On long time behavior of some coagulation processes

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Abstract

We consider an infinite system of particles characterized by their position and mass, in which coalescence occurs. Each particle endures Brownian excitation, and is subjected to the attraction of a potential. We define a stochastic process $(X_t, M_t)_{t\geq 0}$ describing the evolution of the position and mass of a typical particle. We show that under some conditions, the mass process M_t goes almost surely to infinity, while the position process X_t tends almost surely to 0, as time tends to infinity.

Key words : Nonlinear stochastic differential equations with jumps, Coalescence.

MSC 2000 : 60H10, 60K40.

1 Introduction

We consider an infinite system of particles characterized by their position $x \in \mathbb{R}^d$ and their mass $m \in \mathbb{N}^*$. The *mass* of a particle stands for the number of *elementary particles* it contains. Each particle (of size m) endures three phenomena. First, its spatial motion is under the influence of a Brownian excitation, with a coefficient $\alpha(m)$. We naturally assume that the more a particle is large, the more its motion is regular, that is, α is a decreasing function of m. Next, it endures the effect of a potential: each particle is attracted by the origin 0, by a force proportional to its mass so that the speed attraction is independent of the mass. Finally, each particle coalesces with other particles. To be more precise, we will assume that two particles (of masses i and j) of which the locations are at distance smaller than ε coalesce at rate $\varepsilon^{-d}K(i,j)$, to give one particle of mass $i+j$. The delocalisation parameter $\varepsilon > 0$ is fixed, while the *coagulation kernel* K is given.

One may write down an integro-differential equation satisfied by the concentration $f(t,x,i)$ of particles of size i and position x at the instant t (see Laurençot-Mischler [4] for a special case of diffusions). We however adopt, in the present

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paper, a probabilistic approach in the spirit of Tanaka [6] (for the Boltzmann equation), see also Deaconu-Fournier [1] for a coagulation equation. We consider a stochastic process $(X_t, M_t)_{t\geq 0}$, describing the evolution of the location and mass of a *typical particle*. In other words, we follow the location and mass of the particle containing a randomly chosen (at $t = 0$) *elementary particle*. There is an explicit link between $(X_t, M_t)_{t\geq 0}$ and $f(t, x, i)$: the law of (X_t, M_t) is given, for each t, by $Q_t(dx, dm) = \sum_{i\geq 1} if(t, x, i)\delta_i(dm)dx$ (at least formally).

Our aim here is to show that under suitable conditions, almost surely, (i) $\lim_{t\to\infty} M_t = \infty$, and (ii) $\lim_{t\to\infty} X_t = 0$. While point (i) is quite straightforward, point (ii) is much more difficult. Indeed, we have to use that M tends sufficiently quickly to infinity, so that the "noise" coefficient $\alpha(M_t)$ tends sufficiently fast to 0. Thus in large time, the position process X_t behaves as the solution to an ordinary differential equation which is attracted by the origin. It would of course be more interesting to treat the *local* case $\varepsilon = 0$ (see Section 4), but this would be much more difficult.

The paper is organized as follows: we give our notations, definitions and results in Section 2. Then we write the proofs in several steps in Section 3. Finally, we deal with related problems in Section 4.

2 Notations and results

Let us first of all describe the parameters of the equation we will study.

Assumption $(H1)$:

1. The dimension $d \in \mathbb{N}^*$ satisfies $d \geq 2$.
2. The delocalisation parameter is $\varepsilon > 0$.
3. The initial condition Q_0 is a probability measure on $\mathbb{R}^d \times \mathbb{N}^*$ satisfying
$$\int_{\mathbb{R}^d\times\mathbb{N}^*} (|x|^2 + m)Q_0(dx, dm) < \infty.$$
4. The excitation coefficient $\alpha : \mathbb{N}^* \mapsto (0, \infty)$ is non-increasing, (for the simplicity of proofs, we will assume that $\alpha(1) = 1$) and $\lim_{n\to\infty} \alpha(n) = 0$.
5. The coagulation kernel $K : \mathbb{N}^* \times \mathbb{N}^* \mapsto [0, \infty)$ satisfies, for some constant A_0, for all i, j in $\mathbb{N}^*$, $K(i, j) = K(j, i) \leq A_0(i + j)$.
6. The function $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is of class C^2, bounded from above and from below, and its derivative C' is bounded. Finally, $C'(0) = 0$.
7. The function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ is of class C^2, its derivative u' is nonnegative, $u'(0) = 0$.
8. There exists $\eta_0 > 0$ such that for all $x \in \mathbb{R}_+$, $u'(x) \geq \eta_0 x$.
9. For all $i \in \mathbb{N}^*$, $K(i, i) > 0$.

As a simple example, one may note that $C(x) = 1$, $u(x) = x^2$, and $K(i, j) = i+j$ fulfill assumption $(H1)$.
Let us now define the random objects that will drive the equation.

Notation 2.1 *We consider two probability spaces:* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *is an abstract space and* $([0,1], \mathcal{B}[0,1], d\gamma)$ *is an auxiliary space (here* $d\gamma$ *denotes the Lebesgue measure). In order to avoid confusion, the elements on this second space will be called* γ*-elements, and the expectation on* $[0,1]$ *will be denoted* $\mathbb{E}_\gamma$*, the laws* $\mathcal{L}_\gamma$*,...*
We denote by v_d *the volume of the d-dimensional unit ball.*
We will denote by (X_0, M_0) *a* $\mathcal{F}_0$*-measurable* $\mathbb{R}^d \times \mathbb{N}^*$*-valued random variable on* Ω *with law* Q_0*. We also consider a d-dimensional* $(\mathcal{F}_t)_{t\geq 0}$*-adapted Brownian motion* $(B_t)_{t\geq 0} = (B_t^1, ..., B_t^d)_{t\geq 0}$ *on* Ω*. We finally denote by* $N^\varepsilon(dt, d\gamma, du)$ *a* $(\mathcal{F}_t)_{t\geq 0}$*-adapted Poisson measure on* $[0,\infty) \times [0,1] \times [0,\infty)$ *with intensity measure* $(1/v_d \varepsilon^d) dt d\gamma du$*.*
The random objects (X_0, M_0)*,* B*, and* N^ε *are independent.*

Let us finally introduce the dynamics we are interested in.

