A distance for coagulation

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Abstract

We show the link between two distances which allow to obtain some good existence and uniqueness results concerning respectively stochastic coalescence and Smoluchowski's coagulation equation.

Key words : Stochastic coalescence, Coagulation equation, Uniqueness.

1   Introduction

We consider an infinite system of particles characterized by their masses $m \in (0, \infty)$. The only mechanism taken into account by our models is coalescence by pairs: two particles with masses m and m_* merge into a single particle with mass $m + m_*$ at a given rate $K(m, m_*)$. The coagulation kernel K is thus a nonnegative function on $(0, \infty)^2$ satisfying $K(m, m_*) = K(m_*, m) \geq 0$ for all m, m_* in $(0, \infty)$.

Two different situations then have to be separated.
(a) The particles are microscopic (their masses are actually of order 0), and the rate of coalescence is infinitesimal (but proportionnal to K). Then the quantity of interest is the deterministic function $c = (c(t, m))_{t \geq 0, m \in (0, \infty)}$, where $c(t, m)$ stands for the concentration of particles with mass m at time t. In such a case, c solves the famous Smoluchowski coagulation equation, which is a deterministic nonlinear integro-differential equation. The total mass of the system at time t is then given by $\int_0^\infty mc(t, m)dm$.
(b) The particles are macroscopic, and the rate of coalescence is exactly given by K. In such a context, we would like to study the evolution of the stochastic process $M = (M_1(t), M_2(t), ...)_{t \geq 0}$, which describes the evolution of the masses in the system. The total mass of the system at time t is then given by $\sum_{i \geq 1} M_i(t)$.

We refer to Aldous [1] for a review on these topics.

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The object of this paper is to summarize the results of [3], and to show that [4] follows the same spirit. In [3], a distance between two stochastic coalescents M and $\tilde{M}$ is introduced, to derive some existence and uniqueness results. The results of [3] seem to improve previous results (see e.g. those of [2]). In [4], a distance between two solutions c and $\tilde{c}$ to Smoluchowski's equation allows us to obtain also some good existence an uniqueness results, improving for example in many cases those of [6]. While the distance used in the case of stochastic coalescence is intuitively reasonnable, the one used for Smoluchowski's equation is less transparent. We would like to show here why these distances are actually strongly related.

The plan of the paper is the following: in Section 2, we recall the definitions concerning the stochastic coalescent, and the main uniqueness result of [3]. We next give in Section 3 the definition of Smoluchowski's equation, the distance used in [4], and some fundammental inequalities with elements of proofs. We finally show in Section 4 that in some sense, the scheme followed in [4] is the same as that of [3].

We refer to [3] and [4] for more detailed and more general statements and results.

2 Stochastic coalescence

Our aim is to study some infinite stochastic particle systems $(M(t))_{t \geq 0}$ undergoing coalescence. We will order the particles according to their masses, and assume that the total mass of the system is finite, so that the state space of our process will be the following:

$$\mathcal{S} = \left\{ m = (m_n)_{n \geq 0}, \quad m_1 \geq m_2 \geq ... \geq 0, \ \sum_{n=1}^{\infty} m_n < \infty \right\}. \tag{1}$$

For $1 \leq i < j$, the coalescence between the i-th and j-th particles is described by the map $c_{ij} : \mathcal{S} \mapsto \mathcal{S}$, with

$$c_{ij}(m) = \text{reordered}(m_1, m_2, ..., m_{i-1}, m_i + m_j, m_{i+1}, ..., m_{j-1}, m_{j+1}, m_{j+2}, ...). \tag{2}$$

One may now define the stochastic coalescents.

