Support theorem for the solution of a white noise driven parabolic S.P.D.E. with temporal Poissonian jumps.

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Abstract

We study the weak solution X of a parabolic stochastic partial differential equation driven by two independent processes : a gaussian white noise, and a finite Poisson measure. We characterize the support of the law of X as the closure in $\mathbb{D}\left([0,T],\mathbf{C}([0,1])\right)$, endowed with its Skorokhod topology, of a set of weak solutions of ordinary partial differential equations.

Key words : Parabolic stochastic partial differential equations, Support theorem, Poisson measure, White noise.

MSC 91 : 60H15, 60F99, 60G55, 60G57.

1 Introduction.

Consider on $[0,T]\times[0,1]$ a space-time white noise $W(dx,dt)$ based on $dxdt$ (see Walsh, [13], p 269). Denote by (E,d) a Polish space, endowed with a positive finite measure q, and by $N(dt,dz)$ a Poisson measure on $[0,T]\times E$, with intensity measure $dtq(dz)$, independent of W. Our purpose is to study the following stochastic partial differential equation on $[0,T]\times[0,1]$

$$\frac{\partial X}{\partial t}(t,x)=\frac{\partial^2 X}{\partial x^2}(t,x)+b(X(t,x))+\sigma(X(t,x))\dot{W}_{x,t}+\int_E g(X(t-,x),z)\dot{N}_t(dz) \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial X}{\partial x}(t,0)=\frac{\partial X}{\partial x}(t,1)=0 \quad , \quad \forall\, t>0 \tag{1.2}$$

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and deterministic initial condition $\mathcal{X}_0(x) \in \mathbf{C}([0,1])$. The symbol $\dot{N}_t(dz)$ (resp. $\dot{W}_{t,x}$) stands for the heuristical Radon-Nikodym density of $N(dt,dz)$ (resp. $W(dx,dt)$) with respect to the Lebesgue measure dt (resp. $dtdx$). We could also write, with abusive notations, $\dot{N}_t(dz)dtdx = dxN(dt,dz)$ and $\dot{W}_{x,t}dtdx = W(dx,dt)$.

We denote by $I\!D\left([0,T],\mathbf{C}([0,1])\right)$ the set of càdlàg functions from $[0,T]$ into $\mathbf{C}([0,1])$, endowed with the corresponding Skorokhod topology. In this paper, we characterize the support of the law of a weak solution X of equation (1.1) as the closure of a set of weak solutions of ordinary partial differential equations in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$.

Parabolic S.P.D.E.s driven by a white noise, i.e. equation (1.1) with $g \equiv 0$, have been introduced by Walsh, [12] and [13]. In [13], he defines his weak solutions, then he proves a theorem of existence, uniqueness and regularity. Since, various properties of Walsh's equation have been investigated : Malliavin calculus, large deviations, support theorem (see Bally, Millet, Sanz-Solé, [2]) ...
But Walsh builds his equation in order to model a discontinuous neurophysiological phenomenon. In [12], he explains that the white noise W approximates a Poisson point process. This approximation is realistic because there are many jumps, and the jumps are very small, but in any case, the observed phenomenon is discontinuous. However, S.P.D.E.s with jumps are much less known. In the case of temporal and spatial jumps, Saint Loubert Bié have studied in [9] the existence, uniqueness, regularity, and variational calculus. See also [5] for other results on the same subject. Nevertheless, no result about the "joint" regularity of the weak solutions has been proved in this case : we do not really know in which space the weak solution "lives", thus no support theorem may hold for the moment.

In the case of equation (1.1) with $\sigma \equiv 0$, but with $q(E) = \infty$, and with a compensated Poisson measure, Albeverio et al. have checked in [1] the existence and uniqueness of a "modified càdlàg" weak solution $u(t,x)$: a.s., u is continuous in x ; and u is right continuous and has left limits in $L^2(\Omega)$ in the variable t. One more time, we do not know in which space lies a.s. the weak solution.

Since Stroock and Varadhan established in [11] their famous support theorem for diffusion processes, their has been many investigations on the subject. In particular, Millet and Sanz have considerably simplified in [8] the proof of Stroock and Varadhan. But the only support theorem for jump processes seems to be that of Simon in [10], who studies a stochastic differential equation driven by a (compensated or not) infinite Poisson measure. At last, let us mention that as far as we know, no support theorem seems to be known in the case of equations driven by two independent (but different) random elements.

This work is organized as follows. In the second section, we define the weak solutions of (1.1), by following the Walsh ideas, [13]. Using Ikeda and Watanabe's method, see [6], and applying Walsh's results, we sketch the proof of an existence and uniqueness result. We define the "skeleton" associated with equation (1.1), by using the Cameron-Martin space associated with W and the set of finite counting measures associated with N. Finally, we state our support theorem.
The third section is devoted to a simplification of the problem. First, we use a localization

argument, in order to obtain weaker assumptions. Then we prove that it suffices to check two simpler support theorems. The first one is proved in the fourth section, and is related to an equation similar to (1.1), but without white noise : $\dot{W}_{t,x}$ is repaced by $\dot{h}(t,x)$, where h is an element of the Cameron-Martin space associated with W. The second one is proved in the fifth section, and deals with an equation without Poisson measure, but with an additional "jump drift". This concludes the proof of our main result.
The sixth section is devoted to an extension of our result to the case where the Poisson measure is a.s. infinite ($q(E) = \infty$), but where the diffusion coefficient is constant ($\sigma(x) = \sigma$).
Finally, one can find technical results in the Appendix lying at the end of the paper.

2 Framework.

Let us first define the weak solutions of (1.1). To this aim, we need some assumptions :

> Assumption (H) : the functions σ and $b : I\!\!R \mapsto I\!\!R$, satisfy a global Lipschitz condition. The function $g : I\!\!R \times E \mapsto I\!\!R$ is measurable on $I\!\!R \times E$, and for each $z \in E$, the map $g(.,z)$ is continuous on $I\!\!R$.

We define the weak solutions of (1.1) by following the Walsh ideas, [13], p 311-322. Consider the Green kernel $G_t(x,y)$ associated with the deterministic system :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ; \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0 \tag{2.1}$$

This kernel can be explicitely computed :

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in Z} \left[\exp\left(\frac{-(y-x-2nL)^2}{4t} \right) + \exp\left(\frac{-(y+x-2nL)^2}{4t} \right) \right] \tag{2.2}$$

If ϕ belongs to $\mathbf{C}([0,1])$, we set

$$G_t(\phi,x) = \begin{cases} \phi(x) & \text{if} \quad t=0 \\ \int_0^1 G_t(x,y)\phi(y)dy & \text{if} \quad t>0 \end{cases} \tag{2.3}$$

The Appendix of this work contains technical results about this kernel. We endow our probability space $(\Omega, \mathcal{F}, P)$ with the canonical filtration associated with the independent random elements W and N :

$$\mathcal{F}_t = \sigma\left\{ W(A) \; ; \; A \in \mathcal{B}([0,1] \times [0,t]) \right\} \vee \sigma\left\{ N(B) \; ; \; B \in \mathcal{B}([0,t] \times E) \right\} \tag{2.4}$$

A process $X(t,x)$ on $[0,T] \times [0,1]$ is said to be adapted if for all $t \geq 0$, all $x \in [0,1]$, $X(t,x)$ is $\mathcal{F}_t$-measurable.

As Walsh, see also Saint Loubert Bié, [9], or Fournier, [5], we define the weak solutions of (1.1) in the following sense.

Definition 2.1 *Let $\mathcal{X}_0 : [0,1] \mapsto I\!R$ be a continuous deterministic function. Consider an adapted process $X(t,x)$ on $[0,T] \times [0,1]$, lying a.s. in $I\!D([0,T], \mathbf{C}([0,1]))$. Then X is said to be a weak solution of (1.1) if and only if it satisfies the following evolution equation*

$$\begin{aligned} X(t,x) &= G_t(\mathcal{X}_0, x) + \int_0^t \int_0^1 G_{t-s}(x,y) \left[b(X(s,y)) dy ds + \sigma(X(s,y)) W(dy, ds) \right] \\ &\quad + \int_0^t \int_E \int_0^1 G_{t-s}(x,y) g(X(s-,y), z) dy \; N(ds, dz) \end{aligned} \tag{2.5}$$

where $G_t(\mathcal{X}_0, x)$ is defined by (2.3), and with the convention

$$\int_0^1 G_0(x,y) g(X(s-,y), z) dy = g(X(s-,x), z) \tag{2.6}$$

We now establish a result of existence and uniqueness of such a solution. Since $q(E)$ is finite, $N([0,T] \times E)$ is a.s. finite, and thus N can a.s. be written as

$$N(dt, dz) = \sum_{i=1}^{\mu} \delta_{(T_i, Z_i)}(dt, dz) \tag{2.7}$$

with $\mu \in I\!N$, $0 < T_1 < ... < T_\mu < T$, and $Z_1, ..., Z_\mu \in E$. Hence, equation (2.5) can be written as

$$\begin{aligned} X(t,x) &= G_t(\mathcal{X}_0, x) + \int_0^t \int_0^1 G_{t-s}(x,y) \left[b(X(s,y)) dy ds + \sigma(X(s,y)) W(dy, ds) \right] \\ &\quad + \sum_{i=1}^{\mu} \mathbb{I}_{\{t \geq T_i\}} \int_0^1 G_{t-T_i}(x,y) g(X(T_i-, y), Z_i) dy \end{aligned} \tag{2.8}$$

Working recursively on the time intervals $[0, T_1[$, $[T_1, T_2[$,..., $[T_\mu, T]$, as Ikeda and Watanabe (proof of Theorem 9-1 p 231-232 in [6]), using Walsh's Theorems of existence, uniqueness, and regularity for equation (1.1) with $g \equiv 0$ (see [13], Theorem 3-2 p 313 and Corollary 3-4 p 317), and using the well-known estimates of the Green kernel stated in the Appendix, one can prove the following proposition :