Definition 2.2 *A stochastic process* $(X_t, M_t)_{t\geq 0}$ *is said to solve* (SDE) *if the following conditions hold.*

(i) X *is a continuous* $(\mathcal{F}_t)_{t\geq 0}$*-adapted* $\mathbb{R}^d$*-valued process, and* M *is a nondecreasing càdlàg* $(\mathcal{F}_t)_{t\geq 0}$*-adapted* $\mathbb{N}^*$*-valued process.*

(ii) For all $T > 0$*,* $\mathbb{E}\left[\sup_{[0,T]}(|X_t|^2 + M_t)\right] < \infty$*.*

(iii) There exist a γ*-process* $(\tilde{X}_t, \tilde{M}_t)_{t\geq 0}$ *such that*

$$\mathcal{L}_\gamma(\tilde{X}, \tilde{M}) = \mathcal{L}(X, M) \tag{2.1}$$

and some random objects (X_0, M_0)*,* B*, and* N *as in Notation 2.1 such that for all* $t \geq 0$*, a.s.,*

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(\alpha\left(M_s\right) C\left(|X_s|\right)\right)^{\frac{1}{2}} dB_s \\ &+ \frac{1}{2}\int_0^t \alpha\left(M_s\right)\left(1 - \frac{d}{2}\right)\frac{C'\left(|X_s|\right)}{|X_s|} X_s ds - \frac{1}{2}\int_0^t C\left(|X_s|\right) u'\left(|X_s|\right)\frac{X_s}{|X_s|} ds \\ M_t &= M_0 + \int_0^t\int_0^1\int_0^\infty \tilde{M}_{s-}(\gamma)\mathbb{1}_{\left\{u \leq \frac{K\left(M_{s-}, \tilde{M}_{s-}(\gamma)\right)}{\tilde{M}_{s-}(\gamma)}\right\}}\mathbb{1}_{\left\{|X_s - \tilde{X}_s(\gamma)| < \varepsilon\right\}} \\ &\qquad N^\varepsilon(ds, d\gamma, du) \end{aligned} \tag{2.2}$$

Recalling that (X, M) should describe the evolution of the location and mass of a particle containing a typical *elementary particle*, this equation is quite natural. Let us explain it right now.
We assume that each particle, with a given mass m, independently of the others, and without the effect of the potential, would move according to a reversible

diffusion, with a noise coefficient $\alpha(m)$. This is the *Brownian excitation*. In such a case, its location X_t would satisfy the following S.D.E.

$$dX_t = \sqrt{\alpha(m)C\left(|X_s|\right)}dB_s + \frac{\alpha(m)}{2}\left(1-\frac{d}{2}\right)\frac{C'\left(|X_s|\right)}{|X_s|}X_s ds \qquad (2.3)$$

Note that the generator of such a diffusion is simply the Laplace-Beltrami operator corresponding to the (radial) Riemanian metric $g_{ij} = C(|x|)\delta_{ij}$.
Next, our particle is subjected to the effect of a (radial) potential, at a force proportional to its mass, so that the speed is independant of the mass. This explains why we add the term

$$-\frac{1}{2}C\left(|X_s|\right)u'\left(|X_s|\right)\frac{X_s}{|X_s|}ds \qquad (2.4)$$

in the S.D.E. satisfied by X_t. The fact that this term does not depend of the mass is natural, since for instance, we know that particles do fall at a speed independent of their mass in a Newtonian potential.
Now, the masses of particles being non constant in our model, we obtain that the location $\{X_t\}_{t\geq 0}$ of our typical particle, whose mass is $\{M_t\}_{t\geq 0}$, satisfies the first equation in (2.2).
Finally, the second equation in (2.2) explains that our typical particle does coalesce with other (typical) particles: we add $\tilde{M}_{s-}(\gamma)$ to the mass of our particle (that is, we set $M_s = M_{s-} + \tilde{M}_{s-}(\gamma)$), with rate $\varepsilon^{-d}K(M_{s-}, \tilde{M}_{s-}(\gamma))/\tilde{M}_{s-}(\gamma)$ and under the condition that $|\tilde{X}_s(\gamma) - X_s| \leq \varepsilon$. Here $(\tilde{X}_s(\gamma), \tilde{M}_s(\gamma))$ may be seen as the characteristics (location, mass) of another typical particle at the instant s, since we require that $\mathcal{L}_\gamma(\tilde{X}, \tilde{M}) = \mathcal{L}(X, M)$ in (2.1).
The rate $K(i,j)/j$ (instead of $K(i,j)$) comes from the fact that we deal with *particles containing a given elementary particle*, so that each particle (of size j) is represented j times: we have to divide the rate.

For a precise link between $(X_t, M_t)_{t\geq 0}$ and integro-differential equations, see [2], [1] (for slightly different contexts).

Proposition 2.3 *Assume* $(H1)$. *Then there exists a solution* $(X_t, M_t)_{t\geq 0}$ *to* (SDE).

We omit the proof of this proposition. Indeed, using the strong existence and uniqueness of the S.D.E. satisfied by X if α was constant (see Equation (3.7) below), one may adapt easily the method of Fournier-Giet [3]. Although the equations in [3] are spatially homogeneous, the delocalization allows such an extension. Let us now state a first result.

Theorem 2.4 *Assume* $(H1)$, *and consider a solution* $(X_t, M_t)_{t\geq 0}$ *to* (SDE). *Then a.s.,* $\lim_{t\to\infty} M_t = \infty$ *while* $\lim_{t\to\infty} \mathbb{E}[|X_t|^2] = 0$.

We are not able to prove, under $(H1)$, that X_t tends a.s. to 0, and this might be false. Indeed, if α does not tend sufficiently quickly to 0, the process X might be recurrent. We thus add an hypothesis.

Assumption $(H2)$: There exist some constant $c_0 < \infty$ and $\beta_0 \in (3/4, 1)$ such that for all $m \in \mathbb{N}^*$, $\alpha(m) \leq \frac{c_0}{m^{\beta_0}}$. There exists $a_0 > 0$ such that for any $i, j \in \mathbb{N}^*$, $K(i,j)/j \geq a_0$.

Then the following result holds.

Theorem 2.5 *Assume $(H1)$, $(H2)$, and consider a solution $(X_t, M_t)_{t\geq 0}$ to (SDE). Then a.s., $\lim_{t\to\infty} X_t = 0$.*

Before proving these statements, let us explain the main intuition of these convergence results.
First, M tends to infinity, since if not, then M stops to increase, so that the noise coefficient of X becomes constant. In such a case, X will be recurrent. This means that for some i, the concentration of particles of mass i, of which the location is smaller than $\varepsilon/2$, will be bounded below by some $c > 0$ (independently of t sufficiently large). This implies, since $K(i,i) > 0$, that each time X is sufficiently close to 0 (which happens infinitely often since X is recurrent), its coalescence rate is bounded below. Hence M can not stop to increase.
Next X tends to 0, because since $\alpha(M)$ tends to 0, one may hope that for t sufficiently large, X behaves as the (deterministic) solution y of $y'(t) = -\frac{1}{2}C(|y(t)|)u'(|y(t)|)y(t)/|y(t)|$, which tends to 0 as time tends to infinity.

3 Proofs

We break the proofs in several steps. In Subsection 3.1, we study the properties of the motion of particles with constant mass. Subsection 3.2 is devoted to the divergence of the mass process. We next show in Subsection 3.3 that the position process tends to 0 in $L^2(\Omega)$. We introduce a change of time in Subsection 3.4, which will allow to use comparison theorems and to conclude the proof of Theorem 2.5 in Subsection 3.5.

