Monte-Carlo approximations and fluctuations for 2D Boltzmann equations without cutoff

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Abstract

Using the main ideas of Tanaka [18], the measure solution $\{P_t\}_t$ of a 2-dimensional spatially homogeneous Boltzmann equation of Maxwellian molecules without cutoff is related to a Poisson-driven nonlinear stochastic differential equation. Using this tool and a generalized law of large numbers, we present two ways to prove the convergence of the empirical measure associated with an interacting particle system to this measure solution of the Boltzmann equation. Then we give numerical results. We finally discuss about a central limit theorem associated with the above law of large numbers.

Key words : Boltzmann equations without cutoff, Stochastic differential equations, Jump measures, Interacting particle systems, Fluctuation theorems.

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1 Introduction.

Our aim in this paper is to introduce a probabilistic interpretation of 2-dimensional Boltzmann equations without cutoff in order to define a simple Monte-Carlo algorithm for the simulation of the solutions of these equations. We prove the convergence of the empirical measure of underlying interacting stochastic particle systems to this solution, and obtain a central limit theorem describing the rate of the previous approximation. In this setting, deterministic particle methods seem difficult to develop, whereas this stochastic particle method is very natural in a probabilistic point of view and easy to implement.

We consider here spatially homogeneous Boltzmann equations of Maxwellian molecules without cutoff in $I\!R^2$. Following Fournier [8], we describe a stochastic pure jump process, for which the time-marginals flow is measure-solution of the equation. Because of the possible explosions of the jump measure, which is just assumed to have a second order moment,

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this process is defined as a solution of a nonlinear stochastic differential equation driven by a compensated Poisson measure, and we will work in L^2 spaces. The nonlinearity is modeled by adding an auxiliary probability space.
This framework is a generalization in dimension 2 of Desvillettes-Graham-Méléard [4] concerning a Kac equation without cutoff inspired by an original idea of representation due to Tanaka [18]. In [18], the jump measure has a one-order moment and one can directly use Poisson point processes.

As in [4], we approximate the law of the stochastic process by simulable interacting particle systems, proving a generalized law of large numbers on a path space. We use a result of Graham-Méléard [9] who discuss this problem in a general context, but in a cutoff case. We consider first cutoff approximations of our model and associate with each cutoff equation some cutoff approximating interacting particle systems. The cutoff model converges to the model without cutoff with an easily computable rate of convergence. Then one chooses a sequence of cutoff approximations indexed by the size n of the approximating particle systems and converging to 0 as n tends to infinity in a good asymptotics such that finally the empirical measures of the particle systems converge to the solution of the Boltzmann equation without cutoff. We describe the algorithm related with this convergence and give some numerical results. We observe that for the moment of order 4, a uniform (in the cutoff parameter) central limit theorem holds.

In the last part of the paper, we prove that the fluctuation processes related to the particle systems with cutoff converge in law, when the size of the system tends to infinity, to a generalized Ornstein-Uhlenbeck process in an adapted functional space. This is inspired by Ferland, Fernique, Giroud, [5], and by Méléard [13]. We finally discuss about possible uniform controls of these limiting processes, not depending on the cutoff parameter of the equation. By this way, we prove rigorously the numerical observations.

Notations
K will denote a constant which may change from line to line.
For a Polish space E, $\mathcal{P}(E)$ will denote the space of probability measures on E. It will be endowed with the topology of the weak convergence. $\mathcal{P}_2(E)$ will be the subspace of probability measures with a second order moment.
If V is a random variable on a probability space, $\mathcal{L}(V)$ will denote the law of V.

2 A probabilistic interpretation of the 2D Boltzmann equation.

We recall here the main results obtained by Fournier in his thesis [7], see also [8], concerning a probabilistic interpretation of the 2-dimensional spatially homogeneous Boltzmann equation without cutoff for Maxwellian molecules.
The Boltzmann equation we consider describes the evolution of the density $f(t,v)$ of particles with velocity $v \in I\!R^2$ at time t in a rarefied gas:

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{2.1}$$

where Q is a quadratic collision kernel acting on the variables t and v, preserving momentum and kinetic energy, of the form

$$Q(f,f)(t,v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta=-\pi}^{\pi} \Big(f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \Big) \beta(|v-v_*|,\theta) \sin\theta \, d\theta dv_*$$

with

$$v' = v + A(\theta)(v - v_*) \ ; \ v'_* = v - A(\theta)(v - v_*) \tag{2.2}$$

and

$$A(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}$$

Remark 2.1 *For each $\theta \in [-\pi,\pi] \setminus \{0\}$,*

$$|A(\theta)| \leq K|\theta| \tag{2.3}$$

$$|A(\theta) - A(\varphi)| \leq K|\theta - \varphi|. \tag{2.4}$$

The cross section β is an even and positive function. If the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geq 2$, then $\beta(z,\theta) = z^{\frac{s-5}{s-1}} d(|\theta|)$ where $d \in L^\infty_{loc}(]0,\pi])$ and $d(\theta)\sin\theta \sim K(s)\theta^{-\frac{s+1}{s-1}}$ when θ goes to zero, for some $K(s) > 0$. Physically, this explosion comes from the accumulation of grazing collisions. This equation seems very difficult to study and we will restrict our attention to the case of Maxwellian molecules for which the cross section $\beta(z,\theta)\sin\theta = \beta(\theta)$ only depends on θ and is even. The only condition we assume, following the physical behaviour, is that

$$\int_0^\pi \theta^2 \beta(\theta) d\theta < +\infty. \tag{2.5}$$

Equation (2.1) has to be understood in a weak sense. By a standard integration by parts, we define a solution f as satisfying for each $\phi \in C^2_b(\mathbb{R}^2)$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^2} f(t,v)\phi(v) dv = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \int_{-\pi}^{\pi} (\phi(v') - \phi(v)) \beta(\theta) d\theta f(t,v) dv f(t,v_*) dv_*.$$

But here the RHS term may explode, since the function β may have an infinite mass on $[0,\pi]$. Thus we have to compensate this term. Finally, we will use the following definition of the solutions of (2.1).

Definition 2.2 *We say that a probability measure flow $(P_t)_t$ is a measure-solution of the Boltzmann equation (2.1) if for each $\phi \in C^2_b(\mathbb{R}^2)$*

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_0^t \langle K^\phi_\beta(v,v^*), P_s(dv)\, P_s(dv^*) \rangle \, ds, \tag{2.6}$$

where K^ϕ_β is defined in the compensated form

$$K^\phi_\beta(v,v*) = -b(v-v_*).\nabla\phi(v) \tag{2.7}$$

$$+ \int_{-\pi}^{\pi} \Big(\phi(v + A(\theta)(v-v_*)) - \phi(v) - A(\theta)(v-v_*).\nabla\phi(v) \Big) \beta(\theta) d\theta$$

and where

$$b = \int_{-\pi}^{\pi} (1 - \cos\theta)\beta(\theta)d\theta. \tag{2.8}$$

The following result can be deduced from Toscani-Villani, [19], Theorem 5.

Theorem 2.3 *Assume that P_0 is a probability measure on $\mathbb{R}^2$ admitting a moment of order 2, and that β is a cross section satisfying $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$. The uniqueness of a measure solution holds for the Boltzmann equation* (2.6).

We thus deduce the following important remark.

Remark 2.4 *Thanks to Theorem 2.3, we are sure that in the case where the existence of a weak function-solution $f(t,v)$ to (2.1) holds for $P_0 \in \mathcal{P}_2(\mathbb{R}^3)$, the measure-solution P_t we will study in the sequel is given by $P_t(dv) = f(t,v)dv$.*

We have conservation of mass in (2.6), which leads to a probabilistic approach. We consider (2.6) as the evolution equation of the flow of marginals of a Markov process for which the law is defined by a martingale problem.

Definition 2.5 *Let β be a cross section such that $\int_0^\pi \theta^2\beta(\theta)d\theta < +\infty$ and P_0 in $\mathcal{P}_2(\mathbb{R}^2)$. We say that $P \in \mathcal{P}(\mathbb{D}(\mathbb{R}+, \mathbb{R}^2))$ solves the nonlinear martingale problem $MP(\beta, P_0)$ starting at P_0 if for V the canonical process under P, the law of V_0 is P_0 and for any $\phi \in C_b^2(\mathbb{R}^2)$,*

$$\phi(V_t) - \phi(V_0) - \int_0^t \langle K_\beta^\phi(V_s, v_*), P_s(dv_*)\rangle ds \tag{2.9}$$

is a square-integrable martingale. Here, the nonlinearity appears through P_s which denotes the marginal of P at time s.

Remark 2.6 *If P is a solution of $MP(\beta, P_0)$, then its marginal flow $\{P_t\}_t$ is a measure-solution of the Boltzmann equation, in the sense of Definition (2.2).*

1) The simple case of a Boltzmann equation with cutoff

We first consider in this part the simpler so-called cutoff case for which $\int_0^\pi \beta(\theta)d\theta < +\infty$.

Theorem 2.7 *Let β be a cross section such that $\|\beta\|_1 = \int_0^\pi \beta(\theta)d\theta < +\infty$ and $P_0 \in \mathcal{P}(\mathbb{R})$. There exists a unique solution P^β to the nonlinear martingale problem $MP(\beta, P_0)$. Its flow of time-marginals $(P_t^\beta)_{t\geq 0}$ is the unique (probability measure flow) solution of the equation (2.6).*

Proof. The proof is standard and is almost contained in [4] which concerns the one-dimensional case, and follows Shiga-Tanaka, [15], Lemma 2.3. Since β is in $L^1([0,\pi])$ and is even, the jump operator has the simpler form: $\forall \phi \in C_b^2(\mathbb{R}^3)$,

$$K_\beta^\phi(v, v*) = \int_{-\pi}^{\pi} (\phi(v + A(\theta)(v - v_*)) - \phi(v))\,\beta(\theta)d\theta \tag{2.10}$$

and moreover for any flow $(Q_t)_t$ in $I\!D(I\!R+, \mathcal{P}(I\!R^2))$

$$\phi \in L^\infty(I\!R^2) \Rightarrow \langle K_\beta^\phi(.,v*), Q_s(dv_*)\rangle \in L^\infty(I\!R^2). \tag{2.11}$$

Then the operator $\phi \mapsto \langle K_\beta^\phi(.,v*), Q_s(dv_*)\rangle$ is a pure-jump Markov operator generating a unique law P^Q in $\mathcal{P}(I\!D(I\!R+, I\!R^2))$ starting at P_0. Its flow of marginals satisfies a linear evolution equation: for $\phi \in L^\infty(I\!R^2)$,

$$\langle \phi, P_t^Q\rangle = \langle \phi, P_0\rangle + \int_0^t \langle K_\beta^\phi(v,v^*), P_s^Q(dv)\, Q_s(dv^*)\rangle\, ds. \tag{2.12}$$

Let $|\mu| = \sup\{\langle\phi,\mu\rangle : \|\phi\|_\infty \leq 1\}$ denote the variation norm. For $(Q_t^i)_{t\geq 0}$, $i = 1,2$, and the corresponding solutions $(P_t^{Q^i})_{t\geq 0}$, one proves easily that

$$|P_t^{Q^1} - P_t^{Q^2}| \;\leq\; 4\pi\|\beta\|_1 \int_0^t (|P_s^{Q^1} - P_s^{Q^2}| + |Q_s^1 - Q_s^2|)\, ds \tag{2.13}$$

and a standard fixed point argument gives the result. □

By adapting the proof of Desvillettes, [3], Theorem A.1, one can prove

Theorem 2.8 *Let us assume that $\int_0^\pi \beta(\theta)d\theta < +\infty$. Let $f_0 \geq 0$ be an initial density datum such that $\int_{I\!R^3} f_0(v)(1+|v|^2)dv < +\infty$. Then there exists a unique density solution $f^\beta(t,v)$ of (2.1) in $L^\infty(I\!R+, L^1(I\!R^2,(1+|v|^2)dv))$ with initial datum f_0. This solution satisfies the conservation of momentum and energy.*

Using Remark 2.4, we see that under the assumptions of Theorem 2.8, the solution P^β of the martingale problem with initial distribution $P_0(dv) = f_0(v)dv$ satisfies, for each $t > 0$, $P_t^\beta(dv) = f^\beta(t,v)dv$, where f^β is defined in Theorem 2.8.