Proposition 2.2 *Assume (H). Equation (1.1) admits a unique adapted solution $X(t,x)$ on $[0,T] \times [0,1]$, lying a.s. in $I\!D([0,T], \mathbf{C}([0,1]))$. The uniqueness holds in the sense where if Y is another adapted solution lying in $I\!D([0,T], \mathbf{C}([0,1]))$, then a.s.,*

$$\sup_{[0,T] \times [0,1]} |X(t,x) - Y(t,x)| = 0 \tag{2.9}$$

We are now interested in the support of the law of X. Let us first recall the definition of the Skorokhod distance on $I\!D([0,T], \mathbf{C}([0,1]))$. We consider the set of the "changes of time" :

$$\Lambda = \{ \lambda \in \mathbf{C}([0,T]) \; / \; \lambda(0) = 0, \; \lambda(T) = T, \; \lambda \text{ is strictly increasing} \} \tag{2.10}$$

For $\lambda \in \Lambda$, we set

$$|||\lambda||| = \sup_{0 \leq s < t \leq T} \left| \ln \left\{ \frac{\lambda(t) - \lambda(s)}{t - s} \right\} \right| \tag{2.11}$$

The Skorokhod distance between two elements ϕ and ψ of $I\!D\left([0,T],\mathbf{C}([0,1])\right)$ is given by

$$\delta(\phi,\psi)=\inf_{\lambda\in\Lambda}\left\{\sup_{[0,T]\times[0,1]}|\phi(\lambda(t),x)-\psi(t,x)|+|||\lambda|||\right\} \tag{2.12}$$

$I\!D\left([0,T],\mathbf{C}([0,1])\right)$, endowed with δ, is a Polish space (see e.g. Jacod, Shiryaev, [7], p 289).

We now introduce some notations, describing the "supports" of W and N : we denote by

$$\mathcal{H}=\left\{h(t,x)=\int_0^t\int_0^x \dot{h}(s,y)dyds \;\Big/\; \dot{h}\in L^2([0,T]\times[0,1])\right\} \tag{2.13}$$

the Cameron-Martin space associated with W. We also consider the set of the finite counting measures on $[0,T]\times E$, the support of which is contained in $[0,T]\times\text{supp }q$:

$$\mathcal{M}=\left\{m(dt,dz)=\sum_{i=1}^n\delta_{(t_i,z_i)}(dt,dz) \;\Big/\; \begin{array}{c} n\in I\!N,\ 0<t_1<...<t_n<T,\\ z_1,...,z_n\in\text{supp }q\end{array}\right\} \tag{2.14}$$

with the convention $\sum_{i=1}^0=0$. Notice that for all $\omega\in\Omega$, $N(\omega)$ belongs to $\mathcal{M}$. But in general, (with abusive notation) $\dot{W}(\omega)\notin\mathcal{H}$, since $\dot{W}(\omega)$ is not even well-defined.

The following proposition, describes the "skeleton" associated with our evolution equation.

Proposition 2.3 *Assume (H). Let $h\in\mathcal{H}$ and $m\in\mathcal{M}$ be fixed. The following ordinary evolution equation admits a unique solution, which we denote by $S(h,m)$, lying in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$:*

$$\begin{aligned} S(h,m)(t,x) &= G_t(\mathcal{X}_0,x)+\int_0^t\int_0^1 G_{t-s}(x,y)\Big[b(S(h,m)(s,y))dyds \\ &\qquad +\sigma(S(h,m)(s,y))\dot{h}(s,y)dyds\Big] \\ &\quad +\int_0^t\int_E\int_0^1 G_{t-s}(x,y)g(S(h,m)(s-,y),z)dy\; m(ds,dz) \end{aligned} \tag{2.15}$$

This proposition can be proved as Proposition 2.2. Equation (2.15) is the same as (2.5), but we have replaced $W(dy,ds)$ and $N(ds,dz)$ by $\dot{h}(s,y)dyds$ and $m(ds,dz)$.

Finally, we recall the following standard remark :

Remark 2.4 *Let Z be a random variable with values in a Polish space A endowed with a distance α. Recall that the support $\text{supp}_\alpha P\circ Z^{-1}$ of the law of Z related to the distance α is the smaller closed subset F of (A,α) satisfying $P(Z\in F)=1$.*
Let B be a subset of A, and let $\overline{B}^\alpha$ be its closure in (A,α).

1. *If a.s., $Z\in\overline{B}^\alpha$, then*

$$\text{supp}_\alpha P\circ Z^{-1}\subset\overline{B}^\alpha \tag{2.16}$$

2. If for all $b \in B$, all $\epsilon > 0$,

$$P(\alpha(b, Z) < \epsilon) > 0 \tag{2.17}$$

then

$$\overline{B}^{\alpha} \subset \text{supp}_{\alpha}\ P \circ Z^{-1} \tag{2.18}$$

In order to establish a support Theorem, we need the following assumptions.

Assumption $(S1)$: the function σ is C^3 on $I\!R$.

Assumption $(S2)$: for each $z_0 \in E$, each $n \in I\!N$,

$$\sup_{|x|\le n} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{2.19}$$

For each $z_0 \in E$, each $n \in I\!N$, there exists a constant $\xi^n(z_0) > 0$, and a function $\psi^n_{z_0}(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that for all $|x| \le n$, $|y| \le n$,

$$\sup_{d(z,z_0)\le \xi^n(z_0)} |g(x,z) - g(y,z)| \le \psi^n_{z_0}(|x-y|) \tag{2.20}$$

Assumption $(S1)$ is nearly the same as that of Bally, Millet, Sanz, [2], who prove a support theorem in the case where $g \equiv 0$, and comes from a Taylor developpement of order 3. In fact they assume that σ is C^3_b, but a localisation procedure can be done (see the proof of Proposition 3.1 in the next section).
Assumption $(S2)$ says that g is locally uniformly continuous. In the particular case where E is locally compact, $(S2)$ is satisfied as soon as g is continuous on $[0,T] \times E$.

Now we can state our main result :

Theorem 2.5 *Under (H), $(S1)$, and $(S2)$, if X denotes the unique weak solution of equation (1.1),*

$$\text{supp}_{\delta}\ P \circ X^{-1} = \overline{\{S(h,m)\ /\ h \in \mathcal{H}\ ,\ m \in \mathcal{M}\}}^{\delta} \tag{2.21}$$

3 Simplification of the problem.

First, we "delocalize" $(S1)$ and $(S2)$, by using a standard argument. Consider the following assumptions, stronger than $(S1)$ and $(S2)$.

Assumption $(S'1)$: the function σ is C^3 on $I\!R$, bounded with its derivatives.

Assumption $(S'2)$: For all $z_0 \in E$,

$$\sup_{x\in I\!R} |g(x,z_0)| < \infty \quad ; \quad \sup_{x \in I\!R} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{3.1}$$

For all $z_0 \in E$, there exists $\xi(z_0) > 0$, and a function $\psi_{z_0}(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that for all $x, y \in I\!R$,

$$\sup_{d(z,z_0)\le \xi(z_0)} |g(x,z) - g(y,z)| \le \psi_{z_0}(|x-y|) \tag{3.2}$$

Proposition 3.1 *If Theorem 2.5 holds under* (H), $(S'1)$ *and* $(S'2)$, *then it also holds under* (H), $(S1)$ *and* $(S2)$.

We will prove this proposition at the end of the section.

We now would like to check that Theorem 2.5 holds as soon as two easier support theorems are valid. The first one deals with equation 2.5 with a "deterministic" white noise, and the second one with a "deterministic" Poisson measure.

We first introduce some notations. If $h \in \mathcal{H}$ (resp. $m \in \mathcal{M}$), we denote by X_h (resp. X_m) the solution of equation (2.5) where we have replaced $W(dy, ds)$ by $\dot{h}(s, y)dyds$ (resp. $N(dt, dz)$ by $m(dt, dz)$). In other words,

$$\begin{aligned} X_h(t,x) &= G_t(\mathcal{X}_0, x) + \int_0^t \int_0^1 G_{t-s}(x,y) \left[b(X_h(s,y))dyds + \sigma(X_h(s,y))\dot{h}(s,y)dyds \right] \\ &\quad + \int_0^t \int_E \int_0^1 G_{t-s}(x,y) g(X_h(s-,y),z)dy \, N(ds,dz) \end{aligned} \tag{3.3}$$

$$\begin{aligned} X_m(t,x) &= G_t(\mathcal{X}_0, x) + \int_0^t \int_0^1 G_{t-s}(x,y) \left[b(X_m(s,y))dyds + \sigma(X_m(s,y))W(dy,ds) \right] \\ &\quad + \int_0^t \int_E \int_0^1 G_{t-s}(x,y) g(X_m(s-,y),z)dy \, m(ds,dz) \end{aligned} \tag{3.4}$$

We could also write, with abusive notations, $X_h = S(h, N)$, and $X_m = S(\dot{W}, m)$. The next sections are devoted to the proof of the following propositions.

Proposition 3.2 *Assume* (H) *and* $(S'2)$. *Let* $h \in \mathcal{H}$, $m \in \mathcal{M}$, *and* $\epsilon > 0$ *be fixed. Then*

$$P\left(\delta(S(h,m), X_h) \le \epsilon\right) > 0 \tag{3.5}$$

We now denote by $\| u \|_\infty = \sup_{[0,T]\times[0,1]} |u(t,x)|$ the supremum norm on $[0,T] \times [0,1]$.

Proposition 3.3 *Assume* (H), $(S'1)$ *and* $(S'2)$. *Let* $m \in \mathcal{M}$ *be fixed. Then*

$$\mathrm{supp}_{\|\ \|_\infty} P \circ X_m^{-1} = \overline{\{S(h,m) \ / \ h \in \mathcal{H}\}}^{\|\ \|_\infty} \tag{3.6}$$

Let us remark that this second result implies the weaker one :

$$\mathrm{supp}_\delta P \circ X_m^{-1} = \overline{\{S(h,m) \ / \ h \in \mathcal{H}\}}^{\delta} \tag{3.7}$$

Assuming for a moment that these propositions hold, we prove our main result.

Proof of Theorem 2.5 : using Remark 2.4, we break the proof in two parts.

