Local properties of self-similar solutions to Smoluchowski's coagulation equation with sum kernels

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Abstract

The regularity of the scaling profiles ψ to Smoluchowski's coagulation equation is studied when the coagulation kernel K is given by $K(x,y) = x^\lambda + y^\lambda$ with $\lambda \in (0,1)$. More precisely, ψ is $\mathcal{C}^1$-smooth on $(0,\infty)$ and decays exponentially fast for large x. Furthermore, the singular behaviour of $\psi(x)$ as $x \to 0$ is identified, thus giving a rigorous proof of physical conjectures.

Key words. Smoluchowski's coagulation equation, self-similar solutions, mass conservation.

1 Introduction

We investigate the small and large mass behaviour and the regularity of the scaling profile of mass-conserving self-similar solutions to the Smoluchowski coagulation equation [8, 18]

$$\partial_t c(t,x) = L_c(c(t,.))(x), \quad (t,x) \in (0,\infty) \times (0,\infty), \tag{1.1}$$

where the coagulation reaction term L_c is defined by

$$L_c(c)(x) = \frac{1}{2} \int_0^x K(y, x-y) \, c(y) \, c(x-y) \, dy - c(x) \int_0^\infty K(x,y) \, c(y) \, dy \tag{1.2}$$

for $x \in (0,\infty)$. Recall that the Smoluchowski coagulation equation (1.1) is a mean-field model describing the growth of particles by successive binary mergers and $c(t,x)$ denotes the density of particles of mass $x \in (0,\infty)$ at time $t \geq 0$. The coagulation kernel $K(x,y)$ models the likelihood that two particles with respective masses x and y merge into a single one (with mass $x+y$) and is a symmetric and nonnegative function on $(0,\infty) \times (0,\infty)$.

When K is homogeneous of degree $\lambda \in (-\infty, 1)$ (that is, $K(ax, ay) = a^\lambda K(x,y)$), the *dynamical scaling hypothesis* conjectured by physicists predicts that solutions to (1.1) behave in a self-similar way for large times, i.e.

$$c(t,x) \sim c_S(t,x) = s(t)^{-2} \, \psi\left(xs(t)^{-1}\right) \quad \text{as} \quad t \to \infty, \tag{1.3}$$

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where c_S is a self-similar solution to (1.1), see [7, 15] and the references therein. While the validity of (1.3) is still an open problem (except for the constant kernel $K = 1$ and the additive kernel $K(x, y) = x + y$, see [1, 2, 4, 12, 16, 17]), a first step in that direction was recently achieved in [9, 11] where the existence of self-similar solutions c_S to (1.1) as described in (1.3) was proved for a large class of homogeneous coagulation kernels.

Nevertheless, for the so-called "sum" kernel K given by

$$K(x, y) = x^\lambda + y^\lambda, \quad (x, y) \in (0, \infty)^2, \tag{1.4}$$

for some $\lambda \in (0, 1)$, the integrability properties of the scaling profile ψ for small mass obtained in [9, 11] are weaker than that predicted by physicists [7, 15] and one purpose of this work is to fill this gap. More precisely, the scaling profile ψ of the self-similar solution to (1.1) constructed in [9, 11] is such that $\psi \in L^1(0, \infty; x^\sigma \, dx)$ for each $\sigma \geq \lambda$. In this paper, we extend this property to $\sigma \geq \tau - 1$ where $\tau < 1 + \lambda$ is given by (1.8) below. We actually prove that $\psi(x) \sim L_0 \, x^{-\tau}$ as $x \to 0$ for some $L_0 > 0$, which is exactly the small mass behaviour for ψ expected from previous formal computations [7, 15].

The second aim of this paper is to improve the large mass estimates on ψ obtained so far. More precisely, we prove that $\psi(x) \leq C \, e^{-\varrho x}$ for some $C > 0$ and $\varrho > 0$ but also that ψ cannot decay faster than any exponential. These two facts perfectly agree with the conjecture that $\psi(x) \sim A \, x^{-\lambda} \, e^{-\delta x}$ as $x \to \infty$ for some constants $A > 0$ and $\delta > 0$ [7, 15], which we have been yet unable to prove.

Finally, as a by-product of the analysis of the behaviour of ψ for large and small masses, we also study the smoothness of the scaling profile ψ on $(0, \infty)$.

Let us now state precisely our results and first recall the definition of a scaling profile to (1.1).

Definition 1.1 *Consider the coagulation kernel K defined by (1.4) for some $\lambda \in (0, 1)$, and set $\gamma := 1/(1 - \lambda)$. A scaling profile to (1.1) is a strictly positive function $\psi \in L^1(0, \infty; xdx)$ such that*

$$\int_0^\infty x \, \psi(x) \, dx = 1, \quad \psi \in L^1(0, \infty; x^\sigma dx) \quad \text{for each} \quad \sigma \geq \lambda, \tag{1.5}$$

$$\gamma \int_0^\infty x^2 \psi(x) \phi'(x) dx = \int_0^\infty \int_0^\infty x K(x, y) [\phi(x + y) - \phi(x)] \psi(x) \psi(y) dy dx \tag{1.6}$$

for any $\phi \in \mathcal{C}_b^1([0, \infty))$, and

$$\gamma z^2 \psi(z) = \int_0^z \int_{z-x}^\infty K(x, y) x \psi(x) \psi(y) dy dx \quad \text{for a.e.} \quad z \in (0, \infty), \tag{1.7}$$

the right-hand side of (1.7) being finite for almost every $z \in (0, \infty)$.

For the coagulation kernel (1.4), the existence of a scaling profile ψ to (1.1) in the sense of Definition 1.1 follows from [11], see also [9]. It is also shown in these papers that, if ψ is a scaling profile to (1.1) in the sense of Definition 1.1, the function $c_S(t, x) = t^{-2\gamma} \psi(xt^{-\gamma})$, $(t, x) \in (0, \infty) \times (0, \infty)$ solves (1.1) in a weak sense and is thus a self-similar solution to (1.1). Let us also mention at this point that the choice of the constant γ on the left-hand side of (1.7) and of the value 1 for the first moment of ψ is only made for convenience. Indeed, if ψ is a scaling profile to

(1.1) in the sense of Definition 1.1, the function $\psi_{a,b}(x) := a\ \psi(bx)$ is also a scaling profile to (1.1) with $a\gamma b^{-(1+\lambda)}$ instead of γ on the left-hand side of (1.7) and with a first moment equal to ab^{-2}.

We will prove here the following properties of scaling profiles.

Theorem 1.2 *Let ψ be a scaling profile to (1.1) in the sense of Definition 1.1 for the coagulation kernel (1.4). Then $\psi \in \mathcal{C}^1((0,\infty))$ and, setting*

$$\tau := 2 - \frac{1}{\gamma} \int_0^\infty x^\lambda\ \psi(x)\ dx\,, \tag{1.8}$$

we have $\tau \in (1, \min\{3/2, 1+\lambda\})$ and there exists $L_0 > 0$ such that

$$\lim_{z \to 0}\ z^\tau \psi(z)\ =\ L_0\,. \tag{1.9}$$

Moreover, for any $\varrho < 2^{-\gamma-2}\gamma^{\gamma-1}$, there exists a constant $C_0(\varrho)$ such that

$$\psi(z)\ \leq\ C_0(\varrho)\ e^{-\varrho z} \quad for \quad z \in [1, \infty)\,. \tag{1.10}$$

Finally there exists $\varrho_1 > 0$ such that

$$\int_1^\infty \psi(z)\ e^{\varrho_1 z}\ dz = \infty\,, \tag{1.11}$$

so that (1.10) cannot hold true for any $\varrho > 0$.

As already mentioned, the behaviour (1.9) of ψ for small masses has been obtained by formal arguments in the physical literature [7, 15] and we herein provide a rigorous proof of this fact. From a physical point of view, it seems to be quite important that the exponent τ is not determined *a priori* but implicitly defined, which contrasts markedly with other kernels (such as the so-called "product" kernel $K(x, y) = (xy)^{\lambda/2}$, $\lambda \in (0, 1)$, for which it is conjectured that $\tau = 1 + \lambda$ [7, 15]). Notice also that, if (1.3) holds true, we have $c(t, x)/c(t, 1) \sim x^{-\tau}$ for fixed x at large times and the exponent τ thus describes the x-dependence of the solutions c to (1.1) for $x \ll t^\gamma$. In fact, some analytical upper and lower bounds for τ are available [3, 5, 6] and numerical simulations have been performed which allow to compute approximate values of τ [3, 10, 13, 14]. In this direction, we give a rigorous proof of the fact that $\tau < 1 + \lambda$ and also show that $\tau > 1$ for each $\lambda \in (0, 1)$. Seemingly, the latter bound was only known for λ in a neighbourhood of 1 [5].

It is also conjectured in [7, 15] that, for large z, $\psi(z) \sim Az^{-\lambda}e^{-\delta z}$ for some constants $A > 0$ and $\delta > 0$. We only prove the weaker assertions (1.10) and (1.11) but point out that they agree with this conjecture.

We finally mention that the arguments developed below are specific for the analysis of the small mass behaviour of the scaling profile for the sum kernel (1.4). In particular, it seems likely that the study of the scaling profile associated to the "product" kernel $K(x, y) = (xy)^{\lambda/2}$, $\lambda \in (0, 1)$, requires completely different computations.

As already observed in [7, Eq. (4.30c)], it is possible to combine (1.7) and (1.9) to obtain the second term of the expansion of ψ as $z \to 0$.

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Corollary 1.3 *Let ψ be a scaling profile to (1.1) in the sense of Definition 1.1 for the coagulation kernel (1.4). Recalling that τ is defined by (1.8) and introducing*

$$J := \int_0^1 x^{1+\lambda-\tau} \ (1-x)^{1-\tau} \ dx \quad and \quad L_1 := \frac{J \ L_0^2}{\gamma} \ \frac{(\lambda+2-2\tau) \ (3+\lambda-2\tau)}{(\tau-1)(1+\lambda-\tau)^2} \,,$$

we have

$$\psi(z) = L_0 \ z^{-\tau} + L_1 \ z^{1+\lambda-2\tau} + o\left(z^{1+\lambda-2\tau}\right) \quad as \quad z \to 0 \,. \tag{1.12}$$

Observe that J is indeed finite by the bounds on τ obtained in Theorem 1.2 and that the sign of L_1 depends on whether τ is above or below $1 + (\lambda/2)$. According to [5, Table 1], the latter is certainly true for $\lambda \in (0, 0.366)$ so that $L_1 > 0$ in that case. For other values of λ, a negative or vanishing value of L_1 cannot *a priori* be excluded.

