A pure jump Markov process associated with Smoluchowski's coagulation equation

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Abstract

The Smoluchowski coagulation equation models the evolution of the density $n(x,t)$ of the particles of size (or mass) x at the instant $t \geq 0$ for a system in which a coalescence phenomenon occurs. Two versions of this equation exist: the case of discrete sizes (when $x \in \mathbb{N}^*$) and the case of continuous sizes (when $x \in \mathbb{R}^*$).

The aim of the present paper is to construct a stochastic process, whose law is the solution of the Smoluchowski's coagulation equation. This approach is at our knowledge the first in this direction, in that, for the first time the solution of Smoluchowski's coagulation equation is obtained as the law of a stochastic process.

We first introduce a modified equation, dealing with the evolution of the repartition $Q_t(dx)$ of the mass in the system. The advantage we take on this is that we can do an unified study for both continuous and discrete models.

The integro-partial-differential equation satisfied by $\{Q_t\}_{t \geq 0}$ can be interpreted as the evolution equation of the time marginals of a Markov pure jump process. At this end we introduce a nonlinear Poisson driven stochastic differential equation related to the Smoluchowski equation in the following way: if X_t satisfies this stochastic equation, then the law of X_t satisfies the modified Smoluchowski equation. Existence, uniqueness, and pathwise behaviour of a solution to this S.D.E. are studied.

Key words: Smoluchowski's coagulation equations, nonlinear stochastic differential equations, Poisson measures.

MSC 2000: 60H30, 60K35, 60J75.

1   Introduction

Smoluchowski's coagulation equation governs various phenomenons as for example: polymerisation, aggregation of colloidal particles, formation of stars and planets, behaviour of fuel mixtures in engines etc.

We describe this equation as modelling the polymerisation phenomenon.

For $k \in \mathbb{N}^*$, let P_k denote a polymer of mass k, that is a set of k identical particles (monomers). As time advances, the polymers evolve and, if they are sufficiently close, there is some chance that they merge into a single polymer whose mass equals the sum of the two polymers' masses which take part in this binary reaction. We admit here only binary reactions.

Denote by $n(k,t)$ the average number of polymers of mass k per unit volume, at time t so $k\,n(k,t)$ stands for the part of mass consisting on polymers of length k, per unit volume. The coalescence phenomenon of a polymer of mass k with a polymer of mass j, can be written formally as $P_k + P_j \longrightarrow P_{k+j}$, and is proportional to $n(k,t)\,n(j,t)$ with a proportionality constant $K(k,j)$, called coalescence kernel.

Throughout this paper, time t is always continuous, discrete and continuous refer to polymers' masses.

Hereafter (discrete and continuous case), the coagulation kernel K will satisfy the following hypothesis: K is positive (i.e., $K : (\mathbb{N}^*)^2$ or $(\mathbb{R}_+)^2 \to \mathbb{R}_+$) and symmetric (i.e., $K(i,j) = K(j,i)$).

The Smoluchowski coagulation equation, in the discrete case, is the equation on $n(k,t)$, for $k \in \mathbb{N}^*$. It writes:

$$
\begin{cases}
\dfrac{d}{dt}n(k,t) &= \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) \\
& \qquad\qquad\qquad -n(k,t)\displaystyle\sum_{j=1}^{\infty} K(j,k)n(j,t) \qquad (SD) \\
\\
n(k,0) &= n_0(k).
\end{cases}
$$

This system describes a non linear evolution equation of infinite dimension, with initial condition $(n_0(k))_{k\geq 1}$. In the first line of (SD), the first term on the right hand side describes the creation of polymers of mass k by coagulation of polymers of mass j and $k-j$. The coefficient $\frac{1}{2}$ is due to the fact that K is symmetric. The second term corresponds to the depletion of polymers of mass k after coalescence with other polymers.

The continuous analog of the equation (SD) can be written naturally:

$$
\begin{cases}
\dfrac{\partial}{\partial t} n(x,t) & = \dfrac{1}{2} \displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy \\
& \qquad\qquad -n(x,t) \displaystyle\int_0^\infty K(x,y)n(y,t)dy \qquad\qquad (SC) \\
n(x,0) & = n_0(x)
\end{cases}
$$

for all $x \in \mathbb{R}_+$.

We present briefly some recent results on the existence and uniqueness for the Smoluchowski coagulation equation, obtained by employing a probabilistic approach. These results furnish answers to some phenomenon that seems to be accepted as granted by the physicists.

A detailed survey on the present situation of the research on this equation is provided in Aldous [1].

In the discrete case few situations allow to conclude to the existence and uniqueness of the solution of (SD). If the kernel K satisfies:

$$
K(i,j) \le C(i+j), \quad i,j \ge 1. \tag{1.3}
$$

and the initial condition is such that $\displaystyle\sum_{k=1}^\infty kn(k,0) < \infty$ then existence (see Ball and Carr [2]) and uniqueness (see Heilmann [11]) are known.

Jeon [13] approached the solution of a more general equation than (SD), in that, we have also the fragmentation of polymers, by a sequence of finite Markov chains. Jeon gave a general result about the gelification time T_{gel}, i.e. the first instant when particles of infinite mass appear. More precisely, if we have

$$
K(i,j) \ge (ij)^\alpha, \text{ with } \alpha \in \left]\frac{1}{2},1\right[\tag{1.4}
$$

and furthermore

$$
\lim_{i+j\to\infty} \frac{K(i,j)}{ij} = 0 \tag{1.5}
$$

then we have gelification in finite time $(T_{gel} < \infty)$, for a large class of initial conditions.

For the continuous case, Aldous [1], states hypotheses which insure existence and uniqueness of the solution of the Smoluchowski coagulation equation. More precisely existence and uniqueness hold for (SC) if the kernel K satisfies, for all x and y in $\mathbb{R}_+$:

$$
K(x,y) \le C(1+x+y) \tag{1.6}
$$

and the initial condition is such that

$$
\int_0^\infty n(x,0)dx < \infty \text{ and } \int_0^\infty x^2 n(x,0)dx < \infty \tag{1.7}
$$

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and furthermore we have conservation of mass

$$\int_0^\infty xn(x,0)dx = 1. \tag{1.8}$$

Our conditions will be less restrictive because we don't need to impose

$$\int_0^\infty n(x,0)dx < \infty. \tag{1.9}$$

Norris [15], [16] obtains recently some new results on (SC) by generalising to the continuous case the results in Jeon [13]. Norris has two kinds of hypotheses: either $\dfrac{K(x,y)}{xy} \to 0$ as $(x,y) \to \infty$

or $K(x,y) \leq \varphi(x)\varphi(y)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is sub-linear and $\int_0^\infty \varphi^2(x)n(0,x)dx < \infty$.

Under one of this conditions he proved existence and uniqueness of the solution of (SC). The last hypothesis includes the case where $K(x,y)$ explodes as $x \mapsto 0$ or $y \mapsto 0$ and allows also to obtain uniqueness in some cases when the initial condition has no second, or even first, moment. These results are also interesting because one needs no local regularity on K.

Norris constructs a sequence of stochastic processes which converges to a deterministic limit which is a solution of (SC).

Deaconu and Tanré [4] furnish a probabilistic interpretation for the additive, multiplicative and constant kernels in both discrete and continuous cases. They find a duality between the additive and multiplicative solutions which permits to obtain the results for one of this solutions by those on the other one. They have also presented a "long time behaviour" for the solution.

Our approach of (SC) or (SD) is new and purely stochastic. We construct a pure jump stochastic process $(X_t)_{t\geq0}$ whose law is the solution of the Smoluchowski coagulation equation in the following sense: in the discrete case, $P[X_t = k] = kn(k,t)$ for all $t \geq 0$ and all $k \in \mathbb{N}^*$, while in the continuous case, $P[X_t \in dx] = xn(x,t)dx$ for all $t \geq 0$. For each ω, $X_t(\omega)$ may be seen as the evolution of the size of one "mean" particle in the system.

The jump process satisfies a non linear Poisson driven stochastic differential equation.

This approach is strongly inspired by probabilistic works on the Boltzmann equation. The Boltzmann equation deals with the distribution of the speeds in a gas, and can be related to the Smoluchowski equation for two reasons: first, it concerns the evolution of the "density of particles of speed v at the instant t", while the Smoluchowski equation deals with the "density of particles of mass x at the instant t". Second, the phenomenon is discontinuous: in each case, a particle moves instantly from a mass x (or a speed v) to a new mass x' (or a speed v') after a coagulation (or a collision).

Let us first mention Tanaka [18], who first introduced a non linear jump process in order to study the Boltzmann equation of Maxwell molecules. Many

new results about this equation has been obtained thanks to this approach, as regularity, positivity, and numerical approximation: Desvillettes, Graham and Méléard [5], [10], have initiated a new way for studying non linear P.D.E.s, by using recent probabilistic tools, such as Malliavin Calculus and propagation of chaos.

Very recently, Tanaka's approach has been extended, in Fournier, Méléard [7], [8], to the case of non Maxwell molecules, which is technically much less easy. We follow essentially here the approach of [7]. We will transpose it here in order to give a probabilistic interpretation to the Smoluchowski equations.

The main fact that makes the Maxwell molecules easy is that the rate of collisions of a particle does not depend on its speed, which is not the case for non Maxwell molecules. In the Smoluchowski equation, the "rate of coagulation" of a particle depends on its size.

We get rid of this problem by using a sort of "reject" procedure: as in [7], there is, in our stochastic equation, an indicator function which allows to control the rate of coagulation.

Let us finally describe the plan of the present paper.

In Section 2, we introduce our notations, we state a modified Smoluchowski equation (MS) which allows to study equations (SC) and (SD) together. This equation (MS) describes the evolution of the repartition $Q_t(dx)$ (either discrete or continuous) of the sizes: for each t, Q_t is a probability measure on $\mathbb{R}_+^*$. Afterwards we relate (MS) to a nonlinear martingale problem (MP): for Q a solution to (MP), its time marginals Q_t satisfy the modified Smoluchowski equation (MS). We finally exhibit a non linear Poisson driven stochastic differential equation (SDE), which gives a pathwise representation of (MP). If X_t satisfies (SDE), then its law is a solution to (MP). Notice that X_t can be seen as the evolution of a particle chosen randomly in the system, which coagulates randomly with other particles also chosen randomly. In other words, X_t is the evolution of the mass of a "mean particle". In Section 3, we state and prove an existence result for (SDE), under quite general assumptions. The pathwise properties of the solution to (SDE) are briefly discussed in Section 4. Section 5 deals with uniqueness results for (SDE). In Section 6, we study the case where $K(x,y) = xy$. This case drives to very simple computations, and the results we obtain are of course easy and satisfactory. The last section contains an appendix which includes some useful classical results.

In the sequel A and B stand for universal constants whose values may change from line to line.

A forthcoming paper will present a stochastic particle system associated with the process constructed in the present paper, which will permit to solve numerically the Smoluchowski's coagulation equation.

2 Framework

Our probabilistic approach is based on the following remark: there is conservation of mass in (SC) and (SD). This means in the discrete case that a solution $(n(k,t))_{t\geq 0, k\in\mathbb{N}^*}$ of (SD) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \sum_{k\geq 1} k\, n(k,t) = 1 \tag{2.1}$$

and in the continuous one that a solution $(n(x,t))_{t\geq 0, x\in\mathbb{R}_+^*}$ of (SC) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \int_0^\infty x\, n(x,t)dx = 1. \tag{2.2}$$

Thus, either in the discrete or continuous case, the quantity

$$Q_t(dx) = \sum_{k\geq 1} k\, n(k,t)\delta_k(dx) \quad\text{or}\quad Q_t(dx) = x\, n(x,t)\, dx \tag{2.3}$$

(where δ_k denotes the Dirac mass at k) is a probability measure on $\mathbb{R}_+$ for all $t \in [0,T_0[$

Let us first define the weak solution for (SC) (or (SD)).