1) We first check that a.s., X belongs to $\overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}$. Consider the map from $\mathcal{M}$ to $[0,1]$, defined by

$$\phi(\mu) = P\left(X_\mu \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.8}$$

Let us first prove that a.s.,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta} \;\Big|\; \sigma(N)\right) = \phi(N) \tag{3.9}$$

where

$$\sigma(N) = \sigma\left\{N(A)\ ;\ A \in \mathcal{B}([0,T] \times E)\right\} \tag{3.10}$$

In order to understand (3.9), let us work with the canonical product space

$$(\Omega, \mathcal{F}, P) = (\Omega^W, \mathcal{F}^W, P^W) \otimes (\Omega^N, \mathcal{F}^N, P^N) \tag{3.11}$$

associated with W and N. Every element ω of Ω can be written as (ω^W, ω^N), where $\omega^W \in \mathbf{C}([0,T] \times [0,1])$ and $\omega^N \in \mathcal{M}$. Thus,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta} \;\Big|\; \sigma(N)\right)(\omega)$$

$$= \int \mathbb{1}_{\left\{X(\omega^W,\omega^N) \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right\}} dP^W(\omega^W) \tag{3.12}$$

But obviously, $X(\omega) = X(\omega^W, \omega^N) = X_{\omega^N}(\omega^W)$, where X_μ was defined by (3.4) for each $\mu \in \mathcal{M}$. Thus,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta} \;\Big|\; \sigma(N)\right)(\omega)$$

$$= P^W\left(X_{\omega^N} \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.13}$$

Now, we notice, since for each $\mu \in \mathcal{M}$, X_μ is independent of N, that

$$\phi(\mu) = P^W\left(X_\mu \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.14}$$

Comparing (3.13) and (3.14), we deduce (3.9). Hence, we obtain

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) = E(\phi(N)) \tag{3.15}$$

Finally, it is clear from the definition of ϕ and from Proposition 3.3 that $\phi \equiv 1$. The conclusion follows easily.

2) We now fix $h \in \mathcal{H}$, $m \in \mathcal{M}$, and $\epsilon > 0$. We have to check that

$$P_0 = P\left(\delta(X, S(h,m)) \le \epsilon\right) > 0 \tag{3.16}$$

First,

$$P_0 \ge P\left(\delta(X, X_h) \le \epsilon/2\ ;\ \delta(X_h, S(h,m)) \le \epsilon/2\right) \tag{3.17}$$

Noticing that X_h is $\sigma(N)$-measurable, we see that

$$P_0 \geq E\left[\mathbb{1}_{\{\delta(X_h, S(h,m)) \leq \epsilon/2\}} P\left(\delta(X, X_h) \leq \epsilon/2 \mid \sigma(N)\right)\right] \tag{3.18}$$

But we know from Proposition 3.3 that for all $m \in \mathcal{M}$,

$$\psi(m) = P\left(\delta(X_m, S(h,m)) \leq \epsilon/2\right) > 0 \tag{3.19}$$

Working on the canonical product space as in 1), and noticing that for all $\omega = (\omega^W, \omega^N) \in \Omega$, $X(\omega) = X_{\omega^N}(\omega^W)$ and $X_h(\omega) = S(h, \omega^N)$ (all of this **without** abusive notation), we deduce that a.s.,

$$P\left(\delta(X, X_h) \leq \epsilon/2 \mid \sigma(N)\right) = \psi(N) > 0 \tag{3.20}$$

Thus, (3.16) holds as soon as

$$P\left(\delta(X_h, S(h,m)) \leq \epsilon/2\right) > 0 \tag{3.21}$$

which never fails, thanks to Proposition 3.2.

Provided we check Propositions 3.1, 3.2 and 3.3, Theorem 2.5 is be proved.

In order to prove Proposition 3.1, we begin with a Lemma.

Lemma 3.4 *Consider some functions σ, b, g (resp. $\bar{\sigma}$, $\bar{b}$ and $\bar{g}$) satisfying (H), and denote by X (resp. $\bar{X}$) the corresponding unique weak solution of (1.1). Assume that for some $A \in I\!R^+$,*

$$\forall\, |x| \leq A, \quad \forall\, z \in E, \quad \sigma(x) = \bar{\sigma}(x) \quad , \quad b(x) = \bar{b}(x) \quad \text{and} \quad g(x,z) = \bar{g}(x,z) \tag{3.22}$$

Then there exists $\tilde{\Omega} \subset \Omega$ such that $P(\tilde{\Omega}) = 1$ and

$$\{\tilde{\omega} \in \Omega \;/\; \| X(\omega) \|_\infty \leq A\} \subset \left\{\omega \in \tilde{\Omega} \;/\; \| X(\omega) - \bar{X}(\omega) \|_\infty = 0\right\} \tag{3.23}$$

<u>Proof of Lemma 3.4</u> : we consider the stopping time $\tau = \inf\{t \geq 0 \;,\; \sup_x |X(t,x)| \geq A\}$. Then the processes $X^\tau(t,x) = X(t \wedge \tau, x)$ and $\bar{X}^\tau(t,x) = \bar{X}(t \wedge \tau, x)$ satisfy the same evolution equation :

$$\begin{aligned} X^\tau(t,x) &= G_t(\mathcal{X}_0, x) + \int_0^{t\wedge\tau} \int_0^1 G_{t-s}(x,y)\left[\bar{b}(X^\tau(s,y))dyds + \bar{\sigma}(X^\tau(s,y))W(dy,ds)\right] \\ &\quad + \int_0^{t\wedge\tau} \int_E \int_0^1 G_{t-s}(x,y)\bar{g}(X^\tau(s-,y),z)dy\, N(ds,dz) \end{aligned} \tag{3.24}$$

A uniqueness argument yields that a.s., say for all $\omega \in \tilde{\Omega}$, with $P(\tilde{\Omega}) = 1$, $X^\tau = \bar{X}^\tau$ on $[0,T] \times [0,1]$. This yields that for all $\omega \in \tilde{\Omega}$, all $t \leq \tau$, and all $x \in [0,1]$, $X(t,x) = \bar{X}(t,x)$. This implies that

$$\left\{\omega \in \tilde{\Omega} \;/\; \| X(\omega) \|_\infty \leq A\right\} \subset \left\{\omega \in \tilde{\Omega} \;/\; \tau(\omega) > T\right\} \subset \left\{\omega \in \tilde{\Omega} \;/\; \| X(\omega) - \bar{X}(\omega) \|_\infty = 0\right\} \tag{3.25}$$

Proof of Proposition 3.1 : we assume that theorem 2.5 holds under (H), $(S'1)$ and $(S'2)$, and we consider functions b, σ and g satisfying only (H), $(S1)$ and $(S2)$. We need a sequence of C_b^∞ functions $\phi_n : I\!\!R \mapsto [0,1]$, satisfying :

$$\phi_n(x) = \begin{cases} 1 & \text{if} \quad |x| \leq n \\ \\ 0 & \text{if} \quad |x| \geq n+1 \end{cases} \tag{3.26}$$

Then the functions $\sigma_n(x) = \sigma(x)\phi_n(x)$ and $g_n(x,z) = g(x,z)\phi_n(x)$ clearly satisfy $(S'1)$ and $(S'2)$. Denote by X_n the solution of equation (2.5) with σ_n and g_n instead of σ and g. Lemma 3.4 yields that there exists $\tilde{\Omega} \subset \Omega$ such that $P(\tilde{\Omega}) = 1$ and for all $n \in I\!\!N$,

$$\left\{\omega \in \tilde{\Omega} \; / \parallel X(\omega) \parallel_\infty \leq n\right\} \subset \left\{\omega \in \tilde{\Omega} \; / \parallel X(\omega) - X_n(\omega) \parallel_\infty = 0\right\} \tag{3.27}$$

In the same way, we define $S_n(h,m)$, for $h \in \mathcal{H}$ and $m \in \mathcal{M}$, as the solution of equation (2.15) with σ_n and g_n instead of σ and g. We obtain, for all $n \in I\!\!N$,

$$\text{if} \quad \parallel S(h,m) \parallel_\infty \leq n \;\text{ or }\; \parallel S_n(h,m) \parallel_\infty \leq n, \;\text{ then }\; S(h,m) = S_n(h,m) \tag{3.28}$$

Since Theorem 2.5 holds under (H), $(S'1)$, and $(S'2)$, we know that for each $n \in I\!\!N$,

$$\text{supp}_\delta \, P \circ X_n^{-1} = \overline{\{S_n(h,m) \; / \; h \in \mathcal{H} \; , \; m \in \mathcal{M}\}}^\delta \tag{3.29}$$

Using Remark 2.4, Proposition 3.1 will hold if we check that on one hand,

$$P\left(X \in \overline{\{S(h,m) \; / \; h \in \mathcal{H} \; , \; m \in \mathcal{M}\}}^\delta\right) = 1 \tag{3.30}$$

and on the other hand that for all $h \in \mathcal{H}$, all $m \in \mathcal{M}$, all $\epsilon > 0$,

$$P\left(\delta(X, S(h,m)) \leq \epsilon\right) > 0 \tag{3.31}$$

Let us first prove (3.30). Let $\omega \in \tilde{\Omega}$ be fixed. Since $X(\omega)$ belongs to $I\!\!D\left([0,T], \mathbf{C}([0,1])\right)$, it is bounded, and there exists $n \in I\!\!N$ (depending on ω) such that

$$n \geq \parallel X(\omega) \parallel_\infty +1 \tag{3.32}$$

which yields $X(\omega) = X_n(\omega)$. But for all $\epsilon > 0$, we know from (3.29) that for almost all $\omega \in \tilde{\Omega}$, there exists $h \in \mathcal{H}$ and $m \in \mathcal{M}$ (depending on ω) such that

$$\delta(X_n(\omega), S_n(h,m)) \leq \epsilon \tag{3.33}$$

This and (3.32) yield (if $\epsilon \leq 1$), that $\parallel S_n(h,m) \parallel_\infty \leq n$, and thus that $S_n(h,m) = S(h,m)$. Hence,

$$\delta(X(\omega), S(h,m)) \leq \epsilon \tag{3.34}$$

which concludes the proof of (3.30), since $P(\tilde{\Omega}) = 1$.