2) The case of a Boltzmann equation without cutoff

In this case without cutoff, the existence and uniqueness of the nonlinear martingale problem (2.9) is not so natural as in the cutoff case. In order to prove the existence, we associate with this martingale problem a nonlinear stochastic differential equation on a greater probability space. We recall here the results obtained by Fournier [8] concerning the existence of a solution of this SDE on each finite-time interval $[0,T]$, proved by a generalized Picard-iteration and giving as corollary the existence and uniqueness of a solution of the nonlinear martingale problem.

We consider now two probability spaces : the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted E_α, and the laws $\mathcal{L}_\alpha$.
If Q is a probability on $I\!D_T = I\!D([0,T], I\!R^2)$, we will say that $Q \in \mathcal{P}_2(I\!D_T)$ if $\int_{x\in I\!D_T} \sup_{[0,T]} |x(t)|^2 Q(dx) < \infty$. A càdlàg adapted process Y_s on $[0,T]$ will be said to be a L_T^2-process if its law belongs to $\mathcal{P}_2(I\!D_T)$.

Notation 2.9 *Let V_0 be a $\mathbb{R}^2$-valued $\mathcal{F}_0$-measurable random variable, let N be a $\{\mathcal{F}_t\}$-Poisson measure on $[0,T] \times [0,1] \times [-\pi,\pi]$. Let Y be a L^2_T-process and W a L^2_T-α-process, and let $b \in \mathbb{R}$. Then we denote by $X = F(V_0, Y, W, N, b)$ the L^2_T-process defined by*

$$\begin{aligned} X_t(\omega) &= V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(Y_{s-}(\omega) - W_{s-}(\alpha))\tilde{N}(d\theta, d\alpha, ds) \\ &\quad -\frac{b}{2}\int_0^t \int_0^1 (Y_s(\omega) - W_s(\alpha))\, d\alpha ds. \end{aligned} \tag{2.14}$$

where $\tilde{N}(d\theta, d\alpha, ds)$ is the compensated Poisson measure associated with N.

Definition 2.10 *We will say that (V, W, N) is a solution of $SDE(\beta, P_0)$ if V is a L^2_T-process with $\mathcal{L}(V_0) = P_0$, W is a L^2_T-α-process such that $\mathcal{L}_\alpha(W) = \mathcal{L}(V)$, N is a $\{\mathcal{F}_t\}$-Poisson measure on $[0,T] \times [0,1] \times [-\pi,\pi]$ with intensity measure $\beta(\theta)d\theta d\alpha ds$, and $V = F(V_0, V, W, N, b)$, where $b = \int_{-\pi}^{\pi}(1 - \cos\theta)\beta(\theta)d\theta$. That means*

$$\begin{aligned} V_t(\omega) &= V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\left(V_{s-}(\omega) - W_{s-}(\alpha)\right)\tilde{N}(d\theta, d\alpha, ds) \\ &\quad -\frac{b}{2}\int_0^t \int_0^1 (V_s(\omega) - W_s(\alpha))\, d\alpha ds \;;\; \mathcal{L}_\alpha(W) = \mathcal{L}(V). \end{aligned} \tag{2.15}$$

Remark 2.11 *If (V, W, N) is a solution of $SDE(\beta, P_0)$, then $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ is a solution of $MP(\beta, P_0)$, and thus $\{\mathcal{L}(V_s)\}_{s\in[0,T]}$ is the unique measure-solution of* (2.6). *(The uniqueness is recalled in Theorem 2.3).*

Theorem 2.12 *(proved in [8]) Assume that P_0 is a probability measure on $\mathbb{R}^2$ admitting a moment of order 2, and that β is a cross section satisfying $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$. Then*
1) $SDE(\beta, P_0)$ admits a solution (V, W, N) and the law $P^\beta = \mathcal{L}(V) = \mathcal{L}_\alpha(W)$ is unique.
2) $MP(\beta, P_0)$ admits a unique solution, which is given by P^β.

Remark 2.13 *If P_0 is not a Dirac measure and has finite moments of all orders and under some assumption on β, it is also proved in [8] that for each $t >$, P^β_t has a density function of class C^∞, which will then be the unique weak function-solution of the Boltzmann equation. What is interesting is that P_0 can be degenerated; the regularity of the solution, as soon as $t > 0$, is due to the explosions of the jump measure, related to an accumulation of small jumps of the underlying process. The proof uses a stochastic calculus of variations (Malliavin calculus).*

3 Stochastic Approximations for the Boltzmann equation with cutoff.

3.1 The interacting particle systems

Still under the cutoff assumption $\int_0^\pi \beta(\theta)d\theta < +\infty$, and following (2.10), it is natural to introduce mean-field interacting particle systems which approximate in a Monte-Carlo sense the nonlinear martingale problem (2.9). The natural interpretation of the nonlinearity in

(2.9) leads to a simple mean field interacting system but a physical interpretation of the equation leads also naturally to a binary mean field interacting particle system.
In both cases, these n-particles systems are pure-jump Markov processes with values in $(I\!R^2)^n$ and with generators defined for $\phi \in C_b((I\!R^2)^n)$ by

$$\frac{1}{n-1}\sum_{1\leq i\neq j\leq n}\int_{-\pi}^{\pi}\Big(\phi(v^n+\mathbf{e_i}.(A(\theta)(v_i-v_j)))-\phi(v^n)\Big)\beta(\theta)d\theta. \tag{3.1}$$

for the simple mean-field interacting particle system and by:

$$\frac{1}{n-1}\sum_{1\leq i\neq j\leq n}\int_{-\pi}^{\pi}\frac{1}{2}\Big(\phi(v^n+\mathbf{e_i}.(A(\theta)(v_i-v_j))+\mathbf{e_j}.(A(\theta)(v_j-v_i)))-\phi(v^n)\Big)\beta(\theta)d\theta. \tag{3.2}$$

for the binary mean-field interacting particle system. In these formula, $v^n=(v_1,...,v_n)$ denotes the generic point of $(I\!R^2)^n$ and $\mathbf{e_i}: h\in I\!R^2 \mapsto \mathbf{e_i}.h=(0,...,0,h,0,...,0)\in(I\!R^2)^n$ with h at the i-th place.
Both cases can be treated indifferently in a probabilistic point of view. The first particle system can be refered the Nanbu algorithm (cf. [14], [1]) and is as simple as possible. The second one can be related to the Bird algorithm (cf. [20]). Its main interest is that it conserves momentum and kinetic energy. Moreover, a set of numerical experiences (see Subsection 4.3) shows that it looks better. Since it is also slightly more rapid, we consider, from now on, only **the binary mean-field systems**, even if in a theorical point of view, the simple mean-field systems have a similar behaviour.

We denote by $V^{\beta,n}=(V^{\beta,1n},...,V^{\beta,nn})$ the Markov process defined by (3.2) and initial i.i.d. data (V_0^i). Then for each n,i,

$$\begin{aligned} M_t^{\beta,i\phi} &= \phi(V_t^{\beta,in})-\phi(V_0^i) \qquad (3.3)\\ &-\frac{1}{n}\sum_{j=1}^n\int_0^t\int_{-\pi}^{\pi}\Big(\phi(V_s^{\beta,in}+A(\theta)(V_s^{\beta,in}-V_s^{\beta,jn}))-\phi(V_s^{\beta,in})\Big)\beta(\theta)d\theta ds\end{aligned}$$

is a martingale, with Doob-Meyer process given by

$$< M^{\beta,i\phi}>_t=\frac{1}{n}\sum_{j=1}^n\int_0^t\int_{-\pi}^{\pi}\Big(\phi(V_s^{\beta,in}+A(\theta)(V_s^{\beta,in}-V_s^{\beta,jn})))-\phi(V_s^{\beta,in})\Big)^2\beta(\theta)d\theta ds, \tag{3.4}$$

and for $i\neq j$,

$$\begin{aligned} < M^{\beta,i\phi},M^{\beta,j\phi}>_t &= \frac{1}{n}\int_0^t\int_{-\pi}^{\pi}\Big(\phi(V_s^{\beta,in}+A(\theta)(V_s^{\beta,in}-V_s^{\beta,jn})))-\phi(V_s^{\beta,in})\Big)\\ &\Big(\phi(V_s^{\beta,jn}+A(\theta)(V_s^{\beta,jn}-V_s^{\beta,in}))\Big)\beta(\theta)d\theta ds.\end{aligned}$$

Finally, by summing (3.3) over i and if

$$\mu^{\beta,n}=\frac{1}{n}\sum_{i=1}^n\delta_{V^{\beta,in}} \tag{3.5}$$

denotes the empirical measure of the system, the flow of marginals $(\mu_t^{\beta,n})_{t\geq 0}$ is a semi-martingale in the sense where for each bounded function ϕ,

$$\begin{aligned} < \phi, \mu_t^{\beta,n} > &= < \phi, \mu_0^{\beta,n} > + M_t^{\phi,n} \\ &+ \int_0^t \Big\langle \int_{-\pi}^{\pi} (\phi(v + A(\theta)(v - v_*)) - \phi(v))\beta(\theta)d\theta, \mu_s^{\beta,n}(dv)\mu_s^{\beta,n}(dv_*) \Big\rangle \, ds \end{aligned} \tag{3.6}$$

where $M^{\phi,n}$ is a square integrable martingale whose Doob-Meyer process

$$\begin{aligned} < M^{\phi,n} >_t &= \frac{1}{n}\int_0^t \Big\langle \frac{1}{2}\int_{-\pi}^{\pi} \Big(\phi(v + A(\theta)(v - v_*)) - \phi(v) \\ &\quad + \phi(v_* + A(\theta)(v_* - v)) - \phi(v_*)\Big)^2 \beta(\theta)d\theta, \mu_s^{\beta,n}(dv)\mu_s^{\beta,n}(dv_*)\Big\rangle ds. \end{aligned}$$

