On the invariant distribution of a one-dimensional avalanche process

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Abstract

We consider an interacting particle system $(\eta_t)_{t\geq 0}$ with values in $\{0,1\}^{\mathbb{Z}}$, in which each vacant site becomes occupied with rate 1, while each connected component of occupied sites become vacant with rate equal to its size. We show that such a process admits a unique invariant distribution, which is exponentially mixing and can be perfectly simulated. We also prove that for any initial condition, the avalanche process tends to equilibrium exponentially fast, as time increases to infinity. Finally, we consider a related mean-field coagulation-fragmentation model, we compute its invariant distribution, and we show numerically that it is very close to that of the interacting particle system.

Key words : Stochastic interacting particle systems, Equilibrium, Coalescence, Fragmentation, Self organized criticality, Forest-fire model.

MSC 2000 : 60K35.

1 Notations and main results

Consider an independent family $N = ((N_t(i))_{t\geq 0})_{i\in\mathbb{Z}}$ of Poisson processes with rate 1. In the whole paper, such a family will be called an IFPP.
Assume that on each site $i \in \mathbb{Z}$, snow flocks are falling according to the process $(N_t(i))_{t\geq 0}$. When a flock falls on a vacant site of $\mathbb{Z}$, this site becomes occupied. When a flock falls on an occupied site $i \in \mathbb{Z}$, an *avalanche* starts: the whole connected component of occupied sites around i becomes vacant.
We denote by $(\eta_t(i))_{t\geq 0, i\in\mathbb{Z}}$ the process defined, for $t \geq 0$ and $i \in \mathbb{Z}$, by $\eta_t(i) = 1$ (resp. $\eta_t(i) = 0$) if the site i is occupied (resp. vacant) at time t.

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To avoid infinite rates of interaction, we will restrict our study to the case where the initial condition η_0 lies in the following space:

$$E := \left\{ \eta \in \{0,1\}^{\mathbb{Z}};\ \liminf_{i\to-\infty} \eta(i) = \liminf_{i\to+\infty} \eta(i) = 0 \right\}. \tag{1.1}$$

A state η belongs to E if and only if it has no infinite connected component of occupied sites. This condition is not really restrictive: easy considerations show that even if $\eta_0 \in \{0,1\}^{\mathbb{Z}} \backslash E$, $\eta_t \in E$ for all $t > 0$. This comes from the fact that infinite connected components of occupied sites have an infinite death rate.

It is standard and easy to show, using for example a *graphical construction*, that for any initial condition $\eta_0 \in E$, for any IFPP N, the process $(\eta_t)_{t\geq 0}$ exists, is unique, and takes its values in E. It is actually a deterministic function of η_0 and $(N(i))_{i\in\mathbb{Z}}$. We call this process the η_0-avalanche process, or the (η_0, N)-avalanche process when this precision is needed. See [15] for many examples of graphical constructions.

Furthermore, the process $(\eta_t)_{t\geq 0}$ is a strong Markov process, and its infinitesimal generator $\mathcal{A}$ is defined, for $\eta \in E$ and $\Phi : E \mapsto \mathbb{R}$ sufficiently regular (e.g. $\Phi(\eta)$ depending only on a finite number of coordinates of η) by

$$\mathcal{A}\Phi(\eta) = \sum_{i\in\mathbb{Z}} \left[\Phi(a_i(\eta)) - \Phi(\eta)\right], \tag{1.2}$$

where $a_i(\eta) \in E$ is defined in the following way:

- if $\eta(i) = 0$, then $[a_i(\eta)](i) = 1$ and $[a_i(\eta)](k) = \eta(k)$ for all $k \neq i$;
- if $\eta(i) = 1$, set $l_\eta(i) = \sup\{k \leq i; \eta(k) = 0\} + 1$, $r_\eta(i) = \inf\{k \geq i; \eta(k) = 0\} - 1$, and put $[a_i(\eta)](k) = 0$ for $k \in [l_\eta(i), r_\eta(i)]$ and $[a_i(\eta)](k) = \eta(k)$ for all $k \notin [l_\eta(i), r_\eta(i)]$.

Our main result in this paper concerns the invariant distribution of the avalanche process.
For $A \subset \mathbb{Z}$ and $\Gamma \in \mathcal{P}(E)$ we denote by $\Gamma_A = \Gamma \circ p_A^{-1} \in \mathcal{P}(\{0,1\}^A)$ its restriction to A, where $p_A : E \mapsto \{0,1\}^A$ is the canonical projection.
For two probability measures μ, ν on a measurable space $(F, \mathcal{F})$, we denote by $|\mu - \nu|_{TV} = \sup_{G\in\mathcal{F}} |\mu(G) - \nu(G)|$ the total variation between μ and ν.

Theorem 1.1 *(a) The avalanche process admits an unique invariant distribution* $\Pi \in \mathcal{P}(E)$.
(b) The exponential trend to equilibrium holds in the following sense. For $\varphi \in E$, *denote by* Π_t^φ *the law of the* φ*-avalanche process at time* t. *There exist some constants* $C > 0$, $\alpha > 0$ *such that for all* $t \geq 0$, *all* $l \geq 0$,

$$\sup_{\varphi\in E} \left|(\Pi_t^\varphi)_{[-l,l]} - \Pi_{[-l,l]}\right|_{TV} \leq C(1+l)e^{-\alpha t}. \tag{1.3}$$

(c) For $l \geq 0$, *there exists an explicit (see Appendix A) and perfect simulation algorithm for a* $\Pi_{[-l,l]}$*-distributed random variable* $(\eta_0(i))_{i\in[-l,l]}$.

(d) The invariant distribution Π *is exponentially mixing in the following sense: one may find some constants* $C > 0$, $0 < q < 1$, *such that for any* $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$,

$$\left|\Pi_{(-\infty,k]\cup[k+n,\infty)} - \Pi_{(-\infty,k]} \otimes \Pi_{[k+n,\infty)}\right|_{TV} \le Cq^n. \tag{1.4}$$

Let us comment on these results. First, the system is very stable, in the sense that no large clusters of occupied sites may appear. Indeed, large clusters have a large death rate. Clearly, the existence of invariant distributions should easily follow from such an argument. Of course, uniqueness of the invariant distribution and trend to equilibrium are not suprising, but much more work is required, especially to give a rate of convergence. The perfect simulation algorithm we give is quite complicated, but gives, in some sense, an explicit expression of the invariant distribution Π. Finally, point (d) explains that at equilibrium, for two sites i and j, the dependance between $\eta(i)$ and $\eta(j)$ decreases exponentially fast with $|i-j|$. Such a result is also quite natural, but the proof is quite complicated.

At the end of the 80's, the so-called self-organized critical (SOC) systems became rather popular. They are simple models supposed to enlight temporal and spatial randomness observed in a variety of natural phenomena showing *long range correlations*, like sand piles, avalanches, earthquakes, stock market crashes, fire forest, shape of mountains, of clouds, ... Very roughly, the key idea (present in Bak-Tang-Wiesenfeld [2] about sand piles) is that of systems *growing* toward a *critical state* and relaxing through *catastrophic* events (avalanches, crashes, fire, ...) ; if the catastrophic events become more and more probable when approaching the critical state, the system spontaneously reaches an equilibrium *close* to the critical state.
SOC systems commonly share other features as long range correlations, power laws for the amplitude of *catastrophic* events, spatial fractality of observed patterns, lack of typical scale, ... The most classical model is the so-called sand pile model introduced in 1987 in [2], but a lot of variants or related models have been proposed and studied more or less rigorously, describing earthquakes (Olami-Feder-Christensen, [16]) or fire forest (Henley [13] ; Drossel-Schwabl, [8]) to mention a few. For surveys on the subject, see [7] or [3], for instance.

Initially, our process was thought as a very rough simplification of a sand pile model. In sand piles geometric rules describe the structure of a *stable* sand pile. Sand grains fall on a given pile ; if the new pile is *unstable*, it is re-organized to become stable, through (possibly many successive) elementary steps ; such events are called *avalanches*. If the pile lives on a bounded domain, grains falling out of the domain disappear ; if the model is realistic, one can imagine that the number of grains in the pile and the shape of the pile reaches an equilibrium. Frequence and amplitude of *avalanches* at equilibrium are related to the number of grains that disappear. In our much simpler model, a grain falling on an occupied site yields an *avalanche* involving all grains in the connected component (that immediately disappear). It does not pretend to be a good physical description of a sand pile: the purpose is more to catch what is really important in SOC systems.

This simplification is pertinent in that it can also be viewed as a particular case of a maybe more natural simplification of forest fire models. Roughly, fire forest model can be described as follows: on a lattice, trees are born (sites become occupied) at a certain rate, say 1 ; at each tree, a fire may start at some rate, say $\lambda > 0$: the site becomes vacant and fire propagates to neighbouring trees (occupied sites) at a given speed (see [8] for a precise description). Taking an infinite propagation speed means that the whole connected component (of sites occupied by trees) containing the ignited tree burns at once (one may think of lightning). Our model corresponds to the case $\lambda = 1$, infinite propagation speed and a lattice equal to $\mathbb{Z}$. From the point of view of SOC systems, the interesting phenomenon is in the asymptotic regime $\lambda \to 0$. Indeed fires are less frequent, but when they occur, destroyed clusters may be huge. These models have been subject to a lot of numerical and heuristical studies (see [12] for references), but fewer rigorous results. Even existence and uniqueness of the process for a multidimensionnal lattice and given λ has been proved only recently [9, 10]. Limiting rescaling when $\lambda \to 0$ has been studied numerically [8, 12] but attempts to give a rigorous basis, even in dimension 1 are more recent [5, 4, 6]. Still our model had not received a complete rigorous treatment, and as far as we understand, even if results are not surprising they are now quite complete and the approach we propose may be extended.

Consider the model in which *birth* flocks follow Poisson processes with rate 1, while *killing* flocks follow Poisson processes with rate $\lambda > 0$. We believe that our result could be extended without difficulty to the case where $\lambda \geq 1$ (so that the clusters are not very large). In the case where $\lambda < 1$, the method we use probably breaks down, but the refined version of the algorithm described in Appendix A gives hints for further research in this direction

The paper is organized as follows. In Section 2, we show that the avalanche process can be coupled with (and compared to) a very simple system of independent particles which we call a Bernoulli process. The invariant distribution of this particle system is an infinite product of Bernoulli distributions.
In Section 3, we show how to build the invariant distribution of the avalanche process from a stationnary Bernoulli process on an a.s. finite time intervall, provided some cluster (concerning essentially the Bernoulli process) is a.s. finite. We obtain some large-deviation type upperbounds for the width and height of this cluster in Section 4.
This allows us to conclude the proof in Section 5: the invariant distribution exists and can be perfectly simulated. We can estimate the decay of correlations in the invariant distribution of the avalanche process, using the upperbound of the width of the previously cited cluster. The coupling also shows, in some sense, the uniqueness of the invariant distribution and the trend to equilibrium. The rate of return to equilibrium is obtained as a corollary of the upperbound of the height of the cluster. In Appendix A, we write down the *perfect* simulation algorithm for the invariant distribution derived from Sections 3 and 4.

We finally introduce a related coagulation-fragmentation mean-field model in Section 6: assuming that the correlations between the sizes of connected components of occupied sites are neglictable, we write down an infinite system of ordinary differential equations satisfied by the *concentrations* of clusters with size k, for $k \geq 1$: each pair of clusters coalesce at constant rate, while each cluster with size k breaks into clusters with size 0 at rate k. The equilibrium state of the system of O.D.E.s can be computed almost explicitely. Numerical experiments show that this model is an excellent approximation of the avalanche process, at least from a *global* point of view.

2 The coupling with a Bernoulli process

The starting point of our results is that we may deduce a realization of the (possible) equilibrium Π of the avalanche process from that of a much simpler process, which we now describe.