As a final comment, let us mention that one might hope that the qualitative information obtained in Theorem 1.2 could be a small step towards a proof of the uniqueness of the scaling profile ψ and thus towards the proof of (1.3), but this does not seem obvious.

The remainder of the paper is devoted to the proof of Theorem 1.2 and Corollary 1.3. We start with some useful moment estimates in Section 2 where we prove that $\tau < 1 + \lambda$ and that $\psi \in L^1(0, \infty; x^\sigma dx)$ for $\sigma \in (\tau - 1, \lambda)$. These moment estimates then allow us to prove (1.9) in Section 3. At this point, arguing by contradiction enables us to exclude that $\tau = 1$. We next prove (1.10) in Section 4: the first step here is that ψ has some finite exponential moments. Gathering these information, the $\mathcal{C}^1$-smoothness of ψ is shown in Section 5.

From now on, ψ is a scaling profile to (1.1) in the sense of Definition 1.1 for the coagulation kernel (1.4). For $\sigma \in \mathbb{R}$ and $z \in (0, \infty)$, we put

$$M_\sigma := \int_0^\infty x^\sigma \ \psi(x) \ dx \in (0, \infty] \quad and \quad M_\sigma(z) := \int_0^z x^\sigma \ \psi(x) \ dx \in (0, \infty] \,.$$

2 Moment estimates

In this section, we show that we can extend the range of σ for which $\psi \in L^1(0, \infty; x^\sigma dx)$. More precisely, we have the following result:

Proposition 2.1 *We have $\tau < 1 + \lambda$ and $\psi \in L^1(0, \infty; x^\sigma dx)$ for each $\sigma \in (\tau - 1, \lambda)$.*

The fact that $\tau < 1 + \lambda$ follows from the following lower bound for M_λ.

Lemma 2.2 *There holds $M_\lambda \geq 2^\lambda$.*

Proof. We take $\phi(x) = x^{\lambda-1}$ in (1.6) and obtain, thanks to the symmetry of K,

$$\begin{aligned} M_\lambda &= \int_0^\infty \int_0^\infty K(x,y) \ \left[x^\lambda - x \ (x+y)^{\lambda-1}\right] \ \psi(x) \ \psi(y) \ dydx \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty K(x,y) \ \left[x^\lambda + y^\lambda - (x+y)^\lambda\right] \ \psi(x) \ \psi(y) \ dydx \,. \end{aligned}$$

To justify rigorously this equality, consider a sequence of functions $\phi_\varepsilon \in \mathcal{C}_b^1([0,\infty))$ such that $\phi_\varepsilon(x) = x^{\lambda-1}$ for $x > \varepsilon$ and write (1.6) with ϕ_ε. Since $M_\sigma < \infty$ for $\sigma \geq \lambda$ by (1.5), we may let $\varepsilon \to 0$ and obtain the claimed identity. Recalling that

$$x^\lambda + y^\lambda \leq 2^{1-\lambda}(x+y)^\lambda \quad \text{and} \quad x^\lambda + y^\lambda - (x+y)^\lambda \leq \frac{(xy)^\lambda}{(x+y)^\lambda} \tag{2.1}$$

(see Appendix A), we further obtain

$$M_\lambda \leq \frac{1}{2^\lambda} \int_0^\infty \int_0^\infty (xy)^\lambda \, \psi(x) \, \psi(y) \, dydx \leq \frac{M_\lambda^2}{2^\lambda} \, ,$$

and the finiteness of M_λ implies the claim. $\qquad\square$

Proof of Proposition 2.1. We first check that Lemma 2.2 warrants that $\tau < 1 + \lambda$. Indeed, by Lemma 2.2 and the definitions of τ and γ, we have

$$1 + \lambda - \tau = (1-\lambda)(M_\lambda - 1) \geq (1-\lambda)(2^\lambda - 1) > 0 \, .$$

We next fix $\sigma \in (\tau - 1, \lambda)$ and introduce the following approximation of $x^{\sigma-1}$. For $\varepsilon \in (0,1)$, we define $\varphi_\varepsilon \in \mathcal{C}_b^1([0,\infty))$ by

$$\varphi_\varepsilon(x) := \begin{cases} x^{\sigma-1} & \text{if} \quad x \in [\varepsilon, \infty) \, , \\[2mm] (2-\sigma)\,\varepsilon^{\sigma-1} - (1-\sigma)\,\varepsilon^{\sigma-2}\,x & \text{if} \quad x \in [0,\varepsilon] \, . \end{cases}$$

Then

$$\varphi_\varepsilon'(x) := \begin{cases} (\sigma-1)\,x^{\sigma-2} & \text{if} \quad x \in [\varepsilon, \infty) \, , \\[2mm] (\sigma-1)\,\varepsilon^{\sigma-2} & \text{if} \quad x \in [0,\varepsilon] \, , \end{cases}$$

and φ_ε is a non-negative and non-increasing function such that

$$\begin{aligned} x\,|\varphi_\varepsilon'(x)| &\leq (1-\sigma)\,\varphi_\varepsilon(x), \quad x \in [0,\infty) \, , \tag{2.2} \\ \varphi_\varepsilon(x) &\leq x^{\sigma-1}, \quad x \in [0,\infty) \, . \tag{2.3} \end{aligned}$$

Since $\varphi_\varepsilon \in \mathcal{C}_b^1([0,\infty))$ and is non-increasing, we may take $\phi = \varphi_\varepsilon$ in (1.6) and obtain

$$\int_0^\infty \int_0^\infty x\,K(x,y)\,[\varphi_\varepsilon(x) - \varphi_\varepsilon(x+y)]\,\psi(x)\,\psi(y)\,dydx = \gamma \int_0^\infty x^2\,|\varphi_\varepsilon'(x)|\,\psi(x)\,dx \, .$$

Since φ_ε is non-increasing and $K(x,y) \geq y^\lambda$, we have

$$K(x,y)\,[\varphi_\varepsilon(x) - \varphi_\varepsilon(x+y)] \geq y^\lambda\,[\varphi_\varepsilon(x) - \varphi_\varepsilon(x+y)] \, .$$

Thanks to this lower bound and (2.2), we obtain

$$\int_0^\infty \int_0^\infty x\,y^\lambda\,[\varphi_\varepsilon(x) - \varphi_\varepsilon(x+y)]\,\psi(x)\,\psi(y)\,dydx \leq (1-\sigma)\,\gamma \int_0^\infty x\,\varphi_\varepsilon(x)\,\psi(x)\,dx \, ,$$

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whence

$$[M_\lambda - (1 - \sigma)\, \gamma] \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx \leq \int_0^\infty \int_0^\infty x\, y^\lambda\, \varphi_\varepsilon(x + y)\, \psi(x)\, \psi(y)\, dydx\,.$$

Since $\sigma < \lambda < 1$, we may use once more the monotonicity of φ_ε and the fact that $\varphi_\varepsilon(z) = z^{\sigma-1}$ for $z \geq \varepsilon$ to deduce that, for $\delta \in (0,1)$,

$$\begin{aligned}
&[M_\lambda - (1 - \sigma)\, \gamma] \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx \\
\leq\ & \int_0^\infty \int_0^\varepsilon x\, y^\lambda\, \varphi_\varepsilon(x)\, \psi(x)\, \psi(y)\, dydx + \int_0^\infty \int_\varepsilon^\infty x\, y^\lambda\, (x + y)^{\sigma-1}\, \psi(x)\, \psi(y)\, dydx \\
\leq\ & M_\lambda(\varepsilon) \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx + \int_0^\infty \int_\varepsilon^\infty \frac{x}{(x + y)^{1-\lambda}}\, \frac{y^\lambda}{(x + y)^{\lambda-\sigma}}\, \psi(x)\, \psi(y)\, dydx \\
\leq\ & M_\lambda(\varepsilon) \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx + \int_0^\delta \int_\varepsilon^\infty x^\lambda\, y^\sigma\, \psi(x)\, \psi(y)\, dydx \\
+\ & \int_\delta^\infty \int_\varepsilon^\infty x^\lambda\, \frac{y^\lambda}{\delta^{\lambda-\sigma}}\, \psi(x)\, \psi(y)\, dydx \\
\leq\ & (M_\lambda(\varepsilon) + M_\lambda(\delta)) \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx + \frac{M_\lambda^2}{\delta^{\lambda-\sigma}}\,.
\end{aligned}$$

Recalling the definition (1.8) of τ, we have thus shown that, for $\delta \in (0,1)$ and $\varepsilon \in (0, \delta)$, there holds

$$[\gamma\, (\sigma - (\tau - 1)) - 2\, M_\lambda(\delta)] \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx \leq \frac{M_\lambda^2}{\delta^{\lambda-\sigma}}\,.$$

Since $\sigma > \tau - 1$ and $\psi \in L^1(0, \infty; x^\lambda dx)$ by (1.5), there exists $\delta_0 > 0$ such that $4\, M_\lambda(\delta_0) \leq \gamma\, (\sigma - (\tau - 1))$. Therefore, for $\varepsilon \in (0, \delta_0)$, it follows from the above inequality that

$$\frac{\gamma\, (\sigma - (\tau - 1))}{2} \int_0^\infty x\, \varphi_\varepsilon(x)\, \psi(x)\, dx \leq \frac{M_\lambda^2}{\delta_0^{\lambda-\sigma}}\,.$$

In particular,

$$\int_\varepsilon^\infty x^\sigma\, \psi(x)\, dx \leq \frac{2\, \delta_0^{\sigma-\lambda}\, M_\lambda^2}{\gamma\, (\sigma - (\tau - 1))}$$

for $\varepsilon \in (0, \delta_0)$. The Fatou lemma then allows us to complete the proof of Proposition 2.1. $\qquad\square$

We next give some lower and upper bounds for τ which will be helpful to investigate the short and large x behaviour of ψ. These bounds have already been observed in the physical literature [3, 5, 6, 7] with more or less rigorous arguments. We provide a proof below for the sake of completeness.