In order to prove (3.31), we fix $h \in \mathcal{H}$, $m \in \mathcal{M}$, and $\epsilon > 0$. We consider $n \in I\!\!N$ such that

$$n \geq \parallel S(h,m) \parallel_\infty +1 \tag{3.35}$$

This way, if $\epsilon < 1$,

$$\begin{aligned} P\left(\delta(X, S(h,m)) \leq \epsilon\right) &= P\left(\delta(X, S_n(h,m)) \leq \epsilon\right) \\ &= P\left(\| X \|_\infty \leq n, \delta(X, S_n(h,m)) \leq \epsilon\right) \\ &= P\left(\delta(X_n, S_n(h,m)) \leq \epsilon\right) \end{aligned} \tag{3.36}$$

thanks to (3.27). From (3.29), this probability is strictly positive, which yields (3.31). Proposition 3.1 is proved.

4 The case where "W is deterministic".

This section is devoted to the proof of Proposition 3.2. We follow here partially the method of Simon [10], who studies the support of Poisson driven S.D.E.s (without Wiener term). The extension of his method to S.P.D.E.s drives to technical problems, essentially because we have to control the explosion of the Green kernel $G_t(x,y)$. Another new difficulty appears, because we have to add a second drift, in which the term $\dot{h}(s,y)$ belongs only to $L^2([0,T] \times [0,1])$.

In the whole section,

$$h(t,x) = \int_0^t \int_0^x \dot{h}(s,y)dyds \in \mathcal{H} \quad \text{and} \quad m(dt,dz) = \sum_{i=1}^n \delta_{(t_i,z_i)}(dt,dz) \in \mathcal{M} \tag{4.1}$$

are fixed. We set $t_0 = 0$, $t_{n+1} = T$, and

$$\zeta_0 = \inf_{i=0,\ldots,n} |t_{i+1} - t_i| > 0 \tag{4.2}$$

For simplicity, we set $S = S(h,m)$. We denote by $0 < T_1(\omega) < \ldots < T_{\mu(\omega)}(\omega)$ the successive times of jump of $N(\omega)$, and by $Z_1(\omega)$, ..., $Z_{\mu(\omega)}(\omega)$ the size of its jumps. In other words,

$$N(\omega, dt, dz) = \sum_{i=1}^{\mu(\omega)} \delta_{(T_i(\omega), Z_i(\omega))}(dt,dz) \tag{4.3}$$

We recall that for all $\alpha \in]0, \zeta_0[$, and all $\xi > 0$, the set

$$\Omega(\alpha,\xi) = \{\omega \in \Omega \ / \ \mu(\omega) = n, \ t_i - \alpha < T_i(\omega) < t_i, \ d(z_i, Z_i(\omega)) \leq \xi\} \tag{4.4}$$

has a strictly positive probability. We will check that for all $\epsilon > 0$, there exists $\alpha > 0$, and $\xi > 0$ such that for all $\omega \in \Omega(\alpha,\xi)$,

$$\delta(X_h(\omega), S) \leq \epsilon \tag{4.5}$$

which will imply Proposition 3.2.

In the whole section, the constant C depends only on h, m, and on the parameters (σ, b, g, $\mathcal{X}_0$, and T) of equation (1.1).

From now on, we consider $\omega \in \Omega(\alpha, \xi)$.

First, we choose $0 < \alpha < \zeta_0/16$, and $0 < \xi < \xi(z_1) \wedge ... \wedge \xi(z_n)$, where $\xi(z_i)$ was defined in assumption $(S'2)$. For some $\gamma \in]2\alpha, \zeta_0/8[$, which will be chosen later, we define the polygonal change of time $\lambda \in \Lambda$ by $\lambda(0) = 0$, $\lambda(T) = T$, and for all $i \in \{1, ..., n\}$,

$$\lambda(T_i - \gamma) = T_i - \gamma \quad ; \quad \lambda(T_i) = t_i \quad ; \quad \lambda(T_i + \gamma) = t_i + \gamma \qquad \lambda(T_i + 2\gamma) = T_i + 2\gamma \tag{4.6}$$

Notice that all the properties below hold :

$$\text{for all } t \in [T_i, T_i + \gamma], \qquad \lambda(t) - t_i = t - T_i \tag{4.7}$$

$$\int_0^T \mathbb{1}_{\{\lambda(s) \neq s\}} ds \leq 3n\gamma \tag{4.8}$$

$$\text{for all } t \in [0, T], \quad \lambda(t) \geq t \quad \text{and} \quad \mathbb{1}_{\{\lambda(t) \geq t_i\}} = \mathbb{1}_{\{t \geq T_i\}} \tag{4.9}$$

$$\| \lambda - I \|_\infty \leq \alpha \tag{4.10}$$

Furthermore, it is easy to check that

$$|||\lambda||| \leq |\ln(1 - \alpha/\gamma)| \vee |\ln(1 + \alpha/\gamma)| \leq 2\alpha/\gamma \tag{4.11}$$

where the last inequality holds because $\alpha/\gamma \leq 1/2$. We have to prove that if $\alpha > 0$ and $\xi > 0$ are small enough then for some γ well chosen,

$$\| S(\lambda(t), x) - X_h(t, x) \|_\infty + |||\lambda||| \leq \epsilon \tag{4.12}$$

We now set $S_\lambda(t, x) = S(\lambda(t), x)$. Then, using (4.9), we see that for any $\omega \in \Omega(\alpha, \xi)$,

$$\begin{aligned}
S_\lambda(t,x) - X_h(t,x) &= G_{\lambda(t)}(\mathcal{X}_0, x) - G_t(\mathcal{X}_0, x) \\
&+ \int_0^t \int_0^1 \Big(G_{\lambda(t)-s}(x,y) - G_{t-s}(x,y) \Big) \Big[b(S(s,y)) + \sigma(S(s,y)) \dot{h}(s,y) \Big] \, dyds \\
&+ \int_t^{\lambda(t)} \int_0^1 G_{\lambda(t)-s}(x,y) \Big[b(S(s,y)) + \sigma(S(s,y)) \dot{h}(s,y) \Big] \, dyds \\
&+ \int_0^t \int_0^1 G_{t-s}(x,y) \Big[\{ b(S(s,y)) - b(S_\lambda(s,y)) \} \\
&\qquad + \{ \sigma(S(s,y)) - \sigma(S_\lambda(s,y)) \} \dot{h}(s,y) \Big] dyds \\
&+ \int_0^t \int_0^1 G_{t-s}(x,y) \Big[\{ b(S_\lambda(s,y)) - b(X_h(s,y)) \}
\end{aligned}$$

$$+\{\sigma(S_\lambda(s,y)) - \sigma(X_h(s,y))\}\dot{h}(s,y)\Big] dyds$$

$$+\sum_{i=1}^{n} \mathbb{I}_{\{t\geq T_i\}} \int_0^1 \Big(G_{\lambda(t)-t_i}(x,y) - G_{t-T_i}(x,y)\Big)\, g(S(t_i-,y),z_i)dy$$

$$+\sum_{i=1}^{n} \mathbb{I}_{\{t\geq T_i\}} \int_0^1 G_{t-T_i}(x,y)\,[g(S(t_i-,y),z_i) - g(S(t_i-,y),Z_i)]\,dy$$

$$+\sum_{i=1}^{n} \mathbb{I}_{\{t\geq T_i\}} \int_0^1 G_{t-T_i}(x,y)\,[g(S(t_i-,y),Z_i) - g(X_h(T_i-,y),Z_i)]\,dy$$

$$= A(t,x) + ... + H(t,x) \tag{4.13}$$

We compute these terms one by one, still assuming that $\omega \in \Omega(\alpha,\xi)$.

Since $\lambda(t) = t$ for all $t \leq T_1 - \gamma$, and hence for all $t \leq 13\zeta_0/16$

$$|A(t,x)| \leq |A(t,x)|\mathbb{I}_{\{t\geq 13\zeta_0/16\}} \leq \| \mathcal{X}_0 \|_\infty \mathbb{I}_{\{t\geq 13\zeta_0/16\}} \int_0^1 |G_{\lambda(t)}(x,y) - G_t(x,y)|dy \tag{4.14}$$

Using the Appendix, (7.4), then (4.10), we see that

$$|A(t,x)| \leq C\frac{\lambda(t)-t}{(13\zeta_0/16)^{\frac{3}{2}}} \leq C \| \lambda - I \|_\infty \leq C\alpha \tag{4.15}$$

Using Cauchy-Schwarz's inequality, then the Appendix (7.5), and finally (4.10), we obtain

$$\begin{aligned} |B(t,x)| &\leq \left(\int_0^t \int_0^1 \Big[b(S(s,y)) + \sigma(S(s,y))\dot{h}(s,y)\Big]^2 dyds\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \int_0^1 \Big[G_{\lambda(t)-s)}(x,y) - G_{t-s}(x,y)\Big]^2 dyds\right)^{\frac{1}{2}} \\ &\leq C\left(\sqrt{\lambda(t)-t}\right)^{\frac{1}{2}} \leq C\alpha^{\frac{1}{4}} \end{aligned} \tag{4.16}$$

Exactly in the same way, $|C(t,x)| \leq C\alpha^{\frac{1}{4}}$.

Using (H), we see that

$$|D(t,x)| \leq C\int_0^t \int_0^1 G_{t-s}(x,y)\,|S(s,y) - S_\lambda(s,y)|\,\Big(1 + |\dot{h}(s,y)|\Big)\,dyds \tag{4.17}$$

Thanks to Cauchy-Schwarz's inequality, and the Appendix (7.2),

$$\begin{aligned} |D(t,x)| &\leq C\left(\int_0^t \sup_{y\in[0,1]} |S(s,y) - S_\lambda(s,y)|^2\, ds \int_0^1 G^2_{t-s}(x,y)dy\right)^{\frac{1}{2}} \\ &\leq C\left(\int_0^t \mathbb{I}_{\{\lambda(s)\neq s\}}\frac{ds}{\sqrt{t-s}}\right)^{\frac{1}{2}} \end{aligned} \tag{4.18}$$

Using the Hölder inequality with $p = 3$ and $q = 3/2$, we deduce that

$$\begin{aligned} |D(t,x)| &\leq C\left(\int_0^t 1_{\{\lambda(s)\neq s\}} ds\right)^{1/6} \left(\int_0^t \frac{ds}{(t-s)^{3/4}}\right)^{1/3} \\ &\leq C\left(\int_0^t 1_{\{\lambda(s)\neq s\}} ds\right)^{1/6} \leq C(3n\gamma)^{1/6} \leq C\gamma^{1/6} \end{aligned} \quad (4.19)$$

thanks to (4.8).

