

Jumping SDEs: absolute continuity using monotony

Nicolas FOURNIER

Institut Elie Cartan, Campus Scientifique, BP 239
54506 Vandoeuvre-lès-Nancy Cedex, France
fournier@iecn.u-nancy.fr

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Abstract

We study the solution $X = \{X_t\}_{t \in [0, T]}$ to a Poisson-driven S.D.E. This equation is “irregular” in the sense that one of its coefficients contains an indicator function, which allows to generalize the usual situations: the rate of jump of X may depend on X itself. For $t > 0$ fixed, the random variable X_t does not seem to be differentiable (with respect to the *alea*) in a usual sense (see e.g. Bichteler and Jacod, [3]), and actually not even continuous. We thus introduce a new technique, based on a sort of monotony of the map $\omega \mapsto X_t(\omega)$, to prove that under quite stringent assumptions, which make possible comparison theorems, the law of X_t admits a density with respect to the Lebesgue measure on \mathbb{R} .

Key words : Stochastic differential equations, Jump processes, Stochastic calculus of variations.

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1 Introduction and Statements

Consider the following one-dimensional S.D.E., starting at $x \in \mathbb{R}$, on $[0, T]$:

$$X_t = x + \int_0^t \int_O \int_0^\infty \sigma(X_{s-}) \eta(z) \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} N(ds, dz, du) + \int_0^t b(X_{s-}) ds \quad (1.1)$$

where

Assumption (I): $N(ds, dz, du)$ is a Poisson measure on $[0, T] \times O \times [0, \infty[$, where O is an open subset of \mathbb{R} , and its intensity measure is given by $\nu(ds, dz, du) = ds \varphi(z) dz du$ for some strictly positive function $\varphi \in C^1(O)$.

The infinitesimal generator \mathcal{K} associated with this Markov process, defined for $\phi \in C_b^1(\mathbb{R})$ by $\mathcal{K}\phi(x) = \frac{d}{dt}E[\phi(X_t)]|_{t=0}$, is given by:

$$\mathcal{K}\phi(x) = b(x)\phi'(x) + \int_O \{\phi[x + h(x, z)] - \phi[x]\} \delta(x, z) dz \quad (1.2)$$

with $h(x, z) = \sigma(x)\eta(z)$ and $\delta(x, z) = \zeta(x)\varphi(z)$.

The study of such operators seems to be quite important, since they appear in P.D.E.s modeling various phenomena where immediate changes of state occur: collisions, coagulation, fragmentation... See e.g. [8] for the Boltzmann equation and [6] for the Smoluchowski equation. In particular, existence (and smoothness) results for such equations may be obtained by proving existence (and smoothness) of densities for associated stochastic processes, which satisfy nonlinear versions of S.D.E.s of the type of (1.1).

The case where $\delta(x, z)$ (in (1.2)) does not depend (or at least does not “really” depend) on x has been much studied. Let us recall the main known conditions under which the law of X_t admits a density for any $t > 0$, $\{X_t\}_{t \geq 0}$ being a Markov process with generator given by (1.2).

The first result is due to Bismut, [4], (Theorem 4.9), whose main assumptions are the following: $h(x, z) = z$, $O = \mathbb{R}^*$, $\delta(x, z) = [1 + \gamma(x, z)]\varphi(z)$, with b , γ and φ of class C^1 . Furthermore, $\int_O \varphi(z) dz = \infty$ and $\sup_x \int_O \gamma^2(x, z)\varphi(z) dz < \infty$. Notice that $\delta(x, z)$ is not allowed to depend strongly on x , because of the integrability condition on γ . In particular, the case where $\delta(x, z)$ is of the form $\zeta(x)\varphi(z)$ is never contained in [4], except if ζ is constant. Let us mention however that Bismut actually works in \mathbb{R}^n with a more general “compensated” generator.

A second important result is due to Bichteler, Jacod, [3], who essentially assume that: $\delta(x, z) \equiv 1$, h and b are of class C^2 , and for any x in \mathbb{R} , $\int_O \mathbf{1}_{\{h'_x(x, z) \neq 0\}} dz = \infty$. They also assume integrability and boundeness conditions about h , but they work with a compensated generator.

Picard, [13], Carlen, Pardoux, [5], and Denis, [7], have studied the much more difficult case where the “regular” measure $\delta(x, z) dz$ is replaced by any measure $q(dz)$, still independant of x .

However, the case where $\delta(x, z)$ really depends on x is quite difficult. In [9], a substitution in (1.2) has been used, in order to make disappear the dependance in x of δ , in the case of the operator associated with a Boltzmann equation. This approach is not very natural, and drives to quite non-tracktable assumptions, concerning regularity and integrability conditions about the inverse of the repartition function associated with the measure $\delta(x, z) dz$.

In the present work, we show that in certain situations, one may keep the indicator function and obtain absolute continuity results. Let us now state our results. First, we assume the following conditions.

Assumption (H): The functions σ , ζ and b are locally Lipschitz continuous from \mathbb{R} into itself, ζ is nonnegative. The map $|b| + |\sigma|\zeta$ has at most a linear growth. The \mathbb{R} -valued function η on O belongs to $L^1(O, \varphi(z)dz)$.

Then the following result holds.

