

$$g_\zeta(s,\theta,\alpha) = -\frac{1}{\zeta\delta(\theta)} \ln\left[1 - n_\zeta(s)\left(1 - e^{-\zeta\delta(\theta)}\right)\right] 1\!\!1_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1\,T} Y}{\|M_{s-}^{-1\,T} Y\|}\right)}(\theta,\alpha)$$

Hence, for every $\| Y \| = 1$, $t \ge t_0$, $\zeta > 0$,

$$F \times Y^T H_t Y \ge \eta \int_{t_0}^t \int_0^1 \int_{-\pi}^{\pi} g_\zeta(s,\theta,\alpha)\delta(\theta) N_0(d\theta d\alpha ds) = \eta Z_t(\zeta)$$

Using Itô's formula,

$$\begin{aligned} e^{-\zeta Z_t(\zeta)} &= 1 - \zeta \int_0^t e^{-\zeta Z_{s-}(\zeta)} dZ_s(\zeta) + \sum_{s\le t}\left[e^{-\zeta Z_s(\zeta)} - e^{-\zeta Z_{s-}(\zeta)} + \zeta e^{-\zeta Z_{s-}(\zeta)}\Delta Z_s(\zeta)\right] \\ &= 1 - \int_{t_0}^t \int_0^1 \int_{-\pi}^{\pi} e^{-\zeta Z_{s-}(\zeta)} \left(1 - e^{-\zeta g_\zeta(s,\theta,\alpha)\delta(\theta)}\right) N_0(d\theta d\alpha ds) \end{aligned}$$

Taking the expectations, and using the expression of g_ζ, we obtain for every $t \ge t_0$, $\zeta > 0$,

$$\begin{aligned} E(e^{-\zeta Z_t(\zeta)}) &= 1 - E\left(\int_{t_0}^t \int_0^1 \int_{-\pi}^{\pi} e^{-\zeta Z_{s-}(\zeta)} \left(1 - e^{-\zeta g_\zeta(s,\theta,\alpha)\delta(\theta)}\right) \beta_0(\theta) d\theta d\alpha ds\right) \\ &= 1 - q \int_0^{\theta_0} \left(1 - e^{-\zeta\delta(\theta)}\right) \beta_0(\theta) d\theta \times \int_{t_0}^t E(e^{-\zeta Z_s(\zeta)}) ds \end{aligned}$$

Thanks to the Appendix (4.2),

$$E(e^{-\zeta Z_t(\zeta)}) = \exp\left(-q(t-t_0) \int_0^{\theta_0} \left(1 - e^{-\zeta\delta(\theta)}\right) \beta_0(\theta) d\theta\right)$$

and for every $\zeta > 0$, $t \geq t_0$, $\| Y \| = 1$,

$$E\left(\exp\left(-\zeta F \times Y^T H_t Y\right)\right) \leq E\left(e^{-\eta\zeta Z_t(\eta\zeta)}\right) \leq \exp\left(-q(t-t_0)\int_0^{\theta_0}\left(1-e^{-\eta\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta\right)$$

Recall that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r}\mathbb{I}_{|\theta|\leq\theta_0}$. We choose $\delta(\theta) \geq \frac{1}{\eta}e^{-|\theta|^{-2r'}}$ for small θ (with δ even and satisfying (3.13)). Thanks to the Appendix (4.3), there exists $C > 0$ and $\zeta_0 \geq 0$ such that for every $\zeta \geq \zeta_0$,

$$\int_0^{\theta_0}(1-e^{-\eta\zeta\delta(\theta)})\beta_0(\theta)d\theta \geq C(\ln\zeta)^3$$

Thus for every $\zeta \geq \zeta_0$, $t \geq t_0$, and $\| Y \| = 1$,

$$E\left(\exp\left(-\zeta F Y^T H_t Y\right)\right) \leq \exp\left(-Cq(t-t_0)(\ln\zeta)^3\right)$$