Definition 2.1 *We say that $n(x,t)_{\{x,t\geq 0\}}$ is a weak solution of (SC) on $[0,T_0[$ if for all test function $\varphi \in C_0^1(\mathbb{R}_+)$ and all $t \in [0,T_0[$ we have*

$$\int_{\mathbb{R}_+} \varphi(x)n(x,t)\, dx = \int_{\mathbb{R}_+} \varphi(x)n(x,0)\, dx$$

$$+ \int_0^t ds \int_{\mathbb{R}_+} \varphi(x)\Big[\frac{1}{2}\int_0^x n(x-y,s)n(y,s)K(x-y,y)dy$$

$$- \int_{\mathbb{R}_+} n(x,s)n(y,s)K(x,y)dy\Big] dx. \tag{2.4}$$

For any t, $Q_t(dx)$ can be seen as the repartition of the mass of the particles at instant t. This leads us to define a modified Smoluchowski equation. We begin with some notations.

Notation 2.2 *1. We denote by $\mathcal{P}_1$ the set of probability measures Q on $\mathbb{R}_+$ such that*

$$Q\,(]0,\infty[) = 1 \quad;\quad \int_{\mathbb{R}_+} xQ(dx) < \infty. \tag{2.5}$$

2. Let $Q_0 \in \mathcal{P}_1$. We denote by

$$\mathcal{H}_{Q_0} = \overline{\left\{ \sum_{i=1}^n x_i \; ; \; x_i \in \text{Supp } Q_0\,,\, n \in \mathbb{N}^* \right\}}^{\mathbb{R}_+}. \tag{2.6}$$

Notice that $\mathcal{H}_{Q_0}$ is a closed subset of $\mathbb{R}_+$ which contains the support of Q_0.

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Since Q_0 is the repartition of the sizes of the particles in the initial system, $\mathcal{H}_{Q_0}$ simply represents the smallest closed subset of $\mathbb{R}_+$ in which the sizes of the particles will always take their values.

Definition 2.3 *Let Q_0 be a probability measure on $\mathbb{R}_+$ belonging to $\mathcal{P}_1$ and let $T_0 \leq \infty$. We will say that $(Q_t(dx))_{t \in [0,T_0[}$ is a weak solution to (MS) on $[0,T_0[$ with initial condition Q_0 if:*
for all $t \in [0,T_0[$, Supp $Q_t \subset \mathcal{H}_{Q_0}$ and $Q_t \in \mathcal{P}_1$, and for all $\varphi \in C_0^1(\mathbb{R}_+)$ and all $t \in [0,T_0[$,

$$\int_0^\infty \varphi(x) Q_t(dx) = \int_0^\infty \varphi(x) Q_0(dx)$$

$$+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x)] \frac{K(x,y)}{y} Q_s(dy) Q_s(dx) ds. \qquad (2.7)$$

It is obvious the procedure to pass from (SC) to (MS). It suffices to multiply by $\varphi(x)$ and integrate over $\mathbb{R}_+$. This definition allows us to consider together both discrete and continuous cases. To make this assertion clear, let us state the following result:

Proposition 2.4 *Let $(Q_t(dx))_{t \in [0,T_0[}$ be a weak solution to (MS), with initial condition $Q_0 \in \mathcal{P}_1$, for some $T_0 \leq \infty$.*

1. *If Supp $Q_0 \subset \mathbb{N}^*$, then clearly $\mathcal{H}_{Q_0} \subset \mathbb{N}^*$. Thus for all $t \in [0,T_0[$, Supp $Q_t \subset \mathbb{N}^*$, and we can write Q_t as:*

$$Q_t(dx) = \sum_{k \geq 1} \alpha_k(t) \, \delta_k(dx) \quad where \quad \alpha_k(t) = Q_t(\{k\}). \qquad (2.8)$$

   Then, the function $n(k,t) = \alpha_k(t)/k$ is a solution to (SD) on $[0,T_0[$, with initial condition $n_0(k) = \alpha_k(0)/k$

2. *Assume now that for all $t \in [0,T_0[$, the probability measure Q_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. We can thus write $Q_0(dx) = f_0(x)\,dx$ and for any $t \in]0,T_0[$, $Q_t(dx) = f(x,t)\,dx$. Then $n(x,t) = f(x,t)/x$ is a weak solution to (SC) on $[0,T_0[$, with initial condition $n_0(x) = f_0(x)/x$.*

3. *Other cases, as mixed cases, are contained in (MS).*

Notice that the assumption $Q_0 \in \mathcal{P}_1$ simply means that the initial condition to the Smoluchowski equation admits a moment of order 2: in the discrete case this writes $\sum_k k^2 n_0(k) < \infty$, while in the continuous case we have, $\int x^2 n_0(x) dx < \infty$.

Proof 1. Since $Q_t(dx) = \sum_{k \geq 1} \alpha_k(t) \, \delta_k(dx)$, with $\alpha_k(t) = k\, n(k,t)$, is a weak solution to (MS), we may apply (2.7) with $\varphi_k(x) \in C_c^1(\mathbb{R}_+^*)$ such that for some

$k \geq 1$

$$\varphi_k(x) = \begin{cases} 0 \text{ if } x \notin \left[k - \frac{1}{2}, k + \frac{1}{2}\right] \\[2mm] \dfrac{1}{k} \text{ if } x = k. \end{cases} \quad (2.9)$$

We obtain:

$$\frac{\alpha_k(t)}{k} = \frac{\alpha_k(0)}{k} + \int_0^t \frac{1}{k} \sum_{i \geq 1} \alpha_i(s) \sum_{j \geq 1} \alpha_j(s) \left[\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}}\right] \frac{K(i,j)}{j} ds \quad (2.10)$$

and thus

$$\begin{aligned} n(k,t) &= n_0(k) + \int_0^t \Big[\sum_{i=1}^{k-1} \alpha_i(s) n(k-i,s) \frac{K(i,k-i)}{k} \\ &\qquad\qquad - \sum_{j \geq 1} n(k,s) n(j,s) K(k,j) \Big] ds \\ &= n_0(k) + \int_0^t \Big[\frac{1}{2} \sum_{i=1}^{k-1} n(i,s) n(k-i,s) K(i,k-i) \\ &\qquad\qquad - \sum_{j \geq 1} n(k,s) n(j,s) K(k,j) \Big] ds \quad (2.11) \end{aligned}$$

where the last equality comes from the facts that $\alpha_i(s) = in(i,s)$ and $K(i,j)$ is a symmetric kernel.

2. We now assume that $Q_t(dx) = f(x,t)\, dx$ for all $t \in [0,T_0[$; let $\varphi \in C_c^1(\mathbb{R}_+^*)$. Let $\psi(x) = \varphi(x)/x$. Applying (2.7) to ψ, we obtain $Q_0(dx) = f(x,0)dx = xn_0(x)dx$ and :

$$\begin{aligned} \int_{\mathbb{R}_+} \varphi(x) n(x,t)\, dx &= \int_{\mathbb{R}_+} \varphi(x) n_0(x)\, dx \\ &+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x+y)}{x+y} \frac{K(x,y)}{y} f(x,s) f(y,s)\, dxdyds \\ &- \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x)}{x} \frac{K(x,y)}{y} f(x,s) f(y,s)\, dxdyds. \quad (2.12) \end{aligned}$$

Using a symmetry argument, we obtain:

$$
\begin{aligned}
I & = \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x+y)}{x+y} \frac{K(x,y)}{y} f(x,s) f(y,s)\, dx\,dy\,ds \\
& = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(x+y) K(x,y) f(x,s) f(y,s) \\
& \qquad\qquad\qquad \left(\frac{1}{y(x+y)} + \frac{1}{x(x+y)} \right) dx\,dy\,ds \\
& = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(x+y) K(x,y) n(x,s) n(y,s)\, dx\,dy\,ds.
\end{aligned}
$$

(2.13)

Using the substitution $x' = x + y$, $y' = y$, we obtain:

$$
I = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} dx' \int_0^{x'} dy' \varphi(y') K(x'-y', y') n(x'-y', s) n(y', s)\, ds.
$$

(2.14)

We have proved that for any $\varphi \in C_c^1(\mathbb{R}_+^*)$,

$$
\begin{aligned}
\int_{\mathbb{R}_+} \varphi(x) n(x,t)\, dx & = \int_{\mathbb{R}_+} \varphi(x) n(x,0)\, dx \\
& + \int_0^t ds \int_{\mathbb{R}_+} \varphi(x) \Big[\frac{1}{2} \int_0^x n(x-y,s) n(y,s) K(x-y,y) dy \\
& \qquad\qquad - \int_{\mathbb{R}_+} n(x,s) n(y,s) K(x,y) dy \Big]\, dx
\end{aligned}
$$

(2.15)

which is the definition of a weak solution for (SC). This ends the proof. $\qquad \square$

Equation (MS) has to be understood as the evolution equation of the time marginals of a pure jump Markov process. In order to exploit this remark, we will associate to (MS) a martingale problem. We begin with some notations.

Notation 2.5 *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We denote by $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$ the set of positive increasing càdlàg functions from $[0,T_0[$ to $\mathcal{H}_{Q_0}$. We denote by $\mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$ the set of probability measures Q on $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$ such that*

$$
Q\left(\left\{ x \in \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0}) \; ; \; x(0) > 0 \right\} \right) = 1 \qquad (2.16)
$$

and for all $t < T_0$,

$$
\int_{x \in \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})} x(t) Q(dx) = \int_{x \in \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})} \left(\sup_{s \in [0,t]} x(s) \right) Q(dx) < \infty.
$$

(2.17)

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The last equality comes naturally from the fact that $x(t)$ is increasing.

Definition 2.6 *Let $T_0 \leq \infty$, and $Q_0 \in \mathcal{P}_1$ be fixed. Consider $Q \in \mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$. Let Z be the canonical process of $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$. We say that Q is a solution to the martingale problem (MP) on $[0,T_0[$ if for all $\varphi \in C_b^1(\mathbb{R}_+)$ and $t \in [0,T_0[$,*

$$\varphi(Z_t) - \varphi(Z_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(Z_s + y) - \varphi(Z_s)] \frac{K(Z_s,y)}{y} Q_s(dy)ds \qquad (2.18)$$

is a $Q-$martingale, where Q_s denotes the law of Z_s under Q.

Taking expectations in (2.18), we obtain the following remark:

Remark 2.7 *Let Q be a solution to the martingale problem (MP) on $[0,T_0[$. For $t \in [0,T_0[$, let Q_t be its time marginal. Then $(Q_t)_{t\in[0,T_0[}$ is a weak solution of (MS) with initial condition Q_0.*

We are now seeking for a pathwise representation of the martingale problem (MP). To this aim, let us introduce some notations. The main ideas of the following notations and definition are taken from Tanaka [18].

Notation 2.8 *1. We consider two probability spaces: $(\Omega,\mathcal{F},\mathbb{P})$ is an abstract space and $([0,1],\mathcal{B}[0,1],d\alpha)$ is an auxiliary space (here, $d\alpha$ denotes the Lebesgue measure). In order to avoid confusion, the expectation on $[0,1]$ will be denoted E_α, the laws $\mathcal{L}_\alpha$, the processes will be said to be α-processes, etc.*

2. Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. An increasing positive càdlàg process $(X_t(\omega))_{t\in[0,T_0[}$ is said to belong to $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$ if its law belongs to $\mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$.

In the same way, an increasing positive càdlàg α-process $(\tilde{X}_t(x))_{t\in[0,T_0[}$ is said to belong to $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$ if its α-law belongs to $\mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$.

Definition 2.9 *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We say that $(X_0,X,\tilde{X},N)$ is a solution to the problem (SDE) on $[0,T_0[$ if:*

1. $X_0 : \Omega \to \mathbb{R}_+$ is a random variable whose law is Q_0.