Proposition 1.1 *Assume (I), (H). Then there exists a unique càdlàg adapted solution X to (1.1), and this solution satisfies*

$$E \left[\sup_{[0, T]} |X_t| \right] < \infty \quad (1.3)$$

By “adapted”, we mean adapted to the canonical filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ associated with N , defined by

$$\mathcal{F}_t = \sigma(N(A) ; A \in \mathcal{B}([0, t] \times O \times [0, \infty[)) \quad (1.4)$$

In order to obtain an absolute continuity result, we will suppose:

Assumption (AC):

1. The functions σ and ζ are increasing and strictly positive on \mathbb{R} .
2. The map η belongs to $C^2(O)$, is nonnegative, and η'' is bounded.
3. The first order derivative of the jump coefficient η satisfies the non-degeneracy condition:

$$\int_O \mathbf{1}_{\{\eta'(z) \neq 0\}} \varphi(z) dz = \infty \quad (1.5)$$

We now may state our main result.

Theorem 1.2 *Assume (I), (H) and (AC), and consider the unique solution X to (1.1). Then for each $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .*

Of course, this result is not so strong, since the monotony assumptions on σ and ζ are very stringent. Notice however that no smoothness conditions are assumed on σ , b and ζ and that nowhere in this work approximations of these functions by smooth functions have to be done. This comes from the fact that we will not use any integration by parts formula, and we will not have to differentiate $X_t(\omega)$ with respect to ω in any sense.

Let us finally present briefly the organization and main ideas of this paper. In Section 2, we give an idea of the proof of Proposition 1.1, and we show that a localization procedure may be done, in order to obtain bounded coefficients σ , b , and ζ .

In Section 3, we state and prove a new criterion of absolute continuity for the laws of random variables: Y has a density as soon as the map $\omega \mapsto Y(\omega)$ is “strongly” increasing in a certain sense. To be more precise, a \mathbb{R} -valued random variable Y has a density as soon as there exists a family $\{Y^\lambda\}_{\lambda \in [0,1]}$ of random variables of which the laws are absolutely continuous with respect to that of Y , and such that a.s., the map $\lambda \mapsto Y^\lambda(\omega)$ is sufficiently increasing. This is of course less strong than the usual criterion, which says approximatively that Y has a density as soon as the map $\lambda \mapsto Y^\lambda(\omega)$ is of class C^1 , with a strictly positive derivative (see e.g. Nualart, [12], for the Wiener case, and Bichteler, Jacod, [3], for the Poisson case).

In Section 4, we define a family of integer-valued random measures $\{N^\lambda\}_{\lambda \in [0,1]}$, of which the laws are absolutely continuous with respect to that of N . To this aim, we transpose to our context the main ideas of Bichteler, Jacod, [3]: we move slightly the sizes of the jumps of $N(ds, dz, du)$ (in z) according to a well-chosen direction, and we use the Girsanov Theorem. This way, we obtain a family of stochastic processes $\{\{X_t^\lambda\}_{t \in [0,T]}\}_{\lambda \in [0,1]}$ of which the laws are absolutely continuous with respect to that of $\{X_t\}_{t \in [0,T]}$. Then we notice that the map $\lambda \mapsto X_t^\lambda(\omega)$ (for $t \in [0, T]$ and $\omega \in \Omega$ fixed) has no chance to be differentiable: it actually seems to be a.s. almost everywhere discontinuous on $[0, 1]$. Thus no integration by parts may be done, and the use of the “standard” Malliavin Calculus seems to fail. Even in the very weak versions of the Malliavin Calculus non using integration by parts (as developed by Bouleau and Hirsch), some kind of regularity of the map $\lambda \mapsto X_t^\lambda(\omega)$ has to be assumed, see e.g. Nualart, [12], pp 83-87.

In Section 5, we show that for a good choice of the direction (according to which we move the sizes of the jumps of $N(ds, dz, du)$), the map $\lambda \mapsto X_t^\lambda(\omega)$ is a.s. increasing (for $t \in]0, T]$ and $\omega \in \Omega$ fixed), which allows to apply our criterion and thus to conclude the proof of Theorem 1.2.

Finally, Section 6 is devoted to the proof of the following generalization of Theorem 1.2.

Theorem 1.3 *Assume that φ is a C^1 strictly positive function on an open subset O of \mathbb{R} . Let b , σ , ζ , and η satisfy (H) and (AC). Consider also a measurable function $\gamma : \mathbb{R} \times O \mapsto [-1, \infty[$ satisfying*

$$\sup_{x \in \mathbb{R}} \zeta(x) \int_O \gamma^2(x, z) \varphi(z) dz < \infty \quad (1.6)$$

Then there exists a Markov process $\{Y_t\}_{t \in [0, T]}$ with infinitesimal generator \mathcal{L}

defined for all $\phi \in C_b^1(\mathbb{R})$ by

$$\mathcal{L}\phi(x) = b(x)\phi'(x) + \int_O \{\phi(x + \sigma(x)\eta(z)) - \phi(x)\} \zeta(x) \{1 + \gamma(x, z)\} \varphi(z) dz \quad (1.7)$$

and such that for all $t > 0$, the law of Y_t admits a density with respect to the Lebesgue measure on \mathbb{R} .

This result allows to add a “small” but quite irregular dependance in x in the generator: the C^1 function $\varphi(z)$ can be replaced by any measurable function $\delta(x, z)$ which is close, for every x and z , to a C^1 function not depending on x .

2 Existence, uniqueness, and localization

The aim of this short section is to give an idea of the proof of Proposition 1.1 and to show that a localization procedure may be done. We first introduce some notations.