Hence, for every $p \geq 0$, for all $t > t_0$,

$$E\left(\int_{X\in R^2} \| X \|^p \exp\left(-X^T F H_t X\right) dX\right) = \int_{\rho=0}^{\infty}\int_{\|Y\|=1}\rho^p E\left(e^{-\rho^2 F Y^T H_t Y}\right) dY d\rho$$

$$\leq K\int_{\rho=0}^{\sqrt{\zeta_0}}\rho^p d\rho + K\int_{\rho=\sqrt{\zeta_0}}^{\infty}\rho^p \exp\left(-Cq(t-t_0)(\ln\rho^2)^3\right) d\rho < \infty$$

Thanks to the Appendix (4.1), this yields that for every $t > t_0$, $(\det F H_t)^{-1} = (F^2 \det H_t)^{-1}$ is in every L^p. But it is possible to choose k such that F has moments of all orders : $F \leq F_1 \times F_2$, where

$$F_1 = \sup_{[0,T]}\left(1 + \frac{1}{4} \| V_s \|^2 + \frac{\xi}{4}\right) \qquad \text{and} \qquad F_2 = \sup_{[0,T]}\left(k(M_s) \| M_s^{-1^T} \|_{op}^2\right)^{-1}$$

We have already seen that F_1 has moments of all orders. In order to study F_2, let us first recall some norm inequalities for a symmetric positive matrix O :

$$|\det O|^2 \leq \| O \|^4 \leq 1 + \| O \|^8 \qquad |\det O| \times \| O^{-1} \|_{op} = \| O \|_{op} \geq \| O^{-1} \|^{-1}$$

We can choose k such that for every y,

$$k(y) \geq \frac{|\det y|^2}{1 + \| y \|^8}$$

(We still assume that $k(y) = 1$ if $\det y \geq d_0$). Hence,

$$F_2 \leq \sup_{[0,T]}\left(1 + \| M_s \|^8\right) \times \sup_{[0,T]} \| M_s^{-1} \|^2$$

Since M_s and M_s^{-1} are solutions of stochastic differential equations (with initial datum I), it is classical to show that they have moments of all orders, and we can say that F has moments of all orders. Thus :

$$E\left(|\det H_t|^{-p}\right) = E\left(|F|^{2p} \times |\det F H_t|^{-p}\right) \leq E\left(|F|^{4p}\right)^{\frac{1}{2}} E\left(|\det F H_t|^{-2p}\right)^{\frac{1}{2}} < \infty$$

We have proved that for $t > t_0$, $\det H_t$ admits some inverse moments of all orders, and the theorem follows.

Proof of Theorem 3.2 : using Theorem 3.24, Proposition 3.20, Theorem 3.17, Proposition 3.13, Theorem 3.6, the theorem is immediate when $E(V_0) = 0$.
We suppose now that V_0 is not centered. We denote by (V, W) (resp. (V', W')) a solution of the S.D.E. SB) with initial data V_0 (resp. $V_0' = V_0 - E(V_0)$). Since V_0 satisfies (H) and (S), so does V_0'. We thus know that for every $t > 0$, the law of V_t' admits a C^∞ density $f'(t,.)$ on $\mathbb{R}^2$, and that V_t admits a density $f(t,.)$ on $\mathbb{R}^2$. On the other hand, one can check that $(V - E(V_0), W - E(V_0))$ is a solution of (SB) with initial data V_0'. Hence, by Theorem 2.9, $\mathcal{L}(V_t - E(V_0)) = \mathcal{L}(V_t')$. This yields that $f(t,v) = f'(t, v - E(V_0))$, and the theorem follows.

3.6 Joint regularity.

We are now interested in the joint regularity of the weak solution f of (B) built in Theorem 3.1. By Theorems 3.1 and 3.2, and since (H) and (S) hold, we know that for every $t > 0$, the law of V_t admits a C^∞ density $f(t,.)$ with respect to the Lebesgue measure on $I\!R^2$.

In the case of a classical diffusion process X_t, Bichteler, Gravereaux and Jacod give in [1] a method to study the joint smoothness of $f(t,x)$, where $f(t,x)$ is the density of the law of X_t. Their method is based on the Malliavin Calculus, and on the smoothness of the maps $t \longrightarrow E(\psi(X_t))$ for any ψ sufficiently regular. In our case, these maps are only differentiable, because our S.D.E. is not time-homogeneous, and we thus cannot apply their method.