3.1 When α is constant

Notation 3.1 *We first of all introduce, for each $\alpha \in (0,1]$, the operator Γ_α, defined, for all $f : \mathbb{R}^d \mapsto \mathbb{R}$ of class C^2 by*

$$\begin{aligned}\Gamma_\alpha f(x) &= \frac{\alpha}{2}\left[C(|x|)\Delta f(x) + \left(1-\frac{d}{2}\right)\frac{C'(|x|)}{|x|}x.\nabla f(x)\right] \\ &\quad -\frac{1}{2}C(|x|)\frac{u'(|x|)}{|x|}x.\nabla f(x)\end{aligned} \tag{3.5}$$

We also denote by $k_\alpha = [\int_{\mathbb{R}^d} e^{-u(|x|)/\alpha}C(|x|)^{-d/2}dx]^{-1}$, which is positive thanks to $(H1)$ (6 and 7) and by

$$\nu_\alpha(dx) = k_\alpha e^{-u(|x|)/\alpha}C(|x|)^{-d/2}dx \tag{3.6}$$

which is a probability measure on $\mathbb{R}^d$.

We also will consider the motion that one particle would follow independently of the others, if it had a constant mass.

Definition 3.2 *Assume* $(H1)$. *Let* $\alpha \in (0,1]$ *and* $x \in \mathbb{R}^d$ *be fixed. We will denote by* $X^{\alpha,x} = (X_t^{\alpha,x})_{t\geq 0}$ *the unique strong solution of the following S.D.E.:*

$$X_t^{\alpha,x} = x + \int_0^t \left(\alpha C\left(|X_s^{\alpha,x}|\right)\right)^{\frac{1}{2}} dB_s + \frac{\alpha}{2}(1-d/2)\int_0^t \frac{C'\left(|X_s^{\alpha,x}|\right)}{|X_s^{\alpha,x}|} X_s^{\alpha,x}\, ds$$
$$-\frac{1}{2}\int_0^t C\left(|X_s^{\alpha,x}|\right) u'\left(|X_s^{\alpha,x}|\right) \frac{X_s^{\alpha,x}}{|X_s^{\alpha,x}|}\, ds \qquad (3.7)$$

Under $(H1)$, strong existence and uniqueness of such a process is well-known: all the coefficients are locally lipschitz continuous, the diffusion coefficient is bounded, while the drift coefficient is *attracting* to 0 (see $(H1)$-6,7). Then the following proposition is classical.

Proposition 3.3 *Assume* $(H1)$. *Let* $\alpha \in (0,1]$ *and* $x \in \mathbb{R}^d$ *be fixed.*
(i) The generator of the diffusion process $X^{\alpha,x}$ *is* Γ_α.
(ii) The operator Γ_α *is symmetric with respect to* $\nu_\alpha(dx)$. *More precisely, for any* f *and* g *in* $C_b^2(\mathbb{R}^d)$.

$$\int_{\mathbb{R}^d} g(x)\Gamma_\alpha f(x)\nu_\alpha(dx) = -\frac{\alpha}{2}\int_{\mathbb{R}^d} C(|x|)[\nabla f(x).\nabla g(x)]\nu_\alpha(dx) \qquad (3.8)$$

(iii) An ergodic theorem holds for $X^{\alpha,x}$, *i.e. for all Borel subset* Λ *of* $\mathbb{R}^d$, *a.s.,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{1}_{\{X_s^{\alpha,x}\in\Lambda\}} ds = \nu_\alpha(\Lambda) \qquad (3.9)$$

3.2 Almost sure divergence of the mass process

Lemma 3.4 *Assume* $(H1)$. *Consider a solution* $(X_t, M_t)_{t\geq 0}$ *to* (SDE). *Almost surely, as* t *tends to infinity,* M_t *tends to infinity.*

Proof First recall that M is $\mathbb{N}^*$-valued, so that it is in particular always greater than 1, and all its jumps are greater that 1. Furthermore, M is a.s. non-decreasing, so that it a.s. has a finite or infinite limit M_∞. Thus,

$$\{M_\infty < \infty\} = \bigcup_{t\in\mathbb{N}^*}\bigcup_{p\in\mathbb{N}^*} \Gamma_{t,p} \quad \text{where} \quad \Gamma_{t,p} = \{\forall s \geq t, \quad M_s = p\} \qquad (3.10)$$

We have to prove that for each t, p, $\mathbb{P}[\Gamma_{t,p}] = 0$. We assume the contrary, that is there exists t_0, p_0 such that $\mathbb{P}[\Gamma_{t_0,p_0}] > 0$.
Denote by $\tilde{\Gamma}_{t_0,p_0} = \{\gamma \in [0,1]\ ;\ \forall s \geq t_0, \quad \tilde{M}_s = p_0\}$. Then, thanks to (2.1), $\int_0^1 \mathbb{1}_{\tilde{\Gamma}_{t_0,p_0}}(\gamma)d\gamma = \mathbb{P}[\Gamma_{t_0,p_0}] > 0$.
Then we note that on Γ_{t_0,p_0}, X_t coincides for all $t \geq t_0$ with $X_{t-t_0}^{\alpha(p_0),X_{t_0}}$, where

$X^{\alpha,x}$ is defined by (3.7) with the Brownian motion $B_{t_0+\cdot} - B_{t_0}$. Note also that $X_{t-t_0}^{\alpha(p_0),X_{t_0}}$ is independent of $N^{\varepsilon,t_0} = N^{\varepsilon}|_{[t_0,\infty)\times[0,1]\times[0,\infty)}$. For the same reasons, we may write, for $\gamma \in \tilde{\Gamma}_{t_0,p_0}$, $\tilde{X}_t(\gamma)$ as $\tilde{X}_{t-t_0}^{\alpha(p_0),\bar{X}_{t_0}}$. Using (2.2), we deduce that on Γ_{t_0,p_0}, for all $t \geq t_0$,

$$\begin{aligned}
M_t &= p_0 + \int_{t_0}^t \int_0^1 \int_0^\infty \tilde{M}_{s-}(\gamma) \mathbb{1}_{\left\{u \leq \frac{K(M_{s-},\tilde{M}_{s-}(\gamma))}{\tilde{M}_{s-}(\gamma)}\right\}} \\
&\qquad \times \mathbb{1}_{\{|X_s - \bar{X}_s(\gamma)| < \varepsilon\}} N^{\varepsilon}(ds, d\gamma, du) \\
&\geq \int_{t_0}^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{|X_s| < \frac{\varepsilon}{2}\right\}} \mathbb{1}_{\left\{|\bar{X}_s(\gamma)| < \frac{\varepsilon}{2}\right\}} \\
&\qquad \times \mathbb{1}_{\{\gamma \in \bar{\Gamma}_{t_0,p_0}\}} \mathbb{1}_{\left\{u \leq \frac{K(p_0,\tilde{M}_{s-}(\gamma))}{\tilde{M}_{s-}(\gamma)}\right\}} N^{\varepsilon}(ds, d\gamma, du) \\
&= \int_{t_0}^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{|X_{s-t_0}^{\alpha(p_0),X_{t_0}}| < \frac{\varepsilon}{2}\right\}} \mathbb{1}_{\left\{|\tilde{X}_{s-t_0}^{\alpha(p_0),\bar{X}_{t_0}}(\gamma)| < \frac{\varepsilon}{2}\right\}} \\
&\qquad \times \mathbb{1}_{\{\gamma \in \bar{\Gamma}_{t_0,p_0}\}} \mathbb{1}_{\left\{u \leq \frac{K(p_0,p_0)}{p_0}\right\}} N^{\varepsilon}(ds, d\gamma, du) \qquad (3.11)
\end{aligned}$$

