

A weak criterion of absolute continuity for jump processes; Application to the Boltzmann equation

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Abstract

We first prove a general and quite simple criterion of absolute continuity, based on the use of almost sure derivatives, which is available even when no integration by parts may be used. We apply it to Poisson driven stochastic differential equations. Next, using a typically probabilistic substitution in the Boltzmann equation, we extend Tanaka's probabilistic interpretation for spatially homogeneous Boltzmann equations with Maxwell molecules and without angular cutoff to much more general spatially homogeneous 2D Boltzmann equations. We relate a measure-solution $\{Q_t\}_t$ of the equation to a solution V_t of a nonlinear Poisson-driven stochastic differential equation: for each t , Q_t is the law of V_t . We extend our absolute continuity criterion to these nonlinear Poisson functionals and prove that even in the case of degenerated initial distribution, the law of V_t admits a density $f(t, \cdot)$ for each $t > 0$, which is hence solution to the Boltzmann equation. We thus obtain an original existence result.

Key words: Stochastic differential equations with jumps, Stochastic calculus of variations, Boltzmann equations.

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1 Introduction.

The Boltzmann equation we consider describes the evolution of the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^2$ at time t in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t}(t, v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} \left(f(t, v') f(t, v'_*) - f(t, v) f(t, v_*) \right) B(|v - v_*|, \theta) d\theta dv_* \quad (1.1)$$

The post-collisional velocities v' and v'_* are given by

$$v' = v + A(\theta)(v - v_*) ; v'_* = v_* - A(\theta)(v - v_*) \quad (1.2)$$

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where

$$A(\theta) = \frac{1}{2} \begin{pmatrix} \cos \theta - 1 & -\sin \theta \\ \sin \theta & \cos \theta - 1 \end{pmatrix} \quad (1.3)$$

The cross-section B is a positive function, even in the θ -variable. If the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geq 2$, then the physical cross-sections $B(z, \theta)$ tend to infinity when θ goes to zero, but satisfy $\int_{-\pi}^{\pi} |\theta|^2 B(z, \theta) d\theta < +\infty$ for each z . Physically, this explosion near 0 comes from the accumulation of grazing collisions. In this general (spatially homogeneous) setting, the Boltzmann equation is very difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^{\pi} B(z, \theta) d\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross section $B(z, \theta) = \beta(\theta)$ only depends on θ , has been much studied without the cutoff assumption. In the Maxwell context, Tanaka, [15] was considering the case where $\int_0^{\pi} \theta \beta(\theta) d\theta < \infty$, and Horowitz, Karandikar [11] Desvillettes, [5], Desvillettes-Graham-Méléard, [6] and Fournier, [7] have worked under the physical assumption $\int_0^{\pi} \theta^2 \beta(\theta) d\theta < +\infty$.

The case in which B depends on z is really harder and there is just a few results on it. Let us mention Alexandre-Desvillettes-Villani-Wennberg [1] and Fournier-Méléard [9].

In the present paper, we first prove a weak, general and quite simple criterion of absolute continuity that we apply to standard Poisson driven SDEs, and which generalizes in some sense the results of Bichteler, Gravereaux, Jacod [3], [2]. In Section 2, we extend Tanaka's probabilistic interpretation, [15], who was dealing with Maxwell molecules, to much more general spatially homogeneous Boltzmann equations, under the condition $\sup_z \int_0^{\pi} \theta B(z, \theta) d\theta < \infty$. Indeed, using a typically probabilistic substitution in the Boltzmann equation, we relate the solution of the equation to the solution V_t of a Poisson-driven nonlinear stochastic differential equation: the law of V is a measure solution to the Boltzmann equation. Then we develop in Section 3 our weak approach of the stochastic calculus of variations for our nonlinear Poisson functionals, to prove that even when the initial distribution is degenerated, the law of V_t has a density as soon as $t > 0$. This gives a new existence result for the Boltzmann equation.

The reason why we consider 2D equations is technical. However, we are far from being able to prove such a result in the 3D case, see [8] for an idea of technical problems. This limitation is not new: for example, Desvillettes had to consider also 1D or 2D equations to obtain regularization results, see [5].

Let us now comment the probabilistic tools we develop. The stochastic calculus of variations for Poisson processes has been first investigated by Bismut in [4]. Then, Bichteler-Jacod, [3], have rewritten and developed the main ideas of Bismut to prove existence of densities for diffusion processes with jumps. Since, much work have been done. In almost all the cases, existence of densities was based on integration by parts, as in the standard Malliavin Calculus for Wiener functionals. But it is now well known in the Wiener case that the use of integration by parts is not the most efficient when one studies only absolute continuity: it is much easier to use Bouleau-Hirsch's approach, see e.g. Nualart, [14]. However this sort of approach has not been investigated in the case of Poisson functionals.

Unfortunately, we can not use an integration by parts formula in the present study, because

our random variables V_t can not be differentiated in a $L^2(\Omega)$ -sense. Indeed, the ‘‘Malliavin derivative’’ of V_t is not square integrable.

To avoid the use of such a formula, we will use the following weak, general, and quite simple criterion of absolute continuity.

Lemma 1.1 *Let $d \in \mathbb{N}^*$, and let X be a \mathbb{R}^d -valued random variable on a probability space (Ω, \mathcal{F}, P) . Let Λ be a neighborhood of 0 in \mathbb{R}^d . Assume that there exists a family $\{X^\lambda\}_{\lambda \in \Lambda}$ of \mathbb{R}^d -valued random variables such that*

(i) For each $\lambda \in \Lambda$, the law of X^λ is absolutely continuous with respect to that of X . We denote by $G^\lambda = dX/dX^\lambda$ the associated Radon-Nykodym density. The family G^λ satisfies the integrability condition

$$\sup_{\lambda} E(|G^\lambda|^2) < \infty \quad (1.4)$$

(ii) For almost all ω , there exists a neighborhood $\mathcal{V}(\omega)$ of 0 in \mathbb{R}^d on which the map $\lambda \mapsto X^\lambda(\omega)$ is of class C^1 .

(iii) For almost all ω , the derivative $\left. \frac{\partial}{\partial \lambda} X^\lambda \right|_{\lambda=0}$ is invertible.

Then the law of X is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d .

Proof. Let A be a negligible subset of \mathbb{R}^d . We have to prove that $P(X \in A) = 0$.

Applying the inverse local theorem, we deduce from (ii) and (iii) that for almost all ω , there exists a neighborhood $\bar{\mathcal{V}}(\omega)$ of 0 in \mathbb{R}^d on which the map $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism.

We now set, for $n \in \mathbb{N}_*$, $\Omega_n = \left\{ \omega \in \Omega \mid [-1/n, 1/n]^d \subset \bar{\mathcal{V}}(\omega) \right\}$. Then it is clear that Ω_n grows to some $\tilde{\Omega}$, with $P(\tilde{\Omega}) = 1$.

On the other hand, we know from (i) that for each $\lambda \in \Lambda$, $P(X \in A) = E(1_A(X^\lambda)G^\lambda)$. Hence for each n ,

$$P(X \in A) = E \left[\left(\frac{n}{2} \right)^d \int_{[-1/n, 1/n]^d} 1_A(X^\lambda)G^\lambda d\lambda \right] \quad (1.5)$$

It is not hard to conclude, using (1.4), the Cauchy-Schwarz inequality, and the fact that $\lim_n P[\Omega_n] = 1$, that

$$P(X \in A) = \lim_{n \rightarrow \infty} E \left[\left(\frac{n}{2} \right)^d \left\{ \int_{[-1/n, 1/n]^d} 1_A(X^\lambda)G^\lambda d\lambda \right\} \times 1_{\Omega_n} \right] \quad (1.6)$$

To conclude that $P(X \in A) = 0$, it thus suffices to prove that for each n , each $\omega \in \Omega_n$,

$$\int_{[-1/n, 1/n]^d} 1_A(X^\lambda)G^\lambda d\lambda = 0 \quad (1.7)$$

It is of course enough to show that for each n , each $\omega \in \Omega_n$, $I_n(\omega) = 0$, where

$$I_n(\omega) = \int_{[-1/n, 1/n]^d} 1_A(X^\lambda)G^\lambda d\lambda = 0 \quad (1.8)$$

But ω belongs to Ω_n , thus $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism from $[-1/n, 1/n]^d$ into some set $D_n(\omega)$. Substituting $y = X^\lambda(\omega)$ in (1.8), denoting by $J_n(\omega, y)$ the associated Jacobian, we obtain

$$I_n(\omega) = \int_{D_n(\omega)} 1_A(y)J_n(\omega, y)dy \quad (1.9)$$

which of course vanishes since A is Lebesgue-negligible. This concludes the proof. \triangle

We can apply this absolute continuity criterion to standard Poisson driven SDEs and the theorem we obtain (Theorem 1.2 below) generalizes the result of Bichteler-Gravereaux-Jacod [3], [2], in the case of processes with finite variations. Our result is stated for any dimension of space, instead of dimension one in [3] (the multidimensionnal case is treated in [2]). The technical hypotheses on the coefficients are less stringent: instead of boundedness, we assume polynomial growth and the integrability assumption is also weaker. The proof we will propose (in the case of a process related to the Boltzmann equation) is furthermore technically more simple. Let us now state our result.