2. $X_t(\omega) : [0,T_0[\times\Omega \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-process

3. $\tilde{X}_t(\alpha) : [0,T_0[\times[0,1] \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process.

4. $N(\omega,dt,d\alpha,dz)$ is a Poisson measure on $[0,T_0[\times[0,1] \times \mathbb{R}_+$ with intensity measure $dt\,d\alpha\,dz$ and is independent of X_0.

5. X and $\tilde{X}$ have the same law on their respective probability spaces: $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ (this equality holds in $\mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0}))$.

6. Finally, the following S.D.E. is satisfied on $[0,T_0[$:

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \quad (2.19)$$

The motivation of this definition is the following:

Proposition 2.10 *Let $(X_0, X, \tilde{X}, N)$ be a solution to (SDE) on $[0, T_0[$. Then the law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ satisfies the martingale problem (MP) on $[0, T_0[$ with initial condition $Q_0 = \mathcal{L}(X_0)$. Hence $\{\mathcal{L}(X_t)\}_{t \in [0, T_0[}$ is a solution to the modified Smoluchowski equation (MS) with initial condition Q_0.*

Before proving rigorously this result, we explain its main intuition: why is it natural to choose (X_t) satisfying (SDE), in order to obtain a stochastic process whose law satisfies the modified Smoluchowski equation (MS)?
We wish that the law Q_t of X_t describes the evolution of the repartition particles' masses in the system. A natural way to do this is to choose one particle randomly, and to use a random (but natural) coagulation dynamic. Thus, X_t should be understood as the evolution of the size of a sort of "mean" particle. Of course, X_0 has to follow the initial distribution Q_0. Then, at some random instants, which are typically Poissonian instants (for Markovian reasons), coalescence phenomenons occur. Let τ be one of these instants. We choose another particle, randomly, and we denote by $\tilde{X}_\tau(\alpha)$ its size. Then we describe the coagulation as $X_\tau = X_{\tau-} + \tilde{X}_\tau(\alpha)$. The indicator function in (2.19) allows to control the frequency of the coagulations.
Thus, from a time-evolution point of view, $X_{\cdot}$ mimics randomly the evolution of the size of one particle, thus its law is given by the (deterministic) "true" repartition of the sizes in the system at the instant t, which is exactly the solution of (MS).
For fixed t, X_t may be understood as a random variable representing the following experience: we choose randomly one particle in the system, at the instant t, (according to an "uniform law"), and we denote by X_t its size.

Let us now prove Proposition 2.10.

Proof Let φ be a $C_c^1(\mathbb{R}_+)$ function. Then for all $t \in [0, T_0[$,

$$\varphi(X_t) = \varphi(X_0) + \sum_{s \leq t} [\varphi(X_s) - \varphi(X_{s-})] \tag{2.20}$$

$$= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_{s-} + \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} \right) - \varphi(X_{s-}) \right]$$
$$N(ds, d\alpha, dz)$$

$$= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_{s-} + \tilde{X}_{s-}(\alpha) \right) - \varphi(X_{s-}) \right] \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}}$$
$$N(ds, d\alpha, dz).$$

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Hence :

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) \tag{2.21}$$

$$-\int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_s + \tilde{X}_s(\alpha)\right) - \varphi(X_s)\right] \mathbb{1}_{\left\{z \le \frac{K(X_s, \tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} \, dz \, d\alpha \, ds$$

can be written as a stochastic integral with respect to the compensated Poisson measure, and thus is a martingale. But

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) \tag{2.22}$$

$$-\int_0^t E_\alpha \left[\left(\varphi(X_s + \tilde{X}_s(\alpha)) - \varphi(X_s)\right) \frac{K(X_s, \tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right] ds$$

$$= \varphi(X_t) - \varphi(X_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(X_s + y) - \varphi(y)] \frac{K(X_s, y)}{y} Q_s(dy) ds$$

where $Q_s = \mathcal{L}_\alpha(\tilde{X}_s) = \mathcal{L}(X_s)$. We have proved that $\mathcal{L}(X)$ satisfies (MP) on $[0, T_0[$. □

Let us now state hypotheses which will allow to prove existence results for (SDE). In the sequel, we will always suppose that the coagulation kernel K satisfies the following hypothesis :

(H_β): The initial condition Q_0 belongs to $\mathcal{P}_1$. The symmetric kernel $\overline{K : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+}$ is locally Lipschitz continuous on $(\mathcal{H}_{Q_0})^2$, and satisfies, for some constant $C < \infty$,

$$K(x, y) \le C(1 + x + y + x^\beta y^\beta). \tag{2.23}$$

Two different cases will appear according with $\beta = 1/2$ or $\beta = 1$. We will always prove the results for the case $\beta = 1$ the other one being similar and easier to treat. Let us also remark that all results for $\beta = 1/2$ apply also for $0 \le \beta \le 1/2$ and similarly the ones for $\beta = 1$ are true for $1/2 < \beta \le 1$.
Notice that in the discrete case, $\mathcal{H}_{Q_0}$ is contained in $\mathbb{N}^*$, so that we don't need the local Lipschitz continuity condition.

3 Existence results for (SDE)

The aim of this section is to prove the following result.

Theorem 3.1 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β).*
1. If $\beta = 1/2$ then there exists a solution $(X_0, X, \tilde{X}, N)$ to (SDE), on $[0, T_0[$, where $T_0 = \infty$.

2. If $\beta = 1$ then there exists a solution $(X_0, X, \tilde{X}, N)$ to (SDE), on $[0, T_0[$, where $T_0 = 1/C(1 + E(X_0))$.

Remark 3.2 *From now on we state that under (H_β) if $\beta = 1/2$ then $T_0 = \infty$ and if $\beta = 1$ we put $T_0 = 1/C(1 + E(X_0))$.*

Thus, under (H_β), if $\beta = 1/2$ we obtain an existence result on $[0, \infty[$. This is not the case if $\beta = 1$, but this is not a limitation. It is classical that for $\beta = 1$, there may be a gelification time: for example, in the discrete case, Jeon has proved in [13] that if $K(x, y) \geq x^\beta y^\beta$ for some $\beta \in]1/2, 1[$, then a solution $n(k, t)$ to (SD) will satisfy

$$T_{gel} = \inf \left\{ t \geq 0 \; ; \; \sum_{k \geq 1} k^2 n(k, t) = \infty \right\} < \infty \qquad (3.1)$$

which writes, with our notations,

$$T_{gel} = \inf \{ t \geq 0 \; ; \; E(X_t) = \infty \} < \infty. \qquad (3.2)$$

It is thus clear that an existence result on $[0, \infty[$ can not be proved under the general assumption (H_β) for $\beta = 1$.
Finally, notice that for $\beta = 1$, $T_0 = 1/C(1 + E(X_0))$ is not the exact gelification time, except if $K(x, y) = C(1 + x + y + xy)$: since we only assume an upper bound on K, we are able only to prove an existence result for (SDE) on $[0, T_0[$, for some $T_0 \leq T_{gel}$. We however will give exact gelification times corresponding to a class of coagulation kernels for which explicit computations are easy. In such cases, our existence result will easily extend to $[0, T_{gel}[$.

Technically, Theorem 3.1 is not easy to prove, because the coefficients of (SDE) are not globally Lipschitz continuous. Due to the nonlinearity, a direct construction is difficult. Thus, in a first proposition, we prove a result, which combined with Proposition 2.10 shows that the existence (resp. uniqueness in law) for (SDE) is equivalent to existence (resp. uniqueness) for (MP). It will thus be sufficient to prove an existence result for (MP).
Next, we use a cutoff procedure, which transforms the coefficients of our equation globally Lipschitz continuous: we obtain the existence of a solution X^ε to a cutoff equation $(SDE)_\varepsilon$. Then we prove tightness and uniform integrability results, which allow to prove that the family $\mathcal{L}(X^\varepsilon)$ has limiting points, and that these limiting points satisfy (MP).

As said previously, we begin with a proposition, which, combined with Proposition 2.10, shows a sort of equivalence between (MP) and (SDE).

Proposition 3.3 *Let Q_0 belong to $\mathcal{P}_1$. Assume that $Q \in \mathcal{P}_1^{\uparrow}([0, T_0[, \mathcal{H}_{Q_0})$ is a solution to (MP) with initial condition $Q_0 \in \mathcal{P}_1$ on $[0, T_0[$ for some $T_0 \leq \infty$. Consider any $L_1^{T_0, \uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process $\tilde{X}$ such that $\mathcal{L}_\alpha(\tilde{X}) = Q$. Consider also*

the canonical process Z of $\mathbb{D}^{\uparrow}([0,T_0[,\mathcal{H}_{Q_0}))$. Then there exists, on an enlarged probability space (from the canonical one), a Poisson measure $N(\omega,dt,d\alpha,dz)$, independent of Z_0 (all of this under Q), such that $(Z_0,Z,\tilde{X},N)$ is a solution to (SDE) (still under Q).

Proof   It follows from (MP), since $\mathcal{L}_\alpha(\tilde{X}_s) = Q_s$ for all s, that for any $\varphi \in C_b^1(\mathbb{R}_+)$,

$$M_t^\varphi = \varphi(Z_t) - \varphi(Z_0) \tag{3.3}$$

$$- \int_0^t \int_0^1 \left[\varphi(Z_s + \tilde{X}_s(\alpha)) - \varphi(Z_s) \right] \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)} d\alpha\,ds$$

$$= \varphi(Z_t) - \varphi(Z_0)$$

$$- \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(Z_s + \tilde{X}_s(\alpha)\,\mathbb{1}_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} \right) - \varphi(Z_s) \right] dz\,d\alpha\,ds$$

is a martingale under Q. Applying this result with $\varphi(x) = x$, we deduce that

$$Z_t = Z_0 + M_t + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_s(\alpha)\,\mathbb{1}_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz\,d\alpha\,ds \tag{3.4}$$

where M_t is a martingale (under Q). This decomposition is unique, in the sense that if $Z_t = Z_0 + L_t + F_t$, where L_t is a local martingale and F_t has bounded variations, then $L_t = M_t$ and

$$F_t = \int_0^t \int_0^1 \int_0^\infty \tilde{X}_s(\alpha)\,\mathbb{1}_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz\,d\alpha\,ds \tag{3.5}$$

(see Jacod-Shiryaev [12], p 43). Hence, applying the Itô formula for jump processes (see e.g. [12]), we see that for any $\varphi \in C_b^2(\mathbb{R}_+)$ and $t \in [0,T_0[$,

$$\varphi(Z_t) = \varphi(Z_0) + \int_0^t \varphi'(Z_{s-})\,dZ_s + \frac{1}{2} \int_0^t \varphi''(Z_s)\,dM_s^c$$

$$+ \sum_{s \leq t} \left[\varphi(Z_{s-} + \Delta Z_s) - \varphi(Z_{s-}) - \Delta Z_s \varphi'(Z_{s-})\right] \tag{3.6}$$

where M^c denotes the continuous martingale part of M. A comparison with (3.3) shows that $M^c \equiv 0$, and hence that M is a pure jump martingale.