The same computation drives us to

$$|E(t,x)| \leq C\left(\int_0^t \sup_{y\in[0,1]} |S_\lambda(s,y) - X_h(s,y)|^2 \frac{ds}{\sqrt{t-s}}\right)^{\frac{1}{2}} \quad (4.20)$$

Using (4.7), and (3.1) in $(S'2)$, we see that

$$|F(t,x)| \leq C\sum_{i=1}^n \mathbb{1}_{\{t\geq T_i+\gamma\}} \sup_{x,y\in[0,1]} \left|G_{\lambda(t)-t_i}(x,y) - G_{t-T_i}(x,y)\right| \quad (4.21)$$

Thus, thanks to the Appendix (7.4),

$$|F(t,x)| \leq C\sum_{i=1}^n \mathbb{1}_{\{t\geq T_i+\gamma\}} \frac{|(\lambda(t)-t_i) - (t-T_i)|}{[(\lambda(t)-t_i)\wedge(t-T_i)]^{\frac{3}{2}}} \quad (4.22)$$

But $t \geq T_i + \gamma$ implies that $\lambda(t) - t_i \geq \lambda(T_i+\gamma) - t_i = \gamma$. Hence, thanks to (4.10) and since $\omega \in \Omega(\alpha,\xi)$,

$$|F(t,x)| \leq C\frac{\| \lambda - I \|_\infty + \sup_i |t_i - T_i|}{\gamma^{\frac{3}{2}}} \leq C\alpha/\gamma^{\frac{3}{2}} \quad (4.23)$$

Using (7.3) of the appendix, we deduce that

$$|G(t,x)| \leq \sum_{i=1}^n \sup_y |g(S(t_i-,y),z_i) - g(S(t_i-,y),Z_i)| \quad (4.24)$$

Thanks to (3.1) in $(S'2)$, recalling that for all i, $d(z_i,Z_i) \leq \xi$, we see that there exists a function $\varphi(\xi)$ from $\mathbb{R}^+$ into itself, decreasing to 0 when ξ decreases to 0, depending only on h, m, and on the parameters of equation (1.1), such that

$$|G(t,x)| \leq \varphi(\xi) \quad (4.25)$$

In the same way, but using (3.2) and the fact that $\xi \leq \xi(z_1)\wedge ... \wedge \xi(z_n)$, we easily prove the existence of a function $\beta(u) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, decreasing to 0 when u decreases to 0, such that

$$\begin{aligned} |H(t,x)| &\leq \sum_{i=1}^n \mathbb{1}_{\{t\geq T_i\}} \times \beta\left(\sup_{y\in[0,1]} |S(t_i-,y) - X_h(T_i-,y)|\right) \\ &\leq \sum_{i=1}^n \mathbb{1}_{\{t\geq T_i\}} \times \beta\left(\sup_{y\in[0,1]} |S_\lambda(T_i-,y) - X_h(T_i-,y)|\right) \end{aligned} \quad (4.26)$$

since $\lambda(T_i) = t_i$.

Finally, setting

$$I(t) = \sup_{y\in[0,1]} |S_\lambda(t,y) - X_h(t,y)| \tag{4.27}$$

and

$$K(\alpha,\gamma,\xi) = \alpha^{1/4}/\gamma^{3/2} + \gamma^{1/4} + \varphi(\xi) \tag{4.28}$$

we obtain :

$$I(t) \le CK(\alpha,\gamma,\xi) + C\left(\int_0^t I^2(s)\frac{ds}{\sqrt{t-s}}\right)^{\frac{1}{2}} + C\sum_{i=1}^n \mathbb{I}_{\{t\ge T_i\}}\beta(I(T_i-)) \tag{4.29}$$

Hence

$$I^2(t) \le CK^2(\alpha,\gamma,\xi) + C\int_0^t I^2(s)\frac{ds}{\sqrt{t-s}} + C\sum_{i=1}^n \mathbb{I}_{\{t\ge T_i\}}\beta^2(I(T_i-)) \tag{4.30}$$

Iterating one time this formula, we get

$$\begin{aligned} I^2(t) \;\le\; & CK^2(\alpha,\gamma,\xi) + C\sum_{i=1}^n \mathbb{I}_{\{t\ge T_i\}}\beta^2(I(T_i-)) \qquad (4.31)\\ +\; & C\int_0^t \left[CK^2(\alpha,\gamma,\xi) + C\int_0^s I^2(u)\frac{du}{\sqrt{s-u}} + C\sum_{i=1}^n \mathbb{I}_{\{s\ge T_i\}}\beta^2(I(T_i-))\right]\frac{ds}{\sqrt{t-s}} \end{aligned}$$

Using Fubini's Theorem, and noticing that $\displaystyle\int_u^t \frac{ds}{\sqrt{t-s}\sqrt{s-u}} \le 4$, we deduce that

$$I^2(t) \le CK^2(\alpha,\gamma,\xi) + C\int_0^t I^2(u)du + C\sum_{i=1}^n \mathbb{I}_{\{t\ge T_i\}}\beta^2(I(T_i-)) \tag{4.32}$$

We now apply Gronwall's Lemma on $[0,T_1[$. This gives :

$$\sup_{[0,T_1[} I^2(t) \le CK^2(\alpha,\gamma,\xi)e^{CT} \le CK^2(\alpha,\gamma,\xi) \tag{4.33}$$

Thus, on $[0,T_2[$,

$$I^2(t) \le CK^2(\alpha,\gamma,\xi) + \beta^2(CK^2(\alpha,\gamma,\xi)) + C\int_0^t I^2(s)ds \tag{4.34}$$

Thanks to Gronwall's Lemma,

$$\sup_{[0,T_2[} I^2(t) \le \left(CK^2(\alpha,\gamma,\xi) + \beta^2(K^2(\alpha,\gamma,\xi))\right)e^{CT} \tag{4.35}$$

Iterating this argument, we deduce the existence of a function $\eta(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that

$$\sup_{[0,T]} I(t) \le \eta\left(K(\alpha,\gamma,\xi)\right) \tag{4.36}$$

Hence, there exists $\delta > 0$ such that if $K(\alpha,\gamma,\xi) \leq \delta$, then $\sup_{[0,T]} I(t) \leq \epsilon/2$. It now suffices to choose α, γ, ξ small enough, such that

$$K(\alpha,\gamma,\xi) \leq \delta \quad ; \quad 2\alpha/\gamma \leq \epsilon/2 \tag{4.37}$$

which will imply, for all $\omega \in \Omega(\alpha,\xi)$,

$$\delta(X_h(\omega), S) \leq \| I(\omega) \|_\infty + |||\lambda(\omega)||| \leq \epsilon \tag{4.38}$$

First, we choose $\xi \in]0, \xi(z_1) \wedge ... \wedge \xi(z_n)[$ small enough, in order to get $\varphi(\xi) \leq \delta/3$. Then we choose γ in $]0, (\zeta_0/8) \wedge (\delta/3)^6[$. Finally, we choose

$$0 < \alpha < \gamma/2 \wedge \left(\delta\gamma^{\frac{3}{2}}/3\right)^4 \wedge \epsilon\gamma/4 \tag{4.39}$$

Proposition 3.2 is proved.

5 The case where N is "deterministic".

It remains to prove Proposition 3.3. In the whole section,

$$m(dt, dz) = \sum_{i=1}^{n} \delta_{(t_i, z_i)}(dt, dz) \in \mathcal{M} \tag{5.1}$$

is fixed. We set $t_0 = 0$, $t_{n+1} = T$.

We have to establish a support theorem for the solution of equation (3.4). Let us observe that this equation is not much different from that of Walsh [13]. Indeed, it does only contain one additional term, a "jump drift". Nevertheless, it is far from possible to use a method similar to that of Bally, Millet, Sanz-Solé in [2], who proved a support theorem for Walsh's equation, in particular because the solution of (3.4) does not lie in $\mathbf{C}([0,T] \times [0,1])$.

But the times of jump of the solution X_m of equation (3.4) are deterministic, and the associated skeleton $S(h,m)$ (m is fixed) has the same times of jump. Thus we do not need the Skorokhod topology : we will work with the stronger supremum norm on $[0,T] \times [0,1]$.

The method below consists in applying the result of Bally, Millet, and Sanz-Solé on each time interval $[t_i, t_{i+1}[$. To this end, we will define some processes X_m^i, which equal X_m only on $[t_i, t_{i+1}[\times[0,1]$, but also give information about the behaviour of X_m after t_{i+1}. We will also associate with X_m^i some deterministic skeletons $S_m^i(h)$. But we will apply the result of [2] to the conditional law of X_m^i with respect to $\mathcal{F}_{t_i}$ (for each i). Thus, we will have to define a non-deterministic "conditional skeleton" $T_m^i(h)$. Then we will develop a technical way to "paste the pieces".