Proposition 2.3 *There holds $\tau \in [1, 3/2)$.*

Proof. We first establish the lower bound $\tau \geq 1$. Indeed, assume for contradiction that $\tau < 1$. Then $\psi \in L^1(0, \infty)$ by Proposition 2.1 and we can take $\phi(x) = 1/x$ in (1.6) (consider as before a

sequence of functions $\phi_\varepsilon \in C_b^1([0,\infty))$ such that $\phi_\varepsilon(x) = x^{-1}$ for $x > \varepsilon$, write (1.6) with ϕ_ε and pass to the limit as $\varepsilon \to 0$ using the facts that $M_0 < \infty$ and $M_\lambda < \infty$ by (1.5)). We thus obtain

$$
\begin{aligned}
\gamma\, M_0 &= -\gamma \int_0^\infty x^2 \psi(x) \phi'(x) dx \\
&= \int_0^\infty \int_0^\infty x(x^\lambda + y^\lambda)\,(\phi(x) - \phi(x+y))\, \psi(x)\psi(y) dy dx \\
&= \int_0^\infty \int_0^\infty (x^\lambda + y^\lambda)\frac{y}{x+y}\psi(x)\psi(y) dy dx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty (x^\lambda + y^\lambda)\psi(x)\psi(y) dy dx = M_0 M_\lambda,
\end{aligned}
$$

whence $M_\lambda = \gamma$ since M_0 is assumed to be finite. Recalling the definition (1.8) of τ, we would have $\tau = 1$ and a contradiction.

We next turn to the upper bound and follow [5, 6]. If $\lambda \in (0, 1/2]$, the inequality $\tau < 1 + \lambda$ obtained in Proposition 2.1 implies that $\tau < 3/2$. If $\lambda \in (1/2, 1)$, the proof relies on the inequality

$$
(x+y)^{2\lambda} - x^{2\lambda} - y^{2\lambda} \leq \left(2^{2\lambda} - 2\right)\,(xy)^\lambda, \quad (x,y) \in (0,\infty) \times (0,\infty), \tag{2.4}
$$

(see [6, Eq. (5.4)]), a proof of which is given in the Appendix. Since $M_{2\lambda} < \infty$ by (1.5), we may take $\phi(x) = x^{2\lambda - 1}$, $x \in (0,\infty)$, in (1.6) (consider as before an approximating sequence of functions $\phi_\varepsilon \in C_b^1([0,\infty))$) and use (2.4) to obtain

$$
\begin{aligned}
\gamma\,(2\lambda - 1)\,M_{2\lambda} &= \frac{1}{2}\int_0^\infty \int_0^\infty K(x,y)\,[(x+y)\,\phi(x+y) - x\,\phi(x) - y\,\phi(y)]\,\psi(x)\,\psi(y)\,dy dx \\
&\leq \left(2^{2\lambda - 1} - 1\right)\int_0^\infty \int_0^\infty \left(x^\lambda + y^\lambda\right)\,(xy)^\lambda\,\psi(x)\,\psi(y)\,dy dx \\
&\leq \left(2^{2\lambda} - 2\right)\,M_{2\lambda}\,M_\lambda,
\end{aligned}
$$

whence, since $M_{2\lambda} < \infty$,

$$
\frac{2\lambda - 1}{2^{2\lambda} - 2} \leq \frac{M_\lambda}{\gamma} = 2 - \tau.
$$

Now, as $x \longmapsto 2^x$ is strictly convex, $x \longmapsto (2^x - 2)/(x-1)$ is an increasing function. Therefore, since $\lambda \in (1/2, 1)$, we have $2\lambda < 2$ and thus $\left(2^{2\lambda} - 2\right)/(2\lambda - 1) < 2$, whence $\tau < 3/2$. □

3 Small mass behaviour

We now identify the behaviour of $\psi(z)$ as $z \to 0$.

Proposition 3.1 *There exists $L_0 \in (0, \infty)$ such that*

$$
\lim_{z \to 0} z^\tau\,\psi(z) = L_0. \tag{3.1}
$$

The proof of Proposition 3.1 splits in several steps: we first study the behaviour of

$$H(z) := \int_0^z x \, \psi(x) \, dx \tag{3.2}$$

as $z \to 0$ and show that $z^{\tau-2} \, H(z)$ has a non-negative limit $\ell \in [0, \infty)$ as $z \to 0$. This part of the proof relies on the fact that (1.7) can also be written

$$\gamma \, \left(z \, H'(z) + (\tau - 2) \, H(z)\right) = A(z) - B(z), \quad z \in (0, \infty), \tag{3.3}$$

where

$$A(z) := \int_0^z x^{1+\lambda} \, \psi(x) \, \Psi(z - x) \, dx, \quad z \in (0, \infty), \tag{3.4}$$

$$B(z) := \int_0^z x \, \psi(x) \int_0^{z-x} y^\lambda \, \psi(y) \, dy dx, \quad z \in (0, \infty), \tag{3.5}$$

and

$$\Psi(z) := \int_z^\infty \psi(x) \, dx, \quad z \in (0, \infty). \tag{3.6}$$

In a second step, we study the integrability properties of $x \mapsto x \, \psi(x)$ and Ψ and then deduce that $z \mapsto z^\tau \, \psi(z)$ belongs to $L^\infty(0, 1)$. The final step is devoted to the proof that $\ell > 0$.

Lemma 3.2 *There exist $C_1 > 0$ and $\ell \in [0, \infty)$ such that*

$$H(z) \le C_1 \, z^{2-\tau}, \quad z \in (0, \infty), \tag{3.7}$$

$$\lim_{z \to 0} z^{\tau-2} \, H(z) = \ell. \tag{3.8}$$

Proof. We first notice that (1.7) also reads

$$\gamma \, z^2 \, \psi(z) = A(z) + M_\lambda \, H(z) - B(z), \tag{3.9}$$

whence (3.3) by the definition (1.8) of τ. Next, since $\tau \in [1, 1 + \lambda)$ by Propositions 2.1 and 2.3, we may fix $\sigma \in (\tau - 1, (\tau - 1 + \lambda)/2)$ and recall that $M_\sigma < \infty$ by Proposition 2.1. Since $1 - \sigma > \lambda - \sigma > 0$, we have

$$B(z) = \int_0^z \int_0^{z-x} x^\sigma \, x^{1-\sigma} \, y^\sigma \, y^{\lambda-\sigma} \, \psi(x) \, \psi(y) \, dy dx \le M_\sigma^2 \, z^{1+\lambda-2\sigma}$$

for $z \in (0, 1)$. Since $A \ge 0$, we deduce from (3.3) and the previous upper bound on B that, for $z \in (0, 1)$,

$$\gamma \, \frac{d}{dz} \left(z^{\tau-2} \, H(z)\right) = \gamma \, z^{\tau-3} \, \left(z \, H'(z) + (\tau - 2) \, H(z)\right) \ge -M_\sigma^2 \, z^{\tau+\lambda-2\sigma-2},$$

whence, since $\tau + \lambda - 2\sigma - 2 > -1$,

$$\frac{d}{dz} \left(z^{\tau-2} \, H(z) + \frac{M_\sigma^2}{\gamma} \, \frac{z^{\tau+\lambda-2\sigma-1}}{\tau + \lambda - 2\sigma - 1}\right) \ge 0. \tag{3.10}$$

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On the one hand, we infer from (3.10) after integration over $(z, 1)$, $z \in (0, 1)$, that

$$H(1) + \frac{M_\sigma^2}{\gamma} \frac{1}{\tau + \lambda - 2\sigma - 1} \geq z^{\tau-2} \; H(z) + \frac{M_\sigma^2}{\gamma} \frac{z^{\tau+\lambda-2\sigma-1}}{\tau + \lambda - 2\sigma - 1} \;.$$

Since $H(1) \leq M_1 = 1$, we conclude that there is $C_1 > 0$ such that $z^{\tau-2} \; H(z) \leq C_1$ for $z \in (0, 1)$. Now, if $z \geq 1$, the bound $\tau < 1 + \lambda < 2$ (see Proposition 2.1) ensures that $z^{\tau-2} \; H(z) \leq H(z) \leq M_1 = 1$, and we have thus proved (3.7).

On the other hand, it follows from (3.10) that there exists $\ell \in [0, \infty)$ such that

$$\lim_{z \to 0} \left(z^{\tau-2} \; H(z) + \frac{M_\sigma^2}{\gamma} \frac{z^{\tau+\lambda-2\sigma-1}}{\tau + \lambda - 2\sigma - 1} \right) = \ell \;.$$

Since $\tau - 1 + \lambda > 2\sigma$, $\lim_{z \to 0} z^{\tau+\lambda-2\sigma-1} = 0$, from which (3.8) readily follows. □

We now proceed as in [11, Lemma 4.1] to study the integrability of $x \mapsto x \; \psi(x)$ and $x \mapsto \Psi(x)$.

Lemma 3.3 *Consider $\sigma \in (\tau - 1, 1]$. Then $x \mapsto x \; \psi(x)$ and Ψ belong to $L^{1/\sigma}(0, \infty)$ and there exists $C_2(\sigma) > 0$ such that*

$$\Psi(z) \leq C_2(\sigma) \; z^{-\sigma} \;, \quad z \in (0, \infty) \;. \tag{3.11}$$

Proof. For $\sigma = 1$, Lemma 3.3 follows at once from (1.5). Consider next $\sigma \in (\tau - 1, 1)$. We take $\vartheta \in \mathcal{C}_0^\infty((0, \infty))$ and choose

$$\phi(x) = \int_0^x \vartheta(y) \; y^{-1} \; dy \;, \quad x \in [0, \infty) \;,$$

in (1.6). Setting $p = 1/\sigma$ and $p' = p/(p-1)$, it follows from Proposition 2.1 and the Hölder inequality that

$$\begin{aligned}
&\gamma \left| \int_0^\infty x \; \psi(x) \; \vartheta(x) \; dx \right| \\
&= \left| \int_0^\infty \int_0^\infty x \; K(x, y) \int_x^{x+y} \vartheta(z) \; z^{-1} \; dz \; \psi(x) \; \psi(y) \; dy dx \right| \\
&\leq \|\vartheta\|_{L^{p'}} \int_0^\infty \int_0^\infty x \; K(x, y) \left(\int_x^{x+y} z^{-p} \; dz \right)^{1/p} \psi(x) \; \psi(y) \; dy dx \\
&\leq \|\vartheta\|_{L^{p'}} \int_0^\infty \int_0^\infty \left[x^{1+\lambda} \; x^{-1} \; y^{1/p} + x \; y^\lambda \left(\frac{x^{1-p}}{p-1} \right)^{1/p} \right] \psi(x) \; \psi(y) \; dy dx \\
&\leq \|\vartheta\|_{L^{p'}} \left(1 + (p-1)^{-1/p} \right) M_\lambda \; M_\sigma \;.
\end{aligned}$$

A duality argument then yields that $x \mapsto x \; \psi(x)$ belongs to $L^p(0, \infty)$.

