A new regularization possibility for the Boltzmann equation with soft potentials

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Abstract

We consider a simplified Boltzmann equation: spatially homogeneous, two-dimensional, radially symmetric, with Grad's angular cutoff, and linearized around its initial condition. We prove that for a sufficiently singular velocity cross section, the solution may become instantaneously a function, even if the initial condition is a singular measure. To our knowledge, this is the first regularization result in the case with cutoff: all the previous results were relying on the non-integrability of the angular cross section. Furthermore, our result is quite surprising: the regularization occurs for initial conditions that are not too singular, but also not too regular. The objective of the present work is to explain that the singularity of the velocity cross section, which is often considered as a (technical) obstacle to regularization, seems on the contrary to help the regularization.

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1 Introduction

Let $f_t(dv)$ be the velocity distribution in a 2d spatially homogeneous dilute gas at time $t \geq 0$. Then under some physical assumptions, f solves the Boltzmann equation: for some $\gamma \in (-2, 1]$, some angular cross section β (a nonnegative symmetric measure on $(-\pi, \pi)\backslash\{0\}$), for all sufficiently regular functions $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$,

$$\frac{d}{dt} \int_{\mathbb{R}^2} f_t(dv)\varphi(v) = \int_{\mathbb{R}^2} f_t(dv) \int_{\mathbb{R}^2} f_t(dv_*)|v - v_*|^\gamma \int_{-\pi}^{\pi} \beta(d\theta)[\varphi(v') - \varphi(v)], \tag{1}$$

where, for R_θ the rotation of angle θ,

$$v' = v'(v, v_*, \theta) = \frac{v + v_*}{2} + R_\theta \frac{v - v_*}{2}. \tag{2}$$

We refer to Villani [13] and Desvillettes [4] for reviews on this equation. When $\gamma < 0$, we speak of soft potentials. The suject of the present paper is regularization: can f_t be more regular than f_0, as soon as $t > 0$?

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There are many results on this topic: it has been proved by several authors that such a phenomenon occurs in the case without cutoff, that is when $\beta(d\theta) = \beta(\theta)d\theta$ is sufficiently non-integrable near $\theta \sim 0$, say $\beta(\theta) \sim \theta^{-1-\nu}$, with $\nu \in (0, 2)$. There has been essentially three types of results:
• it is shown in Desvillettes-Wennberg [5] that in any dimension, when $\gamma \in [0, 1)$, $\nu \in (0, 2)$, and for an initial condition with finite mass, energy, entropy, for a smoothed interaction kernel (i.e. when $|v - v_*|^\gamma$ is replaced by something like $(1 + |v - v_*|^2)^{\gamma/2}$), then $f_t \in C^\infty$ for all $t > 0$. See also the (older) papers by Desvillettes [3] and Alexandre-El Safadi [1];
• in the case of Maxwellian molecules, i.e. when $\gamma = 0$, $\nu \in (0, 2)$, Graham-Méléard [9] (see also [7]) have proved that $f_t \in C^\infty$ for all $t > 0$, even when the initial condition is a singular measure (but with many moments). This works only in dimension 1 or 2.
• finally, the most general result (but also weaker) is the one of Alexandre-Desvillettes-Villani-Wennberg [2]: in any dimension, for any γ, any $\nu \in (0, 2)$, any initial condition with finite mass, energy, entropy, $\sqrt{f_t}$ instantaneously belongs to $H^{\nu/2}_{loc}$. Let us observe that when tracking the constants in [2], we realize that the result is weaker and weaker when γ becomes more and more negative. These results are based on lowerbounds of the entropy dissipation functionnal. Such an idea was initiated by Lions [10], see also Villani [12].

Our goal in the present paper is to show that the explosion of $|v - v_*|^\gamma$ near $v = v_*$, when $\gamma < 0$, is not an obstacle to regularization: ont the contrary, even in the case with cutoff, such a singular interaction kernel may provide some regularization.

We start with a simplified equation, namely we linearize the Boltzmann equation around its initial condition. We show that for a specific singular initial condition (an uniform distribution on a circle), the solution instantaneously becomes a function, in the case with cutoff, even if the angular cross section is not a function.

The present result is completely new, since all the previous results were relying on the explosion near 0 of the angular cross section β. It is furthermore very surprising, since, as we will show, no regularization may hold if f_0 is too singular **nor too regular**: if f_0 is a measure that is not a function but is too smooth in some sense, then it will never become a function. This kind of phenomenon is fully nonlinear.
However, our result is very weak, since it is qualitative (we only prove that the solution immediately becomes a function), and since we consider a simplified equation.

Regularization is motivated by many other subjects for which regularity estimates are needed: convergence to equilibrium, uniqueness,... For example, it is shown in [8] that uniqueness holds (with $\gamma < 0$ and a possibly non-cutoff angular cross-section) for sufficiently smooth solutions (in some L^p, with p large enough). We are far from such a quantitative regularization.

Let us finally mention that a similar result should hold for the (nonlinear) Boltzmann equation.