Notation 2.1 Assume (I), (H). For each $n \in \mathbb{N}^*$, we set for all $x \in \mathbb{R}$

$$\sigma_n(x) = \sigma(x \wedge n \vee (-n)) ; \zeta_n(x) = \zeta(x \wedge n \vee (-n)) ; b_n(x) = b(x \wedge n \vee (-n)) \quad (2.1)$$

Under (I), (H), one easily proves that for each n , there exists a unique càdlàg adapted solution $\{X_t^n\}_{t \in [0, T]}$ to (1.1) where σ , ζ , and b have been replaced by σ_n , ζ_n , and b_n . Indeed, it suffices to use standard arguments (Gronwall’s Lemma, Picard’s iteration), to notice that b_n is globally Lipschitz continuous, and that for all x, y in \mathbb{R} ,

$$\begin{aligned} & \int_O \int_0^\infty |\sigma_n(x)\eta(z)\mathbf{1}_{\{u \leq \zeta_n(x)\}} - \sigma_n(y)\eta(z)\mathbf{1}_{\{u \leq \zeta_n(y)\}}| \varphi(z) dz du \\ & \leq \int_O |\eta(z)|\varphi(z) dz \times [\|\zeta_n\|_\infty |\sigma_n(x) - \sigma_n(y)| + \|\sigma_n\|_\infty |\zeta_n(x) - \zeta_n(y)|] \\ & \leq C_n |x - y| \quad (2.2) \end{aligned}$$

for some constant C_n .

It is also classically checked, using the fact that for all n (thanks to (H)),

$$\begin{aligned} |b_n(x)| + \int_O \int_0^\infty |\sigma_n(x)\eta(z)\mathbf{1}_{\{u \leq \zeta_n(x)\}}| \varphi(z) dz du \\ \leq |b_n(x)| + C|\sigma_n(x)\zeta_n(x)| \leq C(1 + |x|) \quad (2.3) \end{aligned}$$

the constant C being independant of n , that

$$\sup_n E \left(\sup_{[0, T]} |X_t^n| \right) < \infty \quad (2.4)$$

Let finally

$$\tau_n = \inf \{t \in [0, T] ; |X_t^n| \geq n\} \wedge T \quad (2.5)$$

Due to (2.4), there a.s. exists n such that for any m greater than n , $\tau_m = T$. On the other hand, an uniqueness argument shows that a.s., $X_t^{n+1} = X_t^n$ for all t in $[0, \tau_n]$. This allows to define a solution $\{X_t\}_{t \in [0, T]}$ to (1.1) in the following way: for each ω , we choose n large enough, in order to obtain $\tau_n(\omega) = T$, and then we set $X_t(\omega) = X_t^n(\omega)$ for all $t \in [0, T]$.

The uniqueness is also standard, since any solution Y to (1.1) will satisfy $Y_t = X_t^n$ for all $t \in [0, \tau_n]$ a.s.

We now would like to use a localization procedure, in order to simplify our problem. To this aim, we consider the next stringent hypothesis.

Assumption (H'): The same as (H), but the functions σ , ζ and b are globally Lipschitz continuous and bounded.

The following proposition allows to assume that the coefficients of (1.1) are bounded.

Proposition 2.2 *Assume that the conclusion of Theorem 1.2 holds under (I), (H') and (AC). Then it also holds under (I), (H) and (AC).*

Proof Let σ , ζ , and b satisfy (H) and (AC), and let $t \in]0, T]$ be fixed. Then for each $n \geq 1$, σ_n , ζ_n , and b_n satisfy (H') and (AC). Thus the law of X_t^n admits a density. But we also know that the set $\Omega_n = \{\omega \in \Omega ; X_t^n(\omega) = X_t(\omega)\}$ grows to Ω as n tends to infinity. Hence, for any Lebesgue-null Borel set $A \subset \mathbb{R}$,

$$P(X_t \in A) = \lim_n P(X_t \in A, \Omega_n) = \lim_n P(X_t^n \in A, \Omega_n) \leq \lim_n P(X_t^n \in A) = 0 \quad (2.6)$$

which was our aim. \square

We end this section with an important remark.

Remark 2.3 *We thus will always assume (I), (H') and (AC). Hence ζ will be bounded, and we will denote by M its supremum over \mathbb{R} . Thus the Poisson measure $N(ds, dz, du)$ may and will be considered, from now on, as a Poisson measure on $[0, T] \times O \times [0, M]$. Finally, equation (1.1) can be written:*

$$X_t = x + \int_0^t \int_O \int_0^M \sigma(X_{s-}) \eta(z) \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} N(ds, dz, du) + \int_0^t b(X_{s-}) ds \quad (2.7)$$

3 Absolute continuity using strong monotony

To prove Theorem 1.2, we will use the following absolute continuity criterion. This criterion is inspired from the work of Ben Arous, Léandre, [1].

Theorem 3.1 Let Y be a \mathbb{R} -valued random variable on a probability space (Ω, \mathcal{F}, P) . Assume that there exists a family $\{Y^\lambda\}_{\lambda \in [0,1]}$ of \mathbb{R} -valued random variables such that

1. For each $\lambda \in \Lambda$, the law of Y is absolutely continuous with respect to that of Y^λ .
2. The map $\lambda \mapsto Y^\lambda$ is a.s. strongly increasing on $[0, 1]$, in the sense that there exists an a.s. strictly positive random variable Z such that a.s., for all $0 \leq \lambda < \mu \leq 1$, $Y^\mu - Y^\lambda \geq (\mu - \lambda)Z$.

Then the law of Y is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

To prove this theorem, we need the following lemma.

Lemma 3.2 Let f be a map on $[0, 1]$, which is strongly increasing, in the sense that there exists a constant $c > 0$ such that for all $0 \leq \lambda < \mu \leq 1$, $f(\mu) - f(\lambda) \geq c(\mu - \lambda)$. Then for any Lebesgue-null subset A of \mathbb{R} ,

$$\int_0^1 \mathbf{1}_A(f(\lambda))d\lambda = 0 \quad (3.1)$$

Proof It suffices to check that the nonnegative measure γ on \mathbb{R} , defined by $\gamma(B) = \int_0^1 \mathbf{1}_B(f(\lambda))d\lambda$, is absolutely continuous with respect to the Lebesgue measure.