Recall that thanks to Remark 2.4, we have to prove on one hand that for all $h \in \mathcal{H}$, all $\epsilon > 0$,

$$P\left(\| X_m - S(h,m) \|_\infty \leq \epsilon\right) > 0 \tag{5.2}$$

and on the other hand that

$$P\left(X_m \in \overline{\{S(h,m)\ ;\ h\in\mathcal{H}\}}^{\|\ \|_\infty}\right) = 1 \tag{5.3}$$

To this aim, we introduce some notations. First, if $S(t,x)$ belongs to $I\!D\left([0,T],\mathbf{C}([0,1])\right)$, and if $0\le u<v\le T$,

$$\| S \|_{[u,v]} = \sup_{t\in[u,v],\ x\in[0,1]} |S(t,x)| \tag{5.4}$$

We now define recursively, for i in $\{0,...,n\}$, the processes $X_m^i(t,x)$ on $[t_i,T]\times[0,1]$:

$$\begin{aligned} X_m^0(t,x) &= G_t(\mathcal{X}_0,x) + \int_0^{t_1\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(X_m^0(s,y))dyds + \\ &\qquad \sigma(X_m^0(s,y))W(dy,ds)\Big] \end{aligned} \tag{5.5}$$

and, for $i\in\{1,...,n\}$,

$$\begin{aligned} X_m^i(t,x) &= X_m^{i-1}(t,x) + 1\!\!1_{\{t\ge t_i\}}\int_0^1 G_{t-t_i}(x,y)g(X_m^{i-1}(t_i-,y),z_i)dy \\ &\quad + \int_{t_i}^{t_{i+1}\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(X_m^i(s,y))dyds + \sigma(X_m^i(s,y))W(dy,ds)\Big] \end{aligned} \tag{5.6}$$

Notice that for all i,

$$\text{for all } t\in[t_i,t_{i+1}[,\quad \text{all } x\in[0,1],\quad X_m^i(t,x) = X_m(t,x) \tag{5.7}$$

Indeed, it suffices to use a standard uniqueness argument. In the same way, we define, for $h\in\mathcal{H}$, the functions $S_m^i(h)$ on $[t_i,T]\times[0,1]$, by

$$\begin{aligned} S_m^0(h)(t,x) &= G_t(\mathcal{X}_0,x) + \int_0^{t_1\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(S_m^0(h)(s,y))dyds \\ &\qquad +\sigma(S_m^0(h)(s,y))\dot{h}(s,y)dyds\Big] \end{aligned} \tag{5.8}$$

and, for $i\in\{1,...,n\}$,

$$\begin{aligned} S_m^i(h)(t,x) &= S_m^{i-1}(h)(t,x) + 1\!\!1_{\{t\ge t_i\}}\int_0^1 G_{t-t_i}(x,y)g(S_m^{i-1}(h)(t_i-,y),z_i)dy \\ &\quad + \int_{t_i}^{t_{i+1}\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(S_m^i(h)(s,y))dyds + \sigma(S_m^i(h)(s,y))\dot{h}(s,y)dyds\Big] \end{aligned} \tag{5.9}$$

Then, for all i,

$$\text{for all } t\in[t_i,t_{i+1}[,\quad \text{all } x\in[0,1],\quad S_m^i(h)(t,x) = S(h,m)(t,x) \tag{5.10}$$

Finally, we define the "conditional skeleton" associated with the conditional law of X_m^i with respect to $\mathcal{F}_{t_i}$:

$$\begin{aligned} T_m^i(h)(t,x) &= X_m^{i-1}(t,x) + 1\!\!1_{\{t\ge t_i\}}\int_0^1 G_{t-t_i}(x,y)g(X_m^{i-1}(t_i-,y),z_i)dy \\ &\quad + \int_{t_i}^{t_{i+1}\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(T_m^i(h)(s,y))dyds + \sigma(T_m^i(h)(s,y))\dot{h}(s,y)dyds\Big] \end{aligned} \tag{5.11}$$

The function $T^i_m(h)$ is defined on $[t_i, T] \times [0,1]$. For all $t \in [t_i, T]$, all $x \in [0,1]$, $T^i_m(h)(t,x)$ is $\mathcal{F}_{t_i}$-measurable.

Then one can "nearly" use the Theorem of Bally, Millet, Sanz-Solé, [2] (see also Cardon-Weber, Millet, [3] for a more general setting), which yields the following result.

Proposition 5.1 *Assume (H) and $(S'1)$. Then, with the above notations, for all $i \in \{0, ..., n\}$, the following conditional support theorem on $[t_i, T] \times [0,1]$ holds :*

$$\operatorname{supp}_{\| \|_{[t_i,T]}} \mathcal{L}\left(X^i_m \mid \mathcal{F}_{t_i}\right) = \overline{\{T^i_m(h) \;/\; h \in \mathcal{H}\}}^{\| \|_{[t_i,T]}} \tag{5.12}$$

In fact, the main theorem in [2] only yields the result for $i = 0$, with $\int_0^t$ instead of $\int_0^{t \wedge t_1}$. But conditioning is not a problem, and the initial values we obtain, for example

$$X^{i-1}_m(t,x) + \int_0^1 G_{t-t_i}(x,y) g(X^{i-1}_m(t_i-, y), z_i) dy$$

$$= X^{i-1}_m(t,x) + G_{t-t_i}\left(g(X^{i-1}_m(t_i-, .), z_i), x\right) \tag{5.13}$$

behave on $[t_i, T]$ exactly as $G_t(\mathcal{X}_0, x)$ on $[0, T]$, since they are $\mathcal{F}_{t_i}$-measurable, since $g(X^{i-1}_m(t_i-, .), z_i)$ is continuous on $[0,1]$, and since $X^{i-1}_m(t,x)$ is continuous on $[0,T] \times [0,1]$. Finally, it is clear that considering the integrals from t_i to $t \wedge t_{i+1}$ instead of 0 to t will not change much...

We now establish a Lemma, which will allow to paste the pieces. If $\| X^i_m(\omega) - S^i_m(h) \|_{[t_i,T]}$ is small, then the initial conditions associated with $S^{i+1}_m(h)$ and $T^{i+1}_m(h)(\omega)$ are near, and thus the distance between $S^{i+1}_m(h)$ and $T^{i+1}_m(h)(\omega)$ is small. We need this Lemma, because Proposition 5.1 gives an idea of the distance between $X^i_m(\omega)$ and $T^i_m(h)(\omega)$, but what we need to control is the distance between $S^i_m(h)$ and $X^i_m(\omega)$.

Lemma 5.2 *Assume (H), $(S'2)$. There exists a function $\gamma(x,u) : I\!R^+ \times I\!R^+ \mapsto I\!R^+$, such that for each x, $\gamma(x,u)$ decreases to 0 when u decreases to 0, and such that for all $\epsilon > 0$, all $i \in \{0, ..., n-1\}$,*

$$\left\{\omega \in \Omega \;\middle/ \| X^i_m(\omega) - S^i_m(h) \|_{[t_i,T]} \leq \epsilon\right\}$$

$$\subset \left\{\omega \in \Omega \;\middle/ \| S^{i+1}_m(h) - T^{i+1}_m(h)(\omega) \|_{[t_{i+1},T]} \leq \gamma(\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}, \epsilon)\right\} \tag{5.14}$$

where $\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|^2_{L^2} = \int_{t_{i+1}}^{t_{i+2}} \int_0^1 \dot{h}^2(s,y) dy ds$.

<u>Proof</u> : Let ω belong to $\{\| X^i_m - S^i_m(h) \|_{[t_i,T]} \leq \epsilon\}$. Then, for all t in $[t_{i+1}, T]$, all x in $[0,1]$, using (H),

$$\left|S^{i+1}_m(h)(t,x) - T^{i+1}_m(h)(t,x)\right| \leq \left|S^i_m(h)(t,x) - X^i_m(t,x)\right|$$

$$+ \int_0^1 G_{t-t_{i+1}}(x,y) \left|g\left(X^i_m(t_{i+1}-, y), z_i\right) - g\left(S^i_m(h)(t_{i+1}-, y), z_i\right)\right| dy$$

$$+ C \int_{t_{i+1}}^{t_{i+2} \wedge t} \int_0^1 G_{t-s}(x,y) \left|S^{i+1}_m(h)(s,y) - T^{i+1}_m(h)(s,y)\right| \left(1 + |\dot{h}(s,y)|\right) dy ds \tag{5.15}$$

We now set

$$F(t) = \sup_{x\in[0,1]} \left| S_m^{i+1}(h)(t,x) - T_m^{i+1}(h)(t,x) \right| \tag{5.16}$$

Using the assumption about ω, assumption $(S'2)$, the Appendix (7.3) and (7.2), and Cauchy-Schwarz's inequality, we get :

$$F(t) \leq \epsilon + \psi_{z_i}(\epsilon) + C\left(1+ \parallel \dot{h}|_{[t_{i+1},t_{i+2}]} \parallel_{L^2}\right)\left(\int_{t_{i+1}}^t F^2(s)\frac{ds}{\sqrt{t-s}}\right)^{\frac{1}{2}} \tag{5.17}$$

where ψ_{z_i} was defined in assumption $(S'2)$. Hence,

$$F^2(t) \leq C\epsilon^2 + C\psi_{z_i}^2(\epsilon) + C\left(1+ \parallel \dot{h}|_{[t_{i+1},t_{i+2}]} \parallel_{L^2}\right)^2 \int_{t_{i+1}}^t F^2(s)\frac{ds}{\sqrt{t-s}} \tag{5.18}$$

Iterating once this formula (see the previous section, inequalities (4.30), (4.31), and (4.32) for more precisions), we obtain the existence of a function γ, satisfying the assumptions of the statement, such that

$$F^2(t) \leq \gamma\left(\parallel \dot{h}|_{[t_{i+1},t_{i+2}]} \parallel_{L^2}, \epsilon\right) + C\left(1+ \parallel \dot{h}|_{[t_{i+1},t_{i+2}]} \parallel_{L^2}\right)^2 \int_{t_{i+1}}^t F^2(s)ds \tag{5.19}$$

Gronwall's Lemma allows to conclude.