The pathwise representation

In this cutoff case, we could obtain a representation of the interacting systems in terms of solutions of SDE's driven by Poisson point measures (without compensation), but in view of what follows, we consider as in the previous section a pathwise representation of these processes using compensated Poisson point measures.
We introduce a family of independent Poisson point measures $(N^{ij})_{1\leq i<j\leq n}$ on $[0,T] \times [-\pi,\pi]$ with intensities $\frac{1}{2}\beta(\theta)d\theta ds$ and their compensated parts $(\tilde{N}^{ij})_{1\leq i,j\leq n}$. For $i > j$, we set $N^{ij} = N^{ji}$. Now consider the processes $V^{\beta,in}$ solutions of

$$\begin{aligned} V_t^{\beta,in} &= V_0^{\beta,in} + \frac{1}{n-1}\sum_{j\neq i=1}^{n} \int_0^t \int_{-\pi}^{\pi} \Big(A(\theta)(V_{s-}^{\beta,in} - V_{s-}^{\beta,jn})\Big)\, \tilde{N}^{ij}(d\theta, ds) \\ &\quad - \frac{b}{2(n-1)}\sum_{j\neq i=1}^{n} \int_0^t (V_s^{\beta,in} - V_s^{\beta,jn})ds. \end{aligned} \tag{3.7}$$

3.2 The asymptotics results

We are here in a standard case for which many studies have been done. The specific asymptotic behaviour one obtains in this case is called *propagation of chaos*. It means that the independance of the initial laws propagate: the coordinates of each finite subsystem of the particle system tend to become independent as the size of the system tends to infinity, with common law P^β. This result, due to the mean field interaction, is equivalent to the convergence of the empirical measures of the system to P^β and is then a generalized law of large numbers. Usually the proof follows a standard criterion of tightness-uniqueness, as it can be found in Méléard [12], Section 4, and as it will be developed below. The convergence is then understood as a convergence in law, in the path space $I\!D([0,T], I\!R^2)$. Here, because the dynamics is just a jump dynamics, one can prove a stronger approximation result, due to Graham-Méléard [9] Theorem 3.1. For a given $T > 0$, let us denote by $|.|_T$ the total variation norm in the space of signed measures on $I\!D([0,T], I\!R^2)$. Then we have a propagation of chaos result in variation norm.

Theorem 3.1 *Let $(V_0^i)_{i\geq 1}$ be i.i.d. P_0-distributed random variables. For given $T > 0$ and $k \in I\!N^*$,*

$$|\mathcal{L}(V^{\beta,1n}, ..., V^{\beta,kn}) - (P^\beta)^{\otimes k}|_T \leq K_k \frac{\exp(\|\beta\|_1 T)}{n},$$

with $K_1, K_2 = 6$ and $K_k = 2k(k-1)$ for $k > 2$.

This theorem is proved by using a pathwise representation of a particle during a finite time-interval $[0,T]$, obtained on a random graph which describes the interaction past of this particle. One associates with a couple of particles their random graph and compare it by a coupling to the random graphs we would obtain if the particles were independent. Some computations using the collision time Poisson laws allow us to quantify the probability of a difference between these graphs. The limit law can also be represented on a random tree and the proof consists also in coupling the random graph with the random tree.
The following corollary will be the basis for the numerical approximations in the sequel.

Corollary 3.2 *The empirical measure* $\mu^{\beta,n} = \frac{1}{n}\sum_{i=1}^{n}\delta_{V^{\beta,in}}$ *converges in probability in* $\mathcal{P}(\mathbb{D}([0,T],\mathbb{R}^2))$ *to* P^β*, with the rate* $\sqrt{\frac{\exp(\|\beta\|_1 T)}{n}}$*. (The space* $\mathcal{P}(\mathbb{D}([0,T],\mathbb{R}^2))$ *is endowed with the weak topology for the Skorohod metric on* $\mathbb{D}([0,T],\mathbb{R}^2)$*).*

Then we can prove that under the assumptions of Theorem 2.8, the empirical measure at time t converges for each fixed t to the function $f^\beta(t,.)$ solution of the Boltzmann equation with cutoff. We now consider the case without cutoff which is the original part of the paper.

4 Stochastic particle approximations for the Boltzmann equation without cutoff

4.1 A strong Approach

Convergence of some cutoff approximations. Let us consider the cutoff cross sections β_l defined for $l \geq 1$ by

$$\beta_l(\theta) = \beta(\theta)1_{[\frac{1}{l},\pi]}(|\theta|).$$

We are interested in the convergence, when l tends to infinity, of the solution P^{β_l} of the martingale problem on $\mathbb{D}([0,T],\mathbb{R}^2)$ with cross section β_l to that of the martingale problem P^β with cross section β on $\mathbb{D}([0,T],\mathbb{R}^2)$. Usually (for example for a similar problem in dimension one [4]), one constructs a pathwise coupling of processes, respectively with laws P^{β_l} and P^β, and comparable in a pathwise sense.

Notation 4.1 *We consider on* $\mathcal{P}_2(\mathbb{R}^2)$ *and* $\mathcal{P}_2(\mathbb{D}_T)$ *the Vaserstein metrics:*

$$\rho(q_1,q_2) = \inf\left\{ E\left(|V-W|^2\right)^{1/2} \Big/ \mathcal{L}(V)=q_1 \;,\; \mathcal{L}(W)=q_2 \right\}, \tag{4.1}$$

$$\rho_T(Q_1,Q_2) = \inf\left\{ E\left(\sup_{[0,T]}|V_t-W_t|^2\right)^{1/2} \Big/ \mathcal{L}(V)=Q_1 \;,\; \mathcal{L}(W)=Q_2 \right\} \tag{4.2}$$

We set

$$b = \int_0^\pi (1-\cos\theta)\beta(\theta)d\theta \;, b_l = \int_{1/l}^\pi (1-\cos\theta)\beta(\theta)d\theta. \tag{4.3}$$

Let us remark that $b - b_l$ tends to 0 when l tends to infinity.

Theorem 4.2 *Let $P_0 \in \mathcal{P}_2(\mathbb{R}^2)$. Let β be a cross section satisfying $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$, and let $\beta_l(\theta) = \beta(\theta)1_{[1/l,\pi]}(|\theta|)$. Let P^β (resp. P^{β_l}) be the unique solution of $MP(\beta, P_0)$ (resp. $MP(\beta_l, P_0)$). Then*

$$\rho_T(P^\beta, P^{\beta_l}) \leq \left(\int_{\mathbb{R}^2} |v|^2 P_0(dv)\right)^{1/2} \sqrt{2(b-b_l)+T(b-b_l)^2} 4\sqrt{T}e^{16b+2b^2T} \tag{4.4}$$

Proof. As in [4], we use coupling techniques. Let $l \geq 1$ be fixed. We consider a Poisson measure $N(d\theta, d\alpha, ds)$ with intensity measure $\beta(\theta)d\theta d\alpha ds$. We also consider the Poisson measure with cutoff $N_l = N|_{[0,T]\times[0,1\times\{[-\pi,\pi]\backslash[-1/l,1/l]\}}$, of which the intensity is $\beta_l(\theta)d\theta d\alpha ds$. Then we perform an iteration scheme. We take V_0 of law P_0 and define $V^{l,0} = V^0 = V_0$, When everything is built up to k, we set using Notation 2.9

$$V^{k+1} = F(V_0, V^k, W^k, N, b),\ V^{l,k+1} = F(V_0, V^{l,k}, W^{l,k}, N_l, b_l).$$

and we choose α-processes W^k and $W^{l,k}$ such that

$$\mathcal{L}_\alpha(W^k, W^{l,k}|W^{k-1}, ..., W^0, W^{l,k-1}, ..., W^{l,0}) = \mathcal{L}(V^k, V^{l,k}|V^{k-1}, ..., V^0, V^{l,k-1}, ..., V^{l,0})$$

Then one proves easily by standard stochastic calculus that the sequences $(V^k)_{k\geq0}$, $(W^k)_{k\geq0}$ and $(V^{l,k})_{k\geq0}$, $(W^{l,k})_{k\geq0}$ are Cauchy's sequences and then converge for the L^2-norm and a.s. (using the Borel-Cantelli Lemma) to limits V, W and V^l, W^l, and necessarily, $V = F(V_0, V, W, N, b)$ and $V^l = F(V_0, V^l, W^l, N_l, b_l)$ and $\mathcal{L}_\alpha(W, W^l) = \mathcal{L}(V, V^l)$. Hence (V, W, N) and (V^l, W^l, N_l) are respectively solutions of $SDE(\beta, P_0)$ and $SDE(\beta_l, P_0)$. Thus $\mathcal{L}(V) = \mathcal{L}_\alpha(W) = P^\beta$ and $\mathcal{L}(V^l) = \mathcal{L}_\alpha(W^l) = P^{\beta_l}$. We get, using Doob's Formula and Itô's calculus, setting $\phi_x^l(t) = E(\sup_{[0,t]} |V_s^x - V_s^{l,x}|^2)$ and $\phi_y^l(t) = E(\sup_{[0,t]} |V_s^y - V_s^{l,y}|^2)$,

$$\begin{aligned}
\phi_x^l(t) \quad \leq \quad & 16\int_{-\pi}^{\pi}(\cos\theta - 1)^2\beta(\theta)d\theta \int_0^t \phi_x^l(s)ds + 16\int_{-\pi}^{\pi}(\sin\theta)^2\beta(\theta)d\theta\int_0^t \phi_y^l(s)ds \\
& +4b^2T\int_0^t \phi_x^l(s)ds + \int_{-1/l}^{1/l}(\cos\theta-1)^2\beta(\theta)d\theta\int_0^t E(|V_s^x|^2)ds \\
& + \int_{-1/l}^{1/l}(\sin\theta)^2\beta(\theta)d\theta\int_0^t E(|V_s^y|^2)ds + (b-b_l)^2T\int_0^t E(|V_s^x|^2)ds
\end{aligned}$$

Using the same computation for $\phi_y^l(t)$, we deduce that if $\phi^l(t) = \phi_x^l(t) + \phi_y^l(t)$, using the conservation of the kinetic energy, i.e. $E(|V_t|^2) = E(|V_0^2|)$,

$$\begin{aligned}
\phi^l(t) \quad \leq \quad & [32b + 4b^2T]\int_0^t \phi^l(s)ds \\
& +32(b-b_l)TE(|V_0|^2) + 16(b-b_l)^2T^2E(|V_0|^2)
\end{aligned}$$

One concludes by using Gronwall's Lemma, and the obvious fact that $\phi^l(t) \geq \rho_T^2(P^\beta, P^{\beta_l})$.
□

Convergence rates for the interacting particle systems.
We consider the same cutoff cross section β_l as before. Then with each l, one can associate a particle system $(V^{\beta_l,n})$ as defined in Subsection 2.3.
We can now state our main pathwise convergence result.

Theorem 4.3 *Let β be a cross section. Let us consider a sequence $l(n)$ of integers going to infinity in such a way that*

$$\exp\left(T \parallel \beta_{l(n)} \parallel_1\right) = o(n) \tag{4.5}$$

and let $(V_0^i)_{i\in I\!N}$ be i.i.d. P_0-distributed random variables. Then
1) For every fixed k and every $T > 0$, the sequence of laws $\mathcal{L}(V^{\beta_{l(n)},1n}, ..., V^{\beta_{l(n)},kn})$ of probability measures on the path space $I\!D([0,T],(I\!R^2)^k)$ converges weakly to $(P^\beta)^{\otimes k}$, where P^β is the unique solution of $MP(\beta, P_0)$. Moreover, we have the convergence estimates

$$\sup_{0\leq t\leq T} \rho^2\left(\mathcal{L}(V_t^{\beta_{l(n)},kn}), P_t^\beta\right) \leq \rho_T^2\left(\mathcal{L}(V^{\beta_{l(n)},kn}), P^\beta\right) \tag{4.6}$$

$$\leq\ 6\frac{\exp\left(T \parallel \beta_{l(n)} \parallel_1\right)}{n} + 16Te^{32b+4b^2T}\int |v|^2 P_0(dv)\left[2(b-b_l) + T(b-b_l)^2\right]$$

2) The empirical measures of the system $\mu^{\beta_{l(n)},n}$ converge in probability to P^β in the path space $\mathcal{P}(I\!D([0,T],I\!R^2))$.