Definition 2.1 *Let K be a coagulation kernel. A càdlàg (for the pointwise convergence topology) $\mathcal{S}$-valued Markov process $(M(t))_{t \geq 0}$ is said to be a K-stochastic coalescent if its infinitesimal generator $\mathcal{L}$ is given, for any $\Phi : \mathcal{S} \to \mathcal{S}$ sufficiently regular, any $m \in \mathcal{S}$, by*

$$\mathcal{L}\Phi(m) = \sum_{1 \leq i < j < \infty} K(m_i, m_j) \left[\Phi(c_{ij}(m)) - \Phi(m) \right]. \tag{3}$$

This means that at each time $t \geq 0$, each pair of particles (with masses m and m_*) in the system merge into a single particle (with mass $m + m_*$) at rate $K(m, m_*)$.

One of the main problems is to show the uniqueness (in law) of such a process. To this aim, we consider, for $\lambda \in (0, 1]$ and for two states m and $\tilde{m}$ in $\mathcal{S}$, the distance (and moment of order λ)

$$\delta_\lambda(m, \tilde{m}) = \sum_{i \geq 1} |m_i^\lambda - \tilde{m}_i^\lambda|, \quad \mathcal{M}_\lambda(m) = \sum_{i \geq 1} m_i^\lambda. \tag{4}$$

One of the main results of [3] can be stated in the following way.

Theorem 2.2 *For some $\lambda \in (0, 1]$ and some $a > 0$, assume that the coagulation kernel satisfies, for all x, y, u, v in $(0, \infty)$,*

$$|K(x, y) - K(u, v)| \leq a \left(|x^\lambda - u^\lambda| + |y^\lambda - v^\lambda| \right). \tag{5}$$

Consider two initial conditions $M(0) \in \mathcal{S}$ and $\tilde{M}(0) \in \mathcal{S}$ such that $\mathcal{M}_\lambda(M(0)) + \mathcal{M}_\lambda(\tilde{M}(0)) < \infty$. Then it is possible to build two K-stochastic coalescents $(M(t))_{t \geq 0}$ and $(\tilde{M}(t))_{t \geq 0}$, starting respectively from $M(0)$ and $\tilde{M}(0)$, such that:
1. the maps $t \mapsto \mathcal{M}_\lambda(M(t))$ and $t \mapsto \mathcal{M}_\lambda(\tilde{M}(t))$ are a.s. nonincreasing,
2. for all $t \geq 0$,

$$\frac{d}{dt} E\left[\delta_\lambda(M(t), \tilde{M}(t))\right] \leq 2aE\left[\left(\mathcal{M}_\lambda(M(t)) + \mathcal{M}_\lambda(\tilde{M}(t))\right) \delta_\lambda(M(t), \tilde{M}(t))\right]. \tag{6}$$

A standard consequence of such an inequality is of course existence and uniqueness (in law) of the K-stochastic coalescent for a deterministic initial state admitting a moment of order λ. This improves consequently the results of [2], so that the use of the distance δ_λ seems to be well-adapted.

Sketch of proof We only give the main steps of the proof, refering to [3] for more details. It is possible to build simultaneously the two processes M and $\tilde{M}$ (using a well-chosen Poisson measure), in such a way that the coalescence between the i-th and j-th larger particles do coalesce simultaneously in both systems as soon as possible. These processes satisfy point 1, because the moment of order λ decreases at each coalescence: for all $m \in \mathcal{S}$, all $1 \leq i < j$, $\mathcal{M}_\lambda(c_{ij}(m)) \leq \mathcal{M}_\lambda(m)$. This is due to the fact that, since $\lambda \leq 1$, $(m_i + m_j)^\lambda \leq m_i^\lambda + m_j^\lambda$.

