ON THE UNIQUENESS FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITH A STRONG ANGULAR SINGULARITY

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Abstract. We prove an inequality on the Wasserstein distance with quadratic cost between two solutions of the spatially homogeneous Boltzmann equation without angular cutoff, from which we deduce some uniqueness results. In particular, we obtain a local (in time) well-posedness result in the case of (possibly very) soft potentials. A global well-posedeness result is shown for all regularized hard and soft potentials without angular cutoff. Our uniqueness result seems to be the first one applying to a strong angular singularity, except in the special case of Maxwell molecules.

Our proof relies on the ideas of Tanaka [15]: we give a probabilistic interpretation of the Boltzmann equation in terms of a stochastic process. Then we show how to couple two such processes started with two different initial conditions, in such a way that they almost surely remain close to each other.

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1. Introduction and main results

1.1. **The Boltzmann equation.** Let $f(t,v)$ be the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ in a spatially homogeneous dilute gas. Then under some assumptions, f solves the Boltzmann equation

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v-v_*|,\theta)\big[f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)\big], \tag{1.1}$$

where the pre-collisional velocities are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1.2}$$

and θ is the so-called *deviation angle* defined by $\cos\theta = \frac{(v-v_*)}{|v-v_*|}\cdot\sigma$. The *collision kernel* $B = B(|v-v_*|,\theta) = B(|v'-v'_*|,\theta)$ depends on the nature of the interactions between particles.

This equation is quite natural: it says that for each $v \in \mathbb{R}^3$, new particles with velocity v appear due to a collision between two particles with velocities v' and v'_*, at rate $B(|v'-v'_*|,\theta)$, while particles with velocity v disappear because they collide with another particle with velocity v_*, at rate $B(|v-v_*|,\theta)$. See Desvillettes [3] and Villani [19] for more much more details.

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Since the collisions are assumed to be elastic, conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1), that is for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\,\phi(v)\,dv = \int_{\mathbb{R}^3} f_0(v)\,\phi(v)\,dv, \qquad \phi(v) = 1, v, |v|^2, \tag{1.3}$$

and we classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)\,dv = 1$.

1.2. **Assumptions on the collision kernel.** We will assume that for some functions $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\beta : (0,\pi] \mapsto (0,\infty)$,

$$B(|v-v_*|,\theta)\,\sin\theta = \Phi(|v-v_*|)\,\beta(\theta). \tag{**A1**}$$

In the case of an interaction potential $V(s) = 1/r^s$, with $s \in (2,\infty)$, one has

$$\Phi(z) = z^\gamma, \quad \beta(\theta) \overset{0}{\sim} \text{cst}\,\theta^{-1-\nu}, \quad \text{with } \gamma = \frac{s-5}{s-1} \in (-3,1), \quad \nu = \frac{2}{s-1} \in (0,2). \tag{1.5}$$

On classically names *hard potentials* the case when $\gamma \in (0,1)$ (*i.e.*, $s > 5$), *Maxwellian molecules* the case when $\gamma = 0$ (*i.e.*, $s = 5$), *moderately soft potentials* the case when $\gamma \in (-1,0)$ (*i.e.*, $s \in (3,5)$), and *very soft potentials* the case when $\gamma \in (-3,-1)$ (*i.e.*, $s \in (2,3)$).

In any case, $\int_{0+} \beta(\theta)d\theta = +\infty$, which expresses the affluence of *grazing collisions*, that is collisions with a very small deviation. We will assume here the general physically reasonnable conditions

$$\int_0^\pi \beta(\theta)d\theta = +\infty, \quad \kappa_1 := \int_0^\pi \theta^2\,\beta(\theta)d\theta < +\infty. \tag{**A2**}$$

We now introduce, for $\theta \in (0,\pi]$,

$$H(\theta) := \int_\theta^\pi \beta(x)dx \quad \text{and} \quad G(z) := H^{-1}(z). \tag{1.7}$$

Here H is a continuous deacreasing bijection from $(0,\pi]$ into $[0,+\infty)$, and its inverse function $G : [0,+\infty) \mapsto (0,\pi]$ is defined by $G(H(\theta)) = \theta$, and $H(G(z)) = z$. We will suppose that there exists $\kappa_2 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty (G(z/x) - G(z/y))^2\,dz \leq \kappa_2 \frac{(x-y)^2}{x+y}. \tag{**A3**}$$

Concerning the velocity part of the cross section, we will assume that for all $x, y \in \mathbb{R}_+$,

$$\begin{aligned} \min(x^2,y^2)\frac{[\Phi(x)-\Phi(y)]^2}{\Phi(x)+\Phi(y)} &+ (x-y)^2[\Phi(x)+\Phi(y)] \\ &+ \min(x,y)|x-y||\Phi(x)-\Phi(y)| \quad \leq \quad (x-y)^2[\Psi(x)+\Psi(y)]. \end{aligned} \tag{1.9}$$

for some function $\Psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, with for some $\gamma \in (-3,0]$, some $\kappa_3 > 0$, for all $x \in \mathbb{R}_+$,

$$\Psi(x) \leq \kappa_3 x^\gamma. \tag{**A4**(γ)}$$

Under Assumption ((**A4**)(γ)), we can easy see that necessarily for all $x \in \mathbb{R}_+$, $\Phi(x) \leq \Psi(x)$, and then $\Phi(x) \leq \kappa_3 x^\gamma$.

These assumptions are not very transparent. However, the following lemma, proved in the appendix, shows how they apply. Roughly, **(A3)** is very satisfying, **(A4**(0)**)** corresponds to *regularized* velocity cross sections, while **(A4**(γ)**)** allows us to deal with general soft potentials.

Lemma 1.1. *(i) Assume that there are* $0 < c < C$ *and* $\nu \in (0,2)$ *such that for all* $\theta \in (0,\pi]$, $c\theta^{-\nu-1} \leq \beta(\theta) \leq C\theta^{-\nu-1}$*. Then* **(A2-A3)** *hold.*
(ii) Assume that $\Phi(x) = \min(x^\alpha, A)$ *for some* $A > 0$*, some* $\alpha \in \mathbb{R}$*, or that* $\Phi(x) = (\varepsilon + x)^\alpha$ *for some* $\varepsilon > 0$*,* $\alpha < 0$*. Then* **(A4**(0)**)** *holds.*
(iii) Assume that for some $\gamma \in (-3,0]$*,* $\Phi(x) = x^\gamma$*. Then* **(A4**(γ)**)** *holds.*

1.3. **Goals, existing results and difficulties.** We study in this paper the well-posedness of the spatially homogeneous Boltzmann equation for singular collision kernel as introduced above. In particular we are interested in uniqueness and stability with respect to the initial condition.

In the case of a collision kernel with angular cutoff, that is when $\int_0^\pi \beta(\theta)d\theta < +\infty$, there are some optimal existence and uniqueness results: see Mischler-Wennberg [14] and Lu-Mouhot [13].

The case of collision kernels without cutoff is much more difficult, but is very important, since it corresponds to the previously described physical collision kernels. This difficulty is not surprising: on each compact time interval, each particle collides with infinitely (resp. finitely) many others in the case without (resp. with) cutoff.

In all the previously cited physical situations, global existence of weak solutions has been proved by Villani [18] by using some compactness methods.

Until recently, the only uniqueness result obtained for non cutoff collision kernel was concerning Maxwellian molecules, studied successively by Tanaka [15], Horowitz-Karandikar [12], Toscani-Villani [17]: it was proved in [17] that uniqueness holds for the Boltzmann equation as soon as Φ is constant and **(A2)** is met, for any initial (measure) datum with finite mass and energy, that is $\int_{\mathbb{R}^3}(1 + |v|^2)\, f_0(dv) < +\infty$.

There has been recently three papers in the case where β is non cutoff and Φ is not constant. The case where Φ is bounded (together with additionnal regularity assumptions) was treated in [8], for essentially any initial (measure) datum such that $\int_{\mathbb{R}^d}(1 + |v|)f_0(dv) < \infty$. More realistic collision kernels have been treated by Desvillettes-Mouhot [5] and Fournier-Mouhot [11] (including hard and moderately soft potentials). However, all these results apply only when assuming the following condition, stronger than **(A2)**,

$$\int_0^\pi \theta\beta(\theta)d\theta < \infty. \tag{1.11}$$

In particular, this does not apply to very soft potentials ($s \in (2,3]$). Weighted Sobolev spaces were used in [5], while the results of [11] rely on the Kantorovich-Rubinsten distance.

In the present paper, we obtain the first uniqueness result which can deal with the case where only **(A2)** is supposed. Our result is based on the use of the Wasserstein distance with quadratic cost. The main interest of our paper concerns very soft potentials, for which we obtain the uniqueness of the solution provided it remains in $L^p(\mathbb{R}^3)$, for some p large enough. Since we are only able to propagate locally such a property, we obtain some local (in time) well-posedness result.

Our method certainly applies to the case of hard potentials. We however do not treat this case in the present paper, since there are already some available uniqueness results, as said previously.

Let us also mention that in a companion paper, we use a similar method to get some uniqueness result for the Landau equation with soft potentials, which was still open.

Our proof is probabilistic, and we did not manage to rewrite it in an analytic way. The main idea is quite simple: for two solutions $(f_t)_{t\geq 0}$, $(\tilde{f}_t)_{t\geq 0}$ to the Boltzmann equation, we construct two stochastic processes $(V_t)_{t\geq 0}$ and $(\tilde{V}_t)_{t\geq 0}$ whose time marginal laws are given by $(f_t)_{t\geq 0}$ and $(\tilde{f}_t)_{t\geq 0}$, and which are coupled in such a way that $E[|V_t - \tilde{V}_t|^2]$ is "small" for all times. This bounds from above the Wasserstein distance with quadratic cost between f_t and $\tilde{f}_t$.

1.4. **Notation.** Let us denote by C^2_∞ (resp. C^2_b, resp. C^2_c) the set of C^2-functions $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ of which the second derivative is bounded (resp. of which the derivatives of order 0 to 2 are bounded, resp. which are compactly supported).

Let also $L^p(\mathbb{R}^3)$ be the space of measurable functions f with $\|f\|_{L^p(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} f^p(v)\,dv\right)^{1/p} < +\infty$. Let $\mathcal{P}(\mathbb{R}^3)$ be the set of probability measures on $\mathbb{R}^3$, and

$$\mathcal{P}_2(\mathbb{R}^3) = \left\{f \in \mathcal{P}(\mathbb{R}^3),\ m_2(f) < \infty\right\} \quad \text{with} \quad m_2(f) := \int_{\mathbb{R}^3} |v|^2 f(dv).$$

For $\alpha \in (-3, 0]$, we introduce the space $\mathcal{J}_\alpha$ of probability measures f on $\mathbb{R}^3$ such that

$$J_\alpha(f) \;:=\; \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha f(dv_*) < \infty. \tag{1.12}$$

Of course, for any probability measure f, $J_0(f) = 1$. Let $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3))$, $L^1([0,T], L^p(\mathbb{R}^3))$ and $L^1([0,T], \mathcal{J}_\alpha)$ be the sets of measurable families $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^3$ with $\sup_{[0,T]} m_2(f_t) < +\infty$, $\int_0^T \|f_t\|_{L^p(\mathbb{R}^3)}\,dt < +\infty$, and $\int_0^T J_\alpha(f_t)dt < +\infty$ respectively.

1.5. **Weak solutions.** We follow here [9]. For each $X \in \mathbb{R}^3$, we introduce $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is an orthonormal basis of $\mathbb{R}^3$. We also require that $I(-X) = -I(X)$ and $J(-X) = -J(X)$ for convenience. For $X, v, v_* \in \mathbb{R}^3$, for $\theta \in [0,\pi]$ and $\varphi \in [0, 2\pi)$, we set

$$\begin{cases} \Gamma(X,\varphi) := (\cos\varphi) I(X) + (\sin\varphi) J(X), \\ v' := v'(v, v_*, \theta, \varphi) := v - \frac{1-\cos\theta}{2}(v - v_*) + \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ v'_* := v'_*(v, v_*, \theta, \varphi) := v_* + \frac{1-\cos\theta}{2}(v - v_*) - \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ a := a(v, v_*, \theta, \varphi) := (v' - v) = -(v'_* - v_*), \end{cases} \tag{1.13}$$

which is nothing but a suitable spherical parameterization of (1.2): we write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|}\cos\theta + \frac{I(v-v_*)}{|v-v_*|}\sin\theta\cos\varphi + \frac{J(v-v_*)}{|v-v_*|}\sin\theta\sin\varphi$.

Let us observe at once that

$$v'(v_*, v, \theta, \varphi) = v'_*(v, v_*, \theta, \varphi), \qquad v'_*(v_*, v, \theta, \varphi) = v'(v, v_*, \theta, \varphi), \tag{1.14}$$

$$v'(v, v_*, \pi - \theta, \varphi) = v'_*(v, v_*, \theta, \varphi + \pi), \qquad v'_*(v, v_*, \pi - \theta, \varphi) = v'(v, v_*, \theta, \varphi + \pi). \tag{1.15}$$

Let us define the notion of weak solutions we shall use.

Definition 1.2. *Let B be a collision kernel which satisfies* **(A1-A2)**. *A family $f = (f_t)_{t\in[0,T]} \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3))$ is a weak solution to (1.1) if*

$$\int_0^T dt \int_{\mathbb{R}^3} f_t(dv) \int_{\mathbb{R}^3} f_t(dv_*)\,\Phi(|v - v_*|)\,|v - v_*|^2 < +\infty, \tag{1.16}$$

and if for any $\phi \in C^2_\infty$*, and any* $t \in [0,T]$*,*

$$\int_{\mathbb{R}^3} \phi(v)\, f_t(dv) = \int_{\mathbb{R}^3} \phi(v)\, f_0(dv) + \int_0^t ds \int_{\mathbb{R}^3} f_s(dv) \int_{\mathbb{R}^3} f_s(dv_*)\, \mathcal{A}\phi(v,v_*), \tag{1.17}$$

where

$$\mathcal{A}\phi(v,v_*) = \frac{\Phi(|v-v_*|)}{2} \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi\, [\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)]\,. \tag{1.18}$$

As noted by Villani [18, p 291], one has, for all $v, v_* \in \mathbb{R}^3$, all $\theta \in [0,\pi]$, all $\phi \in C^2_\infty$,

$$\left| \int_0^{2\pi} d\varphi\, [\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)] \right| \le C||\phi''||_\infty \theta^2 |v - v_*|^2, \tag{1.19}$$

so that thanks to assumption **(A2)**, (1.16) ensures that all the terms in (1.17) are well-defined. The proof of (1.19) is given in the appendix for the sake of completeness.

1.6. **A suitable distance.** Let us now introduce the distance on $\mathcal{P}_2(\mathbb{R}^3)$ we shall use. For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g,\tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal g and second marginal $\tilde{g}$. We then set

$$\begin{aligned} W_2(g,\tilde{g}) &= \inf \left\{ \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|^2\, G(dv,d\tilde{v}) \right)^{1/2}, \quad G \in \mathcal{H}(g,\tilde{g}) \right\} \\ &= \min \left\{ \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|^2\, G(dv,d\tilde{v}) \right)^{1/2}, \quad G \in \mathcal{H}(g,\tilde{g}) \right\}. \end{aligned} \tag{1.20}$$

This distance is the so-called Wasserstein distance with quadratic cost. We refer to Villani [20, Chapter 2] for more details on this distance.