Formal result. Assume that $\Lambda := \beta((-\pi, \pi)\backslash\{0\}) \in (0, \infty)$, and that $\gamma \in (-2, -1)$. Consider f_0 the uniform law on the circle $\{|v| = 1\}$. Then for any solution $(f_t)_{t\geq 0}$ to (1), f_t is a function for a.e. $t > 0$. (In dimension 3, take also f_0 uniform on $\{|v| = 1\}$ but $\gamma \in (-3, -2)$).

Formal proof. We assume here that one may apply (1) with any bounded measurable φ. Since f_0 is a radially symmetric probability measure, so is f_t for all $t \geq 0$.

Step 1. First, applying (1) with $\varphi = \mathbf{1}_{\{0\}}$, one easily deduces that $f_t(\{0\}) = 0$ for all $t \geq 0$. Indeed, $f_0(\{0\}) = 0$, and simple considerations using that f_t is radially symetric show that for all v, all

$\theta \neq 0$, $\mathbf{1}_{\{0\}}(v') - \mathbf{1}_{\{0\}}(v) \leq 0$ for f_t-a.e. v_*.

Step 2. Next, for any Lebesgue-null $A \subset \mathbb{R}^2$, one gets convinced that for all $\theta \neq 0$, $\mathbf{1}_A(v') = 0$ for $f_t(dv)f_t(dv_*)$-a.e. v, v_*. Here, one has to use that f_t is radially symmetric and does not give weight to 0. As a consequence,

$$f_t(A) + \Lambda \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \int_{\mathbb{R}^2} f_s(dv_*)|v - v_*|^\gamma \mathbf{1}_A(v) \leq f_0(A). \tag{3}$$

Thus if $f_0(A) = 0$, we deduce that $f_t(A) = 0$ for all $t \geq 0$. This implies that $f_t(dv)$ has a density, except maybe on $C = \{|v| = 1\}$.

Step 3. It remains to check that $f_t(C) = 0$ for a.e. $t > 0$. But (3) applied with C implies that

$$\int_0^\infty ds \int_{\mathbb{R}^2} f_s(dv) \int_{\mathbb{R}^2} f_s(dv_*)|v - v_*|^\gamma \mathbf{1}_C(v) \leq \frac{f_0(C)}{\Lambda} = \frac{1}{\Lambda}.$$

As a consequence, for a.e. $t > 0$, $\iint_{C\times C} f_t(dv)f_t(dv_*)|v - v_*|^\gamma < \infty$. This implies (see Falconer [6, Theorem 4.13 p 64]) that either $f_t(C) = 0$ or that the Haussdorff dimension of C is greater than $|\gamma|$, the latter being excluded since $\dim_H(C) = 1 < |\gamma|$. □

Unfortunately, we are not able to justify the use of such test functions in the nonlinear case.

We state our result in Section 2, we prove it in Section 3. An appendix lies at the end of the paper.

2 Main result

In the whole paper, the angular cross section is supposed to be finite, and to vanish on $\{|\theta| \geq \pi/2\}$. For the nonlinear Boltzmann equation, this last condition is not restrictive, for symmetrical reasons, see the introduction of [2]. We also impose that β vanishes near 0 for simplicity.

Assumption $(A1)$**:** β is a nonnegative symmetric (even) measure on $[-\pi/2, \pi/2]\backslash\{0\}$, with total mass $\Lambda = \beta([-\pi/2, \pi/2]) \in (0, \infty)$.
Assumption $(A2)$**:** there is $\theta_0 \in (0, \pi/2)$ such that $\beta((-\theta_0, \theta_0)) = 0$.

We now define the notion weak solutions we will use. We denote by $Lip(\mathbb{R}^2)$ the set of globally Lipschitz functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Definition 1. *Let $\gamma \in (-2, 0)$ be fixed, consider β satisfying $(A1)$. A family $(f_t)_{t\geq 0}$ of probability measures on $\mathbb{R}^2$ is said to solve $LB(f_0, \gamma, \beta)$ if for all $\varphi \in Lip(\mathbb{R}^2)$, all $t \geq 0$,*

$$\int_{\mathbb{R}^2} f_t(dv)\varphi(v) = \int_{\mathbb{R}^2} f_0(dv)\varphi(v) + \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v, v_*), \tag{4}$$

$$\text{where} \quad \mathcal{A}\varphi(v, v_*) = \mathbf{1}_{\{v\neq v_*\}}|v - v_*|^\gamma \int_{-\pi/2}^{\pi/2} \beta(d\theta) \left[\varphi(v') - \varphi(v)\right] \tag{5}$$

with $v' = v'(v, v_*, \theta)$ defined in (2).

We will check later that in our situation, all the terms make sense in (4). The indicator $v \neq v_*$ is written for convenience, since $v = v_*$ implies $v' = v$. Our main result writes as follows.