The method we use here is based on the weak continuity of $t \to \mathcal{L}(V_t)$ and on Theorem 3.2. As in the proof of Theorem 3.2, **we assume that** $E(V_0) = 0$, the generalization beeing immediate by the uniqueness in law for the S.D.E. (SB) (see Theorem 2.9). We also fix $t_0 > 0$, and we prove Theorem 3.3 on $[t_0, T] \times I\!R^2$, which of course suffices. We begin with a lemma.

Lemma 3.25 *Assume (H), (S), and $E(V_0) = 0$. For every multi-index α, there exists a constant C_{α,t_0} such that for every $g \in C_b^\infty(I\!R^2)$, for every $t \in [t_0, T]$,*

$$E\left(\partial_\alpha g(V_t)\right) \le C_{\alpha,t_0} \parallel g \parallel_\infty \tag{3.19}$$

Proof : we just have to study the proof of Theorem 3.6 (which can be found in [1]). Let ϕ be a random variable with values in $I\!R^2$ satisfying the assumptions of Theorem 3.6, with the same notations. Then Bichteler et al. prove that for every multi-index α, there exists a constant K_α such that for every $g \in C_b^\infty(I\!R^2)$, $E\left(\partial_\alpha g(\phi)\right) \le K_\alpha \parallel g \parallel_\infty$. Following closely their proof, one can check that the constants K_α depends only on the moments of the elements of C_n $(n \in I\!N)$, and on the inverse moments of $\det \sigma$.
Let us come back to our problem : here we have a family $\phi_t = V_t$ of random variables satisfying the conditions of Theorem 3.6, with $\sigma_t = DV_t$. The sets C_n^t are composed with the derivatives of all orders of V and G. Then one can check that for any n, for every $X_t \in C_n^t$, for all $p \ge 1$, $\sup_{[0,T]} E\left(|X_t|^p\right) < \infty$. Furthermore, following closely the proof of Theorems 3.24 and 3.20, one can see that for every p, $\sup_{[t_0,T]} E\left(|\det DV_t|^{-p}\right) < \infty$, and the lemma follows.

We now prove that our weak solution f is equicontinuous :

Proposition 3.26 *For every v in $I\!R^2$,*

$$\sup_{s\in[t_0,T]} |f(s, v + k) - f(s, v)| \longrightarrow_{\|k\|\to 0} 0 \tag{3.20}$$

Proof : following Nualart [8] Lemma 2.1.5 p 88-89, and using Lemma 3.25, one can show that if $\mathcal{L}(V_t) = P_t$, and if $\hat{P}_t$ is the Fourier transform of P_t, then for every $t \in [t_0, T]$, $\left|\hat{P}_t(v)\right| \le \dfrac{C_{(2,2),t_0}}{v_x^2 v_y^2} \wedge 1$ (it suffices to apply Lemma 3.25 with $\alpha = (2,2)$ and with $g(y) = e^{i\langle v,y\rangle}$). Furthermore, f is the following inverse Fourier transform :

$$f(t,v) = \left(\frac{1}{2\pi}\right)^2 \int_{R^2} e^{-i\langle y,v\rangle} \hat{P}_t(y) dy \tag{3.21}$$

Using Lebesgue's theorem and the uniform upperbound of $\hat{P}_t$, the proposition is immediate.

The proof of Theorem 3.3 is a simple application of Proposition 3.26 and of the weak continuity of the map $t \longrightarrow f(t,v)dv$.

4 Appendix.

We begin this annex with a lemma that can be found in [1], p 92 :

Lemma 4.1 *For every $p > 0$, there exists a constant C_p such that for every 2×2 symmetric positive matrix A,*

$$(det\ A)^{-p} \leq C_p \int_{X \in R^2} \| X \|^{4p-2} e^{-X^T AX} dX$$

The following lemma is well-known, and can be shown as Gronwall's Lemma.

Lemma 4.2 *Let $0 \leq \epsilon < T < \infty$. Let g be a bounded function on $[\epsilon, T]$, and let a be a real number. Assume that for every $t \in [\epsilon, T]$,*

$$g(t) = 1 - a \int_{\epsilon}^{t} g(s) ds$$

Then $g(t) = e^{-a(t-\epsilon)}$ on $[\epsilon, T]$.