Theorem 1.2 *Let us consider on a probability space (Ω, \mathcal{F}, P) a Poisson point measure $N(\omega, dt, dz)$ on $[0, T] \times \mathbb{R}$ with intensity measure $m(dt, dz) = dt dz$, and consider the \mathbb{R}^d -valued stochastic differential equation*

$$X_t = x_0 + \int_0^t \int_{\mathbb{R}} \gamma(X_{s-}, z) N(ds, dz) + \int_0^t b(X_{s-}) ds \quad (1.10)$$

where $x_0 \in \mathbb{R}^d$, and where the coefficients γ and b satisfy the following hypotheses:

i) the maps $\gamma(X, z) : \mathbb{R}^d \times \mathbb{R}^* \mapsto \mathbb{R}^d$ and b are of class C^2 . There exist $p \in \mathbb{N}$, $K \in \mathbb{R}^+$, and a bounded positive function $\eta : \mathbb{R}^* \mapsto \mathbb{R}^+$, satisfying the integrability condition

$$\bar{\eta}(z) = \sup_{|u-z| \leq |z|/2 \wedge 1/|z|} \eta(u) \in L^1(\mathbb{R}^*, dz) \quad (1.11)$$

and such that for $X \in \mathbb{R}^d$ and $z \in \mathbb{R}$

$$|\gamma(X, z)| \leq (1 + |X|) \eta(z) ; |b(X)| \leq K(1 + |X|) \quad (1.12)$$

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leq (1 + |X|^p) \eta(z) ; |b'(X)| + |b''(X)| \leq K(1 + |X|^p) \quad (1.13)$$

$$|\gamma'_z(X, z)| + |\gamma''_{zz}(X, z)| + |\gamma''_{Xz}(X, z)| \leq K(1 + |X|^p) \quad (1.14)$$

Notice that the integrability condition (1.11) is not much more stringent than the simple condition $\eta \in L^1(\mathbb{R}^*, dz)$.

ii) the following non-degeneracy condition holds: for each $X \in \mathbb{R}^d$, for each $Y \in \mathbb{R}^d / \{0\}$,

$$\int_{\mathbb{R}} \mathbf{1}_{\{Y^t \gamma'_z(X, z) (\gamma'_z(X, z))^t Y \neq 0\}}(z) dz = \infty.$$

Then there exists a solution $\{X_t\}_{t \in [0, T]}$ to (1.10), and for all $t > 0$ the law of X_t admits a density with respect to the Lebesgue measure on \mathbb{R}^d .

We don't give the proof of this result here, since it will be exposed later (Section 3) for the more complicated case (since nonlinear) of Boltzmann processes.

In [3] and [2], the equivalent to assumptions (1.13) and (1.14) are given by

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leq \eta(z) ; |b'(X)| + |b''(X)| \leq K \quad (1.15)$$

$$|\gamma'_z(X, z)| + |\gamma''_{zz}(X, z)| + |\gamma''_{Xz}(X, z)| \leq K \quad (1.16)$$

Finally notice that a localization procedure may be used to generalize directly the results of [3] and [2]. But we could probably not obtain such weak assumptions.

Furthermore, localization can not be used in nonlinear settings, such as that of the Boltzmann equation.

2 Transformation of the Boltzmann equation, main results.

First of all, let us precise the family of cross sections we will study.

Hypothesis 2.1 For each $x \in \mathbb{R}_+$, $B(x, \theta)$ is an even strictly positive function on $[-\pi, \pi] \setminus \{0\}$ satisfying

$$\text{for all } x \in \mathbb{R}_+, \quad \int_{-\pi}^{\pi} B(x, \theta) d\theta = \infty ; \text{ and } \sup_{x \in \mathbb{R}_+} \int_{-\pi}^{\pi} |\theta| B(x, \theta) d\theta < \infty \quad (2.1)$$

For $X \in \mathbb{R}^2$, we will denote by $B(X, \theta)$ the quantity $B(|X|, \theta)$.

Equation (1.1) has to be understood in a weak sense. Integrating (1.1) against test functions, and making standard integration by parts, see e.g. Desvillettes, [5], we obtain the following weak formulation.

First of all, we define, for each probability measure $q \in \mathcal{P}(\mathbb{R}^2)$, each $\phi \in C_b^1(\mathbb{R}^2)$,

$$L_q \phi(v) = \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} \left(\phi(v + A(\theta)(v - v_*)) - \phi(v) \right) B(v - v_*, \theta) d\theta q(dv_*) \quad (2.2)$$

This kernel is well defined since $|A(\theta)| \leq K|\theta|$ and thanks to (2.1).

Definition 2.2 Assume Hypothesis 2.1. Consider Q_0 a probability measure on \mathbb{R}^2 . We say that a probability measure family $\{Q_t\}_{t \in [0, T]}$ is a measure-solution of the Boltzmann equation (1.1) with initial data Q_0 if for each $\phi \in C_b^1(\mathbb{R}^2)$, all $t \in [0, T]$,

$$\langle \phi, Q_t \rangle = \langle \phi, Q_0 \rangle + \int_0^t \langle L_{Q_s} \phi(v), Q_s(dv) \rangle ds, \quad (2.3)$$

If furthermore for all $t \in]0, T]$, the probability measure Q_t admits a density $f(t, \cdot)$ with respect to the Lebesgue measure on \mathbb{R}^2 , the obtained function $f(t, v):]0, T] \times \mathbb{R}^2 \mapsto \mathbb{R}_+$ is said to be a function-solution of the Boltzmann equation (1.1).

The probabilistic approach will consist in considering (2.3) as the evolution equation of the flow of time-marginals of a Markov process.

This whole work is based on the following substitution in L_q .

Notation 2.3 For each $X \in \mathbb{R}^2$, we consider the function h_X defined on $[-\pi, \pi] \setminus \{0\}$ by

$$h_X(\theta) = \int_{\theta}^{\pi} B(X, \varphi) d\varphi \text{ if } \theta > 0 ; h_X(\theta) = - \int_{-\pi}^{\theta} B(X, \varphi) d\varphi \text{ if } \theta < 0 \quad (2.4)$$

Thanks to Hypothesis 2.1, it is clear that for each X , $h_X(\theta)$ is strictly decreasing from 0 to $-\infty$ between $\theta = -\pi$ and $\theta = 0^-$, and from $+\infty$ to 0 between $\theta = 0^+$ and $\theta = \pi$. We thus can set, for each $X \in \mathbb{R}^2$ and each $z \in \mathbb{R}^*$,

$$g(X, z) = h_X^{-1}(z), \quad \text{i.e. } h_X(g(X, z)) = z \quad (2.5)$$

Notice that for each X, z , the derivative $g'_z(X, z) = -1/B(X, g(X, z)) < 0$, thanks to Hypothesis 2.1. The function $g(X, z)$ is thus strictly decreasing from 0 to $-\pi$ between $-\infty$ and 0^- , and from π to 0 between 0^+ and $+\infty$. Notice also that $g(X, \cdot)$ is odd and depends only on $|X|$. Finally remark that (2.1) can be written as

$$\sup_{X \in \mathbb{R}^2} \int_{\mathbb{R}^*} |g(X, z)| dz < +\infty \quad (2.6)$$

We introduce again some notations.

Notation 2.4 For $X \in \mathbb{R}^2$ and $z \in \mathbb{R}^*$, we set

$$\gamma(X, z) = A(g(X, z)).X : \mathbb{R}^2 \times \mathbb{R}^* \mapsto \mathbb{R}^2 \quad (2.7)$$

By this way, we obtain a new expression of the operator L_q .