The proof is immediate by associating Theorem 3.1 and Theorem 4.2.

In the case of potential interactions in $1/r^s$, $s > 2$, we know that

$$\beta(|\theta|) \leq C|\theta|^{-\alpha}, C > 0, \alpha \in]1,3[. \tag{4.7}$$

Then one has to choose the sequence $l(n)$ in such a way that

$$\exp\left(T\frac{2C}{\alpha-1}l(n)^{\alpha-1}\right) = o(n) \tag{4.8}$$

Of couse, this choice of sequences $l(n)$ is not very good in a practical point of view. Its interest is to give a "bound" for the choice of $(l(n))$ in the method. This strong convergence is thus stringent about the correlation of l and n. We will see in the following subsection that if we accept a less good convergence, no correlation is needed between the convergence of l and n to infinity.

4.2 A weak Approach

We consider the same cutoff cross section β_l as before and associate as before with each l the particle system $(V^{\beta_l,n})$ as defined in Subsection 2.3.
Let us introduce the sequence of laws $(\pi^{l,n})_n$ of $\mu^{\beta_l,n}$, which are probability measures on $\mathcal{P}(I\!D([0,T],I\!R^2)$.

Theorem 4.4 *Assume that the initial particles are independent with a two order moments common law P_0. Then if n tends to infinity and l tends to infinity, the sequence $(\pi^{l,n})_{l,n}$ converges for the weak topology to δ_{P^β}, and thus the sequence $(\mu^{\beta_l,n})$ converges in law and in probability to P^β.*

Proof. To prove this theorem we follow the classical trilogy of arguments:
1) Tightness of $(\pi^{l,n})_{l,n}$ in $\mathcal{P}(\mathcal{P}(I\!D([0,T],I\!R^2))$,
2) Identification of the limiting values of $(\pi^{l,n})_n$ as solution of a nonlinear martingale problem,
3) Uniqueness of the solution of the martingale problem.
The third point has already be done.
One knows (cf. [12], Lemma 4.5), that the tightness of $(\pi^{l,n})_{l,n}$ is equivalent to the tightness of the laws of the semimartingales $V^{\beta_l,1n}$ belonging to $\mathcal{P}(I\!D([0,T],I\!R^2)$. This tightness can be proved by showing the tightness of the law of the supremum of $V_t^{\beta_l,1n}$ on $[0,T]$, and the Aldous criterion for $V^{\beta_l,1n}$.
One proves easily by a good use of Doob's inequality and Burkholer-Davis-Gundy inequality for (3.7) that

$$\sup_{l,n} E(\sup_{t\leq T} |V_t^{\beta_l,1n}|^2) < +\infty, \tag{4.9}$$

from which we deduce without difficulty the tightness of the laws of $V^{\beta_l,1n}$ and the the tightness of the sequence $(\pi^{l,n})$.
Let us now prove that all the limit values are solutions of the nonlinear martingale problem (2.9).
Consider $\pi^\infty \in \mathcal{P}(\mathcal{P}(I\!D([0,T],I\!R^2)))$ be an accumulation point of $(\pi^{l,n})_{l,n}$. It is the limit point of a subsequence we still denote by $(\pi^{l,n})_{l,n}$. Our aim is to prove that $\pi^\infty = \delta_{P^{\beta_l}}$. For $\phi \in C_b^2(I\!R^2)$, $0 \leq s_1, ..., s_q \leq s \leq t, g_1, ..., g_q \in C_b(I\!R^2), Q \in \mathcal{P}(I\!D([0,T],I\!R^2))$, set, for X the canonical process on $I\!D([0,T],I\!R^2)$.

$$F(Q) = \left\langle Q, \left(\phi(X_t) - \phi(X_0) - \int_s^t \langle K_\beta^\phi(X_u, v_*), Q_u(dv_*)\rangle\right) g_1(X_{s_1})...g_q(X_{s_q})\right\rangle.$$

We have to prove that $< \pi^\infty, |F| >= 0$. The mapping F is not continuous since the projection $X \mapsto X_t$ are not continuous for the Skorohod topology. However, for any $Q \in \mathcal{P}(I\!D([0,T],I\!R^2))$, $X \mapsto X_t$ is Q-almost surely continuous for all t outside an at most countable set D_Q, and then F is continuous at point Q if $s, t, s_1, ..., s_q$ are not in D_Q.We use here the continuity and boundedness of ϕ, $g_1, ..., g_q$ and also the continuity of $(q, v) \mapsto \langle K_\beta^\phi(v, v_*), q(dv_*)\rangle$ on $\mathcal{P}(I\!D([0,T],I\!R^2)) \times I\!R^2$.
Now one can show that the set D of all t for which $\pi^\infty(Q, t \in D_Q) > 0$ is again at most countable. Thus, if $s, t, s_1, ..., s_q$ are in D^c, F is π^∞-a.s. continuous. Then,

$$\left\langle \pi^\infty, F^2 \right\rangle = \lim_{l,n} \left\langle \pi^{l,n}, F^2 \right\rangle$$

But $\left\langle \pi^{l,n}, |F| \right\rangle \leq \left\langle \pi^{l,n}, |F^l| \right\rangle + \left\langle \pi^{l,n}, |F - F^l| \right\rangle$, where F^l is defined by

$$F^l(Q) = \left\langle Q, \left(\phi(X_t) - \phi(X_0) - \int_s^t \langle K_{\beta_l}^\phi(X_u, v_*), Q_u(dv_*)\rangle\right) g_1(X_{s_1})...g_q(X_{s_q})\right\rangle.$$

Firstly,

$$\left\langle \pi^{l,n}, (F^l)^2 \right\rangle = E((F^l)^2(\mu^{\beta_l,n})) = E\left(\left(\frac{1}{n}\sum_{i=1}^n (M_t^{\beta_l,i\phi} - M_s^{\beta_l,i\phi}) g_1(V_{s_1}^{\beta_l,in})...g_q(V_{s_q}^{\beta_l,in})\right)^2\right)$$

$$= \frac{1}{n}E\left(\left((M_t^{\beta_l,1\phi} - M_s^{\beta_l,1\phi})g_1(V_{s_1}^{\beta_l,n})...g_q(V_{s_q}^{\beta_l,1n})\right)^2\right)$$
$$+\frac{n-1}{n}E\Big((M_t^{\beta_l,1\phi} - M_s^{\beta_l,1\phi})(M_t^{\beta_l,2\phi} - M_s^{\beta_l,2\phi})g_1(V_{s_1}^{\beta_l,1n})...g_q(V_{s_q}^{\beta_l,1n})$$
$$g_1(V_{s_1}^{\beta_l,2n})...g_q(V_{s_q}^{\beta_l,2n})\Big).$$

The first term goes to zero because of the uniform integrability given by the L^2-bounds and the second term tends to zero, as it is easy to see on (3.5). Let us remark that the convergences are uniform on l. Hence

$$\lim_n \left\langle \pi^{l,n}, |F^l| \right\rangle = 0, \text{ uniformly in } l.$$

Next,

$$\begin{aligned}
\left\langle \pi^{l,n}, |F - F^l| \right\rangle &= E(|F - F^l|(\mu^{\beta_l,n})) \\
&= E\left(\left|\left\langle \mu^{\beta_l,n}, \int_s^t \int_{-\pi}^{\pi} \left(K_\beta^\phi(X_u, v_*) - K_{\beta_l}^\phi(X_u, v_*)\right) \mu_u^{\beta_l,n} du \right\rangle\right|\right) \\
&\leq K_l \sup_{n,l} E\left(\sup_{u\leq T} \langle \mu_u^{\beta_l,n}, |v|^2 \rangle\right)
\end{aligned}$$

by using the form of K_β^ϕ with $\phi \in C_b^2(I\!R^2)$ and (2.8). The second term of the product is finite by (4.9) and $K_l = C^{te} \int_{-\pi}^{\pi} \theta^2 |\beta(\theta) - \beta_l(\theta)| d\theta$ tends to zero as l tends to infinity.
We have then proved that

$$\langle \pi^\infty, |F| \rangle = 0.$$

Thus, $F(Q)$ is π^∞-a.s. equal to zero, for every $s, t, s_1, ..., s_q$ outside of the countable set D. It is sufficient to assure that π^∞-a.s., Q is solution of the nonlinear martingale problem (2.9). This problem as a unique solution P^β as seen in Section 2, and π^∞ is the Dirac mass at P^β.
Therefore, we have finally proved that $(\mathcal{L}(\mu^{\beta_l,n}))$ converge when l and n tend to infinity, to the Dirac mass at P^β and the theorem is proved. □

4.3 Numerical Results

First of all, let us mention that the simulation algorithm related with the Bird system looks better than the one related with Nanbu's system. Hence, we will study the binary algorithm. We choose a typical cross section without cutoff, which does not admit a moment of order 1 :

$$\beta(\theta) = \frac{1}{2\pi \sin^2 \theta} 1_{\{0<|\theta|<\pi/2\}} \tag{4.1}$$

and we consider the following initial distribution of the velocities :

$$P_0(dv) = 1_{[-1/2,1/2]^2}(v)dv \tag{4.2}$$

We want to deal with known quantities, thus we consider the moments of order 4 :

$$m_4(t) = \int_{R^2} |v|^4 P_t^\beta(dv) \quad ; \quad m_4^l(t) = \int_{R^2} |v|^4 P_t^{\beta_l}(dv) \tag{4.3}$$

which we would like to compare with

$$m_4^{l,n}(t) = \int_{R^2} |v|^4 \mu_t^{n,\beta_l}(dv) \tag{4.4}$$

where μ_t^{n,β_l} is the empirical measure defined by (3.5). A simple computation shows that

$$\begin{aligned} m_4(t) &= \frac{1}{18}\left(1 - e^{-t/4}\right) + \frac{7}{180} e^{-t/4} \\ m_4^l(t) &= \frac{1}{18}\left(1 - e^{-t/4+t/2\pi l}\right) + \frac{19}{240} e^{-t/4+t/2\pi l} \end{aligned} \tag{4.5}$$

We choose $t = t_0 = 2$. Then one easily checks that

$$|m_4(t_0) - m_4^l(t_0)| \times 100/m_4(t_0) \sim 7.1/l\% \tag{4.6}$$

Then, using simulation, we see that

$$|m_4^{l,n}(t_0) - m_4^l(t_0)| \times 100/m_4(t_0) \sim 103/\sqrt{n}\% \tag{4.7}$$

and in particular does not depend on l, as shows figure 1 below.
This suggests that a central limit theorem might hold for the moments of each particle system $\mu^{\beta_l,n}$ (l fixed), with constants not depending too much on l.
This question is studied in the next section.