Hence, still on Γ_{t_0,p_0}, $M_\infty \geq N^{\varepsilon,t_0}(E)$, where

$$\begin{aligned}
E = \Big\{ &(s,\gamma,u),\ s \geq t_0,\ u \leq K(p_0,p_0)/p_0, \gamma \in \tilde{\Gamma}_{t_0,p_0}, \\
&|X_{s-t_0}^{\alpha(p_0),X_{t_0}}| < \frac{\varepsilon}{2}, |\tilde{X}_{s-t_0}^{\alpha(p_0),\bar{X}_{t_0}}(\gamma)| < \frac{\varepsilon}{2} \Big\} \qquad (3.12)
\end{aligned}$$

One easily deduces from Proposition 3.3 (iii) (since $\nu_{\alpha(p_0)}(|x| < \varepsilon/2) > 0$), $(H1)$-9 and the fact that X and $\tilde{X}$ are independent (since they are defined on different probability spaces) that if $\mathbb{P}_\gamma[\tilde{\Gamma}_{t_0,p_0}] > 0$, then almost surely,

$$\int_{t_0}^\infty ds \int_0^1 d\gamma \int_0^\infty du \mathbb{1}_{\{(s,\gamma,u)\in E\}} = \infty \qquad (3.13)$$

We also know that E is independent of N^{ε,t_0} (since $X_{t-t_0}^{\alpha(p_0),X_{t_0}}$ is independent of N^{ε,t_0}). Recalling that $N^{\varepsilon,t_0}(ds,d\gamma,du)$ is a Poisson measure with intensity measure $(v_d^{-1}\varepsilon^{-d})dsd\gamma du$, we deduce that almost surely, $N^{\varepsilon,t_0}(E) = \infty$. We obtain that on Γ_{t_0,p_0}, $M_\infty = \infty$, which is a contradiction. This concludes the proof. □

3.3 Convergence in L^2 of the position process

We now prove the second part of Theorem 2.4.

Lemma 3.5 *Assume* $(H1)$. *Consider a solution* $(X_t, M_t)_{t\geq 0}$ *to* (SDE). *Then* $\lim_{t\to\infty} \mathbb{E}[|X_t|^2] = 0$.

Proof First note that thanks to $(H1)$ (6 and 8), there exists $b > 0$ such that for all $r \in \mathbb{R}_+$, $C(r)u'(r) \geq 2br$. Next denote by Y the unique strong solution of (here the mass process is considered as a parameter)

$$Y_t = X_0 + \int_0^t \left(\alpha\left(M_s\right) C\left(|Y_s|\right)\right)^{\frac{1}{2}} dB_s$$
$$+\frac{1}{2}\int_0^t \alpha\left(M_s\right)\left(1-\frac{d}{2}\right)\frac{C'\left(|Y_s|\right)}{|Y_s|}Y_s ds - b\int_0^t Y_s ds \tag{3.14}$$

In other words, Y satisfies the same equation as X replacing the drift term $-C(X_s)u'(X_s)/2$ by $-bY_s$. Writing the SDEs satisfied by $|X_t|^2$ and $|Y_t|^2$, and using the standard comparison theorem (see Revuz-Yor, [5]), one deduces that a.s., for all $t \geq 0$, $|X_t|^2 \leq |Y_t|^2$.
We thus just have to check that $\lim_{t\to\infty} \mathbb{E}[|Y_t|^2] = 0$. But Y can be written as

$$Y_t = e^{-bt}\Big[X_0 + \int_0^t e^{bs}\left(\alpha\left(M_s\right) C\left(|Y_s|\right)\right)^{\frac{1}{2}} dB_s$$
$$+\tfrac{1}{2}\int_0^t e^{bs}\alpha\left(M_s\right)\left(1-\frac{d}{2}\right)\frac{C'\left(|Y_s|\right)}{|Y_s|}Y_s ds\Big] \tag{3.15}$$

We obtained this formula using the method of "variation of constants". One might however check directly that the process defined by (3.15) satisfies (3.14) and use a uniqueness argument.
Using finally the facts that C and C' are bounded (see $(H1)$), that α is smaller than 1, and that, thanks to the Lebesgue Theorem and Lemma 3.4, $\mathbb{E}[\alpha(M_t)]$ and $\mathbb{E}[\alpha^2(M_t)]$ decrease to 0 as t tends to infinity leads to the conclusion: for some constant A whose values changes from line to line,

$$\begin{aligned}
\mathbb{E}\left[|Y_t|^2\right] &\leq Ae^{-2bt}\left[1+\int_0^t e^{2bs}\mathbb{E}[\alpha(M_s)]ds + \mathbb{E}\left(\left[\int_0^t e^{bs}\alpha(M_s)ds\right]^2\right)\right] \\
&\leq Ae^{-2bt}\left[1+\int_0^{t/2} e^{2bs}ds + \left(\int_0^{t/2} e^{bs}ds\right)^2\right] \\
&\quad + Ae^{-2bt}\mathbb{E}[\alpha(M_{t/2})+\alpha^2(M_{t/2})]\left[\int_{t/2}^t e^{2bs}ds + \left(\int_{t/2}^t e^{bs}ds\right)^2\right] \\
&\leq A\left(e^{-2bt}+e^{-bt}+\mathbb{E}[\alpha(M_{t/2})+\alpha^2(M_{t/2})]\right) \qquad (3.16)
\end{aligned}$$

which tends to 0 as t tends to infinity. □

As an immediate corollary, we deduce that

Corollary 3.6 *Assume* $(H1)$. *Then a.s.,* $\liminf_{t\to\infty} |X_t| = 0$.

3.4 A change of time

A difficulty to study the almost sure asymptotics of the location process X is that we can not compare the solutions $X^{\alpha,x}$ to (3.7) with different values of α, since α appears in the diffusion coefficient. A way to overcome this problem is to introduce a change of time, in order to make α appear in the drift term, as follows.

Lemma 3.7 *Assume (H1), and consider a solution $(X_t, M_t)_{t\geq 0}$ to (2.2). Let $R_t = |X_t|$. Then*

$$\begin{aligned} R_t = R_0 + \int_0^t (\alpha(M_s)C(R_s))^{\frac{1}{2}}\, d\tilde{B}_s + \frac{d-1}{2}\int_0^t \frac{\alpha(M_s)C(R_s)}{R_s} ds \\ + \frac{1}{2}\int_0^t \left[\left(1-\frac{d}{2}\right)\alpha(M_s)C'(R_s) - C(R_s)u'(R_s)\right] ds \end{aligned} \tag{3.17}$$

where $\tilde{B}_t = \sum_{i=1}^d \int_0^t \frac{X_i dB_s^i}{R_s}$ is a one-dimensional Brownian motion, while

$$\begin{aligned} R_t^2 = R_0^2 + 2\int_0^t (\alpha(M_s)C(R_s))^{\frac{1}{2}} R_s d\tilde{B}_s + \int_0^t \alpha(M_s)\left(1-\frac{d}{2}\right)C'(R_s)R_s\, ds \\ - \int_0^t C(R_s)u'(R_s)R_s\, ds + d\int_0^t \alpha(M_s)C(R_s)\, ds \end{aligned} \tag{3.18}$$

The proof is a straightforward application of the Itô formula.