Our result is based on the use of this distance. A remarkable result, due to Tanaka [15], is that in the Maxwellian case, that is when $\Phi \equiv 1$, $t \mapsto W_2(f_t, \tilde{f}_t)$ is nonincreasing for each pair of reasonnable solutions $f, \tilde{f}$ to the Boltzmann equation.

1.7. **The main results.** Our main result is the following inequality.

Theorem 1.3. *Assume* **(A1-A2-A3-A4(**γ**))** *for some* $\gamma \in (-3,0]$*. Let us consider two weak solutions* $(f_t)_{t\in[0,T]}, (\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) lying in* $L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T],\mathcal{J}_\gamma)$*. Assume furthermore than for all* $t\in[0,T]$*,* f_t *(or* $\tilde{f}_t$*) has a density with respect to the Lebesgue measure on* $\mathbb{R}^3$*. There exists a constant* $K = K(\kappa_1,\kappa_2,\kappa_3)$ *such that for all* $t \in [0,T]$*,*

$$W_2(f_t,\tilde{f}_t) \le W_2(f_0,\tilde{f}_0) \exp\left(K \int_0^t J_\gamma(f_s + \tilde{f}_s) ds \right). \tag{1.21}$$

Observe here that the technical assumption that f_t has a density can easily be removed, provided one has some uniform estimates on $J_\gamma(f_t)$, as will be the case in the applications below.

We first give some application to the case of mollified velocity cross sections.

Corollary 1.4. *Assume* **(A1-A2-A3-A4(**0**))***. For any* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$*, any* $T>0$*, there exists a unique weak solution* $(f_t)_{t\in[0,T]} \in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$ *to (1.1). Furthermore, there exists a constant* $K = K(\kappa_1,\kappa_2,\kappa_3)$ *such that for any pair of weak solutions* $(f_t)_{t\in[0,T]}$ *and* $(\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) in* $L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$*, any* $t>0$*,*

$$W_2(f_t,\tilde{f}_t) \le W_2(f_0,\tilde{f}_0)e^{Kt}. \tag{1.22}$$

We now apply our inequality to the case of soft potentials.

Corollary 1.5. *Assume* **(A1-A2-A3-A4**(γ)**)** *for some* $\gamma \in (-3, 0]$*, and let* $p \in (3/(3+\gamma), \infty)$.
(i) For any pair of weak solutions $(f_t)_{t\in[0,T]}$, $(\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) lying in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$*, there holds*

$$\forall\, t \in [0,T], \qquad W_2(f_t, \tilde{f}_t) \le W_2(f_0, \tilde{f}_0) \exp\left(K_p \int_0^t [1 + \|f_s\|_{L^p(\mathbb{R}^3)} + \|\tilde{f}_s\|_{L^p(\mathbb{R}^3)}] ds \right) \tag{1.23}$$

where K_p *depends only on* $\gamma, p, \kappa_1, \kappa_2, \kappa_3$. *Uniqueness and stability with respect to the initial condition thus hold in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$.
(ii) For any $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$*, there exists* $T_* = T_*\big(\|f_0\|_{L^p(\mathbb{R}^3)}, p, \gamma, \kappa_1, \kappa_2, \kappa_3\big) > 0$ *such that there exists a unique weak solution* $(f_t)_{t\in[0,T_*)}$ *to (1.1) lying in* $L^\infty_{\text{loc}}\big([0,T_*), \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)\big)$.

The rest of the paper is dedicated to the proof of these results. We first state some preliminary lemmas in Section 2. Since the rigorous proof of Theorem 1.3, handled in Section 4, is quite complicated, we first give some formal arguments in Section 3. Corollaries 1.4 and 1.5 are checked in Section 5. Finally, an appendix containing technical computations lies at the end of the paper.

2. Preliminaries

We start by a suitable way to rewrite the collision operator. The main interest of the following expression is that we make disappear the velocity-dependance $\Phi(|v - v_*|)$ in the *rate*. Such a trick was already used in [10].

Lemma 2.1. *Assume* **(A1-A2)** *and set*

$$\kappa_0 := \pi \int_0^\pi (1 - \cos\theta)\beta(\theta) d\theta. \tag{2.1}$$

Recalling (1.7) and (1.13), define, for $z \in (0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$,

$$c(v, v_*, z, \varphi) := a[v, v_*, G(z/\Phi(|v - v_*|)), \varphi]. \tag{2.2}$$

We have $\mathcal{A}\phi(v, v_*) = \frac{1}{2}[\tilde{\mathcal{A}}\phi(v, v_*) + \tilde{\mathcal{A}}\phi(v_*, v)]$ *for all* $v, v_* \in \mathbb{R}^3$ *and* $\phi \in C^2_\infty$*, where*

$$\begin{aligned} \tilde{\mathcal{A}}\phi(v, v_*) &= \int_0^\infty dz \int_0^{2\pi} d\varphi \Big(\phi[v + c(v, v_*, z, \varphi)] - \phi[v] - c[v, v_*, z, \varphi].\nabla\phi[v] \Big) \\ &\quad -\kappa_0 \Phi(|v - v_*|) \nabla\phi(v).(v - v_*) \\ &= \int_0^\infty dz \int_0^{2\pi} d\varphi \Big(\phi[v + c(v, v_*, \varphi + \varphi_0)] - \phi[v] - c[v, v_*, z, \varphi + \varphi_0].\nabla\phi[v] \Big) \\ &\quad -\kappa_0 \Phi(|v - v_*|) \nabla\phi(v).(v - v_*), \end{aligned} \tag{2.3}$$

the second equality holding for any $\varphi_0 \in [0, 2\pi)$ *(which may depend on* v, v_*, z*). As a consequence, we may replace* $\mathcal{A}$ *by* $\tilde{\mathcal{A}}$ *in (1.17).*

This lemma is proved in the appendix. Let us now recall a fundamental remark by Tanaka [15], slighlty precised in [9, Lemma 2.6]. We use here notation (1.13).

Lemma 2.2. *There exists a measurable function* $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$*, such that for all* $X, Y \in \mathbb{R}^3$*, all* $\varphi \in [0, 2\pi)$*,*

$$|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \le 3|X - Y|. \tag{2.4}$$

The following fundammental estimates, on which our results rely, are proved in the appendix.

Lemma 2.3. *Assume* **(A1-A2-A3)** *and (1.9). There exists a constant* $C = C(\kappa_1, \kappa_2)$ *such that the following estimates hold.*
(i) For $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,

$$\int_0^\infty dz \int_0^{2\pi} d\varphi |c(v, v_*, z, \varphi)|^2 \leq C|v - v_*|^2 \Phi(|v - v_*|), \tag{2.5}$$

$$\int_0^\infty dz \int_0^{2\pi} d\varphi |c(v, v_*, z, \varphi) - c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 \\ \leq C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)(\Psi(|v - v_*|) + \Psi(|\tilde{v} - \tilde{v}_*|)), \tag{2.6}$$

$$\int_0^\infty dz \left| \int_0^{2\pi} d\varphi \; c(v, v_*, z, \varphi) \right| \leq C|v - v_*|\Phi(|v - v_*|), \tag{2.7}$$

$$\int_0^\infty dz \left| \int_0^{2\pi} d\varphi [c(v, v_*, z, \varphi) - c(\tilde{v}, \tilde{v}_*, z, \varphi)] \right| \\ \leq C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)(\Psi(|v - v_*|) + \Psi(|\tilde{v} - \tilde{v}_*|)). \tag{2.8}$$

(ii) For any $\phi \in C_b^2$, *any* $v, v_* \in \mathbb{R}^3$,

$$|\tilde{\mathcal{A}}\phi(v, v_*)| \leq C\Phi(|v - v_*|) \left(|v - v_*|.||\phi'||_\infty + |v - v_*|^2 ||\phi''||_\infty \right). \tag{2.9}$$

(iii) For any $\phi \in C_c^2$ *with* $\text{supp}\, \phi \subset \{|v| \leq x\}$, *for all* $v, v_* \in \mathbb{R}^3$,

$$|\tilde{\mathcal{A}}\phi(v, v_*)| \leq C \left(||\phi'||_\infty |v - v_*| + ||\phi''||_\infty |v - v_*|^2 \right) \Phi(|v - v_*|) 1\!\!1_{\{|v| \leq 2x\}} \\ + C||\phi||_\infty \frac{|v - v_*|^2 \Phi(|v - v_*|)}{|v|^2} 1\!\!1_{\{|v| \geq 2x\}}. \tag{2.10}$$

We now state again some estimates that will be usefull when passing to the limit in some cutoff Boltzmann equations.

Lemma 2.4. *We assume* **(A1-A2-A3)** *and (1.9). For* $k \geq 1$ *and* $x \in \mathbb{R}_+$, *we set*

$$h_0^k(x) := \pi \int_0^k dz [1 - \cos(G[z/\Phi(x)])] \text{ and } \varepsilon_0^k(x) := \int_0^{G[k/\Phi(x)]} \theta^2 \beta(\theta) d\theta \tag{2.11}$$

There exists a constant $C = C(\kappa_1, \kappa_2)$ *such that for all* $v, v_* \in \mathbb{R}^3$, *all* $x, y \in \mathbb{R}_+$, *all* $k \geq 1$,

$$\int_k^\infty dz \int_0^{2\pi} d\varphi |c(v, v_*, z, \varphi)|^2 \leq C|v - v_*|^2 \Phi(|v - v_*|) \varepsilon_0^k(|v - v_*|) \tag{2.12}$$

$$|x h_0^k(x) - y h_0^k(y)| \leq C|x - y|(\Psi(x) + \Psi(y)), \tag{2.13}$$

$$|\kappa_0 x \Phi(x) - x h_0^k(x)| \leq C x \Phi(x) \varepsilon_0^k(x). \tag{2.14}$$

Furthermore, ε_0^k *is bounded by* κ_1, *and for all* $x \in \mathbb{R}_+$, $\lim_k \varepsilon_0^k(x) = 0$.

This Lemma will be checked in the appendix, as the following continuity property of $\tilde{\mathcal{A}}$.

Lemma 2.5. *Assume* **(A1-A2-A3-A4)**(γ), *for some* $\gamma \in (-3, 0]$, *and consider* $g \in \mathcal{P}_2(\mathbb{R}^3) \cap \mathcal{J}_\gamma$. *Then for any* $\phi \in C_c^2$, $v \mapsto \int_{\mathbb{R}^3} g(dv_*) \tilde{\mathcal{A}}\phi(v, v_*)$ *is continuous on* $\mathbb{R}^3$.

3. A short and unrigorous proof

We give here the main idea of this paper in the cutoff case. In the case without cutoff, we are not able to give a direct proof (not relying on the use of Poisson measures, martingale problems,... see the next section). We consider a solution $(f_t)_{t\in[0,T]}$ to the Boltzmann equation. Then

$$\frac{d}{dt}\int_{\mathbb{R}^3}\phi(v)f_t(dv)=\int_{\mathbb{R}^3}f_t(dv)\int_{\mathbb{R}^3}f_t(dv_*)\tilde{\mathcal{A}}\phi(v,v_*),\tag{3.15}$$

and we can formally write

$$\tilde{\mathcal{A}}\phi(v,v_*)=2\pi\int_0^\infty dz\int_0^{2\pi}\frac{d\varphi}{2\pi}[\phi(v+c(v,v_*,z,\varphi))-\phi(v)].\tag{3.16}$$

This roughly means the following: take a particle at random at time t, and call its velocity V_t. Then V_t is f_t-distributed. Then for all $z\in\mathbb{R}_+$, it will collide, at rate $2\pi dz$, with another particle with velocity V_t^* (independant and also f_t-distributed), it will choose α uniformly in $[0,2\pi)$, and its new velocity after the collision will be $Z_t(z):=V_t+c(V_t,V_t^*,z,\alpha)$. Let us call $\Delta(f_t,z)$ the law of $Z_t(z)$.
Then if we have two solutions $(f_t)_{t\in[0,T]},(\tilde{f}_t)_{t\in[0,T]}$ to (1.1), it is natural to think that

$$\frac{d}{dt}W_2^2(f_t,\tilde{f}_t)\le\int_0^\infty 2\pi dz[W_2^2(\Delta(f_t,z),\Delta(\tilde{f}_t,z))-W_2^2(f_t,\tilde{f}_t)].\tag{3.17}$$

Indeed, at each time t, for each z, f_t and $\tilde{f}_t$ are replaced by $\Delta(f_t,z)$ and $\Delta(\tilde{f}_t,z)$ at rate $2\pi dz$. Such an inequality can be rigorously and easily obtained when truncating the integral $\int_0^\infty dz$ into $\int_0^k dz$, by using the dual formulation of the Wasserstein distance (see Villani [20]).
We then claim that for all pair of laws $f,\tilde{f}$ on $\mathbb{R}^3$,

$$\int_0^\infty 2\pi dz[W_2^2(\Delta(f,z),\Delta(\tilde{f},z))-W_2^2(f,\tilde{f})]\le CW_2^2(f,\tilde{f})[J_\gamma(f)+J_\gamma(\tilde{f})],\tag{3.18}$$

where C depends only on $\kappa_1,\kappa_2,\kappa_3$, see **(A1-A2-A3-A4)**$(\gamma)$. Gathering (3.17) and (3.18), Theorem 1.3 would follow immediately from the generalized Gronwall Lemma 6.1.
Let us prove (3.18). Consider thus $f,\tilde{f}$ two probability distributions on $\mathbb{R}^3$, and two couples $(V,\tilde{V})$ and $(V_*,\tilde{V}_*)$ with V and V_* f-distributed, $\tilde{V}$ and $\tilde{V}_*$ $\tilde{f}$-distributed, with $(V,\tilde{V})$ independent of $(V_*,\tilde{V}_*)$, and such that $E[|V-\tilde{V}|^2]=E[|V_*-\tilde{V}_*|^2]=W_2^2(f,\tilde{f})$. Choose α uniformly distributed on $[0,2\pi)$ (independent of everything else), and set $\tilde{\alpha}=\alpha+\varphi_0(V-V_*,\tilde{V}-\tilde{V}_*)$ (modulo 2π), where φ_0 was introduced in Lemma 2.2. Then $\tilde{\alpha}$ is also uniformly distributed on $[0,2\pi)$, and is also independent of $(V,\tilde{V},V_*,\tilde{V}_*)$. As a consequence, $Z(z)=V+c(V,V_*,z,\alpha)$ is $\Delta(f,z)$-distributed, and $\tilde{Z}(z)=\tilde{V}+c(\tilde{V},\tilde{V}_*,z,\tilde{\alpha})$ is $\Delta(\tilde{f},z)$-distributed, so that

$$W_2^2(\Delta(f,z),\Delta(\tilde{f},z))-W_2^2(f,\tilde{f})\le E[|Z(z)-\tilde{Z}(z)|^2-|V-\tilde{V}|^2]=:\delta(z).\tag{3.19}$$

But a simple computation using (2.6) and (2.8) shows that for some constant $C = C(\kappa_1, \kappa_2, \kappa_3)$,

$$\begin{aligned}\int_0^\infty dz\delta(z) &= \int_0^\infty dz \int_0^{2\pi} \frac{d\varphi}{2\pi} E\Big[|c(V, V_*, z, \varphi) - c(\tilde{V}, \tilde{V}_*, z, \varphi + \varphi_0(V - V_*, \tilde{V} - \tilde{V}_*))|^2 \\ &\quad +2(V - \tilde{V})(c(V, V_*, z, \varphi) - c(\tilde{V}, \tilde{V}_*, z, \varphi + \varphi_0(V - V_*, \tilde{V} - \tilde{V}_*)))\Big] \\ &\le CE\left[(|V - \tilde{V}|^2 + |V_* - \tilde{V}_*|^2)(\Psi(|V - V_*|) + \Psi(|\tilde{V} - \tilde{V}_*|))\right] \\ &\le CE\left[|V - \tilde{V}|^2(|V - V_*|^\gamma + |\tilde{V} - \tilde{V}_*|^\gamma)\right] \qquad (3.20)\end{aligned}$$

by a symmetry argument and **(A4)**(γ). Using finally the definition of J_γ and the independance of $(V, \tilde{V})$ and $(V_*, \tilde{V}_*)$, one easily deduces that

$$\begin{aligned}\int_0^\infty dz\delta(z) &\le CE\left[|V - \tilde{V}|^2\right] (\sup_v E[|v - V_*|^\gamma] + \sup_{\tilde{v}} E[|\tilde{v} - \tilde{V}_*|^\gamma]) \\ &= CW_2^2(f, \tilde{f})[J_\gamma(f) + J_\gamma(\tilde{f})]. \qquad (3.21)\end{aligned}$$

This concludes the proof of (3.18).