We then obtain that for all $t \geq 0$,

$$\frac{d}{dt} E\left[\delta_\lambda(M(t), \tilde{M}(t))\right] = A(t) + B_1(t) + B_2(t), \tag{7}$$

where

$$A(t) = E\Big[\sum_{i<j}\min\Big(K(M_i(t),M_j(t)),K(\tilde{M}_i(t),\tilde{M}_j(t))\Big)$$

$$\Big\{\delta_\lambda(c_{ij}(M(t)),c_{ij}(\tilde{M}(t)))-\delta_\lambda(M(t),\tilde{M}(t))\Big\}\Big],$$

$$B_1(t) = E\Big[\sum_{i<j}\Big(K(M_i(t),M_j(t))-K(\tilde{M}_i(t),\tilde{M}_j(t))\Big)_+$$

$$\Big\{\delta_\lambda(c_{ij}(M(t)),\tilde{M}(t))-\delta_\lambda(M(t),\tilde{M}(t))\Big\}\Big], \qquad (8)$$

with $(x)_+ = \max(x,0)$, and where the expression of B_2 is the same as that of B_1 permuting the roles of M and $\tilde{M}$.

These three terms are quite clear: for example, A and B_1 express the evolution of the distance between the two processes respectively when
(a) in both systems, the i-th and j-th particles do coalesce, which occurs with rate $\min\Big\{K(M_i(t),M_j(t)),K(\tilde{M}_i(t),\tilde{M}_j(t))\Big\}$,
(b) coalescence between the i-th and j-th particles occurs only in the first system, which occurs with rate $\Big(K(M_i(t),M_j(t))-K(\tilde{M}_i(t),\tilde{M}_j(t))\Big)_+$.

Some computations (using that $\lambda \in (0,1]$, see [3]) show that for all m and $\tilde{m}$ in $\mathcal{S}$, for any $i<j$,

$$\delta_\lambda(c_{ij}(m),c_{ij}(\tilde{m}))\leq\delta_\lambda(m,\tilde{m}),\quad \delta_\lambda(c_{ij}(m),\tilde{m})\leq\delta_\lambda(m,\tilde{m})+2m_j^\lambda. \quad (9)$$

Note that the first inequality is very satisfying: when both systems coalesce simultaneously, the distance δ_λ can not increase. We deduce that the term $A(t)$ is always nonpositive. Next, we deduce from (5), the second inequality of (9) and some symmetry arguments, that

$$B_1(t) \leq 2aE\left(\sum_{i<j}\Big\{|M_i^\lambda(t)-\tilde{M}_i^\lambda(t)|+|M_j^\lambda(t)-\tilde{M}_j^\lambda(t)|\Big\}M_j^\lambda(t)\right)$$

$$= aE\Big(\sum_{i\neq j}\Big\{|M_i^\lambda(t)-\tilde{M}_i^\lambda(t)|+|M_j^\lambda(t)-\tilde{M}_j^\lambda(t)|\Big\}$$

$$\min\big(M_i^\lambda(t),M_j^\lambda(t)\big)\Big)$$

$$\leq 2aE\Big(\delta_\lambda(M(t),\tilde{M}(t))\mathcal{M}_\lambda(M(t))\Big). \qquad (10)$$

The same computation holding for B_2, the proof is finished. $\qquad\square$

3 Smoluchowski's coagulation equation

We start with a standard definition of *measure* solutions.

Definition 3.1 *A family $(c_t)_{t \geq 0}$ of nonnegative measures on $(0, \infty)$ is said to be a measure solution to Smoluchowski's equation if for a sufficiently large class of test functions $\phi : (0, \infty) \mapsto \mathbb{R}$ (see [5, 4]), for all $t \geq 0$,*

$$\frac{d}{dt} \int_0^\infty \phi(m) c_t(dm) = \frac{1}{2} \int_0^\infty \int_0^\infty \{\phi(m + m_*) - \phi(m) - \phi(m_*)\}$$
$$K(m, m_*) c_t(dm) c_t(dm_*). \qquad (11)$$

This equation is quite clear, recalling that c_t stands for the concentration distribution of masses in the system: the right hand side member describes the appearance (resp. disappearance) of particles with mass $m + m_*$ (resp. m and m_*) at rate $K(m, m_*)$, proportionally to $c_t(dm)$ and $c_t(dm_*)$. The factor $1/2$ avoids to count twice each pair of masses $\{m, m_*\}$.