The next lemma is a simple computation :

Lemma 4.3 *Let $r \in]1, 3[$, let $r'' = \frac{1}{4}(r-1)$, and let $\epsilon > 0$. We set $\delta(\theta) = e^{-\theta^{-r''}}$. There exists a constant $C > 0$, a real number $\zeta_0 \geq 0$, such that for every $\zeta \geq \zeta_0$,*

$$\int_0^{\epsilon} \left(1 - e^{-\zeta\delta(\theta)}\right) \frac{d\theta}{\theta^r} \geq C(\ln \zeta)^3$$

<u>Proof</u> : we first notice that for every $x \in [0, 1]$, one has $1 - e^{-x} \geq \frac{x}{2}$. Furthermore, for every $\theta < 1$, $\delta^{-1}(\theta) = (\ln \theta^{-1})^{-\frac{1}{r''}}$. Hence, if ζ_0 is large enough (we need $\zeta_0^{-1} < 1$ and $\delta^{-1}(\zeta_0^{-1}) < \epsilon$), then for all $\zeta \geq \zeta_0$,

$$I(\zeta) = \int_0^{\epsilon} \left(1 - e^{-\zeta\delta(\theta)}\right) \frac{d\theta}{\theta^r} \geq \frac{\zeta}{2} \int_0^{\delta^{-1}(\zeta^{-1})} \frac{\delta(\theta)}{\theta^r} d\theta \geq \frac{\zeta}{2r''} \int_0^{\delta^{-1}(\zeta^{-1})} \frac{r''}{\theta^{r''+1}} \delta(\theta) \times \theta^{r''+1-r} d\theta$$

Since $r - r'' - 1 = \frac{3}{4}(r-1) > 0$, and since $\delta'(\theta) = \frac{r''}{\theta^{r''+1}} \delta(\theta)$, we obtain :

$$I(\zeta) \geq \frac{\zeta}{2r''} \times \left(\delta^{-1}(\zeta^{-1})\right)^{-\frac{3}{4}(r-1)} \times [\delta(\theta)]_0^{\delta^{-1}(\zeta^{-1})} = \frac{1}{2r''} (\ln \zeta)^3$$

which was our aim.

The following lemma is adapted from a lemma in the Appendix of [2]. We state it for N and β, but it can be obviously adapted to N_0 and β_0 or N_1 and β_1.

Lemma 4.4 *Let $Y(s, \alpha, \theta)$ be a predictable process such that $|Y(s, \alpha, \theta)| \leq |X(s, \alpha)| z(\theta)$. Then*

- *if z is in $\cap_{p \geq 2} L^p(\beta(\theta) d\theta)$, for every $p = 2^q$,*

$$E\left(\sup_{[0,t]} \left| \int_0^s \int_0^1 \int_{-\pi}^{\pi} Y(u, \alpha, \theta) \tilde{N}(d\theta d\alpha du) \right|^p \right) \leq C_p(z) \int_0^t \int_0^1 E\left(|X(s, \alpha)|^p\right) d\alpha ds$$

- *if z is in $L^1(\beta(\theta) d\theta)$, then for every $p < \infty$,*

$$E\left(\sup_{[0,t]} \left| \int_0^s \int_0^1 \int_{-\pi}^{\pi} Y(u, \alpha, \theta) d\theta d\alpha du \right|^p \right) \leq C_p(z) \int_0^t \int_0^1 E\left(|X(s, \alpha)|^p\right) d\alpha ds$$

- *if z is in $\cap_{p \geq 1} L^p(\beta(\theta) d\theta)$, for every $p = 2^q$,*

$$E\left(\sup_{[0,t]} \left| \int_0^s \int_0^1 \int_{-\pi}^{\pi} Y(u, \alpha, \theta) N(d\theta d\alpha du) \right|^p \right) \leq C_p(z) \int_0^t \int_0^1 E\left(|X(s, \alpha)|^p\right) d\alpha ds$$

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Nicolas FOURNIER,
Laboratoire de Probabilités, UMR 7599,
Université Paris VI,
4, Place Jussieu, Tour 56, 3^o étage,
F-75252 Paris Cédex 05, FRANCE
fournier@proba.jussieu.fr.