5 Study of the fluctuations

5.1 The fluctuation process associated with the cutoff kernel

In Méléard [13], we study the fluctuations associated with a cutoff Boltzmann model, under a uniform moment hypothesis on the jump measure. The techniques used to obtain the convergence of the fluctuation processes consist in immersing these fluctuations in weighted Sobolev spaces and in obtaining compactness by using uniform bounds and Hilbert-Schmidt embeddings between some of these spaces.
Here, we need first to cutoff the jump measure to hope some results, and even under this hypothesis, the moment condition on the jump measures assumed in [13] is not satisfied. Then we do not know how to obtain good estimates in weighted Sobolev spaces. Here we use a simpler space introduced by Ferland, Fernique, Giroux [5] for the Kac equation and which is adequate here since we are in a spatially homogeneous situation.
We are interested here in the behaviour of the fluctuations of the empirical measures $\mu^{\beta_l,n}$ to P^{β_l} when $l > 0$ is fixed and n tends to infinity. More precisely we define the fluctuation process $\eta^{n,l}$ defined for bounded functions ϕ by

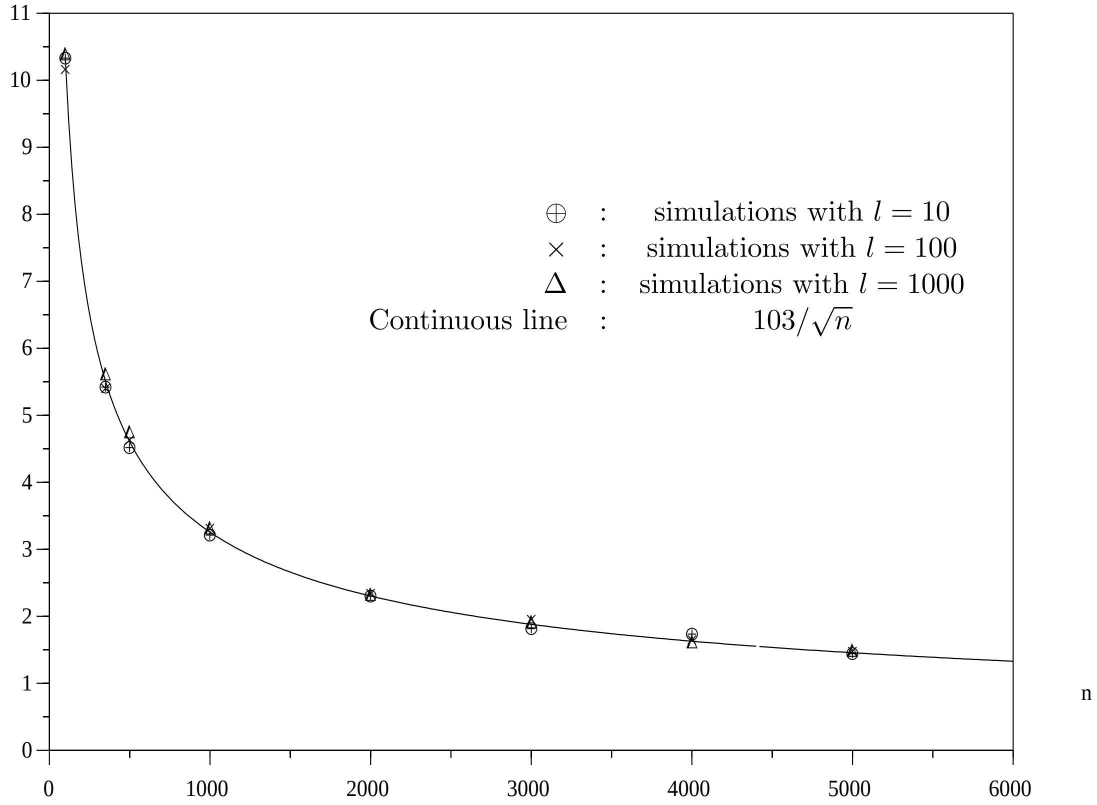
$$< \phi, \eta_t^{n,l} >= \sqrt{n}(< \phi, \mu_t^{\beta_l,n} > - < \phi, P_t^{\beta_l} >). \tag{5.1}$$

For simplicity we will denote $\mu^{\beta_l,n} = \mu^{n,l}$ in all what follows and we assume that

Hypothesis (M): *The initial law P_0 has first order moments.*

It is easy to prove by standard arguments that this moment condition propagates.

Figure 1: error (in %) of one simulation as a function of n



Proposition 5.1 *Under hypothesis* (M)*, for each* $T > 0$*,*

$$\sup_n E\left(\sup_{t\leq T}|V_t^{\beta_l,in}|\right) < +\infty. \tag{5.2}$$

$$\sup_n E\left(\sup_{t\leq T} < |v|, \mu_t^{n,l} >\right) < +\infty \ ; \ \int_{x\in \mathbb{D}([0,T],\mathbb{R}^2)} \sup_{[0,T]} |x(t)| P^{\beta_l}(dx) < +\infty. \tag{5.3}$$

The stochastic equation (3.6) satisfied by the flow of empirical measures and the evolution equation (2.6) satisfied by the limit flow P^{β_l} allow us to get the flow of fluctuation processes as solution of the stochastic differential equation

$$\begin{aligned} < \phi, \eta_t^{n,l} > &= < \phi, \eta_0^{n,l} > + N_t^{n,l}(\phi) + \int_0^t \left\langle K_{\beta_l}^{\phi}(v, v_*), \eta_s^{n,l}(dv)\mu_s^{n,l}(dv_*)\right\rangle ds \\ &+ \int_0^t \left\langle K_{\beta_l}^{\phi}(v, v_*), P_s^{\beta}(dv)\eta_s^{n,l}(dv_*)\right\rangle ds \end{aligned} \tag{5.4}$$

where $N^{n,l}(\phi)$ is a square integrable martingale whose Doob-Meyer process is

$$< N^{n,l}(\phi) >_t \;=\; \int_0^t \Big\langle \frac{1}{2}\int_{-\pi}^{\pi} (\phi(v + A(\theta)(v - v_*) - \phi(v) \tag{5.5}$$

$$+\phi(v_* + A(\theta)(v_* - v)) - \phi(v_*))^2\beta(\theta)d\theta, \mu_s^{n,l}(dv)\mu_s^{n,l}(dv_*)\Big\rangle ds.$$

Since $\eta_t^{n,l}$ is, for each n, l, t, a signed measure, the first difficulty to overcome is to find a suitable space in which $\eta_t^{n,l}$ and its eventual limits can be immersed. Here we consider the space H defined as follows.

Definition 5.2 *For any function such that this term makes sense, one defines*

$$\|f\| = \left(\int_{I\!R^2} |\nabla f(v)|^2 dv\right)^{\frac{1}{2}}. \tag{5.6}$$

This is a seminorm on the space $\mathcal{K}$ of bounded functions with a bounded derivative in $L^2(I\!R^2)$. The seminorm is Hilbertian and the corresponding Hilbert space is denoted by H. We will denote by H' its topological dual space, with the norm $\|.\|_{-1}$.

Lemma 5.3 *If $\phi \in H$, then for $x, y \in I\!R^2$,*

$$|\phi(x) - \phi(y)| \leq \|\phi\|\sqrt{|x-y|};$$

and then for each orthonormal basis (ϕ_p) in H, for fixed $v, w \in I\!R^2$,

$$\sum_{p\geq 1}(\phi_p(v) - \phi_p(w))^2 \leq |v - w|.$$

Since each function of H satisfies $\phi(0) = 0$, we deduce that

$$\sum_{p\geq 1}(\phi_p(v))^2 \leq |v|.$$

Proof. The proof just consists in writing

$$|\phi(x) - \phi(y)| = |h(1) - h(0)| = |\int_0^1 h'(t)dt| \leq \left(\int_0^1 (h'(t))^2 dt\right)^{\frac{1}{2}}$$

where $h(t) = \phi(y + t(x-y))$; $h'(t) = (x-y).\nabla\phi(y + t(x-y))$. Then one applies the Parseval inequality to the linear mapping $\phi \mapsto \phi(v) - \phi(w)$ □

We can deduce the

Corollary 5.4 *The fluctuation process takes its values in H, as soon as the condition (M) is satisfied.*

Proof. Let ϕ be in H. We write

$$\begin{aligned} |<\phi, \eta_t^{n,l}>| &\leq \sqrt{n}(<\phi, \mu_t^{n,l}> + <\phi, P_t^{\beta_l}>) \\ &\leq \|\phi\|\sqrt{n}\left(\int_{I\!R^2}\sqrt{|v|}\mu_t^{n,l}(dv) + \int_{I\!R^2}\sqrt{|v|}P_t^{\beta_l}(dv)\right) \end{aligned}$$

since all functions of H are null at 0 and by using Lemma 5.3. Under (M), this quantity is finite. □

We have moreover

Proposition 5.5 *Assume* (M)*. Then*

$$\sup_n E\left(\|\eta_0^{n,l}\|_{-1}^2\right) < +\infty. \tag{5.7}$$

and for each n *and* T*, there exists a constant* K_n *depending on* n *and* T *such that*

$$E\left(\sup_{t\leq T}\|\eta_t^{n,l}\|_{-1}^2\right) \leq K_n. \tag{5.8}$$

Proof. Let us consider an orthonormal basis (ϕ_p) of the Hilbert space H. Since the initial values $V_0^{\beta_l,in}$ of the particle system are independent,

$$\begin{aligned}
E\left(\|\eta_0^{n,l}\|_{-1}^2\right) &= E\left(\sum_p <\phi_p,\eta_0^{n,l}>^2\right) = \sum_p E\left(\frac{1}{n}\left(\sum_{i=1}^n \phi_p(V_0^{\beta_l,in}) - <\phi_p,P_0^{\beta_l}>\right)^2\right)\\
&\leq \frac{1}{n}\sum_{i=1}^n E\left(\sum_p \phi_p^2(V_0^{\beta_l,in})\right) \text{ by independence}\\
&\leq \frac{K}{n}\sum_{i=1}^n E\left(|V_0^{\beta_l,in}|\right) < +\infty, \text{ by Lemma 5.3 and } (M).
\end{aligned}$$

The second assertion is proved in the same way, except we loose the independence property.

$$\begin{aligned}
E\left(\sup_{t\leq T}\|\eta_t^{n,l}\|_{-1}^2\right) &= nE\left(\sup_{t\leq T}\sum_p\left(<\phi_p,\mu_t^{n,l}> - <\phi_p,P_t^{\beta_l}>\right)^2\right)\\
&\leq 2nE\left(\sup_{t\leq T}\left(\sum_p <\phi_p,\mu_t^{n,l}>^2 + \sum_p <\phi_p,P_t^{\beta_l}>^2\right)\right)\\
&\leq KnE\left(\sup_{t\leq T}\left(<|v|,\mu_t^{n,l}> + <|v|,P_t^{\beta_l}>\right)\right) \leq K_n,
\end{aligned}$$

by Lemma 5.3 and Proposition 5.1. □

Let us introduce, for each probability measure $\mu \in \mathcal{P}(I\!\!R^2)$ the drift operators

$$\mathcal{L}^1(\mu)\phi(v) = \int_{I\!\!R^2} K_{\beta_l}^{\phi}(v,v_*)\mu(dv_*)\ ;\ \mathcal{L}^2(\mu)\phi(v) = \int_{I\!\!R^2} K_{\beta_l}^{\phi}(v_*,v)\mu(dv_*).$$