Lemma 3.8 *Assume (H1), and consider a solution $(X_t, M_t)_{t\geq 0}$ to (2.2). Denote by $A_t = \int_0^t \alpha(M_s)ds$, and by $\tau_t = \inf\{s,\ A_s > t\}$ its inverse. Then $\beta_t := \int_0^{\tau_t} \sqrt{\alpha(M_s)}d\tilde{B}_s$ is a Brownian motion, and*

$$\begin{aligned} R_{\tau_t} &= R_0 + \int_0^t \sqrt{C(R_{\tau_s})}d\beta_s + \frac{d-1}{2}\int_0^t \frac{C(R_{\tau_s})}{R_{\tau_s}}ds \\ &\quad + \frac{1}{2}\int_0^t \left[\left(1-\frac{d}{2}\right)C'(R_{\tau_s}) - \frac{C(R_{\tau_s})u'(R_{\tau_s})}{\alpha(M_{\tau_s})}\right] ds \end{aligned} \tag{3.19}$$

The proof is again immediate. The fact that β is a Brownian motion comes from Theorem 1.7 (p. 182) in Revuz-Yor [5]. We now introduce another process, which corresponds to particles with constant mass.

Notation 3.9 *Assume (H1). Let $\alpha \in (0,1]$ and $\rho > 0$ be fixed. Consider a one-dimensional Brownian motion W. We denote by $Z^{\alpha,\rho} = (Z_t^{\alpha,\rho})_{t\geq 0}$ the unique strong solution of*

$$\begin{aligned} Z_t^{\alpha,\rho} &= \rho + \int_0^t \sqrt{C(Z_s^{\alpha,\rho})}dW_s + \frac{d-1}{2}\int_0^t \frac{C(Z_s^{\alpha,\rho})}{Z_s^{\alpha,\rho}}ds \\ &\quad + \frac{1}{2}\int_0^t \left[\left(1-\frac{d}{2}\right)C'(Z_s^{\alpha,\rho}) - \frac{C(Z_s^{\alpha,\rho})u'(Z_s^{\alpha,\rho})}{\alpha}\right] ds \end{aligned} \tag{3.20}$$

Equation (3.20) is obtained by replacing the non-constant function $\alpha(M_s)$ by the constant α in (3.19).

3.5 Almost sure convergence of the position process

We finally prove Theorem 2.5. We begin with a straightforward consequence of Lemma 3.5.

Lemma 3.10 *Assume* $(H1)$. *There exists* $s_0 > 0$ *such that for all* $s \geq s_0$, $\mathbb{P}[X_s < \varepsilon/2] \geq 1/2$.

Lemma 3.11 *Assume* $(H1)$. *There exist* $k > 0$ *and* $\alpha_0 > 0$ *such that for all* $\sigma > 0$, *all* $\lambda > 0$, *all* $\alpha \in (0, \alpha_0]$, *all* $\rho \in (0, 1]$, *for* $Z^{\alpha,\rho}$ *defined in Lemma 3.9, with a Brownian motion* W,

$$\mathbb{P}\left\{ \sup_{t\in[0,\sigma]} \left| \int_0^t Z_s^{\alpha,\rho} \sqrt{C(Z_s^{\alpha,\rho})} dW_s \right| > \lambda \right\} \leq \frac{k(1+\sigma)\alpha}{\lambda^2} \tag{3.21}$$

Proof We break the proof in two steps.
Step 1 We first check that there exist some constants $\alpha_0 > 0$, $A > 0$, $B > 0$ such that for all $\alpha \in (0, \alpha_0]$, all $\rho \in (0, 1]$, all $t \geq 0$,

$$\phi(t) = \mathbb{E}\left([Z_t^{\alpha,\rho}]^2 \right) \leq A\alpha + Ae^{-(B/\alpha)t} \tag{3.22}$$

where the first equality stands for a definition. Let θ the function on $\mathbb{R}_+$ be defined by

$$\theta(z) := dC(z) + \left(1 - \frac{d}{2}\right) zC'(z) - \frac{C(z)u'(z)z}{\alpha} \tag{3.23}$$

Then a simple computation using the Itô formula leads to

$$\mathbb{E}\left([Z_t^{\alpha,\rho}]^2 \right) = \rho^2 + \int_0^t \mathbb{E}\left(\theta[Z_s^{\alpha,\rho}]\right) ds \tag{3.24}$$

Using $(H1)$ (6 and 8), we obtain that for some constants $A > 0$, $B > 0$,

$$\theta(z) \leq A + Az - Bz^2/\alpha \tag{3.25}$$

for all $z \geq 0$. We thus have for α small enough (say for $\alpha \leq \alpha_0$), for some constants $A > 0$, $B > 0$,

$$\theta(z) \leq A - \frac{B}{\alpha} z^2 \tag{3.26}$$

for all $z \geq 0$. We thus deduce from (3.24) and the Jensen inequality that $\phi'(t) \leq A - \frac{B}{\alpha}\phi(t)$, from which one easily deduces that for all $t \geq 0$,

$$\phi(t) \leq \rho^2 e^{-(B/\alpha)t} + \frac{A\alpha}{B}\left[1 - e^{-(B/\alpha)t}\right] \tag{3.27}$$

Hence (3.22) holds.
Step 2 Using Doob's inequality, the fact that C is bounded (see $(H1)$), and Step 1, we obtain the existence of a constant k (whose value changes from line to line) such that for all $\sigma > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,\sigma]}\left|\int_0^t Z_s^{\alpha,\rho}\sqrt{C(Z_s^{\alpha,\rho})}dW_s\right| > \lambda\right) \leq \frac{k}{\lambda^2}\int_0^\sigma \mathbb{E}\left[(Z_s^{\alpha,\rho})^2\right]ds$$
$$\leq \frac{k}{\lambda^2}\int_0^\sigma [A\alpha + Ae^{-(B/\alpha)s}]ds \leq \frac{k}{\lambda^2}\alpha(1+\sigma) \qquad (3.28)$$

which ends the proof. □

Notation 3.12 *Assume $(H1)$. For $\alpha \in (0,\alpha_0]$, $\lambda > 0$, $\sigma > 0$, $\rho > 0$, and $(W_t)_{t\in[0,\sigma]}$ a one dimensional Brownian motion, we consider the event*

$$A_{\alpha,\rho}^{\lambda,\sigma}(W) = \left\{\sup_{t\in[0,\sigma]}\left|\int_0^t Z_s^{\alpha,\rho}\sqrt{C(Z_s^{\alpha,\rho})}dW_s\right| \leq \lambda\right\} \qquad (3.29)$$

the process $Z^{\alpha,\rho}$ being defined by Lemma 3.9 with the Brownian motion W. We have a lower bound of the probability of this event, thanks to Lemma 3.11.