First notice that $\text{supp } \gamma \subset [f(0), f(1)]$. Then denote by f^{-1} the inverse of f , defined on $[f(0), f(1)]$ by $f^{-1}(x) = \inf\{\lambda \in [0, 1] ; f(\lambda) \geq x\}$. It is easily seen that f^{-1} is Lipschitz continuous on $[f(0), f(1)]$, with a Lipschitz constant equal to $1/c$. Thus, for any $x \leq y$ in $[f(0), f(1)]$,

$$\gamma([x, y]) = \int_0^1 \mathbf{1}_{\{f^{-1}(x) \leq \lambda \leq f^{-1}(y)\}}d\lambda = f^{-1}(y) - f^{-1}(x) \leq c^{-1}(y - x) \quad (3.2)$$

Thus γ is smaller than (and thus absolutely continuous with respect to) $c^{-1}dx$, dx standing for the Lebesgue measure on \mathbb{R} . The lemma follows. \square

Proof of Theorem 3.1 Let A be a Lebesgue-null subset of \mathbb{R} . We have to prove that $P(Y \in A) = 0$.

Using 1., we know that for each λ in $[0, 1]$, $P(Y \in A) = E(\mathbf{1}_A(Y^\lambda)G^\lambda)$, G^λ standing for the Radon-Nikodym density dY/dY^λ . Hence,

$$P(Y \in A) = \int_0^1 E(\mathbf{1}_A(Y^\lambda)G^\lambda) d\lambda = E\left(\int_0^1 \mathbf{1}_A(Y^\lambda)G^\lambda d\lambda\right) \quad (3.3)$$

The result will thus be proved if a.s., $\int_0^1 \mathbf{1}_A(Y^\lambda)G^\lambda d\lambda = 0$. But this is a consequence of the fact that a.s., $\int_0^1 \mathbf{1}_A(Y^\lambda)d\lambda = 0$, which holds thanks to Lemma 3.2, since for almost all $\omega \in \Omega$, the map $\lambda \mapsto Y^\lambda(\omega)$ is strongly increasing. \square

4 Perturbation of the solution

In order to prove Theorem 1.2, we will apply the previous criterion to the random variable X_t , for $t > 0$ fixed. Actually, we will use the Girsanov Theorem, which will allow us to build a family of stochastic processes $\{\{X_t^\lambda\}_{t \in [0, T]}\}_{\lambda \in [0, 1]}$ of which the laws will be absolutely continuous with respect to that of $\{X_t\}_{t \in [0, T]}$. All what follows is a generalization of what Bichteler and Jacod have introduced in [3] in the case of Poisson measures of the form $N(ds, dz)$ on $[0, T] \times O$, with intensity measure $dsdz$. Bichteler and Jacod were inspired by the work of Bismut, [4].

First of all, we define a class of “directions” in which we will be allowed to “perturbe” the Poisson measure N , and thus the solution $\{X_t\}_{t \in [0, T]}$ to (1.1).

Definition 4.1 *Let α be a C^1 function from O into \mathbb{R} . We will say that α belongs to \mathcal{D} if the following conditions hold.*

1. $\alpha \in L^1 \cap L^\infty(O, \varphi(z)dz)$, and α goes to 0 at the boundary of O .
2. *Setting,*

$$\xi(z) = |\alpha'(z)| + 2 \frac{|\alpha(z)|}{\varphi(z)} \sup_{w \in [z - |\alpha(z)|, z + |\alpha(z)|]} |\varphi'(w)| \quad (4.1)$$

ξ belongs to $L^1(O, \varphi(z)dz)$ and is smaller than 1/2.

Let now $\alpha \in \mathcal{D}$ be a fixed “direction”. For each $\lambda \in [0, 1]$, the map $z \mapsto \gamma^\lambda(z) = z + \lambda\alpha(z)$ is an increasing C^1 bijection from O to O (thanks to the facts that $|\alpha'| \leq 1/2$ and $\lim_{z \rightarrow \partial O} \alpha(z) = 0$). This allows us to consider, for each $\lambda \in [0, 1]$ fixed, the image measure $N^\lambda = \gamma^\lambda(N)$ of N by γ^λ , which is still a measure on $[0, T] \times O \times [0, M]$: for all Borel subset A of $[0, T] \times O \times [0, M]$,

$$N^\lambda(A) = \int_0^T \int_O \int_0^M 1_A(s, \gamma^\lambda(z), u) N(ds, dz, du) \quad (4.2)$$

We also denote by θ^λ the shift on Ω defined by $N \circ \theta^\lambda(\omega) = N^\lambda(\omega)$.

We now would like to find a probability measure P^λ , under which the law of N^λ is the same as that of N under P . We first introduce the following function

$$Y^\lambda(z) = (1 + \lambda\alpha'(z)) \times \frac{\varphi(\gamma^\lambda(z))}{\varphi(z)} \quad (4.3)$$

A simple computation shows that for all $\lambda \in [0, 1]$, $|Y^\lambda(z) - 1| \leq \lambda\xi(z)$, and thus in particular, $Y^\lambda - 1$ belongs to $L^1 \cap L^\infty(O, \varphi(z)dz)$.