Consider as before an IFPP N, and an initial state $\zeta_0 \in \{0,1\}^{\mathbb{Z}}$. Assume that the snow flocks are falling on each site i according to $N(i)$, but that the avalanche is restricted to the site i: if i was vacant, it becomes occupied as before, but if it was occupied, it becomes vacant, letting its neighbors enjoy their own life. Denote, for each $i \in \mathbb{Z}$, each $t \geq 0$, by $\zeta_t(i) = 1$ (resp. $\zeta_t(i) = 0$) if the site i is occupied (resp. vacant) at time t.
The process $(\zeta_t)_{t\geq 0}$ is obviously well-defined, unique, and explicit: for $i \in \mathbb{Z}$, $t \geq 0$, $\zeta_t(i) = \zeta_0(i)$ (resp. $\zeta_t(i) = 1 - \zeta_0(i)$) if $N_t(i)$ is even (resp. odd). In other words,

$$\zeta_t(i) = \frac{1}{2}[1 - (-1)^{\zeta_0(i) + N_t(i)}]. \tag{2.1}$$

We call it the ζ_0-Bernoulli process (or if necessary the (ζ_0, N)-Bernoulli process). Let us now describe its trend to equilibrium.

Lemma 2.1 *Let $\Gamma = \otimes_{i\in\mathbb{Z}} \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)$ be the infinite product of Bernoulli laws with parameter $1/2$. For $\zeta_0 \in \{0,1\}^{\mathbb{Z}}$, denote by $\Gamma_t^{\zeta_0}$ the law of the ζ_0-Bernoulli process at time t. Then for all $l \geq 0$, all $t \geq 0$,*

$$\sup_{\zeta_0 \in \{0,1\}^{\mathbb{Z}}} \left| (\Gamma_t^{\zeta_0})_{[-l,l]} - \Gamma_{[-l,l]} \right|_{TV} \leq (2l+1)e^{-2t}. \tag{2.2}$$

As a consequence, Γ is the only invariant distribution of the Bernoulli process.

Proof Let thus $\zeta_0 \in \{0,1\}^{\mathbb{Z}}$ be fixed and $(\zeta_t)_{t\geq 0}$ be the (ζ_0, N)-Bernoulli process, for some given IFPP N. First of all observe that for any $t \geq 0$, any $i \in \mathbb{Z}$, using the explicit expression of $\zeta_t(i)$ leads us to

$$\begin{aligned} P[\zeta_t(i) = 0] &= \mathbb{1}_{\{\zeta_0(i)=0\}} P[N_t(i) \text{ is even}] + \mathbb{1}_{\{\zeta_0(i)=1\}} P[N_t(i) \text{ is odd}] \\ &= \frac{1}{2}\left[\mathbb{1}_{\{\zeta_0(i)=0\}}(1 + e^{-2t}) + \mathbb{1}_{\{\zeta_0(i)=1\}}(1 - e^{-2t})\right]. \end{aligned} \tag{2.3}$$

This implies that $|P[\zeta_t(i)=0]-\frac{1}{2}|\leq e^{-2t}/2$. By the same way, $|P[\zeta_t(i)=1]-\frac{1}{2}|\leq e^{-2t}/2$, so that we get $\left|(\Gamma_t^{\zeta_0})_{\{i\}}-\Gamma_{\{i\}}\right|_{TV}\leq e^{-2t}$. The result follows, since $[-l,l]$ contains $2l+1$ sites and since the coordinates of $(\zeta_t(i))_{i\in[-l,l]}$ are independent. $\square$

Next, we explain how to reverse time in the stationnary Bernoulli process. This will be usefull to build *from the past* the invariant distribution of the avalanche process.

Lemma 2.2 *Let ζ_0 be a $\{0,1\}^{\mathbb{Z}}$-valued random variable with law Γ. Consider N an IFPP, and let $(\zeta_t)_{t\geq 0}$ be the (stationnary) (ζ_0,N)-Bernoulli process.*
Consider the process $(\tilde{\zeta}_t)_{t\in(-\infty,0]}$ defined by $\tilde{\zeta}_t=\zeta_{(-t)-}$ for all $t\leq 0$. Then this is again a (stationnary) Bernoulli process, in the sense that for any $T\leq 0$, the process $(\tilde{\zeta}_t)_{t\in[T,0]}$ is a $(\tilde{\zeta}_T,N^T)$-Bernoulli process with $\tilde{\zeta}_T$ independent of N^T, with $\tilde{\zeta}_T\sim\Gamma$, and where the IFPP N^T on $[T,0]$ is defined by $N_t^T(i)=N_{(-T)-}(i)-N_{(-t)-}(i)$ for $t\in[T,0]$ and $i\in\mathbb{Z}$.

Proof Let $T=-S<0$ be fixed. Using the explicit formula, we know that for all $i\in\mathbb{Z}$, all $t\in[T,0]$,

$$\begin{aligned}\tilde{\zeta}_t(i) &= \zeta_{(-t)-}(i)=\frac{1}{2}[1-(-1)^{\zeta_0(i)+N_{(-t)-}(i)}]\\ &= \frac{1}{2}[1-(-1)^{\zeta_0(i)+N_{(-T)-}(i)-N_t^T(i)}]=\frac{1}{2}[1-(-1)^{\tilde{\zeta}_T(i)+N_t^T(i)}], \quad (2.4)\end{aligned}$$

since one easily checks that $\zeta_0(i)+N_{(-T)-}(i)$ and $\tilde{\zeta}_T(i)=\zeta_{(-T)-}(i)$ have the same parity. Thus, we just have to prove that (i) $\tilde{\zeta}_T\sim\Gamma$, (ii) N^T is an IFPP on $[T,0]$, (iii) $\tilde{\zeta}_T$ and N^T are independent.
Point (i) is obvious from the stationnarity of $(\zeta_t)_{t\geq 0}$, while point (ii) is a well-known fact about Poisson processes. To prove point (iii), it suffices to notice that for any $i\in\mathbb{Z}$, $x\in\{0,1\}$,

$$\begin{aligned}&P\left[\tilde{\zeta}_T(i)=x\,\middle|\,\sigma((N_t^T(i))_{t\in[T,0]}\right]=P\left[\zeta_{S-}(i)=x\,\middle|\,\sigma((N_t(i))_{t\in[0,S]}\right]\\ &=\mathbb{1}_{\{N_{S-}(i)\text{ is even}\}}P[\zeta_0(i)=x]+\mathbb{1}_{\{N_{S-}(i)\text{ is odd}\}}P[\zeta_0(i)=1-x]\\ &\qquad=\frac{1}{2}=P\left[\tilde{\zeta}_T(i)=x\right]. \quad (2.5)\end{aligned}$$

This ends the proof. $\square$

We will also need later the following monotonicity result about the Bernoulli process.

Lemma 2.3 *Consider N and V two independent IFPPs. Let $\zeta_0^1,\zeta_0^2\in\{0,1\}^{\mathbb{Z}}$. Consider the (ζ_0^1,N)-Bernoulli process $(\zeta_t^1)_{t\geq 0}$.*
There exists M an IFPP such that, denoting by $(\zeta_t^2)_{t\geq 0}$ the (ζ_0^2,M)-Bernoulli process, a.s., for all $t\geq 0$, all $i\in\mathbb{Z}$,

(i) $M_t(i) = \int_0^t 1\!\!1_{\{\zeta^1_{s-}(i)=\zeta^2_{s-}(i)\}} dN_s(i) + \int_0^t 1\!\!1_{\{\zeta^1_{s-}(i)\neq\zeta^2_{s-}(i)\}} dV_s(i)$*;*
(ii) if $\gamma_i := \inf\{t \geq 0; \zeta^1_t(i) = \zeta^2_t(i)\}$, $P[\gamma_i \geq t] \leq e^{-2t}$, *and* $(M_{\gamma_i+t} - M_{\gamma_i})_{t\geq 0} = (N_{\gamma_i+t} - N_{\gamma_i})_{t\geq 0}$*;*
(iii) if $\zeta^1_t(i) = \zeta^2_t(i)$ *then* $\zeta^1_{t+s}(i) = \zeta^2_{t+s}(i)$ *for all* $s \geq 0$*;*
(iv) if $\zeta^1_0(i) \leq \zeta^2_0(i)$, *then* $\zeta^1_t(i) \leq \zeta^2_t(i)$ *a.s. for all* $t \geq 0$.
We will say that $(\zeta^1_t, \zeta^2_t)_{t\geq 0}$ *are the* $(\zeta^1_0, \zeta^2_0, N, V)$*-coupled Bernoulli processes.*

Of course, the more natural coupling consisting in building the two Bernoulli processes with the same IFPP would not preserve order as time evolves.

Proof The coupling we use here consists in choosing the same Poisson process $N(i)$ for both processes when $\zeta^1_0(i) = \zeta^2_0(i)$, so that they will appear or die simultaneously, and will remain equal for all times. But if $0 = \zeta^1_0(i) < \zeta^2_0(i) = 1$ (resp. $0 = \zeta^2_0(i) < \zeta^1_0(i) = 1$), we use first independent Poisson processes: $\zeta^2_t(i)$ dies using $V_t(i)$, while $\zeta^1_t(i)$ appears following $N_t(i)$: after this first jump, they become equal, and we then use the same Poisson process $N_t(i)$, and they remain equal for all times.

More rigorously, for $i \in \mathbb{Z}$, denote by T_i (resp. S_i) the first instant of jump of $N(i)$ (resp. $V(i)$), and put $\tau_i = T_i \wedge S_i$. It is immediate that τ_i follows an exponential distribution with parameter 2. Define the process $M(i)$ by

$$M_t(i) = 1\!\!1_{\{\zeta^1_0(i)=\zeta^2_0(i)\}} N_t(i) + 1\!\!1_{\{\zeta^1_0(i)\neq\zeta^2_0(i)\}} \left[V_{t\wedge\tau_i}(i) + (N_t - N_{\tau_i}) 1\!\!1_{\{t>\tau_i\}}\right]. \tag{2.6}$$

Then $M(i)$ is classically a Poisson process with rate 1. We thus may define the (ζ^2_0, M)-Bernoulli process $(\zeta^2_t)_{t\geq 0}$.
Let us check points (i), (ii), (iii), and (iv). If $\zeta^1_0(i) = \zeta^2_0(i)$, these points are obvious and $\gamma_i = 0$, since then $M(i) = N(i)$ and $\zeta^1_t(i) = \zeta^2_t(i)$ for all times.
If $0 = \zeta^1_0(i) < \zeta^2_0(i) = 1$, then $\gamma_i = \tau_i$ and $0 = \zeta^1_t(i) < \zeta^2_t(i) = 1$ for $t \in [0, \gamma_i)$. Easy considerations show that for $t \geq \gamma_i$, $M_t(i)$ and $N_t(i)$ have an opposite parity, which implies that $\zeta^1_t(i) = \zeta^2_t(i)$. This shows points (ii), (iii), and (iv). Since $\gamma_i = \inf\{t \geq 0, \zeta^1_t(i) = \zeta^2_t(i)\}$, point (i) can be written as $M_t(i) = V_{t\wedge\gamma_i} + (N_t - N_{\gamma_i}) 1\!\!1_{\{t>\gamma_i\}}$, which achieves the proof. □

We now describe the coupling between the avalanche and Bernoulli processes.

Proposition 2.4 *Consider* N *and* V *two independent IFPPs. Let* $\eta_0 \in E$ *and* $\zeta_0 \in \{0,1\}^{\mathbb{Z}}$. *Assume that for all* $i \in \mathbb{Z}$, $\eta_0(i) \leq \zeta_0(i)$. *Consider the* (ζ_0, N)*-Bernoulli process* $(\zeta_t)_{t\geq 0}$.
There exists M *an IFPP such that, denoting by* $(\eta_t)_{t\geq 0}$ *the* (η_0, M)*-avalanche process, a.s., for all* $t \geq 0$, *all* $i \in \mathbb{Z}$,
(i) $\eta_t(i) \leq \zeta_t(i)$*;*
(ii) $M_t(i) = \int_0^t 1\!\!1_{\{\eta_{s-}(i)=\zeta_{s-}(i)\}} dN_s(i) + \int_0^t 1\!\!1_{\{\eta_{s-}(i)<\zeta_{s-}(i)\}} dV_s(i)$.
We will say that $(\zeta_t, \eta_t)_{t\geq 0}$ *is the* (ζ_0, η_0, N, V)*-coupled Bernoulli-avalanche process.*

Again here, building the Bernoulli and avalanche processes with the same IFPP would not preserve the order.