Lemma 3.13 *Assume $(H1)$. There exists a constant $a_1 > 0$ such that for all $\alpha \in (0,\alpha_0]$, $\lambda > 0$, $\sigma > 0$, $\rho > 0$, and $(W_t)_{t\in[0,\sigma]}$ a Brownian motion, the process $Z^{\alpha,\rho}$ being defined by Lemma 3.9 with the Brownian motion W,*

$$A_{\alpha,\rho}^{\lambda,\sigma}(W) \subset \left\{\sup_{t\in[0,\sigma]}[Z_t^{\alpha,\rho}]^2 \leq (\rho^2 \vee [a_1\alpha + \lambda]) + \lambda\right\} \qquad (3.30)$$

Proof First note that, thanks to the Itô formula,

$$[Z_t^{\alpha,\rho}]^2 = \rho^2 + \int_0^t \theta(Z_s^{\alpha,\rho})ds + \int_0^t Z_s^{\alpha,\rho}\sqrt{C(Z_s^{\alpha,\rho})}dW_s \qquad (3.31)$$

where θ was defined by (3.23). Note also that $\theta(z) \leq 0$ for $z^2 \geq a_1\alpha$, the constant a_1 not depending on α. Fix $\omega \in A_{\alpha,\rho}^{\lambda,\sigma}(W)$, $\alpha < \alpha_0$, $\lambda > 0$, and $\rho > 0$. Denote by $\varphi(t) = \displaystyle\int_0^t Z_s^{\alpha,\rho}\sqrt{C(Z_s^{\alpha,\rho})}dW_s$, and by $y(t) = [Z_t^{\alpha,\rho}]^2 - \varphi(t)$. Then y satisfies

$$y(t) = \rho^2 + \int_0^t \zeta(s, y(s))ds \qquad (3.32)$$

where $\zeta(s,x) = \theta(\sqrt{x} + \varphi(s))$. But since ω belongs to $A_{\alpha,\rho}^{\lambda,\sigma}(W)$, we deduce that $|\varphi(s)|$ is bounded by λ (for $s \leq \sigma$), so that $\zeta(s,x) \leq 0$ for all $s \in [0,\sigma]$, $x \geq \lambda + a_1\alpha$. A classical argument shows that for each $t \in [0,\sigma]$, $y(t) \leq \rho^2 \vee [\lambda + a_1\alpha]$. Hence, $[Z_t^{\alpha,\rho}]^2 \leq \lambda + \rho^2 \vee [\lambda + a_1\alpha]$, which was our aim.

□

Lemma 3.14 *Let N_t a standard Poisson process with parameter $\mu > 0$. For all $x < 1 - 1/e$, all $t \geq 0$, $\mathbb{P}(N_t \leq x\mu t) \leq \exp(-\mu t[1 - 1/e - x])$.*

Proof A simple computation shows that

$$\mathbb{P}(N_t \leq x\mu t) = \mathbb{P}\left(e^{-N_t} \geq e^{-x\mu t}\right) \leq e^{x\mu t}\mathbb{E}\left(e^{-N_t}\right) = e^{x\mu t}e^{-\mu t[1-1/e]} \quad (3.33)$$

□

Lemma 3.15 *Assume $(H1)$. Recall the notations of Lemma 3.8. On the set where $A_\infty < \infty$, $\lim_{t\to\infty} X_t = 0$ a.s.*

Proof It of course suffices to show that R_t tends a.s. to 0 on the set where $A_\infty < \infty$. We thus consider ω to be fixed in $\{A_\infty < \infty\}$ in the whole proof below. Thanks to (3.18), for all $t \geq 0$,

$$\int_0^t \left[C(R_s)u'(R_s) - (1 - d/2)\alpha(M_s)c'(R_s)\right] R_s ds \quad (3.34)$$
$$\leq R_0^2 + 2\int_0^t [\alpha(M_s)C(R_s)]^{1/2} R_s d\tilde{B}_s + d\int_0^t \alpha(M_s)C(R_s)ds$$

But, since we know that $\alpha(M_s)$ tends to 0, we deduce from $(H1)$ (6 and 8) that for t sufficiently large, $C(R_s)u'(R_s) - (1 - d/2)\alpha(M_s)c'(R_s) \geq 0$. Hence the left hand side of (3.34) is nondecreasing (for t sufficiently large), so that it is a.s. bounded below. On the other hand, since C is bounded, it is immediate to obtain that (since $A_\infty < \infty$) $\int_0^\infty \alpha(M_s)C(R_s)ds < \infty$.
We deduce that the stochastic integral $\int_0^t [\alpha(M_s)C(R_s)]^{1/2} R_s d\tilde{B}_s$ is bounded below. Hence it does converge, so that $\int_0^\infty \alpha(M_s)C(R_s)R_s^2 ds < \infty$. Hence the right hand side of (3.34) converges, so that the left hand side, which is nondecreasing (for t sufficiently large) does also converge. Thus, we obtain, using (3.18), that R_t^2 (and thus also R_t) does a.s. converge as t tends to infinity.
Let R_∞ be its limit. We know that a.s., $\lim_{t\to\infty} \sup_{h>0} |R_{t+h} - R_t| = 0$. Assume that $R_\infty > 0$. Then

$$R_{t+h} - R_t = \Delta^1_{t,t+h} + \Delta^2_{t,t+h} + \Delta^3_{t,t+h} \quad (3.35)$$

where $\Delta^1_{t,t+h} = \int_t^{t+h} [\alpha(M_s)C(R_s)]^{1/2} R_s d\tilde{B}_s$ tends to 0 uniformly in h since C is bounded and since $\int_0^\infty \alpha(M_s)C(R_s)R_s^2 ds < \infty$;
where $\Delta^2_{t,t+h} = \int_t^{t+h} \frac{\alpha(M_s)C(R_s)}{R_s} ds$ also tends to 0 uniformly in h, since C is bounded, since $R_\infty > 0$, and since $A_\infty < \infty$;

and where $\Delta^3_{t,t+h} = \int_t^{t+h} [(1-d/2)\alpha(M_s)C'(R_s) - C(R_s)u'(R_s)]\,ds$ behaves as $-C(R_\infty)u'(R_\infty)h$ for t large enough since C' is bounded and since $A_\infty < \infty$. Since $C(r)u'(r)$ does not vanish except for $r = 0$, (see $(H1)$ 6 and 8), this contradicts the fact that $R_\infty > 0$, and ends the proof. □

Lemma 3.16 *Assume $(H1)$ and $(H2)$. Recall the notations of Lemma 3.8. On the set where $A_\infty = \infty$, $\lim_{t\to\infty} X_t = 0$ a.s.*