We next notice that we have

$$\Psi(z) = z^{-\sigma} \int_z^\infty z^\sigma \; \psi(x) \; dx \leq z^{-\sigma} \int_z^\infty x^\sigma \; \psi(x) \; dx \leq M_\sigma \; z^{-\sigma}$$

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by Proposition 2.1, whence (3.11). Finally, we deduce from (3.11), Proposition 2.1 and the Fubini theorem that

$$
\begin{aligned}
\int_0^\infty \Psi(z)^{1/\sigma} \, dz &\leq \int_0^\infty \left(C_2(\sigma) \, z^{-\sigma} \right)^{(1/\sigma)-1} \int_z^\infty \psi(x) \, dx dz \\
&\leq C_2(\sigma)^{(1/\sigma)-1} \int_0^\infty \psi(x) \int_0^x z^{\sigma-1} \, dz dx \\
&\leq C_2(\sigma)^{(1/\sigma)-1} \frac{M_\sigma}{\sigma},
\end{aligned}
$$

and the proof of Lemma 3.3 is complete. $\square$

Lemma 3.4 *There is a constant C_3 such that*

$$
\psi(z) \leq C_3 \, z^{-\tau}, \quad z \in (0,1). \tag{3.12}
$$

Proof. We infer from (1.5) and (3.7) that, for $z \in (0,\infty)$,

$$
\int_0^z x \, \psi(x) \int_{z-x}^\infty y^\lambda \, \psi(y) \, dy dx \leq M_\lambda \, H(z) \leq C \, \min\left\{ z^{2-\tau}, 1 \right\}. \tag{3.13}
$$

Next recall that $\tau - 1 < \min\{\lambda, 1/2\}$ by Proposition 2.1 and Proposition 2.3. Hence $\sigma = \min\{1/2, (\lambda + \tau - 1)/2\} > \tau - 1$. Consequently, $x \mapsto x \, \psi(x)$ and Ψ belong to $L^{1/\sigma}(0,\infty)$ by Lemma 3.3. Since $1 - 2\sigma \geq 0$, the Hölder inequality (with $p = 1/\sigma$, $q = 1/\sigma$ and $r = 1/(1-2\sigma)$) yields

$$
\int_0^z x^{1+\lambda} \, \psi(x) \, \Psi(z-x) \, dx \leq z^\lambda \left(\int_0^z [x \, \psi(x)]^{1/\sigma} \, dx \right)^\sigma \|\Psi\|_{L^{1/\sigma}} \, z^{1-2\sigma} \leq C \, z^{1+\lambda-2\sigma}
$$

for $z \in (0,\infty)$. Owing to (3.13) and the above estimate, we deduce from (1.7) that, for $z \in (0,\infty)$,

$$
\psi(z) \leq C \left(z^{\lambda-1-2\sigma} + \min\{z^{-\tau}, z^{-2}\} \right).
$$

Since $\lambda - 1 - 2\sigma \geq -\tau$, the above estimate implies that $\psi(z) \leq C \, z^{-\tau}$ for $z \in (0,1)$, whence (3.12). $\square$

We next turn to the proof of the positivity of ℓ. To this end, we first prove that $M_{-\sigma}$ cannot be finite for large values of σ.

Lemma 3.5 *If $\sigma \in (-1,\infty)$ is such that $\gamma \, (\sigma + 1) > \left(1 + (\sigma+1)^\lambda \right) M_\lambda$, then $M_{-\sigma} = \infty$.*

Proof. Assume for contradiction that $M_{-\sigma} < \infty$ and put $\phi(x) = x^{-(1+\sigma)}$ for $x \in (0,\infty)$. Since $\sigma > -1$, we have

$$
\begin{aligned}
\phi(x) - \phi(x+y) &\leq \phi(x), \\
\phi(x) - \phi(x+y) &\leq (\sigma+1) \, y \, x^{-(\sigma+2)}
\end{aligned}
$$

for $(x,y) \in (0,\infty) \times (0,\infty)$, from which we deduce that

$$
\begin{aligned}
x^{1+\lambda} \, (\phi(x) - \phi(x+y)) &\leq x^{1+\lambda} \, \phi(x)^{1-\lambda} \, (\phi(x) - \phi(x+y))^\lambda \\
&\leq (\sigma+1)^\lambda \, x^{-\sigma} \, y^\lambda
\end{aligned} \tag{3.14}
$$

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and

$$x\, y^\lambda \left(\phi(x) - \phi(x + y)\right) \le x^{-\sigma}\, y^\lambda\,. \tag{3.15}$$

Since $M_{-\sigma}$ and M_λ are both finite, (3.14) and (3.15) actually imply that we may take ϕ as a test function in (1.6) (consider as before an approximating sequence of functions $\phi_\varepsilon \in C_b^1([0,\infty))$). Using (3.14) and (3.15), we further obtain that

$$
\begin{aligned}
\gamma\,(\sigma + 1)\, M_{-\sigma} &= -\gamma \int_0^\infty x^2 \phi'(x)\psi(x)dx \\
&= \int_0^\infty \int_0^\infty \left(x^{1+\lambda} + xy^\lambda\right)\left(\phi(x) - \phi(x+y)\right)\psi(x)\psi(y)dydx \\
&\le (\sigma + 1)^\lambda M_\lambda\, M_{-\sigma} + M_\lambda\, M_{-\sigma} \\
&\le \left[1 + (\sigma + 1)^\lambda\right]\, M_\lambda\, M_{-\sigma}\,,
\end{aligned}
$$

whence a contradiction when $\gamma\,(\sigma + 1) > \left[1 + (\sigma + 1)^\lambda\right]\, M_\lambda$. $\qquad\square$

We show in the next lemma that, if $\ell = 0$, ψ enjoys some regularizing properties for small z.

Lemma 3.6 *Assume that there are $\alpha \ge -\tau$ and $C > 0$ such that*

$$\psi(z) \le C\, z^\alpha\,, \quad z \in (0, 1/2)\,, \tag{3.16}$$

and recall that ℓ is defined in (3.8). Then there exists a constant $C(\alpha) > 0$ such that

$$\psi(z) \le \frac{M_\lambda}{\gamma}\, \ell\, z^{-\tau} + C(\alpha)\, \omega_\alpha(z)\,, \quad z \in (0, 1/2)\,, \tag{3.17}$$

where $\omega_\alpha(z) = z^{1+\lambda+2\alpha}$ if $\alpha < -1$, $\omega_{-1}(z) = z^{\lambda-1}\, |\ln(z)|$ and $\omega_\alpha(z) = z^{\lambda+\alpha}$ if $\alpha > -1$.

Proof. We split the proof into three cases.

Case 1: $\alpha < -1$. Since $M_1 = 1$ it follows from (3.16) that, for $z \in (0, 1/2)$,

$$\Psi(z) = \int_z^\infty \psi(x)dx \le C \int_z^{1/2} x^\alpha\, dx + 2 \int_{1/2}^\infty x\, \psi(x) \le C(\alpha)\, z^{1+\alpha} + 2 \le C(\alpha)\, z^{1+\alpha}\,.$$

Recalling that A is given by (3.4), the previous upper bound on Ψ and (3.16) imply that, for $z \in (0, 1/2)$,

$$A(z) \le C(\alpha) \int_0^z x^{1+\lambda+\alpha}\, (z - x)^{1+\alpha}\, dx \le C(\alpha)\, z^{3+\lambda+2\alpha}\,.$$

Here we have used the fact that $1 + \alpha > -1$ since $\alpha \ge -\tau > -2$. We then infer from (3.3) that

$$\frac{d}{dz}\left(z^{\tau-2}\, H(z)\right) \le \frac{z^{\tau-3}}{\gamma}\, A(z) \le C(\alpha)\, z^{\tau+\lambda+2\alpha}\,.$$

Since $\tau + \lambda + 2\alpha \ge \lambda - \tau > -1$, we may integrate the above inequality between 0 and $z \in (0, 1/2)$ and use (3.8) to deduce that

$$
\begin{aligned}
z^{\tau-2}\, H(z) &\le \ell + C(\alpha)\, z^{\tau+\lambda+2\alpha+1}\,, \\
H(z) &\le \ell\, z^{2-\tau} + C(\alpha)\, z^{3+\lambda+2\alpha}\,.
\end{aligned}
$$

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Using the previous upper bounds on H and A, we finally conclude from (3.3) that

$$\psi(z) \leq \frac{M_\lambda}{\gamma} \frac{H(z)}{z^2} + \frac{A(z)}{\gamma z^2} \leq \frac{M_\lambda}{\gamma} \ell z^{-\tau} + C(\alpha) z^{1+\lambda+2\alpha},$$

whence (3.17) for $\alpha < -1$.

Case 2: $\alpha = -1$. In that case, it follows from (1.5) and (3.16) that $\Psi(z) \leq C |\ln(z)|$ for $z \in (0, 1/2)$. We then proceed as in Case 1 to conclude that (3.17) holds true for $\alpha = -1$.

Case 3: $\alpha > -1$. In that case, $\psi \in L^1(0, 1/2)$, which, together with (1.5), implies that $\Psi \in L^\infty(0, \infty)$. We then proceed as in Case 1 to complete the proof of Lemma 3.6. $\qquad\square$

Lemma 3.7 *There holds $\ell > 0$, where $\ell \in [0, \infty)$ is defined in (3.8).*

Proof. Assume for contradiction that $\ell = 0$. We first prove that

$$\psi(z) \leq C z^{-1}, \quad z \in (0, 1/2). \tag{3.18}$$

Since (3.18) is clearly true if $\tau = 1$, we consider now the case $\tau > 1$. Introducing the sequence $(\alpha_k)_{k \geq 0}$ defined by $\alpha_0 = -\tau$ and $\alpha_{k+1} = 2\alpha_k + \lambda + 1$ for $k \geq 0$, we notice that $\alpha_k = 2^k(\lambda + 1 - \tau) - (\lambda + 1)$ for $k \geq 0$ and, since $\tau < 1 + \lambda$ by Proposition 2.1, $(\alpha_k)_{k \geq 0}$ is an increasing sequence such that $\alpha_k \to \infty$ as $k \to \infty$. In particular, there exists a unique $k_0 \geq 0$ such that $\alpha_{k_0} < -1 \leq \alpha_{k_0+1}$.