Theorem 2. *Let $\gamma \in (-2,-1)$ be fixed, consider β satisfying $(A1-A2)$. Assume that for some $r_0 > 0$, f_0 is a uniform distribution on the circle $\{|v| = r_0\}$. Then the exists a solution $(f_t)_{t\geq 0}$ to $LB(f_0, \gamma, \beta)$ such that f_t has a density w.r.t. the Lebesgue measure on $\mathbb{R}^2$ for almost every $t > 0$.*

Let us comment on this result. Consider $\gamma \in (-2, 0)$, and an initial condition f_0 satisfying

$$\text{for } f_0\text{-a.e. } v \in \mathbb{R}^2, \quad \lambda_0(v) := \int_{\mathbb{R}^2} f_0(dv_*)|v - v_*|^\gamma \mathbf{1}_{\{v_* \neq v\}} < \infty. \tag{6}$$

Then we consider a solution $(f_t)_{t\geq 0}$ to $LB(f_0, \gamma, \beta)$. Applying (4) with some nonnegative Lipschitz function φ and using $(A1)$, we immediately get

$$\frac{d}{dt}\int_{\mathbb{R}^2} f_t(dv)\varphi(v) \geq -\Lambda \int_{\mathbb{R}^2} f_t(dv)\lambda_0(v)\varphi(v),$$

whence, at least formally, $f_t(dv) \geq e^{-\Lambda t \lambda_0(v)} f_0(dv)$ for all $t \geq 0$: no regularization may occur.

This result is fully nonlinear and quite surprising: if f_0 is regular enough to satisfy (6) but is not a function, then it does never become a function. Such examples can easily be built: as shown in Falconer [6, Theorem 4.13 p 64], for any Borel subset $A \subset \mathbb{R}^2$ with Haussdorff dimension strictly greater than $|\gamma|$, we may find a probability measure f_0 on $\mathbb{R}^2$ with $f_0(A) = 1$ and such that such that $\int_{\mathbb{R}^2} f_0(dv) \int_{\mathbb{R}^2} f_0(dv_*)|v - v_*|^\gamma < \infty$, which of course imply (6).
Let us insist on the fact that initial conditions satisfying (6) are more regular than the uniform distribution on $\{|v| = 1\}$: the latter gives positive weight to some sets with lower dimension.
Note also that on the contrary, f_0 has to be sufficiently regular. If for example we assume that $f_0 = \frac{1}{2}(\delta_{v_0} + \delta_{v_1})$, no regularization may hold (due to the indicator function in (6)). The same argument applies to $f_0 = \frac{1}{2}(\delta_{v_0} + g_0)$, for some bounded probability density g_0.

Finally, let us mention that our result holds even if $\beta = \delta_{\theta_0} + \delta_{-\theta_0}$, for some fixed $\theta_0 \in (0, \pi/2)$: the regularization does really not follow from the regularity of the angular cross section.

3 Proof

The aim of this section is to prove Theorem 2. We assume in the whole section that $(A1 - A2)$ hold, that $r_0 = 1$ (for simplicity), and that $\gamma \in (-2,-1)$ is fixed. Thus our initial condition f_0 is defined by

$$\int_{\mathbb{R}^2} \varphi(v) f_0(dv) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \varphi(\mathbf{e}_\alpha), \tag{7}$$

for all measurable $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}_+$, where $\mathbf{e}_\alpha := (\cos\alpha, \sin\alpha)$.

Proposition 3. (i) *There exists a radially symmetric solution $(f_t)_{t\geq 0}$ to $LB(f_0, \gamma, \beta)$.*
(ii) For $t \geq 0$, define the probability measure λ_t on $\mathbb{R}_+$ by $\lambda_t(A) = f_t(\{|v| \in A\})$. Then we have, for all $\varphi \in L^\infty(\mathbb{R}^2)$, all $t \geq 0$,

$$\int_{\mathbb{R}^2} \varphi(v) f_t(dv) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \int_0^\infty \lambda_t(dr)\varphi(r\mathbf{e}_\alpha). \tag{8}$$

(iii) Consider the class $\mathcal{F}$ of functions of the form $\psi = \varphi + \delta$, where $\varphi \in Lip(\mathbb{R}_+)$, and with $\delta \in L^\infty(\mathbb{R})$, with $1 \notin \text{supp }\delta$. Then

$$\int_0^\infty \lambda_t(dr)\psi(r) = \psi(1) + \int_0^t ds \int_0^\infty \lambda_s(dr) \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi}(r^2 + 1 - 2r\cos\alpha)^{\gamma/2}[\psi(r') - \psi(r)] \quad (9)$$

where $r' = r'(r, \theta, \alpha) = \left(\frac{1+\cos\theta}{2}r^2 + \frac{1-\cos\theta}{2} - r\sin\theta\sin\alpha\right)^{1/2}$.

To understand (9), observe that $r'(r,\theta,\alpha) = |v'(r\mathbf{e}_{\alpha_0+\alpha}, \mathbf{e}_{\alpha_0}, \theta)|$ and that $(r^2 + 1 - 2r\cos\alpha)^{\gamma/2} = |r\mathbf{e}_{\alpha_0+\alpha} - \mathbf{e}_{\alpha_0}|^\gamma$ for any α_0. This result is routine and will be checked at the end of the section. We now start the proof of Theorem 2. First, 0 cannot be reached by the radius distribution λ_t.