In order to simplify the notations, we assume in the sequel that $n=2$, i.e. that

$$m(dt,dz) = \delta_{(t_1,z_1)} + \delta_{(t_2,z_2)} \tag{5.20}$$

1) We fix $h\in\mathcal{H}$, and $\epsilon>0$, and we check that

$$P_0 = P\left(\parallel X_m - S(h,m) \parallel_\infty \leq \epsilon\right) > 0 \tag{5.21}$$

First, using (5.7) and (5.10), we see that

$$P_0 \geq P\Big(\parallel X_m^0 - S_m^0(h) \parallel_{[0,T]} \leq \epsilon/3, \ \parallel X_m^1 - S_m^1(h) \parallel_{[t_1,T]} \leq \epsilon/3, \\ \parallel X_m^2 - S_m^2(h) \parallel_{[t_2,T]} \leq \epsilon/3\Big) \tag{5.22}$$

Noticing that for each i, X_m^i is $\mathcal{F}_{t_{i+1}}$-measurable and $S_m^i(h)$ is deterministic, we obtain, by conditionning our probability with respect to $\mathcal{F}_{t_2}$,

$$P_0 \geq E\Big[\mathbb{1}_{\{\parallel X_m^0 - S_m^0(h) \parallel_{[0,T]} \leq \epsilon/3\}}\mathbb{1}_{\{\parallel X_m^1 - S_m^1(h) \parallel_{[t_1,T]} \leq \epsilon/3\}} \\ \times P\left(\parallel X_m^2 - S_m^2(h) \parallel_{[t_2,T]} \leq \epsilon/3 \,\middle|\, \mathcal{F}_{t_2}\right)\Big] \tag{5.23}$$

On the other hand,

$$P\left(\| X_m^2 - S_m^2(h) \|_{[t_2,T]} \le \epsilon/3 \;\Big|\; \mathcal{F}_{t_2} \right)$$

$$\ge P\left(\| X_m^2 - T_m^2(h) \|_{[t_2,T]} \le \epsilon/6, \;\; \| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \le \epsilon/6 \Big| \; \mathcal{F}_{t_2} \right)$$

$$\ge \mathbb{I}_{\{\| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \le \epsilon/6\}} P\left(\| X_m^2 - T_m^2(h) \|_{[t_2,T]} \le \epsilon/6 \Big| \; \mathcal{F}_{t_2} \right) \quad (5.24)$$

since $S_m^2(h)$ is deterministic and $T_m^2(h)$ is $\mathcal{F}_{t_2}$-measurable. Using Proposition 5.1, we also know that a.s.,

$$P\left(\| X_m^2 - T_m^2(h) \|_{[t_2,T]} \le \epsilon/6 \Big| \; \mathcal{F}_{t_2} \right) > 0 \quad (5.25)$$

Hence, it suffices that $P_1 > 0$, where

$$P_1 = P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \le \epsilon/3, \;\; \| X_m^1 - S_m^1(h) \|_{[t_1,T]} \le \epsilon/3$$

$$\| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \le \epsilon/6 \Big) \quad (5.26)$$

Thanks to Lemma 5.2, we know that for $\alpha > 0$ small enough,

$$\| X_m^1 - S_m^1(h) \|_{[t_1,T]} < \alpha \Longrightarrow \| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \le \epsilon/6 \quad (5.27)$$

Thus,

$$P_1 \ge P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \le \epsilon/3, \;\; \| X_m^1 - S_m^1(h) \|_{[t_1,T]} \le \alpha \wedge \epsilon/3 \Big) \quad (5.28)$$

Iterating this argument, we see that P_0 is strictly positive as soon as $P_2 > 0$, where

$$P_2 = P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \le \beta \Big) \quad (5.29)$$

for some $\beta > 0$ small enough. But it is clear that $S_m^0(h)$ identically equals $T_m^0(h)$. Thus, Proposition 5.1 implies that P_2 is strictly positive, and hence that (5.21) holds, which was our aim.

2) We still have to check that

$$P\left(X_m \in \overline{\{S(h,m)\;,\; h \in \mathcal{H}\}}^{\|\;\|_\infty} \right) = 1 \quad (5.30)$$

We know from Proposition 5.1 that for almost all ω, say for all $\omega \in \bar{\Omega}$, with $P(\bar{\Omega}) = 1$,

$$X_m^0(\omega) \in \overline{\{T_m^0(h),\; h \in \mathcal{H}\}}^{\|\;\|_\infty} \quad ; \quad X_m^1(\omega) \in \overline{\{T_m^1(h)(\omega),\; h \in \mathcal{H}\}}^{\|\;\|_\infty}$$

$$X_m^2(\omega) \in \overline{\{T_m^2(h)(\omega),\; h \in \mathcal{H}\}}^{\|\;\|_\infty} \quad (5.31)$$

We now fix $\omega \in \bar{\Omega}$. There exists $h_n^0 \in \mathcal{H}$, $h_n^1 \in \mathcal{H}$, $h_n^2 \in \mathcal{H}$, (depending on ω) such that, for $i \in \{0,1,2\}$, when n goes to infinity,

$$\| X_m^i(\omega) - T_m^i(h_n^i)(\omega) \|_{[t_i,T]} \longrightarrow 0 \quad (5.32)$$

We now set

$$h_{n,k,q}(t,x) = h_n^0(t,x)\mathbb{1}_{[0,t_1]}(t) + h_k^1(t,x)\mathbb{1}_{[t_1,t_2]}(t) + h_q^2(t,x)\mathbb{1}_{[t_2,T]}(t) \tag{5.33}$$

We fix $\epsilon > 0$, and we prove that for n, k, q large enough,

$$\| X_m(\omega) - S(h_{n,k,q}, m) \|_{[0,T]} \leq \epsilon \tag{5.34}$$

which will suffice. One can easily check, using (5.7) and (5.10), that

$$\| X_m(\omega) - S(h_{n,k,q}, m) \|_{[0,T]} \leq A_n^0(\omega) + A_k^1(\omega) + A_q^2(\omega) + B_n^0(\omega) + B_k^1(\omega) + B_q^2(\omega) \tag{5.35}$$

where (if $i = 0, 1, 2$ and $l \in I\!N$)

$$A_l^i(\omega) = \| X_m^i(\omega) - T_m^i(h_l^i)(\omega) \|_{[t_i,T]} \tag{5.36}$$

and

$$B_l^i(\omega) = \| T_m^i(h_l^i)(\omega) - S_m^i(h_l^i) \|_{[t_i,T]} \tag{5.37}$$

First notice that B_n^0 vanishes identically. Thanks to Lemma 5.2, we know that

$$B_k^1(\omega) \leq \gamma\left(\| \dot{h}_k^1|_{[t_1,t_2]} \|_{L^2}, A_n^0(\omega)\right) \tag{5.38}$$

$$B_q^2(\omega) \leq \gamma\left(\| \dot{h}_q^2|_{[t_2,T]} \|_{L^2}, A_k^1(\omega) + B_k^1(\omega)\right) \tag{5.39}$$

i) First, we choose q large enough, in order that

$$A_q^2(\omega) \leq \epsilon/6 \tag{5.40}$$

Now that q is fixed, we consider $\alpha > 0$ such that

$$\gamma\left(\| \dot{h}_q^2|_{[t_2,T]} \|_{L^2}, \alpha\right) \leq \epsilon/6 \tag{5.41}$$

ii) Then we choose k in such a way that

$$A_k^1(\omega) \leq \epsilon/6 \wedge \alpha/2 \tag{5.42}$$

and we consider $\beta > 0$ such that

$$\gamma\left(\| \dot{h}_k^1|_{[t_1,t_2]} \|_{L^2}, \beta\right) \leq \epsilon/6 \wedge \alpha/2 \tag{5.43}$$

iii) Finally, we choose n such that

$$A_n^0(\omega) \leq \epsilon/6 \wedge \beta \tag{5.44}$$

We deduce from (5.44), (5.38), and (5.43) that

$$B_k^1(\omega) \leq \epsilon/6 \wedge \alpha/2 \tag{5.45}$$

Thanks to (5.45), (5.42), (5.41), and (5.39), we also see that

$$B_q^2(\omega) \leq \epsilon/6 \tag{5.46}$$

Finally, using (5.35), (5.44), (5.42), (5.40), (5.45), (5.46), we deduce (5.34). We thus have checked that for each $\omega \in \bar{\Omega}$, all $\epsilon > 0$, there exists $h \in \mathcal{H}$ such that

$$\| X_m(\omega) - S(h, m) \|_\infty \leq \epsilon \tag{5.47}$$

Since $P(\bar{\Omega}) = 1$, (5.30) holds, and Proposition 3.3 is proved.

6 Extension to the case of an a.s. infinite number of jumps when the diffusion coefficient is constant.

We now consider equation (1.1) in the following new setting : the diffusion coefficient is constant, $\sigma(x) \equiv \sigma$; but the positive measure q on E is only assumed to be σ-finite (*a priori*, $q(E) = \infty$). N is still a Poisson measure on $[0,T] \times E$, with intensity measure $dtq(dz)$. The evolution equation associated to equation (1.1) is still given by (2.5).

We also consider an increasing sequence of subsets E_p of E satisfying

$$q(E_p) < \infty \quad ; \quad \cup_{p \in I\!N} E_p = E \tag{6.1}$$

In order to obtain a result of existence and uniqueness, we state the following hypothesis :

> Assumption (A) : the function σ is constant. The function b satisfy a global Lipschitz condition. There exists $\eta \in L^1(E,q)$ such that for all $x, y \in I\!R$, all $z \in E$,
>
> $$|g(0,z)| \leq \eta(z) \quad ; \quad |g(x,z) - g(y,z)| \leq |x-y|\eta(z) \tag{6.2}$$

Proposition 2.2 yields that equation (1.1) with E_p instead of E admits a unique weak solution X^p lying in $I\!D\left([0,T], \mathbf{C}([0,1])\right)$. Under (A), using strongly the fact that σ is constant, it is easy to check that there exists an adapted process X such that, when p goes to infinity,

$$E\left(\sup_{[0,T]\times[0,1]} |X(t,x) - X^p(t,x)|\right) \longrightarrow 0 \tag{6.3}$$

This way, we obtain the existence of an adapted weak solution X of equation (1.1) with our new setting. See Remark 6.6 for the case where σ is not a constant.
The uniqueness is straightforward under (A), and we can state the following proposition.