Proof The coupling is the following. For each $i \in \mathbb{Z}$, at time $t \geq 0$, we use:
(a) the same IFPP $N_t(i)$ to make appear a flock in η and ζ if $\eta_{t-}(i) = \zeta_{t-}(i) = 0$;
(b) the same IFPP $N_t(i)$ to make die the flock at i (in ζ) or the whole connected component of flocks around i (in η) if $\eta_{t-}(i) = \zeta_{t-}(i) = 1$;
(c) the IFPP $N_t(i)$ to make die the flock at i (in ζ) and the independent IFPP $V_t(i)$ to make appear a flock (in η) if $0 = \eta_{t-}(i) < \zeta_{t-}(i) = 1$.
This construction guarantees that for all $t \geq 0$, all $i \in \mathbb{Z}$, $\eta_t(i) \leq \zeta_t(i)$. The rigorous proof is similar to that of Lemma 2.3. □

This coupling is illustrated by Figure 1, and can be represented graphically in the following way.

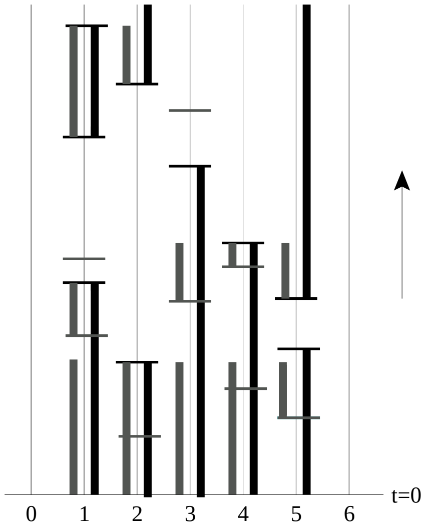
Graphical construction 2.5 *(a) Initially, each site of $\mathbb{Z}$ is occupied or not according to ζ_0 or η_0. We draw black (resp. grey) segments to represent the marks of N (resp. V) above each site of $\mathbb{Z}$.*
(b) Next, we deduce the Bernoulli process ζ:
when an occupied site encounters a black mark, it becomes vacant;
when a vacant site encounters a black mark, it becomes occupied.
(The Bernoulli process is not concerned with the grey marks).
(c) Finally, we deduce the avalanche process η:
when an occupied site, say i, encounters a black mark, this makes become vacant the whole connected component of occupied sites around i;
when a vacant site (say the site i, at time t) encounters a black mark, it becomes occupied (and so does it in the process ζ) if and only if the Bernoulli process satisfies $\zeta_{t-}(i) = 0$;
when a vacant site (say the site i, at time t), encounters a grey mark, then it becomes occupied if and only if $\zeta_{t-}(i) = 1$.

This graphical construction is possible because $\eta_0 \in E$, which guarantees that for any $T > 0$, there are a.s. infinitely many sites i for which $\eta_0(i) = N_T(i) = V_T(i) = 0$, and since such sites *cut* the interactions.
An immediate consequence of Proposition 2.4 is the following, which we will use later.

Corollary 2.6 *Let Π be an invariant distribution of the avalanche process. Recall that Γ is the invariant distribution of the Bernoulli process. Then Π is stochastically smaller than Γ. This implies that for any random variable $\zeta \sim \Gamma$, we may find a random variable $\eta \sim \Pi$ such that a.s., for all $i \in \mathbb{Z}$, $\eta(i) \leq \zeta(i)$.*

Proof First, Supp $\Pi \subset E$, since the rate of death for each site is bounded below by 1. Consider $\eta_0 = \zeta_0 \sim \Pi$. Using Proposition 2.4, consider a η_0-avalanche process and a ζ_0-Bernoulli process such that a.s., for all $t \geq 0$, $i \in \mathbb{Z}$, $\eta_t(i) \leq \zeta_t(i)$. Of course, $\eta_t \sim \Pi$ for all t, while ζ_t goes in law to Γ as t tends to infinity, due to Lemma 2.1. We conclude that for any $\gamma \in \{0,1\}^{\mathbb{Z}}$, setting $F_\gamma = \{\alpha \in \{0,1\}^{\mathbb{Z}},\ \forall\, i \in \mathbb{Z},\ \alpha(i) \geq \gamma(i)\}$, $\Pi(F_\gamma) \leq \Gamma(F_\gamma)$. This says exactly that Π is stochastically smaller than Γ. □

Figure 1: Coupled avalanche and Bernoulli processes.



The Bernoulli (resp. avalanche) process is represented in black on the right (resp. in grey on the left) of each site. Initially, the Bernoulli (resp. avalanche) process is occupied on the sites $1, 2, 3, 4, 5$ (resp. $1, 2, 3, 4$), and vacant on the sites $0, 6$ (resp. $1, 5, 6$). The Bernoulli process is easily constructed from the black marks: each time a site encounters a black mark, its state changes. Next, we have to build the avalanche process. The sites 2 and 4 are not affected by the first grey marks, since they are occupied. On the contrary, 5 becomes occupied when it encounters its first grey mark, since it is vacant and the Bernoulli process is occupied (at this time on this site). Next, the site 2 encounters a black mark, which kills him and its whole connected component of occupied sites, that is $1, 2, 3, 4, 5$. Next, the (vacant) site 5 encounters a black mark, but it does not become ocuupied, because the Bernoulli process is occupied. Next, the site 3 encounters a grey mark: since it is vacant and the Bernoulli process is occupied, it becomes occupied. But it is killed again by the site 4, which becomes vacant because it encounters a black mark. And so on...

3 Coupling the invariant distributions

Our aim in this section is to describe a way to build the invariant distribution of the avalanche process from that of the Bernoulli process, using the coupling introduced in Proposition 2.4. Our method is based on the ideas of the famous Propp-Wilson algorithm, [17], which concerns Markov chains with finite state space. In the sequel, we will denote, for $\zeta \in \{0, 1\}^{\mathbb{Z}}$,

$$E_\zeta := \{\eta \in E; \forall\, i \in \mathbb{Z},\ \eta(i) \le \zeta(i)\}\,. \tag{3.1}$$

Proposition 3.1 *Let V and N be two independent IFPPs, and $\zeta_0 \sim \Gamma$ (recall Lemma 2.1). Consider the (ζ_0, N)-Bernoulli process $(\zeta_t)_{t\ge 0}$, and its time-reversed $(\tilde{\zeta}_t)_{t\in(-\infty,0]}$ built in Lemma 2.2.*
For $T \in (-\infty, 0]$ and $\varphi \in E_{\zeta_T}$, we denote by $(\tilde{\zeta}_t, \eta_t^{T,\varphi})_{t\in[T,0]}$ the $(\tilde{\zeta}_T, \varphi, N^T, V^T)$-

coupled Bernoulli avalanche process with $N_t^T(i) = N_{(-T)-}(i) - N_{(-t)-}(i)$ *and* $V_t^T(i) = V_{(-T)-}(i) - V_{(-t)-}(i)$ *for* $t \in [T, 0]$ *and* $i \in \mathbb{Z}$.
Observe that a.s., due to Proposition 2.4, for all $S \le T \le t \le 0$, *all* $\varphi \in E_{\tilde{\zeta}_S}$,

$$\eta_T^{S,\varphi} \in E_{\tilde{\zeta}_T} \text{ and } \eta_t^{S,\varphi} = \eta_t^{T,\eta_T^{S,\varphi}}. \tag{3.2}$$

Denote, for each $i \in \mathbb{Z}$, *by (here* $\mathbf{0} \in E$ *is the state with all sites vacant)*

$$\tau_i = \sup\{T \le 0; \forall \varphi \in E_{\tilde{\zeta}_T},\ \eta_0^{T,\varphi}(i) = \eta_0^{T,\mathbf{0}}(i)\}. \tag{3.3}$$

Assume for a moment that a.s., for all $i \in \mathbb{Z}$, $\tau_i > -\infty$. *Notice that we have a.s., for all* $i \in \mathbb{Z}$, *all* $s_1 \le s_2 < 0$, *all* $\varphi_1 \in E_{\tilde{\zeta}_{\tau_i+s_1}}$, $\varphi_2 \in E_{\tilde{\zeta}_{\tau_i+s_2}}$,

$$\eta_0^{\tau_i+s_1,\varphi_1}(i) = \eta_0^{\tau_i+s_2,\varphi_2}(i). \tag{3.4}$$

We thus may define $\eta_0(i) = \eta_0^{\tau_i+s,\mathbf{0}}(i)$ *(which does not depend on* $s < 0$*).*
Then $\Pi := \mathcal{L}(\eta_0)$ *is the unique invariant distribution of the avalanche process.*

It seems that the Bernoulli process is almost unusefull in this statement. However, it allows us to couple all the avalanche processes (with different initial conditions) together. Furthermore, the behaviour of τ_i will be studied through the Bernoulli process. For example, notice that $\tau_i = 0 > -\infty$ if $\zeta_0(i) = 0$. Indeed, due to Proposition 2.4, we know that for all $T < 0$, all $\varphi \in E_{\tilde{\zeta}_T}$, all $s \in [T, 0]$, $\eta_t^{T,\varphi}(i) \le \tilde{\zeta}_t(i)$, which implies that if $\tilde{\zeta}_0(i) = 0$ (i.e. $\zeta_0(i) = 0$), then $\eta_0^{T,\varphi}(i) = 0$. Hence $\tau_i = 0$ and $\eta_0(i) = 0$. When $\zeta_0(i) = 1$, it is much less clear that $\tau_i > -\infty$.

Proof We split the proof into three parts.
Step 1. Let us first explain (3.2) and (3.4). First, the fact that for $S \le T \le 0$ and $\varphi \in E_{\tilde{\zeta}_S}$, $\eta_T^{S,\varphi} \in E_{\tilde{\zeta}_T}$ is straightforward from Proposition 2.4. Then the second equality in (3.2) follows from the construction. Next, consider $i \in \mathbb{Z}$, $s_1 \le s_2 < 0$, $\varphi_1 \in E_{\tilde{\zeta}_{\tau_i+s_1}}$ and $\varphi_2 \in E_{\tilde{\zeta}_{\tau_i+s_2}}$. Due to the definition of τ_i, we deduce that $\eta_0^{\tau_i+s_1,\varphi_1}(i) = \eta_0^{\tau_i+s_1,\mathbf{0}}(i)$. On the other hand, we get from (3.2) that $\eta_{\tau_i+s_2}^{\tau_i+s_1,\mathbf{0}} \in E_{\tilde{\zeta}_{\tau_i+s_2}}$ and $\eta_0^{\tau_i+s_1,\mathbf{0}}(i) = \eta_0^{\tau_i+s_2,\eta_{\tau_i+s_2}^{\tau_i+s_1,\mathbf{0}}}(i)$, the latter being equal to $\eta_0^{\tau_i+s_2,\mathbf{0}}(i)$ due to the definition of τ_i. But using again the definition of τ_i, we deduce that $\eta_0^{\tau_i+s_2,\varphi_2}(i) = \eta_0^{\tau_i+s_2,\mathbf{0}}(i)$. This shows (3.4).

Step 2. Let us now show that $\Pi = \mathcal{L}(\eta_0)$ is an invariant distribution for the avalanche process. To this aim, call $(\eta_t^{\mathbf{0}})_{t\ge0}$ the $\mathbf{0}$-avalanche process. Consider also a bounded function $\Phi : E \mapsto \mathbb{R}$ depending only on a finite number of coordinates, say $\Phi(\eta) = \Phi((\eta(k))_{|k|\le n})$ for some $n \ge 0$. We will show that $\lim_{T\to+\infty} E[\Phi(\eta_T^{\mathbf{0}})] = E[\Phi(\eta_0)]$, which classically suffices to conclude.

Consider now the processes coupled as in the statement. First, $E[\Phi(\eta_T^{\mathbf{0}})] = E[\Phi(\eta_0^{-T,\mathbf{0}})]$ for all $T \ge 0$. Next, on the set $\Omega_n(T) = \{\forall\ |i| \le n,\ \tau_i > -T\}$, $\Phi(\eta_0^{-T,\mathbf{0}}) = \Phi(\eta_0)$ a.s. Since $P[\Omega_n(T)]$ increases to 1 as T increases to infinity (because a.s., $\tau_{-n} \vee ... \vee \tau_n > -\infty$), we deduce that $\lim_{T\to+\infty} E[\Phi(\eta_0^{-T,\mathbf{0}})] = E[\Phi(\eta_0)]$. This concludes the second step.