Proposition 2.5 Assume Hypothesis 2.1. Then for each $q \in \mathcal{P}(\mathbb{R}^2)$, each $\phi \in C_b^1(\mathbb{R}^2)$,

$$L_q \phi(v) = \int_{\mathbb{R}^2} \int_{z \in \mathbb{R}^*} \left(\phi(v + \gamma(v - v_*, z)) - \phi(v) \right) dz q(dv_*) \quad (2.8)$$

Proof. It suffices to use the substitution $\theta = g(v - v_*, z)$ in (2.2), which implies that $z = h_{v-v_*}(\theta)$ and thus $dz = -B(v - v_*, \theta)d\theta$. \triangle

Let us now explain why this substitution is interesting. Tanaka, [15], was dealing with the much more simple case of Maxwell molecules (i.e. $B(X, \theta) = \beta(\theta)$). In this case, the jump measure appearing in (2.2) is $\beta(\theta)d\theta q(dv_*)$, and does not depend on v . The main interest of the substitution described above is to transform the jump measure $B(v - v_*, \theta)d\theta q(dv_*)$ in a measure of the form $dz q(dv_*)$, independent of v . This will allow us to have a probabilistic interpretation in terms of Poisson measure.

We finally want to introduce a nonlinear stochastic differential equation associated with our Boltzmann equation. To this aim, we follow the main ideas of Tanaka. We first consider two probability spaces: the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ and the second one is the auxiliary space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ introduced to model the nonlinearity. In order to avoid any confusion, the processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ will be called α -processes, the expectation under $d\alpha$ will be denoted by E_α , and the laws \mathcal{L}_α .

Definition 2.6 Assume Hypothesis 2.1. We will say that (V, W, N, V_0) is a solution of (SDE) if

(i) $\{V_t(\omega)\}_{t \in [0, T]}$ is a \mathbb{R}^2 -valued càdlàg adapted process on Ω such that $E \left(\sup_{t \in [0, T]} |V_t|^2 \right) < +\infty$,

(ii) $\{W_t(\alpha)\}_{t \in [0, T]}$ is a \mathbb{R}^2 -valued càdlàg α -process on $[0, 1]$,

(iii) $N(\omega, dt, d\alpha, dz)$ is a Poisson measure on $[0, T] \times [0, 1] \times \mathbb{R}^*$ with intensity measure

$$m(dt, d\alpha, dz) = dt d\alpha dz \quad (2.9)$$

(iv) $V_0(\omega)$ is a square integrable variable independent of N ,

(v) The laws of V and W are the same, i.e. $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$,

(vi) The following S.D.E. is satisfied:

$$V_t = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma(V_{s-} - W_{s-}(\alpha), z) N(ds, d\alpha, dz) \quad (2.10)$$

The following remark shows the connection between (SDE) and the Boltzmann equation (1.1).

Remark 2.7 *Let (V, W, N, V_0) be a solution of (SDE), and set $Q_t = \mathcal{L}(V_t) = \mathcal{L}_\alpha(W_t)$ for each $t \in [0, T]$. Then one easily proves by using the Itô formula, that the family $\{Q_t\}_{t \in [0, T]}$ is a measure-solution of (2.3) with initial data Q_0 .*

Let us now state an hypothesis, which, combined with Hypothesis 2.1, will be sufficient to obtain existence of a function solution to the Boltzmann equation.

Hypothesis 2.8 (i) *The initial distribution $Q_0 = \mathcal{L}(V_0)$ admits moments of all orders, and is not a Dirac mass.*

(ii) *The map $\gamma(X, z) : \mathbb{R}^2 \times \mathbb{R}^* \mapsto \mathbb{R}^2$ is of class C^2 . There exist $p \in \mathbb{N}$, $K \in \mathbb{R}^+$, and a bounded positive function $\eta : \mathbb{R}^* \mapsto \mathbb{R}^+$, satisfying the integrability condition*

$$\bar{\eta}(z) = \sup_{|u-z| \leq (|z|/2) \wedge (1/|z|)} \eta(u) \in L^1(\mathbb{R}^*, dz) \quad (2.11)$$

and such that

$$|\gamma(X, z)| \leq (1 + |X|) \eta(z) \quad (2.12)$$

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leq (1 + |X|^p) \eta(z) \quad (2.13)$$

$$|\gamma'_z(X, z)| + |\gamma''_{zz}(X, z)| + |\gamma''_{Xz}(X, z)| \leq K(1 + |X|^p) \quad (2.14)$$

Notice that the integrability condition (2.11) is not much more stringent than the simple condition $\eta \in L^1(\mathbb{R}^*, dz)$.

The following existence result for (SDE) is an easy but fastidious exercise, and can be proved by following the main ideas of [9].

Proposition 2.9 *Assume Hypotheses 2.1 and 2.8. Then there is weak existence for (SDE). This means that there exist a probability space (Ω, \mathcal{F}, P) , on which there exists a solution (V, W, N, V_0) to (SDE). Furthermore, we have for any $q \in \mathbb{N}$,*

$$E \left(\sup_{[0, T]} |V_t|^q \right) = E_\alpha \left(\sup_{[0, T]} |W_t|^q \right) < \infty \quad (2.15)$$

Finally, the conservations of momentum and kinetic energy hold, i.e. for all $t \in [0, T]$, $E(V_t) = E(V_0)$ and $E(|V_t|^2) = E(|V_0|^2)$.

Then the following regularization result holds:

Theorem 2.10 *Assume Hypotheses 2.1 and 2.8. Consider a solution (V, W, N, V_0) of (SDE). Then for all $t > 0$ the law of V_t admits a density $f(t, \cdot)$ with respect to the Lebesgue measure on \mathbb{R}^2 .*

The next corollary, immediately deduced from Theorem 2.10, states the analytical version of our result.

Corollary 2.11 *Assume Hypotheses 2.1 and 2.8. Then there exists a function-solution*

$$f \in L^\infty([0, T], L^1((1 + |v|^2)dv)) \quad (2.16)$$

to the Boltzmann equation without cutoff, for non Maxwell molecules, with initial data Q_0 , and $f(t, \cdot)$ is for each $t > 0$ a probability density function.

We now give examples of application.

Remark 2.12 *Assume that the cross section is of the form $B(X, \theta) = \psi(X)/|\theta|^\alpha$, with ψ positive and $\alpha \in [1, 2[$. Then Hypotheses 2.1 and 2.8 are satisfied if ψ is bounded from above and from below, is of class C^2 on \mathbb{R}^2 , and if ψ' and ψ'' have at most a polynomial growth.*

Proof. Observing that when $\alpha = 1$, $g(X, z) = \text{sign}(z)e^{-|z|/\psi(X)}$, and when $\alpha > 1$, $g(X, z) = \text{sign}(z) \left(\frac{\pi^{\alpha-1}\psi(X)}{(\alpha-1)|z|\pi^{\alpha-1}+\psi(X)} \right)^{\frac{1}{\alpha-1}}$, the remark can be proved by using simple computations. \triangle

3 Existence of a function-solution.

In this section we will prove Theorem 2.10. We assume from now Hypotheses 2.1 and 2.8. We consider a fixed solution (V, W, N, V_0) of (SDE) . Our aim is to prove that the law of V_T admits a density with respect to the Lebesgue measure on \mathbb{R}^2 , for $T > 0$ the fixed terminal time, which of course suffices since $T > 0$ is arbitrarily fixed.

To prove such a result, we will apply Lemma 1.1 to the random variable $X = V_T$. First, we will build in Subsection 3.1 some absolutely continuous changes of measures, on our Poisson space, which will allow to define the perturbed processes V^λ . In fact, we will define a "class" of changes of measure, depending on the "direction" in which we want to perturb our process. The a.s. differentiability of V_T^λ with respect to λ will be studied in Subsection 3.2. In Subsection 3.3, we will choose a "direction", and we prove that the associated $\frac{\partial}{\partial \lambda} V_T^\lambda \Big|_{\lambda=0}$ is a.s. invertible. We will finally conclude in Subsection 3.4.

3.1 Absolutely continuous changes of measure.

Following the ideas of Bichteler-Jacod, [3], we now build a family of shifts S^λ on Ω , such that the family $V_T^\lambda = V_T \circ S^\lambda$ satisfies the assumptions of the criterion given in Lemma 1.1. We begin with a definition, which describes in which "directions" we are authorized to perturb our process.