4. Coupling Boltzmann processes

To prove Theorem 1.3, we will use some probabilistic arguments, which is of course a natural way to couple two solutions of the Boltzmann equation. We follow the line of Tanaka [15] (see also [9]), who was dealing with the Maxwellian case, that is $\Phi \equiv 1$.

In the whole section, C (resp. C_T) stands for a constant whose value may change from line to line, and which depend only on $\kappa_1, \kappa_2, \kappa_3, \gamma$ (resp. $\kappa_1, \kappa_2, \kappa_3, \gamma, T$).

Recall that $\mathbb{D}_T = \mathbb{D}([0, T], \mathbb{R}^3)$ stands for the Skorokhod space of càdlàg functions, see Ethier-Kurtz [6] for many details on this topic. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$.

Notation 4.1. *Let $g = (g_t)_{t\in[0,T]}$ be a measurable family of probability measures on $\mathbb{R}^3$.*
(i) We say that N is a g-Poisson measure if it is a $(\mathcal{F}_t)_{t\in[0,T]}$-Poisson measure on $[0, T] \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi)$ with intensity measure $dsg_s(dv)dzd\varphi$. We denote by $\tilde{N}$ its compensated Poisson measure.
(ii) For $k \ge 1$, V_0 a $\mathcal{F}_0$-measurable $\mathbb{R}^3$-valued random variable and N a g-Poisson measure, we define $(V_t^k)_{t\in[0,T]}$ the unique solution to

$$V_t^k = V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^k \int_0^{2\pi} c(V_{s-}^k, v, z, \varphi) N(ds, dv, dz, d\varphi). \qquad (4.1)$$

Then $(V_t^k)_{t\in[0,T]}$ is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and belongs a.s. to $\mathbb{D}_T$. We will refer to $(V_t^k)_{t\in[0,T]}$ as the (V_0, g, k, N)-process. Its law does not depend on the choice of the probability space, on N, and depends on V_0 only through its law.

The existence and uniqueness of V^k is obvious, because $N([0, T] \times \mathbb{R}^3 \times [0, k] \times [0, 2\pi))$ is a.s. finite, so that (4.1) is nothing but a recursive equation.
We will show the following result at the end of this section.

Lemma 4.2. *Assume* **(A1-A2-A3-A4**(γ)**)***, for some $\gamma \in (-3, 0]$. Consider a weak solution $(f_t)_{t\in[0,T]} \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0, T], \mathcal{J}_\gamma)$ to (1.1). Consider any $\mathcal{F}_0$-measurable random* *variable* $V_0 \sim f_0$. *Consider a* f*-Poisson measure* N, *and for each* $k \geq 1$, *the* (V_0, f, k, N)*-process* $(V_t^k)_{t\in[0,T]}$. *For each* $t \in [0,T]$, *denote by* f_t^k *the law of* V_t^k. *Then*

$$\lim_{k\to\infty} \sup_{[0,T]} W_2^2(f_t, f_t^k) = 0. \tag{4.2}$$

Thus we will study a solution f to (1.1) through its related stochastic process V_t^k. We start with some moment computations.

Lemma 4.3. *Assume* **(A1-A2-A3-A4)**(γ) *for some* $\gamma \in (-3, 0]$. *Let* $g \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], \mathcal{J}_\gamma)$. *There exists a constant* $K_T(g)$ *depending only on* $T, \gamma, \kappa_1, \kappa_2, \kappa_3, g$ *such that for each* $k \geq 1$, $V_0 \in L^2$, *each* g*-Poisson measure* N, *the* (V_0, g, k, N)*-process* $(V_t^k)_{t\in[0,T]}$ *satisfies*

$$E\left[\sup_{[0,T]} |V_t^k|^2\right] \leq K_T(g)\{1 + E[|V_0|^2]\}. \tag{4.3}$$

Proof. Let $k \geq 1$ be fixed. Writing the Poisson measure N as $\tilde{N} + ds g_s(dv)dzd\varphi$, we obtain, using the Doob inequality, that for $t \in [0,T]$, $E[\sup_{[0,t]} |V_s^k|^2] \leq C\{E[|V_0|^2] + A_t + B_t\}$, where

$$\begin{aligned} A_t &:= E\left[\int_0^t ds \int_{\mathbb{R}^3} g_s(dv) \int_0^k dz \int_0^{2\pi} d\varphi |c(V_s^k, v, z, \varphi)|^2\right], \\ B_t &:= E\left[\sup_{[0,t]} \left|\int_0^t ds \int_{\mathbb{R}^3} g_s(dv) \int_0^k dz \int_0^{2\pi} d\varphi c(V_s^k, v, z, \varphi)\right|^2\right]. \end{aligned} \tag{4.4}$$

Using now (2.5) and then **(A4**(γ)**)**, we get

$$A_t \leq CE\left[\int_0^t ds \int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{2+\gamma}\right], \tag{4.5}$$

while (2.7) and **(A4**(γ)**)** yield

$$B_t \leq CE\left[\left|\int_0^t ds \int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{\gamma+1}\right|^2\right]. \tag{4.6}$$

We then have to divide the study into several cases.

Case $\gamma \in [-1, 0]$. Then $\gamma + 2 \in [0,2]$, so that $|V_s^k - v|^{2+\gamma} \leq C(1 + |V_s^k|^2 + |v|^2)$, and one easily checks that $A_t \leq C_T(1 + \int_0^t m_2(g_s)ds + \int_0^t E[|V_s^k|^2]ds)$. Furthermore $\gamma + 1 \in [0,1]$, so that $|V_s^k - v|^{1+\gamma} \leq C(1 + |V_s^k| + |v|)$. Thus $B_t \leq C_T(1 + \int_0^t m_2(g_s)ds + \int_0^t E[|V_s^k|^2]ds)$ by the Cauchy-Schwarz inequality. We finally find $E[\sup_{[0,t]} |V_s^k|^2] \leq C_T(1+E[|V_0|^2]+\int_0^t m_2(g_s)ds+\int_0^t E[|V_s^k|^2]ds)$ and the conclusion follows by the Gronwall Lemma, since $\int_0^T m_2(g_s)ds < \infty$ by assumption.

Case $\gamma \in [-2, -1]$. Since $\gamma + 2 \in [0, 2]$, we obtain as previously $A_t \leq C(1 + \int_0^t m_2(g_s)ds + \int_0^t E[|V_s^k|^2]ds)$. On the other hand, $\gamma < \gamma + 1 \leq 0$, so that $|V_s^k - v|^{\gamma+1} \leq 1 + |V_s^k - v|^\gamma$. Recalling (1.12), we deduce that $\int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{\gamma+1} \leq 1 + \int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^\gamma \leq 1 + J_\gamma(g_s)$, and thus $B_t \leq C|\int_0^t(1 + J_\gamma(g_s))ds|^2$. We finally get $E[\sup_{[0,t]} |V_s^k|^2] \leq C_T(1 + E[|V_0|^2] + \int_0^t m_2(g_s)ds + \int_0^t E[|V_s^k|^2]ds + |\int_0^t(1 + J_\gamma(g_s))ds|^2)$, and the conclusion follows by the Gronwall Lemma, since $\int_0^T m_2(g_s)ds + \int_0^T J_\gamma(g_s)ds < \infty$ by assumption.

Case $\gamma\in(-3,-2]$. Since $\gamma<\gamma+1\leq 0$, we obtain as previously that $B_t\leq C|\int_0^t(1+J_\gamma(g_s))ds|^2$. A similar argument, using that $\gamma<\gamma+2\leq 0$ (and thus $x^{\gamma+2}\leq 1+x^\gamma$), yields $A_t\leq C\int_0^t(1+J_\gamma(g_s))ds$. We finally find $E[\sup_{[0,t]}|V_s^k|^2]\leq C_T(E[|V_0|^2]+\int_0^t(1+J_\gamma(g_s))ds+|\int_0^t(1+J_\gamma(g_s))ds|^2)$, and the conclusion follows, since $\int_0^T J_\gamma(g_s)ds<\infty$ by assumption. □

Tanaka [15] (see also [9, Lemma 4.7]) observed the following elementary fact.

Lemma 4.4. *Consider a $(\mathcal{F}_t)_{t\in[0,T]}$-Poisson measure $\mu(ds,dx,d\varphi)$ on $[0,T]\times F\times[0,2\pi)$ with intensity measure $ds\nu(dx)d\varphi$, for some measurable space space F endowed with a nonnegative measure ν. Then for any predictable map $\varphi_*:\Omega\times[0,T]\times F\mapsto[0,2\pi)$, the random measure $\mu_*(ds,dx,d\varphi)$ on $[0,T]\times F\times[0,2\pi)$ defined by*

$$\mu_*(A)=\int_0^T\int_F\int_0^{2\pi}1\!\!1_A(s,x,\varphi+\varphi_*(s,x))\mu(ds,dx,d\varphi)\quad\forall\, A\subset[0,T]\times F\times[0,2\pi) \tag{4.7}$$

is again a $(\mathcal{F}_t)_{t\in[0,T]}$-Poisson measure with intensity measure $ds\nu(dx)d\varphi$. Of course, we write $\varphi+\varphi_*(s,x)$ for its value modulo 2π.

Our main result will be based on the following proposition.

Proposition 4.5. *Assume* **(A1-A2-A3-A4(γ))**, *for some $\gamma\in(-3,0]$. Let $k\geq 1$, $V_0,\tilde V_0$ two $\mathcal{F}_0$-measurable $\mathbb{R}^3$-valued random variables. We also consider g and $\tilde g$ in $L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))\cap L^1([0,T],\mathcal{J}_\gamma)$. Let us finally consider, for each $s\in[0,T]$, $R_s\in\mathcal{H}(g_s,\tilde g_s)$ such that $s\mapsto R_s$ is measurable. We may find a g-Poisson measure N and a $\tilde g$-Poisson measure M such that, for V^k the (V_0,g,k,N)-process and $\tilde V^k$ the $(\tilde V_0,\tilde g,k,M)$-process, the following property holds.*
(i) If $\gamma\in(-3,0)$, set $\alpha(\gamma)=\min(1/|\gamma|,|\gamma|/2)>0$. For all $L\geq 1$, all $t\in[0,T]$,

$$\begin{aligned}&E[|V_t^k-\tilde V_t^k|^2]\leq E[|V_0-\tilde V_0|^2]+K_T(g,\tilde g,V_0,\tilde V_0)L^{-\alpha(\gamma)}\\ &\quad+C\int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3}R_s(dv,d\tilde v)\left\{|V_s^k-\tilde V_s^k|^2+|v-\tilde v|^2\right\}\min(|V_s^k-v|^\gamma+|\tilde V_s^k-\tilde v|^\gamma,L)\right\}\right],\end{aligned} \tag{4.8}$$

where $K_T(V_0,\tilde V_0,g,\tilde g)$ depends only on $T,\gamma,\kappa_1,\kappa_2,\kappa_3,g,\tilde g$ and $E[|V_0|^2],E[|\tilde V_0|^2]$.
(ii) If $\gamma=0$, for all $t\in[0,T]$,

$$E[|V_t^k-\tilde V_t^k|^2]\leq E[|V_0-\tilde V_0|^2]+C\int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3}R_s(dv,d\tilde v)\left\{|V_s^k-\tilde V_s^k|^2+|v-\tilde v|^2\right\}\right]. \tag{4.9}$$

Proof. Let thus $k\geq 1$, $g,\tilde g$, $V_0,\tilde V_0$, and $(R_s)_{s\in[0,T]}$ be as in the statement. We introduce a Poisson measure Δ on $[0,T]\times(\mathbb{R}^3\times\mathbb{R}^3)\times[0,\infty)\times[0,2\pi)$ with intensity measure $dsR_s(dv,d\tilde v)dzd\varphi$. Then, since the restriction of this measure to $z\in[0,k]$ is a.s. finite, there exists a unique pair of processes $(V_t^k)_{t\in[0,T]}$ and $(\tilde V_t^k)_{t\in[0,T]}$, solution of

$$\begin{aligned}V_t^k&=V_0+\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^k\int_0^{2\pi}c(V_{s-}^k,v,z,\varphi)\Delta(ds,d(v,\tilde v),dz,d\varphi),\\ \tilde V_t^k&=\tilde V_0+\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^k\int_0^{2\pi}c(\tilde V_{s-}^k,\tilde v,z,\varphi+\varphi_0(V_{s-}^k-v,\tilde V_{s-}^k-\tilde v))\Delta(ds,d(v,\tilde v),dz,d\varphi),\end{aligned} \tag{4.10}$$

where φ_0 was introduced in Lemma 2.2. Consider now the random measures N and M defined on $[0,T]\times\mathbb{R}^3\times[0,\infty)\times[0,2\pi)$ by

$$N(A)=\int_0^T\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}1\!\!1_A(s,v,z,\varphi)\Delta(ds,d(v,\tilde v),dz,d\varphi),$$

$$M(A)=\int_0^T\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}1\!\!1_A(s,\tilde v,z,\varphi+\varphi_0(V^k_{s-}-v,\tilde V^k_{s-}-\tilde v))\Delta(ds,d(v,\tilde v),dz,d\varphi). \tag{4.11}$$

Then N is classically a g-Poisson measure, since for each s, g_s is the first marginal of R_s. Furthemore, M is a $\tilde g$-Poisson-measure, since for each s, $\tilde g_s$ is the second marginal of R_s, and due Lemma (4.4).