Lemma 5.6 *For* $i = 1,2$, $\mathcal{L}^i(\mu)$ *are continuous linear operators on* $\mathcal{K}$ *and*

$$\|\mathcal{L}^i(\mu)\| \leq K\|\beta_l\|_1, \tag{5.9}$$

where the constant K *does not depend on* μ*.*

Proof.

$$\begin{aligned}
&\int_{I\!\!R^2} dv\left(\frac{d}{dv}\int_{I\!\!R^2} K_{\beta_l}^{\phi}(v,v_*)\mu(dv_*)\right)^2\\
&= \int_{I\!\!R^2} dv\left(\int_{I\!\!R^2}\int_{-\pi}^{\pi}\Big((I+A(\theta))\nabla\phi(v') - \nabla\phi(v)\Big)\beta_l(\theta)d\theta\mu(dv_*)\right)^2\\
&\leq \|\beta_l\|_1\int_{I\!\!R^2}\mu(dv_*)\int_{I\!\!R^2} dv\int_{-\pi}^{\pi}\Big(((I+A(\theta))\nabla\phi(v'))^2 + (\nabla\phi(v))^2\Big))\beta_l(\theta)d\theta\\
&\leq K\|\beta_l\|_1^2\int_{I\!\!R^2}(\nabla\phi(v))^2 dv
\end{aligned}$$

where we have used the change of variable $v \mapsto v'$ in the first part of the integral. We thus deduce that $\mathcal{L}^1(\mu)$ is continuous. A similar computation gives the same result for $\mathcal{L}^2(\mu)$. $\square$

Proposition 5.7 *Let (ϕ_p) be an orthonormal basis of H, then for each $T > 0$,*

$$\sup_n E\left(\sum_{p\geq 1}\sup_{t\leq T} N_t^{n,l}(\phi_p)^2\right) < +\infty. \tag{5.10}$$

In particular,

$$\sup_n E\left(\sup_{t\leq T}\|N_t^{n,l}\|_{-1}^2\right) < +\infty. \tag{5.11}$$

Proof. By Doob's inequality,

$$E\left(\sum_{p\geq 1}\sup_{t\leq T} N_t^{n,l}(\phi_p)^2\right) \leq K\sum_{p\geq 1} E\left(N_T^{n,l}(\phi_p)^2\right) = K\sum_{p\geq 1} E\left(< N^{n,l}(\phi_p) >_T\right).$$

Here we have

$$\begin{aligned}
&\sum_{p\geq 1} E\left(< N^{n,l}(\phi_p) >_T\right) \\
&= \int_0^T E\Big\langle \frac{1}{2}\int_{-\pi}^{\pi}\sum_{p\geq 1}(\phi_p(v + A(\theta)(v - v_*)) - \phi_p(v) + \phi_p(v_* + A(\theta)(v_* - v)) - \phi_p(v_*))^2 \\
&\qquad\qquad \beta_l(\theta)d\theta, \mu_s^{n,l}(dv)\mu_s^{n,l}(dv_*)\Big\rangle ds.
\end{aligned}$$

By Lemma 5.3, we deduce

$$\begin{aligned}
&\int_{-\pi}^{\pi}\sum_{p\geq 1}(\phi_p(v + A(\theta)(v - v_*)) - \phi_p(v) + \phi_p(v_* + A(\theta)(v_* - v)) - \phi_p(v_*))^2\beta_l(\theta)d\theta \\
\leq\ & K|v - v_*|\int_{-\pi}^{\pi}|\theta|\beta_l(\theta)d\theta.
\end{aligned}$$

Then the proposition follows by Proposition 5.1. $\square$

5.2 Tightness and convergence results

Theorem 5.8 *Let us consider an orthonormal basis (ϕ_p) of H with functions ϕ_p belonging to $\mathcal{K}$. Then*

$$\sup_n E(\sup_{t\leq T}\|\eta_t^{n,l}\|_{-1}^2) \leq K e^{K\|\beta_l\|_1 T}. \tag{5.12}$$

We deduce that

$$\sup_n E(\sum_p \sup_{t\leq T} < \phi_p, \eta_t^{n,l} >^2) < +\infty. \tag{5.13}$$

Proof. Let us consider an orthonormal basis (ϕ_p) of H with functions ϕ_p belonging to $\mathcal{K}$. For each p, one has

$$< \phi_p, \eta_t^{n,l} >=< \phi_p, \eta_0^{n,l} > +N_t^{n,l}(\phi_p) + \int_0^t < \mathcal{L}^1(\mu_s^{n,l})\phi_p + \mathcal{L}^2(P_s^{\beta_l})\phi_p, \eta_s^{n,l} > ds. \quad (5.14)$$

Then,

$$\begin{aligned} E\Big(\sup_{t\leq T}\sum_p < \phi_p, \eta_t^{n,l} >^2\Big) &\leq K\Big(E\Big(\sum_p < \phi_p, \eta_0^{n,l} >^2\Big) + E\Big(\sum_p < N^{n,l}(\phi_p) >_T^2\Big) \\ &+ \int_0^T E\Big(\sup_{u\leq s}\sum_p < \mathcal{L}^1(\mu_u^{n,l})\phi_p + \mathcal{L}^2(P_u^{\beta_l})\phi_p, \eta_u^{n,l} >^2\Big)ds\Big). \end{aligned}$$

Thanks to Lemma 5.6, the linear operator L on $\mathcal{K}$ which with ϕ associates $< \mathcal{L}^1(\mu_u^{n,l})\phi + \mathcal{L}^2(P_u^{\beta_l})\phi, \eta_u^{n,l} >$ satisfies

$$|L(\phi)| \leq K\|\eta_u^{n,l}\|_{-1}\|\beta_l\|_1\|\phi\|$$

where the constant K is independent of l, $\mu_u^{n,l}$ and $P_u^{\beta_l}$. Thus $\|L\| \leq K\|\eta_u^{n,l}\|_{-1}\|\beta_l\|_1$ and $\sum_p L(\phi_p)^2 \leq K^2\|\eta_u^{n,l}\|_{-1}^2\|\beta_l\|_1^2$. Using this property and Proposition 5.7, we deduce that

$$E\Big(\sup_{t\leq T}\sum_p < \phi_p, \eta_t^{n,l} >^2\Big) \leq K\Big(E(\sum_p < \phi_p, \eta_0^{n,l} >^2) + 1 + \|\beta_l\|_1^2\int_0^t E(\sup_{u\leq s}\|\eta_u^{n,l}\|_{-1}^2)ds\Big).$$

Gronwall's inequality gives (5.12). Now, we come back to (5.14) and obtain similarly that

$$E\Big(\sum_p \sup_{t\leq T} < \phi_p, \eta_t^{n,l} >^2\Big) \leq K\Big(E(\sum_p < \phi_p, \eta_0^{n,l} >^2) + 1 + \|\beta_l\|_1^2\int_0^t E(\sup_{u\leq s}\|\eta_u^{n,l}\|_{-1}^2)ds\Big).$$

which allows us to conclude. □

Thanks to Theorem 5.8, we are now able to prove that the fluctuation trajectories are almost surely càdlàg in H' and that their laws are tight when H' is endowed with its weak topology.

Proposition 5.9 *The trajectories of the fluctuation processes are almost surely strongly càdlàg in* H'

Proof. For every function $\phi \in H$, the process $< \phi, \eta^{n,l} >$ is càdlàg. Let us consider an orthonormal complete basis (ϕ_p) in H. Thanks to (5.13) and for every fixed n, we can find for every $\varepsilon > 0$ a positive real number M such that $\sum_{p>M}\sup_{t\leq T} < \phi_p, \eta_t^{n,l} >^2 < \frac{\varepsilon^2}{6}$ a.s. Let (t_m) be a sequence of real numbers greater than t which tends to t when m tends to infinity. For m sufficiently large,

$$\begin{aligned} \|\eta_{t_m}^{n,l} - \eta_t^{n,l}\|_{-1}^2 &= \sum_{p\geq 1}\Big(< \phi_p, \eta_{t_m}^{n,l} > - < \phi_p, \eta_t^{n,l} > \Big)^2 \\ &\leq \sum_{1\leq p\leq M}\Big(< \phi_p, \eta_{t_m}^{n,l} > - < \phi_p, \eta_t^{n,l} > \Big)^2 \\ &\quad +2\sum_{p>M}\Big(< \phi_p, \eta_{t_m}^{n,l} >^2 + < \phi_p, \eta_t^{n,l} >^2 \Big) \\ &< \sum_{1\leq p\leq M}\frac{\varepsilon^2}{3M} + \frac{4\varepsilon^2}{6} = \varepsilon^2. \end{aligned}$$

In the first term, we have used that for every function ϕ_p, the process $< \phi_p, \eta_t^{n,l} >$ is càdlàg. We deduce that the mapping $t \mapsto \eta_t^{n,l}$ is càd in H' and a similar proof implies that it is also làg. □

Theorem 5.10 *Under (M), the laws of the fluctuation processes are relatively compact on $I\!D([0,T],H')$, where H' is endowed with the weak topology and any limiting process has strongly continuous paths.*

Proof. We are here inspired by Ferland-Fernique-Giroux [5]. Since the space H' is endowed with the weak topology, it is no longer a Polish space but a Lusin space. Then, as proved in Fernique, [6], the space $I\!D([0,T],H')$ is a Lusin space and any probability measure on this path space is tight. That allows the author to obtain a general compactness criterion for the laws of processes with values in a Lusin space. In the context of $I\!D([0,T],H')$, the sequence of laws of $(\eta^{n,l})$ will be relatively compact as soon as the two following conditions are satisfied:
1) There exists a sequence (K_m) of weakly compact sets of H' such that

$$\forall m \geq 1, \forall n \geq 2, P(\exists t \in [0,T] \mid \eta_t^{n,l} \notin K_m) \leq 2^{-m}.$$

2) For all $\phi \in \mathcal{K}$, the laws of the real processes $(< \phi, \eta_t^{n,l} >)$ are relatively compact.