Definition 3.1 *We say that a predictable function $v(\omega, s, \alpha, z): \Omega \times [0, T] \times [0, 1] \times \mathbb{R}^* \mapsto \mathbb{R}^2$ is a **direction** if it is of class C^1 in z , and if there exists a deterministic positive function $\rho(z): \mathbb{R}^* \mapsto \mathbb{R}^+$ such that*

$$|v(\omega, s, \alpha, z)| + |v'(\omega, s, \alpha, z)| \leq \rho(z) \quad (3.1)$$

(where $v' = v'_z$), and

$$\rho \in L^1(\mathbb{R}^*, dz) ; \rho(z) \leq (|z|/2) \wedge (1/|z|) ; \text{ and } \forall z \in \mathbb{R}^*, \rho(z) \leq 1/2 \quad (3.2)$$

Let now v be a fixed direction. We associate with v many objects.

We consider a neighborhood Λ of 0 in $B(0, 1) \subset \mathbb{R}^2$. For $\lambda \in \Lambda$, we define the following perturbation:

$$\Gamma^\lambda(\omega, t, z, \alpha) = z + \lambda.v(\omega, t, z, \alpha) = z + \lambda_x.v_x(\omega, t, z, \alpha) + \lambda_y.v_y(\omega, t, z, \alpha) \quad (3.3)$$

One can check, using (3.1) that for every $\lambda \in \Lambda$, for every ω, t, α , the map $z \mapsto \Gamma^\lambda(\omega, t, z, \alpha)$ is an increasing bijection from \mathbb{R}^* into itself.

For $\lambda \in \Lambda$, we set $N^\lambda = \Gamma^\lambda(N)$: if A is a Borel set of $[0, T] \times [0, 1] \times \mathbb{R}^*$,

$$N^\lambda(A) = \int_0^T \int_0^1 \int_{\mathbb{R}^*} \mathbf{1}_A(s, \Gamma^\lambda(\omega, s, z, \alpha), \alpha) N(\omega, dz, d\alpha, ds) \quad (3.4)$$

We consider the shift S^λ defined by

$$V_0 \circ S^\lambda(\omega) = V_0(\omega), \quad N \circ S^\lambda(\omega) = N^\lambda(\omega) \quad (3.5)$$

We now look for a family of probability measures P^λ on Ω satisfying $P^\lambda \circ (S^\lambda)^{-1} = P$. To this end, we consider the following predictable real-valued function on $\Omega \times [0, T] \times \mathbb{R}^* \times [0, 1]$

$$Y^\lambda(\omega, s, z, \alpha) = 1 + \lambda_x.v'_x(\omega, s, z, \alpha) + \lambda_y.v'_y(\omega, s, z, \alpha). \quad (3.6)$$

We have

$$|Y^\lambda(\omega, s, z, \alpha) - 1| \leq |\lambda|\rho(z). \quad (3.7)$$

Then we consider the following square integrable Doléans-Dade martingale:

$$G_t^\lambda = 1 + \int_0^t \int_0^1 \int_{\mathbb{R}^*} G_{s-}^\lambda (Y^\lambda(s, z, \alpha) - 1) \tilde{N}(dz, d\alpha, ds). \quad (3.8)$$

Proposition 3.2 G_t^λ is strictly positive for every $t \in [0, t]$. If P^λ is the probability measure defined by $P^\lambda = G_T^\lambda.P$, then $P^\lambda \circ (S^\lambda)^{-1} = P$. Furthermore,

$$\sup_\lambda E \left[(G_T^\lambda)^2 \right] < \infty \quad (3.9)$$

Proof. Recall that if

$$M_t^\lambda = \int_0^t \int_0^1 \int_{\mathbb{R}^*} (Y^\lambda(\omega, s, z, \alpha) - 1) \tilde{N}(dz, d\alpha, ds) \quad (3.10)$$

then (see Jacod-Shiryaev [13] p.59), $G_t^\lambda = e^{M_t^\lambda} \prod_{s \leq t} (1 + \Delta M_s^\lambda) e^{-\Delta M_s^\lambda}$.

Since by construction, $|Y^\lambda(\omega, s, z, \alpha) - 1| \leq \frac{1}{2}$ for $z \in \mathbb{R}^*_+$, the jumps of M^λ are greater than $-\frac{1}{2}$, and thus G_t^λ is strictly positive. Now, using the definition of the shift S^λ and the Girsanov theorem (see Jacod-Shiryaev [13] p.157), we see that the compensator of N under P^λ is $\Gamma^\lambda(Y^\lambda.m)$. But Y^λ has been chosen such that $\Gamma^\lambda(Y^\lambda.m) = m$. Indeed, considering a Borel set A of $[0, T] \times \mathbb{R}^* \times [0, 1]$, we have

$$\Gamma^\lambda(Y^\lambda.m)(A) = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \mathbf{1}_A(s, \Gamma^\lambda(s, z, \alpha), \alpha) Y^\lambda(s, z, \alpha) dz d\alpha ds. \quad (3.11)$$

The substitution $z' = \Gamma^\lambda(s, z, \alpha)$ implies that $\Gamma^\lambda(Y^\lambda.m)(A) = m(A)$. Hence since the law of a Poisson point measure is characterized by its intensity, we deduce that $\mathcal{L}(N^\lambda|P^\lambda) = \mathcal{L}(N|P)$. Finally, since V_0 is independent of G^λ , it is clear that $\mathcal{L}(V_0|P^\lambda) = \mathcal{L}(V_0|P)$. We have shown that $P^\lambda \circ (S^\lambda)^{-1} = P$.

We deduce (3.9) from (3.8), (3.7), the fact that $\rho \in L^1 \cap L^\infty(\mathbb{R}^*, dz)$, and the Gronwall Lemma. \triangle

3.2 Perturbation and derivation of V_t .

In this subsection, we consider a fixed direction v , we use the notations of the previous subsection, and we study the smoothness of the map $\lambda \mapsto V_t^\lambda = V_t \circ S^\lambda$. Here the α -process W is fixed, deterministic (from the point of view of the probability space Ω), and thus behaves as a parameter.

Proposition 3.3 *Assume Hypotheses 2.1 and 2.8. Let $\lambda \in \Lambda$ be fixed. The perturbed process V^λ , defined by $V_t^\lambda = V_t \circ S^\lambda$, satisfies the following equation:*

$$V_t^\lambda = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) N(dz, d\alpha, ds) \quad (3.12)$$

Proof. It suffices to replace everywhere ω by $S^\lambda(\omega)$ in equation (2.10). \triangle

We will need the following Lemma.

Lemma 3.4 *Assume Hypotheses 2.1 and 2.8. For each λ , equation (3.12) admits a unique solution that is a.s. càdlàg from $[0, T]$ into \mathbb{R}^2 . We furthermore have a.s.*

$$\sup_{\lambda} \sup_{0 \leq t \leq T} |V_t^\lambda| < \infty \quad (3.13)$$

We omit the proof of this lemma, because it can be done in the same way as that of the next one.

The following lemma deals with the possible derivative of V_t^λ , which should satisfy the equation obtained by differentiating formally equation (3.12) with respect to λ .

Lemma 3.5 *Assume Hypotheses 2.1 and 2.8. For each λ , the equation*

$$\begin{aligned} D_t^\lambda &= \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) D_{s-}^\lambda N(ds, d\alpha, dz) \\ &\quad + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) v(s, \alpha, z) N(ds, d\alpha, dz) \end{aligned} \quad (3.14)$$

admits a unique solution which is a.s. càdlàg from $[0, T]$ into $\mathcal{M}_{2 \times 2}(\mathbb{R})$. We have furthermore almost surely

$$\sup_{\lambda} \sup_{0 \leq t \leq T} |D_t^\lambda| < \infty \quad (3.15)$$

Remark that there is no reason why that for some λ fixed, say for $\lambda = 0$, D_T^0 belongs to L^2 (or even L^1). The only assumption that makes D_T^0 belonging to L^2 easily is the Maxwell assumption $B(X, \theta) = \beta(\theta)$, which yields that $\gamma(X, z) = A(g(z)) \cdot X$, with g no more depending on X , and thus $\gamma'_X(X, z) = A(g(z))$. In any other case, D_T^0 behaves almost as the Doléans-Dade exponential of a pure jump process with finite variations, belonging to all the L^q s, but this does not imply that D_T^0 belongs to L^1 . (One easily builds Poisson driven semimartingales which belong to all the L^q s, and of which the Doléans-Dade exponential is not in L^1). This is the reason why we have to use the a.s. derivatives and the weak criterion given by Lemma 1.1.