Thus $(V^k_t)_{t\in[0,T]}$ (resp. $(\tilde V^k)_{t\in[0,T]}$) is the (V_0,g,k,N)-process (resp. the $(\tilde V_0,\tilde g,k,M)$-process). Next, setting for simplicity $c:=c(V^k_{s-},v,z,\varphi)$ and $\tilde c:=c(\tilde V^k_{s-},\tilde v,z,\varphi+\varphi_0(V^k_{s-}-v,\tilde V^k_{s-}-\tilde v))$, we get

$$|V^k_t-\tilde V^k_t|^2=|V_0-\tilde V_0|^2+\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^k\int_0^{2\pi}\left\{|V^k_{s-}+c-\tilde V^k_{s-}-\tilde c|^2-|V^k_{s-}-\tilde V^k_{s-}|^2\right\}\Delta(ds,d(v,\tilde v),dz,d\varphi). \tag{4.12}$$

Hence, taking expectations,

$$E[|V^k_t-\tilde V^k_t|^2]=E[|V_0-\tilde V_0|^2]+\int_0^t dsE\left\{\int_{\mathbb{R}^3\times\mathbb{R}^3}R_s(dv,d\tilde v)\int_0^k dz\int_0^{2\pi}d\varphi[|c-\tilde c|^2+2(V^k_s-\tilde V^k_s).(c-\tilde c)]\right\}. \tag{4.13}$$

Now, using (2.6), (2.8) and **(A4)**(γ), we easily deduce that a.s.,

$$\int_0^k dz\int_0^{2\pi}d\varphi[|c-\tilde c|^2+2(V^k_s-\tilde V^k_s).(c-\tilde c)]\leq C(|V^k_s-\tilde V^k_s|^2+|v-\tilde v|^2)(|V^k_s-v|^\gamma+|\tilde V^k_s-\tilde v|^\gamma), \tag{4.14}$$

while using (2.5), (2.7) and **(A4)**(γ), we obtain a.s.

$$\int_0^k dz\int_0^{2\pi}d\varphi[|c-\tilde c|^2+2(V^k_s-\tilde V^k_s).(c-\tilde c)]\leq C|V^k_s-v|^{2+\gamma}+C|\tilde V^k_s-\tilde v|^{2+\gamma}+C|V^k_s-\tilde V^k_s|\left\{|V^k_s-v|^{1+\gamma}+|\tilde V^k_s-\tilde v|^{1+\gamma}\right\}. \tag{4.15}$$

If $\gamma=0$, (4.9) follows immediately from (4.13) and (4.14). We thus now assume that $\gamma\in(-3,0)$. Let $L\geq 1$ be fixed. We insert (4.14) (resp. (4.15)) in (4.13) when $|V^k_s-v|^\gamma+|\tilde V^k_s-\tilde v|^\gamma\leq L$ (resp. $|V^k_s-v|^\gamma+|\tilde V^k_s-\tilde v|^\gamma>L$), and we obtain

$$E[|V^k_t-\tilde V^k_t|^2]\leq E[|V_0-\tilde V_0|^2]+C\sum_{i=1}^4(I^{i,L}_t+\tilde I^{i,L}_t)+C\int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3}R_s(dv,d\tilde v)\left\{|V^k_s-\tilde V^k_s|^2+|v-\tilde v|^2\right\}\min(|V^k_s-v|^\gamma+|\tilde V^k_s-\tilde v|^\gamma,L)\right\}\Bigg], \tag{4.16}$$

where

$$
\begin{aligned}
I_t^{1,L} &:= \int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|V_s^k - v|^{2+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right],\\
I_t^{2,L} &:= \int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|\tilde V_s^k - \tilde v|^{2+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right],\\
I_t^{3,L} &:= \int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|V_s^k - \tilde V_s^k||V_s^k - v|^{1+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right],\\
I_t^{4,L} &:= \int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|V_s^k - \tilde V_s^k||\tilde V_s^k - \tilde v|^{1+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right], \qquad (4.17)
\end{aligned}
$$

and where $\tilde I_t^{1,L},...,\tilde I_t^{4,L}$ have the same expressions replacing $V_s^k, \tilde V_s^k, v, \tilde v$ by $\tilde V_s^k, V_s^k, \tilde v, v$.
We first treat the case of $I^{1,L}$. Since $R_s \in \mathcal{H}(g_s,\tilde g_s)$ and $\gamma \in (-3,0)$, and using notation (1.12) one has

$$
\begin{aligned}
I_t^{1,L} &= \int_0^t dsE\left[\int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{2+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right]\\
&\le CL^{2/\gamma}\int_0^t dsE\left[\int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{\gamma}\right] \le CL^{2/\gamma}\int_0^t dsJ_\gamma(g_s) \le K_T(g)L^{-\alpha(\gamma)}, \qquad (4.18)
\end{aligned}
$$

since $2/|\gamma| \ge \alpha(\gamma)$. Similarly, since $1/|\gamma| \ge \alpha(\gamma)$

$$
\begin{aligned}
I_t^{3,L} &= \int_0^t dsE\left[|V_s^k - \tilde V_s^k|\int_{\mathbb{R}^3} g_s(dv)|V_s^k - v|^{1+\gamma}1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right]\\
&\le CL^{1/\gamma}E\left[\sup_{[0,T]}(|V_s^k| + |\tilde V_s^k|)\right]\int_0^t dsJ_\gamma(g_s) \le K_T(g,\tilde g, V_0, \tilde V_0)L^{-\alpha(\gamma)}, \qquad (4.19)
\end{aligned}
$$

where we used Lemma 4.3. We now study $I_t^{2,L}$ when $\gamma \in [-2,0)$, so that $\gamma+2 \in [0,2)$. Using the Hölder inequality, we get

$$
\begin{aligned}
I_t^{2,L} &\le E\left[\left(\int_0^t ds\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|\tilde V_s^k - \tilde v|^2\right)^{\frac{2+\gamma}{2}}\left(\int_0^t ds\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{|\gamma|}{2}}\right]\\
&= E\left[\left(\int_0^t ds\int_{\mathbb{R}^3} \tilde g_s(d\tilde v)|\tilde V_s^k - \tilde v|^2\right)^{\frac{2+\gamma}{2}}\left(\int_0^t ds\int_{\mathbb{R}^3} g_s(dv)1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{|\gamma|}{2}}\right]\\
&\le CE\left[\left(\int_0^t ds(m_2(\tilde g_s) + |\tilde V_s^k|^2)\right)^{\frac{2+\gamma}{2}}\left(\int_0^t ds\int_{\mathbb{R}^3} g_s(dv)\frac{|V_s^k - v|^\gamma}{L/2}\right)^{\frac{|\gamma|}{2}}\right]\\
&\le CL^{\gamma/2}\{1 + \sup_{[0,T]} m_2(\tilde g_s) + E[\sup_{[0,T]}|\tilde V_s^k|^2]\}\left(\int_0^t dsJ_\gamma(g_s)ds\right)^{\frac{|\gamma|}{2}} \le K_T(g,\tilde g,\tilde V_0)L^{-\alpha(\gamma)}, \qquad (4.20)
\end{aligned}
$$

by Lemma 4.3, since $\frac{\gamma+2}{\gamma} \in [0,1]$ and $|\gamma|/2 \geq \alpha(\gamma)$. We next study $I_t^{2,L}$ when $\gamma \in (-3,-2)$, so that $\gamma+2 \in (-1,0)$. The Hölder inequality yields

$$\begin{aligned} I_t^{2,L} &\leq E\left[\left(\int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde{v})|\tilde{V}_s^k-\tilde{v}|^\gamma\right)^{\frac{2+\gamma}{\gamma}} \left(\int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde{v})1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{2}{|\gamma|}}\right] \\ &= E\left[\left(\int_0^t ds \int_{\mathbb{R}^3} \tilde{g}_s(d\tilde{v})|\tilde{V}_s^k-\tilde{v}|^\gamma\right)^{\frac{2+\gamma}{\gamma}} \left(\int_0^t ds \int_{\mathbb{R}^3} g_s(dv)1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{2}{|\gamma|}}\right] \\ &\leq C\left(\int_0^t ds J_\gamma(\tilde{g}_s)\right)^{\frac{2+\gamma}{\gamma}} \left(\int_0^t ds J_\gamma(g_s)\right)^{\frac{2}{|\gamma|}} L^{2/\gamma} \leq K_T(g,\tilde{g})L^{-\alpha(\gamma)}, \end{aligned} \tag{4.21}$$

since $2/|\gamma| \geq \alpha(\gamma)$. Let us now upperbound $I_t^{4,L}$ in the case $\gamma \in [-1,0)$, so that $1+\gamma \in [0,1)$. Using the Hölder inequality, one finds as usual (since $(1+\gamma)/2 \leq 1/2$),

$$\begin{aligned} I_t^{4,L} &\leq E\Big[\sup_{[0,T]}|V_s^k-\tilde{V}_s^k| \times \left(\int_0^t ds \int_{\mathbb{R}^3} \tilde{g}_s(d\tilde{v})|\tilde{V}_s^k-\tilde{v}|^2\right)^{\frac{1+\gamma}{2}} \\ &\qquad\qquad \times \left(\int_0^t ds \int_{\mathbb{R}^3} g_s(dv)1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{1-\gamma}{2}}\Big] \\ &\leq CE\left[\left\{\sup_{[0,T]}(|V_s^k|+|\tilde{V}_s^k|)\right\}\left\{1+\int_0^t ds(m_2(g_s)+|V_s^k|^2)\right\}^{\frac{1}{2}}\left\{\frac{1}{L}\int_0^t ds J_\gamma(g_s)\right\}^{\frac{1-\gamma}{2}}\right] \\ &\leq CL^{\frac{\gamma-1}{2}}\{1+\sup_{[0,T]} m_2(\tilde{g}_s)+E[\sup_{[0,T]}(|V_s^k|^2+|\tilde{V}_s^k|^2)]\}\left(\int_0^t ds J_\gamma(g_s)\right)^{\frac{1-\gamma}{2}} \\ &\leq K_T(g,\tilde{g},V_0,\tilde{V}_0)L^{-\alpha(\gamma)}, \end{aligned} \tag{4.22}$$

since $(1+|\gamma|)/2 \geq \alpha(\gamma)$ and by Lemma 4.3. Finally we consider $I_t^{4,L}$ in the case $\gamma \in (-3,-2)$, so that $1+\gamma \in (\gamma,0)$:

$$\begin{aligned} I_t^{4,L} &\leq E\Big[\sup_{[0,T]}|V_s^k-\tilde{V}_s^k| \times \left(\int_0^t ds \int_{\mathbb{R}^3} \tilde{g}_s(d\tilde{v})|\tilde{V}_s^k-\tilde{v}|^\gamma\right)^{\frac{1+\gamma}{\gamma}} \\ &\qquad\qquad \times \left(\int_0^t ds \int_{\mathbb{R}^3} g_s(dv)1\!\!1_{\{|V_s^k-v|^\gamma>L/2\}}\right)^{\frac{1}{|\gamma|}}\Big] \\ &\leq CL^{1/\gamma}E[\sup_{[0,T]}(|V_s^k|+|\tilde{V}_s^k|)]\left(\int_0^t ds J_\gamma(\tilde{g}_s)\right)^{\frac{1+\gamma}{\gamma}}\left(\int_0^t ds J_\gamma(g_s)\right)^{\frac{1}{|\gamma|}} \\ &\leq K_T(g,\tilde{g},V_0,\tilde{V}_0)L^{-\alpha(\gamma)}. \end{aligned} \tag{4.23}$$

since $1/|\gamma| \geq \alpha(\gamma)$. Using the same computations for $\tilde{I}_t^{1,L}, ...\tilde{I}_t^{4,L}$, (4.8) follows immediately. □

Admitting for a moment Lemma 4.2, we give the

Proof of Theorem 1.3. We thus assume **(A1-A2-A3-A4**(γ)**)** for some $\gamma \in (-3,0)$, the case $\gamma = 0$ is easier and left to the reader. We consider two weak solutions $(f_t)_{t\in[0,T]}$ and $(\tilde f_t)_{t\in[0,T]}$ to (1.1) lying in $L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T]),\mathcal{J}_\gamma)$.
We consider two $\mathcal{F}_0$-measurable random variables $V_0 \sim f_0$ and $\tilde V_0 \sim \tilde f_0$, such that $W_2^2(f_0,\tilde f_0) = E[|V_0 - \tilde V_0|^2]$, and for each $s \in [0,T]$, we consider $R_s \in \mathcal{H}(f_s,\tilde f_s)$ such that $W_2^2(f_s,\tilde f_s) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |v - \tilde v|^2 R_s(dv,d\tilde v)$. Due to Fontbona-Guérin-Méléard [7, Theorem 1.3], $s \mapsto R_s$ is measurable: indeed, since by assumption f_t has a density for all $t \in [0,T]$, the minimizer R_t is unique (see e.g. Villani [20, Theorem 2.12]).
Finally, for each $k \ge 1$, we consider some (V_0,f,k,N)-process $(V_t^k)_{t\in[0,T]}$ and some $(\tilde V_0,\tilde f,k,M)$ process $(\tilde V_t^k)_{t\in[0,T]}$, coupled as in Proposition 4.5.
We set $w_t^k := E[|V_t^k - \tilde V_t^k|^2]$ for each $k \ge 1$, each $t \in [0,T]$. Using Lemma 4.2, we deduce that for all $t \in [0,T]$,

$$u_t := W_2^2(f_t,\tilde f_t) \le \limsup_k w_t^k =: w_t. \tag{4.24}$$

We observe at once that due to Lemma 4.3 and by assumption on $f,\tilde f$,

$$\sup_k \sup_{[0,T]} w_t^k + \sup_{[0,T]} w_t < \infty. \tag{4.25}$$

Due to Proposition 4.5, we know that for all $L \ge 1$, all $k \ge 1$, all $t \in [0,T]$,

$$\begin{aligned}
w_t^k \quad \le \quad & u_0 + K(T,f,\tilde f,V_0,\tilde V_0)L^{-\alpha(\gamma)} \\
& + C\int_0^t dsE\left[|V_s^k - \tilde V_s^k|^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)(|V_s^k - v|^\gamma + |\tilde V_s^k - \tilde v|^\gamma)\right] \\
& + C\int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|v-\tilde v|^2 \min(|V_s^k - v|^\gamma, L)\right] \\
& + C\int_0^t dsE\left[\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|v-\tilde v|^2 \min(|\tilde V_s^k - \tilde v|^\gamma, L)\right] \\
=: \quad & u_0 + K(T,f,\tilde f,V_0,\tilde V_0)L^{-\alpha(\gamma)} + CA_k(t,L) + CB_k(t,L) + C\tilde B_k(t,L).
\end{aligned} \tag{4.26}$$

First, recalling (1.12) and that $R_s \in \mathcal{H}(f_s,\tilde f_s)$, we observe that

$$\begin{aligned}
A_k(t,L) \quad = \quad & \int_0^t dsE\left[|V_s^k - \tilde V_s^k|^2 \int_{\mathbb{R}^3} (f_s(dv)|V_s^k - v|^\gamma + \tilde f_s(d\tilde v)|\tilde V_s^k - \tilde v|^\gamma)\right] \\
\le \quad & \int_0^t ds\ w_s^k J_\gamma(f_s + \tilde f_s).
\end{aligned} \tag{4.27}$$

Hence for all $L \ge 1$, using (4.25), the Lebesgue Theorem and that $J_\gamma(f_s + \tilde f_s)$ belongs to $L^1([0,T])$ by assumption,

$$\limsup_k A_k(t,L) \le \int_0^t ds\ w_s J_\gamma(f_s + \tilde f_s). \tag{4.28}$$

Next, one easily checks that for each $s \in [0,T]$, the map $v_* \mapsto \alpha_s(v_*) := \int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|v - \tilde v|^2 \min(|v_* - v|^\gamma, L)$ is continuous on $\mathbb{R}^3$ and bounded by $2L(m_2(f_s) + m_2(\tilde f_s))$. Since furthermore

$\int_0^T (m_2(f_s)+m_2(\tilde f_s))ds<\infty$ by assumption, we easily deduce from Lemma 4.2 and the Lebesgue Theorem that

$$\begin{aligned}\lim_k B_k(t,L) &= \int_0^t ds\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|v-\tilde v|^2\int_{\mathbb{R}^3} f_s(dv_*)\min(|v_*-v|^\gamma, L)\\ &\le \int_0^t ds\int_{\mathbb{R}^3\times\mathbb{R}^3} R_s(dv,d\tilde v)|v-\tilde v|^2\int_{\mathbb{R}^3} f_s(dv_*)|v_*-v|^\gamma\\ &\le \int_0^t ds\, u_s J_\gamma(f_s)\le \int_0^t ds\, w_s J_\gamma(f_s). \qquad (4.29)\end{aligned}$$

Using the same computation for $\tilde B_k(t,L)$, we finally obtain, for all $L\ge 1$,

$$w_t\le u_0+K(T,f,\tilde f,V_0,\tilde V_0)L^{-\alpha(\gamma)}+C\int_0^t w_s J_\gamma(f_s+\tilde f_s)ds. \qquad (4.30)$$

Making L tend to infinity, and using the generalized Gronwall Lemma 6.1, we deduce that for $t\in[0,T]$, $w_t\le u_0\exp(C\int_0^t J_\gamma(f_s+\tilde f_s)ds)$. Since $W_2^2(f_t,\tilde f_t)=u_t\le w_t$, this concludes the proof. □

It remains to prove the convergence Lemma 4.2. To this aim, we first prove a uniqueness result for a linearized Boltzmann equation.