Proposition 6.1 *Assume (A). Equation (1.1) admits a unique weak solution $X(t,x)$, lying a.s. in $I\!D\left([0,T], \mathbf{C}([0,1])\right)$, and bounded in L^1.*

We now consider

$$\mathcal{M}_p = \left\{ m(dt,dz) = \sum_{i=1}^{n} \delta_{(t_i,z_i)}(dt,dz) \;\Big/\; \begin{array}{c} n \in I\!N,\ 0 < t_1 < \ldots < t_n < T, \\ z_1, \ldots, z_n \in \text{supp } q \cap E_p \end{array} \right\} \tag{6.4}$$

and we set $\mathcal{M} = \cup_p \mathcal{M}_p$. The Cameron-Martin space $\mathcal{H}$ associated with W is still defined by (2.13). For each $m \in \mathcal{M}$ and $h \in \mathcal{H}$, we denote by $S(h,m)$ the unique solution of equation (2.15) (there is no difference with Proposition 2.3, since there exists p such that $m \in \mathcal{M}_p$). Since g is already Lipschitz, we assume (T) below instead of $(S2)$,

> Assumption (T) : for each $z_0 \in E$, each $n \in I\!N$,
>
> $$\sup_{|x| \leq n} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{6.5}$$
>
> For each z_0 in E, there exists $\xi(z_0) > 0$ such that
>
> $$\sup_{d(z,z_0) \leq \xi(z_0)} \eta(z) < \infty \tag{6.6}$$

A function $g(x,z) = \alpha(z)\eta(z)$ clearly satisfies (A) and (T) if α is lipschitz, and $\eta \in L^1(E,q)$ is continuous. The aim of this section is to prove the following result.

Theorem 6.2 *Under (A) and (T), if X denotes the unique weak solution of equation (1.1),*

$$\mathrm{supp}_\delta\, P \circ X^{-1} = \overline{\{S(h,m)\ /\ h \in \mathcal{H}\ ,\ m \in \mathcal{M}\}}^\delta \tag{6.7}$$

Since the method of Simon [10], combined with the previous sections, applies easily, we will only sketch the Proof.

First, for the same reasons as in the previous sections, see Proposition 3.1, we can assume, additionally to (A) and (T), that for all $x \in I\!R$, all $z \in E$, $|g(x,z)| \leq \eta(z)$ and for each $z_0 \in E$,

$$\sup_{x \in I\!R} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{6.8}$$

Then, we notice that the direct inclusion $(\subset)$ of Theorem 6.2 is immediate, thanks to Theorem 2.5 (for X^p) and thanks to the convergence (6.3).

We now fix $p \in I\!N$, $h \in \mathcal{H}$, $m = \sum_{i=1}^n \delta_{(t_i,z_i)} \in \mathcal{M}_p$, and $\epsilon > 0$. We have to prove that

$$P\left(\delta(X, S(h,m)) \leq \epsilon\right) > 0 \tag{6.9}$$

To prove this, we will use three lemmas. The first one is a very particular case of the result of Bally, Millet, Sanz, [2].

Lemma 6.3 *Let $\alpha > 0$ be fixed, and let*

$$\Omega_0(\alpha) = \left\{\omega \in \Omega \ \Big/ \sup_{t,x} \left| \int_0^t \int_0^1 G_{t-s}(x,y)\left\{W(dy,ds) - \dot{h}(s,y)dyds\right\}\right| \leq \alpha \right\} \tag{6.10}$$

Then $P(\Omega_0(\alpha)) > 0$.

We now write the restriction $N^p = N|_{[0,T]\times E_p}$ (recall that p is fixed) as

$$N^p(ds,dz) = \sum_{i=1}^{\mu} \delta_{(T_i,Z_i)}(ds,dz) \tag{6.11}$$

The second lemma can be proved by using the same method as that of Proposition 3.2 (see Section 4). The only difference comes from the fact that $X^p(\omega)$ depends on W, but since σ is constant, Lemma 6.3 allows to deal easily with this problem.

Lemma 6.4 *Let $\beta > 0$ be fixed. There exists a set*

$$\Omega_1(\beta) \in \sigma\left\{N(A)\ ;\ A \in \mathcal{B}([0,T]\times E_p)\right\} \tag{6.12}$$

such that $P(\Omega_1(\beta)) > 0$, such that for each $\omega \in \Omega_1(\beta)$,

$$\mu(\omega) = n \quad ;\forall i, \qquad d(z_i, Z_i(\omega)) \leq \xi(z_i) \tag{6.13}$$

and such that for some $\alpha > 0$ small enough, every $\omega \in \Omega_0(\alpha) \cap \Omega_1(\beta)$ satisfies

$$\delta(X^p(\omega), S(h,m)) \leq \beta \tag{6.14}$$

We will finally use the following result :

Lemma 6.5 *Let $\gamma > 0$ be fixed, and let*

$$\Omega_2(\gamma) = \left\{ \omega \in \Omega \ \Big/ \int_0^T \int_{E/E_p} \eta(z) N(ds,dz) \leq \gamma \right\} \tag{6.15}$$

Then $P(\Omega_2(\gamma)) > 0$.

<u>Proof of Lemma 6.5</u> : we set

$$\theta_p = \int_0^T \int_{E/E_p} \eta(z) N(ds,dz) \tag{6.16}$$

and, for $q > p$,

$$\theta_p^q = \int_0^T \int_{E_q/E_p} \eta(z) N(ds,dz) \tag{6.17}$$

We see that $\theta_p = \theta_p^q + \theta_q$, that for all q, $P(\theta_p^q = 0) > 0$, and that when q goes to infinity, θ_q goes to 0 in probability. Since for each q, θ_p^q is independent of θ_q, we can write

$$P(\theta_p \leq \gamma) \geq P(\theta_p^q = 0) P(\theta_q \leq \gamma) \tag{6.18}$$

and the lemma follows easily.

We finally sketch the proof of Theorem 6.2. An easy independance argument yields that for every $\alpha > 0$, $\beta > 0$, $\gamma > 0$, the set

$$\Omega_3(\alpha,\beta,\gamma) = \Omega_0(\alpha) \cap \Omega_1(\beta) \cap \Omega_2(\gamma) \tag{6.19}$$

has a strictly positive probability. We now have to to choose α, β, γ in such a way that for all $\omega \in \Omega_3(\alpha,\beta,\gamma)$,

$$\delta(X(\omega), S(h,m)) \leq \epsilon \tag{6.20}$$

Let $\omega \in \Omega_3(\alpha,\beta,\gamma)$ be fixed. If α is small enough, we know from Lemma 6.4 that

$$\begin{aligned} \delta(X(\omega), S(h,m)) &\leq \| X(\omega) - X^p(\omega) \|_\infty + \delta(X^p(\omega), S(h,m)) \\ &\leq \| X(\omega) - X^p(\omega) \|_\infty + \beta \end{aligned} \tag{6.21}$$

We now set

$$V^p(t) = \sup_{x \in [0,1]} |X(t,x) - X^p(t,x)| \tag{6.22}$$

Using the Appendix, (A), (T), since $|g(x,z)| \leq \eta(z)$, and since ω belongs to $\Omega_3(\alpha,\beta,\gamma)$, we see that

$$\begin{aligned} V^p(t) &\leq C \int_0^t V^p(s) ds + C \sum_{i=1}^n 1_{\{t \geq T_i\}} V^p(T_i-) \eta(Z_i) \\ &\quad + \int_0^T \int_{E \setminus E_p} \eta(z) N(ds,dz) \\ &\leq C \int_0^t V^p(s) ds + C \sum_{i=1}^n 1_{\{t \geq T_i\}} V^p(T_i-) + \gamma \end{aligned} \tag{6.23}$$

For the second term, we have used (6.6) in Assumtion (T), and the fact that for all i, Z_i belongs to $\{z \in E\ ,\ d(z_i,z) \leq \xi(z_i)\}$.
Applying successively Gronwall's Lemma on the time intervals $[0,T_1[$, ..., $[T_n,T]$, we deduce that for all $\omega \in \Omega_3(\alpha,\beta,\gamma)$,

$$\sup_{[0,T]} V^p(\omega,t) \leq C\gamma \tag{6.24}$$

The conclusion follows easily.

Remark 6.6 *Of course, we are also interested in the case where $q(E) = \infty$ and σ is a function. In this case, it is possible to prove (under assumptions) that the sequence X_p of weak solutions of (1.1) where we have replaced E by E_p, converges to an adapted process $X(t,x)$ in the following sense :*

$$\sup_{t,x} E\left(|X(t,x) - X^p(t,x)|\right) \longrightarrow 0 \tag{6.25}$$

Once X is built, it might be possible to check that it admits a modification lying in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$, by using the fact that X satisfies the evolution equation, but this is not immediate. If so, it seems natural to think that our support theorem extends to this case. However, everything will become much more technical. In particular, the direct inclusion is not obvious any more, since (6.3) does not seem to hold any more.

7 Appendix

We collect here well-known estimates about the Green kernel $G_t(x,y)$ associated with the deterministic system (2.1), and which has the expression (2.2). In all the inequalities below, the constant C depends only on the terminal time value T. The three first estimates can be found in [13], and the next ones are either easy consequences or classical estimates.

First, for all $x,y \in [0,1]$ and all $t \in [0,T]$,

$$\frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\} \leq G_t(x,y) \leq \frac{C}{\sqrt{t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\} \tag{7.1}$$

For all $0 < t < T$, all $x \in [0,1]$,

$$\int_0^1 G_t^2(x,y)dy \leq \frac{C}{\sqrt{t}} \tag{7.2}$$

and

$$\int_0^1 G_t(x,y)dy = 1 \tag{7.3}$$

For all $0 < s < t < T$, all $x,y \in [0,1]$, (see Lemma A3 in [4])

$$|G_t(x,y) - G_s(x,y)| \leq C\frac{|t-s|}{s^{\frac{3}{2}}} \tag{7.4}$$

and (see Lemma B1 in [2])

$$\int_0^s \int_0^1 \left(G_{t-r}(x,y) - G_{s-r}(x,y)\right)^2 dydr + \int_s^t \int_0^1 G_{t-r}^2(x,y)dydr \leq C\sqrt{t-s} \tag{7.5}$$

Finally, for all $\phi \in \mathbf{C}([0,1])$, the map

$$(t,x) \mapsto G_t(\phi,x) \tag{7.6}$$

is continuous on $[0,T] \times [0,1]$ (see Lemma A2 in [2] for a similar result).

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