A second comparison between (3.3) and (3.6) shows that the compensator of the jump measure $\mu = \sum_{s \leq T_0} \delta_{(s,\Delta M_s)}$ of M is the image measure of the Lebesgue measure $ds\,d\alpha\,dz$ by the map:

$$(s,\alpha,z) \rightarrow \tilde{X}_{s-}(\alpha)\,\mathbb{1}_{\left\{z \leq \frac{K(Z_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} \cdot \tag{3.7}$$

Using a representation theorem for point processes (see El Karoui, Lepeltier [6]) we see that there exists, on an enlarged probability space, a Poisson measure $N(\omega, dt, d\alpha, dz)$, with intensity measure $dt\, d\alpha\, dz$, such that:

$$M_t = \int_0^t \int_0^1 \int_{\mathbb{R}_+} \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{ z \leq \frac{K(Z_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} \bar{N}(ds, d\alpha, dz), \qquad (3.8)$$

$\bar{N}(ds, d\alpha, dz)$ denoting the compensated Poisson measure of N, i.e.

$$\bar{N}(ds, d\alpha, dz) = N(ds, d\alpha, dz) - ds\, d\alpha\, dz. \qquad (3.9)$$

We finally obtain:

$$Z_t = Z_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{ z \leq \frac{K(Z_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} N(ds, d\alpha, dz). \qquad (3.10)$$

Since we work under Q and since $\mathcal{L}_\alpha(\tilde{X}) = Q$, we deduce that $(Z_0, Z, \tilde{X}, N)$ satisfies (SDE). This was our aim. $\qquad \square$

In order to prove Theorem 3.1, we first consider a simpler problem with cutoff.
For Q_0 in $\mathcal{P}_1$, we define a solution $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ to $(SDE)_\varepsilon$ exactly in the same way as in Definition 2.9, but replacing (2.19) by

$$X_t^\varepsilon = X_0 + \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^\varepsilon(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} N(ds, d\alpha, dz)$$

$$(3.11)$$

with the conditions that $\mathcal{L}(X^\varepsilon) \in \mathcal{P}_1^\uparrow([0, T_0[, \mathcal{H}_{Q_0})$ and that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$.

We begin with an important remark:

Remark 3.4 *We need that for each $\varepsilon > 0$ and for $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ a solution to $(SDE)_\varepsilon$, X^ε takes its values in $\mathcal{H}_{Q_0}$. Indeed, the regularity assumption (H_β) on K holds only on $\mathcal{H}_{Q_0}$. Hence, in (3.11), $x \vee \varepsilon \wedge (1/\varepsilon)$ is only a notation, of which the rigorous definition is, for any $x \in \mathcal{H}_{Q_0}$, any $\varepsilon > 0$,*

$$x \vee \varepsilon \wedge (1/\varepsilon) = \begin{cases} \inf\{y \in \mathcal{H}_{Q_0} \; ; \; y \geq \varepsilon\} & \text{if} \quad 0 \leq x \leq \varepsilon \\ x & \text{if} \quad x \in [\varepsilon, 1/\varepsilon] \\ \sup\{y \in \mathcal{H}_{Q_0} \; ; \; y \leq 1/\varepsilon\} & \text{if} \quad 1/\varepsilon \leq x. \end{cases} \qquad (3.12)$$

Of course, $x \wedge (1/\varepsilon)$ is defined in the same way. With these definitions, $x \vee \varepsilon \wedge (1/\varepsilon)$ and $x \wedge (1/\varepsilon)$ belong to $\mathcal{H}_{Q_0}$ for any $x \in \mathcal{H}_{Q_0}, \varepsilon > 0$.
We now prove an existence result for $(SDE)_\varepsilon$.

Proposition 3.5 *Let $Q_0 \in \mathcal{P}_1$ and $\varepsilon > 0$. Assume (H_β). Let X_0 be a random variable whose law is Q_0 and N be a Poisson measure independent of X_0. Then there exists a solution $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ to $(SDE)_\varepsilon$.*

Proof The proof mimics that of Tanaka, who proved in [18] a similar result in the case of a nonlinear S.D.E. related to a Boltzmann equation (and with globally Lipschitz coefficients). To this end, we introduce the following non-classical Picard approximations. First, we consider the process $X^{0,\varepsilon} \equiv X_0$, and any α-process $\tilde{X}^{0,\varepsilon}$ such that $\mathcal{L}_\alpha(\tilde{X}^{0,\varepsilon}) = \mathcal{L}(X^{0,\varepsilon})$.

Once everything is built up to n, we set

$$X_t^{n+1,\varepsilon} = X_0 + \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) \mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}}$$

$$N(ds,d\alpha,dz) \tag{3.13}$$

and we consider any α-process $\tilde{X}^{n+1,\varepsilon}$ such that

$$\mathcal{L}_\alpha(\tilde{X}^{n+1,\varepsilon}|\tilde{X}^{0,\varepsilon},\ldots,\tilde{X}^{n,\varepsilon}) = \mathcal{L}(X^{n+1,\varepsilon}|X^{0,\varepsilon},\ldots,X^{n,\varepsilon}). \tag{3.14}$$

One easily checks recursively that for each n, $X^{n,\varepsilon}$ is an $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-process. Let us show now that the sequence $\{X^{n,\varepsilon}\}_n$ is Cauchy in $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$. A simple computation gives

$$X_t^{n+1,\varepsilon} - X_t^{n,\varepsilon} = \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} - \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right)$$

$$\mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} N(ds,d\alpha,dz)$$

$$+ \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) .$$

$$\left(\mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} - \mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} \right) N(ds,d\alpha,dz).$$

$$\tag{3.15}$$

Let us for example present the proof under (H_β) when $\beta = 1$. Then

$$\left| X_t^{n+1,\varepsilon} - X_t^{n,\varepsilon} \right| \le \int_0^t \int_0^1 \int_0^\infty \left| \tilde{X}_{s-}^{n,\varepsilon}(\alpha) - \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \right| \tag{3.16}$$

$$\mathbb{1}_{\left\{ z \le \frac{C}{\varepsilon}(1 + 2/\varepsilon + 1/\varepsilon^2) \right\}} N(ds,d\alpha,dz)$$

$$+ \frac{1}{\varepsilon} \int_0^t \int_0^1 \int_0^\infty \left| \mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} - \mathbb{1}_{\left\{ z \le \frac{K(X_{s-}^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} \right|$$

$$N(ds,d\alpha,dz).$$

By setting $\varphi_n(t) := E\left[\sup_{s \in [0,t]} \left| X_s^{n+1,\varepsilon} - X_s^{n,\varepsilon} \right|\right]$, we obtain, for some constant A, depending only on ε,

$$\varphi_n(t) \le A \int_0^t \varphi_{n-1}(s) \, ds \tag{3.17}$$

$$+ A \int_0^t E E_\alpha \left| \frac{K\left(X_s^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_s^{n,\varepsilon} \wedge \frac{1}{\varepsilon}\right)}{\tilde{X}_s^{n,\varepsilon} \vee \varepsilon \wedge \frac{1}{\varepsilon}} - \frac{K\left(X_s^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_s^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}\right)}{\tilde{X}_s^{n-1,\varepsilon} \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right| \, ds.$$

Since K is locally Lipschitz continuous on $(\mathcal{H}_{Q_0})^2$, it is clear that the map

$$(x,y) \mapsto \frac{K\left(x \wedge \frac{1}{\varepsilon}, y \wedge \frac{1}{\varepsilon}\right)}{x \vee \varepsilon \wedge \frac{1}{\varepsilon}} \tag{3.18}$$

is globally Lipschitz continuous on $(\mathcal{H}_{Q_0})^2$. Hence, using the fact that $\int_0^1 |\tilde{X}^{n,\varepsilon}(\alpha) - \tilde{X}^{n-1,\varepsilon}(\alpha)| \, d\alpha \le \varphi_{n-1}(s)$, we obtain

$$\varphi_n(t) \le A \int_0^t \varphi_{n-1}(s) \, ds. \tag{3.19}$$

We conclude, thanks to the usual Picard Lemma, that there exists a $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-process X^ε such that, for any $T < \infty$, when n tends to infinity,

$$E\left[\sup_{t \in [0,T]} |X_t^{n,\varepsilon} - X_t^\varepsilon| \right] \longrightarrow 0. \tag{3.20}$$

By construction, the α-law of the sequence of processes $\tilde{X}^{0,\varepsilon}, \ldots, \tilde{X}^{n,\varepsilon}, \ldots$ is the same as the law of the sequence $X^{0,\varepsilon}, \ldots, X^{n,\varepsilon}, \ldots$. We thus deduce the existence of an $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process $\tilde{X}^\varepsilon$ such that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, and such that for all $T < \infty$, when n tends to infinity,

$$E_\alpha\left[\sup_{t \in [0,T]} |\tilde{X}_t^{n,\varepsilon} - \tilde{X}_t^\varepsilon| \right] \longrightarrow 0. \tag{3.21}$$

Letting n go to infinity in (3.13) concludes the proof. $\qquad\square$

We now prove the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$.

Lemma 3.6 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β). For $\beta = 1/2$ set $T_0 = \infty$, while for $\beta = 1$ set $T_0 = 1/C \left(1 + \int x Q_0(dx)\right)$, where C is the constant which appear in the hypothesis (H_β).*
Consider a family $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ of solutions to $(SDE)_\varepsilon$. Then, for all $T < T_0$,

$$\sup_{\varepsilon > 0} E\left[\sup_{t \in [0,T]} |X_t^\varepsilon| \right] = \sup_{\varepsilon > 0} E_\alpha\left[\sup_{t \in [0,T]} |\tilde{X}_t^\varepsilon| \right] < \infty. \tag{3.22}$$

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Furthermore, the family $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ of probability measures on $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$ is tight, and any limiting point Q of a convergent subsequence is the law of a quasi-left continuous process (for the definition see Jacod, Shiryaev [12]). .

Proof   Let us again prove the result under (H_β) for $\beta = 1$, the case of $\beta = 1/2$ being similar. We first check (3.22). Setting

$$f_\varepsilon(t) = E\left[\sup_{s \in [0,t]} |X_s^\varepsilon|\right] \tag{3.23}$$

it is immediate, since the processes are positive and increasing and since for each ε, $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, that:

$$f_\varepsilon(t) = E\left[X_t^\varepsilon\right] = E_\alpha\left[\tilde{X}_t^\varepsilon\right] \tag{3.24}$$

A simple computation, using (3.11), yields that

$$f_\varepsilon(t) = E(X_0) + \int_0^t EE_\alpha\left[K\left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\right)\right] ds. \tag{3.25}$$

But since we are under (H_β) with $\beta = 1$, it is clear that

$$EE_\alpha\left[K\left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\right)\right] \leq C\left(1 + 2f_\varepsilon(s) + f_\varepsilon^2(s)\right) = C\left(1 + f_\varepsilon(s)\right)^2. \tag{3.26}$$

Lemma 7.3 of the appendix, applied to the function $g_\varepsilon = 1 + f_\varepsilon$, which is clearly continuous thanks to (3.25) allows to conclude that for any $t < T_0 = 1/C(1 + E(X_0))$,

$$f_\varepsilon(t) \leq \frac{1 + E(X_0)}{1 - t/T_0} - 1 \tag{3.27}$$

from which (3.22) is straightforward.

In order to obtain the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$, we use the Aldous criterion, which is recalled in the appendix (Theorem 7.1).
We just have to check (for example) that for all $T < T_0$ fixed, there exists a constant A_T such that for all $\delta > 0$, all couple of stopping times S and S' satisfying *a.s.* $0 \leq S \leq S' \leq (S + \delta) \wedge T$, and all ε,

$$E|X_{S'}^\varepsilon - X_S^\varepsilon| \leq A_T \delta \tag{3.28}$$

the constant A_T being independent of ε, δ, S and S'. This is not hard. Indeed,

$$|X_{S'}^\varepsilon - X_S^\varepsilon| = \int_S^{S'} \left(\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^\varepsilon(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}}\right\}} N(ds, d\alpha, dz). \tag{3.29}$$

Hence

$$
\begin{aligned}
E\left[|X_{S'}^{\varepsilon} - X_{S}^{\varepsilon}|\right] &= EE_{\alpha}\left[\int_{S}^{S'} K(X_s^{\varepsilon},\tilde{X}_s^{\varepsilon}(\alpha))ds\right] \\
&\leq \delta \sup_{s\in[0,T]} EE_{\alpha}\left[K(X_s^{\varepsilon},\tilde{X}_s^{\varepsilon})\right]. \qquad (3.30)
\end{aligned}
$$

But thanks to (H_{β}) for $\beta = 1$ and to (3.22) (since $T < T_0$),

$$
\begin{aligned}
\sup_{s\in[0,T]} EE_{\alpha}\left[K(X_s^{\varepsilon},\tilde{X}_s^{\varepsilon})\right] &\leq C \sup_{s\in[0,T]} EE_{\alpha}\left[1 + X_s^{\varepsilon} + \tilde{X}_s^{\varepsilon} + X_s^{\varepsilon}\tilde{X}_s^{\varepsilon}\right] \\
&\leq A_T \qquad (3.31)
\end{aligned}
$$

which concludes the proof. $\qquad\square$

To prove that any limiting point Q of $\mathcal{L}(X^{\varepsilon})$ satisfies (MP), we will also need a property of uniform integrability, which will be obtained in the next lemma.