Lemma 4.6. *Assume* **(A1-A2-A3-A4**(γ)**)**, *for some* $\gamma\in(-3,0]$. *Consider a weak solution* $f=(f_t)_{t\in[0,T]}\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))\cap L^1([0,T],\mathcal{J}_\gamma)$ *to (1.1). Assume that for some* $g=(g_t)_{t\in[0,T]}\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$ *for all* $\phi\in C_c^2$, *all* $t\in[0,T]$,

$$\int_{\mathbb{R}^3}\phi(v)g_t(dv) = \int_{\mathbb{R}^3}\phi(v)f_0(dv)+\int_0^t ds\int_{\mathbb{R}^3} g_s(dv)\int_{\mathbb{R}^3} f_s(dv_*)\tilde{\mathcal{A}}\phi(v,v_*), \qquad (4.31)$$

with $\tilde{\mathcal{A}}\phi$ *defined by (2.3). Then* $g=f$.

Proof. We thus assume **(A1-A2-A3-A4**(γ)**)** for some $\gamma\in(-3,0]$, and (unfortunately) use some martingale problems techniques. We consider a weak solution $f=(f_t)_{t\in[0,T]}\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))\cap L^1([0,T],\mathcal{J}_\gamma)$ to (1.1). We also consider, for each $t\ge 0$ the operator $\tilde{\mathcal{A}}_t$ defined for $\phi\in C_\infty^2$ and $v\in\mathbb{R}^3$ by

$$\tilde{\mathcal{A}}_t\phi(v)=\int_{\mathbb{R}^3} f_t(dv_*)\tilde{\mathcal{A}}\phi(v,v_*). \qquad (4.32)$$

We will prove that for any $\mu\in\mathcal{P}_2(\mathbb{R}^3)$, there exists at most one $g\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$ such that for all $t\ge 0$, all $\phi\in C_c^2$,

$$\int_{\mathbb{R}^3}\phi(v)g_t(dv)=\int_{\mathbb{R}^3}\phi(v)\mu(dv)+\int_0^t ds\int_{\mathbb{R}^3} g_s(dv)\tilde{\mathcal{A}}_s\phi(v). \qquad (4.33)$$

Since by assumption, f and g solve this equation with $\mu=f_0$, this will conclude the proof.

Step 1. Let $\mu\in\mathcal{P}_2(\mathbb{R}^3)$. A càdlàg adapted $\mathbb{R}^3$-valued stochastic process $(V_t)_{t\in[0,T]}$ on some filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$ is said to solve the martingale problem $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\mu)$ if $P\circ V_0^{-1}=\mu$ and if for all $\phi\in C_c^2$, $(M_t^\phi)_{t\in[0,T]}$ is a $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$-martingale, where

$$M_t^\phi=\phi(V_t)-\int_0^t\tilde{\mathcal{A}}_s\phi(V_s)ds. \qquad (4.34)$$

Assume for a moment that:
(i) there exists a countable subset $(\phi_k)_{k\geq 1}\subset C^2_c$ such that for all $t\in[0,T]$, the closure (for the bounded pointwise convergence) of $\{(\phi_k,\tilde{\mathcal{A}}_t\phi_k),k\geq 1\}$ contains $\{(\phi,\tilde{\mathcal{A}}_t\phi),\phi\in C^2_c\}$,
(ii) for each $v_0\in\mathbb{R}^3$, there exists a solution to $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0})$,
(iii) for each $v_0\in\mathbb{R}^3$, uniqueness (in law) holds for $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0})$.
Then, due to Bhatt-Karandikar [2, Theorem 5.2] (see also Remark 3.1 and Theorem 5.1 in [2] and Theorem B.1 in [12]), uniqueness for (4.33) holds.

Step 2. First, (i) holds: consider any countable subset $(\phi_k)_{k\geq 1}\subset C^2_c$ dense in C^2_c, in the sense that for $\psi\in C^2_c$ with supp $\psi\subset\{|v|\leq x\}$, there exists a subsequence ϕ_{k_n} such that supp $\phi_{k_n}\subset\{|v|\leq 2x\}$, and $\lim_{n\to\infty}(||\psi-\phi_{k_n}||_\infty+||\psi'-\phi'_{k_n}||_\infty+||\psi''-\phi''_{k_n}||_\infty)=0$.
We then have to prove that, for $t\in[0,T]$,
(a) $\tilde{\mathcal{A}}_t\phi_{k_n}(v)$ tends to $\tilde{\mathcal{A}}_t\psi(v)$ for all $v\in\mathbb{R}^3$,
(b) and that $\sup_n||\tilde{\mathcal{A}}_t\phi_{k_n}||_\infty<\infty$.
First, using Lemma 2.3-(ii) and **(A4)**(γ), we get, for $v\in\mathbb{R}^3$,

$$
\begin{aligned}
&|\tilde{\mathcal{A}}_t\phi_{k_n}(v)-\tilde{\mathcal{A}}_t\psi(v)|=|\tilde{\mathcal{A}}_t[\phi_{k_n}-\psi](v)|\\
&\leq C(||\psi'-\phi'_{k_n}||_\infty+||\psi''-\phi''_{k_n}||_\infty)\int_{\mathbb{R}^3}f_t(dv_*)(|v-v_*|+|v-v_*|^2)|v-v_*|^\gamma,
\end{aligned}
\tag{4.35}
$$

which tends to 0 as n tends to infinity, provided $\alpha_t:=\displaystyle\int_{\mathbb{R}^3}f_t(dv_*)(|v-v_*|+|v-v_*|^2)|v-v_*|^\gamma<\infty$. But $\alpha_t\leq\displaystyle\int_{\mathbb{R}^3}f_t(dv_*)(1+2|v-v_*|^2)|v-v_*|^\gamma\leq J_\gamma(f_t)+2\int_{\mathbb{R}^3}f_t(dv_*)|v-v_*|^{2+\gamma}$. If $\gamma\in[-2,0]$, then $|v-v_*|^{2+\gamma}\leq 1+2|v|^2+2|v_*|^2$, so that $\alpha_t\leq J_\gamma(f_t)+C(m_2(f_t)+1+|v|^2)<\infty$ by assumption. If $\gamma\in(-3,-2)$, then $|v-v_*|^{2+\gamma}\leq 1+|v-v_*|^\gamma$, so that $\alpha_t\leq 3J_\gamma(f_t)+1<\infty$. Thus (a) holds, and it remains to prove (b). Set $M:=\sup_n(||\phi_{k_n}||_\infty+||\phi'_{k_n}||_\infty+||\phi''_{k_n}||_\infty)$. If $\gamma\in(-3,-2]$, then one easily deduces from **(A4)**(γ) and Lemma 2.3-(ii) that for all t, all n, all v, $|\tilde{\mathcal{A}}_t\phi_{k_n}(v)|\leq CM(1+J_\gamma(f_t))$, which implies (b). If $\gamma\in(-2,0]$, we use Lemma 2.3-(iii), and get, since $|v-v_*|^{2+\gamma}\leq 1+2|v|^2+2|v_*|^2$,

$$
\begin{aligned}
|\tilde{\mathcal{A}}_t\phi_{k_n}(v)| &\leq CM\int_{\mathbb{R}^3}f_t(dv_*)\left[1+|v-v_*|^{2+\gamma}1\!\!1_{\{|v|\leq 4x\}}+\frac{|v-v_*|^{2+\gamma}}{|v|^2}1\!\!1_{\{|v|\geq 4x\}}\right]\\
&\leq CM\int_{\mathbb{R}^3}f_t(dv_*)\left[1+x^2+|v_*|^2+x^{-2}(1+|v_*|^2)\right]\\
&\leq CM(1+x^2+x^{-2}+m_2(f_t)(1+x^{-2})).
\end{aligned}
\tag{4.36}
$$

Step 3. Classical arguments (see e.g. Tanaka [16, Section 4] or Desvillettes-Graham-Méléard [4, Theorem 3.8]) yield that a process $(V_t)_{t\in[0,T]}$ on some filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$ is a solution to $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0})$ if and only if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t\in[0,T]}$-adapted f-Poisson measure $N(dt,dv,dz,d\varphi)$ such that (recall the expression (2.3) of $\tilde{\mathcal{A}}$, and that $\tilde{N}$ stands for the compensated Poisson measure)

$$
\begin{aligned}
V_t=v_0&+\int_0^t\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}c(V_{s-},v,s,\varphi)\tilde{N}(ds,dv,dz,d\varphi)\\
&-\kappa_0\int_0^t ds\int_{\mathbb{R}^3}f_s(dv)\Phi(|V_s-v|)(V_s-v).
\end{aligned}
\tag{4.37}
$$

We thus just have to prove the existence and uniqueness in law for solutions to (4.37).

Step 4. We now check that for $(V_t)_{t\in[0,T]}$ a solution to (4.37),

$$E\left[\sup_{[0,T]}|V_t|^2\right]<\infty. \tag{4.38}$$

We introduce, for $n\geq 1$, the stopping time $\tau_n=\inf\{t\geq 0, |V_t|\geq n\}$. Using the Doob and Cauchy-Schwarz inequalities, thanks to (2.5) and **(A4)**(γ) we get

$$\begin{aligned}E\left[\sup_{[0,t\wedge\tau_n]}|V_s|^2\right]&\leq C|v_0|^2+CE\left[\int_0^{t\wedge\tau_n}ds\int_{\mathbb{R}^3}f_s(dv)\int_0^\infty dz\int_0^{2\pi}d\varphi|c(V_{s-},v,s,\varphi)|^2\right]\\&\quad+C_TE\left[\left|\int_0^{t\wedge\tau_n}ds\int_{\mathbb{R}^3}f_s(dv)\Phi(|V_{s-}-v|)|V_{s-}-v|\right|^2\right] \qquad (4.39)\\&\leq C|v_0|^2+CE\left[\int_0^{t\wedge\tau_n}ds\int_{\mathbb{R}^3}f_s(dv)|V_s-v|^{2+\gamma}\right]+C_TE\left[\left|\int_0^{t\wedge\tau_n}ds\int_{\mathbb{R}^3}f_s(dv)|V_s-v|^{1+\gamma}\right|^2\right].\end{aligned}$$

Separating as usual the cases $\gamma\in(-3,-2]$, $\gamma\in(-2,-1]$ and $\gamma\in(-1,0]$ (see e.g. the proof of Lemma 4.3), we obtain in any case

$$\begin{aligned}E\left[\sup_{[0,t\wedge\tau_n]}|V_s|^2\right]&\leq C|v_0|^2+C_TE\left[\int_0^{t\wedge\tau_n}ds|V_s|^2\right]+C_T\int_0^T J_\gamma(f_s)ds\\&\quad+C_T\left(\int_0^T J_\gamma(f_s)ds\right)^2+C_T\int_0^T dsm_2(f_s)\\&\leq C_T(v_0,f)+C_T\int_0^t dsE\left[\sup_{[0,s\wedge\tau_n]}|V_u|^2\right]. \qquad (4.40)\end{aligned}$$

The Gronwall Lemma ensures us that for all $n\geq 1$, $E\left[\sup_{[0,T\wedge\tau_n]}|V_s|^2\right]\leq C_T(v_0,f)e^{TC_T}$. We immediately deduce that a.s., $\lim_n\tau_n=\infty$, and then that (4.38) holds.

Step 5. Let $(V_t)_{t\in[0,T]}$ be a càdlàg adapted solution to (4.37), for some f-Poisson measure N. Recall Lemma 2.2, and define $(\tilde V_t^k)_{t\in[0,T]}$ as the solution (which clearly exists and is unique since $1\!\!1_{\{z\leq k\}}N(ds,dv,dz,d\varphi)$ is a.s. finite)

$$\tilde V_t^k=v_0+\int_0^t\int_{\mathbb{R}^3}\int_0^k\int_0^{2\pi}c(\tilde V_{s-}^k,v,z,\varphi+\varphi_0(V_{s-}-v,\tilde V_{s-}^k-v))N(ds,dv,dz,d\varphi). \tag{4.41}$$

The map $\varphi_0(V_{s-}-v,\tilde V_{s-}^k-v)$ being predictable, we deduce from Lemma 4.4 that N_0 defined by $N_0(A)=\int_0^T\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}1\!\!1_A(s,v,z,\varphi+\varphi_0(V_{s-}-v,\tilde V_{s-}^k-v))N(ds,dv,dz,d\varphi)$ is still a f-Poisson measure. Hence $(\tilde V_t^k)_{t\in[0,T]}$ is nothing but the (v_0,f,k,N_0)-process, and its law is entirely determined by v_0 and f, see Notation 4.1.

We will now show that $(\tilde V_t^k)_{t\geq 0}$ goes in probability to $(V_t)_{t\geq 0}$, which will yield the uniqueness of the law of $(V_t)_{t\geq 0}$ and thus will end the proof of (iii). To this end, we first observe that due to

Step 4 and Lemma 4.3,

$$C(T,f,v_0) := E\left[\sup_{[0,T]}|V_t|^2\right] + \sup_k E\left[\sup_{[0,T]}|\tilde{V}_t^k|^2\right] < \infty. \tag{4.42}$$

Then, we may rewrite, recalling (2.11)

$$\begin{aligned} \tilde{V}_t^k &= v_0 + \int_0^t\int_{\mathbb{R}^3}\int_0^k\int_0^{2\pi} c(\tilde{V}_{s-}^k, v, z, \varphi + \varphi_0(V_{s-}-v, \tilde{V}_{s-}^k - v))\tilde{N}(ds,dv,dz,d\varphi) \\ &\quad + \int_0^t ds\int_{\mathbb{R}^3} f_s(dv)h_0^k(|\tilde{V}_s^k - v|)(\tilde{V}_s^k - v). \end{aligned} \tag{4.43}$$

This is due to the fact that $\int_0^k dz\int_0^{2\pi} d\varphi c(V,v,z,\varphi) = -(V-v)h_0^k(|V-v|)$. Hence,

$$(V_t - \tilde{V}_t^k) \quad = \quad A_t^{1,k} + A_t^{2,k} + A_t^{3,k} + A_t^{4,k}, \tag{4.44}$$

where

$$\begin{aligned} A_t^{1,k} &:= \int_0^t\int_{\mathbb{R}^3}\int_0^k\int_0^{2\pi}[c(V_{s-},v,z,\varphi) - c(\tilde{V}_{s-}^k, v, z, \varphi+\varphi_0(V_{s-}-v, \tilde{V}_{s-}^k - v))]\tilde{N}(ds,dv,dz,d\varphi), \\ A_t^{2,k} &:= \int_0^t\int_{\mathbb{R}^3}\int_k^\infty\int_0^{2\pi} c(V_{s-},v,z,\varphi)\tilde{N}(ds,dv,dz,d\varphi), \\ A_t^{3,k} &:= \int_0^t ds\int_{\mathbb{R}^3} f_s(dv)[h_0^k(|V_s - v|)(V_s - v) - h_0^k(|\tilde{V}_s^k - v|)(\tilde{V}_s^k - v)], \\ A_t^{4,k} &:= \int_0^t ds\int_{\mathbb{R}^3} f_s(dv)[\kappa_0\Phi(|V_s - v|)(V_s - v) - h_0^k(|V_s - v|)(V_s - v)]. \end{aligned}$$

First, we immediately deduce from the Doob inequality, (2.6) and **(A4)**(γ), that

$$\begin{aligned} E\left[\sup_{[0,t]}|A_s^{1,k}|^2\right] &\leq C\int_0^t ds\int_{\mathbb{R}^3} f_s(dv)E\left[|V_s - \tilde{V}_s^k|^2(|V_s - v|^\gamma + |\tilde{V}_s^k - v|^\gamma)\right] \\ &\leq C\int_0^t dsE\left[|V_s - \tilde{V}_s^k|^2\right]J_\gamma(f_s). \end{aligned} \tag{4.45}$$

Next, Doob's inequality, (2.12) and **(A4)**(γ) yield

$$E\left[\sup_{[0,t]}|A_s^{2,k}|^2\right] \leq C\int_0^T ds\int_{\mathbb{R}^3} f_s(dv)E\left[|V_s - v|^{2+\gamma}\varepsilon_0^k(|V_s - v|)\right] \to 0 \tag{4.46}$$

as k tends to infinity, since due to Lemma 2.4, ε_0^k is bounded and tends simply to 0, and since $|V_s - v|^{2+\gamma}$ belongs to $L^1(dsf_s(dv)P(d\omega))$ (as usual, if $2+\gamma \geq 0$, this follows from (4.42) and the fact that $f \in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$, while if $2+\gamma < 0$, we just use that $f \in L^1([0,T],\mathcal{J}_\gamma)$).