Step 3. Consider another invariant distribution Π' of the avalanche process. Let $T \geq 0$, and consider, using Lemma 2.6, a random variable $\varphi_T \sim \Pi'$ such that $\varphi_T \in E_{\zeta_T}$ a.s. Consider, as in Step 2, a bounded function $\Phi : E \mapsto \mathbb{R}$ depending only on a finite number of coordinates, say $\Phi(\eta) = \Phi((\eta(k))_{|k|\leq n})$ for some $n \geq 0$, and set $\Omega_n(T) = \{\forall\ |i| \leq n,\ \tau_i > -T\}$. Then on $\Omega_n(T)$, $\Phi(\eta_0^{T,\varphi_T}) = \Phi(\eta_0)$. On the other hand, $\eta_0^{T,\varphi_T} \sim \Pi'$, since Π' is invariant. Using that $P[\Omega_n(T)]$ increases to 1 as T tends to ∞, we easily conclude that $\int \Phi d\Pi' = E[\Phi(\eta_0)]$. Thus $\Pi' = \Pi$. □

4 The contour process

Our aim in this section is to define and study a process which will allow us to estimate τ_i, for $i \in \mathbb{Z}$, and to bound the number of sites involved in the construction of $\eta_0(i)$, in order to estimate the decay of correlations.
The first idea is the following: consider the occupied zone in $\mathbb{Z} \times [0, \infty)$ of the Bernoulli process. Clearly, if this occupied zone has no infinite connected components, then τ_i is finite a.s. for all $i \in \mathbb{Z}$. Indeed, each site i would then a.s. be encompassed by a vacant zone of the Bernoulli process, which implies that the avalanche process is also vacant, and *cuts* the interaction in some sense, which would allow us to build $\eta_0(i)$ from the stationnary process, using the graphical construction 2.5.
But such a consideration would probably lead to a *fat tail* estimate of the distribution of τ_i, because we are in a critical case (the proportion of space occupied by the Bernoulli process is 1/2). A way to overcome this difficulty is to make use of the grey marks (recall Figure 1), which also give us some information about $\eta_0(i)$.
Let us now define the left and right *contour* processes, keeping in mind the coupling between stationnary measures built in Proposition 3.1.

Definition 4.1 *Let $\zeta_0 \in E$, and N, V be two independent IFPPs. We consider the (ζ_0, N)-Bernoulli process $(\zeta_t)_{t\geq 0}$ and we introduce the filtration $\mathcal{G}_t = \sigma\{\zeta_0(i), N_s(i), V_s(i);\ s \in [0,t], i \in \mathbb{Z}\}$.*
For $i \in \mathbb{Z}$, we define the (ζ_0, N, V)-right contour process $(R_t^i)_{t\geq 0}$ around i, with values in $\mathbb{Z} + \frac{1}{2} \cup \{\infty\}$ (see Figure 2 for an illustration) by

$$R_t^i = \sum_{n\geq 0} R_{T_n^i}^i \mathbb{1}_{\{t\in[T_n^i, T_{n+1}^i)\}}, \tag{4.1}$$

where:
Initially, $T_0^i = 0$, $R_0^i = \inf\{k \geq i,\ \zeta_0(k) = 0\} - \frac{1}{2}$. For $n \geq 0$,

$$T_{n+1}^i = \inf\{t > T_n^i, \Delta N_t(R_{T_n^i}^i + \frac{1}{2}) + \Delta N_t(R_{T_n^i}^i - \frac{1}{2}) + \Delta V_t(R_{T_n^i}^i - \frac{1}{2}) > 0\}. \tag{4.2}$$

Then
(a) if $\Delta N_{T^i_{n+1}}(R^i_{T^i_n}+\frac{1}{2})>0$, *then* $R^i_{T^i_{n+1}}=\inf\{k>R^i_{T^i_n},\ \zeta_{T^i_{n+1}}(k)=0\}-\frac{1}{2}$*;*
(b) if $\Delta N_{T^i_{n+1}}(R^i_{T^i_n}-\frac{1}{2})>0$, *then* $R^i_{T^i_{n+1}}=\sup\{k<R^i_{T^i_n},\ \zeta_{T^i_{n+1}}(k)=1\}+\frac{1}{2}$*;*
(c) if $\Delta V_{T^i_{n+1}}(R^i_{T^i_n}-\frac{1}{2})>0$, *then*
  (i) if $\zeta_{T^i_{n+1}}(R^i_{T^i_n}-\frac{3}{2})=1$, $R^i_{T^i_{n+1}}=R^i_{T^i_n}$,
  (ii) if $\zeta_{T^i_{n+1}}(R^i_{T^i_n}-\frac{3}{2})=0$, *then* $R^i_{T^i_{n+1}}=\sup\{k<R^i_{T^i_n},\ \zeta_{T^i_{n+1}}(k)=1\}+\frac{1}{2}$.
The left contour process $(L^i_t)_{t\geq 0}$ *around* i *is defined symmetrically.*

Remark that the sequence $(T^i_n)_{n\geq 0}$ contains all the instants of jumps of $(R^i_t)_{t\geq 0}$, but it contains also fictitious jumps (case (c)-(i)). We explain how to build graphically these contour processes, as illustrated by Figure 2.

Graphical construction 4.2 *Draw above each site* $i\in\mathbb{Z}$ *the marks of* N *in black and those of* V *in grey. Draw in black the Bernoulli process corresponding to a given initial data* ζ_0.
A time 0*, the right contour process* R^0_0 *lies on the left of the first vacant site of* ζ_0 *on the right of* 0 *(e.g., if* $\zeta_0(0)=\zeta_0(1)=1$ *and* $\zeta_0(2)=0$*, then* $R^0_0=1.5$*).*
Next the dynamics of R^0 *are the following:*
(a) each time it encounters a black mark on its right, it jumps to the left of the first vacant site on its right;
(b) when it encounters a black mark on its left, it jumps to the right of the first occupied site on its left;
(c) when it encounters a grey mark on its left (say that $R^0_{t-}=i+0.5$*), and if* $\zeta_{t-}(i-1)=0$*, then it jumps to the right of the first occupied site on the left of* $i-1$*.*
The process $(L^0_t)_{t\geq 0}$ *follows the same dynamics, permuting the roles of* left *and* right*.*

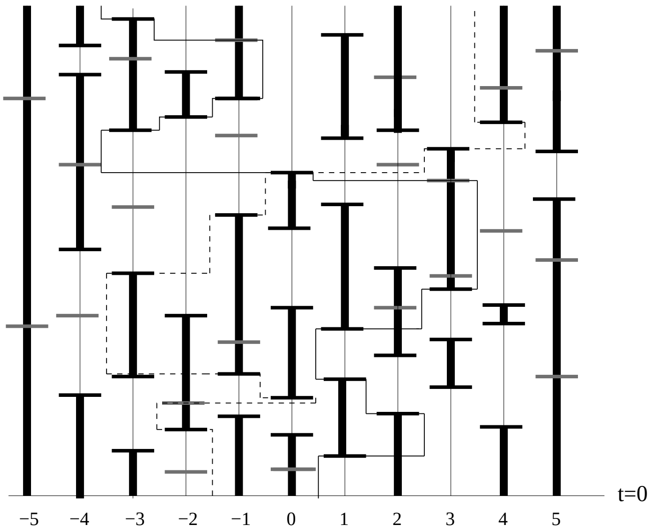
We will see in the next section that it is possible to build $\eta_0\sim\Pi$ (where Π is the invariant distribution of the avalanche process) in such a way that $\eta_0(i)$ depends only on ζ_0, N and V in the *box* delimited by L^i_t and R^i_t until they first meet (if they do). The main reason for this is the following property, which says that in some sense, the contour processes encompass a given site i by a vacant zone of the Bernoulli process.

Lemma 4.3 *We adopt the notations of Definition 4.1. A.s., for all* $t\geq 0$*, all* $i\in\mathbb{Z}$*,* $\zeta_t(R^i_t+\frac{1}{2})=\zeta_t(L^i_t-\frac{1}{2})=0$*.*

Proof It is clear from the construction. □

Remark here that case (c)-(i) in Definition 4.1 is considered to have this Lemma. Indeed, if we want the right contour process to have only vacant sites (of the Bernoulli process) on its right, we can use grey marks to jump to the left only when there is at least one vacant site on its strict left.

Figure 2: The contour processes $(R_t^0)_{t\geq 0}$ and $(L_t^0)_{t\geq 0}$ around 0.



The process R_t^0 (resp. L_t^0) is represented in plain (resp. dashed) line. First, $R_0^0 = 0.5$, since the first vacant site (of the Bernoulli process at time 0) on the right of 0 is 1. By the same way, $L_0^0 = -1.5$. Next, R^0 encounters a grey mark on its left, but since at this time $\zeta_t(-1) = 1$, it does not jump. Then R^0 encounters a black mark on its right, so that it jumps to 2.5, i.e. the left of 3, which is (at this time) the first vacant site on its right. Next, it encounters a black mark on its left and jumps to 1.5, which is the right of 1, i.e. the first vacant site on its left, and so on... As we see on the picture, when it encouters its fourth grey mark on its left, we have $R_{t-}^0 = 3.5$, and since $\zeta_{t-}(2) = 0$, it jumps to 0.5, which is the right of 0, i.e. the first occupied site on the left of 2.

To study the decay of correlations, we have to estimate the width of the *box*, while to study the rate of trend to equilibrium, we have to estimate its height. The following estimates, central in our proof, will provide some bounds on these quantities.

Proposition 4.4 *Let $\zeta_0 \sim \Gamma$, let N, V be two independent IFPPs. Consider the right and left (ζ_0, N, V)-contour processes $(R_t^0)_{t\geq 0}$ and $(L_t^0)_{t\geq 0}$ around 0. Consider the stopping time (for the filtration $(\mathcal{G}_t)_{t\geq 0}$)*

$$\rho^0 = \inf\{t \geq 0;\ R_t^0 < L_t^0\} \tag{4.3}$$

and the random variable

$$\overline{R}_\infty^0 = \sup_{t\geq 0} R_t^0. \tag{4.4}$$

(a) There exists $\beta > 0$ such that $E[e^{\beta\rho^0}] < \infty$.
(b) There exists $\gamma > 0$ such that $E[e^{\gamma \overline{R}_\infty^0}] < \infty$.

The remainder of this section is devoted to the proof of these estimates. They seem quite natural, since the process $(R_t^0)_{t\geq 0}$ is a sort of random walk with negative *jump size expectation*: two types of events allow $(R^0_t)_{t\geq 0}$ to jump to the left, while only one type allows him to jump to the right. Furthermore, some symmetry seems to hold between the jumps to the right (due to N) and those to left due to N. Thus, the result seems almost obvious, and intuitively very clear. However, we have not found a simple proof. Of course, the main difficulty is that $(R^0_t)_{t\geq 0}$ is not a continuous-time random walk: its sizes of jumps are not independent. Thus quite a precise study has to be done. Our strategy consists in bounding from above $(R^0_t)_{t\geq 0}$ by a continuous-time random walk with negative jump size expectation. We first describe some immediate properties of the contour processes.

Lemma 4.5 *We adopt the notations of Definition 4.1. Let $i \in \mathbb{Z}$ be fixed.*
(a) If $\zeta_0 \sim \Gamma$, then the processes $(i - L^i_t)_{t\geq 0}$, $(R^i_t - i)_{t\geq 0}$, and $(R^0_t)_{t\geq 0}$ have the same law (but are far from being independent).
(b) If $R^i_0 < \infty$, then $R^i_t < \infty$ for all $t \geq 0$ a.s.
(c) For $j \leq i$, $R^j_t \leq R^i_t$ for all $t \geq 0$ a.s. Furthermore, if $R^j_t = R^i_t$ for some t, then a.s., $R^j_{t+s} = R^i_{t+s}$ for all $s \geq 0$.
(d) The counting processes

$$\begin{aligned}
Z^{1,i}_t &= \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^{1,i}_n\}} := \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^i_n\}} \mathbb{1}_{\{\Delta N_{T^i_n}(R^i_{T^i_{n-1}}+\frac{1}{2})>0\}} = N_t(R^i_{t-} + \frac{1}{2}) \\
Z^{2,i}_t &= \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^{2,i}_n\}} := \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^i_n\}} \mathbb{1}_{\{\Delta N_{T^i_n}(R^i_{T^i_{n-1}}-\frac{1}{2})>0\}} = N_t(R^i_{t-} - \frac{1}{2}) \\
Z^{3,i}_t &= \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^{3,i}_n\}} := \sum_{n\geq 1} \mathbb{1}_{\{t\geq T^i_n\}} \mathbb{1}_{\{\Delta V_{T^i_n}(R^i_{T^i_{n-1}}-\frac{1}{2})>0\}} = V_t(R^i_{t-} - \frac{1}{2}) \quad (4.5)
\end{aligned}$$

are three independent Poisson processes with rate 1. They are $(\mathcal{G}_t)_{t\geq 0}$-adapted, and independent of ζ_0.