Lemma 4. *Consider the family $(\lambda_t)_{t\geq 0}$ introduced in Proposition 3. For all $t \geq 0$, $\lambda_t(\{0\}) = 0$.*

Proof. Since $\mathbf{1}_{\{0\}}$ belongs to $\mathcal{F}$, we may apply (9). We realize that
(a) initially, $\mathbf{1}_{\{0\}}(1) = 0$;
(b) (when $r = 0$) $\mathbf{1}_{\{0\}}(r'(0,\theta,\alpha)) - \mathbf{1}_{\{0\}}(0) \leq 0$ for all α, θ;
(c) when $r > 0$, for all $\theta \in [-\pi/2, \pi/2]\backslash\{0\}$, $\mathbf{1}_{\{0\}}(r'(r,\theta,\alpha)) = 0$ for $d\alpha$-a.e. $\alpha \in [-\pi, \pi]$ (use here that $r\sin\theta \neq 0$).
Hence (9) yields that for all $t \geq 0$, $\int_0^\infty \lambda_t(dr)\mathbf{1}_{\{0\}}(r) \leq 0$, and the result follows. ☐

Then we may prove that the radius distribution λ_t has a density, except maybe at 1.

Lemma 5. *Consider the family $(\lambda_t)_{t\geq 0}$ introduced in Proposition 3. Consider a Lebesgue-null subset $A \subset \mathbb{R}_+$ with $1 \notin A$. For all $t \geq 0$, $\lambda_t(A) = 0$.*

Proof. We first assume that $1 \notin \bar{A}$. Then $\mathbf{1}_A$ belongs to $\mathcal{F}$, so that we may use (9). Since initially $\mathbf{1}_A(1) = 0$ and due to Lemma 4, it suffices to prove that for all $r > 0$, for all $\theta \in [-\pi/2, \pi/2]\backslash\{0\}$, $\mathbf{1}_A(r'(r,\theta,\alpha)) = 0$ for $d\alpha$-a.e. $\alpha \in [-\pi, \pi]$. But this is immediate, using that $r\sin\theta \neq 0$, that A is Lebesgue-null, and the substitution $\alpha \mapsto r'(r,\theta,\alpha)$.
As previously, (9) yields that for all $t \geq 0$, $\int_0^\infty \lambda_t(dr)\mathbf{1}_A(r) \leq 0$, and the result follows.

Now if $1 \in \bar{A}$, we consider $A_n = A \cap \{|r-1| \geq 1/n\}$, which increases to A (because $1 \notin A$ by assumption). Since $1 \notin \bar{A}_n$, we know that for all $t \geq 0$, all $n \geq 1$, $\lambda_t(A_n) = 0$. Making n tend to infinity, we get $\lambda_t(A) = 0$ for all $t \geq 0$ by the Beppo-Levi Theorem. ☐

Finally, we prove that our solution leaves immediately the unit circle.

Lemma 6. *Consider the family $(\lambda_t)_{t\geq 0}$ introduced in Proposition 3. For a.e. $t > 0$, $\lambda_t(\{1\}) = 0$.*

Proof. We divide the proof into two steps.

Step 1. We first show that for all $T > 0$, there is $\kappa_T > 0$ such that for all $\varepsilon > 0$ small enough,

$$\int_0^T ds\lambda_s(\{1\}) \leq \kappa_T\varepsilon^{|\gamma|-1} + \kappa_T\varepsilon^{|\gamma|}\int_0^T ds\int_0^\infty \lambda_s(dr)|r^2-1|^\gamma \mathbf{1}_{\{|r^2-1|>\varepsilon\}}. \quad (10)$$

We consider $\varepsilon \in (0,1)$, and we apply (9) with $\psi(r) = \mathbf{1}_{\{|r^2-1|\leq\varepsilon\}}$, which belongs to $\mathcal{F}$. We get

$$\lambda_0(\{|r^2-1| \leq \varepsilon\}) - \lambda_T(\{|r^2-1| \leq \varepsilon\}) = \int_0^T ds \int_0^\infty \lambda_s(dr)[A_\varepsilon(r) - B_\varepsilon(r)], \quad (11)$$

where, using $(A1-A2)$ and setting $D_{\theta_0}=[-\pi/2,\pi/2]\backslash(-\theta_0,\theta_0)$,

$$\begin{aligned} A_\varepsilon(r) &= \mathbf{1}_{\{|r^2-1|\le\varepsilon\}}\int_{D_{\theta_0}}\beta(d\theta)\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\mathbf{1}_{\{|r'^2-1|>\varepsilon\}}(r^2+1-2r\cos\alpha)^{\gamma/2},\\ B_\varepsilon(r) &= \mathbf{1}_{\{|r^2-1|>\varepsilon\}}\int_{D_{\theta_0}}\beta(d\theta)\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\mathbf{1}_{\{|r'^2-1|\le\varepsilon\}}(r^2+1-2r\cos\alpha)^{\gamma/2}.\end{aligned}$$