Lemma 3.7 *Assume that $Q_0 \in \mathcal{P}_1$, and that $\int x^2 Q_0(dx) < \infty$. Assume (H_{β}), and following the value of β consider the associated T_0. Consider a family $(X_0, X^{\varepsilon}, \tilde{X}^{\varepsilon}, N)$ of solutions to $(SDE)_{\varepsilon}$. Then for all $T < T_0$ fixed,*

$$
\sup_{\varepsilon > 0} E\left[\sup_{t\in[0,T]} |X_t^{\varepsilon}|^2\right] < \infty. \qquad (3.32)
$$

Proof For $k \in \mathbb{N}^*$, we define

$$
g_k^{\varepsilon}(t) = E\left[\sup_{t\in[0,T]} |X_t^{\varepsilon}|^k\right] = E\left[(X_T^{\varepsilon})^k\right]. \qquad (3.33)
$$

For all $t < T_0$,

$$
\begin{aligned}
(X_t^{\varepsilon})^2 &= (X_0)^2 + \sum_{s\leq t}\left(\left(X_{s-}^{\varepsilon} + \Delta X_s^{\varepsilon}\right)^2 - \left(X_{s-}^{\varepsilon}\right)^2\right) \qquad (3.34) \\
&= (X_0)^2 + \sum_{s\leq t}\left(2X_{s-}^{\varepsilon}\Delta X_s^{\varepsilon} + (\Delta X_s^{\varepsilon})^2\right) \\
&= (X_0)^2 + \int_0^t \int_0^1 \int_0^{\infty}\left(2X_{s-}^{\varepsilon}\left(\tilde{X}_{s-}^{\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right) + \left(\tilde{X}_{s-}^{\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right)^2\right) \\
&\qquad \mathbb{1}_{\left\{z \leq \frac{K\left(X_{s-}^{\varepsilon}\wedge\frac{1}{\varepsilon},\tilde{X}_{s-}^{\varepsilon}(\alpha)\wedge\frac{1}{\varepsilon}\right)}{\tilde{X}_{s-}^{\varepsilon}(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}}\right\}} N(ds,d\alpha,dz).
\end{aligned}
$$

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Hence

$$g_2^\varepsilon(t) \;=\; E(X_0^2) + 2 \int_0^t EE_\alpha \left[X_s^\varepsilon K \left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon} \right) \right] ds$$

$$+ \int_0^t EE_\alpha \left[(\tilde{X}_s^\varepsilon \vee \varepsilon) K \left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon} \right) \right] ds \qquad (3.35)$$

Let us finish the proof for (H_β) with $\beta = 1$, the other case being similar. Using (H_β) with $\beta = 1$, the fact that $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ and (3.22), we obtain the existence of a constant A_T, not depending on ε, such that for all $t \leq T$,

$$g_2^\varepsilon(t) \;\leq\; E(X_0^2) + 3C \int_0^t EE_\alpha \left[\left(\tilde{X}_s^\varepsilon + \varepsilon \right) \left(1 + X_s^\varepsilon + \tilde{X}_s^\varepsilon + X_s^\varepsilon \tilde{X}_s^\varepsilon \right) \right] ds$$

$$\leq\; E(X_0^2) + A_T \int_0^t \left[1 + g_2^\varepsilon(s) \right] ds. \qquad (3.36)$$

The usual Gronwall Lemma allows to conclude. $\qquad\qquad\square$

The following lemma, associated with Proposition 3.3, will conclude the proof of Theorem 3.1.

Lemma 3.8 *Let Q_0 belong to $\mathcal{P}_1$ and satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Consider a family $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ of solutions to $(SDE)_\varepsilon$, and a limiting point Q of the tight family $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$. Then Q is a solution to (MP) on $[0, T_0[$, with initial condition $Q_0 = \mathcal{L}(X_0)$.*

Proof We prove the result for $\beta = 1$. The other case is simpler. Let Q be the limit of a sequence of $Q^k = \mathcal{L}(X^{\varepsilon_k})$, ε_k being a sequence of positive real numbers decreasing to 0.
We have to check that for any $\phi \in C_b^1(\mathbb{R}_+)$, any $g_1, \ldots, g_l \in C_b(\mathbb{R}_+)$ and any $0 \leq s_1 \leq \cdots \leq s_l < s < t < T_0$,

$$\langle Q \otimes Q, F \rangle = 0 \qquad (3.37)$$

where F is the map from $\mathbb{D}^\uparrow ([0, T_0[, \mathcal{H}_{Q_0}) \times \mathbb{D}^\uparrow ([0, T_0[, \mathcal{H}_{Q_0})$ defined by

$$F(x, y) = g_1(x(s_1)) \times \cdots \times g_l(x(s_l)) \times$$

$$(3.38)$$

$$\left\{ \phi(x(t)) - \phi(x(s)) - \int_s^t \left[\phi(x(u) + y(u)) - \phi(x(u)) \right] \frac{K(x(u), y(u))}{y(u)} du \right\}.$$

It is clear from the definition of the process X^{ε_k} that for any k,

$$\langle Q^k \otimes Q^k, F^k \rangle = 0, \qquad (3.39)$$

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where F^k is defined by:

$$F^k(x,y) = g_1(x(s_1)) \times \cdots \times g_l(x(s_l)) \times \Big\{ \phi(x(t)) - \phi(x(s)) $$

$$\text{(3.40)}$$

$$- \int_s^t \left[\phi(x(u) + y(u) \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \phi(x(u)) \right] \frac{K(x(u) \wedge \frac{1}{\varepsilon_k}, y(u) \wedge \frac{1}{\varepsilon_k})}{y(u) \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} du \Big\} \quad .$$

It thus suffices to prove that $\langle Q^k \otimes Q^k, F^k \rangle$ tends to $\langle Q \otimes Q, F \rangle$ as k tends to infinity. We split the proof in two steps.

Step 1: Let us first check that, as k goes to infinity,

$$\langle Q^k \otimes Q^k, |F - F^k| \rangle \longrightarrow 0. \qquad \text{(3.41)}$$

By definition,

$$\langle Q^k \otimes Q^k, |F - F^k| \rangle = EE_\alpha \Big[\Big| g_1(X^{\varepsilon_k}(s_1)) \times \cdots \times g_l(X^{\varepsilon_k}(s_l))$$

$$\int_s^t \Big\{ \left[\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k}) \right] \frac{K(X_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}}$$

$$- \left[\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k}) \right] \frac{K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \Big\} du \Big| \Big]. \qquad \text{(3.42)}$$

Hence, for some constant A, $\langle Q^k \otimes Q^k, |F - F^k| \rangle$ is smaller than

$$AEE_\alpha \Big[\int_s^t \left| \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \right|$$

$$K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) du \Big]$$

$$+ AEE_\alpha \Big[\int_s^t \left| \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} \right| \qquad \text{(3.43)}$$

$$\left| K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) - K(X_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}) \right| du \Big]$$

$$= A(I_{\varepsilon_k} + J_{\varepsilon_k}),$$

with obvious notations for I_{ε_k} and J_{ε_k}. Since φ' is bounded, we obtain, using (H_β) for $\beta = 1$,

$$
\begin{aligned}
J_{\varepsilon_k} \quad &\leq \quad 2\|\varphi'\|_\infty EE_\alpha \Big[\int_s^t \Big(\mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} + \mathbb{1}_{\{\bar{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} \Big) \\
&\hspace{6cm} (1 + X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} + X_u^{\varepsilon_k}\tilde{X}_u^{\varepsilon_k})du \Big] \\
&\leq \quad A\int_s^t \Big\{ \mathbb{P}(X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}) + E_\alpha(\tilde{X}_u^{\varepsilon_k})\mathbb{P}(X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}) + E(X_u^{\varepsilon_k}\mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}}) \\
&\hspace{4cm} + E(X_u^{\varepsilon_k}\mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}})E_\alpha(\tilde{X}_u^{\varepsilon_k})) \Big\} du. \tag{3.44}
\end{aligned}
$$

Since the processes are increasing, and thanks to (3.22), we deduce that

$$
J_{\varepsilon_k} \leq A \left[\mathbb{P}(X_t^{\varepsilon_k} > 1/\varepsilon_k) + E\left[X_t^{\varepsilon_k}\mathbb{1}_{\{X_t^{\varepsilon_k} > 1/\varepsilon_k\}} \right] \right]. \tag{3.45}
$$

The uniform integrability obtained in Lemma 3.7 allows to conclude that J_{ε_k} tends to 0.

Let us now bound I_{ε_k} from above. First,

$$
\begin{aligned}
I_{\varepsilon_k} \quad &\leq \quad AEE_\alpha \Big[\int_s^t \Big\{ \mathbb{1}_{\{\bar{X}_u^{\varepsilon_k} < \varepsilon_k\}} K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) \\
&\hspace{2cm} \left| \frac{\varphi(X_u^{\varepsilon_k} + \varepsilon_k) - \varphi(X_u^{\varepsilon_k})}{\varepsilon_k} - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \right| \\
&\hspace{2cm} + \mathbb{1}_{\{\bar{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) \\
&\hspace{2cm} \left| \frac{\varphi(X_u^{\varepsilon_k} + \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\frac{1}{\varepsilon_k}} - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \right| \Big\} du \Big] \\
&= \quad I_{\varepsilon_k}^1 + I_{\varepsilon_k}^2, \tag{3.46}
\end{aligned}
$$

with obvious notations. The second term is similar to J_{ε_k}, and thus goes to 0 as k tends to infinity. Using (H_β) with $\beta = 1$ and (3.22), we see that the first

term is smaller than

$$I_{\varepsilon_k}^1 \quad \leq 2A||\varphi'||_\infty \int_s^t EE_\alpha \left[\mathbb{1}_{\{\bar X_u^{\varepsilon_k} < \varepsilon_k\}} (1 + X_u^{\varepsilon_k} + \tilde X_u^{\varepsilon_k} + \tilde X_u^{\varepsilon_k} X_u^{\varepsilon_k}) \right] du$$

$$\leq A \int_s^t EE_\alpha \left[(1 + X_u^{\varepsilon_k})(1 + \varepsilon_k) \mathbb{1}_{\{\bar X_u^{\varepsilon_k} < \varepsilon_k\}} \right] du$$

$$\leq A \int_s^t E(1 + X_u^{\varepsilon_k}) E_\alpha (\mathbb{1}_{\{\bar X_u^{\varepsilon_k} < \varepsilon_k\}}) du$$

$$\leq A \int_0^t \mathbb{P}(X_u^{\varepsilon_k} < \varepsilon_k) du$$

$$\leq At\, \mathbb{P}(X_0 < \varepsilon_k)$$

where the last inequality comes from the fact that the process X^{ε_k} is increasing. This goes to 0, because $X_0 > 0$ a.s. Step 1 is finished.