Proof. 1) We first prove the uniqueness. We will use Lemma 4.1 of the Appendix, for λ and ω fixed. Let thus λ be fixed, and let D and E be two càdlàg solutions of (3.14). A simple computation shows that

$$|D_t - E_t| \leq \int_0^t \int_0^1 \int_{\mathbb{R}^*} |D_{s-} - E_{s-}| \times \left| \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \right| N(ds, d\alpha, dz) \quad (3.16)$$

Since $\Gamma^\lambda(s, \alpha, z) = z + \lambda.v(s, \alpha, z)$, we deduce from (3.1) and (3.2) that $|\Gamma^\lambda(s, \alpha, z) - z| \leq (|z|/2) \wedge (1/|z|)$. Hence, using (2.11) and (2.13) in Hypothesis 2.8, we obtain the existence of a constant C such that

$$\left| \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \right| \leq C \left(1 + |V_{s-}^\lambda|^p + |W_{s-}(\alpha)|^p \right) \bar{\eta}(z) \quad (3.17)$$

We set $\tilde{\eta}(s, \alpha, z) = (1 + |W_{s-}(\alpha)|^p) \bar{\eta}(z)$. Then $\tilde{\eta}$ belongs to $L^1(ds, d\alpha, dz)$, thanks to Hypothesis 2.8 and (2.15), and hence $\tilde{\eta}$ belongs a.s. to $L^1(N(ds, d\alpha, dz))$. We also set $c = 1 + \sup_\lambda \sup_{s \in [0, T]} |V_{s-}^\lambda|^p$, which is a.s. finite thanks to Lemma (3.4). We finally obtain

$$|D_t - E_t| \leq Kc \int_0^t \int_0^1 \int_{\mathbb{R}^*} |D_{s-} - E_{s-}| \times \tilde{\eta}(s, \alpha, z) N(ds, d\alpha, dz) \quad (3.18)$$

Applying Lemma (4.1), we finally deduce that $\sup_{[0, T]} |D_t - E_t| = 0$ a.s., which was our aim.

2) We now prove the existence. We still fix λ . We first consider the simpler equation, for $n \in \mathbb{N}_*$ fixed,

$$\begin{aligned} \bar{D}_t^n &= \int_0^t \int_0^1 \int_{|z| \leq n} \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \bar{D}_{s-}^n N(ds, d\alpha, dz) \\ &+ \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) v(s, \alpha, z) N(ds, d\alpha, dz) \end{aligned} \quad (3.19)$$

We denote by U_t the last term of this equation. Notice that thanks to (2.14) in Hypothesis 2.8 and to (3.1), a.s., $\sup_{[0, T]} |U_t| \leq A$, where

$$A = K \left(1 + \sup_{\lambda, u} |V_u^\lambda|^p \right) \int_0^t \int_0^1 \int_{\mathbb{R}^*} (1 + |W_{s-}(\alpha)|^p) \rho(z) N(ds, d\alpha, dz) \quad (3.20)$$

which is a.s. finite thanks to (3.2), Lemma 3.4 and (2.15).

Since $N|_{[0, T] \times [0, 1] \times \{|z| \leq n\}}$ is a finite counting measure, it can be written (for each ω) as a (finite) sum of n Dirac measures at some points (T_i, α_i, z_i) , and one may assume that $0 < T_1 < T_2 < \dots < T_n < T$. Thus equation (3.19) can be solved by working recursively on the time intervals $[T_i, T_{i+1}[$:

for $t \in [0, T_1[$, we set $\bar{D}_t^n = U_t$

for $t \in [T_1, T_2[$, we set $\bar{D}_t^n = \gamma'_X(V_{T_1-}^\lambda - W_{T_1-}(\alpha_1), \Gamma^\lambda(T_1 s, \alpha_1, z_1)) \bar{D}_{T_1-}^n + U_t$

and so on...

Then we have to prove that for almost all ω ,

$$\sup_n \sup_{t \in [0, T]} |\bar{D}_t^n| < \infty \quad (3.21)$$

Using (3.20) and the same arguments and notations as in the proof of uniqueness, we obtain:

$$|\bar{D}_t^n| \leq A + Kc \int_0^t \int_0^1 \int_{\mathbb{R}^*} |\bar{D}_{s-}^n| \tilde{\eta}(s, \alpha, z) N(ds, d\alpha, dz) \quad (3.22)$$

Lemma 4.1 allows us to conclude that

$$\sup_{[0, T]} |\bar{D}_t^n| \leq A \exp \left(\int_0^T \int_0^1 \int_{\mathbb{R}^*} \ln(1 + Kc\tilde{\eta}(s, \alpha, z)) N(ds, d\alpha, dz) \right) \quad (3.23)$$

and (3.21) is proved. We finally check that the family \bar{D}^n is Cauchy for the supremum norm on $[0, T]$ (for almost all ω fixed). Let $n < n'$ be fixed. Then

$$\begin{aligned} |\bar{D}_t^n - \bar{D}_t^{n'}| &\leq \int_0^t \int_0^1 \int_{\mathbb{R}^*} |\gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))| |\bar{D}_{s-}^n - \bar{D}_{s-}^{n'}| N(ds, d\alpha, dz) \\ + \sup_{l, u \in [0, T]} |\bar{D}_u^l| &\times \int_0^T \int_0^1 \int_{n < |z| < n'} |\gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))| N(ds, d\alpha, dz) \end{aligned} \quad (3.24)$$

Still using the same notations as in the proof of uniqueness, we obtain

$$|\bar{D}_t^n - \bar{D}_t^{n'}| \leq Kc \int_0^t \int_0^1 \int_{\mathbb{R}^*} \tilde{\eta}(s, \alpha, z) |\bar{D}_{s-}^n - \bar{D}_{s-}^{n'}| N(ds, d\alpha, dz) + Z^{n, n'} \quad (3.25)$$

where

$$Z^{n, n'} = \sup_{l, u \in [0, T]} |\bar{D}_u^l| \times cK \int_0^T \int_0^1 \int_{n < |z| < n'} \tilde{\eta}(s, \alpha, z) N(ds, d\alpha, dz) \quad (3.26)$$

Since $\tilde{\eta}$ belongs (a.s.) to $L^1(N)$, it is clear that when n, n' go to infinity, $Z^{n, n'}$ goes to 0. Lemma 4.1 yields immediately that

$$\sup_{[0, T]} |\bar{D}_t^n - \bar{D}_t^{n'}| \leq B \times Z^{n, n'} \quad (3.27)$$

where B is an a.s. finite random variable. The family \bar{D}_t^n is thus a.s. Cauchy for the supremum norm on $[0, T]$, and hence admits a limit \bar{D}_t . Making n tend to infinity in (3.19), using (3.21), we show that \bar{D} satisfies (3.14). This concludes the proof of the existence.

3) We finally check (3.15). Still using the same arguments and notations, we obtain

$$|D_t^\lambda| \leq Kc \int_0^t \int_0^1 \int_{\mathbb{R}^*} \tilde{\eta}(s, \alpha, z) |D_{s-}^\lambda| N(ds, d\alpha, dz) + A \quad (3.28)$$

and Lemma 4.1 allows to conclude as usual that

$$\sup_{\lambda} \sup_{[0, T]} |D_t^\lambda| \leq A \exp \left(\int_0^T \int_0^1 \int_{\mathbb{R}^*} \ln(1 + Kc\tilde{\eta}(s, \alpha, z)) N(ds, d\alpha, dz) \right) \quad (3.29)$$

which implies (3.15) △

To check that $\lambda \mapsto V_T^\lambda$ is a.s. differentiable, we first need a Lipschitz estimate.