Remark here that $Z^{1,i}$ counts the jumps to the right of R^i, while $Z^{2,i}$ counts its jumps to the left due to N (black marks on Figure 2) and $Z^{3,i}$ counts its *possible* jumps to the left due to V (grey marks on Figure 2).

Proof Point (a) is obvious by symmetry and invariance by translation. Point (b) follows from the fact that the Bernoulli a.s. process belongs to E for all $t > 0$, even if it does not at time 0. Point (c) is clear from the construction. Point (e) follows from classical properties on Poisson processes. □

We carry on with a natural monotonicity property.

Lemma 4.6 *We consider three independent IFPPs N, V and W. Let also $\zeta^1_0, \zeta^2_0 \in \{0,1\}^{\mathbb{Z}}$ satisfy, for all $i \in \mathbb{Z}$, $\zeta^1_0(i) \leq \zeta^2_0(i)$. Then we build, recalling Lemma 2.3, the $(\zeta^1_0, \zeta^2_0, N, W)$-coupled Bernoulli process $(\zeta^1_t, \zeta^2_t)_{t\geq 0}$. As stated in Lemma 2.3, $(\zeta^1_t)_{t\geq 0}$ is the (ζ^1_0, N)-Bernoulli process, while $(\zeta^2_t)_{t\geq 0}$ is the (ζ^2_0, M)-Bernoulli process for some IFPP M. We denote by $(R^{0,1}_t)_{t\geq 0}$ (resp. $(R^{0,2}_t)_{t\geq 0}$) the (ζ^1_0, N, V) (resp. (ζ^2_0, M, V)) right contour process around* 0.

We will say that $(R_t^{0,1}, R_t^{0,2})_{t\geq 0}$ *are the* $(\zeta_0^1, \zeta_0^2, N, W, V)$*-coupled right contour processes around* 0*. We have a.s., for all* $t \geq 0$*,* $R_t^{0,1} \leq R_t^{0,2}$*.*

Proof The proof is obvious from the definition of the contour process, since we know from Lemma 2.3 that a.s., for all $t \geq 0$, all $i \in \mathbb{Z}$, $\zeta_t^1(i) \leq \zeta_t^2(i)$. □

We consider the following initial condition.

Notation 4.7 *We say that a* $\{0,1\}^{\mathbb{Z}}$*-valued random variable* $\tilde{\zeta}_0$ *has the distibution* Ξ *is if* $\tilde{\zeta}_0(i) = 1$ *for* $i \leq 0$*,* $\tilde{\zeta}_0(1) = 0$*, and if* $(\tilde{\zeta}_0(i))_{i\geq 2}$ *is a family of i.i.d. Bernoulli random variables with parameter* $1/2$*.*

Let us now explain our strategy to bound the right contour process by a random walk:
we will first upperbound the initial configuration ζ_0 of the Bernoulli process by a (possibly shifted) realization $\tilde{\zeta}_0$ of Ξ, we thus upperbound our contour process by the corresponding contour process $\tilde{R}^0$,
then we will wait for the first instant $\tilde{T}_1^{1,0}$ of jump to the right of $\tilde{R}_t^0$ to the right; this yields a total jump which we will call $Y_1 := \tilde{R}_{\tilde{T}_1^{1,0}} - \tilde{R}_0$, and whose expectation will be shown to be negative;
we will also observe that at $\tilde{T}_1^{1,0}$, we may again bound the configuration of the Bernoulli process by a realization $\tilde{\zeta}_0^1$ of Ξ (shifted around $\tilde{R}_{\tilde{T}_1^{1,0}}$) independent of Y_1;
this last renewal argument allows us to build, recursively, a random walk with negative mean jump size, bounding from above our contour process.

Lemma 4.8 *Let* $\tilde{\zeta}_0 \sim \Xi$*, and consider two independent IFPPs* N, V*. Consider the* $(\tilde{\zeta}_0, N, V)$*-right contour process* $(\tilde{R}_t^0)_{t\geq 0}$ *around* 0*, observe that* $\tilde{R}_0^0 = 1/2$*, and denote by* $\tilde{T}_1^{1,0} := \inf\{t > 0, \Delta\tilde{R}_t^0 > 0\}$ *the first instant where it jumps to the right. We also consider* $(\tilde{\zeta}_t)_{t\geq 0}$ *the* $(\tilde{\zeta}_0, N)$*-Bernoulli process. We set* $Y_1 = \tilde{R}^0_{\tilde{T}_1^{1,0}} - 1/2$*.*
(i) Then $E[Y_1] < 0$*.*
(ii) For all $\varepsilon \in (0, \ln 2)$*,* $E[e^{\varepsilon Y_1}] < \infty$*.*
Furthermore, then there exists $\tilde{\zeta}_0^1 \sim \Xi$ *such that*
(iii) a.s., $\tilde{\zeta}_{\tilde{T}_1^{1,0}}(\tilde{R}^0_{\tilde{T}_1^{1,0}} + i - 1/2) \leq \tilde{\zeta}_0^1(i)$ *for all* $i \in \mathbb{Z}$*,*
(iv) $\tilde{\zeta}_0^1$ *is independent of* $\mathcal{H}_{\tilde{T}_1^{1,0}}$*, where* $\mathcal{H}_t = \sigma\{\tilde{R}_s^0, s \leq t\}$*.*

Proof To simplify the notation, we omit the superscript 0 (which says that we are dealing with the contour process around 0) in this proof. We consider the three independent Poisson processes with rate 1 (see Lemma 4.5-(d)) $\tilde{Z}_t^1 = N_t(\tilde{R}_{t-} + 1/2)$, $\tilde{Z}_t^2 = N_t(\tilde{R}_{t-} - 1/2)$ and $\tilde{Z}_t^3 = V_t(\tilde{R}_{t-} - 1/2)$, and we denote by $(\tilde{T}_i^1)_{i\geq 1}$, $(\tilde{T}_i^2)_{i\geq 1}$, $(\tilde{T}_i^3)_{i\geq 1}$, respectively, their successive instants of jumps. We also denote by $A_j = \cup_{i\geq 1}\{\tilde{T}_i^j\}$, for $j = 1, 2, 3$. We set $\tilde{Z}_t = \tilde{Z}_t^1 + \tilde{Z}_t^2 + \tilde{Z}_t^3$, which is a Poisson process with rate 3, we denote by $(\tilde{T}_i)_{i\geq 1}$ its successive instants of jumps, and we set $A = \cup_{i\geq 1}\{\tilde{T}_i\} = A_1 \cup A_2 \cup A_3$. Finally, we also set for convenience $\tilde T_0 = \tilde T_0^1 = \tilde T_0^2 = \tilde T_0^3 = 0$. Recall that we want to study $Y_1 = \tilde R_{\tilde T_1^1} - 1/2$.

Step 1. For $n \geq 1$, the event $\Omega_n = \{\tilde T_1 \notin A_1, ..., \tilde T_{n-1} \notin A_1, \tilde T_n \in A_n\}$ occurs with probability $p_n := \frac{2^{n-1}}{3^n}$, since A_1, A_2, A_3 are the sets of jumps of three independent Poisson processes with same rate. Notice also that on Ω_n, $\tilde T_1^1 = \tilde T_n$, and we may write $Y_1 = -\sum_{i=1}^{n-1} X_i + X_n$, where $X_1, ..., X_n$ are the successive sizes of the (possibly fictitious) jumps of $\tilde R$ (at the instants $\tilde T_1 < ... < \tilde T_n$), with $X_1 \geq 0$, ..., $X_n \geq 0$. We obtain

$$E[Y_1] = \sum_{n\geq 1} E[\{-(X_1 + ... + X_{n-1}) + X_n\}\mathbb{1}_{\Omega_n}]. \tag{4.6}$$

Step 2. Let us now bound from below $C_{i,n} := E[X_i \mathbb{1}_{\Omega_n}]$, for $1 \leq i \leq n-1$. We denote by Z_i the number of vacant sites of the Bernoulli process on the strict left of $R_{\tilde T_i -} - 1/2$ at time T_i-, that is

$$Z_i := \tilde R_{\tilde T_i -} - 3/2 - \sup\{j \leq \tilde R_{\tilde T_i -} - 3/2,\ \tilde\zeta_{\tilde T_i -}(j) = 1\}. \tag{4.7}$$

Then, due to the definition of $\tilde R$, we know that
(a) on $F_i^2 := \{\tilde T_i \in A_2\}$, $X_i = -\Delta \tilde R_{\tilde T_i} = 1 + Z_i$,
(b) on $F_i^3 := \{\tilde T_i \in A_3\}$, $X_i = -\Delta \tilde R_{\tilde T_i} = (1 + Z_i)\mathbb{1}_{\{Z_i \geq 1\}}$.
Observe that $P[F_i^2|\Omega_n] = P[F_i^3|\Omega_n] = 1/2$, and that F_i^2, F_i^3 are independent of $(Z_i, \tilde T_i)$ conditionally to Ω_n. These are standard properties of Poisson processes. Hence,

$$\begin{aligned} C_{i,n} &= \frac{1}{2} E\left[(1+Z_i)\mathbb{1}_{\Omega_n} + (1+Z_i)\mathbb{1}_{\{Z_i\geq 1\}}\mathbb{1}_{\Omega_n}\right] \\ &= E\left[(1+Z_i)\mathbb{1}_{\Omega_n}\right] - \frac{1}{2}P[Z_i = 0, \Omega_n] \\ &= P[\Omega_n] + \sum_{k\geq 1} P[Z_i \geq k, \Omega_n] - \frac{1}{2} P[Z_i = 0, \Omega_n]. \end{aligned} \tag{4.8}$$

Let now $k \geq 1$ be fixed. We have

$$\begin{aligned} P[Z_i \geq k, \Omega_n] &= P\Big(\tilde\zeta_{\tilde T_i -}(\tilde R_{\tilde T_i -} - \frac{3}{2}) = 0, ..., \tilde\zeta_{\tilde T_i -}(\tilde R_{\tilde T_i -} - \frac{1}{2} - k) = 0, \Omega_n\Big) \\ &= E\left[\prod_{l=1}^{k} P\left(\tilde\zeta_{\tilde T_i -}(\tilde R_{\tilde T_i -} - \frac{1}{2} - l) = 0 \middle| \Omega_n, \tilde T_i, \tilde T_{i-1}\right)\mathbb{1}_{\Omega_n}\right]. \end{aligned} \tag{4.9}$$

Indeed, recalling that on Ω_n, $\tilde R$ has had only jumps to the left before $\tilde T_i$, we easily deduce that on Ω_n, the values of the Bernoulli process at sites $j \leq \tilde R_{\tilde T_i -} - \frac{3}{2}$ are mutually independent conditionnally to $\tilde T_i, \tilde T_{i-1}$.
Let us set $p_s := (1 - e^{-2s})/2 = P[N_s \in 2\mathbb{N} + 1]$ for $s \geq 0$ (for $(N_t)_{t\geq 0}$ is a standard Poisson process with rate 1).