Then, a simple substitution shows that $\gamma^\lambda(Y^\lambda.\nu) = \nu$, i.e. that for all Borel set $A \subset [0, T] \times O \times [0, M]$,

$$\begin{aligned} & \int_0^T \int_O \int_0^M 1_A(s, \gamma^\lambda(z), u) Y^\lambda(z) du \varphi(z) dz ds \\ &= \int_0^T \int_O \int_0^M 1_A(s, z', u) du \varphi(z') dz' ds \end{aligned} \quad (4.4)$$

We finally consider the following martingale

$$M_t^\lambda = \int_0^t \int_O \int_0^M (Y^\lambda(z) - 1) [N(ds, dz, du) - ds\varphi(z)dzdu] \quad (4.5)$$

and its Doléans-Dade exponential (see Jacod, Shiryaev, [11], p 59)

$$G_t^\lambda = 1 + \int_0^t G_{s-}^\lambda dM_s^\lambda = e^{M_t^\lambda} \prod_{0 \leq s \leq t} (1 + \Delta M_s^\lambda) e^{-\Delta M_s^\lambda} \quad (4.6)$$

which clearly is a square integrable martingale.

Notice that it is here the only place where we use our localization procedure. If ζ was not bounded, we would have to consider an integral on $[0, \infty[$ instead of $[0, M]$ in (4.5), and M^λ would not be well-defined.

Let us come back to our problem. Then, thanks to the Girsanov Theorem for random measures (see Jacod, Shiryaev, [11], p 157), the compensator of N under the probability measure $G_T^\lambda.P$ is given by $Y^\lambda.\nu$. Hence, the compensator of $N^\lambda = \gamma^\lambda(N)$ under $G_T^\lambda.P$ is $\gamma^\lambda(Y^\lambda.\nu) = \nu$. Thus, still under $G_T^\lambda.P$, N^λ is a random integer-valued measure on $[0, T] \times O \times [0, M]$ with deterministic compensator ν , thus is a Poisson measure, and finally has the same law than N under P .

In other words, we have proved that $(G_T^\lambda.P) \circ (\theta^\lambda)^{-1} = P$.

We finally have to apply this family of “absolutely continuous shifts” $\{\theta^\lambda\}_{\lambda \in [0,1]}$ to the solution of (1.1).

Proposition 4.2 *Assume (I) and (H'). Let α belong to \mathcal{D} . For $\lambda \in [0, 1]$, consider the shift θ^λ associated with α . Then, setting $X_t^\lambda = X_t \circ \theta^\lambda$, we deduce from (1.1) that X^λ satisfies the following stochastic differential equation on $[0, T]$:*

$$X_t^\lambda = x + \int_0^t \int_O \int_0^M \sigma(X_{s-}^\lambda) \eta(z + \lambda \alpha(z)) \mathbf{1}_{\{u \leq \zeta(X_{s-}^\lambda)\}} N(ds, dz, du) + \int_0^t b(X_{s-}^\lambda) ds \quad (4.7)$$

Then for G_T^λ defined by (4.6), the law of X^λ under $G_T^\lambda.P$ is the same as that of X under P . In particular, for any $t \in [0, T]$, the law of X_t is absolutely continuous with respect to that of X_t^λ .

Thus the first part of criterion 3.1 is satisfied, and it remains to prove that for a good choice of α , the map $\lambda \mapsto X_t^\lambda(\omega)$ is strongly increasing (for t and ω fixed).

The following remark shows the originality of this work, and why the new criterion of absolute continuity we have introduced is useful in our situation.

Remark 4.3 *Consider any direction $\alpha \in \mathcal{D}$ is any direction which does not vanish too much, in the sense that $\int_O \mathbf{1}_{\{\alpha(z) \neq 0, \eta'(z) \neq 0\}} \varphi(z) dz = \infty$ (all the directions built by Bichteler, Gravereaux, Jacod, [3], [2] satisfy this condition,*

which looks necessary to obtain a result). Assume also that ζ is not too much constant, suppose e.g. that it is C^1 with a strictly positive derivative.

Then it seems that the map $\lambda \mapsto X_t^\lambda$ is a.s. almost everywhere discontinuous (and a fortiori nowhere differentiable) on $[0, 1]$. This remark seems to hold even if we assume that σ , b , and ζ are smooth.

We are of course not able to give a rigorous proof of this remark, but let us explain the main intuition.

Denote by $\{(T_i, Z_i, U_i)\}_{i \geq 1}$ the points in the support of N , and let ω be fixed. We will assume that for some $t > 0$, the map $\lambda \mapsto X_t^\lambda$ is C^1 on a neighborhood $[0, \epsilon(\omega)]$ of 0, with a strictly positive derivative at 0 (which would be the case in traditional situations), and try to explain that for any $\delta > 0$, the map $\lambda \mapsto X_{t+\delta}^\lambda$ becomes a.e. discontinuous, at least for λ in a neighborhood of 0.

At the instant t , we thus know that there exists $\epsilon'(\omega) > 0$ and $a(\omega) > 0$ such that for all $\lambda \leq \mu$ in $[0, \epsilon'(\omega)]$, $X_t^\mu - X_t^\lambda \geq a(\mu - \lambda)$. But then, between t and $t + \delta$, for any $\lambda < \mu$ in $[0, \epsilon']$, there will be infinitely many jumps of N such that $U_i \leq \zeta(X_{T_i}^\mu)$ but $U_i > \zeta(X_{T_i}^\lambda)$. This means that X^μ will “accept” infinitely many jumps that X^λ will “reject”, which of course makes many discontinuities.

5 Choice of the direction and conclusion

We have to choose a direction $\alpha \in \mathcal{D}$ such that the map $\lambda \mapsto X_t^\lambda$ is a.s. increasing, for any $t > 0$ fixed. It is quite clear that this will hold, since $\sigma > 0$, and $\zeta > 0$ are increasing and since η is nonnegative, as soon as for each $z \in O$, the map $\lambda \mapsto \eta(z + \lambda\alpha(z))$ is increasing on $[0, 1]$. Thus the main idea is that $\alpha(z)$ has to be of the same sign than $\eta'(z)$: by this way, $\lambda \mapsto \eta(z + \lambda\alpha(z))$ will be increasing on a neighborhood $[0, \epsilon]$ of 0. To obtain the increasing property on the whole interval $[0, 1]$, it suffices to choose α small enough. Let us now become rigorous.