We next claim that

$$\psi(z) \leq C(k) z^{\alpha_k+1}, \quad z \in (0, 1/2), \tag{3.19}$$

for each $k \in \{0, \ldots, k_0\}$. Indeed, we argue by induction and first consider the case $k = 0$. Owing to Lemma 3.4, the bound (3.16) holds true for $\alpha = \alpha_0 < -1$ and, since we have assumed that $\ell = 0$, Lemma 3.6 implies that the assertion (3.19) is true for $k = 0$. Assume now that (3.19) holds true for some $k \in \{0, \ldots, k_0 - 1\}$. Then ψ enjoys the property (3.16) with $\alpha = \alpha_{k+1} < -1$ and Lemma 3.6 (with $\ell = 0$) ensures that the assertion (3.19) is true for $k + 1$.

Having proved (3.19), we apply (3.19) with $k = k_0$ and conclude that

$$\psi(z) \leq C z^{\alpha_{k_0}+1} \leq C z^{-1}$$

for $z \in (0, 1/2)$. Therefore, (3.18) is also valid for $\tau > 1$.

Now, thanks to (3.18), we are in a position to apply Lemma 3.6 with $\alpha = -1$ and deduce that, since $\ell = 0$,

$$\psi(z) \leq C z^{\lambda-1} |\ln(z)|, \quad z \in (0, 1/2).$$

Choosing $\lambda' \in (\lambda/2, \lambda)$, the previous upper bound yields

$$\psi(z) \leq C z^{\lambda'-1}, \quad z \in (0, 1/2). \tag{3.20}$$

Introducing the sequence $(\beta_k)_{k \geq 0}$ defined by $\beta_k = \lambda' - 1 + k\lambda$, $k \geq 0$, we claim that

$$\psi(z) \leq C z^{\beta_k}, \quad z \in (0, 1/2), \tag{3.21}$$

for each $k \geq 0$. By (3.20), the assertion (3.21) is clearly true for $k = 0$. We next argue by induction and assume that (3.21) is satisfied for some $k \geq 0$. Since $\beta_k > \beta_0 > -1$, we infer from Lemma 3.6 with $\alpha = \beta_k$ (and $\ell = 0$) that (3.21) holds true for $k + 1$, which completes the proof of (3.21).

Since $\beta_k \to \infty$ as $k \to \infty$, it readily follows from (3.21) that $\psi \in L^1(0, 1/2; x^{-\sigma} dx)$ for each $\sigma > 0$, which, together with (1.5), implies that $M_{-\sigma} < \infty$ for each $\sigma > 0$ and contradicts Lemma 3.5. Therefore, $\ell > 0$. $\qquad\square$

We are now in a position to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. By Proposition 2.1, we may fix $\delta \in (0, 1 + \lambda - \tau)$. Due to Proposition 2.3, $\sigma = \tau + \delta > 1$. For $z \in (0, 1/2)$, we infer from (3.11) and (3.12) that, since $\sigma - 1 \in (\tau - 1, 1)$,

$$z^{\tau-2} \, A(z) \leq C(\sigma) \, z^{\tau-2} \, \int_0^z x^{1+\lambda-\tau} \, (z-x)^{1-\sigma} \, dx \leq C(\sigma) \, z^{1+\lambda-\sigma} \, .$$

We used here that $1 + \lambda - \tau > -1$, while $1 - \sigma > -1$. Thanks to the choice of δ, $1 + \lambda - \sigma > 0$ and we realize that $z^{\tau-2} \, A(z) \to 0$ as $z \to 0$. Similarly, since $1 - \tau > \lambda - \tau > -1$, it follows from (3.12) that, for $z \in (0, 1/2)$,

$$z^{\tau-2} \, B(z) \leq C z^{\tau-2} \int_0^z x^{1-\tau} dx \int_0^{z-x} y^{\lambda-\tau} dy \leq C \, z^{\tau-2} \int_0^z x^{1-\tau} \, (z-x)^{1+\lambda-\tau} \, dx \leq C \, z^{1+\lambda-\tau} \, ,$$

and $\lim_{z\to 0} z^{\tau-2} \, B(z) = 0$. We now multiply (3.9) by $z^{\tau-2}$ and pass to the limit as $z \to 0$ with the help of (3.8) to obtain (3.1) with $L_0 := (M_\lambda \, \ell)/\gamma$. $\qquad\square$

As a consequence of Proposition 3.1, we can exclude that τ is equal to 1. This has already been shown in [5] for $\lambda \geq 0.7$ by obtaining an explicit lower bound for τ. The proof we give now does not provide such a lower bound but warrants that $\tau > 1$ for every $\lambda \in (0, 1)$.

Proposition 3.8 *There holds $\tau > 1$.*

Proof. Assume for contradiction that $\tau = 1$ and fix $z \in (0, \infty)$. We take $\phi(x) = \max\{x, z\}^{-1}$, $x \in (0, \infty)$, in (1.6) to obtain

$$
\begin{aligned}
\gamma \, \Psi(z) &= \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \, [x \, \phi(x) + y \, \phi(y) - (x+y) \, \phi(x+y)] \, \psi(x) \, \psi(y) \, dy dx \\
&= \frac{1}{2} \int_z^\infty \int_z^\infty K(x, y) \, \psi(x) \, \psi(y) \, dy dx \\
&\quad + \frac{1}{2z} \int_z^\infty \int_0^z y \, K(x, y) \, \psi(x) \, \psi(y) \, dy dx \\
&\quad + \frac{1}{2z} \int_0^z \int_z^\infty x \, K(x, y) \, \psi(x) \, \psi(y) \, dy dx \\
&\quad + \frac{1}{2z} \int_0^z \int_{z-x}^z (x+y-z) \, K(x, y) \, \psi(x) \, \psi(y) \, dy dx \\
&= \Psi(z) \left(\int_z^\infty x^\lambda \, \psi(x) \, dx + \frac{1}{z} \int_0^z x^{1+\lambda} \, \psi(x) \, dx \right) \\
&\quad + \frac{1}{z} \left(\int_0^z x \, \psi(x) \, dx \right) \left(\int_z^\infty y^\lambda \, \psi(y) \, dy \right) + \frac{1}{z} \int_0^z \int_{z-x}^z (x+y-z) \, x^\lambda \, \psi(x) \, \psi(y) \, dy dx \, .
\end{aligned}
$$

Since $\tau = 1$, it follows from (1.8) that $\gamma = M_\lambda$ and the above equality becomes

$$\left(\int_0^z x^\lambda \, \psi(x) \, dx - \frac{1}{z} \int_0^z x^{1+\lambda} \, \psi(x) \, dx \right) \Psi(z) = \frac{R(z)}{z} \tag{3.22}$$

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with

$$R(z) := \left(\int_0^z x\ \psi(x)\ dx \right) \left(\int_z^\infty y^\lambda\ \psi(y)\ dy \right) + \int_0^z \int_{z-x}^z (x+y-z)\ x^\lambda\ \psi(x)\ \psi(y)\ dydx\,.$$

On the one hand, for $z \in (0,1)$, we have

$$\int_0^z \int_{z-x}^z (x+y-z)\ x^\lambda\ \psi(x)\ \psi(y)\ dydx \quad \le \quad C \int_0^z \int_{z-x}^z y\ x^\lambda\ x^{-1}\ y^{-1}\ dydx$$
$$\le \quad C\ z^{1+\lambda}$$

by Lemma 3.4 and the assumption that $\tau = 1$, and

$$\lim_{z\to 0} \frac{1}{z} \left(\int_0^z x\ \psi(x)\ dx \right) \left(\int_z^\infty y^\lambda\ \psi(y)\ dy \right) = L_0\ M_\lambda$$

by (1.5) and Proposition 3.1. Therefore,

$$\lim_{z\to 0} \frac{R(z)}{z} = L_0\ M_\lambda\,. \tag{3.23}$$

On the other hand, the assumption that $\tau = 1$ and Proposition 3.1 entail that, as $z \to 0$,

$$\left(\int_0^z x^\lambda\ \psi(x)\ dx - \frac{1}{z} \int_0^z x^{1+\lambda}\ \psi(x)\ dx \right) \quad \sim \quad L_0 \int_0^z \left(1 - \frac{x}{z} \right) x^{\lambda-1}\ dx$$
$$\sim \quad \frac{L_0}{\lambda\ (\lambda+1)}\ z^\lambda. \tag{3.24}$$

Therefore, by (3.22), (3.23) and (3.24) we have

$$\lim_{z\to 0} z^\lambda\ \Psi(z) = \lambda\ (\lambda+1)\ M_\lambda > 0\,. \tag{3.25}$$

But $\psi(z) \sim L_0\ z^{-1}$ as $z \to 0$ implies that $\Psi(z) \sim L_0\ |\ln z|$ as $z \to 0$, which clearly contradicts (3.25) since $\lambda > 0$. Consequently, $\tau > 1$. $\qquad\square$

We end this section by identifying the second term of the expansion of $\psi(z)$ as $z \to 0$.

Proof of Corollary 1.3. Since $\tau > 1$ by Proposition 3.8, we deduce from (3.1) that

$$(A-B)(z) \sim J\ L_0^2\ \frac{(\lambda+2-2\tau)}{(\tau-1)(1+\lambda-\tau)}\ z^{3+\lambda-2\tau} + o\left(z^{3+\lambda-2\tau} \right)$$

as $z \to 0$, where J is defined in Corollary 1.3 and A and B by (3.4) and (3.5), respectively. Since

$$\gamma\ \frac{d}{dz} \left(z^{\tau-2}\ H(z) \right) = z^{\tau-3}\ (A-B)(z)$$

by (3.3) and $\tau < 1 + \lambda$, we further deduce from (3.8) that

$$H(z) = \ell\ z^{2-\tau} + \frac{J\ L_0^2}{\gamma}\ \frac{(\lambda+2-2\tau)}{(\tau-1)(1+\lambda-\tau)^2}\ z^{3+\lambda-2\tau} + o\left(z^{3+\lambda-2\tau} \right)$$

as $z \to 0$. Inserting the expansions of $A - B$ and H just obtained in (3.9), we are led to (1.12). $\square$

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4 Large mass behaviour

We first establish the finiteness of some exponential moments of ψ.