Lemma 3.6 *Assume Hypotheses 2.1 and 2.8. There exists an a.s. finite random variable A such that for all $0 \leq t \leq T$, all $\lambda, \mu \in \Lambda$,*

$$|V_t^\lambda - V_t^\mu| \leq A|\lambda - \mu| \quad (3.30)$$

Proof. Let λ, μ be fixed. Notice that thanks to Hypothesis 2.8, to the definition of Γ^λ , and to the properties of the direction v , there exists a constant C such that

$$\begin{aligned}
& |\gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) - \gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z))| \\
& \leq C \left(1 + |V_{s-}^\lambda|^p + |V_{s-}^\mu|^p + |W_{s-}(\alpha)|^p\right) \bar{\eta}(z) |V_{s-}^\lambda - V_{s-}^\mu| \\
& \quad + C \left(1 + |V_{s-}^\mu|^p + |W_{s-}(\alpha)|^p\right) |\Gamma^\lambda(s, \alpha, z) - \Gamma^\mu(s, \alpha, z)| \\
& \leq C \sup_{\lambda', u} \left(1 + |V_u^{\lambda'}|^p\right) \times \left(\bar{\eta}(z)(1 + |W_{s-}(\alpha)|^p) \times |V_{s-}^\lambda - V_{s-}^\mu| + |\lambda - \mu|\rho(z)\right) \\
& = Cc \left(\bar{\eta}(s, \alpha, z) |V_{s-}^\lambda - V_{s-}^\mu| + |\lambda - \mu|\rho(z)\right) \quad (3.31)
\end{aligned}$$

where the last inequality defines some notations. As in the previous proofs, $\bar{\eta} \in L^1(N)$ almost surely, and c is a.s. finite. We thus deduce that

$$\begin{aligned}
|V_t^\lambda - V_t^\mu| & \leq Cc \int_0^t \int_0^1 \int_{\mathbb{R}^*} |V_{s-}^\lambda - V_{s-}^\mu| \bar{\eta}(s, \alpha, z) N(ds, d\alpha, dz) \\
& \quad + Cc |\lambda - \mu| \int_0^T \int_0^1 \int_{\mathbb{R}^*} \rho(z) N(ds, d\alpha, dz) \quad (3.32)
\end{aligned}$$

Lemma 4.1 allows one more time to conclude that (3.30) holds with

$$A = Cc \int_0^T \int_0^1 \int_{\mathbb{R}^*} \rho(z) N(ds, d\alpha, dz) \exp \left(\int_0^T \int_0^1 \int_{\mathbb{R}^*} \ln[1 + Cc\bar{\eta}(s, \alpha, z)] N(ds, d\alpha, dz) \right) \quad (3.33)$$

\triangle

We finally can prove the differentiability of V_T^λ .

Proposition 3.7 *Assume Hypotheses 2.1 and 2.8. For almost all ω , the map $\lambda \mapsto V_T^\lambda$ is differentiable on Λ , and $\frac{\partial}{\partial \lambda} V_T^\lambda = D_T^\lambda$.*

Proof. We will check the existence of an a.s. finite random variable B such that a.s., for all $0 \leq s \leq T$, all $\lambda, \mu \in \Lambda$,

$$|V_s^\lambda - V_s^\mu - D_s^\lambda(\lambda - \mu)| \leq B|\lambda - \mu|^2 \quad (3.34)$$

which will of course suffice. Let thus $\lambda, \mu \in \Lambda$ be fixed. Set $\Delta_s(\lambda, \mu) = V_s^\lambda - V_s^\mu - D_s^\lambda(\lambda - \mu)$. Using Hypothesis 2.8, the definition of Γ^λ , the properties of the direction v , the notations of the proof of Lemma 3.5, and Lemma 3.6, we deduce the existence of a constant K such that for all $s \leq T$,

$$\begin{aligned}
& \left| \gamma \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) - \gamma \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z) \right) \right. \\
& \quad \left. - \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) D_{s-}^\lambda(\lambda - \mu) \right|
\end{aligned}$$

$$\begin{aligned}
& -\gamma'_z \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) v(s, \alpha, z) (\lambda - \mu) \Big| \\
\leq & \left| \gamma \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) - \gamma \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \right. \\
& \left. - \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) (V_{s-}^\lambda - V_{s-}^\mu) \right| \\
& + \left| \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \Delta_{s-}(\lambda, \mu) \right| \\
& + \left| \gamma \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z) \right) - \gamma \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \right. \\
& \left. - \gamma'_z \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) v(s, \alpha, z) (\lambda - \mu) \right| \\
& + \left| \left[\gamma'_z \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) - \gamma'_z \left(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z) \right) \right] v(s, \alpha, z) (\lambda - \mu) \right| \\
\leq & K \left[c\tilde{\eta}(s, \alpha, z) |V_{s-}^\lambda - V_{s-}^\mu|^2 + c\tilde{\eta}(s, \alpha, z) |\Delta_{s-}(\lambda, \mu)| \right. \\
& \left. + c(1 + |W_{s-}(\alpha)|^p) \rho(z) |\lambda - \mu| \left\{ |\lambda - \mu| + |V_{s-}^\lambda - V_{s-}^\mu| \right\} \right] \\
\leq & B_1 |\lambda - \mu|^2 [\tilde{\eta}(s, \alpha, z) + (1 + |W_{s-}(\alpha)|^p) \rho(z)] + B_1 \tilde{\eta}(s, \alpha, z) |\Delta_{s-}(\lambda, \mu)| \tag{3.35}
\end{aligned}$$

where B_1 is an a.s. finite random variable. Since $\tilde{\eta}(s, \alpha, z) + (1 + |W_{s-}(\alpha)|^p) \rho(z)$ belongs a.s. to $L^1(N(ds, d\alpha, dz))$, it is easily deduced from (2.15) and (3.30) that for all $t \leq T$, a.s.,

$$|\Delta_t(\lambda, \mu)| \leq B_2 |\lambda - \mu|^2 + B_1 \int_0^t \int_0^1 \int_{\mathbb{R}^*} \tilde{\eta}(s, \alpha, z) |\Delta_{s-}(\lambda, \mu)| N(ds, d\alpha, dz) \tag{3.36}$$

where B_2 is an a.s. finite random variable. Lemma 4.1 allows to deduce that (3.34) holds, which concludes the proof. \triangle

3.3 Choice of v and invertibility of D_T^0 .

We still have to check that for a good choice of v , D_T^0 is a.s. invertible (that will provide the condition (iii) in Lemma 1.1). To this aim, we adapt to our context the ideas of Bichteler-Jacod, [3]. Thanks to (3.14), we may write

$$D_t^0 = \int_0^t dX_s \cdot D_{s-}^0 + H_t \tag{3.37}$$

where

$$X_t = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_X(V_{s-} - W_{s-}(\alpha), z) N(ds, d\alpha, dz) \tag{3.38}$$

$$H_t = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V_{s-} - W_{s-}(\alpha), z) v(s, \alpha, z) N(ds, d\alpha, dz) \tag{3.39}$$

Using Jacod [12], we compute explicitly D_T^0 . First, we denote by K_t the Doléans-Dade exponential of X : for I the unit matrix of $\mathcal{M}_{2 \times 2}(\mathbb{R})$,

$$K_t = \mathcal{E}(X)_t = I + \int_0^t dX_s \cdot K_{s-} = \prod_{s \leq t} (I + \Delta X_s) \quad (3.40)$$

Then we consider the following sequence of stopping times:

$$S_0 = 0 \quad ; \quad S_{n+1} = \inf \{t \in]S_n, T] \mid \det(I + \Delta X_t) = 0\} \quad (3.41)$$

with the convention $\inf \emptyset = \infty$. Then the sequence S_n is totally inaccessible, and we have, a.s., for all n , $T \neq S_n$. Furthermore, it is clear that for all n , all $t \in]S_n, S_{n+1}[$, the Doléans-Dade exponential $\mathcal{E}(X - X^{S_n})_t = \prod_{S_n < s \leq t} (I + \Delta X_s)$ is invertible. We thus know, still from [12], that if ω satisfies $S_n < T < S_{n+1} = \infty$, then

$$D_T^0 = \mathcal{E}(X - X^{S_n})_T \cdot \left[\Delta H_{S_n} + \int_{]S_n, T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \Delta X_s)^{-1} dH_s \right] \quad (3.42)$$

We finally rewrite (3.42) explicitly:

Proposition 3.8 *For almost all ω , there exists n such that $S_n < T < S_{n+1}$, and*

$$\begin{aligned} D_T^0 = & \mathcal{E}(X - X^{S_n})_T \cdot \left[\Delta H_{S_n} + \right. \\ & \left. \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z))^{-1} \gamma'_z(V_{s-} - W_{s-}(\alpha), z) \right. \\ & \left. v(s, \alpha, z) N(ds, d\alpha, dz) \right] \end{aligned} \quad (3.43)$$

We now choose v . First of all, we denote by k a function from $\mathcal{M}_{2 \times 2}(\mathbb{R})$ into $[0, 1]$ such that

$$k(M) = 0 \quad \iff \quad \det M = 0 \quad (3.44)$$

and such that the map

$$M \mapsto \begin{cases} k(M)(M^{-1})^T & \text{if } \det M \neq 0 \\ 0 & \text{if } \det M = 0 \end{cases} \quad (3.45)$$

is of class C_b^∞ from $\mathcal{M}_{2 \times 2}(\mathbb{R})$ into itself.