Using (2.13), **(A4)**(γ), and then the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
E\left[\sup_{[0,t]}|A^{3,k}_s|^2\right] &\leq CE\left[\left(\int_0^t ds\int_{\mathbb{R}^3} f_s(dv)|V_s-\tilde{V}^k_s|(|V_s-v|^\gamma+|\tilde{V}^k_s-v|^\gamma)\right)^2\right]\\
&\leq CE\left[\left(\int_0^t ds|V_s-\tilde{V}^k_s|J_\gamma(f_s)\right)^2\right]\\
&\leq C\left(\int_0^t dsE[|V_s-\tilde{V}^k_s|^2]J_\gamma(f_s)\right)\left(\int_0^T dsJ_\gamma(f_s)\right). \quad (4.47)
\end{aligned}
$$

Finally, (2.14) and **(A4)**(γ) allow us to obtain,

$$
E\left[\sup_{[0,t]}|A^{4,k}_s|^2\right] \leq CE\left[\left(\int_0^T ds\int_{\mathbb{R}^3} f_s(dv)|V_s-v|^{1+\gamma}\varepsilon^k_0(|V_s-v|)\right)^2\right]\to 0 \quad (4.48)
$$

using similar arguments as for the study of $A^{1,k}$. We thus obtain, for some η_k going to 0, some constant $C(T,f)$,

$$
E\left[|V_t-\tilde{V}^k_t|^2\right]\leq \eta_k + C(T,f)\int_0^t dsE[|V_s-\tilde{V}^k_s|^2]J_\gamma(f_s). \quad (4.49)
$$

The generalized Gronwall Lemma 6.1 and the fact that $f\in L^1([0,T],\mathcal{J}_\gamma)$ by assumption allow us to conclude that

$$
E\left[\sup_{[0,T]}|V_t-\tilde{V}^k_t|^2\right]\leq \eta_k\exp[C(T,f)\int_0^T dsJ_\gamma(f_s)]\to 0 \quad (4.50)
$$

as k tends to infinity. Hence $(\tilde{V}^k_t)_{t\in[0,T]}$ goes in probability to $(V_t)_{t\in[0,T]}$.

Step 6. It remains to prove (ii), i.e. the existence for $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0})$. We use to this aim a Picard iteration. Let N be a f-Poisson measure as in Step 2. We consider the constant process $V^0_t\equiv v_0$, we set $\varphi^*_0=0$ and we define recursively

$$
\begin{aligned}
V^{n+1}_t &= v_0+\int_0^t\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} c(V^n_{s-},v,z,\varphi+\varphi^*_n(s,v))\tilde{N}(ds,dv,dz,d\varphi)\\
&\quad -\kappa_0\int_0^t ds\int_{\mathbb{R}^3} f_s(dv)\Phi(|V^n_s-v|)(V^n_s-v), \quad (4.51)
\end{aligned}
$$

and $\varphi^*_{n+1}(s,v)=\varphi^*_n(s,v)+\varphi_0(V^{n+1}_{s-}-v,V^n_{s-}-v)$, where φ_0 is defined by Lemma 2.2. A computation based on Doob's inequality using (2.6) and that $|x\Phi(x)-y\Phi(y)|\leq \min(x,y)|\Phi(x)-\Phi(y)|+$

$|x-y|(\Phi(x)+\Phi(y)) \leq C|x-y|(x^\gamma+y^\gamma)$ yields that for all $t\in[0,T]$, all $n\geq 1$,

$$\begin{aligned}
E\left[\sup_{[0,t]}|V_s^{n+1}-V_s^n|^2\right] &\leq C\int_0^t ds\int_{\mathbb{R}^3} f_s(dv)E[|V_s^n-V_s^{n-1}|^2(|V_s^n-v|^\gamma+|V_s^{n-1}-v|^\gamma)]\\
&\quad +CE\left[\left(\int_0^t ds\int_{\mathbb{R}^3} f_s(dv)|V_s^n-V_s^{n-1}|(|V_s^n-v|^\gamma+|V_s^{n-1}-v|^\gamma)]\right)^2\right]\\
&\leq C\int_0^t dsE[|V_s^n-V_s^{n-1}|^2]J_\gamma(f_s)\\
&\quad +C\left(\int_0^T dsJ_\gamma(f_s)\right)\left(\int_0^t dsE[|V_s^n-V_s^{n-1}|^2]J_\gamma(f_s)\right)\\
&\leq C(T,f)\int_0^t dsE[|V_s^n-V_s^{n-1}|^2]J_\gamma(f_s). \qquad (4.52)
\end{aligned}$$

Using Lemma 6.1, we deduce that $\sum_n E\left[\sup_{[0,T]}|V_s^{n+1}-V_s^n|^2\right]<\infty$, so that there exists a càdlàg adapted process $(V_t)_{t\in[0,T]}$ such that

$$\lim_n E\left[\sup_{[0,T]}|V_s-V_s^n|^2\right]=0 \text{ and } E\left[\sup_{[0,T]}|V_s|^2\right]<\infty. \qquad (4.53)$$

To show that $(V_t)_{t\geq 0}$ satisfies $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0}))$, we need to check that for all $0\leq s_1\leq ...\leq s_l\leq s\leq t\leq T$, all $\phi_1,...,\phi_l\in C_c(\mathbb{R}^3)$, and all $\phi\in C_c^2$,

$$E\left[\left(\prod_{i=1}^l\phi_i(V_{s_i})\right)\left(\phi(V_t)-\phi(V_s)-\int_s^t\tilde{\mathcal{A}}_u\phi(V_u)du\right)\right]=0. \qquad (4.54)$$

But we know from (4.51) that for all $n\geq 1$,

$$E\left[\left(\prod_{i=1}^l\phi_i(V_{s_i}^n)\right)\left(\phi(V_t^{n+1})-\phi(V_s^{n+1})-\int_s^t\tilde{\mathcal{A}}_u\phi(V_u^n)du\right)\right]=0. \qquad (4.55)$$

It remains to pass to the limit in (4.55) to obtain (4.54). It suffices to use (4.53), and to observe that the map $(v_u)_{u\in[0,T]}\mapsto\mathcal{K}((v_u)_{u\in[0,T]}):=\left(\prod_{i=1}^l\phi_i(v_{s_i})\right)\left(\phi(v_t)-\phi(v_s)-\int_s^t\tilde{\mathcal{A}}_u\phi(V_u)du\right)$ is continuous on $\mathbb{D}_T$ (endowed here with the uniform convergence) and bounded. First, we have shown in Step 2-(b) that $\tilde{\mathcal{A}}_t\phi$ is bounded by $C_\phi(1+m_2(f_t)+J_\gamma(f_t))$, and we easily deduce that $\mathcal{K}$ is bounded since $f\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))\cap L^1([0,T],\mathcal{J}_\gamma)$. Next, the only difficulty concerning the continuity of $\mathcal{K}$ is to check that of $(v_u)_{u\in[0,T]}\mapsto\int_s^t ds\tilde{\mathcal{A}}_u\phi(v_u)du$. Since $\tilde{\mathcal{A}}_u\phi$ is bounded by $C_\phi(1+m_2(f_u)+J_\gamma(f_u))\in L^1([0,T])$, it suffices to check that for each $u\in[0,T]$, $\tilde{\mathcal{A}}_u\phi$ is continuous on $\mathbb{R}^3$. This follows from Lemma 2.5. □

We finally conclude the section with the

Proof of Lemma 4.2. The proof is actually almost contained in that of Lemma 4.6. Indeed, consider a weak solution $f\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))\cap L^1([0,T],\mathcal{J}_\gamma)$ to (1.1), and define, for each $t\in[0,T]$, the operator $\tilde{\mathcal{A}}_t$ by (4.32). We have checked in Step 6 the existence of a solution to $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]},\delta_{v_0})$. Of course, the same arguments allow us to prove the existence of a solution

$(V_t)_{t\in[0,T]}$ to $MP((\tilde{\mathcal{A}}_t)_{t\in[0,T]}, f_0)$, since $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. Consider now the law g_t of V_t. Then taking expectations in (4.34), one easily deduces that $(g_t)_{t\in[0,T]}$ solves (4.31), so that due to Lemma 4.6, $g = f$. Hence for each $t \in [0,T]$, the law of V_t is nothing but f_t. Next, using Steps 3 and 5, we have shown how to build some f-Poisson measures N^k in such a way that for $(\tilde{V}_t^k)_{t\in[0,T]}$ the (V_0, f, k, N^k)-process, $\lim_k E[\sup_{[0,T]} |V_t - \tilde{V}_t^k|^2] = 0$. Denoting by f_t^k the law of $\tilde{V}_t^k$, this of course implies that $\lim_k \sup_{[0,T]} W_2(f_t, f_t^k) = 0$, which was our goal. □

5. Applications

We now prove our well-posedness results. We start with the case of regularized velocity cross sections.

Proof of Corollary 1.4. We assume **(A1-A2-A3-A4**(0)**)**. We oberve that for any $g \in \mathcal{P}_2(\mathbb{R}^3)$, $J_0(g) \leq 1$. Hence for any pair of solutions (f_t) and $(\tilde{f}_t)_{t\in[0,T]}$, and such that one of them has a density for all times, (1.22) follows immediately from Theorem 1.3.

Let us now prove the existence result. We found no reference about such an existence result, but it is completely standard. The case where $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ has a finite entropy, that is when $\int_{\mathbb{R}^3} f_0(v)\log(1+f_0(v))dv < \infty$ can be treated following the line of Villani [18] (and is much more easy since we assume here that Φ is bounded, while true soft potentials were treated there). The obtained solution has furthermore a finite entropy (and thus a density) for all times. Then the existence result for any $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ is a straightforward consequence of (1.22): it suffices to consider a sequence $f_0^n \in \mathcal{P}_2(\mathbb{R}^3)$ with finite entropy, tending to f_0 for the distance W_2, and the associated weak solutions $(f_t^n)_{t\in[0,T]}$ to (1.17). Then (1.22) ensures us that there exists $(f_t)_{t\in[0,T]}$ such that $\lim_{n\to\infty} \sup_{[0,T]} W_2(f_t^n, f_t) = 0$. It is then not hard to pass to the limit in (1.17), and to deduce that $(f_t)_{t\in[0,T]}$ is indeed a weak solution to (1.17).

Let us now extend (1.22) to any pair of solutions $(f_t)_{t\in[0,T]}$, $(\tilde{f}_t)_{t\in[0,T]})$, without assuming that f_t (or $\tilde{f}_t$) has a density for all times: consider f_0^n with a finite entropy, such that $W_2(f_0, f_0^n)$ tends to 0, and the associated solution $(f_t^n)_{t\in[0,T]}$. Since f_t^n has a finite entropy (and thus a density) for all times, we deduce that for all $n \geq 1$, all $t \in [0,T]$,

$$\begin{aligned} W_2(f_t,\tilde{f}_t) &\leq W_2(f_t,f_t^n) + W_2(f_t^n,\tilde{f}_t) \leq [W_2(f_0,f_0^n) + W_2(f_0^n,\tilde{f}_0)]e^{Kt} \\ &\leq [2W_2(f_0,f_0^n) + W_2(f_0,\tilde{f}_0)]e^{Kt} \end{aligned} \tag{5.1}$$

by the triangular inequality. Letting n tend to infinity, we obtain (1.22). The uniqueness result is now straightforward. □

We now study the case of soft potentials.

Proof of Corollary 1.5. We consider $\gamma \in (-3,0)$, and assume **(A1)-(A2)-(A3)-(A4)**(γ).

First note that we consider only solutions with densities here, since we work in L^p with $p > 3/(3+\gamma) > 1$.

We also observe that for $\alpha \in (-3,0)$, and for $p \in (3/(3+\alpha),\infty]$, there exists a constant $C_{\alpha,p}$ such that for any $g \in \mathcal{P}_2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$, any $v \in \mathbb{R}^d$,

$$\begin{aligned} J_\alpha(g) &= \sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3} g(v_*)\,|v-v_*|^\alpha\,dv_* \\ &\le \sup_{v\in\mathbb{R}^3}\int_{|v_*-v|<1} g(v_*)\,|v-v_*|^\alpha\,dv_* + \sup_{v\in\mathbb{R}^3}\int_{|v_*-v|\ge 1} g(v_*)\,dv_* \\ &\le C_{\alpha,p}\,\|g\|_{L^p(\mathbb{R}^3)}+1, \end{aligned} \tag{5.2}$$

where $C_{\alpha,p} = \sup_{v\in\mathbb{R}^3}[\int_{|v_*-v|\le 1}|v-v_*|^{\alpha p/(p-1)}dv_*]^{(p-1)/p} = [\int_{|v_*|\le 1}|v_*|^{\alpha p/(p-1)}dv_*]^{(p-1)/p} < \infty$, since by assumption, $\alpha p/(p-1) > -3$.

Step 1. We first observe that point (i) is an immediate consequence of Theorem 1.3 and (5.2), since we deal with solutions with densities.