Lemma 4.1 *Set $\alpha_0 := 2^{-\gamma}\gamma^{\gamma-1}$. Then for all $\alpha \in [0, \alpha_0)$,*

$$\int_1^\infty e^{\alpha x}\ \psi(x)\ dx < \infty\,. \tag{4.1}$$

Proof. For $a > 1$ and $\alpha \in (0, \infty)$, we define

$$\Phi_a(\alpha) := \int_0^\infty \frac{x^2}{x \wedge a} e^{\alpha(x \wedge a)}\psi(x)dx \tag{4.2}$$

with the notation $x \wedge a := \min\{x, a\}$. By (1.5), Φ_a is well-defined and differentiable on $[0, \infty)$ and satisfies

$$1 = M_1 \le \Phi_a(0) = \int_0^a x\psi(x)dx + a^{-1}\int_a^\infty x^2\psi(x)dx \le 1 + M_2/a\,. \tag{4.3}$$

Furthermore, using (1.6) with $\phi'(x) = e^{\alpha(x \wedge a)}$, we obtain

$$\begin{aligned}
\Phi_a'(\alpha) &= \int_0^\infty x^2 e^{\alpha(x \wedge a)}\psi(x)dx \\
&= \gamma^{-1}\int_0^\infty \int_0^\infty (x^{1+\lambda} + xy^\lambda)\int_x^{x+y} e^{\alpha(z \wedge a)}dz\ \psi(y)\psi(x)dydx \\
&\le \gamma^{-1}\int_0^\infty \int_0^\infty (x^{1+\lambda} + xy^\lambda)y e^{\alpha(x \wedge a)}e^{\alpha(y \wedge a)}\psi(y)\psi(x)dydx \\
&\le 2\gamma^{-1}\int_0^\infty x^{1+\lambda}e^{\alpha(x \wedge a)}\psi(x)dx \int_0^\infty y e^{\alpha(y \wedge a)}\psi(y)dy \\
&\le 2\gamma^{-1}\Phi_a(\alpha)\int_0^\infty x^\lambda\ \mu_{\alpha,a}(x)\ dx\,,
\end{aligned}$$

where $\mu_{\alpha,a}(x) := x\ e^{\alpha(x \wedge a)}\ \psi(x)$, $x \in (0, \infty)$. Clearly, $\mu_{\alpha,a} \in L^1(0, \infty)$ by (1.5) and we deduce from the Jensen inequality that

$$\begin{aligned}
\|\mu_{\alpha,a}\|_{L^1}\int_0^\infty x^\lambda \frac{\mu_{\alpha,a}(x)}{\|\mu_{\alpha,a}\|_{L^1}}\ dx &\le \|\mu_{\alpha,a}\|_{L^1}^{1-\lambda}\left\{\int_0^\infty x\ \mu_{\alpha,a}(x)\ dx\right\}^\lambda \\
&\le \Phi_a(\alpha)^{1-\lambda}\Phi_a'(\alpha)^\lambda.
\end{aligned}$$

Therefore, $[\Phi_a'(\alpha)]^{1-\lambda} \le 2\gamma^{-1}\Phi_a(\alpha)^{2-\lambda}$, whence, since $\gamma = 1/(1 - \lambda)$,

$$\Phi_a'(\alpha)\Phi_a(\alpha)^{-1-\gamma} \le 2^\gamma\gamma^{-\gamma}.$$

After integration, we obtain that, for all $\alpha \in [0, \alpha_0(a))$ with $\alpha_0(a) := 2^{-\gamma}\gamma^{\gamma-1}\Phi_a(0)^{-\gamma}$,

$$\Phi_a(\alpha) \le \left[\Phi_a(0)^{-\gamma} - 2^\gamma\gamma^{1-\gamma}\ \alpha\right]^{-1/\gamma}.$$

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Now, fix $\alpha \in (0, \alpha_0)$. Since $\lim_{a \to \infty} \Phi_a(0) = 1$ by (4.3), the right-hand side of the above inequality is bounded from above by a constant which does not depend on a large enough. Consequently, the Fatou Lemma implies that

$$\int_1^\infty e^{\alpha x} \, \psi(x) \, dx \; \leq \; \limsup_{a \to \infty} \int_1^a e^{\alpha x} \, \psi(x) \, dx \leq \limsup_{a \to \infty} \Phi_a(\alpha) < \infty,$$

which completes the proof. $\qquad\square$

From (1.7) and (4.1), we deduce that ψ decays at least exponentially fast at infinity.

Proposition 4.2 *Set $\varrho_0 := \alpha_0/4 = 2^{-\gamma-2}\gamma^{\gamma-1}$. Then for any $\varrho \in (0, \varrho_0)$, there exists $C(\varrho) > 0$ such that*

$$\psi(z) \leq C(\varrho) \, e^{-\varrho z} \quad for \quad z \in [1, \infty) \, .$$

Proof. Since $\tau - 1 < 1/2$ by Proposition 2.3, Lemma 3.3 ensures that both $x \mapsto x \, \psi(x)$ and Ψ belong to $L^2(0, \infty)$. We then infer from (1.7), (3.13) and the Hölder inequality that, for $z \in (0, \infty)$,

$$\gamma \, z^2 \, \psi(z) \leq z^\lambda \, \left(\int_0^z [x \, \psi(x)]^2 \, dx \right)^{1/2} \, \|\Psi\|_{L^2} + C \, z^{1-\lambda} \, \min\{z^{1+\lambda-\tau}, z^{\lambda-1}\} \, ,$$

whence

$$\psi(z) \leq C \, \left(z^{\lambda-2} + z^{-(1+\lambda)} \right) \, , \quad z \in (0, \infty) \, , \tag{4.4}$$

since $\tau - 1 < \lambda < 1$. Note also that it follows easily from Lemma 4.1 and (1.5) that for every $\varepsilon > 0$, $\alpha \in [0, \alpha_0)$ and $p \geq 0$, we have

$$A(\varepsilon, \alpha, p) = \int_\varepsilon^\infty x^p e^{\alpha x} \psi(x) dx < \infty. \tag{4.5}$$

Next, for $z \geq 1$, it follows from (1.7), Lemma 3.3 and (4.4) that for $\alpha \in [0, \alpha_0)$,

$$
\begin{aligned}
\gamma \, z^2 \, \psi(z) \;\leq\; & \int_0^{z/2} x^{1+\lambda}\psi(x)dx \int_{z/2}^\infty \psi(y)dy + \int_0^{z/2} x\psi(x)dx \int_{z/2}^\infty y^\lambda\psi(y)dy \\
& + \int_{z/2}^z x^{1+\lambda} \, \psi(x) \, \Psi(z-x) \, dx + \int_{z/2}^z x \, \psi(x) \, dx \int_{z-x}^\infty y^\lambda\psi(y)dy \\
\;\leq\; & M_{1+\lambda}e^{-(\alpha z)/2}A(1/2, \alpha, 0) + M_1 e^{-(\alpha z)/2} A(1/2, \alpha, \lambda) \\
& + z^\lambda \, \left(\int_{z/2}^z [x \, \psi(x)]^2 \, dx \right)^{1/2} \|\Psi\|_{L^2} + M_\lambda \, e^{-(\alpha z)/2} \, A(1/2, \alpha, 1) \\
\;\leq\; & C(\alpha) \, e^{-(\alpha z)/2} + C \, z^\lambda \, \left(\int_{z/2}^z x^2 \, \left[x^{\lambda-2} + x^{-(1+\lambda)} \right] \psi(x) \, dx \right)^{1/2} \\
\;\leq\; & C(\alpha) \, e^{-(\alpha z)/2} + z^\lambda e^{-(\alpha z)/4} \left[A(1/2, \alpha, \lambda) + A(1/2, \alpha, 1 - \lambda) \right]^{1/2} \\
\;\leq\; & C(\alpha)z^\lambda e^{-(\alpha z)/4} \, .
\end{aligned}
$$

The above inequality readily implies that, for any $\alpha \in [0, \alpha_0)$, there exists a constant $C(\alpha) > 0$ such that $\psi(z) \leq C(\alpha) \, e^{-(\alpha z)/4}$ for $z \geq 1$, whence the expected result. $\qquad\square$

Corollary 4.3 *For each $p \geq \tau$, there exists $C(p) > 0$ such that*

$$\psi(z) \leq C(p) \, z^{-p}, \quad z \in (0, \infty). \tag{4.6}$$

Proof. Since

$$\psi(z) \leq C \, z^{-\tau} \leq C \, z^{-p} \quad \text{for} \quad p \geq \tau \quad \text{and} \quad z \in (0, 1)$$

by Lemma 3.4, Corollary 4.3 is a straightforward consequence of Proposition 4.2. $\qquad\square$

We end this section with the proof of (1.11).