For $\lambda \in (0, 1]$ and for two nonnegative measures c and $\tilde{c}$ on $(0, \infty)$, we consider the distance (and moment of order λ)

$$d_\lambda(c, \tilde{c}) = \int_0^\infty m^{\lambda - 1} \left| \int_m^\infty (c(dm_*) - \tilde{c}(dm_*)) \right| dm, \quad M_\lambda(c) = \int_0^\infty m^\lambda c(dm). \tag{12}$$

Easy considerations show that $d_\lambda(c, \tilde{c})$ is well-defined as soon as $M_\lambda(c)$ and $M_\lambda(\tilde{c})$ are finite.

One of the main results of [4] is the following.

Theorem 3.2 *Let $\lambda \in (0, 1]$. Assume that the coagulation kernel K is continuous on $(0, \infty)^2$, that $\partial_m K$ exists almost everywhere, and that for some constants a and b, for all m, m_* in $(0, \infty)$,*

$$K(m, m_*) \leq a(m^\lambda + m_*^\lambda), \quad \min(m^\lambda, m_*^\lambda)|\partial_m K(m, m_*)| \leq b\lambda m^{\lambda - 1} m_*^\lambda. \tag{13}$$

Consider two measure solutions $(c_t)_{t \geq 0}$ and $(\tilde{c}_t)_{t \geq 0}$ to Smoluchowski's equation such that $M_\lambda(c_0) + M_\lambda(\tilde{c}_0) < \infty$. Then
1. the maps $t \mapsto M_\lambda(c_t)$ and $t \mapsto M_\lambda(\tilde{c}_t)$ are nonincreasing,
2. for all $t \geq 0$,

$$\frac{d}{dt} d_\lambda(c_t, \tilde{c}_t) \leq b\left(M_\lambda(c_t) + M_\lambda(\tilde{c}_t)\right) d_\lambda(c_t, \tilde{c}_t) \tag{14}$$

Since all the terms in (14) make sense as soon as $M_\lambda(c_0) + M_\lambda(\tilde{c}_0) < \infty$, an easy consequence is that for any initial condition c_0 such that $M_\lambda(c_0) < \infty$, there exists a unique measure solution to Smoluchowski's equation starting from c_0. This very weak requirement on the initial condition seems to improve previous results, which were always assuming the finiteness of two (or more) moments of the initial condition (except when the kernel K is bounded, see Norris [6], and for the solvable kernels $K(x, y) = x + y$ and $K(x, y) = xy$, see Menon-Pego [5]). Thus the use of the distance d_λ seems to be particularly well-adapted.
Note that for example, then kernels $m^\lambda + m_*^\lambda$ and $(mm_*)^{\lambda/2}$ do satisfy (13).

See [4] for a list of more or less physical kernels satisfying (13).

Sketch of proof We omit here all the technical issues, and give only the main steps of the proof, see [4] for more details. Applying (11) to the solutions c and $\tilde{c}$, making the difference, and using a symmetry argument, we easily check that

$$\frac{d}{dt}\int_0^\infty \phi(m)(c_t - \tilde{c}_t)(dm) = \frac{1}{2}\int_0^\infty \int_0^\infty (\phi(m + m_*) - \phi(m) - \phi(m_*))$$
$$K(m, m_*)(c_t - \tilde{c}_t)(dm)(c_t + \tilde{c}_t)(dm_*). \quad (15)$$

Let us set $F(t, x) = \int_x^\infty c_t(dm)$, $\tilde{F}(t, x) = \int_x^\infty \tilde{c}_t(dm)$, and $E(t, x) = F(t, x) - \tilde{F}(t, x)$. We choose the test function $\phi(m) = \int_0^m x^{\lambda-1}sign(E(t, x))dx$. A first integration by parts shows that

$$\int_0^\infty \phi(m)(c_t - \tilde{c}_t)(dm) = \int_0^\infty m^{\lambda-1}|E(t, m)|dm = d_\lambda(c_t, \tilde{c}_t). \quad (16)$$