Now for $l \geq 2$, the site $\tilde{R}_{\tilde{T}_i-} - \frac{1}{2} - l$ was occupied at time 0, and its evolution is obviously independent of $(\tilde{R}_t)_{t\in[0,\tilde{T}_i)}$, so that

$$P\left(\tilde{\zeta}_{\tilde{T}_i-}(\tilde{R}_{\tilde{T}_i-} - \frac{1}{2} - l) = 0 \middle| \Omega_n, \tilde{T}_i, \tilde{T}_{i-1}\right) = p_{\tilde{T}_i}. \qquad (4.10)$$

Next, the same argument holds for $l = 1$ on the set $\{X_{i-1} > 0\}$, which indicates that the previous jump to the left was not fictitious: we have

$$P\left(\tilde{\zeta}_{\tilde{T}_i-}(\tilde{R}_{\tilde{T}_i-} - \frac{3}{2}) = 0 \middle| \Omega_n, \tilde{T}_i, \tilde{T}_{i-1}, X_{i-1} > 0\right) = p_{\tilde{T}_i}. \qquad (4.11)$$

But on the event $\{X_{i-1} = 0\}$, we know that $\tilde{\zeta}_{\tilde{T}_{i-1}}(\tilde{R}_{\tilde{T}_i-} - \frac{3}{2}) = 1$. Hence we get

$$P\left(\tilde{\zeta}_{\tilde{T}_i-}(\tilde{R}_{\tilde{T}_i-} - \frac{3}{2}) = 0 \middle| \Omega_n, \tilde{T}_i, \tilde{T}_{i-1}, X_{i-1} = 0\right) = p_{\tilde{T}_i - \tilde{T}_{i-1}}. \qquad (4.12)$$

Noting that $p_{\tilde{T}_i} \geq p_{\tilde{T}_i - \tilde{T}_{i-1}}$, that $\{X_{i-1} = 0\} \subset \{\tilde{T}_{i-1} \in A_3\}$, we deduce that

$$P\left(\tilde{\zeta}_{\tilde{T}_i-}(\tilde{R}_{\tilde{T}_i-} - \frac{3}{2}) = 0 \middle| \Omega_n, \tilde{T}_i, \tilde{T}_{i-1}\right) \geq \mathbb{1}_{\{\tilde{T}_{i-1}\in A_2\}} p_{\tilde{T}_i} + \mathbb{1}_{\{\tilde{T}_{i-1}\in A_3\}} p_{\tilde{T}_i - \tilde{T}_{i-1}}. \qquad (4.13)$$

Gathering the estimates obtained for $l \geq 2$ and $l = 1$, we obtain, for $k \geq 1$,

$$P[Z_i \geq k, \Omega_n] \geq E\left[\mathbb{1}_{\Omega_n} p_{\tilde{T}_i}^{k-1}\left(\mathbb{1}_{\{\tilde{T}_{i-1}\in A_2\}} p_{\tilde{T}_i} + \mathbb{1}_{\{\tilde{T}_{i-1}\in A_3\}} p_{\tilde{T}_i - \tilde{T}_{i-1}}\right)\right] \qquad (4.14)$$

Using finally classical properties of Poisson processes, we see that $\tilde{T}_{i-1}, \tilde{T}_i$ are independent of $\Omega_n, \{\tilde{T}_{i-1} \in A_2\}, \{\tilde{T}_{i-1} \in A_3\}$ and that $P[\Omega_n \cap \{\tilde{T}_{i-1} \in A_2\}] = P[\Omega_n \cap \{\tilde{T}_{i-1} \in A_3\}] = P[\Omega_n]/2 = 2^{n-1}/2.3^n$, so that

$$P[Z_i \geq k, \Omega_n] \geq \frac{2^{n-1}}{2.3^n} E\left[\left(p_{\tilde{T}_i} + p_{\tilde{T}_i - \tilde{T}_{i-1}}\right) p_{\tilde{T}_i}^{k-1}\right]. \qquad (4.15)$$

Next,

$$P[Z_i = 0, \Omega_n] = P[\Omega_n] - P[Z_i \geq 1, \Omega_n] \leq \frac{2^{n-1}}{2.3^n} E\left[2 - p_{\tilde{T}_i} - p_{\tilde{T}_i - \tilde{T}_{i-1}}\right]. \qquad (4.16)$$

Thus, recalling (4.8), for any $1 \leq i \leq n-1$,

$$\begin{aligned} C_{i,n} &\geq \frac{2^{n-1}}{3^n} E\left[1 + \frac{p_{\tilde{T}_i} + p_{\tilde{T}_i - \tilde{T}_{i-1}}}{2 - 2p_{\tilde{T}_i}} - \frac{2 - p_{\tilde{T}_i} - p_{\tilde{T}_i - \tilde{T}_{i-1}}}{4}\right] \\ &\geq \frac{2^{n-1}}{2.3^n} E\left[1 + \frac{p_{\tilde{T}_i}}{1 - p_{\tilde{T}_i}} + \frac{p_{\tilde{T}_i - \tilde{T}_{i-1}}}{1 - p_{\tilde{T}_i}} + \frac{1}{2} p_{\tilde{T}_i} + \frac{1}{2} p_{\tilde{T}_i - \tilde{T}_{i-1}}\right] \\ &=: \frac{2^{n-1}}{2.3^n}(1 + B_i), \end{aligned} \qquad (4.17)$$

where the last equality stands for a definition.

Step 3. We now upperbound $C_{n,n} := E[X_n \mathbb{1}_{\Omega_n}]$. We denote by Z_n the number of occupied sites on the strict right of $\tilde{R}_{\tilde{T}_n-}$, that is

$$Z_n = \inf\{j \geq \tilde{R}_{\tilde{T}_n-} + 3/2, \tilde{\zeta}_{\tilde{T}_n-}(j) = 0\} - \tilde{R}_{\tilde{T}_n-} - 3/2. \tag{4.18}$$

By construction, we have $X_n = 1 + Z_n$ on Ω_n. For $k \geq 1$, we set $J_k := \tilde{R}_{\tilde{T}_n-} + k + \frac{1}{2}$ and $\xi_k := \tilde{\zeta}_{\tilde{T}_n-}(J_k)$. By construction, we have $X_n = 1 + Z_n$ on Ω_n, so that for $k \geq 1$

$$P[Z_n \geq k, \Omega_n] = P\left[\xi_1 = 1, ..., \xi_k = 1, \Omega_n\right]. \tag{4.19}$$

We now introduce the σ-field generated by the path of $(\tilde{R}_t)_{t\in[0,\tilde{T}_1^1)}$, containing also the fictitious jumps, that is, for ν defined by $\tilde{T}_\nu = \tilde{T}_1^1$ ($\nu = n$ on Ω_n),

$$\mathcal{H} := \sigma\left(\nu, \tilde{T}_1, ..., \tilde{T}_\nu, \Delta\tilde{R}_{\tilde{T}_1}, ..., \Delta\tilde{R}_{\tilde{T}_{\nu-1}}\right). \tag{4.20}$$

Observe that obvisouly, $(J_k)_{k\geq 1}$ and Ω_n are $\mathcal{H}$-measurable.
We will show that conditionnally to $\mathcal{H}$, the sequence $(\xi_k)_{k\geq 1}$ is a family of independent random variables on Ω_n, and that for each $k \geq 1$, $P[\xi_k = 1|\mathcal{H}, \Omega_n] \leq 1/2$. Since Ω_n belongs to $\mathcal{H}$, for all $n \geq 1$, we will deduce that

$$P[Z_n \geq k, \Omega_n] \leq \frac{1}{2^k} P[\Omega_n] = \frac{1}{2^k}\frac{2^{n-1}}{3^n}, \tag{4.21}$$

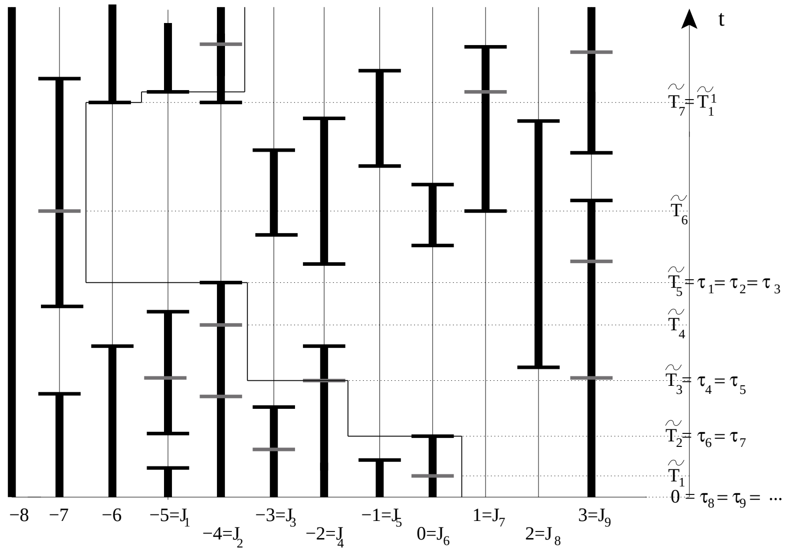
so that (since $X_n = 1 + Z_n$ on Ω_n),

$$C_{n,n} = \sum_{k\geq 1} P[1 + Z_n \geq k, \Omega_n] = \sum_{k\geq 0} P[Z_n \geq k, \Omega_n] \leq \frac{2^n}{3^n}. \tag{4.22}$$

Let us thus check the announced properties of the sequence $(\xi_k)_{k\geq 1}$. For each $k \geq 1$, let $\tau_k = 0$ if $J_k \geq 2$, and let τ_k be the unique instant $t \in [0, \tilde{T}_1^1)$ such that $J_k - 1 \in \{\tilde{R}_t, \tilde{R}_{t-}\}$ if $J_k \leq 1$. We refer to Figure 3 for an illustration. Roughly, τ_k is the last instant before $\tilde{T}_1^1$ where we get some information (from $\mathcal{H}$) about the site J_k. Of course, $(\tau_k)_{k\geq 1}$ is $\mathcal{H}$-measurable.

We will show that the family of random variables $\tilde{\zeta}_{\tau_k}(J_k)$ is mutually independent on Ω_n conditionnally to $\mathcal{H}$, and that for each $k \geq 1$, $P[\tilde{\zeta}_{\tau_k}(J_k) = 1|\mathcal{H}, \Omega_n] \leq 1/2$, This will imply the announced properties, for two reasons:
(i) conditionnally to $\mathcal{H}$ and Ω_n, the evolution of the Bernoulli process at two different sites J_k (on $[\tau_k, \tilde{T}_1^1)$) and J_l (on $[\tau_l, \tilde{T}_1^1)$) are independent, since they concern independent Poisson processes,
(ii) for each $k \geq 1$, $P[\tilde{\zeta}_{\tilde{T}_1^1-}(J_k) = 1|\mathcal{H}, \Omega_n] = P[\tilde{\zeta}_{\tau_k}(J_k) = 1|\mathcal{H}, \Omega_n](1 - p_{\tilde{T}_1^1-\tau_k}) + P[\tilde{\zeta}_{\tau_k}(J_k) = 0|\mathcal{H}, \Omega_n] p_{\tilde{T}_1^1-\tau_k} \leq 1/2$. Indeed, recall that $p_s = (1 - e^{-2s})/2 \leq 1/2$ stands for the probability that a standard Poisson process at time s is odd, and that for $a, b \in [0, 1/2]$, $a(1 - b) + (1 - a)b \leq 1/2$.

Figure 3: Illustration of Step 3 (and Step 6).



With this realization, we have $\tilde{R}_{\tilde{T}^1_1-} = -6.5$ and $G = \{2, 4, 6\}$. We remark that the only site which is occupied by the Bernoulli process when crossed by $\tilde{R}$ is the site $J_4 = -2$: it is crossed through a grey mark.

Consider now the random set, measurable with respect to $\mathcal{H}$,

$$G := \{k \geq 1; \tau_{k+1} > \tau_{k+2}\} = \{k \geq 1; J_k = \tilde{R}_{\tau_k-} - 1/2\}. \tag{4.23}$$

We notice that $\{J_k, k \in G\} \subset [\tilde{R}_{\tilde{T}^1_1}, 0]$.
We observe (see Figure 3) that conditionnally to $\mathcal{H}$, Ω_n, for all $k \geq 1$, we have,

$$\tilde{\zeta}_{\tau_k}(J_k) = 1\!\!1_{\{J_k \leq 1\}} 1\!\!1_{\{k \in G\}} 1\!\!1_{\{\tau_k \in A_3\}} + 1\!\!1_{\{J_k \geq 2\}} \tilde{\zeta}_0(J_k). \tag{4.24}$$

Indeed, if $J_k \geq 2$, the formula is obvious because then $\tau_k = 0$. If $J_k \leq 1$, this comes from the fact that the only way for $\tilde{R}$ to jump to the left through an occupied site is that the jump follows from a grey mark (i.e. $\tau_k \in A_3$) and that the concerned site is just on the left of $\tilde{R}_{\tau_k-}$ (i.e. $k \in G$).