Lemma 4.4 *There exists $\alpha_1 > 0$ such that*

$$\int_1^\infty e^{\alpha_1 x} \psi(x) dx = \infty. \tag{4.7}$$

Proof. We argue by contradiction. Assume thus that $\psi \in L^1(1, \infty; e^{\alpha x} dx)$ for every $\alpha > 0$. Then, for $i = 0, 1$, the map

$$\Phi_i(\alpha) := \int_1^\infty x^i e^{\alpha x} \psi(x) dx$$

is well-defined and belongs to $\mathcal{C}^1([0, \infty))$. The strict positivity of ψ ensures that $\Phi_0(0) > 0$ while we observe at once that for each $\alpha \geq 0$,

$$\Phi_0(\alpha) \geq \Phi_0(0) e^\alpha. \tag{4.8}$$

Next, an easy computation using (1.6) shows that

$$
\begin{aligned}
\Phi_1'(\alpha) &= \int_1^\infty x^2 e^{\alpha x} \psi(x) dx \\
&= \gamma^{-1} \int_0^\infty \int_0^\infty (x^{1+\lambda} + xy^\lambda) \int_x^{x+y} e^{\alpha z} \mathbf{1}_{\{z \geq 1\}} dz \; \psi(y) \psi(x) dy dx \\
&\geq \gamma^{-1} \int_1^\infty x^{1+\lambda} e^{\alpha x} \psi(x) dx \int_1^\infty \frac{e^{\alpha y} - 1}{\alpha} \psi(y) dy
\end{aligned}
$$

On the one hand, the Jensen inequality implies that

$$
\begin{aligned}
\int_1^\infty x^{1+\lambda} e^{\alpha x} \psi(x) dx &\geq \Phi_0(\alpha) \int_1^\infty x^{1+\lambda} \left[\Phi_0(\alpha)^{-1} e^{\alpha x} \psi(x) \right] dx \\
&\geq \Phi_0(\alpha) \left[\int_1^\infty x \Phi_0(\alpha)^{-1} e^{\alpha x} \psi(x) dx \right]^{1+\lambda} \\
&\geq \Phi_0(\alpha)^{-\lambda} \Phi_1(\alpha)^{1+\lambda}
\end{aligned}
$$

On the other hand, since $e^x - 1 \geq e^x/2$ for $x \geq 1$, we have

$$\int_1^\infty \frac{e^{\alpha y} - 1}{\alpha} \psi(y) dy \geq \frac{1}{2\alpha} \int_1^\infty e^{\alpha y} \psi(y) dy = \frac{1}{2\alpha} \Phi_0(\alpha)$$

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for $\alpha \geq 1$. Combining the above inequalities and using (4.8), we end up with

$$\Phi_1'(\alpha) \geq \Phi_1(\alpha)^{1+\lambda}\frac{\Phi_0(\alpha)^{1-\lambda}}{2\gamma\alpha} \geq \Phi_1(\alpha)^{1+\lambda}\frac{\Phi_0(0)^{1-\lambda}e^{(1-\lambda)\alpha}}{2\gamma\alpha} \geq \varepsilon\Phi_1(\alpha)^{1+\lambda},$$

for $\alpha \geq 1$, where $\varepsilon = \inf_{\alpha \geq 1}(\Phi_0(0)^{1-\lambda}e^{(1-\lambda)\alpha})/(2\gamma\alpha) > 0$. Since $\Phi_1(1) > 0$ (recall that ψ is strictly positive on $(0,\infty)$), this classically implies that there exists $\alpha_1 \in (0,\infty)$ such that $\lim_{\alpha \nearrow \alpha_1} \Phi(\alpha) = \infty$, whence a contradiction. $\qquad\square$

5 Regularity of ψ in $(0,\infty)$

We finally study the smoothness of ψ. The main difficulty we face here is the singularity of ψ for small mass and the proof of the $\mathcal{C}^1$-smoothness of ψ turns out to be rather technical. One could probably show that ψ is $\mathcal{C}^\infty$-smooth on $(0,\infty)$ as conjectured by the physicists, but this could be rather technical and we have been unable to prove it.

Theorem 5.1 *The function ψ is $\mathcal{C}^1$-smooth on $(0,\infty)$.*

We first prove that ψ is Hölder continuous.

Lemma 5.2 *The function ψ is continuous on $(0,\infty)$. More precisely, there is a constant C such that for any $z \in (0,\infty)$ and $h \in (0,\infty)$,*

$$\left|(z+h)^2\psi(z+h) - z^2\psi(z)\right| \leq C\,h^{2-\tau}\,. \tag{5.1}$$

Proof. The identity (1.7) reads

$$\gamma z^2\psi(z) = F(z) + G(z)\,, \quad z \in (0,\infty)\,, \tag{5.2}$$

with

$$F(z) = \int_0^z x^{1+\lambda}\,\psi(x)\int_{z-x}^\infty \psi(y)\,dydx\,,$$

$$G(z) = \int_0^z x\,\psi(x)\int_{z-x}^\infty y^\lambda\,\psi(y)\,dy\,.$$

Since $\tau \in (1,2)$ by Proposition 3.8, it follows from (4.6) with $p = 1+\lambda$ and $p = \tau$ that,

$$\begin{aligned}
|F(z+h) - F(z)| &\leq \int_z^{z+h} x^{1+\lambda}\,\psi(x)\int_{z+h-x}^\infty \psi(y)dydx \\
&\quad + \int_0^z x^{1+\lambda}\,\psi(x)\int_{z-x}^{z+h-x} \psi(y)\,dy \\
&\leq C\int_z^{z+h}(z+h-x)^{1-\tau}\,dx \\
&\quad + C\int_0^z \left[(z-x)^{1-\tau} - (z+h-x)^{1-\tau}\right]dx \\
&\leq C\,h^{2-\tau} + C\left(h^{2-\tau} + z^{2-\tau} - (z+h)^{2-\tau}\right) \\
&\leq C\,h^{2-\tau}\,.
\end{aligned}$$

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Similarly, we infer from (1.5) and (4.6) with $p = 1 + \lambda$ and $p = \tau$ that

$$
\begin{aligned}
|G(z+h) - G(z)| \;\le\; & \int_z^{z+h} x\,\psi(x) \int_{z+h-x}^\infty y^\lambda\,\psi(y)\,dydx \\
& + \int_0^z x\,\psi(x) \int_{z-x}^{z+h-x} y^\lambda\,\psi(y)\,dydx \\
\le\; & C\,M_\lambda \int_z^{z+h} x^{1-\tau}\,dx + C \int_0^z x^{-\lambda}\,(z-x)^{\lambda-1} \int_{z-x}^{z+h-x} y\,\psi(y)\,dydx \\
\le\; & C\,h^{2-\tau} + C \int_0^z x^{-\lambda}\,(z-x)^{\lambda-1} \left[(z+h-x)^{2-\tau} - (z-x)^{2-\tau}\right]\,dx \\
\le\; & C\,h^{2-\tau}\left(1 + \int_0^z x^{-\lambda}\,(z-x)^{\lambda-1}\,dx\right) \\
\le\; & C\,h^{2-\tau}\left(1 + \int_0^1 u^{-\lambda}\,(1-u)^{\lambda-1}\,du\right) \\
\le\; & C\,h^{2-\tau}.
\end{aligned}
$$

Therefore, (5.1) holds true, so that $z \mapsto z^2\psi(z)$ is continuous on $(0,\infty)$ and the proof of Lemma 5.2 is complete. $\qquad\square$

Proof of Theorem 5.1. Obviously, it suffices to show that $z \mapsto z^2\psi(z) \in \mathcal{C}^1((0,\infty))$. Differentiating (1.7) formally, we see that the first derivative D of $x \mapsto \gamma\,x^2\psi(x)$ (if any) should be given by $D = D_1 + D_2 - D_3$ where

$$
D_1(x) \;:=\; \int_0^x d_1(x,y)\,dy\,, \tag{5.3}
$$

$$
d_1(x,y) \;:=\; \mathbf{1}_{(0,x)}(y)\,\psi(y)\,\left(x^{1+\lambda}\,\psi(x) - (x-y)^{1+\lambda}\,\psi(x-y)\right)\,, \tag{5.4}
$$

$$
D_2(x) \;:=\; x^{1+\lambda}\,\psi(x)\,\Psi(x) + M_\lambda\,x\,\psi(x)\,, \tag{5.5}
$$

$$
D_3(x) \;:=\; \int_0^x y^\lambda\,(x-y)\,\psi(y)\,\psi(x-y)\,dy \tag{5.6}
$$

for $x \in (0,\infty)$.

Step 1: We prove that $D(x)$ is well-defined for $x \in (0,\infty)$ and that $D \in L^1(0,z)$ for $z \in (0,\infty)$. For that purpose, we first note the following consequence of Lemma 5.2. For $\alpha \in (0,2)$, there exists a constant $C(\alpha) > 0$ such that

$$
|z_2^\alpha\,\psi(z_2) - z_1^\alpha\,\psi(z_1)| \le C(\alpha)\,\left(z_2^{\alpha-2} + z_1^{2-\tau}\,\min\{z_1,z_2\}^{\alpha+\tau-4}\right)\,|z_2 - z_1|^{2-\tau} \tag{5.7}
$$

for $(z_1, z_2) \in (0,\infty) \times (0,\infty)$. Indeed, we infer from (4.6) (with $p = \tau$) and Lemma 5.2 that

$$
\begin{aligned}
|z_2^\alpha\,\psi(z_2) - z_1^\alpha\,\psi(z_1)| \;\le\; & z_2^{\alpha-2}\,\left|z_2^2\,\psi(z_2) - z_1^2\,\psi(z_1)\right| + z_1^2\,\psi(z_1)\,\left|z_2^{\alpha-2} - z_1^{\alpha-2}\right| \\
\le\; & C\,z_2^{\alpha-2}\,|z_2 - z_1|^{2-\tau} + C\,z_1^{2-\tau}\,\min\{z_1,z_2\}^{\alpha+\tau-4}\,\left|z_2^{2-\tau} - z_1^{2-\tau}\right|\,,
\end{aligned}
$$

whence (5.7).

We now fix $z \in (0,\infty)$. By (4.6) (with $p = \tau$) and (5.7) (with $\alpha = 1 + \lambda$), we have

$$
|d_1(x,y)| \le C\,y^{-\tau}\,\left(x^{\lambda-1} + (x-y)^{\lambda-1}\right)\,y^{2-\tau}\,\mathbf{1}_{(0,x)}(y) \le C\,y^{2-2\tau}\,\left(x^{\lambda-1} + (x-y)^{\lambda-1}\right)\,\mathbf{1}_{(0,x)}(y)
$$

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for $x \in (0, z)$, so that

$$d_1 \in L^1((0, z) \times (0, z)), \tag{5.8}$$

since $2 - 2\tau > -1$ by Proposition 2.3 and $\lambda > 0$. As a straightforward consequence of (5.8), we deduce that $D_1 \in L^1(0, z)$. Next, since $\tau > 1$ by Proposition 3.8, we infer from (4.6) (with $p = \tau$) that

$$D_2(x) \leq C \, x^{2+\lambda-2\tau} + C \, x^{1-\tau},$$

and $D_2 \in L^1(0, z)$. Similarly, since $1 - \tau > \lambda - \tau > -1$, it follows from (4.6) with $p = \tau$ that

$$D_3(x) \leq C \int_0^x y^{\lambda-\tau}(x-y)^{1-\tau} dx \leq C x^{2+\lambda-2\tau} \in L^1(0, z).$$

Consequently, $D \in L^1(0, z)$.