We also consider a C^1 function f from \mathbb{R}^* into $]0, 1]$ such that for some $c \in]0, 1]$

$$|f| + |f'| \leq c \quad ; \quad |f(z)| + |f'(z)| \leq (|z|/2) \wedge (1/|z|) \quad ; \quad |f| + |f'| \in L^1(\mathbb{R}^*, dz) \quad (3.46)$$

Definition 3.9 *We set*

$$\begin{aligned} v(s, \alpha, z) = & \frac{\gamma'_z(V_{s-} - W_{s-}(\alpha), z)^T}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p} \\ & \frac{(I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z))^{-1, T} \times k(I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z))}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p} \\ & \cdot \mathcal{E}(X - X^{S_n})_{s-}^{-1, T} k(\mathcal{E}(X - X^{S_n})_{s-}) \times f(z) \end{aligned} \quad (3.47)$$

Then the following result is straightforward.

Lemma 3.10 *If c (see (3.46)) is small enough, which we assume, then the map v defined in Definition 3.9 is a direction in the sense of Definition 3.1.*

In view of (3.43), the first interest of this direction is the following.

Lemma 3.11 *With our choice for v , $\Delta H_{S_n} = 0$ for all n and almost all ω such that $S_n < T$.*

Proof. The stopping time S_n is a time of jump of the Poisson measure. Let us denote by (α_{S_n}, z_{S_n}) the associated jump. We know, from the definition of S_n , that $\det(I + \Delta X_{S_n}) = 0$, which implies that $\det(I + \gamma'_X(V_{S_n} - W_{S_n}(\alpha_{S_n}), z_{S_n})) = 0$. Hence, thanks to the definition of v and k , we deduce that $v(S_n, \alpha_{S_n}, z_{S_n}) = 0$, which clearly implies the result thanks to (3.39). \triangle

Remark 3.12 (i) *We deduce from the lemma above that in order to prove that D_T^0 is a.s. invertible, it suffices to check that for any n , for almost all ω satisfying $S_n < T < S_{n+1}$, Δ_T^n is invertible, where*

$$\begin{aligned} \Delta_T^n &= \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z))^{-1} \\ &\quad \gamma'_z(V_{s-} - W_{s-}(\alpha), z) v(s, \alpha, z) N(ds, d\alpha, dz) \end{aligned} \quad (3.48)$$

(ii) *We can also write, using the explicit expression of v ,*

$$\Delta_T^n = \int_{]S_n, T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1} dR_s \mathcal{E}(X - X^{S_n})_{s-}^{-1, T} \quad (3.49)$$

where

$$R_t = \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} J(V_{s-} - W_{s-}(\alpha), z) \times h(s, \alpha, z) \times f(z) N(ds, d\alpha, dz) \quad (3.50)$$

with, for $X \in \mathbb{R}^2$,

$$J(X, z) = (I + \gamma'_X(X, z))^{-1} \gamma'_z(X, z) \gamma'_z(X, z)^T (I + \gamma'_X(X, z))^{-1, T} \quad (3.51)$$

and

$$h(s, \alpha, z) = \frac{1}{(1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p)^2} \times k(I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z)) k(\mathcal{E}(X - X^{S_n})_{s-}) \quad (3.52)$$

For all X, z , $J(X, z)$ is a symmetric nonnegative matrix. The function h is always nonnegative. Hence R_t is nonnegative, symmetric, and increasing for the strong order. Since h does never vanish, and since $\mathcal{E}(X - X^{S_n})_{s-}^{-1}$ is invertible for all $s \in]S_n, T]$, it suffices to prove that a.s., for all $0 \leq s < t \leq T$, $\bar{R}_t - \bar{R}_s$ is invertible, where

$$\bar{R}_t = \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} J(V_{s-} - W_{s-}(\alpha), z) \times f(z) N(ds, d\alpha, dz) \quad (3.53)$$

One may for example check that a.s., for all $0 \leq s < t \leq T$, all $Y \in \mathbb{R}^2 / \{0\}$, $Y^T(\bar{R}_t - \bar{R}_s)Y > 0$.

Before concluding that D_T^0 is a.s. invertible, we state and prove a last lemma.

Lemma 3.13 *Assume Hypotheses 2.1 and 2.8. Then for all $t \in]0, T]$, the law of V_t (and thus that of W_t) is not a Dirac mass.*

Proof. Using the conservations of momentum and kinetic energy, see Proposition 2.9, one easily proves that for any a in \mathbb{R}^2 , the quantity $E(|V_t - a|^2)$ does not depend on t . Hence, if the law of V_t was a Dirac mass at some $a \in \mathbb{R}^2$, we would deduce that the law of V_0 , i.e. Q_0 , is also a Dirac mass at a . This contradicts Hypothesis 2.8. \triangle

We finally prove that condition (iii) of Lemma 1.1 is satisfied by V_T .

Proposition 3.14 *Assume Hypotheses 2.1 and 2.8. With our choice of v , D_T^0 is a.s. invertible.*

Proof. We of course use Remark 3.12. The proof necessitates several steps.

Step 1: Let X and Y be two non zero vectors of \mathbb{R}^2 . Then

$$\int_{\mathbb{R}^*} 1_{\{Y^T \gamma'_z(X, z) \gamma'_z(X, z)^T Y \neq 0\}} dz = \infty \quad (3.54)$$

To prove this, we first set $I(X, z) = \gamma'_z(X, z) \gamma'_z(X, z)^T$. Notice that, by definition of γ ,

$$I(X, z) = (g'_z(X, z))^2 A'(g(X, z)) X X^T A'(g(X, z))^T \quad (3.55)$$

But it is clear, see Section 2, that g'_z does never vanish. Hence, thanks to the substitution $\theta = g(X, z)$, we obtain (see Section 2 again)

$$\int_{\mathbb{R}^*} 1_{\{Y^T I(X, z) Y \neq 0\}} dz = \int_{-\pi}^{\pi} 1_{\{Y^T A'(\theta) X X^T A'(\theta)^T Y \neq 0\}} B(X, \theta) d\theta \quad (3.56)$$

But a simple computation shows that $Y^T A'(\theta) X X^T A'(\theta)^T Y$ does $1_{[-\pi, \pi]}(\theta) d\theta$ -almost never vanish (for $X \neq 0$ and $Y \neq 0$ fixed). Since $\int_{-\pi}^{\pi} B(X, \theta) d\theta = \infty$, the proof of Step 1 is finished.

Step 2: For all $s \in [0, T]$, for almost all ω ,

$$\int_0^1 1_{\{V_{s-} - W_{s-}(\alpha) \neq 0\}} d\alpha > 0 \quad (3.57)$$

Indeed, we know from Lemma 3.13 that $\mathcal{L}_\alpha(W_s)$ is not a Dirac mass. Hence, for any deterministic $X \in \mathbb{R}^2$,

$$\int_0^1 1_{\{X - W_{s-}(\alpha) \neq 0\}} d\alpha = P_\alpha(W_s \neq X) > 0 \quad (3.58)$$

Since ω is fixed, $V_{s-}(\omega)$ is " α -deterministic", and hence (3.58) holds for $X = V_{s-}(\omega)$, which drives immediately to (3.57).

Step 3: Associating Steps 1 and 2, we finally deduce: for all non-zero vector $Y \in \mathbb{R}^2$, all $s \in [0, T]$, a.s.,

$$\int_0^1 \int_{\mathbb{R}^*} 1_{\{Y^T \gamma'_z(V_{s-} - W_{s-}(\alpha) X, z) \gamma'_z(V_{s-} - W_{s-}(\alpha), z)^T Y \neq 0\}} d\alpha dz = \infty \quad (3.59)$$

Step 4: Let $s > 0$ and $Y \in \mathbb{R}^2/\{0\}$ be fixed. We now prove that on the set $S_n < T < S_{n+1} = \infty$, for all $s > S_n$ a.s. for all $t \in]s, T]$,

$$Y^T(\bar{R}_t - \bar{R}_s)Y > 0 \quad (3.60)$$

To this end, it suffices to show that the stopping time defined by

$$\tau(Y) = \inf \left\{ u > s \ / \int_s^u \int_{\mathbb{R}^*} \int_0^1 \mathbf{1}_{\{Y^T J(V_{s-} - W_{s-}(\alpha)X)Y > 0\}} N(ds, d\alpha, dz) > 0 \right\} \quad (3.61)$$

satisfies a.s. $\tau(Y) = s$. We have, by construction,

$$\int_s^{\tau(Y)} \int_{\mathbb{R}^*} \int_0^1 \mathbf{1}_{\{Y^T J(V_{s-} - W_{s-}(\alpha)X)Y > 0\}} N(ds, d\alpha, dz) \leq 1 \quad (3.62)$$

Taking the expectation in this expression, we obtain

$$E \left(\int_s^{\tau(Y)} \int_{\mathbb{R}^*} \int_0^1 \mathbf{1}_{\{Y^T J(V_{s-} - W_{s-}(\alpha)X)Y > 0\}} ds d\alpha dz \right) \leq 1 \quad (3.63)$$

and, we deduce that a.s.,

$$\int_s^{\tau(Y)} \int_{\mathbb{R}^*} \int_0^1 \mathbf{1}_{\{Y^T J(V_{s-} - W_{s-}(\alpha)X)Y > 0\}} ds d\alpha dz < \infty \quad (3.64)$$

Due to (3.59), this is not possible, except if $\tau(Y) = s$ a.s.