$\underline{\text{Step 2}}$: It remains to prove that as k goes to infinity,

$$\langle Q^k \otimes Q^k, F \rangle \longrightarrow \langle Q \otimes Q, F \rangle. \tag{3.48}$$

This convergence would be obvious if F was continuous and bounded on $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0}) \times \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$, thanks to the definition of the convergence in law. The map F is not continuous on $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0}) \times \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$, but only on $\mathcal{C} \times \mathcal{C}$, where

$$\mathcal{C} = \left\{ x \in \mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0}) \; ; \; \Delta x(s_1) = \ldots \Delta x(s_l) = \Delta x(s) = \Delta x(t) = 0 \right\}. \tag{3.49}$$

Thanks to Lemma 3.6, Q is the law of a quasi-left continuous process, thus $Q(\mathcal{C}) = 1$, and hence F is $Q \otimes Q$-a.e. continuous. This implies that for any positive constant A,

$$\langle Q^k \otimes Q^k, F \wedge A \vee (-A) \rangle \longrightarrow \langle Q \otimes Q, F \wedge A \vee (-A) \rangle \tag{3.50}$$

because $F \wedge A \vee (-A)$ is $Q \otimes Q$-a.e. continuous and bounded. Thus (3.48) will hold if we prove that

$$\sup_k \langle Q^k \otimes Q^k, |F| \mathbb{1}_{|F| \geq A} \rangle \longrightarrow 0 \tag{3.51}$$

as A tends to infinity. One can check, after many but easy computations, that

$$\langle Q^k \otimes Q^k, |F| \mathbb{1}_{|F| \geq A} \rangle \leq BE \left[X_t^\varepsilon \mathbb{1}_{\{X_t^\varepsilon > \zeta(A)\}} \right] \tag{3.52}$$

for some constants B and some function $\zeta(A)$ tending to infinity with A. The uniform integrability obtained in Lemma 3.7 allows to conclude that (3.51) holds. Hence (3.48) is valid. This concludes the proof of Step 2 and thus the

proof of the lemma. $\qquad \square$

Let us finally give the proof of the main result of this section.

Proof of Theorem 3.1 Thanks to Lemma 3.5, there exists a solution $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ to $(SDE)_\varepsilon$ for each ε. Due to Lemma 3.6, the sequence $\{\mathcal{L}(X^\varepsilon)\}$ is tight, and in particular there exists a sequence ε_k decreasing to 0 such that $\{\mathcal{L}(X^{\varepsilon_k})\}$ tends to some Q. From Lemma 3.8, Q satisfies (MP). Finally, Proposition 3.3 allows us to build a solution $(X_0,X,\tilde{X},N)$ to (SDE). $\qquad \square$

4 Pathwise behaviour of (SDE)

In this short section, we would like to give an idea on the pathwise properties of X_t, for $(X_0,X,\tilde{X},N)$ a solution to (SDE). We have very few results on this topic, and the study seems to be difficult. However, we hope that new results will arise in a forthcoming paper. Let us begin with a remark concerning the long time behaviour.

Remark 4.1 *Let Q_0 belong to $\mathcal{P}_1$, and satisfy $\int x^2 Q_0(dx) < \infty$. Let us assume (H_β) with $\beta = 1/2$, and let $(X_0,X,\tilde{X},N)$ be a solution to the corresponding (SDE). A natural question is the following. Does the size of every particle in the system tend to infinity when the time grows to infinity? In other words, does X_t tend to infinity a.s. with t? This result, which seems to hold in the cases where $K(x,y) = 1$ and $K(x,y) = x + y$ (for which explicit computations are easy), is not obvious. It is clear that a lower bound of K has to be supposed. Indeed, if we assume for example that $K(x,y)$ vanishes for all x,y such that $x \vee y \geq n$ then it is clear that a.s., $\lim_t X_t < \infty$. This comes from the fact that in such a case, if we set*

$$\tau = \inf\{t > 0 \ ; \ X_t \geq n\} \tag{4.1}$$

then either $\tau = \infty$ (and hence $\lim_t X_t < \infty$) or τ is finite, and then it is easily deduced from (2.19) that $\lim_t X_t = X_\tau < \infty$.
We are not able yet to express properly the lower bound which has to be assumed on K.

We now present an idea about the frequency of the jumps of X_t. How often does a particle in the system coagulate?
The following result, which says that the number of jumps is finite on every compact interval, is not *a priori* obvious in the continuous case.

Proposition 4.2 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Let $(X_0,X,\tilde{X},N)$ be a solution to the correspon-*

ding (SDE). Assume furthermore that

$$\int_{\mathbb{R}_+} \frac{1}{x} Q_0(dx) < \infty \tag{4.2}$$

which always holds in the discrete case, and which simply means, in the continuous case, that $\int n_0(x) dx < \infty$.

Denote by $J_t = \sum_{s \leq t} \mathbb{1}_{\{\Delta X_s \neq 0\}}$ the number of jumps of X on $[0,t]$. Then for all $t < T_0$, $E[J_t] < \infty$.

Proof Let us again prove the result for $\beta = 1$. Thanks to (2.19), we see that

$$J_t = \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} N(ds, d\alpha, dz) \tag{4.3}$$

and hence

$$
\begin{aligned}
E[J_t] &= E\left[\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{ z \leq \frac{K(X_s, \tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)} \right\}} dz \, d\alpha \, ds \right] \\
&= \int_0^t E E_\alpha \left[\frac{K(X_s, \tilde{X}_s)}{\tilde{X}_s} \right] ds.
\end{aligned}
\tag{4.4}
$$

Using (H_β) with $\beta = 1$, we obtain

$$
\begin{aligned}
E[J_t] &\leq C \int_0^t E E_\alpha \left[1/\tilde{X}_s + X_s/\tilde{X}_s + 1 + X_s \right] ds \\
&\leq C \int_0^t \left[E[1/X_s] + E[X_s] E[1/X_s] + 1 + E[X_s] \right] ds \\
&\leq Ct \left[E[1/X_0] + E[X_t] E[1/X_0] + 1 + E[X_t] \right]
\end{aligned}
\tag{4.5}
$$

where the last inequality comes from the fact that X is *a.s.* increasing. This last upper bound is clearly finite, since $t < T_0$, and since we have assumed that $E(1/X_0) < \infty$. The proof is complete. $\qquad \square$

Remark 4.3 *If we do not assume (4.2), we do not know what happens. It however seems that in the (non explosive) case where $K(x,y) = 1$ and where $E(1/X_0) = \infty$, then $X.$ has infinitely many jumps immediately after 0, but that for any $0 < s < t < \infty$, $X.$ has an a.s. finite number of jumps on $[s,t]$.*

Let us finally talk about the gelification time:

$$T_{gel} = \inf \{ t \geq 0 \; ; \; E(X_t) = \infty \}. \tag{4.6}$$

This quantity, which can be seen as a L^1-gelification time, has been much studied by the analysts and physicists. It is easily deduced from Theorem 3.1 that under

(H_β) with $\beta = 1/2$, $T_{gel} = \infty$ for any initial condition (satisfying $Q_0 \in \mathcal{P}_1$ and $\int x^2 Q_0(dx) < \infty$).

In the case of $\beta = 1$, under the same assumptions on Q_0, Theorem 3.1 yields that $T_{gel} \geq T_0 = 1/C(1 + \int x Q_0(dx))$. Of course, we have only proved the existence for (SDE) on $[0, T_0[$, because we have only assumed an upper bound for K. But in any particular case where explicit computations can be done, solutions to (SDE) may be built on $[0, T_{gel}[$. For example, the following proposition holds.

Proposition 4.4 *Assume that $Q_0 \in \mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume that $K(x,y) = A + B(x + y) + Cxy$, for some nonnegative constants A and B, and for some $C > 0$. Then Theorem 3.1 still holds replacing T_0 by T_{gel}, where, if $a_0 = \int x Q_0(dx)$,*

1. *If $\Delta = 4(B^2 - AC) = 0$, then $T_{gel} = \dfrac{1}{C(a_0 + B)}$,*

2. *If $\Delta = 4(B^2 - AC) < 0$, then $T_{gel} = \dfrac{2\pi C}{-\Delta} - \dfrac{4C}{-\Delta} \arctan\left(4C \dfrac{a_0 + B}{-\Delta}\right)$,*

3. *If $\Delta = 4(B^2 - AC) > 0$, then $T_{gel} = \dfrac{1}{2\Delta} \ln\left(\dfrac{a_0 + B/C + \sqrt{\Delta}/C}{a_0 + B/C - \sqrt{\Delta}/C}\right)$.*

Proof　This is not hard. It suffices to replace the use of the extended Gronwall Lemma 7.3 by solving classical ODEs. The result is easily understood *a posteriori*. If $(X_0, X, \tilde{X}, N)$ is a solution to (SDE), one easily checks that

$$E[X_t] = a_0 + \int_0^t E E_\alpha \left[K(X_s, \tilde{X}_s)\right] ds. \tag{4.7}$$

In the present case, by setting $f(t) = E[X_t]$, one easily gets

$$f(t) = a_0 + \int_0^t \left[A + 2Bf(s) + Cf^2(s)\right] ds. \tag{4.8}$$

This equation has an unique solution, which can be explicitly computed, and T_{gel}, which corresponds to its explosion time, can also be computed. We obtain the expressions given in the statement. □

¿From a probabilistic point of view, the L^1-gelification time is of course important, but we want also to study the stochastic gelification time:

$$\tau_{gel} = \inf\{t \geq 0 ; X_t = \infty\}. \tag{4.9}$$

Obviously, $\tau_{gel} \geq T_{gel}$ *a.s.* An interesting question is the following. Under which conditions on Q_0 and K do we have

$$P(\tau_{gel} > T_{gel}) \in]0,1[, \ P(\tau_{gel} > T_{gel}) = 0 \text{ or } P(\tau_{gel} > T_{gel}) = 1 ? \tag{4.10}$$

In other words, are there particles of which the mass is finite (resp. infinite) at the instant T_{gel}? Do all particles have a finite (resp. infinite) mass at the instant T_{gel}?

We are not able to give a complete answer for the moment. Let us however state and prove the following result.

Proposition 4.5 *Let $Q_0 \in \mathcal{P}_1$ with $\int x^2 Q_0(dx) < \infty$, and let us assume (H_β) with $\beta = 1$. Assume furthermore that $T_{gel} < \infty$, and that there exists a function $\zeta : \mathrm{Supp}\, Q_0 \mapsto \mathbb{R}_+$ such that for all $x \in \mathrm{Supp}\, Q_0$,*

$$\sup_{y \in \mathcal{H}_{Q_0}} \frac{K(x,y)}{y} \leq \zeta(x). \tag{4.11}$$

Consider a solution $(X_0, X, \tilde{X}, N)$ to (SDE). Then for any $t \in [0, \infty[$,

$$P(\tau_{gel} > t) > 0. \tag{4.12}$$

This means in particular that there are many particles which have a finite mass at the instant T_{gel}.

Notice that (4.11) is always satisfied in the discrete case, and more generally for any kernel satisfying (H_β) with $\beta = 1$ if $[0, \varepsilon[\cap \mathrm{Supp}\, Q_0 = \emptyset$ for some $\varepsilon > 0$. Notice also that (4.11) is satisfied with any initial condition, if $K(x,y) \leq Cxy$ for some constant $C \in \mathbb{R}_+$.

Proof   We will prove a much stronger result: for any $t > 0$,

$$P(X_t = X_0) > 0. \tag{4.13}$$

To this end, we study the first jump time

$$T_1 = \inf\left\{ s \geq 0 \; ; \; \Delta X_s \neq 0 \right\}. \tag{4.14}$$

By remarking that thanks to (4.11) and (2.19),

$$X_0 \leq X_t \leq X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\{z \leq \zeta(X_{s-})\}} N(ds, d\alpha, dz) \tag{4.15}$$

we deduce that $T_1 \geq S_1$ a.s., where

$$S_1 = \inf\left\{ s \geq 0 \; ; \; \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} N(ds, d\alpha, dz) > 0 \right\}. \tag{4.16}$$

Since N is a Poisson measure independent of X_0, the random variable

$$\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} N(ds, d\alpha, dz) \tag{4.17}$$

follows, conditionally to X_0, a Poisson distribution of parameter

$$\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} ds\, d\alpha\, dz = t\zeta(X_0). \tag{4.18}$$

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Hence
$$P(S_1 > t) = E\left[P\left(S_1 \geq t | X_0\right)\right] = E\left[e^{-t\zeta(X_0)}\right] > 0. \qquad (4.19)$$

Finally, we conclude that

$$P(\tau_{gel} > t) \geq P(X_t = X_0) = P(T_1 > t) \geq P(S_1 > t) > 0 \qquad (4.20)$$

which was our aim. □

This concludes the section.