We thus obtain, using now an integration by parts (with respect to the variable m) in the right hand side of (15),

$$\frac{d}{dt}d_\lambda(c_t, \tilde{c}_t) = A(t) + B(t), \quad (17)$$

where

$$A(t) = \frac{1}{2}\int_0^\infty \int_0^\infty K(m, m_*)\left(\phi'(m + m_*) - \phi'(m)\right)$$
$$E(t, m)dm(c_t + \tilde{c}_t)(dm_*),$$

$$B(t) = \frac{1}{2}\int_0^\infty \int_0^\infty \partial_m K(m, m_*)\left(\phi(m + m_*) - \phi(m) - \phi(m_*)\right)$$
$$E(t, m)dm(c_t + \tilde{c}_t)(dm_*). \quad (18)$$

First, since $\phi'(m) = m^{\lambda-1}sign(E(t, m))$ and since $m \mapsto m^{\lambda-1}$ is nonincreasing,

$$A(t) = \frac{1}{2}\int_0^\infty \int_0^\infty \left((m + m_*)^{\lambda-1}sign(E(t, m + m_*)E(t, m)) - m^{\lambda-1}\right)$$
$$K(m, m_*)|E(t, m)|dm(c_t + \tilde{c}_t)(dm_*) \le 0. \quad (19)$$

This is very satisfying: this first term is again nonpositive. Next, since

$$|\phi(m + m_*) - \phi(m) - \phi(m_*)| \le \int_m^{m+m_*} x^{\lambda-1}dx + \int_0^{m_*} x^{\lambda-1}dx \le \frac{2}{\lambda}m_*^\lambda, \quad (20)$$

so that by symmetry,

$$|\phi(m + m_*) - \phi(m) - \phi(m_*)| \le \frac{2}{\lambda}\min(m^\lambda, m_*^\lambda), \quad (21)$$

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we deduce, using (13),

$$
\begin{aligned}
B(t) &\leq b \int_0^\infty \int_0^\infty m^{\lambda-1} m_*^\lambda |E(t,m)| dm(c_t + \tilde{c}_t)(dm_*) \\
&= b \left(M_\lambda(c_t) + \tilde{M}_\lambda(\tilde{c}_t) \right) d_\lambda(c_t, \tilde{c}_t).
\end{aligned} \tag{22}
$$

The conclusion follows. The main difficulty omitted here consists in proving rigorously that in the integrations by parts, all the boundary terms vanish. $\square$

4 A link between d_λ and δ_λ

We of course note at once that while the proofs of Theorems 2.2 and 3.2 are very different, their statements are quite similar. Furthermore, comparing (7) and (17), we observe that in both formulae, the first term is nonpositive while the other(s) involve the smoothness of the kernel K (in terms of $|K(m_i, m_j) - K(\tilde{m}_i, \tilde{m}_j)|$ for the stochastic coalescent, and in terms of $\partial_m K(m, m_*)$ for the Smoluchowski equation).

The aim of this section is to show that in some sense, Theorem 3.2 is the *microscopic particles* version of Theorem 2.2. The motivation for this is to uncover the understanding of the statement (and proof) of Theorem 3.2. To this aim, we will only show that d_λ and δ_λ coincide, in some sense, on $\mathcal{S}$.

Remark 4.1 *Let $\lambda \in (0,1]$ be fixed. Consider two states $m = (m_1, m_2, ...)$ and $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, ...)$ in $\mathcal{S}$. Then the corresponding concentration distributions c and $\tilde{c}$ are given by $c(dx) = \sum_{i \geq 1} \delta_{m_i}(dx)$, and $\tilde{c}(dx) = \sum_{i \geq 1} \delta_{\tilde{m}_i}(dx)$. Recalling the notations of (4) and (12), there holds that $\mathcal{M}_\lambda(m) = \tilde{M}_\lambda(c)$, and*

$$
\delta_\lambda(m, \tilde{m}) = \lambda d_\lambda(c, \tilde{c}). \tag{23}
$$

Proof We just have to check (23), for which we have unfortunately not found an easier proof. We first of all denote by

$$
F(x) = \int_x^\infty c(dy) = \sum_{i \geq 1} \mathbb{1}_{\{m_i \geq x\}} = \sum_{i \geq 1} i \mathbb{1}_{x \in (m_{i+1}, m_i]}. \tag{24}
$$