Definition 5.1 Assume (I), (H') and (AC). Consider any direction $\beta \in \mathcal{D}$ satisfying the additional condition that $0 < \beta \leq 1/4\|\eta''\|_\infty$ on O .

We consider the perturbation $\alpha \in \mathcal{D}$ defined on O by

$$\alpha(z) = \beta(z) \frac{\eta'(z)}{1 + [\eta'(z)]^2} \quad (5.1)$$

The fact that α belongs to \mathcal{D} is not hard to check: since η is C^2 , α is clearly C^1 . On the other hand, it is easily proved that $|\alpha(z)| \leq \beta(z)/2$, and that $|\alpha'(z)| \leq |\beta'(z)|/2 + \|\eta''\|_\infty \beta(z)$, which immediately yield that α satisfies the conditions of Definition 4.1.

Lemma 5.2 Assume (I), (H'), and (AC), and consider the direction α built in Definition 5.1. Then for all z in O , all $0 \leq \lambda < \lambda + \mu \leq 1$,

$$\eta[z + (\lambda + \mu)\alpha(z)] - \eta[z + \lambda\alpha(z)] \geq \frac{1}{2}\mu\beta(z) \times \frac{[\eta'(z)]^2}{1 + [\eta'(z)]^2} \quad (5.2)$$

We deduce in particular that the map $\lambda \mapsto \eta(z + \lambda\alpha(z))$ is increasing on $[0, 1]$ for each z in O .

Proof Let us compute, by using the Taylor-Lagrange Theorem. There exist y and \bar{y} in O such that:

$$\begin{aligned}
& \eta[z + (\lambda + \mu)\alpha(z)] - \eta[z + \lambda\alpha(z)] \\
&= \mu\alpha(z)\eta'[z + \lambda\alpha(z)] + \frac{1}{2}\mu^2\alpha^2(z)\eta''(y) \\
&= \mu\alpha(z)[\eta'(z) + \lambda\alpha(z)\eta''(\bar{y})] + \frac{1}{2}\mu^2\alpha^2(z)\eta''(y) \\
&= \mu\frac{[\eta'(z)]^2}{1 + [\eta'(z)]^2}\beta(z) \left[1 + \frac{\lambda\beta(z)\eta''(\bar{y})}{1 + [\eta'(z)]^2} + \frac{\mu\beta(z)\eta''(y)}{2(1 + [\eta'(z)]^2)} \right] \\
&\geq \mu\frac{[\eta'(z)]^2}{1 + [\eta'(z)]^2}\beta(z) \times \left[1 - \beta(z)\|\eta''\|_\infty - \frac{1}{2}\beta(z)\|\eta''\|_\infty \right] \\
&\geq \mu\frac{[\eta'(z)]^2}{1 + [\eta'(z)]^2}\beta(z) [1 - 1/4 - 1/8] \tag{5.3}
\end{aligned}$$

The lemma is proved. \square

As a corollary, we obtain that the map $\lambda \mapsto X_t^\lambda$ is a.s. strongly increasing.

Proposition 5.3 Assume (I), (H') and (AC), and consider the direction α built in Definition 5.1. Denote by b_{ip} the Lipschitz constant of b .

1. Almost surely, for all $t \in [0, T]$, all $0 \leq \lambda < \lambda + \mu \leq 1$

$$X_t^{\lambda+\mu} - X_t^\lambda \geq \frac{e^{-b_{ip}T}}{2}\mu Z_t \tag{5.4}$$

where

$$Z_t = \int_0^t \int_O \int_0^M \sigma(X_{s-}) \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} \beta(z) \frac{(\eta'(z))^2}{1 + (\eta'(z))^2} N(ds, dz, du) \tag{5.5}$$

2. As soon as $t > 0$, a.s., $Z_t > 0$.

Thus for any $t > 0$, the map $\lambda \mapsto X_t^\lambda$ is a.s. strongly increasing on $[0, 1]$.

Proof 1. Let $0 \leq \lambda < \lambda + \mu \leq 1$ be fixed. Our aim is to write a linear equation satisfied by the process $X_t^{\lambda+\mu} - X_t^\lambda$, and then to solve this equation in terms of positive exponentials. First notice that

$$X_t^{\lambda+\mu} - X_t^\lambda = \int_0^t (X_{s-}^{\lambda+\mu} - X_{s-}^\lambda) dY_s^{\lambda,\mu} + H_t^{\lambda,\mu} \tag{5.6}$$