5 About the uniqueness for (SDE)

In this section, we deal with the uniqueness in law for (SDE), which is equivalent to the uniqueness for (MP) (see Propositions 2.10 and 3.3). We are not able to prove such uniqueness results by ourselves (except in the case where $K(x,y) = xy$, see the next section). However, we may prove uniqueness by using the results of the analysts. In other words, we may prove uniqueness in law for (SDE) once we know the uniqueness for the Smoluchowski equation.

We first consider the discrete case.

Proposition 5.1 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0.*
Assume that $Q_0(\mathbb{N}^*) = 1$, and write Q_0 as $\sum_{k \geq 1} \alpha_k \delta_k(dx)$. Set $n_0(k) = \alpha_k/k$. Assume that the uniqueness of a solution to (SC) with the kernel K and the initial condition n_0 holds on $[0,T_0[$. Then the uniqueness of a solution Q to (MP), on $[0,T_0[$ holds. Hence the uniqueness in law holds for (SDE), in the sense that any solution $(X_0,X,\tilde{X},N)$ to (SDE) with $\mathcal{L}(X_0) = Q_0$, satisfies $\mathcal{L}(X) = Q$.

As we will prove below a similar result in the continuous case, we just sketch the proof.

Proof Let $(X_0,X,\tilde{X},N)$ be any solution to (SDE) corresponding to the initial condition Q_0 and to the kernel K. It is clear that for all $t \in [0,T_0[$, $\mathcal{L}(X_t)$ has its support in $\mathbb{N}^*$, and thus can be written as $\sum_{k \geq 1} f(k,t)\delta_k(dx)$. Then, $n(k,t) = f(k,t)/k$ satisfies (SD), thanks to Proposition 2.4 and Remark 2.7. Thus $\mathcal{L}(X_t)$ is completely determined for each $t \in [0,T_0[$, since the uniqueness holds for (SD). This is of course not sufficient, but one can conclude exactly as in the proof of Proposition 5.4 below. □

The following corollary is immediately deduced from Proposition 5.1 and from Heilmann [11].

Corollary 5.2 *Assume that $Q_0 \in \mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume also that Q_0 is discrete, i.e. that its support is contained in $\mathbb{N}^*$. Then, if $K(x,y) \leq$*

$C(1 + x + y)$ for all x, y in $\mathbb{N}^*$, uniqueness holds for (MP), and we have uniqueness in law for (SDE).

In order to use the results of the analysts in the continuous case, we first have to check that for $(X_0, X, \tilde{X}, N)$ a solution to (SDE), $\mathcal{L}(X_t)$ is really a modified solution to (SC): we have to prove that if Q_0 has a density, then for all $t \geq 0$, the law of X_t admits a density.

Proposition 5.3 *Assume that $X_0 > 0$ is a random variable of which the law Q_0 belongs to $\mathcal{P}_1$, and such that $E(X_0^2) < \infty$.*
Assume (H_β) and consider the corresponding T_0. Assume also that Q_0 is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$, and that $K(x,y)$ is (not necessarily) increasing (for example in x when y is fixed).
Consider a solution $(X_0, X, \tilde{X}, N)$ to (SDE). Then for all $t \in [0, T_0[$, the law of X_t is also absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$. Hence the law of X_t is really a weak solution to (SC), in the sense that if $f(x,t)$ denotes the density of X_t, then $n(x,t) = f(x,t)/x$ is a weak solution to (SC).

Proof   Let us denote by $f_0(x)$ the density of the law of X_0. Let $t \in]0, T_0[$ be fixed. Consider a Lebesgue-null set $\mathcal{A}$. Our aim is to check that $\mathbb{P}(X_t \in \mathcal{A}) = 0$. First notice that

$$
\begin{aligned}
\mathbb{P}(X_t \in \mathcal{A}) &= \int_0^\infty \mathbb{P}(X_t \in \mathcal{A} \mid X_0 = x) f_0(x) dx \\
&= \int_0^\infty \mathbb{P}(X_t^x \in \mathcal{A}) f_0(x) dx \\
&= E\left(\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x) dx \right) \quad (5.1)
\end{aligned}
$$

where X^x is a solution, on $[0, T_0[$, of the following standard S.D.E. (here $\tilde{X}$ is known, is fixed and behaves as a parameter):

$$
X_t^x = x + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} N(ds, d\alpha, dz). \quad (5.2)
$$

We will prove that for almost all ω, the map $x \mapsto X_t^x(\omega)$ can be written as $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. This will allow to conclude, thanks to Lemma 7.2 of the appendix, that for almost all ω,

$$
\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) dx = 0 \quad (5.3)
$$

thus that

$$
\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x) dx = 0 \quad (5.4)
$$

and hence, using (5.1) that $\mathbb{P}(X_t \in \mathcal{A}) = 0$, which was our aim.

It remains to check that for almost all ω, $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. It of course suffices to prove that for all $x > y$, $X_t^x - X_t^y \geq x - y$.
Let thus $x > y$ be fixed. Consider the following stopping time:

$$\tau = \inf \left\{ s \in [0,T_0[\mid X_s^x < X_s^y \right\}. \tag{5.5}$$

Then it is clear that for all $t < \tau$, since K is increasing,

$$\int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha)\mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} N(ds,d\alpha,dz)$$

$$\geq \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha)\mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^y, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)} \right\}} N(ds,d\alpha,dz) \tag{5.6}$$

from which we deduce that for all $s < \tau$,

$$X_s^x - X_s^y \geq x - y \tag{5.7}$$

It remains to prove that $\tau = T_0$. Let us assume that for some ω, $\tau(\omega) < T_0$. We deduce from (5.7) that
$$X_{\tau-}^x - X_{\tau-}^y \geq x - y. \tag{5.8}$$
Hence, still using the fact that K is increasing, we obtain that, for some random $\alpha_\tau \in [0,1]$, $z_\tau \in [0,\infty[$,

$$\begin{aligned}
\Delta X_\tau^x &= \tilde{X}_{\tau-}(\alpha_\tau)\mathbb{1}_{\left\{ z_\tau \leq \frac{K(X_{\tau-}^x, \tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)} \right\}} \\
&\leq \tilde{X}_{\tau-}(\alpha_\tau)\mathbb{1}_{\left\{ z_\tau \leq \frac{K(X_{\tau-}^y, \tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)} \right\}} \\
&= \Delta X_\tau^y. \tag{5.9}
\end{aligned}$$

We deduce that

$$X_\tau^x = X_{\tau-}^x + \Delta X_\tau^x \geq x - y + X_{\tau-}^y + \Delta X_\tau^y \geq x - y + X_\tau^y \tag{5.10}$$

which contradicts the definition of τ. $\qquad\square$

Thanks to the previous Proposition, we are able to state the following uniqueness result:

Proposition 5.4 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Assume also that K is increasing and satisfies the regularity condition: there exists a locally bounded function ζ on $[0,\infty[^2$ such that for all $x,x',y \in \mathbb{R}_+$,*

$$|K(x,y) - K(x',y)| \leq |x - x'|\zeta(x,x')(1 + y^2). \tag{5.11}$$

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Assume also that Q_0 admits a density $f_0(x)$, and set $n_0(x) = f_0(x)/x$. Assume that the uniqueness of a weak solution to (SC) with initial condition n_0 and kernel K holds. Then there exists a unique solution Q to (MP) with initial condition Q_0. Thus uniqueness in law holds for (SDE), in the sense that any solution $(X_0,X,\hat{X},N)$ to (SDE) with $\mathcal{L}(X_0) = Q_0$ satisfies $\mathcal{L}(X) = Q$.

Notice that (5.11) always holds when $K(x,y)$ is of the form $A+B(x+y)+Cxy$, for some nonnegative constants A,B,C.

Proof   Let Q be a solution to (MP). Thanks to Propositions 5.3 and 3.3, we know that for all t, $Q_t(dx) = f(t,x)\,dx$, for some function $f : [0,T_0[\times\mathbb{R}_+ \to \mathbb{R}_+$. Hence, Proposition 2.4 (2) and Remark 2.7 show that $f(x,t) = xn(x,t)$, where n is the unique solution of (SC). Since $Q_0 \in \mathcal{P}_1$ and $\int x^2 Q_0(dx) < \infty$, it is easily deduced that for all $T < T_0$,

$$\sup_{t\in[0,T]} \int_0^\infty \left(x + x^2 + x^3\right) n(x,t)dx$$

$$= \sup_{t\in[0,T]} \left[1 + E(X_t) + E(X_t^2)\right] < \infty \qquad (5.12)$$

The uniqueness of $\{Q_t\}_{t\in[0,T_0[}$ is proved, but we need more: we want to prove the uniqueness of $Q \in \mathcal{P}_1^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$.

As Q satisfies (MP) it also satisfies the simple (because linear) martingale problem (MPS): for all $\phi \in C_b^1(\mathbb{R}_+)$,

$$\phi(Z_t) - \phi(Z_0) - \int_0^t \int_{\mathbb{R}_+} \left(\phi(Z_s + y) - \phi(Z_s)\right) K(Z_s,y)n(y,s)\,dy\,ds \qquad (5.13)$$

is a Q-martingale, Z standing for the canonical process of $\mathbb{D}^\uparrow([0,T_0[,\mathcal{H}_{Q_0})$. We will prove the uniqueness for (MPS). In this way, we will deduce that Q is entirely determined, since any solution to (MP) satisfies also (MPS). This will conclude the proof.

The uniqueness for (MPS) is equivalent to the uniqueness in law for the following S.D.E.:

$$Y_t = X_0 + \int_0^t \int_{\mathbb{R}_+} \int_0^\infty y\mathbb{1}_{\left\{z \le \frac{K(Y_{s-},y)}{y}\right\}}\mu(ds,dy,dz) \qquad (5.14)$$

$\mu(ds,dy,dz)$ being a Poisson measure on $[0,T_0[\times\mathbb{R}^+ \times [0,\infty[$ with intensity measure $ds\,(yn(y,s)dy)\,dz$. But the strong uniqueness (which implies the uniqueness in law) holds for this equation, thanks to standard arguments: local Lipschitz continuity and at most linear growth. Indeed, for all $u \ge 0$, all $T < T_0$, we obtain, using (H_β) and (5.12),

$$\sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty y\mathbb{1}_{\left\{z \le \frac{K(u,y)}{y}\right\}}\,dz\,yn(y,s)\,dy$$

$$\le A(1+u) \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y + y^3)n(y,s)\,dy \qquad (5.15)$$

$$\le A_T(1+u)$$

the constant A_T depending only on T. We also have, for all u,u' in $[0,\infty[$, all $T < T_0$, by using (5.11) and (5.12),

$$
\begin{aligned}
\sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty & \left| y\,\mathbb{1}_{\left\{z\leq\frac{K(u,y)}{y}\right\}} - y\,\mathbb{1}_{\left\{z\leq\frac{K(u',y)}{y}\right\}} \right| dz\, yn(y,s)\, dy \\
&\leq \sup_{s\in[0,T]} \int_{\mathbb{R}_+} |K(u,y) - K(u',y)|\, yn(y,s)\, dy \\
&\leq \zeta(u,u')|u - u'| \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y + y^3)n(y,s)dy \\
&\leq A_T\zeta(u,u')|u - u'|.
\end{aligned}
\tag{5.16}
$$

Using these properties, the strong uniqueness is easily checked for equation (5.14). This implies the uniqueness for (MPS) and concludes the proof. $\quad\square$

We finally deduce the following corollary from Aldous [1], Principle 1.