Proof We break the proof in several steps. In the whole proof, we consider ω to be fixed in $\{A_\infty = \infty\}$.
Step 1 We introduce in this step the notations.
First note that on the set where $A_\infty = \infty$, $\tau_t < \infty$ for all t. However, $\lim_{t\to\infty} \tau_t = \infty$, since the map α is smaller than 1 (see $(H1)$-4).
We consider $0 < \eta < \varepsilon/2$ to be fixed. We will show that a.s. (on $\{A_\infty = \infty\}$),

$$\limsup_{t\to\infty} R_{\tau_t} \le \eta \tag{3.36}$$

which of course suffices. For each $n \ge 1$, we consider a set of numbers $m_n \in \mathbb{N}^*$ (an increasing sequence of masses), $\rho_n \in (0,\eta)$ (a nondecreasing sequence of initial points), $\sigma_n > 0$ (a sequence of *widthes* of time intervals), and a sequence $\lambda_n > 0$ (of *fluctuations controls*). We will choose these sequences conveniently at the end of the proof. Let us however right now assume that (recall that α_0 and a_1 were defined in Lemmas 3.11 and 3.13), setting $\mu_0 = (a_0/2)(v_d\varepsilon^d)^{-1}$ (a_0 is defined in $(H2)$),

$$\alpha(m_1) \le \alpha_0 \tag{3.37}$$

$$\sum_{i=1}^{\infty} \sigma_i = \infty \tag{3.38}$$

$$\forall n \ge 1, \quad \rho_{n+1}^2 = \rho_n^2 \vee [a_1\alpha(m_n) + \lambda_n] + \lambda_n \le \eta^2 \tag{3.39}$$

$$\forall n \ge 1, \quad 0 \le \frac{m_{n+1} - m_n}{\mu_0\sigma_n/\alpha(m_n)} < 1 - 1/e - 1/2 \tag{3.40}$$

We will consider here only times greater than s_0 defined in Lemma 3.10. Note that a.s., $\tau_s \ge s_0$ for all $s \ge s_0$, by definition of τ (see Lemma 3.8) and since the map α is smaller than 1 (see $(H1)$-4). Recall that the Brownian motion β was defined in Lemma 3.8.
Step 2 We introduce the following random times and events defined recursively by (recall Notation 3.12)

$$T_1 = \inf\{t \ge s_0;\ M_{\tau_t} \ge m_1, R_{\tau_t} \le \rho_1\} \tag{3.41}$$

$$A_1 = A^{\lambda_1,\sigma_1}_{\alpha(m_1),\rho_1}(\beta_{T_1+\cdot} - \beta_{T_1}) \cap \left\{M_{\tau_{T_1+\sigma_1}} > m_2\right\} \tag{3.42}$$

and, for $n \ge 2$,

$$T_n = \inf\{t \ge T_{n-1} + \sigma_{n-1};\ M_{\tau_t} \ge m_n, R_{\tau_t} \le \rho_n\} \tag{3.43}$$

$$A_n = A^{\lambda_n,\sigma_n}_{\alpha(m_n),\rho_n}(\beta_{T_n+\cdot} - \beta_{T_n}) \cap \left\{M_{\tau_{T_n+\sigma_n}} > m_{n+1}\right\} \tag{3.44}$$

Note that since $\lim_n \alpha(n) = 0$, since $\lim_t \tau_t = \infty$, Lemma 3.4 and Corollary 3.6 ensure that T_n is a.s. finite for all n.
Our aim is to apply the Borel-Cantelli Lemma, in order to show that a.s., there exists n_0 such that for all $n \geq n_0$, A_n holds. This will allow us to conclude.
First note that thanks to Lemma 3.13, (since α is non-increasing while m_n is increasing, (3.37) ensures that $\alpha(m_n) \leq \alpha_0$ for all n)

$$\begin{aligned} A_n &\subset A^{\lambda_n,\sigma_n}_{\alpha(m_n),\rho_n}(\beta_{T_n+\cdot} - \beta_{T_n}) \\ &\subset \left\{ \sup_{t\in[0,\sigma_n]} \left| Z^{\alpha(m_n),\rho_n}_t \right|^2 \leq \rho_n \vee [a_1\alpha(m_n) + \lambda_n] + \lambda_n \right\} \end{aligned} \tag{3.45}$$

$Z^{\alpha(m_n),\rho_n}_t$ being defined with the Brownian motion $\beta_{T_n+\cdot} - \beta_{T_n}$. Since $R_{\tau_{T_n}} \leq \rho_n$ and since for all $t \geq T_n$, $\alpha(M_{\tau_t}) \leq \alpha(m_n)$, one may deduce from the comparison Theorem (see [5]) that a.s., for all $t \geq 0$, $R_{\tau_{T_n+t}} \leq Z^{\alpha(m_n),\rho_n}_t$. Hence

$$\begin{aligned} A_n &\subset \left\{ \sup_{t\in[T_n,T_n+\sigma_n]} R^2_{\tau_t} \leq \rho_n^2 \vee [a_0\alpha(m_n) + \lambda_n] + \lambda_n \right\} \\ &\subset \left\{ \sup_{t\in[T_n,T_n+\sigma_n]} R_{\tau_t} \leq \frac{\varepsilon}{2} \right\} \end{aligned} \tag{3.46}$$

Next, with the notation $(A_n)^c = \Omega/A_n$, we obtain, using (3.46) and Lemma 3.11,

$$\begin{aligned} \mathbb{P}\left[(A_n)^c\right] &\leq \mathbb{P}\left[\left(A^{\lambda_n,\sigma_n}_{\alpha(m_n),\rho_n}(\beta_{T_n+\cdot} - \beta_{T_n})\right)^c\right] \\ &\quad + \mathbb{P}\left[A^{\lambda_n,\sigma_n}_{\alpha(m_n),\rho_n}(\beta_{T_n+\cdot} - \beta_{T_n}) \cap \left\{M_{\tau_{T_n+\sigma_n}} > m_{n+1}\right\}^c\right] \\ &\leq \frac{k(1+\sigma_n)\alpha(m_n)}{\lambda_n^2} + I_n \end{aligned} \tag{3.47}$$

where

$$I_n = \mathbb{P}\left[\sup_{t\in[T_n,T_n+\sigma_n]} R_{\tau_t} \leq \varepsilon/2,\ M_{\tau_{T_n+\sigma_n}} - M_{\tau_{T_n}} \leq m_{n+1} - m_n \right] \tag{3.48}$$

One easily understand, using (2.2), Lemma 3.10, the fact that $\tilde{M}$ is always greater than 1, and $(H2)$ $(K(i,j)/j \geq a_0)$, that since $\tau_{T_n+t} \geq s_0$ and $\partial_t \tau_{T_n+t} \geq \alpha(m_n)^{-1}$ for all $t \geq 0$, the process $(M_{\tau_{T_n+\sigma_n}} - M_{\tau_{T_n}})_{t\in[0,\sigma_n]}$ is bounded below (on the event $\sup_{t\in[T_n,T_n+\sigma_n]} R_{\tau_t} \leq \varepsilon/2$), by $N_{t/\alpha(m_n)}$, N being a standard Poisson process with rate $\mu_0 = (a_0/2)(v_d\varepsilon^d)^{-1}$. Hence, Lemma 3.14 allows to conclude, using (3.40), that

$$I_n \leq \exp[-\mu_0\sigma_n/2\alpha(m_n)] \tag{3.49}$$

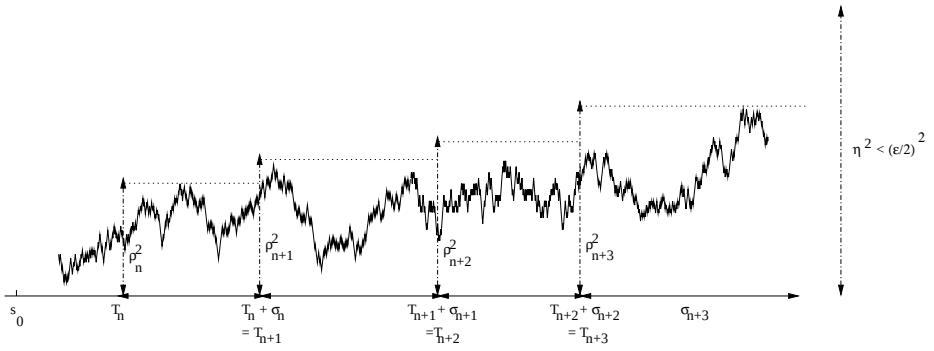

Figure 1: A typical path of $R^2_{\tau_\cdot}$ for $t > T_n$ ($n \geq n_0$).