where

$$\begin{aligned}
Y_t^{\lambda,\mu} &= \int_0^t \int_0^t \int_0^M \frac{\sigma(X_{s-}^{\lambda+\mu}) - \sigma(X_{s-}^\lambda)}{X_{s-}^{\lambda+\mu} - X_{s-}^\lambda} \mathbf{1}_{\{X_{s-}^{\lambda+\mu} \neq X_{s-}^\lambda\}} \eta(z + \lambda\alpha(z)) \\
&\quad \mathbf{1}_{\{u \leq \zeta(X_{s-}^\lambda)\}} N(ds, dz, du) \\
&\quad + \int_0^t \int_0^t \int_0^M \sigma(X_{s-}^{\lambda+\mu}) \frac{\mathbf{1}_{\{u \leq \zeta(X_{s-}^{\lambda+\mu})\}} - \mathbf{1}_{\{u \leq \zeta(X_{s-}^\lambda)\}}}{X_{s-}^{\lambda+\mu} - X_{s-}^\lambda} \mathbf{1}_{\{X_{s-}^{\lambda+\mu} \neq X_{s-}^\lambda\}} \\
&\quad \eta(z + \lambda\alpha(z)) N(ds, dz, du) \\
&\quad + \int_0^t \frac{b(X_{s-}^{\lambda+\mu}) - b(X_{s-}^\lambda)}{X_{s-}^{\lambda+\mu} - X_{s-}^\lambda} \mathbf{1}_{\{X_{s-}^{\lambda+\mu} \neq X_{s-}^\lambda\}} ds
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
H_t^{\lambda,\mu} &= \int_0^t \int_0^t \int_0^M \sigma(X_{s-}^{\lambda+\mu}) [\eta(z + (\lambda + \mu)\alpha(z)) - \eta(z + \lambda\alpha(z))] \\
&\quad \mathbf{1}_{\{u \leq \zeta(X_{s-}^{\lambda+\mu})\}} N(ds, dz, du)
\end{aligned} \tag{5.8}$$

Then, we consider the Doléans-Dade exponential (see Jacod, Shiryaev, [11])

$$\begin{aligned}
\mathcal{E}(Y^{\lambda,\mu})_t &= e^{Y_t^{\lambda,\mu}} \prod_{s \leq t} (1 + \Delta Y_s^{\lambda,\mu}) e^{-\Delta Y_s^{\lambda,\mu}} \\
&= \exp \left(\int_0^t \frac{b(X_{s-}^{\lambda+\mu}) - b(X_{s-}^\lambda)}{X_{s-}^{\lambda+\mu} - X_{s-}^\lambda} \mathbf{1}_{\{X_{s-}^{\lambda+\mu} \neq X_{s-}^\lambda\}} ds \right) \\
&\quad \times \prod_{s \leq t} (1 + \Delta Y_s^{\lambda,\mu})
\end{aligned} \tag{5.9}$$

This Doléans-Dade exponential is always strictly positive, since it is clear from the facts that σ and ζ are increasing and that η is nonnegative that all the jumps of $Y^{\lambda,\mu}$ are nonnegative, and since b is globally Lipschitz continuous. Thus, using the work of Jacod, [10], we know that

$$X_t^{\lambda+\mu} - X_t^\lambda = \mathcal{E}(Y^{\lambda,\mu})_t \int_0^t \mathcal{E}(Y^{\lambda,\mu})_{s-}^{-1} (1 + \Delta Y_s^{\lambda,\mu})^{-1} dH_s^{\lambda,\mu} \tag{5.10}$$

One easily concludes that $X_t^{\lambda+\mu} - X_t^\lambda$ is a nonnegative process, using the fact that $H^{\lambda,\mu}$ is an increasing process (since σ is nonnegative and thanks to Lemma 5.2). We thus in particular deduce that $\lambda \mapsto X_t^\lambda$ is increasing for all $t \in [0, T]$. We still have to prove (5.4). For all $0 \leq s \leq t \leq T$, we easily deduce from (5.9)

that

$$\begin{aligned} & \mathcal{E}(Y^{\lambda,\mu})_t \mathcal{E}(Y^{\lambda,\mu})_{s-}^{-1} (1 + \Delta Y_s^{\lambda,\mu})^{-1} \\ & \geq \exp(-Tb_{lip}) \times \prod_{s \leq u \leq t} (1 + \Delta Y_u^{\lambda,\mu}) \geq \exp(-Tb_{lip}) \end{aligned} \quad (5.11)$$

since the jumps of $Y^{\lambda,\mu}$ are always nonnegative. We thus obtain from (5.10), since $H^{\lambda,\mu}$ is an increasing process, that

$$X_t^{\lambda+\mu} - X_t^\lambda \geq \exp(-Tb_{lip}) H_t^{\lambda,\mu} \quad (5.12)$$

But, since we have already seen that $\lambda \mapsto X_t^\lambda$ is increasing, we know that $X_{s-}^{\lambda+\mu} \geq X_{s-}^0 = X_{s-}$ for all s . Using furthermore the fact that σ and ζ are increasing and Lemma 5.2, it is immediately seen that

$$H_t^{\lambda,\mu} \geq \frac{\mu}{2} Z_t \quad (5.13)$$

Associating (5.12) and (5.13) concludes the proof of 1.

2. Since σ and β are always strictly positive, it suffices to check that a.s.,

$$U_t = \int_0^t \int_O \int_0^M \mathbf{1}_{\{\eta'(z) \neq 0\}} \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} N(ds, dz, du) > 0 \quad (5.14)$$

for any $t > 0$. It thus suffices to prove that $\tau = 0$ a.s., where τ is the stopping time defined by

$$\tau = \inf \{s \geq 0 ; U_s > 0\} \quad (5.15)$$

By definition of U , it is clear that $U_\tau \leq 1$, which implies that $E[U_\tau] \leq 1$. On the other hand, one may compute the expectation of U_τ , this gives

$$\begin{aligned} E(U_\tau) &= E \left[\int_0^\tau \int_O \int_0^M \mathbf{1}_{\{\eta'(z) \neq 0\}} \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} du \varphi(z) dz ds \right] \\ &= E \left[\int_0^\tau \zeta(X_{s-}) ds \int_O \mathbf{1}_{\{\eta'(z) \neq 0\}} \varphi(z) dz \right] \end{aligned} \quad (5.16)$$

which is infinite thanks to (AC), except if $\tau = 0$ a.s. The proof is complete. \square

We are now able to conclude.