Step 2. We now check point (ii). We only have to prove the existence of solutions, since uniqueness follows from point (i). Using some results of Villani [18, Theorems 1 and 3], we know that for $\gamma \in (-3,0)$, for any $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} f_0(v)\log(1+f_0(v))dv < \infty, \tag{5.3}$$

there exists a weak solution $(f_t)_{t\ge0} \in L^\infty([0,\infty), L^1(\mathbb{R}^3,(1+|v|^2)\,dv))$ to (1.1) starting from f_0. Recall that if $f_0 \in \mathcal{P}_2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ for some $p>1$, then (5.3) holds. Then the existence result of point (ii) follows immediately from the following *a priori* estimate, which guarantees that if $f_0 \in L^p(\mathbb{R}^d)$ for some $p > 3/(3+\gamma)$, then this bound propagates locally (in time): there exists $C = C(p,\gamma,\kappa_1,\kappa_2,\kappa_3)$ such that any weak solution to (1.1) *a priori* satisfies

$$\frac{d}{dt}\|f_t\|_{L^p} \le C\left(1+\|f_t\|^2_{L^p(\mathbb{R}^3)}\right). \tag{5.4}$$

This will guarantee that for $0 \le t < T_* := \frac{1}{C}(\pi/2 - \arctan||f_0||_{L^p})$, we have

$$\|f_t\|_{L^p} \le \tan\left(\arctan\|f_0\|_{L^p} + C\,t\right). \tag{5.5}$$

Thus point (ii) will be proved.

To obtain (5.4), we follow the method of Desvillettes-Mouhot, see [5, Proposition 3.2]. First, one classically may replace, in $\mathcal{A}\phi$ taken in the form (1.18), $\beta(\theta)$ by $\hat\beta(\theta) = [\beta(\theta)+\beta(\pi-\theta)]\mathbb{1}_{\{\theta\in(0,\pi/2]\}}$, see e.g. [1, Introduction] or [5, Section 2]. This relies on the use of (1.15). Next, following the line of [5, proof of Proposition 3.2], we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}|f_t(v)|^p dv \le (p-1)\int_{\mathbb{R}^3} f_t(v_*)dv_*\int_{\mathbb{R}^3} dv\Phi(|v-v_*|)\int_0^{\pi/2}\hat\beta(\theta)d\theta\int_0^{2\pi}d\varphi[f_t^p(v')-f_t^p(v)], \tag{5.6}$$

where v' is given by (1.13). Using now the cancelation Lemma of Alexandre-Desvillettes-Villani-Wennberg [1, Lemma 1] (with $N=3$, f given by f_t^p, and $B(|v-v_*|,\cos\theta)\sin\theta = \hat\beta(\theta)\Phi(|v-v_*|)$), we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3}|f_t(v)|^p dv \le 2\pi(p-1)&\int_{\mathbb{R}^3} f_t(v_*)dv_*\int_{\mathbb{R}^3} f_t^p(v)dv\int_0^{\pi/2}\hat\beta(\theta)d\theta \\ &\left|\cos^{-3}(\theta/2)\Phi(|v-v_*|\cos^{-1}(\theta/2)) - \Phi(|v-v_*|)\right|. \end{aligned} \tag{5.7}$$

Due to **(A4)**(γ), we know that $|\Phi(x)-\Phi(y)|\leq \kappa_3(x^\gamma+y^\gamma)|x-y|/\min(x,y)$. One easily deduces that for some constant $C=C(\kappa_3)$, for all $x\in\mathbb{R}_+$, all $\theta\in(0,\pi/2]$,

$$\begin{aligned}\left|\cos^{-3}(\theta/2)\Phi(x\cos^{-1}(\theta/2))-\Phi(x)\right| &\leq \left|\cos^{-3}(\theta/2)-1\right|\Phi(x)\\ &\quad+\cos^{-3}(\theta/2)\left|\Phi(x\cos^{-1}(\theta/2))-\Phi(x)\right|\\ &\leq C\theta^2x^\gamma+C|x\cos^{-1}(\theta/2)-x|(x^\gamma+(x\cos^{-1}(\theta/2))^\gamma)/x\\ &\leq C\theta^2x^\gamma. \qquad (5.8)\end{aligned}$$

we used here that $|\cos^{-3}(\theta/2)-1|+|\cos^{-1}(\theta/2)-1|\leq \text{cst}\theta^2$ for all $\theta\in(0,\pi/2]$.
Since $\int_0^{\pi/2}\theta^2\hat\beta(\theta)d\theta\leq\kappa_1$, we finally get, with $C=C(\gamma,p,\kappa_1,\kappa_2,\kappa_3)$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}|f_t(v)|^pdv &\leq C\int_{\mathbb{R}^3}f_t^p(v)dv\int_{\mathbb{R}^3}|v-v_*|^\gamma f_t(v_*)dv_*\\ &\leq C\left(\int_{\mathbb{R}^3}f_t^p(v)dv+C_{\gamma,p}\left[\int_{\mathbb{R}^3}f_t^p(v)dv\right]^{1+1/p}\right) \qquad (5.9)\end{aligned}$$

the last inequality using (5.2). This yields

$$\frac{d}{dt}||f_t||_{L^p} \leq C\left(||f_t||_{L^p}+C_{\gamma,p}||f_t||^2_{L^p}\right), \qquad (5.10)$$

from which (5.4) immediately follows. □

6. Appendix

We start this annex by recalling the generalized Gronwall Lemma and the associated Picard Lemma.

Lemma 6.1. *Let $T\geq 0$ and a nonnegative function v on $[0,T]$ such that $\int_0^T v(s)ds<\infty$.*
(i) Any nonnegative bounded function on $[0,T]$ satisfying $u(t)\leq a+\int_0^t u(s)v(s)ds$ for all $t\in[0,T]$ also satisfies $u(t)\leq a\exp(\int_0^t v(s)ds)$ for all $t\in[0,T]$.
(ii) Consider a sequence of nonnegative functions u_n on $[0,T]$, with u_0 bounded and $u_{n+1}(t)\leq \int_0^t u_n(s)v(s)ds$ for all $n\geq 0$, all $t\in[0,T]$. Then $\sum_{n\geq 0}\sup_{[0,T]}u_n(t)\leq ||u_0||_\infty\exp(\int_0^T v(s)ds)$.

We carry on with the

Proof of Lemma 1.1. We first prove (i). Recall that $H(\theta)=\int_\theta^\pi\beta(x)dx$, that G is its inverse function, and that $c\theta^{-1-\nu}\leq\beta(\theta)\leq C\theta^{-1-\nu}$ for some $\nu\in(0,2)$. Since $H(\theta)\leq a(\theta^{-\nu}-\pi^{-\nu})$ (with $a=C/\nu$), we deduce that $G(z)\leq(z/a+\pi^{-\nu})^{-1/\nu}$. Now for $0\leq z\leq w$

$$\begin{aligned}0\leq G(z)-G(w) &= -\int_z^w G'(u)du=\int_z^w\frac{du}{\beta(G(u))}\leq\frac{1}{c}\int_z^w G(u)^{\nu+1}\\ &\leq \frac{1}{c}\int_z^w(u/a+\pi^{-\nu})^{-1-1/\nu}\leq A\left[(z/a+\pi^{-\nu})^{-1/\nu}-(w/a+\pi^{-\nu})^{-1/\nu}\right]\\ &\leq B\left[(1+\varepsilon z)^{-1/\nu}-(1+\varepsilon w)^{-1/\nu}\right] \qquad (6.11)\end{aligned}$$

for some constants A, B, $\varepsilon>0$. We set $\alpha=1/\nu>1/2$, and first treat the

Case $\alpha \in (1/2, 1]$. Let thus $x \geq y > 0$, and $z \in (0, \infty)$. Using the inequality $|u^\alpha - v^\alpha| \leq$ cst $|u-v|/(u^{1-\alpha}+v^{1-\alpha})$, we obtain, the value of B changing from line to line,

$$\begin{aligned}
|G(z/x) - G(z/y)|^2 &\leq B|(1+\varepsilon z/x)^{-1} - (1+\varepsilon z/y)^{-1}|^2((1+\varepsilon z/x)^{\alpha-1} + (1+\varepsilon z/y)^{\alpha-1})^{-2} \\
&\leq B\left|\frac{x}{x+\varepsilon z} - \frac{y}{y+\varepsilon z}\right|^2 \left(\frac{x+\varepsilon z}{x}\right)^{2-2\alpha} \\
&\leq B(x-y)^2 z^2 x^{2\alpha-2}(x+\varepsilon z)^{-2\alpha}(y+\varepsilon z)^{-2} \\
&\leq B(x-y)^2 x^{2\alpha-2}(x+\varepsilon z)^{-2\alpha}. \qquad (6.12)
\end{aligned}$$

Integrating this inequality, we get

$$\int_0^\infty dz|G(z/x) - G(z/y)|^2 \leq B(x-y)^2 x^{2\alpha-2} x^{1-2\alpha} \leq B\frac{(x-y)^2}{x} \leq B\frac{(x-y)^2}{x+y}, \qquad (6.13)$$

since $x \geq y$ by assumption.

Case $\alpha \in [1, \infty)$. Let $x \geq y > 0$. Then

$$\begin{aligned}
|G(z/x) - G(z/y)|^2 &\leq B|(1+\varepsilon z/x)^{-\alpha} - (1+\varepsilon z/y)^{-\alpha}|^2 \\
&\leq B|(1+\varepsilon z/x)^{-1} - (1+\varepsilon z/y)^{-1}|^2, \qquad (6.14)
\end{aligned}$$

and we may use the same computation as previously with $\alpha = 1$.

We leave the proof (ii) to the reader, and finally prove (iii). We thus assume that $\Phi(x) = x^\gamma$ for some $\gamma \in (-3, 0)$, and show that (1.9) holds with $\Psi(x) =$cst.x^γ. First, it is of course immediate that $(x-y)^2[\Phi(x)+\Phi(y)] \leq (x-y)^2[x^\gamma + y^\gamma]$. Next,

$$\begin{aligned}
\min(x,y)|x-y||\Phi(x) - \Phi(y)| &\leq |\gamma| \min(x,y)(x-y)^2 \min(x,y)^{\gamma-1} \\
&\leq |\gamma|(x-y)^2 \min(x,y)^\gamma \leq |\gamma|(x-y)^2(x^\gamma + y^\gamma). \qquad (6.15)
\end{aligned}$$

Finally,

$$\begin{aligned}
\min(x^2,y^2)\frac{|\Phi(x)-\Phi(y)|^2}{\Phi(x)+\Phi(y)} &\leq |\gamma|^2 \min(x^2,y^2)(x-y)^2 \frac{\min(x,y)^{2\gamma-2}}{x^\gamma + y^\gamma} \\
&\leq |\gamma|^2 (x-y)^2 \min(x,y)^\gamma \leq |\gamma|^2(x-y)^2(x^\gamma+y^\gamma). \qquad (6.16)
\end{aligned}$$

As a conclusion, (1.9) holds with $\Psi(x) := (1+|\gamma|+\gamma^2)x^\gamma$. □

Next, we give the

Proof of (1.19). Let thus $\phi \in C^2_\infty$, denote by ϕ'' its Hessian matrix, and set $\Delta = \Delta(v, v_*, \theta, \varphi) = \phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)$, where we used the shortened notation (1.13). Recalling that $v' = v + a$ while $v'_* = v_* - a$, a Taylor expansion yields that for some $w_1, w_2 \in \mathbb{R}^3$, $\Delta = a.[\nabla\phi(v) - \nabla\phi(v_*)] + \frac{1}{2}a.[\phi''(w_1) + \phi''(w_2)]a$. Recall now that $\int_0^{2\pi} a d\varphi = -\frac{1-\cos\theta}{2}(v - v_*)$, and that $|a|^2 = \frac{1-\cos\theta}{2}|v-v_*|^2$. Hence

$$\begin{aligned}
\left|\int_0^{2\pi} \Delta d\varphi\right| &\leq \frac{1-\cos\theta}{2}|v-v_*|.|\nabla\phi(v) - \nabla\phi(v_*)| + 2\pi||\phi''||_\infty \frac{1-\cos\theta}{2}|v-v_*|^2 \\
&\leq C(1-\cos\theta)||\phi''||_\infty |v-v_*|^2, \qquad (6.17)
\end{aligned}$$

which yields the desired result, since $1 - \cos\theta \leq \theta^2$. □

Next, we treat the

Proof of Lemma 2.1. First, the second equality in (2.3) is obvious, since $c(v, v_*, z, .)$ is 2π-peridodic. Next, we consider $\phi \in C^2_\infty$. We have already seen that $\mathcal{A}\phi$ is well-defined (see (1.19)), and $\tilde{\mathcal{A}}\phi$ is also well-defined, since, setting $c = c(v, v_*, z, \varphi)$ for simplicity, $|\phi(v+c)-\phi(v)-c.\nabla\phi(v)| \leq |c|^2||\phi''||_\infty/2$. Using the substitution $\theta = G(z/\Phi(|v-v_*|))$, which yields $dz = -\Phi(|v-v_*|)\beta(\theta)d\theta$, we get,

$$\int_0^\infty dz \int_0^{2\pi} d\varphi |c(v, v_*, z, \varphi)|^2 = \Phi(|v-v_*|)\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi |a(v, v_*, \theta, \varphi)|^2$$

$$= \Phi(|v-v_*|)\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi |v-v_*|^2 \frac{1-\cos\theta}{2} \leq C|v-v_*|^2\Phi(|v-v_*|) \quad (6.18)$$

thanks to **(A2)**, where C depends only on κ_1. Thus $\tilde{\mathcal{A}}\phi$ is well-defined for $\phi \in C^2_\infty$, and if $\phi \in C^2_b$,

$$|\tilde{\mathcal{A}}\phi(v, v_*)| \leq C||\phi''||_\infty |v-v_*|^2\Phi(|v-v_*|) + C||\phi'||_\infty|v-v_*|\Phi(|v-v_*|). \quad (6.19)$$

Next, we consider again $\phi \in C^2_\infty$. Using the substitution $\theta = G(z/\Phi(|v-v_*|))$, we observe that (using the shortened notation (1.13))

$$\tilde{\mathcal{A}}\phi(v, v_*) = \Phi(|v-v_*|)\left[\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi[\phi(v') - \phi(v) - a.\nabla\phi(v)] - \kappa_0(v-v_*).\nabla\phi(v)\right]. \quad (6.20)$$

Using now (1.14) and that $\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi a = -\kappa_0(v-v_*)$, we obtain

$$\frac{\tilde{\mathcal{A}}\phi(v, v_*) + \tilde{\mathcal{A}}\phi(v_*, v)}{2} = \frac{\Phi(|v-v_*|)}{2}\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi[\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)], \quad (6.21)$$

which was our goal. □

Proof of Lemma 2.3. First (2.5) has already been proved, see (6.18). Using again the substitution $\theta = G(z/\Phi(|v-v_*|))$, we obtain (2.7):

$$\int_0^\infty dz \left|\int_0^{2\pi} d\varphi c(v, v_*, z, \varphi)\right| = \Phi(|v-v_*|)\int_0^\pi \beta(\theta)d\theta \left|\int_0^{2\pi} d\varphi a(v, v_*, \theta, \varphi)\right|$$

$$= \pi|v-v_*|\Phi(|v-v_*|)\int_0^\pi \beta(\theta)d\theta\,[1-\cos\theta] \leq C|v-v_*|\Phi(|v-v_*|). \quad (6.22)$$

Next, we observe that

$$\begin{aligned}
\Delta &:= |c(v, v_*, z, \varphi) - c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v-v_*, \tilde{v}-\tilde{v}_*)|^2 \\
&\leq 4\left|\frac{1-\cos G(z/\Phi(|v-v_*|))}{2}[(v-v_*) - (\tilde{v}-\tilde{v}_*)]\right|^2 \\
&\quad +4\left|\frac{\cos G(z/\Phi(|\tilde{v}-\tilde{v}_*|)) - \cos G(z/\Phi(|v-v_*|))}{2}[\tilde{v}-\tilde{v}_*]\right|^2 \\
&\quad +4\left|\frac{\sin G(z/\Phi(|v-v_*|))}{2}[\Gamma(v-v_*, \varphi) - \Gamma(\tilde{v}-\tilde{v}_*, \varphi + \varphi_0(v-v_*, \tilde{v}-\tilde{v}_*))]\right|^2 \\
&\quad +4\left|\frac{\sin G(z/\Phi(|\tilde{v}-\tilde{v}_*|)) - \sin G(z/\Phi(|v-v_*|))}{2}\Gamma(\tilde{v}-\tilde{v}_*, \varphi + \varphi_0(v-v_*, \tilde{v}-\tilde{v}_*))\right|^2. \quad (6.23)
\end{aligned}$$