We finally deduce that for all $n \geq 1$,

$$\mathbb{P}\left[(A_n)^c\right] \leq \frac{k(1+\sigma_n)\alpha(m_n)}{\lambda_n^2} + \exp[-\mu_0\sigma_n/2\alpha(m_n)] \tag{3.50}$$

Step 3 Recall now that the constants c_0, β_0, a_1, α_0 and μ_0 are defined in $(H2)$, Lemmas 3.13 and 3.11 and before (3.37).
We first of all consider some exponent $p \in (3/\beta_0, 1/[1-\beta_0])$, which is possible since $\beta_0 \in (3/4, 1)$. Next, we choose σ_n constant $\sigma_n = \sigma$, $m_n = \delta_1 n^p$, $\lambda_n = \delta_2\eta^2/[n\log^2(n+1)]$, and $\rho_n = \sqrt{\eta^2/2 + \sum_{i=1}^n \lambda_i}$. Choosing δ_1 large enough ($\delta_1 \geq (c_0/\alpha_0)^{1/\beta_0}$ and $\delta_1 \geq (2a_1c_0/\eta^2)^{1/\beta_0}$), δ_2 small enough ($\delta_2 \leq 1/2\sum_{n\geq 1}[1/n\log^2(n+1)]$), and finally σ large enough ($\sigma \geq \delta_1^{1-\beta_0}c_0p2^{p+1}/\mu_0$), we deduce that the conditions (3.37)–(3.40) are satisfied (with, for each n, $\rho_n^2 \vee [a_1\alpha(m_n) + \lambda_n] = \rho_n^2$), and, thanks to $(H2)$,

$$\sum_{n\geq 1} \mathbb{P}\left[(A_n)^c\right] < \infty \tag{3.51}$$

Using the Borel-Cantelli Lemma, we deduce that there a.s. exists n_0 such that for all $n \geq n_0$, A_n holds. This implies that

$$\sup_{t\geq T_{n_0}} R_{\tau_t} \leq \eta \tag{3.52}$$

Indeed, if A_n holds for all $n \geq n_0$, then on one hand, for all $n \geq n_0$, $T_{n+1} = T_n + \sigma_n$ (thanks to the first line of (3.46) and to condition (3.39)), and on the other hand, for all $n \geq n_0$, $\sup_{[T_n, T_n+\sigma_n]} R_{\tau_t} \leq \eta$ (thanks to the first line of (3.46) and to condition (3.39)). This ends the proof.

On Figure 1, one can see a typical path of $R^2_{\tau_t}$ for $n \geq n_0$. Note that during the time interval $[T_n, T_n + \sigma_n]$, the increment of mass process $M_{\tau_\cdot}$ is at least $m_{n+1} - m_n$.

□

4 Related problems

We finally would like to talk about related problems. Points 1. and 2. might be interesting for applied physics, while points 3. and 4. are rather theoretical questions.

1. It would be more interesting to study the local case, where $\varepsilon = 0$. In such a case, the characteristics $(X_t, M_t)_{t\geq 0}$ of a typical particle would satisfy a different and more complicated equation. Indeed, the location process X would still satisfy the first equation in (2.2), but the mass process would satisfy

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}^d \times \mathbb{N}^*} \int_0^\infty m \mathbb{1}_{\left\{u \leq \frac{K(M_{s-},m)}{m}\right\}} \mu(ds, d(x,m), du) \quad (4.53)$$

   the counting random measure μ on $[0,\infty) \times (\mathbb{R}^d \times \mathbb{N}^*) \times [0,\infty)$ having the intensity measure $ds f_s(X_s, dm)du$, the probability measure $Q_s(dx, dm) = dx f_s(x, dm)$ standing for the law of (X_s, M_s) (for each s). In other words, a particle does coalesce with others at a rate depending on the density of particles which have the same location. This is of course more delicate, but we hope that Proposition 2.3 and Theorems 2.4 and 2.5 would still hold in such a context.

2. Consider now the *standard* case where the location of each particle (of mass m) is Brownian motion reflected in a bounded smooth domain $D \subset \mathbb{R}^d$, with a coefficient $\alpha(m)$ (see [4]). Then two behaviors may be possible. On one hand, if α decreases slowly to 0, then one may hope that each particle has a recurrent motion: X_t does converge in law as t tends to infinity, but does not converge almost surely, and visits infinitely often each open subset of D. On the converse, if α decreases quickly to 0, then it is reasonable to think that X_t will converge a.s. as time tends to infinity, to a random position X_∞.
   Note that the standard P.D.E. approach does not seem to allow such considerations.

3. Assume now that we are in the local case $\varepsilon = 0$ (see point 1. above), and that the effect of the potential increases as the mass of particles increase. In other words, replace $u'(|X_s|)$ by $\beta(M_s)u'(|X_s|)$, for some function $\beta(m) \geq 0$ which goes to infinity with m. Then the more a particle is large, the more it visits neighborhoods of 0, so that it encounters many other particles, increases more and more fast, and so on... Is there a *gelation* phenomenon in such a case? That is, does it exist $T < \infty$ such that $\mathbb{P}[M_T = \infty, X_T = 0] > 0$?

4. Finally, assume that the location process $\{X_t\}_{t\geq 0}$ of a particle of size m is a diffusion depending on m, such that:
   if m was constant and smaller than some m_0, X_t would be transient,

if m was constant and larger than m_0 , X_t would be recurrent.
Think, for example, to Bessel processes of dimension $\alpha(m)$, for some non-increasing function α. Coupling such a motion with coalescence, the masses would increase, so that we might observe the following behavior:

(i) with positive probability, the mass of our typical particle does not increase too much, so that its location X_t will be transient, hence it will never encounter other particles, ... In other words,
$\mathbb{P}[\lim_{t\to\infty} M_t < m_0, \lim_{t\to\infty} |X_t| = \infty] > 0$.

(ii) with positive probability, the mass of our typical particle does increase, so that its location X_t will be recurrent, so that it will encounter many other particles, ... In other words,
$\mathbb{P}[\lim_{t\to\infty} M_t = \infty, \lim_{t\to\infty} |X_t| = 0] > 0$.

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