We first give a lowerbound of A_ε. We only consider the case where $r=1$. Since $|(r'(1,\theta,\alpha))^2-1|=|\sin\theta\sin\alpha|\ge(\sin\theta_0)\alpha/2$, for $\alpha\in(0,\pi/2)$, and since $(2-2\cos\alpha)\le\alpha^2$, we obtain

$$A_\varepsilon(r)\ge\mathbf{1}_{\{r=1\}}\frac{\beta(D_{\theta_0})}{2\pi}\int_0^{\pi/2}d\alpha\mathbf{1}_{\{\alpha>\frac{2\varepsilon}{\sin\theta_0}\}}\alpha^\gamma\ge c_0\mathbf{1}_{\{r=1\}}\varepsilon^{\gamma+1} \tag{12}$$

for some constant $c_0>0$, at least for $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0:=(\sin\theta_0)\pi/8$.
We now upperbound B_ε. Some easy considerations allow us to get

$$B_\varepsilon(r)\le\frac{2}{\pi}\mathbf{1}_{\{|r^2-1|>\varepsilon\}}\int_{\theta_0}^{\pi/2}\beta(d\theta)\int_{-\pi/2}^{\pi/2}d\alpha\mathbf{1}_{\{|r'^2-1|\le\varepsilon\}}(r^2+1-2r\cos\alpha)^{\gamma/2}.$$

We of course have $(r^2+1-2r\cos\alpha)^{\gamma/2}\le|r-1|^\gamma$, and a computation shows that $|r'^2-1|\le\varepsilon$ implies that $\sin\alpha\in[\frac{(1+\cos\theta)(r^2-1)-2\varepsilon}{2r\sin\theta},\frac{(1+\cos\theta)(r^2-1)+2\varepsilon}{2r\sin\theta}]$. This yields

$$B_\varepsilon(r)\le\frac{2}{\pi}|r-1|^\gamma\mathbf{1}_{\{|r^2-1|>\varepsilon\}}\int_{\theta_0}^{\pi/2}\beta(d\theta)\int_{-\pi/2}^{\pi/2}d\alpha\mathbf{1}_{\{\sin\alpha\in[\frac{(1+\cos\theta)(r^2-1)\pm2\varepsilon}{2r\sin\theta}]\}}.$$

Consider first $a_1>0$ such that $|r^2-1|\le a_1$ implies $r\in[1/2,2]$ and $\frac{(1+\cos\theta)(r^2-1)}{2r\sin\theta}\in[-\pi/8,\pi/8]$ for all $\theta\in[\theta_0,\pi/2]$.
Consider $\varepsilon_1>0$ such $|r^2-1|\le a_1$ implies $[\frac{(1+\cos\theta)(r^2-1)\pm2\varepsilon_1}{2r\sin\theta}]\subset[-\pi/4,\pi/4]$ for all $\theta\in[\theta_0,\pi/2]$.
Then for $|r^2-1|\le a_1$, we get, for all $\varepsilon\in(0,\varepsilon_1)$, for some constants $\kappa_1,c_1>0$, (since then $2r\sin\theta\ge\sin\theta_0>0$ and $|r^2-1|=|r-1|(r+1)\le3|r-1|$),

$$B_\varepsilon(r)\le\kappa_1\varepsilon|r-1|^\gamma\mathbf{1}_{\{|r^2-1|>\varepsilon\}}\le c_1\varepsilon|r^2-1|^\gamma\mathbf{1}_{\{|r^2-1|>\varepsilon\}}. \tag{13}$$

On the other hand, it is immediate that for some $c_2>0$, for $|r^2-1|\ge a_1$ (so that $|r-1|\ge a_2>0$),

$$B_\varepsilon(r)\le\frac{2}{\pi}\beta([\theta_0,\pi/2])\pi|r-1|^\gamma\le c_2. \tag{14}$$

Using that $\lambda_0(\{|r^2-1|\le\varepsilon\})-\lambda_T(\{|r^2-1|\le\varepsilon\})\le1$ and gathering (11-12-13-14), we obtain for all $\varepsilon\in(0,\varepsilon_2)$, with $\varepsilon_2=\min(\varepsilon_0,\varepsilon_1)$,

$$1\ge\int_0^T ds\int_0^\infty\lambda_s(dr)\left[c_0\mathbf{1}_{\{r=1\}}\varepsilon^{\gamma+1}-c_1\varepsilon|r^2-1|^\gamma\mathbf{1}_{\{|r^2-1|>\varepsilon\}}-c_2\right],$$

whence (10).

Step 2. We now conclude. Consider the measure $\mu_T(dr)=\int_0^T ds\lambda_s(dr)\mathbf{1}_{\{r\neq1\}}$. Since μ_T is finite and $\mu_T(\{1\})=0$, the de la Vallée Poussin Lemma 7 ensures us that there exists a function $g:\mathbb{R}_+\mapsto$

$\mathbb{R}_+$, with $g(\infty)=\infty$, such that $x\mapsto xg(1/x)$ is nondecreasing on $\mathbb{R}_+$, and $\int_0^\infty \mu_T(dr)g(1/|r^2-1|)<\infty$. Since $|\gamma|\geq 1$ by assumption, we deduce that

$$\mathbf{1}_{\{|r^2-1|>\varepsilon\}} \leq \frac{|r^2-1|^{|\gamma|}g(1/|r^2-1|)}{\varepsilon^{|\gamma|}g(1/\varepsilon)},$$

so that (10) becomes

$$\int_0^T ds\lambda_s(\{1\}) \leq \kappa_T\varepsilon^{|\gamma|-1} + \frac{\kappa_T}{g(1/\varepsilon)}\int_0^\infty \mu_T(dr)g(1/|r^2-1|).$$