Let us show that these two properties are satisfied in our context. Proposition 5.8 shows that $M = \sup_n E(\sup_{t\leq T} \|\eta_t^{n,l}\|_{-1}^2) < \infty$. Then the sets $K_m = \{\eta \in H' \mid \|\eta\|_{-1}^2 \leq M2^m\}$ are weakly compact sets and satisfy the condition (1).
Let us now prove the second point and fix a function $\phi \in \mathcal{K}$. Again by Proposition 5.8, we obtain that the laws of $< \phi, \eta_t^{n,l} >$ are relatively compact. We prove now that the Aldous condition is satisfied. Since $< \phi, \eta_t^{n,l} >$ is a semimartingale, we use the Rebolledo theorem and prove that the Aldous condition is proved for the drift term and for the Doob-Meyer process associated with the martingale part. Let τ be a stopping time.

$$\begin{aligned}
&\sup_n \sup_{r\leq\delta} P(| < N^{n,l}(\phi) >_{\tau+r} - < N^{n,l}(\phi) >_\tau | > \epsilon) \\
&\leq \frac{1}{\epsilon} E\left(\int_\tau^{\tau+r} \int_{-\pi}^{\pi} \Big(\phi(v + A(\theta)(v - v_*)) - \phi(v) + \phi(v_* + A(\theta)(v_* - v)) - \phi(v_*)\Big)^2 \beta_l(\theta)d\theta\right) \\
&\leq \delta \frac{K\|\phi\|^2}{\epsilon} \int_{-\pi}^{\pi} |\theta|\beta_l(\theta)d\theta \sup_n E(\sup_{t\leq T} < |v|, \mu_t^{n,l} >)
\end{aligned}$$

which tends to 0 as δ tends to 0, uniformly in n. We have used here Lemma 5.3 and Proposition 5.1.
Let us show now the Aldous condition for the drift term.

$$\begin{aligned}
&\sup_n \sup_{r\leq\delta} P(| \int_\tau^{\tau+r} < \mathcal{L}^1(\mu_s^{n,l})\phi + \mathcal{L}^2(P_s^{\beta_l})\phi), \eta_s^{n,l} > ds| > \epsilon) \\
&\leq K\|\beta_l\|_1 \frac{\|\phi\|}{\epsilon} E\left(\int_\tau^{\tau+r} \|\eta_s^{n,l}\|_{-1} ds\right) \\
&\leq \delta \frac{K\|\phi\|\|\beta_l\|_1}{\epsilon} \sup_n E(\sup_{s\leq T} \|\eta_s^{n,l}\|_{-1})
\end{aligned}$$

which tends to 0 as δ tends to 0, uniformly in n, thanks to (5.13). The sequence $(\eta^{n,l})$ is then uniformly tight in $I\!D([0,T],H')$. Moreover, by (5.13), we deduce that each limit process η^l satisfies $E(\sum_p \sup_{t\leq T} <\phi_p,\eta^l>^2)<\infty$ and that $t\mapsto <\phi_p,\eta^l>$ is a.s. continuous for any p. This is enough to prove as in Proposition 5.9 that any limiting process has strongly continuous path. □

Theorem 5.11 *Under assumption (M), the sequence $(N^{n,l})_n$ converges in law in $I\!D([0,T],H')$ to a continuous Gaussian process W^l with quadratic variation given for every $\phi\in H$ and $t\in[0,T]$ by*

$$<W^l(\phi)>_t \;=\; \int_0^t \Big\langle \frac{1}{2}\int_{-\pi}^{\pi}(\phi(v+A(\theta)(v-v_*))-\phi(v)+\phi(v_*+A(\theta)(v_*-v))-\phi(v_*))^2 \beta_l(\theta)d\theta, P_s^{\beta_l}(dv)P_s^{\beta_l}(dv_*)\Big\rangle ds. \tag{5.15}$$

Proof. The proof is given in two steps.
1) We first prove the tightness of the laws of $(N^{n,l})$ in $I\!D([0,T],H')$, where H' is endowed with its weak topology. We proceed exactly as before, using Proposition 5.7.
We prove moreover that the accumulations points of the laws of $(N^{n,l})$ charge only $C([0,T],H')$ (one says that the laws are C-tight). Following [11], it suffices to prove that the sequence $\sup_{s\leq T}\|N_s^{n,l}-N_{s^-}^{n,l}\|_{-1}$ converges in probability to 0. It is easy to remark that for each $\phi\in H$, the jumps of $N_.^{n,l}(\phi)$ and of $<\phi,\mu_.^{n,l}>$ are at the same time and just two particles jump at every jump time. Then, if the jump takes place at time t for particles i and j,

$$|N_s^{n,l}(\phi)-N_{s^-}^{n,l}(\phi)|^2=\frac{1}{n}|\phi(V_{s^-}^{\beta_l,in}+\Delta V_s^{\beta_l,in})-\phi(V_{s^-}^{\beta_l,in})+\phi(V_{s^-}^{\beta_l,jn}+\Delta V_s^{\beta_l,jn})-\phi(V_{s^-}^{\beta_l,jn})|^2.$$

By Lemma 5.3, $|\phi(z+h)-\phi(z)|\leq K\sqrt{|h|}\|\phi\|$ and then

$$|N_s^{n,l}(\phi)-N_{s^-}^{n,l}(\phi)|^2\leq \frac{K}{n}\sup_{s\leq t}\left(|V_s^{\beta_l,in}|+|V_s^{\beta_l,jn}|\right)\|\phi\|.$$

Hence

$$E\left(\sup_{s\leq T}\|N_s^{n,l}-N_{s^-}^{n,l}\|_{-1}^2\right)\leq\frac{K}{n}$$

using (5.2).
2) Now we are able to study the convergence in law of $N^{n,l}$. We have already seen that the laws of $N^{n,l}$ are C-tight. Since moreover the sequence $N^{n,l}$ is uniformly integrable as elements of H' by (5.11), each limit of a subsequence is a continuous square integrable martingale. Otherwise, since the empirical measures $\mu^{n,l}$ converge in law to the deterministic measure P^{β_l}, then for each $\phi\in H$, the quadratic variation $<N^{n,l}(\phi)>$ defined in (5.5) converge in law and then in probability to the deterministic limit defined in (5.15). Hence each limit value (in law) of $N^{n,l}$ is a continuous square integrable martingale with values in H', with the deterministic Doob-Meyer process characterized by (5.15). The theorem is then proved. □

We now give the main theorem of the section.

Theorem 5.12 *Assume (M). Then the sequence $(\eta^{n,l})_n$ converges in law in $\mathbb{D}([0,T],H')$ to a continuous process η^l which satisfies: for each $\phi \in H$,*

$$< \phi, \eta_t^l > = < \phi, \eta_0^l > + W^l(\phi) + \int_0^t < \mathcal{L}^1(P_s^{\beta_l})\phi + \mathcal{L}^2(P_s^{\beta_l})\phi, \eta_s^l > ds. \tag{5.16}$$

where η_0^l is a random variable with values in H' such that for all $\phi \in H$, $< \phi, \eta_0^l >$ has a gaussian law $\mathcal{N}\left(0, \int_{\mathbb{R}^2} |v|^2 P_0(dv) - \left(\int_{\mathbb{R}^2} v P_0(dv)\right)^2\right)$

Proof. Let us consider an orthonormal basis (ϕ_p) of H with functions ϕ_p belonging to $\mathcal{K}$. We have seen that the sequence $(\eta^{n,l})_n$ is uniformly tight in $\mathbb{D}([0,T],H')$, where H' is endowed with the weak topology and that each limiting process η^l has strongly continuous path. Let us consider a limit process η^l and then there exists a subsequence of $(\eta^{n,l})_n$, that we will again denote by $(\eta^{n,l})_n$ for simplicity, and which converges in law to η^l.
For each function $\phi \in H$, we first introduce the function F_ϕ defined from $C([0,T],H')$ to $\mathbb{R}$ by

$$F_\phi(\alpha) = < \phi, \alpha_t > - < \phi, \alpha_0 > - \int_0^t < \mathcal{L}^1(P_s^{\beta_l})\phi + \mathcal{L}^2(P_s^{\beta_l})\phi, \alpha_s > ds.$$

The function F_ϕ is continuous and then the sequence $(F_\phi(\eta^{n,l}))_n$ converges to $F_\phi(\eta^l)$.
Let us now prove that $\int_0^t < \mathcal{L}^1(\mu_s^{n,l})\phi - \mathcal{L}^1(P_s^{\beta_l})\phi, \eta_s^l > ds$ tends in L^1 (and thus in law) to 0 when n tends to infinity. We use (5.12) which implies in particular that for each $\phi \in H$,

$$E(\sup_{t \le T} | < \phi, \mu_t^{n,l} > - < \phi, P_t^{\beta_l} > |^2)^{\frac{1}{2}} \le \frac{K_l \|\phi\|}{\sqrt{n}}.$$

Thus, since $v_* \to K_{\beta_l}^\phi(v, v_*)$ belongs to H,

$$\begin{aligned}
& E\left(\left|\int_0^t < \mathcal{L}^1(\mu_s^{n,l})\phi - \mathcal{L}^1(P_s^{\beta_l})\phi, \eta_s^l > ds\right|\right) \\
& \le E\left(\int_0^t | < \int_{\mathbb{R}^2} K_{\beta_l}^\phi(v, v_*)(\mu_s^{n,l}(dv_*) - P_s^{\beta_l}(dv_*)), \eta_s^l(dv) > | ds\right) \\
& \le \int_0^t E(\|\eta_s^l\|_{-1}^2)^{\frac{1}{2}} E(\| \int_{\mathbb{R}^2} K_{\beta_l}^\phi(v, v_*)(\mu_s^{n,l}(dv_*) - P_s^{\beta_l}(dv_*))\|^2)^{\frac{1}{2}} ds \\
& \le \frac{K_l \|K_{\beta_l}^\phi\|}{\sqrt{n}} \int_0^t E(\|\eta_s^l\|_{-1}^2)^{\frac{1}{2}} ds
\end{aligned}$$

and always by (5.13), we deduce that this term tends to 0 as n tends to infinity.
Then since we have already seen the convergence of the martingale term and by adding the previous results, we finally obtain that each limit point is solution of (5.16).
Now let us prove that such a solution is unique in $\mathbb{D}([0,T],H')$. The white noise W^l is a Gaussian martingale with respect to the filtration generated by (W, η^l) (cf. [11] Prop. 1.12 p.484). Then we adapt to our context the Yamada-Watanabe theorem and the pathwise uniqueness of (5.16) will imply the uniqueness in law.
Now, let η^1 and η^2 two solutions of the equation. Then for (ϕ_p) an orthonormal basis of H, and $t \le T$,

$$\begin{aligned}
\|\eta_t^1 - \eta_t^2\|^2 = \sum_p < \phi_p, \eta_t^1 - \eta_t^2 >^2 & \le T \int_0^t < \mathcal{L}^1(P_s^{\beta_l})\phi_p + \mathcal{L}^2(P_s^{\beta_l})\phi_p, \eta_s^1 - \eta_s^2 >^2 ds \\
& \le K \|\beta_l\|_1^2 \int_0^t \|\eta_s^1 - \eta_s^2\|^2 ds
\end{aligned}$$

as seen in the proof of Theorem 5.8. Then, always by (5.12), we conclude by Gronwall's lemma that $\eta^1 = \eta^2$.
The theorem is then proved. $\square$

5.3 A uniform control of the limit fluctuation processes ?

We know from the previous subsection that for each l, the fluctuation process $\eta^{n,l}$ goes to a process η^l when n tends to infinity. We now wonder if this limit process η^l can be controlled uniformly in l. This would allow us to say that the speed of convergence of the empirical measure $\mu^{n,l}$ to the solution of the martingale problem P^l does not depend too much on l. However, we are not able to prove, for example, that for all test function ϕ (in a set of regular bounded functions),

$$\sup_l E\left[\sup_{[0,T]} | < \eta^l_t, \phi > |\right] < \infty \tag{5.17}$$

It might even not be true. We have showed that μ^{n,β_l} goes to P^{β_l} with a speed of convergence in $e^{\|\beta_l\|_1}/\sqrt{n}$. This may not be optimal, but it seems natural that this rate of convergence becomes less and less good when $\| \beta_l \|_1$ increases. Indeed, $\| \beta_l \|_1$ is the mean number of collisions for one particle (on the time interval $[0,1]$). It is thus clear that the more $\| \beta_l \|_1$ will be large, the less the propagation of chaos will be fast.