Step 5: The previous step shows that on the set $S_n < T < S_{n+1} = \infty$, for all $s \in]S_n, T]$, a.s., for all $u \in]s, T]$, $\bar{R}_u - \bar{R}_s$ is invertible.

What we have to prove is that on the set $S_n < T < S_{n+1} = \infty$, a.s., for all $s \in]S_n, T]$, for all $u \in]s, T]$, $\bar{R}_u - \bar{R}_s$ is invertible.

This extension is not hard, by using the fact that \bar{R} is increasing. The proof is complete \triangle

3.4 Conclusion.

We finally are able to conclude the

Proof of Theorem 2.10. Since $T > 0$ is arbitrarily fixed, it of course suffices to prove that the law of V_T admits a density. We thus apply Lemma 1.1 with $X = V_T$. The family X^λ is defined by $V_T^\lambda = V_T \circ S^\lambda$, the shift S^λ being defined by (3.5), relatively to the direction v chosen in Definition 3.9. Condition (i) of Lemma 1.1 is satisfied thanks to Proposition 3.2. Condition (ii) holds thanks to Proposition 3.7. Finally, Proposition 3.14 shows that condition (iii) is met. Hence the law of V_T admits a density, which was our aim. \triangle

4 Appendix.

Our purpose is to prove the following Gronwall type lemma.

Lemma 4.1 *Let \mathcal{X} be a measurable space. We consider a counting σ -finite measure $\mu(dt, dx)$ on $[0, T] \times \mathcal{X}$. Let $\eta(s, x)$ be a positive function belonging to $L^1(\mu)$. Then every bounded positive function φ_t on $[0, T]$, satisfying, for all $t > 0$,*

$$\varphi_t \leq a + \int_0^t \int_{\mathcal{X}} \varphi_{s-} \eta(s, x) \mu(ds, dx) \quad (4.1)$$

is bounded by

$$\sup_{[0,T]} \varphi_t \leq a \exp \left(\int_0^T \int_{\mathcal{X}} \ln(1 + \eta(s, x)) \mu(ds, dx) \right) \quad (4.2)$$

Proof. We divide the proof in two steps.

Step 1: We begin with the case where $\mu(\eta \neq 0) < \infty$. In this case, we can consider that the support of μ is finite, and thus that μ is of the form $\sum_{i=1}^n \delta_{(T_i, X_i)}$, with $0 < T_1 < T_2 < \dots < T_n < T$. Then we use (4.1). First, for all $t < T_1$, $\varphi_t \leq a$, from which we deduce, for all $t \in [T_1, T_2[$,

$$\varphi_t \leq a + a\eta(T_1, X_1) \leq a(1 + \eta(T_1, X_1)) \quad (4.3)$$

which clearly also holds for all $t \in [0, T_2[$. And so on... We finally obtain that for all $t \in [0, T]$,

$$\begin{aligned} \varphi_t &\leq a(1 + \eta(T_1, X_1)) \times \dots \times (1 + \eta(T_n, X_n)) \\ &\leq a \exp \left(\sum_{i=1}^n \ln(1 + \eta(T_i, X_i)) \right) \\ &\leq a \exp \left(\int_0^T \int_{\mathcal{X}} \ln(1 + \eta(s, x)) \mu(ds, dx) \right) \end{aligned} \quad (4.4)$$

which was our aim.

Step 2: If $\mu(\eta \neq 0) = \infty$, then we split the space \mathcal{X} into $\mathcal{X}_\epsilon \cup \mathcal{X}_\epsilon^c$, in such a way that for all ϵ , $\mu([0, T] \times \mathcal{X}_\epsilon) < \infty$, and such that \mathcal{X}_ϵ grows to \mathcal{X} when ϵ goes to 0. Then we rewrite (4.1) as

$$\varphi_t \leq (a + u_\epsilon) + \int_0^t \int_{\mathcal{X}_\epsilon} \varphi_{s-} \eta(s, x) \mu(ds, dx) \quad (4.5)$$

where $u_\epsilon = \|\varphi\|_\infty \int_0^t \int_{\mathcal{X}_\epsilon^c} \eta(s, x) \mu(ds, dx)$ clearly goes to 0 since $\eta \in L^1(\mu)$. Applying Step 1, we obtain for each ϵ

$$\sup_{[0,T]} \varphi_t \leq (a + u_\epsilon) \exp \left(\int_0^T \int_{\mathcal{X}_\epsilon} \ln(1 + \eta(s, x)) \mu(ds, dx) \right) \quad (4.6)$$

Making ϵ tend to 0 drives immediately to (4.2). \triangle

References

- [1] Alexandre, R.; Desvillettes, L.; Villani, C.; Wennberg, B.: Entropy dissipation and long range interactions, to appear in ARMA, Section Applications (2000).
- [2] K. Bichteler, J.B. Gravereaux, J. Jacod *Malliavin calculus for processes with jumps*, Number 2 in Stochastic monographs. Theory and application of stochastic processes. Gordon and Breach, (1987).
- [3] Bichteler, K., Jacod, J. : Calcul de Malliavin pour les diffusions avec sauts: existence d'une densité dans le cas unidimensionnel, Séminaire de Probabilités XVII, L.N.M. 986, 132-157, Springer (1983).

- [4] Bismut, J.M., Calcul des variations stochastique et processus de sauts, *Z. Wahrsch. Verw. Gebiete* 63, no. 2, 147-235, (1983).
- [5] Desvillettes, L.: Regularization properties of the 2-dimensional non-radially symmetric non-cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transport Theory Statist. Phys.* 26 no 3, 341-357, (1997).
- [6] Desvillettes, L.; Graham, C.; Méléard, S.: Probabilistic interpretation and numerical approximation of a Kac equation without cutoff, *Stoch. Proc. and Appl.*, 84, 1, 115-135 (1999).
- [7] Fournier, N.: Existence and regularity study for a 2-dimensional Kac equation without cutoff by a probabilistic approach, *Annals of applied probability*, Vol 10, No 2, 434-462 (2000).
- [8] Fournier, N.; Méléard, S.: A stochastic particle numerical method for 3D Boltzmann equations without cutoff, to appear in *Mathematics of Computation* (2001).
- [9] Fournier, N.; Méléard, S.: A Markov process associated with a Boltzmann equation without cutoff and for non Maxwell molecules, to appear in *Journal of Statistical Physics* (2001).
- [10] Graham, C.; Méléard, S.: Existence and regularity of a solution of a Kac equation without cutoff using the stochastic calculus of variations, *Commun. Math. Phys.* 205, 551-569 (1999).
- [11] Horowitz, J.; Karandikar, R.L.: Martingale problem associated with the Boltzmann equation, *Seminar on Stochastic Processes, 1989* (E. Cinlar, K.L. Chung, R.K. Gettoor, eds.), Birkhäuser, Boston (1990).
- [12] Jacod, J.; *Equations différentielles linéaires, la variation des constantes*, Séminaire de Probabilités XVI, L.N.M. 920, 442-448, Springer-verlag (1982).
- [13] Jacod, J.; Shiryaev, A.N.: *Limit theorems for stochastic processes*, Springer-Verlag (1987).
- [14] Nualart, D. : *Malliavin Calculus and related topics*, Springer Verlag, (1995).
- [15] Tanaka, H.: Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, *Z. Wahrsch. Verw. Geb.* 46, 67-105 (1978).