F is a nonincreasing function (from $(0, \infty)$ into $[0, \infty)$). The generalized inverse G of F is given by

$$
G(x) = \sup \{y, F(y) \geq x\} = \sum_{i \geq 1} m_i \mathbb{1}_{\{x \in (i-1, i]\}}. \tag{25}
$$

One easily checks that for all x, y,

$$
G(x) < y \Leftrightarrow x > F(y). \tag{26}
$$

We finally define $\tilde{F}(x)$ and $\tilde{G}(x)$ in the same way, replacing everywhere m and c by $\tilde{m}$ and $\tilde{c}$. We may now start the computations. By definition and due to (25),

$$\delta_\lambda(m,\tilde{m}) \;=\; \sum_{i\geq 1}|m_i^\lambda - \tilde{m}_i^\lambda| = \int_0^\infty |G^\lambda(x) - \tilde{G}^\lambda(x)|dx = \mathrm{Area}(\Delta), \quad (27)$$

where Δ is the domain of $\mathbb{R}^2$ defined by

$$\Delta = \Big\{(x,y),\ x>0,\ y\in \Big(\min(G^\lambda(x),\tilde{G}^\lambda(x)),\max(G^\lambda(x),\tilde{G}^\lambda(x))\Big]\Big\}. \quad (28)$$

But one easily checks that

$$\Delta = \Big\{(x,y),\ y>0,\ x\in \Big(\min(F(y^{1/\lambda}),\tilde{F}(y^{1/\lambda})),\max(F(y^{1/\lambda}),\tilde{F}(y^{1/\lambda}))\Big]\Big\}. \tag{29}$$

Indeed, considering for example the case where $G(x)\leq \tilde{G}(x)$, using (26) and its negation, we obtain that

$$G^\lambda(x) < y \leq \tilde{G}^\lambda(x) \Leftrightarrow G(x) < y^{1/\lambda} \leq \tilde{G}(x)$$
$$\Leftrightarrow x > F(y^{1/\lambda}) \text{ and } x \leq \tilde{F}(y^{1/\lambda}). \quad (30)$$

This implies that

$$\delta_\lambda(m,\tilde{m}) \;=\; \int_0^\infty |F(y^{1/\lambda}) - \tilde{F}(y^{1/\lambda})|dy = \int_0^\infty |F(z) - \tilde{F}(z)|\lambda z^{\lambda-1}dz$$
$$=\; \lambda \int_0^\infty z^{\lambda-1}\left|\int_x^\infty c(dy) - \int_x^\infty \tilde{c}(dy)\right|dz = \lambda d_\lambda(c,\tilde{c}) \quad (31)$$

by definition. We have performed the subsitution $z = y^{1/\lambda}$. $\qquad\square$

References

[1] D.J. Aldous, *Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory for Probabilists*, Bernoulli, 5, p 3-48, 1999.

[2] S.N. Evans, J. Pitman, *Construction of Markovian coalescents*, Ann. Inst. H. Poincaré Probab. Statist. 34, no. 3, p 339-383, 1998.

[3] N. Fournier, *On some stochastic coalescents*, Preprint 2005/11 de l'Institut Elie Cartan.

[4] N. Fournier, Ph. Laurençot, *Uniqueness results for some Smoluchowski coagulation equations*, in preparation.

[5] G. Menon,R.L. Pego, *Approach to self-similarity in Smoluchowski's coagulation equations,* Comm. Pure Appl. Math. 57, no. 9, 1197–1232, 2004.

[6] J.R. Norris, *Smoluchowski's coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent*, Ann. Appl. Probab., vol. 9, no1, 78-109, 1999.