Conditionnally to $\mathcal{H}$, Ω_n, the sequence of events $\{\tau_k \in A_3\}_{k \in G}$ is independent, this assertion makes sense since G is itself $\mathcal{H}$-measurable. Indeed, we know that conditionnally to $\mathcal{H}$, for $k \in G$, $\{\tau_k \in A_3\}$ depends only on the Poisson processes $N_t(J_k), V_t(J_k)$ (and on $\tilde{\zeta}_{\tau_k-}(J_k - 3/2)$ which is $\mathcal{H}$-measurable). The conditionnal independence (with respect to $\mathcal{H}$, Ω_n) of the family $\{\tau_k \in A_3\}_{k \in G}$ follows then from the fact that for $k_1 < k_2$ in G, $J_{k_1} > J_{k_2}$ (see Figure 3), and from the independence of the Poisson processes $(N_t(i), V_t(i))$ and $(N_t(j), V_t(j))$ for $i \neq j$.

Recall now (4.24). Using the conditionnal independence (with respect to $\mathcal{H}$, Ω_n) of the family $\{\tau_k \in A_3\}_{k \in G}$, the fact that the sequence $(J_k)_{k \geq 1}$ is $\mathcal{H}$-measurable, and the fact that the family $(\tilde{\zeta}_0(i))_{i\geq 2}$ is mutually independent and independent of $\mathcal{H}$, Ω_n (these are i.i.d. Bernoulli random variables with parameter 1/2), we obtain that conditionnally to $\mathcal{H}$, Ω_n, the sequence $(\tilde{\zeta}_{\tau_k}(J_k))_{k\geq 1}$ is mutually independent.

We finally conclude by noting that for any $k \geq 1$,

$$\begin{aligned} P\left[\tilde{\zeta}_{\tau_k}(J_k) = 1 \middle| \mathcal{H}, \Omega_n\right] &= \mathbb{1}_{\{J_k \leq 1\}} \mathbb{1}_{\{k \in G\}} P\left[\tau_k \in A_3 \middle| \mathcal{H}, \Omega_n\right] \\ &\quad + \mathbb{1}_{\{J_k \geq 2\}} P\left[\tilde{\zeta}_0(J_k) = 1 \middle| \mathcal{H}, \Omega_n\right] \leq 1/2. \end{aligned} \tag{4.25}$$

The last inequality comes from the fact that $P\left[\tilde{\zeta}_0(J_k) = 1 | \mathcal{H}, \Omega_n\right] = 1/2$ if $J_k \geq 2$ as was previously noticed, while for $k \in G$, $P\left[\tau_k \in A_3 | \mathcal{H}, \Omega_n\right] \leq 1/2$. Indeed, having a look at Figure 3, we realize that for $k \in G$, $\{\tau_k \in A_3\} \subset \{\tilde{\zeta}_{\tau_k -}(J_k - 3/2) = 0\}$, and that due to classical properties of Poisson processes,

$$P\left[\tau_k \in A_3 | \mathcal{H}, \Omega_n, \{\tilde{\zeta}_{\tau_k -}(J_k - 3/2) = 0\}\right] = 1/2. \tag{4.26}$$

Step 4. Gathering (4.6), (4.17) and (4.22), we get

$$\begin{aligned} E[Y_1] &= \sum_{n\geq 1} \left\{ C_{n,n} - \sum_{i=1}^{n-1} C_{i,n} \right\} \\ &\leq \sum_{n\geq 1} \left\{ \frac{2^n}{3^n} - \sum_{i=1}^{n-1} \frac{2^{n-1}}{2.3^n}(1+B_i) \right\} = 2 - \sum_{i\geq 1} \frac{2^{i-1}}{3^i}(1+B_i) \\ &\leq 1 - \sum_{i\geq 1} \frac{2^{i-1}}{3^i} B_i. \end{aligned} \tag{4.27}$$

To conclude that $E[Y_1] < 0$, we thus have to prove that $I = \sum_{i\geq 1} \frac{2^{i-1}}{3^i} B_i > 1$. But, we may write $I = I_1 + I_2 + I_3/2 + I_4/2$, with

$$\begin{aligned} I_1 &:= \sum_{i\geq 1} \frac{2^{i-1}}{3^i} E\left[\frac{p_{\tilde{T}_i}}{1 - p_{\tilde{T}_i}}\right], \qquad I_2 := \sum_{i\geq 1} \frac{2^{i-1}}{3^i} E\left[\frac{p_{\tilde{T}_i - \tilde{T}_{i-1}}}{1 - p_{\tilde{T}_i}}\right] \\ I_3 &:= \sum_{i\geq 1} \frac{2^{i-1}}{3^i} E\left[p_{\tilde{T}_i}\right], \qquad I_4 := \sum_{i\geq 1} \frac{2^{i-1}}{3^i} E\left[p_{\tilde{T}_i - \tilde{T}_{i-1}}\right]. \end{aligned} \tag{4.28}$$

Since $\tilde{T}_i - \tilde{T}_{i-1}$ is exponentially distributed with parameter 3 (for all $i \geq 1$),

$$I_4 = \sum_{i\geq 1} \frac{2^{i-1}}{3^i} \int_0^\infty ds 3e^{-3s} \frac{1 - e^{-2s}}{2} = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{5}. \tag{4.29}$$

Next, since $\tilde{T}_i$ follows a $\Gamma(i, 3)$-distribution,

$$I_3 = \sum_{i\geq 1} \frac{2^{i-1}}{3^i} \int_0^\infty ds \frac{3^i}{(i-1)!} s^{i-1} e^{-3s} \frac{1 - e^{-2s}}{2} = \int_0^\infty ds e^{-s} \frac{1 - e^{-2s}}{2} = \frac{1}{3}, \tag{4.30}$$

and (using the substitution $u = e^{-s}$)

$$\begin{aligned} I_1 &= \sum_{i\geq 1} \frac{2^{i-1}}{3^i} \int_0^\infty ds \frac{3^i}{(i-1)!} s^{i-1} e^{-3s} \frac{1-e^{-2s}}{1+e^{-2s}} = \int_0^\infty ds e^{-s} \frac{1-e^{-2s}}{1+e^{-2s}} \\ &= \int_0^1 du \frac{1-u^2}{1+u^2} = 2\arctan 1 - 1 = \frac{\pi}{2} - 1. \end{aligned} \tag{4.31}$$

Finally, using the independence between $\tilde{T}_{i-1}$ and $\tilde{T}_i - \tilde{T}_{i-1}$, we get

$$\begin{aligned} I_2 &= \frac{1}{3} E\left[\frac{p_{\tilde{T}_1}}{1-p_{\tilde{T}_1}}\right] \\ &\quad + \sum_{i\geq 2} \frac{2^{i-1}}{3^i} \int_0^\infty ds \frac{3^{i-1}}{(i-2)!} s^{i-2} e^{-3s} \int_0^\infty dt 3e^{-3t} \frac{1-e^{-2t}}{1+e^{-2s-2t}} \\ &= \frac{1}{3} \int_0^\infty ds 3e^{-3s} \frac{1-e^{-2s}}{1+e^{-2s}} + \int_0^\infty ds \int_0^\infty dt 2e^{-s} e^{-3t} \frac{1-e^{-2t}}{1+e^{-2s-2t}} \\ &= \int_0^1 du \frac{u^2(1-u^2)}{1+u^2} + 2 \int_0^1 du \int_0^1 dv v^2 \frac{1-v^2}{1+u^2v^2} \\ &= \left(\frac{5}{3} - \frac{\pi}{2}\right) + 2\left(\frac{\pi}{4} - \frac{2}{3}\right) = \frac{1}{3}. \end{aligned} \tag{4.32}$$

We finally get that $I = \frac{\pi}{2} - 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} = \frac{\pi}{2} - \frac{2}{5} > 1$. Thus $E[Y_1] < 0$.

Step 5. We still have to prove the exponential moment estimate. Using the same notation as previously, we will just use that for each $n \geq 1$, $Y_1 \leq X_n = 1 + Z_n$ on Ω_n. Recalling (4.21) and that $\sum_{n\geq 1} P[\Omega_n] = 1$, we get, for any $k \geq 1$

$$P[Y_1 \geq k] \leq \sum_{n\geq 1} P[Z_n \geq k-1, \Omega_n] \leq \frac{1}{2^{k-1}}. \tag{4.33}$$

We classically conclude that for $\varepsilon \in (0, \ln 2)$, $E[e^{\varepsilon Y_1}] < \infty$.

Step 6. We finally have to build $\tilde{\zeta}_0^1$. First note that obviously, $\tilde{\zeta}_{\tilde{T}_1^1}(i + \tilde{R}_{\tilde{T}_1^1} - 1/2) \leq 1 = \tilde{\zeta}_0^1(i)$ for $i \leq 0$, while $\tilde{\zeta}_{\tilde{T}_1^1}(1 + \tilde{R}_{\tilde{T}_1^1} - 1/2) = 0 = \tilde{\zeta}_0^1(1)$ due to Lemma 4.3. Hence we just have to build $\tilde{\zeta}_0^1(i)$ for $i \geq 2$. We write for simplicity $K_i = i + \tilde{R}_{\tilde{T}_1^1} - 1/2$.

Using the same arguments as in Step 3, one may check that conditionally to $\mathcal{H}_{\tilde{T}_1^1}$, the family $(\tilde{\zeta}_{\tilde{T}_1^1}(K_i))_{i\geq 2}$ is independent, and that for all $i \geq 2$,

$$P\left[\tilde{\zeta}_{\tilde{T}_1^1}(K_i) = 1 \middle| \mathcal{H}_{\tilde{T}_1^1}\right] \leq 1/2. \tag{4.34}$$

We consider a family $(U_i)_{i\geq 2}$ of i.i.d. random variables uniformly distributed on $[0,1]$ (independent of everything else) and we set, for each $i \geq 2$,

$$\tilde{\zeta}_0^1(i) := \mathbb{1}_{\{\tilde{\zeta}_{\tilde{T}_1^1}(K_i)=1\}} + \mathbb{1}_{\{\tilde{\zeta}_{\tilde{T}_1^1}(K_i)=0\}} \mathbb{1}_{\{U_i < \epsilon(i)\}}, \tag{4.35}$$

where, due to (4.34),

$$\epsilon(i) = \frac{1 - 2P[\tilde{\zeta}_{\tilde{T}_1^1}(K_i) = 1|\mathcal{H}_{\tilde{T}_1^1}]}{2P[\tilde{\zeta}_{\tilde{T}_1^1}(K_i) = 0|\mathcal{H}_{\tilde{T}_1^1}]} \in [0, 1]. \tag{4.36}$$

We next observe that for all $i \geq 2$,

$$\begin{aligned} P\left[\tilde{\zeta}_0^1(i) = 1|\mathcal{H}_{\tilde{T}_1^1}\right] &= E\left[P[\tilde{\zeta}_0^1(i) = 1|\mathcal{H}_{\tilde{T}_1^1}]\right] \\ &= E\left[P[\tilde{\zeta}_{\tilde{T}_1^1}(K_i) = 1|\mathcal{H}_{\tilde{T}_1^1}] + \varepsilon(i)P[\tilde{\zeta}_{\tilde{T}_1^1}(K_i) = 0|\mathcal{H}_{\tilde{T}_1^1}]\right] = \frac{1}{2} \end{aligned} \tag{4.37}$$

due to our choice for $\varepsilon(i)$ and to the independence of U_i of everything else. We deduce that for each $i \geq 2$, $\tilde{\zeta}_0^1(i)$ is a Bernoulli random variable with parameter $1/2$, and that it is independent of $\mathcal{H}_{\tilde{T}_1^1}$. This and the conditionnal (to $\mathcal{H}_{\tilde{T}_1^1}$) independence of the family $(\tilde{\zeta}_0^1(i))_{i\geq 2}$ clearly imply that finally, $(\tilde{\zeta}_0^1(i))_{i\geq 2}$ is an i.i.d. sequence of Bernoulli random variables with parameter $1/2$, independent of $\mathcal{H}_{\tilde{T}_1^1}$. Finally, it is clear from (4.35) that for all $i \geq 2$, $\tilde{\zeta}_0^1(i) \geq \tilde{\zeta}_{\tilde{T}_1^1}(K_i)$. This concludes the proof. □

The following lemma shows a way to bound from above the right contour process started with the initial condition $\zeta_0 \sim \Gamma$ by a continuous-time random walk.