Letting ε tend to 0, we get $\int_0^T ds\lambda_s(\{1\})=0$. Since T is arbitrarily large, this ends the proof. □

We may now conclude the

Proof of Theorem 2. We consider the solution $(f_t)_{t\geq 0}$ built in Proposition 3, and the associated radius distribution $(\lambda_t)_{t\geq 0}$. Owing to the Radon-Nikodym Theorem, to Lemmas 5 and 6, we deduce that for a.e. $t\geq 0$, $\lambda_t(dr)$ has a density $\lambda_t(r)$ with respect to the Lebesgue measure on $\mathbb{R}_+$. Then we deduce from (8) that $f_t(dv)$ has the density $f_t(v)=\lambda_t(|v|)/(2\pi|v|)\mathbf{1}_{\{|v|\neq 0\}}$, and thus is indeed a function. (The case $v=0$ is not a problem, since $f_t(\{0\})=\lambda_t(\{0\})=0$). □

We conclude the section with the

Proof of Proposition 3. We split the proof into 4 steps.

Step 1. We first check the existence of a solution. We introduce, for $n\geq 1$, the operator $\mathcal{A}_n$, of which the expression is the same as (5) with $\min(|v-v_*|^\gamma, n)$ instead of $|v-v_*|^\gamma$. Then we observe that with our choice for f_0, as shown in the appendix, we have for all $\varphi\in Lip(\mathbb{R}^2)$,

$$\left|\textstyle\int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v,v_*)\right| \leq C(\Lambda,\gamma)||\varphi||_{lip}, \tag{15}$$

$$v\mapsto \textstyle\int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v,v_*) \text{ is continuous on } \mathbb{R}^2, \tag{16}$$

$$\sup_{n\geq 1}\left|\textstyle\int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}_n\varphi(v,v_*)\right| \leq C(\Lambda,\gamma)||\varphi||_{lip}, \tag{17}$$

$$\lim_{n\to\infty}\sup_{v\in\mathbb{R}^2}\left|\textstyle\int_{\mathbb{R}^2} f_0(dv_*)(\mathcal{A}-\mathcal{A}_n)\varphi(v,v_*)\right| = 0. \tag{18}$$

In particular, (15) implies that all the terms make sense in (4).
One easily checks, by classical methods (Gronwall Lemma and Picard iteration using the total variation norm), that there exists a unique solution $(f_t^n)_{t\geq 0}$ to $LB_n(f_0,\gamma,\beta)$, where $\mathcal{A}$ is replaced by $\mathcal{A}_n$. The obtained solution f^n is clearly radially symmetric.
Next, using (4) and (17), we deduce that
(a) using $\varphi(v)=|v|$, $C_T:=\sup_n\sup_{[0,T]}\int_{\mathbb{R}^2} f_t^n(dv)|v| \leq 1+C(\Lambda,\gamma)T$ for all $T>0$,
(b) for any $\varphi\in Lip(\mathbb{R}^2)$, for all $0\leq s\leq t$, $\left|\int_{\mathbb{R}^2}(f_t^n-f_s^n)(dv)\varphi(v)\right| \leq C(\Lambda,\gamma)||\varphi||_{lip}|t-s|$.
Point (a) ensures that for each $t\geq 0$, $(f_t^n)_{n\geq 1}$ is tight, while (b) gives some equicontinuity estimates. It is then standard that up to extraction of a (not relabelled) subsequence, $(f_t^n)_{t\geq 0}$ tends to some family of (radially symmetric) probability measures $(f_t)_{t\geq 0}$, in the sense that for all $\varphi\in Lip(\mathbb{R}^2)$, for all $T\geq 0$, $\lim_n\sup_{[0,T]}|\int_{\mathbb{R}^2}(f_t^n-f_t)(dv)\varphi(v)|=0$. This also implies that for all $t\geq 0$, all $\varphi:\mathbb{R}^2\mapsto\mathbb{R}$ continuous and bounded, $\lim_n\int_{\mathbb{R}^2} f_t^n(dv)\varphi(v)=\int_{\mathbb{R}^2} f_t(dv)\varphi(v)$. We deduce that $(f_t)_{t\geq 0}$ solves $LB(f_0,\gamma,\beta)$, by passing to the limit in $LB_n(f_0,\gamma,\beta)$, using the convergence properties of f^n to f, the Lebesgue dominated convergence Theorem, as well as (15-16-17-18).

Step 2. Next, point (ii) of the statement is a simple consequence of the radial symmetry of $(f_t)_{t\geq 0}$.