Proof of Theorem 1.2 Thanks to Proposition 2.2, we may assume (I), (H') and (AC). Let $t \in]0, T]$ be fixed. We have to apply Theorem 3.1 with $Y = X_t$. The family Y^λ is defined by $X_t^\lambda = X_t \circ \theta^\lambda$, the shift θ^λ being defined as in Section 4 relatively to the direction α introduced in Definition 5.1. Condition 1. of Theorem 3.1 is satisfied thanks to Proposition 4.2. Finally, condition 2. holds thanks to Proposition 5.3. Hence the law of X_t admits a density, which was our aim. \square

6 Extension

We finally would like to give a proof of Theorem 1.3, which relies on the use of the Girsanov Theorem for random measures.

Proof of Theorem 1.3 First of all, we consider a Poisson measure $N(ds, dz, du)$ satisfying assumption (I), and the solution $\{X_t\}_{t \in [0, T]}$ to (1.1). Since (I), (H) and (AC) hold, we know from Theorem 1.2 that for each $t > 0$, the law of X_t has a density. Our aim is to check that there exists a nonnegative square integrable exponential martingale $\{D_t\}_{t \in [0, T]}$ such that under the probability measure $D_T.P$, $\{X_t\}_{t \in [0, T]}$ is a Markov process with generator \mathcal{L} (defined in (1.7)). The existence of a density for the law of X_t under $D_T.P$ will of course be straightforward.

Let us now build D_T . We first consider the martingale

$$C_t = \int_0^t \int_O \int_0^\infty \gamma(X_{s-}, z) \mathbf{1}_{\{u \leq \zeta(X_{s-})\}} [N(ds, dz, du) - du\varphi(z)dzds] \quad (6.1)$$

Then we denote by $D = \mathcal{E}(C)$ the Doléans-Dade exponential of C , see Jacod, Shiryaev, [11], p 59. Since γ is greater than -1 , D takes its values in $[0, \infty[$, and (1.6) ensures that D is square integrable, and in particular that $E(D_T) = 1$.

We denote by $P_D = D_T.P$, and by E_D the corresponding expectation.

The Girsanov Theorem for random measures (see Jacod, Shiryaev, [11], p 157) says that under P_D , the integer-valued random measure N has its compensator given by

$$[1 + \gamma(X_{s-}, z) \mathbf{1}_{\{u \leq \zeta(X_{s-})\}}] du\varphi(z)dzds \quad (6.2)$$

Under P_D , $\{X_t\}_{t \in [0, T]}$ is thus still a Markov process, and for any $\phi \in C^1(\mathbb{R})$,

$$\begin{aligned} E_D(\phi(X_t)) &= \phi(x) + E_D \left[\int_0^t b(X_s) \phi'(X_s) ds \right] \\ &+ E_D \left[\int_0^t \int_O \int_0^\infty \{ \phi(X_s + \sigma(X_s)\eta(z)) \mathbf{1}_{\{u \leq \zeta(X_s)\}} - \phi(X_s) \} \right. \\ &\quad \left. [1 + \gamma(X_s, z) \mathbf{1}_{\{u \leq \zeta(X_s)\}}] du\varphi(z)dzds \right] \\ &= \phi(x) + \int_0^t E_D \left[b(X_s) \phi'(X_s) \right. \\ &\quad \left. + \int_O \{ \phi(X_s + \sigma(X_s)\eta(z)) - \phi(X_s) \} [1 + \gamma(X_{s-}, z)] \zeta(X_s) \varphi(z) dz \right] ds \end{aligned} \quad (6.3)$$

from which it is not hard to deduce that under P_D , X has the generator \mathcal{L} defined by (1.7). This concludes the proof. \square

References

- [1] G. Ben Arous, R. Léandre, *Décroissance exponentielle du noyau de la chaleur sur la diagonale (II)*, Probability Theory and Related Fields 90, p 377-402, 1991.
- [2] K. Bichteler, J.B. Gravereaux, J. Jacod, *Malliavin calculus for processes with jumps*, Number 2 in Stochastic monographs, Gordon and Breach, 1987.
- [3] K. Bichteler, J. Jacod, *Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité dans le cas unidimensionnel*, Séminaire de Probabilités XVII, L.N.M. 986, p 132-157, Springer, 1983.
- [4] J.M. Bismut, *Calcul des variations stochastiques et processus de sauts*, Z.W. 63, 147-235, 1983.
- [5] E. Carlen, E. Pardoux, *Differential calculus and integration by parts on Poisson space*, in Kluwer ed, Stochastic algebra and analysis in classical and quantum dynamics, p63-73, 1990.
- [6] M. Deaconu, N. Fournier, E. Tanré, *A pure jump Markov process associated with Smoluchowski's coagulation equation*, Prépublication no 2001/6 de l'IECN, 2001.
- [7] L. Denis, *A criterion of density for solutions of Poisson-driven SDEs*, Probability Theory and Related Fields 118, no. 3, 406-426, 2000.
- [8] N. Fournier, S. Méléard, *A Markov process associated with a Boltzmann equation without cutoff and for non Maxwell molecules*, to appear in Journal of Statistical Physics.
- [9] N. Fournier, S. Méléard, *Existence results for 2D homogeneous Boltzmann equations without cutoff and for non Maxwell molecules by use of Malliavin calculus*, Prépublication no 622 du Laboratoire PMA de Paris 6, 2000.
- [10] J. Jacod, *Equations différentielles linéaires, la méthode de variation des constantes*, Séminaire de Probabilités XVI, L.N.M. 920, p 442-448, Springer Verlag, 1982.
- [11] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, 1987.
- [12] D. Nualart, *Malliavin Calculus and related topics*, Springer, 1995.
- [13] J. Picard, *On the existence of smooth densities for jump processes*, Probability Theory and Related Fields 105, no. 4, 481-511, 1996.