It seems anyway that (5.17) holds for the moments of the Boltzmann equation, i.e. for $\phi(v) = |v|^{2n}$, $n \geq 1$. In particular, the experiences we have presented in Subsection 4.3 show that it should hold for the moment of order 4, which we now prove, in a special case which simplifies the computations. As usual, V_0 denotes a random variable of which the law is the initial distibution $P_0 \in \mathcal{P}_2(\mathbb{R}^2)$ of our Boltzmann equation.

Proposition 5.13 *We assume that for some $p_0 \in \mathcal{P}_2(\mathbb{R})$,*

$$P_0 = p_0 \otimes p_0, \quad E(|V_0|^8) < \infty, \quad E(V_0) = 0 \tag{5.18}$$

Then, if $\phi(v) = |v|^4$,

$$\sup_l E\left[\sup_{[0,T]} | < \eta^l_t, \phi > |\right] < \infty \tag{5.19}$$

We begin with a lemma.

Lemma 5.14 *1. Assume that $E(|V_0|^4) < \infty$. Then, $\psi(v) = |v|^2$,*

$$\sup_l E\left[\sup_{[0,T]} | < \eta^l_t, \psi > |\right] < \infty \tag{5.20}$$

2. We suppose the assumptions of Proposition 5.13. For $X \in \mathbb{R}^2$, we set $\zeta^X(v) = (v.X)^2$. There exists a family of deterministic functions γ^l_t, uniformly bounded on $[0,T]$, such that for each l,

$$< P^{\beta_l}_t, \zeta^X >= \gamma^l_t \times |X|^2 \tag{5.21}$$

Proof. Let us first prove 1. One easily checks that $K^{\psi}_{\beta_l}(v,v_*) = \frac{1}{2}b_l(|v_*|^2 - |v|^2)$ and thus that $K^{\psi}_{\beta_l}(v,v_*) + K^{\psi}_{\beta_l}(v_*,v) = 0$. Hence, we know from equation (5.16) that

$$\begin{aligned} < \eta^l_t, \psi > &= < \eta_0, \psi > + \int_0^t \int_{-\pi}^{\pi} \int_{I\!R^2} \int_{I\!R^2} \Big[|v + A(\theta)(v - v_*)|^2 - |v|^2 \\ &\qquad + |v_* + A(\theta)(v_* - v)|^2 - |v_*|^2 \Big] W^l(ds, d\theta, dv, dv_*) \end{aligned}$$

where W^l is a white noise with intensity $\frac{1}{2}ds\beta_l(\theta)d\theta P^{\beta_l}_s(dv)P^{\beta_l}_s(dv_*)$. On the other hand, one can classically check that

$$\sup_l \sup_{[0,T]} \int_{I\!R^2} |v|^4 P^{\beta_l}_s(dv) < \infty \tag{5.22}$$

and (5.20) is easily deduced.

We are now interested in 2. Computing explicitely $K^{\zeta^X}_{\beta_l}(v,v_*)$, applying the fact that $\{P^{\beta_l}_t\}$ is a measure solution of the Boltzmann equation in the sense of Definition (2.2), using the conservation of the momentum (for all $t > 0$, $\int v P^{\beta_l}_t(dv) = 0$) and of the kinetic energy (for all $t > 0$, $\int |v|^2 P^{\beta_l}_t(dv) = E(|V_0|^2)$), one proves that $< P^{\beta_l}_t, \zeta^X >$ solves an ordinary differential equation, that can be explicitely solved. One obtains

$$\begin{aligned} < P^{\beta_l}_t, \zeta^X > &= 4\frac{d_l}{b_l + d_l}|X|^2 E(|V_0|^2) \\ &\quad + \left[E\left[(X.V_0)^2\right] - 4\frac{d_l}{b_l + d_l}|X|^2 E(|V_0|^2) \right] e^{-\frac{d_l}{8(b_l+d_l)}t} \end{aligned} \tag{5.23}$$

where $d_l = \int_{-\pi}^{\pi} \sin^2\theta d\theta < \infty$. Since under our assumptions, $E\left[(X.V_0)^2\right] = E(|V_0|^2)|X|^2/2$, the second part of the lemma follows. □

Proof. of Proposition 5.13. We set $Z^l_t =< \eta^l_t, \phi >$. Then we know, from the expression of η^l, that $Z^l_t = Z_0 + U^l_t + X^l_t$, where

$$\begin{aligned} U^l_t &= \int_0^t \int_{I\!R^2} \int_{I\!R^2} \left[K^{\beta_l}_{\phi}(v,v_*) + K^{\beta_l}_{\phi}(v_*,v) \right] P^{\beta_l}_s(dv_*)\eta^l_s(dv)ds \\ X^l_t &= \int_0^t \int_{-\pi}^{\pi} \int_{I\!R^2} \int_{I\!R^2} \Big[|v + A(\theta)(v - v_*)|^4 - |v|^4 \\ &\qquad + |v_* + A(\theta)(v_* - v)|^4 - |v_*|^4 \Big] W^l(ds, d\theta, dv, dv_*) \end{aligned}$$

Using the fact that $E(|V_0|^8) < \infty$, one easily proves, using the Itô stochastic calculus, that there exists a constant K, independent of l, such that $\sup_l E[\sup_{[0,T]} |X^l_t|] < \infty$.
A simple but tedious computation shows that

$$K^{\phi}_{\beta_l}(v,v_*) = \alpha^l_1|v|^2|v_*|^2 + \alpha^l_2|v|^4 + \alpha^l_3|v_*|^4 + \alpha^l_4(v.v_*)^2 \tag{5.24}$$

for some uniformly bounded constants α^l_i. Hence

$$\begin{aligned} U^l_t &= 2\alpha^l_1 \int_0^t < P^{\beta_l}_s, |v|^2 >< \eta^l_s, |v|^2 > ds + (\alpha^l_2 + \alpha^l_3)\int_0^t < P^{\beta_l}_s, 1 >< \eta^l_s, |v|^4 > ds \\ &\quad + (\alpha^l_2 + \alpha^l_3)\int_0^t < P^{\beta_l}_s, |v|^4 >< \eta^l_s, 1 > ds + \alpha^l_4 \int_0^t \left\langle \eta^l_s(dv), < P^{\beta_l}_s(dv_*), (v.v_*)^2 > \right\rangle ds \\ &= U^{l,1}_t + ... + U^{l,4}_t \end{aligned}$$

We deduce from the conservation of the kinetic energy and from Lemma 5.14-1. that $\sup_l E[\sup_{[0,T]} |U_t^{l,1}|] < \infty$.
Since $< P_s^{\beta_l}, 1 >= 1$, it is obvious that for a constant K independant of l,

$$E\left[\sup_{[0,t]} |U_s^{l,2}|\right] \leq K \int_0^t E\left[|Z_s^l|\right] ds \tag{5.25}$$

It is clear that for each l, $< \eta_s^l, 1 >= 0$, and thus $U_t^{l,3}$ vanishes identically. Finally, using Lemma 5.14-2.,

$$U_t^{l,4} = 2\alpha_4^l \int_0^t \gamma_s^l \times < \eta_s^l, |v|^2 > ds \tag{5.26}$$

Since α_4^l and γ_t^l are uniformly bounded, we deduce, using Lemma 5.14-1., that $\sup_l E[\sup_{[0,T]} |U_t^{l,4}|] < \infty$. We have proved that

$$E\left[\sup_{[0,t]} |Z_s^l|\right] \leq K + K \int_0^t E(|Z_s^l|) ds \tag{5.27}$$

where K does not depend on l. Gronwall's Lemma allows to conclude. □

Notice that using exactly the same arguments, one should be able to prove recursively that for all $n \geq 1$, $\sup_l E\left[\sup_{[0,T]} | < \eta_t^l, |v|^{2n} > |\right] < \infty$, but the computations should be much more complicated.

References

[1] Babovsky, H.; Illner, R.: A convergence proof for Nanbu's simulation method for the full Boltzmann equation, SIAM J. Num. Anal. 26 1, 46-65 (1989).

[2] Cercignani, C.; Illner, R.; Pulvirenti, M.: The mathematical theory of dilute gases, Applied Math. Sciences, Springer-Verlag, berlin (1994).

[3] Desvillettes, L.: About the regularizing properties of the non-cut-off Kac equation. Comm. Math. Physics 168, 416-440, (1995).

[4] Desvillettes, L.; Graham, C.; Méléard, S.: Probabilistic interpretation and numerical approximation of a Kac equation without cutoff, Stoch. Proc. and Appl., 84, 1, 115-135 (1999).

[5] Ferland, R.; Fernique, X.; Giroux G.: Compactness of the fluctuations associated with some generalized nonlinear Boltzmann equations, Canad. J. Math. 44, 1192-1205 (1992).

[6] Fernique, X.: Convergence en loi de fonctions aleatoires continues ou càdlàg, propriétés de compacité des lois, Séminaire de Probabilités XXV, LNM 1485, 178-195, (1991).

[7] Fournier, N.: Calcul des variations stochastiques sur l'espace de Poisson, applications à des EDPS paraboliques avec sauts et à certaines équations de Boltzmann, Thèse de Doctorat de l'Université Paris 6 (1999).

[8] Fournier, N.: Existence and regularity study for a 2-dimensional Kac equation without cutoff by a probabilistic approach, to appear in Annals of applied probability (2000).

[9] Graham, C.; Méléard, S.: Stochastic particle approximations for generalized Boltzmann models and convergence estimates, Ann. Prob.25, 115-132 (1997).

[10] Graham, C.; Méléard, S.: Existence and regularity of a solution of a Kac equation without cutoff using the stochastic calculus of variations, Commun. Math. Phys. 205, 551-569 (1999).

[11] Jacod, J.; Shiryaev, A.N.: Limit theorems for stochastic prcesses, Springer-Verlag (1987).

[12] Méléard, S.: Asymptotic behaviour of some interacting particle systems, McKean-Vlasov and Boltzmann models, cours du CIME 95, Probabilistic models for nonlinear pde's, L.N. in Math. 1627, Springer (1996).

[13] Méléard, S.: Convergence of the fluctuations for interacting diffusion with jumps associated with Boltzmann equations, Stochastics and Stoch. Rep., Vol. 63, 195-225 (1998).

[14] Nanbu, K.: Interrelations between various direct simulation methods for solving the Boltzmann equation, J. Phys. Soc. Japan 52, 3382-3388 (1983).

[15] Shiga, T.; Tanaka, H.: Central limit theorem for a system of Markovian particules with mean-field interactions, Z. Wahrsch. Verw. Geb. 69, 439-459 (1985).

[16] Sznitman, A.S.: Équations de type de Boltzmann, spatialement homogènes, Z. Wahrsch. verw. Geb. 66, 559-592 (1984).

[17] Tanaka, H.: On the uniqueness of Markov process associated with the Boltzmann equation of Maxwellian molecules, Proc. Intern. Symp. SDE, Kyoto, 409-425 (1976).

[18] Tanaka, H.: Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, Z. Wahrsch. Verw. Geb. 46, 67-105 (1978).

[19] Toscani, G.; Villani, C.: Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas, J. Stat. Phys., 619-637 (1999).

[20] Wagner, W.: A convergence proof for Bird's direct simulation method for the Boltzmann equation, J. Stat. Phys. 66, 1011-1044 (1992).