Step 3. We now check point (iii) when $\psi \in Lip(\mathbb{R})$. Then $\varphi(v) = \psi(|v|) \in Lip(\mathbb{R}^2)$, and we thus may apply (4). Using several times (8) and the expression (7) of f_0, we get

$$\int_0^\infty \lambda_t(dr)\psi(r) = \int_{\mathbb{R}^2} f_t(dv)\varphi(v) = \int_{\mathbb{R}^2} f_0(dv)\varphi(v) + \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v, v_*)$$
$$= \psi(1) + \int_0^t ds \int_0^\infty \lambda_s(dr) \int_{-\pi}^{\pi} \frac{d\alpha_*}{2\pi} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \mathcal{A}\varphi(r\mathbf{e}_\alpha, \mathbf{e}_{\alpha_*}). \qquad (19)$$

Then a simple computation shows that

$$\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \mathcal{A}\varphi(r\mathbf{e}_\alpha, \mathbf{e}_{\alpha_*}) = \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} |r\mathbf{e}_\alpha - \mathbf{e}_{\alpha_*}|^\gamma [\psi(|v'(r\mathbf{e}_\alpha, \mathbf{e}_{\alpha_*}, \theta)|) - \psi(|r\mathbf{e}_\alpha|)]$$
$$= \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} (r^2 + 1 - 2r\cos(\alpha - \alpha_*))^{\gamma/2} [\psi(r'(r, \theta, \alpha - \alpha_*) - \psi(r)]$$
$$= \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} (r^2 + 1 - 2r\cos\alpha)^{\gamma/2} [\psi(r'(r, \theta, \alpha) - \psi(r)], \qquad (20)$$

and in particular does not depend on α_*. Gathering (19) and (20), we obtain (4).

Step 4. Finally, we have to prove that (4) still holds when $\psi \in L^\infty([0,\infty))$, such that there exists $\varepsilon > 0$ with $\psi = \psi \mathbf{1}_{\{|r^2-1|\geq 2\varepsilon\}}$.
To this end, we consider the finite Borel measure μ_t on $\mathbb{R}_+$ defined by $\mu_t(A) := \lambda_t(A) + \int_0^t ds[\lambda_s(A) + \int \beta(d\theta) \int_{-\pi}^{\pi} d\alpha \lambda_s(\{r' \in A\})]$. We consider $\psi_n \in Lip(\mathbb{R}_+)$, uniformly bounded by $2||\psi||_\infty$, satisfying $\psi_n = \psi_n \mathbf{1}_{\{|r^2-1|\geq \varepsilon\}}$ and such that $\psi_n(r)$ tends to $\psi(r)$ for μ_t-a.e. $r \in [0, \infty)$. Such an approximating sequence can be found, due to the Lusin Theorem (see e.g. Rudin [11]).
Then we may apply (9) for each $n \geq 1$, and get

$$\int_0^\infty \lambda_t(dr)\psi_n(r) = \int_0^t ds \int_0^\infty \lambda_s(dr) \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} (r^2 + 1 - 2r\cos\alpha)^{\gamma/2} [\psi_n(r') - \psi_n(r)].$$

To pass to the limit in this equation, we will use the Lebesgue dominated convergence Theorem. First, ψ_n is uniformly bounded, so that the left hand side is not a problem (recall that ψ_n goes to ψ μ_t-a.e., and thus λ_t-a.e.). Next, using the properties of ψ_n, we get $|\psi_n(r') - \psi_n(r)| \leq c[\mathbf{1}_{\{|r'^2-1|\geq\varepsilon\}} + \mathbf{1}_{\{|r^2-1|\geq\varepsilon\}}]$ and the proof will be finished if we show that

$$O_\varepsilon(t) = \int_0^t ds \int_0^\infty \lambda_s(dr) \int_{-\pi/2}^{\pi/2} \beta(d\theta) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} (r^2 + 1 - 2r\cos\alpha)^{\gamma/2} (\mathbf{1}_{\{|r'^2-1|\geq\varepsilon\}} + \mathbf{1}_{\{|r^2-1|\geq\varepsilon\}}) < \infty,$$

because since ψ_n goes to ψ μ_t-a.e., $[\psi_n(r') - \psi_n(r)]$ goes to $[\psi(r') - \psi(r)]$, $ds\lambda_s(dr)d\alpha\beta(d\theta)$-a.e.
To show that $O_\varepsilon(t) < \infty$, we observe that $|r'^2 - 1| = |(1 + \cos\theta)(r^2 - 1) - 2r\sin\theta\sin\alpha|/2 \leq |r^2 - 1| + r|\sin\alpha|$. Thus $|r'^2 - 1| \geq \varepsilon$ implies that either $|r^2 - 1| \geq \varepsilon/2$ or $r|\sin\alpha| \geq \varepsilon/2$. Hence, recalling $(A1)$ and since $(r^2 + 1 - 2r\cos\alpha)^{\gamma/2} \leq ((r-1)^2 + 2r\alpha^2/5)^{\gamma/2}$, (because $1 - \cos\alpha \geq \alpha^2/5$ on $[-\pi, \pi]$),

$$O_\varepsilon(t) \leq \Lambda t \sup_{r\geq 0, |\alpha|\leq\pi} (2\mathbf{1}_{\{|r^2-1|\geq\varepsilon/2\}} + \mathbf{1}_{\{r|\sin\alpha|\geq\varepsilon/2\}})((r-1)^2 + 2r\alpha^2/5)^{\gamma/2}.$$

This last quantity is bounded for each $\varepsilon > 0$ fixed (separate the cases $\{|r^2-1| \geq \varepsilon/2\}$, $\{r > 2\}$, and $\{r \leq 2, r|\sin\alpha| \geq \varepsilon/4\} \subset \{2r\alpha^2/5 \geq \varepsilon^2/80\}$). This concludes the proof. $\square$

4 Appendix

We start with a result in the spirit of de la Vallée Poussin, adapted to our problem.