Using now Lemma 2.2, that $|\Gamma(X,\varphi)| = |X|$, and easy estimates about cosinus and sinus functions, we deduce that

$$\begin{aligned}\Delta &\le C|(v-v_*)-(\tilde v-\tilde v_*)|^2 G^2(z/\Phi(|v-v_*|))\\ &\quad +C|\tilde v-\tilde v_*|^2\,|G(z/\Phi(|v-v_*|))-G(z/\Phi(|\tilde v-\tilde v_*|))|^2.\end{aligned}\tag{6.24}$$

On the one hand, the substitution $\theta = G(z/\Phi(|v-v_*|))$ yields that $\int_0^\infty dz\int_0^{2\pi} d\varphi G^2(z/\Phi(|v-v_*|)) = 2\pi\Phi(|v-v_*|)\int_0^\pi \theta^2\beta(\theta)d\theta = 2\pi\kappa_1\Phi(|v-v_*|)$, and on the other hand we may use **(A3)**. We thus get

$$\begin{aligned}\int_0^\infty dz\int_0^{2\pi} d\varphi\Delta &\le C|(v-v_*)-(\tilde v-\tilde v_*)|^2\Phi(|v-v_*|)\\ &\quad +C|\tilde v-\tilde v_*|^2\frac{(\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|))^2}{\Phi(|v-v_*|)+\Phi(|\tilde v-\tilde v_*|)}.\end{aligned}\tag{6.25}$$

But using a symmetry argument, we easily deduce that

$$\begin{aligned}\int_0^\infty dz\int_0^{2\pi} d\varphi\Delta &\le C|(v-v_*)-(\tilde v-\tilde v_*)|^2(\Phi(|v-v_*|)+\Phi(|\tilde v-\tilde v_*|))\\ &\quad +C\min(|v-v_*|^2,|\tilde v-\tilde v_*|^2)\frac{(\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|))^2}{\Phi(|v-v_*|)+\Phi(|\tilde v-\tilde v_*|)}.\end{aligned}\tag{6.26}$$

Then (1.9) leads us to

$$\int_0^\infty dz\int_0^{2\pi} d\varphi\Delta \le C|(v-v_*)-(\tilde v-\tilde v_*)|^2[\Psi(|v-v_*|)+\Psi(|\tilde v-\tilde v_*|)]\tag{6.27}$$

which yields (2.6). We finally check (2.8). Integrating first against $d\varphi$, we get

$$\begin{aligned}D &:= \int_0^\infty dz\left|\int_0^{2\pi} d\varphi[c(v,v_*,z,\varphi)-c(\tilde v,\tilde v_*,z,\varphi)]\right|\\ &= \pi\int_0^\infty dz\Big|(v-v_*)[1-\cos G(z/\Phi(|v-v_*|))]-(\tilde v-\tilde v_*)[1-\cos G(z/\Phi(|\tilde v-\tilde v_*|))]\Big|\\ &\le \pi|(v-v_*)-(\tilde v-\tilde v_*)|\int_0^\infty dz[1-\cos G(z/\Phi(|v-v_*|))]\\ &\quad +\pi|\tilde v-\tilde v_*|\int_0^\infty dz\,|\cos G(z/\Phi(|v-v_*|))-\cos G(z/\Phi(|\tilde v-\tilde v_*|))|\\ &\le \pi|(v-v_*)-(\tilde v-\tilde v_*)|\int_0^\infty dz G^2(z/\Phi(|v-v_*|))\\ &\quad +\pi|\tilde v-\tilde v_*|\int_0^\infty dz\Big|G^2(z/\Phi(|v-v_*|))-G^2(z/\Phi(|\tilde v-\tilde v_*|))\Big|.\end{aligned}\tag{6.28}$$

The monotonicity of G ensures us that for any $x,y>0$,

$$\int_0^\infty dz\Big|G^2(z/x)-G^2(z/y)\Big| = \Big|\int_0^\infty dz G^2(z/x)-\int_0^\infty dz G^2(z/y)\Big|.\tag{6.29}$$

On the other hand, $\int_0^\infty dz G^2(z/x) = x\kappa_1$ (recall **(A2)**), thanks to the substitution $\theta = G(z/x)$. We thus obtain

$$D \le \kappa_1\pi|(v-v_*)-(\tilde v-\tilde v_*)|\Phi(|v-v_*|)+\kappa_1\pi|\tilde v-\tilde v_*|.|\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|)|.\tag{6.30}$$

A symmetry argument and then (1.9) thus yields

$$\begin{aligned} D &\leq \kappa_1\pi|(v-v_*)-(\tilde v-\tilde v_*)|(\Phi(|v-v_*|)+\Phi(|\tilde v-\tilde v_*|)) \\ &\quad +\kappa_1\pi\min(|v-v_*|,|\tilde v-\tilde v_*|)|\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|)| \\ &\leq \kappa_1\pi|(v-v_*)-(\tilde v-\tilde v_*)|(\Psi(|v-v_*|)+\Psi(|\tilde v-\tilde v_*|)), \end{aligned} \tag{6.31}$$

from which (2.8) follows.

We have already checked point (ii), see (6.19). Let us finally prove point (iii), following the line of [12, Lemma 4.1]. We thus consider $\phi\in C^2_c$, with supp $\phi\subset\{|v|\leq x\}$. Recalling (6.20), we see that

$$\begin{aligned} |\tilde{\mathcal{A}}\phi(v,v_*)| &\leq \kappa_0\Phi(|v-v_*|)|v-v_*|||\phi'||_\infty 1\!\!1_{\{|v|\leq x\}} \\ &\quad +\Phi(|v-v_*|)1\!\!1_{\{|v|\leq 2x\}}\int_0^\pi\beta(\theta)d\theta\int_0^{2\pi}d\varphi|\phi(v')-\phi(v)-a.\nabla\phi(v)| \\ &\quad +\Phi(|v-v_*|)1\!\!1_{\{|v|\geq 2x\}}\int_0^\pi\beta(\theta)d\theta\int_0^{2\pi}d\varphi|\phi(v')| =: A_1+A_2+A_3. \end{aligned} \tag{6.32}$$

Recalling that $v'=v+a$, we deduce that $|\phi(v')-\phi(v)-a.\nabla\phi(v)|\leq|a|^2||\phi''||_\infty/2$, and then recalling (6.18), we deduce that $A_2\leq C||\phi''||_\infty\Phi(|v-v_*|)|v-v_*|^2 1\!\!1_{\{|v|\leq 2x\}}$. Next, we observe that $|\phi(v')|\leq||\phi||_\infty 1\!\!1_{\{|v'|\leq x\}}$. Since $v'=v+a$, we deduce that $1\!\!1_{\{|v|\geq 2x,|v'|\leq x\}}\leq 1\!\!1_{\{|v|\geq 2x,|a|\geq|v|/2\}}\leq 1\!\!1_{\{|v|\geq 2x\}}\frac{4|a|^2}{|v|^2}$. Thus, using (6.18) again,

$$\begin{aligned} A_3 &\leq \Phi(|v-v_*|)1\!\!1_{\{|v|\geq 2x\}}\frac{4||\phi||_\infty}{|v|^2}\int_0^\pi\beta(\theta)d\theta\int_0^{2\pi}d\varphi|a|^2 \\ &\leq C\Phi(|v-v_*|)1\!\!1_{\{|v|\geq 2x\}}\frac{||\phi||_\infty|v-v_*|^2}{|v|^2}. \end{aligned} \tag{6.33}$$

As a conclusion, (2.10) holds, which concludes the proof. □

Let us now give the

Proof of Lemma 2.4. First, similarly to (6.18), we get

$$\begin{aligned} \int_k^\infty dz\int_0^{2\pi}d\varphi|c(v,v_*,z,\varphi)|^2 &= \Phi(|v-v_*|)\int_0^{G[k/\Phi(|v-v_*|)]}\beta(\theta)d\theta\int_0^{2\pi}d\varphi|a(v,v_*,\theta,\varphi)|^2 \\ &= 2\pi\Phi(|v-v_*|)|v-v_*|^2\int_0^{G[k/\Phi(|v-v_*|)]}\beta(\theta)\frac{1-\cos\theta}{2}d\theta \\ &\leq C|v-v_*|^2\Phi(|v-v_*|)\varepsilon_0^k(|v-v_*|) \end{aligned} \tag{6.34}$$

by definition of ε_0^k, and since $(1-\cos\theta)\leq\theta^2$. This is nothing but (2.12). Next, one easily gets $|xh_0^k(x)-yh_0^k(y)|\leq|x-y|(h_0^k(x)+h_0^k(y))+\min(x,y)|h_0^k(x)-h_0^k(y)|$. On the one hand, the definition of h_0^k and the substitution $\theta=G(z/\Phi(x))$ yields $h_0^k(x)\leq\pi\Phi(x)\int_0^\pi(1-\cos\theta)\beta(\theta)d\theta\leq\pi\kappa_1\Phi(x)$, and on the other hand, $|h_0^k(x)-h_0^k(y)|\leq\pi\int_0^\infty dz|\cos G(z/\Phi(x))-\cos G(z/\Phi(y))|\leq C|\Phi(x)-\Phi(y)|$, recall the computations in (6.28-6.29-6.30). Hence (1.9) yields $|xh_0^k(x)-yh_0^k(y)|\leq C|x-y|(\Phi(x)+\Phi(y))+C\min(x,y)|\Phi(x)-\Phi(y)|\leq C|x-y|(\Psi(x)+\Psi(y))$, i.e. (2.13). Next, an easy computation shows that $h_0^k(x)=\pi\Phi(x)\int_{G[k/\Phi(x)]}^\pi(1-\cos\theta)\beta(\theta)d\theta$. Hence, recalling (2.1), $|\kappa_0x\Phi(x)-xh_0^k(x)|= x\Phi(x)\pi\int_0^{G[k/\Phi(x)]}(1-\cos\theta)\beta(\theta)d\theta\leq x\Phi(x)\pi\varepsilon_0^k(x)$.

Finally, the fact that ε_0^k is bounded by κ_1 is obvious from **(A2)**, and for $x \geq 0$ fixed, $k/\Phi(x)$ tends to infinity, so that $G[k/\Phi(x)]$ tends to 0, and thus $\varepsilon_0^k(x)$ tends to 0. □

We conclude the paper with the

Proof of Lemma 2.5. We thus assume **(A1-A2-A3-A4)**(γ) for some $\gamma \in (-3, 0]$, and consider $\phi \in C_c^2$, and $g \in \mathcal{P}_2(\mathbb{R}^3) \cap \mathcal{J}_\gamma$. We consider a sequence $v_n \to v$ in $\mathbb{R}^3$, and we have to show that $h(v_n) \to h(v)$, where $h(v) := \int_{\mathbb{R}^3} g(dv_*)\tilde{\mathcal{A}}\phi(v, v_*)$. Recalling (6.20), we write $h(v) = h_1(v) - \kappa_0 \nabla\phi(v).h_2(v)$, with

$$\begin{aligned} h_1(v) &:= \int_{\mathbb{R}^3} g(dv_*)\Phi(|v - v_*|) \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \Delta(v, v_*, \theta, \varphi), \\ \Delta(v, v_*, \theta, \varphi) &:= \phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi).\nabla\phi(v), \\ h_2(v) &:= \int_{\mathbb{R}^3} g(dv_*)(v - v_*)\Phi(v - v_*). \end{aligned} \tag{6.35}$$

First, due to **(A4)**(γ), one has $|x\Phi(x) - y\Phi(y)| \leq |x - y|(\Phi(x) + \Phi(y)) + \min(x, y)|\Phi(x) - \Phi(y)| \leq C|x - y|(x^\gamma + y^\gamma)$. Thus,

$$|h_2(v_n) - h_2(v)| \leq C \int_{\mathbb{R}^3} g(dv_*)|v_n - v|(|v_n - v_*|^\gamma + |v - v_*|^\gamma) \leq C|v_n - v|J_\gamma(g) \to 0 \tag{6.36}$$

as n tends to infinity, since $g \in \mathcal{J}_\gamma$ by assumption. Next, we use the map φ_0 introduced in Lemma 2.2, and write

$$h_1(v_n) = \int_{\mathbb{R}^3} g(dv_*)\Phi(|v_n - v_*|) \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \Delta(v_n, v_*, \theta, \varphi + \varphi_0(v - v_*, v_n - v*)). \tag{6.37}$$

We now introduce, for $\varepsilon > 0$, h_1^ε, which is defined as h_1 but replacing Φ by $\Phi_\varepsilon(x) := \Phi(\max(x, \varepsilon))$. First, $\lim_n h_1^\varepsilon(v_n) = h_1^\varepsilon(v)$ for each $\varepsilon > 0$, due to the Lebesgue Theorem and the following facts:
(i) Φ_ε is continuous and bounded due to **(A4)**(γ) ;
(ii) $\lim_n \Delta(v_n, v_*, \theta, \varphi + \varphi_0(v - v_*, v_n - v*)) = \Delta(v, v_*, \theta, \varphi)$ for all v_*, θ, φ (because due to Lemma 2.2, $\lim_n a(v_n, v_*, \theta, \varphi + \varphi_0(v - v_*, v_n - v*)) = a(v, v_*, \theta, \varphi)$);
(iii) $|\Delta(v_n, v_*, \theta, \varphi + \varphi_0(v - v_*, v_n - v*))| \leq C_\phi |v_n - v_*|^2\theta^2 \leq C_\phi(\sup_n |v_n|^2 + |v_*|^2)\theta^2$ which belongs to $L^1(g(dv_*)\beta(\theta)d\theta d\varphi)$ due to **(A2)** and since $g \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$.
We thus just have to prove that $\lim_{\varepsilon\to 0} \limsup_n |h_1(v_n) - h_1^\varepsilon(v_n)| = 0$ and $\lim_{\varepsilon\to 0} |h_1(v) - h_1^\varepsilon(v)| = 0$. Using point (iii) above and then **(A2)-(A4)**(γ),

$$\begin{aligned} |h_1(v_n) - h_1^\varepsilon(v_n)| &\leq C_\phi \int_{\mathbb{R}^3} g(dv_*)\Phi(|v_n - v_*|)\mathbb{1}_{\{|v_n - v_*| \leq \varepsilon\}} \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi |v_n - v_*|^2\theta^2 \\ &\leq C_\phi \int_{\mathbb{R}^3} g(dv_*)|v_n - v_*|^{2+\gamma}\mathbb{1}_{\{|v_n - v_*| \leq \varepsilon\}} \leq C_\phi J_\gamma(g)\varepsilon^2. \end{aligned} \tag{6.38}$$

This implies that $\limsup_n |h_1(v_n) - h_1^\varepsilon(v_n)| \leq C_\phi J_\gamma(g)\varepsilon^2 \to 0$ as $\varepsilon \to 0$. The same computation shows that $\lim_{\varepsilon\to 0} |h_1(v) - h_1^\varepsilon(v)| = 0$, and this concludes the proof. □

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