Corollary 5.5 *Assume that Q_0 belongs to $\mathcal{P}_1$ and that $\int x^2Q_0(dx) < \infty$. Assume that $K(x,y) \leq C(1+x+y)$, that K is increasing, and that the regularity condition (5.11) holds.*
In addition, assume that Q_0 admits a density $f_0(x)$ and that $\int \frac{1}{x}Q_0(dx) < \infty$. Then uniqueness in law holds for (SDE), and so does uniqueness for (MP).

6 Study of the exact multiplicative kernel

In this short section, we will make explicit computations for the case $K(x,y) = xy$. In this explicit case, we obtain very satisfying results. In particular, we get rid of the assumption $\int x^2Q_0(dx) < \infty$. We build directly a solution by using a Picard iteration without cutoff. Uniqueness for (SDE) is proved without using the results of the analysts. Let us begin with the statement.

Theorem 6.1 *Assume that $K(x,y) = xy$. Let Q_0 belong to $\mathcal{P}_1$ and $T_0 = 1/\int xQ_0(dx)$. Then the following results hold.*
1. *For any random variable X_0 of law Q_0, any independent Poisson measure $N(dt,d\alpha,dz)$ with intensity measure $dtd\alpha dz$, there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE) on $[0,T_0[$.*
2. *The obtained law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ is unique, and depends only on Q_0.*
3. *Hence existence and uniqueness for (MP) hold.*

Proof 1. Let X_0 and N be fixed. We consider the following Picard iterations: first, we consider the process $X^0 \equiv X_0$. Then we consider any α-process $\tilde{X}^0$ such that $\mathcal{L}_\alpha(\tilde{X}^0) = \mathcal{L}(X^0)$. Once everything is built up to n, we consider

$$
X_t^{n+1} = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}^n(\alpha)\,\mathbb{1}_{\left\{z\leq X_{s-}^n\right\}} N(ds,d\alpha,dz)
\tag{6.1}
$$

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and we build an α-process $\tilde{X}^{n+1}$ such that

$$\mathcal{L}_\alpha(\tilde{X}^{n+1}|\tilde{X}^0,...,\tilde{X}^n) = \mathcal{L}(X^{n+1}|X^0,...,X^n). \tag{6.2}$$

One can check that if $f_n(t) = E(X_t^n) = E_\alpha(\tilde{X}_t^n)$, then $f_0(t) = a = E(X_0)$, and for all $n \geq 0$,

$$f_{n+1}(t) = a + \int_0^t f_n^2(s)ds \tag{6.3}$$

which easily implies that for all $t \leq T_0 = 1/a$,

$$\sup_n f_n(t) \leq \frac{a}{1-at}. \tag{6.4}$$

Let now $g_n(t) = E\left[\sup_{s\in[0,t]} |X_s^{n+1} - X_s^n|\right] = E_\alpha\left[\sup_{s\in[0,t]} |\tilde{X}_s^{n+1} - \tilde{X}_s^n|\right]$. A simple computation shows that

$$
\begin{aligned}
g_n(t) &\leq \int_0^t EE_\alpha\left(\int_0^\infty \left|\tilde{X}_s^n \mathbb{1}_{\{z\leq X_s^n\}} - \tilde{X}_s^{n-1}\mathbb{1}_{\{z\leq X_s^{n-1}\}}\right|dz\right)ds \\
&\leq \int_0^t EE_\alpha\left(X_s^n\left|\tilde{X}_s^n - \tilde{X}_s^{n-1}\right| + \tilde{X}_s^{n-1}\left|X_s^n - X_s^{n-1}\right|\right)ds \\
&\leq \int_0^t \frac{2a}{1-as} \times g_{n-1}(s)ds. \tag{6.5}
\end{aligned}
$$

It is now clear that for all $T < T_0$,

$$\sum_{n\geq 1} g_n(T) < \infty. \tag{6.6}$$

Hence there exist a process $X \in L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$ and an α-process $\tilde{X}$ such that for all $T < T_0$,

$$E\left[\sup_{s\in[0,T]} |X_s - X_s^n|\right] = E_\alpha\left[\sup_{s\in[0,T]} |\tilde{X}_s - \tilde{X}_s^n|\right] \tag{6.7}$$

goes to 0 when n tends to infinity. One easily concludes that $(X_0,X,\tilde{X},N)$ satisfies (SDE).

2. The uniqueness is much more difficult to prove. We can follow the proof of Desvillettes, Graham, Méléard [5] which concerns the Boltzmann equation, and we only give the main steps of the proof.

Step 1: it is clear that in the existence proof, the obtained law $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$ does not depend on the possible choices for Ω, X_0, N and $\tilde{X}^n(\alpha)$, but only on the law of the initial condition $\mathcal{L}(X_0) = Q_0$.

$\underline{\text{Step 2}}$: let thus Ω, X_0 and N be fixed. Consider two solutions $(X_0,X,\tilde{X},N)$ and $(X_0,Y,\tilde{Y},N)$ of (SDE). We have to prove that $\mathcal{L}(X) = \mathcal{L}(Y)$. Let us denote $Q = \mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ and $Q' = \mathcal{L}(Y) = \mathcal{L}_\alpha(\tilde{Y})$. For $T < T_0$, we consider the quantity

$$\rho_T(Q,Q') = \inf_{\tilde{Z},\tilde{Z}'} \left\{ E_\alpha \left(\sup_{s \in [0,T]} |\tilde{Z}_s - \tilde{Z}'_s| \right) \; ; \; \mathcal{L}_\alpha(\tilde{Z}) = Q, \, \mathcal{L}_\alpha(\tilde{Z}') = Q' \right\}.$$
(6.8)

For some $\varepsilon > 0$ fixed, we consider α-processes $\tilde{X}^\varepsilon$ and $\tilde{Y}^\varepsilon$ such that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = Q$, $\mathcal{L}_\alpha(\tilde{Y}^\varepsilon) = Q'$, and

$$\rho_T(Q,Q') \le E_\alpha \left(\sup_{s \in [0,T]} |\tilde{X}^\varepsilon_s - \tilde{Y}^\varepsilon_s| \right) + \varepsilon.$$
(6.9)

Then we build X^ε and Y^ε, in such a way that $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ and $(X_0,Y^\varepsilon,\tilde{Y}^\varepsilon,N)$ be solutions to (SDE). This can be done by solving linear S.D.E.s (because $\tilde{X}^\varepsilon$ and $\tilde{Y}^\varepsilon$ are fixed processes). For all $T < T_0$, one easily obtains the existence of a constant A_T, not depending on ε, such that

$$E \left[\sup_{s \in [0,T]} X^\varepsilon_s \right] + E \left[\sup_{s \in [0,T]} Y^\varepsilon_s \right] \le A_T.$$
(6.10)

Finally, we obtain, for any $t \le T < T_0$,

$$E \left(\sup_{s \in [0,t]} |X^\varepsilon_s - Y^\varepsilon_s| \right) \quad \le \quad \int_0^t E E_\alpha \left(X^\varepsilon_s \left| \tilde{X}^\varepsilon_s - \tilde{Y}^\varepsilon_s \right| + \tilde{Y}^\varepsilon_s |X^\varepsilon_s - Y^\varepsilon_s| \right) ds$$

$$\le \quad A_T \int_0^t \left[\rho_T(Q,Q') + \varepsilon + E(|X^\varepsilon_s - Y^\varepsilon_s|) \right] ds \quad (6.11)$$

which yields that for any $t \le T$

$$E \left(\sup_{s \in [0,t]} |X^\varepsilon_s - Y^\varepsilon_s| \right) \le A_T T (\rho_T(Q,Q') + \varepsilon) e^{A_T T}.$$
(6.12)

The left hand side member is greater than $\rho_T(Q,Q')$. We thus obtain, making ε go to 0,

$$\rho_T(Q,Q') \le A_T T \rho_T(Q,Q') e^{A_T T}.$$
(6.13)

But X_T is increasing in T. Hence, we can choose τ small enough, such that $A_\tau \tau e^{A_\tau \tau} < 1$ and thus $\rho_\tau(Q,Q') = 0$. The Markov property of the Poisson measure allows to prove that the result remains true on $[0,T_0[$, i.e. that $\rho_T(Q,Q') = 0$ for all $T < T_0$. $\qquad\square$

7 Appendix

First, we recall the Aldous criterion for tightness (see Jacod, Shiryaev [12]).

Theorem 7.1 *Let* $\{X_t^n\}_{t\in[0,T_0[}$ *be a family of càdlàg adapted processes on* $[0,T_0[$, *for some* $T_0 \leq \infty$. *Denote by* $Q^n \in \mathcal{P}\left(\mathbb{D}([0,T_0[,\mathbb{R})\right)$ *the law of* X^n. *Assume that:*

1. For all $T < T_0$,
$$\sup_n E\left[\sup_{t\in[0,T]}|X_t^n|\right] < \infty \tag{7.1}$$

2. For all $T < T_0$, *all* $\eta > 0$,
$$\sup_n \sup_{(S,S')\in ST_T(\delta)} P\left[|X_{S'}^n - X_S^n| \geq \eta\right] \longrightarrow 0 \tag{7.2}$$

when δ *goes to* 0, *where* $ST_T(\delta)$ *is the set of couples* (S,S') *of stopping times satisfying a.s.* $0 \leq S \leq S' \leq (S+\delta) \wedge T$.

Then the family $\{Q^n\}$ *is tight. Furthermore, any limiting point* Q *of this family is the law of a quasi-left continuous process, i.e. for all* $t \in [0,T_0[$ *fixed,*
$$\int_{\mathbb{D}([0,T_0[,\mathbb{R})} \mathbb{1}_{\{\Delta x(t)\neq 0\}} Q(dx) = 0. \tag{7.3}$$

We now prove an easy absolute continuity result.

Lemma 7.2 *Let* φ *be an increasing map from* $\mathbb{R}_+$ *into itself. Let* $\mathcal{A}$ *be a Lebesgue-null subset of* $\mathbb{R}_+$. *Then*
$$\int_0^\infty \mathbb{1}_{\mathcal{A}}\left(x + \varphi(x)\right)dx = 0. \tag{7.4}$$

Proof We set $f(x) = x + \varphi(x)$. Since φ is increasing, the Stieljes measure $df^{-1}(x)$ is clearly smaller than the Lebesgue measure dx on $\mathbb{R}_+$. In particular, $df^{-1}(x) << dx$. Hence,
$$\int_0^\infty \mathbb{1}_{\mathcal{A}}\left(x + \varphi(x)\right)dx = \int_{\mathcal{A}} df^{-1}(x) = 0 \tag{7.5}$$
which was our aim. $\qquad\qquad\square$

We carry on with a generalised Gronwall Lemma (see Beesack [3]).

Lemma 7.3 *Let* $a,b \geq 0$. *Consider a continuous function* g *on* $[0,T]$, *satisfying for all* $t \in [0,T]$,
$$g(t) \leq a + b\int_0^t g^2(s)ds. \tag{7.6}$$

Then, for all $t < T_0 = 1/ab$,
$$g(t) \leq \frac{a}{1-abt}. \tag{7.7}$$

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Proof Let us denote
$$U(t) = a + b \int_0^t g^2(s)ds.$$

Clearly for all t, $x(t) \leq U(t)$. Let $G(x) = -\frac{1}{x}$. We have:

$$\frac{d}{ds}G(U(s)) = b\frac{x^2(s)}{U^2(s)} \leq b$$

thus:

$$G(U(t)) \leq G(a) + bt$$

which yields

$$U(t) \leq \frac{a}{1 - abt}.$$

This inequality ends the proof. $\qquad\square$

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