Lemma 7. *Let μ be a nonegative finite measure on $\mathbb{R}_+$ such that $\mu(\{1\}) = 0$. Then there exists a function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_\infty g = \infty$ and $\int_0^\infty \mu(dr) g(1/|r^2-1|) < \infty$. Furthermore, g can be chosen in such a way that $x \mapsto xg(1/x)$ is non-decreasing on $\mathbb{R}_+$.*

Proof. Since μ is finite and since $\mu(\{1\}) = 0$, we may find an increasing sequence $(a_k)_{k\geq 1} \subset (0,\infty)$, with $\mu(\{|r^2-1| \leq 1/a_k\}) \leq 2^{-k}$. We also set $a_0 = 0$, and define the non-decreasing function $f : \mathbb{R}_+ \mapsto [1,\infty)$ by $f(x) = k+1$ if $x \in [a_k, a_{k+1})$. Then $\lim_\infty f = \infty$, and

$$\int_0^\infty \mu(dr) f(1/|r^2-1|) = \sum_{k\geq 0}(k+1)\mu(\{|r^2-1| \in (1/a_{k+1}, 1/a_k]\}) \leq \mu(\mathbb{R}_+) + \sum_{k\geq 1}(k+1)2^{-k} < \infty.$$

We now set $g(x) := x \inf_{[0,x]}(f(y)/y) \leq f(x)$. Hence $\int_0^\infty \mu(dr) g(1/|r^2-1|) < \infty$. Moreover, $xg(1/x) = \inf_{[0,1/x]} \frac{f(y)}{y}$ is clearly non-decreasing. We finally have to check that $\lim_\infty g = \infty$. But for each $\varepsilon > 0$, for all $x \geq 0$,

$$g(x) \geq \min(x \inf_{[0,\varepsilon x]} f(y)/y, x \inf_{[\varepsilon x, x]} f(y)/y) \geq \min(1/\varepsilon, \inf_{[\varepsilon x, x]} f(y))$$

since $f \geq 1$. Using that $\lim_\infty f = \infty$, we obtain $\liminf_\infty g \geq 1/\varepsilon$. This holds for all $\varepsilon > 0$, and thus allows us to conclude. $\square$

Before proving (15-16-17-18), we observe that for $\mathbf{e}_\alpha = (\cos\alpha, \sin\alpha)$, the function

$$h_\delta(v) := \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha |v - \mathbf{e}_\alpha|^\delta$$

is bounded on $\mathbb{R}^2$ if $\delta \in (-1, 0)$.

Proof of (15). We first observe, recalling (2), that $|v'-v| \leq |v-v_*|$. Thus, due to $(A1)$, (5) and (7), we deduce that $|\int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v,v_*)| \leq \Lambda ||\varphi||_{lip} \int_{\mathbb{R}^2} f_0(dv_*)|v-v_*|^{\gamma+1} = h_{\gamma+1}(v)$, which is bounded since $\gamma+1 \in (-1,0)$ by assumption. $\square$

Proof of (17). It is the same as that of (15). $\square$

Proof of (18). Using the same arguments as in the proof of (15), we get

$$\begin{aligned}|\int_{\mathbb{R}^2} f_0(dv_*)(\mathcal{A}-\mathcal{A}_n)\varphi(v,v_*)| &\leq \Lambda ||\varphi||_{lip} \int_{\mathbb{R}^2} f_0(dv_*)|v-v_*|^{\gamma+1}\mathbf{1}_{\{|v-v_*|^\gamma \geq n\}} \\ &\leq \Lambda ||\varphi||_{lip} \int_{\mathbb{R}^2} f_0(dv_*)|v-v_*|^{\gamma/2}|v-v_*|^{1+\gamma/2}\mathbf{1}_{\{|v-v_*| \leq n^{1/\gamma}\}} \\ &\leq \Lambda ||\varphi||_{lip} n^{(2+\gamma)/2\gamma} \int_{\mathbb{R}^2} f_0(dv_*)|v-v_*|^{\gamma/2} = \Lambda ||\varphi||_{lip} n^{(2+\gamma)/2\gamma} h_{\gamma/2}(v).\end{aligned}$$

We used here that $1 + \gamma/2 > 0$. Since $h_{\gamma/2}$ is bounded (because $\gamma/2 \in (-1, 0)$), and since $(2+\gamma)/2\gamma < 0$, the result follows. □

Proof of (16). For $\varphi \in Lip(\mathbb{R}^2)$, we set $h_\varphi(v) = \int_{\mathbb{R}^2} f_0(dv_*)\mathcal{A}\varphi(v, v_*)$. We wish to show that h_φ is continuous on $\mathbb{R}^2$. This follows from (18). Indeed, consider h^n_φ, where $\mathcal{A}$ is replaced by $\mathcal{A}_n$. Then for each $n \geq 1$, h^n_φ is obviously continuous on $\mathbb{R}^2$, by the Lebesgue Theorem (because for all v_*, θ, $v \mapsto \min(|v - v_*|^\gamma, n)$ and $v \mapsto v'(v, v_*, \theta)$ are continuous). But (18) implies that h^n_φ goes uniformly to h_φ on $\mathbb{R}^2$. □

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