Step 2: We now check that D is indeed the first derivative of $z \mapsto \gamma \, z^2 \, \psi(z)$. We fix $z \in (0, \infty)$. Since $d_1 \in L^1((0, z) \times (0, z))$ by (5.8), the Fubini theorem yields

$$\begin{aligned}
\int_0^z D_1(x) \, dx &= \int_0^z \int_y^z d_1(x, y) \, dxdy \\
&= \int_0^z \psi(y) \int_y^z \left(M'_{1+\lambda}(x) - M'_{1+\lambda}(x-y) \right) \, dxdy \\
&= \int_0^z \psi(y) \left(M_{1+\lambda}(z) - M_{1+\lambda}(z-y) - M_{1+\lambda}(y) \right) \, dy \\
&= \int_0^z \psi(y) \int_{z-y}^z x^{1+\lambda} \, \psi(x) \, dxdy - \int_0^z \psi(y) \, M_{1+\lambda}(y) \, dy.
\end{aligned}$$

Owing to (4.6) and the bounds on τ, we may use again the Fubini theorem and obtain

$$\int_0^z D_1(x) \, dx = \int_0^z x^{1+\lambda} \, \psi(x) \int_{z-x}^z \psi(y) \, dydx - \int_0^z x^{1+\lambda} \, \psi(x) \int_x^z \psi(y) \, dydx. \tag{5.9}$$

It also follows from the Fubini theorem that

$$\begin{aligned}
\int_0^z D_3(x) \, dx &= \int_0^z y^\lambda \, \psi(y) \int_y^z (x-y) \, \psi(x-y) \, dxdy \\
&= \int_0^z y^\lambda \, \psi(y) \int_0^{z-y} x \, \psi(x) \, dxdy \\
&= \int_0^z x \, \psi(x) \int_0^{z-x} y^\lambda \, \psi(y) \, dydx. \tag{5.10}
\end{aligned}$$

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Now, by (5.9) and (5.10) we have

$$
\begin{aligned}
\int_0^z D(x)\ dx &= \int_0^z x^{1+\lambda}\ \psi(x) \int_{z-x}^z \psi(y)\ dydx - \int_0^z x^{1+\lambda}\ \psi(x) \int_x^z \psi(y)\ dydx \\
&\quad + \int_0^z x^{1+\lambda}\ \psi(x) \int_x^\infty \psi(y)\ dy + \int_0^z x\ \psi(x)\ dx \int_0^\infty y^\lambda \psi(y) dy \\
&\quad - \int_0^z x\ \psi(x) \int_0^{z-x} y^\lambda\ \psi(y)\ dydx \\
&= \int_0^z x^{1+\lambda}\ \psi(x) \int_{z-x}^\infty \psi(y)\ dydx + \int_0^z x\ \psi(x) \int_{z-x}^\infty y^\lambda\ \psi(y)\ dydx \\
&= \gamma\ z^2\ \psi(z)
\end{aligned}
$$

by (1.7).

Step 3: We finally show that $D \in \mathcal{C}((0,\infty))$ and study separately D_1, D_2 and D_3.
Let $z \in (0,\infty)$ and $h \in (0,\infty)$. On the one hand, it follows from (4.6) (with $p = \tau$) and (5.7) (with $\alpha = 1 + \lambda$) that

$$
\begin{aligned}
\int_z^{z+h} |d_1(z+h,x)|\ dx &\leq C \int_z^{z+h} x^{-\tau} \left((z+h)^{\lambda-1} + (z+h-x)^{\lambda-1} \right) x^{2-\tau}\ dx \\
&\leq C\ z^{2-2\tau} \left(h\ z^{\lambda-1} + h^\lambda \right).
\end{aligned} \tag{5.11}
$$

On the other hand, the continuity of ψ implies that

$$
\lim_{h \to 0} d_1(z+h,x) = d_1(z,x)
$$

for $x \in (0,z)$, while a further use of (4.6) (with $p = \tau$) and (5.7) (with $\alpha = 1 + \lambda$) entails that, for $x \in (0,z)$ and $h \in (0,\infty)$,

$$
\begin{aligned}
|d_1(z+h,x) - d_1(z,x)| &\leq (|d_1(z+h,x)| + |d_1(z,x)|) \\
&\leq C\ x^{-\tau} \left((z+h)^{\lambda-1} + (z+h-x)^{\lambda-1} + z^{\lambda-1} + (z-x)^{\lambda-1} \right) x^{2-\tau} \\
&\leq C\ x^{2-2\tau}\ (z-x)^{\lambda-1} \in L^1(0,z).
\end{aligned}
$$

We then deduce from the Lebesgue dominated convergence theorem that

$$
\lim_{h \to 0} \int_0^z |d_1(z+h,x) - d_1(z,x)|\ dx = 0.
$$

Noting that

$$
|D_1(z+h) - D_1(z)| \leq \int_z^{z+h} |d_1(z+h,x)|\ dx + \int_0^z |d_1(z+h,x) - d_1(z,x)|\ dx,
$$

and recalling (5.11), we conclude that $\lim_{h \to 0} |D_1(z+h) - D_1(z)| = 0$. Consequently, $D_1 \in \mathcal{C}((0,\infty))$.

Next, the continuity of D_2 on $(0, \infty)$ obviously follows from that of ψ and Ψ.

Finally, for $z \in (0, \infty)$ and $h \in (0, \infty)$, we have

$$|D_3(z+h) - D_3(z)| \le \int_z^{z+h} |d_3(z+h, x)| \; dx + \int_0^z |d_3(z+h, x) - d_3(z, x)| \; dx \,,$$

where $d_3(z, x) := x^\lambda \, (z - x) \, \psi(x) \, \psi(z - x) \, \mathbf{1}_{(0,z)}(x)$. Owing to the continuity of ψ, we have

$$\lim_{h \to 0} d_3(z+h, x) = d_3(z, x)$$

for $x \in (0, z)$. We then infer from (4.6) (with $p = \tau$) that, for $x \in (0, z)$,

$$
\begin{aligned}
|d_3(z+h, x) - d_3(z, x)| &\le |d_3(z+h, x)| + |d_3(z, x)| \\
&\le C \, x^{\lambda - \tau} \, \left((z + h - x)^{1-\tau} + (z-x)^{1-\tau} \right) \\
&\le C \, x^{\lambda - \tau} \, (z - x)^{1-\tau} \in L^1(0, z) \,,
\end{aligned}
$$

recalling that $1 - \tau > \lambda - \tau > -1$ by Proposition 2.1. We are thus in a position to apply the Lebesgue dominated convergence theorem and obtain that

$$\lim_{h \to 0} \int_0^z |d_3(z+h, x) - d_3(z, x)| \; dx = 0 \,.$$

We finally notice that, by (4.6) (with $p = \tau$), we have

$$\int_z^{z+h} |d_3(z+h, x)| \; dx \le C \int_z^{z+h} x^{\lambda - \tau} \, (z-x)^{1-\tau} \; dx \le C \, z^{\lambda - \tau} \, h^{2-\tau} \underset{h \to 0}{\longrightarrow} 0 \,.$$

Consequently, $\lim_{h \to 0} |D_3(z+h) - D_3(z)| = 0$, whence $D_3 \in \mathcal{C}((0, \infty))$ and the proof of Theorem 5.1 is complete. □

Gathering the outcome of Proposition 2.1, Proposition 2.3, Proposition 3.1, Proposition 3.8, Proposition 4.2, Lemma 4.4 and Theorem 5.1, we conclude that Theorem 1.2 holds true.

A   Some useful inequalities

We devote this last section to a sketch of the proofs of the inequalities (2.1) and (2.4).

Proof of the first inequality of (2.1). It of course suffices to show this inequality when $0 < x < y$. Dividing this inequality by y^λ, we realize that it is enough to show that $f(u) = 2^{1-\lambda} \, (1+u)^\lambda - 1 - u^\lambda \ge 0$ for $u \in (0, 1)$. This is straightforward since f is a non-increasing function on $(0, 1)$ and $f(1) = 0$. □

Proof of the second inequality of (2.1). We observe that

$$\frac{(xy)^\lambda}{(x+y)^\lambda} + (x+y)^\lambda - x^\lambda - y^\lambda = \left((x+y)^\lambda - y^\lambda \right) \, \left(1 - \frac{x^\lambda}{(x+y)^\lambda} \right) \ge 0$$

for $(x, y) \in (0, \infty) \times (0, \infty)$. □

Proof of (2.4). We fix $\lambda \in (1/2, 1)$. By symmetry, it suffices to consider the case where $0 < y \leq x$. Dividing (2.4) by $x^{2\lambda}$, we conclude that it is enough to check that

$$f(x) := 1 + x^{2\lambda} + (2^{2\lambda} - 2)x^{\lambda} - (1 + x)^{2\lambda} \geq 0, \quad x \in (0, 1] \,.$$

We first obtain

$$g(x) := x^{1-\lambda} f'(x)/2\lambda = x^{\lambda} + 2^{2\lambda-1} - 1 - x^{1-\lambda}(1+x)^{2\lambda-1} \,.$$

Differentiating again, we get

$$h(x) := x^{1-\lambda} g'(x) = \beta(x/(1+x)) \,,$$

where, for $u \in [0, 1/2]$,

$$\beta(u) = \lambda - (1 - \lambda)/u^{2\lambda-1} - (2\lambda - 1)u^{2-2\lambda}$$

Easy computations show that $\beta'(u) > 0$ for $u \in (0, 1/2)$ with $\beta(0) = -\infty$ and $\beta(1/2) > 0$. We deduce that there exists $u_0 \in (0, 1/2)$ such that $\beta(u) < 0$ on $(0, u_0)$, $\beta(u_0) = 0$, while $\beta(u) > 0$ on $(u_0, 1/2)$. The map $x \mapsto x/(1+x)$ being an increasing one-to-one mapping from $(0, 1)$ onto $(0, 1/2)$, we deduce that there exists $x_0 \in (0, 1)$ such that $h(x) < 0$ on $(0, x_0)$, $h(x_0) = 0$, while $h(x) > 0$ on $(x_0, 1)$. This implies that $g'(x) < 0$ on $(0, x_0)$, $g'(x_0) = 0$, while $g'(x) > 0$ on $(x_0, 1)$. Since $g(0) > 0 = g(1)$, we deduce that there exists $x_1 \in (0, 1)$ such that $g(x) > 0$ on $(0, x_1)$, $g(x_1) = 0$, while $g(x) < 0$ on $(x_1, 1)$. This of course ensures that $f'(x) > 0$ on $(0, x_1)$, $f'(x_1) = 0$, while $f'(x) < 0$ on $(x_1, 1)$. Since $f(0) = f(1) = 0$, the conclusion follows. $\square$

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