Lemma 4.9 *Let $\zeta_0 \sim \Gamma$, let N, V be two independent IFPPs, and consider the (ζ_0, N, V)-right contour process $(R_t^0)_{t\geq 0}$ around 0. Then for $k \geq 0$, $P[R_0^0 = k - 1/2] = (1/2)^{k+1}$. Furthermore, we may find a Poisson process $(Z_t)_{t\geq 0}$ with rate 1, a family of i.i.d. random variables $(Y_i)_{i\geq 1}$ distributed as Y_1 (see Lemma 4.8) in such a way that R_0^0 and $((Z_t)_{t\geq 0}, (Y_i)_{i\geq 1})$ are independent, while a.s., for all $t \geq 0$,*

$$R_t^0 \leq R_0^0 + \sum_{i=1}^{Z_t} Y_i. \tag{4.38}$$

Proof We omit as in the proof of Lemma 4.8 the superscript 0. We consider $\zeta_0 \sim \Gamma$ to be fixed, and a ζ_0-right contour process $(R_t)_{t\geq 0}$ around 0. First, it is obvious that for $k \geq 0$,

$$P[R_0 = k - 1/2] = P[\zeta_0(0) = 1, ..., \zeta_0(k-1) = 0, \zeta_0(k) = 1] = (1/2)^{k+1}, \tag{4.39}$$

since $\zeta_0 \sim \Gamma$. Next, let us explain (4.38). The main ideas are the following: we first bound our contour process by a contour process $\tilde{R}^1$ which starts from a (shifted) Ξ-distributed initial data. When this process first jumps to the right, at some instant τ_1, we bound the Bernoulli process at this time by a shifted Ξ-distributed data, independent of $(\tilde{R}_t^1)_{t\in[0,\tau_1]}$. Thus we make start again a contour process $\tilde{R}^2$ from this Ξ-distributed initial data in such a way that it dominates (with a shift) $\tilde{R}^1$, and thus R. And so on... The advantage of this method is that the *increments* (between two *renewal times*) are independent.

We define ζ_0^1 by $\zeta_0^1(i) = 1$ if $i \le 0$, $\zeta_0^1(1) = 1$, and $\zeta_0^1(i) = \zeta_0(i + R_0 - 1/2)$ for $i \ge 2$. We observe that $\zeta_0(i + R_0 - 1/2) \le \zeta_0^1(i)$ for all $i \in \mathbb{Z}$, using Lemma 4.3. We also notice that ζ_0^1 is independent of R_0, and is Ξ-distributed.
We thus may find, using Lemma 4.6, a contour process $(\tilde{R}_t^1)_{t\ge 0}$ around 0, starting from ζ_0^1, independent of R_0, such that for all times, $R_t - R_0 \le \tilde{R}_t^1 - 1/2$ (recall that $R_t - R_0$ starts from $1/2$, and that for all $i \in \mathbb{Z}$, $\zeta_0(i + R_0 - 1/2) \le \zeta_0^1(i)$).
On the other hand, we consider $\tau_1 = \inf\{t \ge 0; \Delta \tilde{R}_t^1 > 0\}$ the first instant where $\tilde{R}^1$ jumps to the right, so that $\tilde{R}_t^1 \le \tilde{R}_{\tau_1}^1$ for all $t \in [0, \tau_1]$. Hence setting $Y_1 := \tilde{R}_{\tau_1}^1 - 1/2$, we finally obtain that a.s., for all $t \in [0, \tau_1]$, $R_t \le R_0 + Y_1$. We also observe that R_0 and (τ_1, Y_1) are independent. Finally, τ_1 is exponentially distributed with parameter 1, due to Lemma 4.5-(d), and Y_1 is distributed as in Lemma 4.8 by construction.

Due to Lemma 4.8, we may find $\zeta_0^2 \sim \Xi$, independent of R_0 and Y_1, such that a.s., for all $i \in \mathbb{Z}$, $\zeta_{\tau_1}^1(i + \tilde{R}_{\tau_1}^1 - 1/2) \le \zeta_0^2(i)$, where $(\zeta_t^1)_{t\ge 0}$ is the Bernoulli process starting from ζ_0^1 associated with the contour process $(\tilde{R}_t^1)_{t\ge 0}$.
Using Lemma 4.6, we thus may build a contour process $(\tilde{R}_t^2)_{t\ge 0}$ around 0, independent of R_0 and Y_1, such that for all times $\tilde{R}_{\tau_1+t}^1 - \tilde{R}_{\tau_1}^1 \le \tilde{R}_t^2 - 1/2$. As a consequence, we observe that for all $t \ge 0$,

$$R_{\tau_1+t} - R_0 \le \tilde{R}_{\tau_1+t}^1 - 1/2 \le (\tilde{R}_{\tau_1}^1 - 1/2) + (\tilde{R}_t^2 - 1/2) \le Y_1 + (\tilde{R}_t^2 - 1/2). \quad (4.40)$$

Denote by $\tau_2 = \inf\{t \ge 0; \Delta \tilde{R}_t^2 > 0\}$ the first instant where $\tilde{R}^2$ jumps to the right, so that $\tilde{R}_t^2 \le \tilde{R}_{\tau_2}^2$ for all $t \in [0, \tau_2]$. Hence setting $Y_2 := \tilde{R}_{\tau_2}^2 - 1/2$, we finally obtain that a.s., for all $t \in [\tau_1, \tau_1 + \tau_2]$, $R_t \le R_0 + Y_1 + Y_2$. We also observe that (τ_2, Y_2) is independent of R_0 and (τ_1, Y_1). Finally, τ_2 is exponentially distributed with parameter 1, due to Lemma 4.5-(d), and Y_2 is distributed as Y_1 by construction.

Iterating the procedure, we find an i.i.d. family $(\tau_k, Y_k)_{k\ge 1}$ of random variables, independent of R_0, such that Y_1 is distributed as in Lemma 4.8 and τ_1 is exponentially distributed with parameter 1, such that for all $t \ge 0$,

$$R_t - R_0 \le \sum_{k\ge 1} Y_k \mathbb{1}_{\{t \ge \tau_1 + \ldots + \tau_k\}}. \quad (4.41)$$

This ends the proof. □

We finally conclude the proof of the main estimates of this section.

Proof of Proposition 4.4. We omit the superscript 0 for simplicity. The proof is based on the use of Lemmas 4.9 and 4.8. We thus write, according to (4.38), $R_t \le R_0 + S_{Z_t}$, with $S_n = Y_1 + \ldots + Y_n$.
First of all, we deduce from Lemma 4.8-(i)-(ii) that there exists $\gamma \in (0, \ln 2)$ such that $q := E[e^{\gamma Y_1}] < 1$.
Since $\gamma \in (0, \ln 2)$, we also deduce from Lemma 4.9 that $E[e^{\gamma R_0}] < \infty$.
Next, we recall that since $(Z_t)_{t\ge 0}$ is a Poisson process with rate 1, for all $t \ge 0$, $P[Z_t \le t/2] \le e^{-\delta t}$, where $\delta := (1 - \ln 2)/2 > 0$ (any $\delta > 0$ would work as well).

We have $\overline{R}_\infty = \sup_{t\geq 0} R_t \leq R_0 + \sup_{n\geq 1} S_n$. This implies that $e^{\gamma \overline{R}_\infty} \leq e^{\gamma R_0} \sum_{n\geq 1} e^{\gamma S_n}$. Thus, since $q \in (0,1)$, and since S_n and R_0 are independent,

$$E[e^{\gamma \overline{R}_\infty}] \leq E[e^{\gamma R_0}] \sum_{n\geq 1} E[e^{\gamma S_n}] \leq C \sum_{n\geq 1} E[e^{\gamma Y_1}]^n = C \sum_{n\geq 1} q^n < \infty. \tag{4.42}$$

Next we want to upperbound ρ. First, $\rho \leq \inf\{t \geq 0, R_t < 0 \text{ and } L_t > 0\}$, so that by symmetry, for any $t \geq 0$,

$$P[\rho \geq t] \leq P[R_t > 0 \text{ or } L_t < 0] \leq 2P[R_t > 0]. \tag{4.43}$$

But, since $R_t \leq R_0 + S_{Z_t}$,

$$\begin{aligned} P[\rho \geq t] &\leq 2P[R_t > 0] \leq 2P[Z_t \leq t/2] + 2P[R_0 + \sup_{n>t/2} S_n > 0] \\ &\leq 2e^{-\delta t} + 2E\left[e^{\gamma(R_0 + \sup_{n>t/2} S_n)}\right] \leq 2e^{-\delta t} + 2E\left[e^{\gamma R_0}\right] \sum_{n>t/2} E\left[e^{\gamma S_n}\right] \\ &\leq 2e^{-\delta t} + C \sum_{n>t/2} q^n \leq 2e^{-\delta t} + Cq^{t/2} \leq Ae^{-at} \end{aligned} \tag{4.44}$$

for some constants $A > 0$, $a > 0$. We classically conclude that for any $\beta \in (0,a)$, $E[e^{\beta\rho}] < \infty$. $\square$

5 Proof of Theorem 1.1

Our aim in this section is to conclude the proof of our main result.

Proof of Theorem 1.1 We divide the proof into several steps. Let us recall briefly the notation of Proposition 3.1 and Definition 4.1: we consider two independent IFPPs N, V, and $\zeta_0 \sim \Gamma$. Let $(\zeta_t)_{t\geq 0}$ be the (ζ_0, N)-Bernoulli process $(\zeta_t)_{t\geq 0}$, and let $(\tilde{\zeta}_t)_{t\in(-\infty,0]}$ be its time-reversed built in Lemma 2.2.
For $T \in (-\infty, 0]$ and $\varphi \in E_{\zeta_T}$ (recall (3.1)), we denote by $(\tilde{\zeta}_t, \eta_t^{T,\varphi})_{t\in[T,0]}$ the $(\tilde{\zeta}_T, \varphi, N^T, V^T)$-coupled Bernoulli avalanche process with $N_t^T(i) = N_{(-T)-}(i) - N_{(-t)-}(i)$ and $V_t^T(i) = V_{(-T)-}(i) - V_{(-t)-}(i)$ for $t \in [T, 0]$ and $i \in \mathbb{Z}$. Recall that

$$\tau_i = \sup\{T \leq 0; \forall \varphi \in E_{\zeta_T},\ \eta_0^{T,\varphi}(i) = \eta_0^{T,\mathbf{0}}(i)\}. \tag{5.1}$$

We then may put $\eta_0(i) := \eta_0^{\tau_i+s,\mathbf{0}}(i)$ (for some $s < 0$, recall Proposition 3.1) provided $\tau_i > -\infty$. Recall also that by definition of τ_i, we have, for all $T < 0$, all $\varphi \in E_{\zeta_T}$, $\eta_0^{T,\varphi}(i) = \eta_0(i)$ on the event $\{T < \tau_i\}$.
Next, we consider the (ζ_0, N, V)-left and right contour processes $(L_t^i)_{t\geq 0}$ and $(R_t^i)_{t\geq 0}$ around i, for each $i \in \mathbb{Z}$, and we adopt the notation

$$\overline{R}_t^i = \sup_{s\in[0,t]} R_s^i, \quad \underline{L}_t^i = \inf_{s\in[0,t]} L_s^i, \quad \rho^i = \inf\{t \geq 0;\ R_t^i < L_t^i\}. \tag{5.2}$$

Finally, for $k < l \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $t \in [0, \infty]$, we consider the σ-field

$$\mathcal{G}_{k,l,t} = \sigma\left\{\zeta_0(j),\ V_s(j),\ N_s(j),\ s \in [0,t],\ l \leq j \leq k\right\}. \tag{5.3}$$

Step 1. We first show that a.s., $-\tau_i \leq \rho^i$ a.s., and that $\eta_0(i) = \Phi_i(Z^i)$, for some deterministic function Φ_i, where

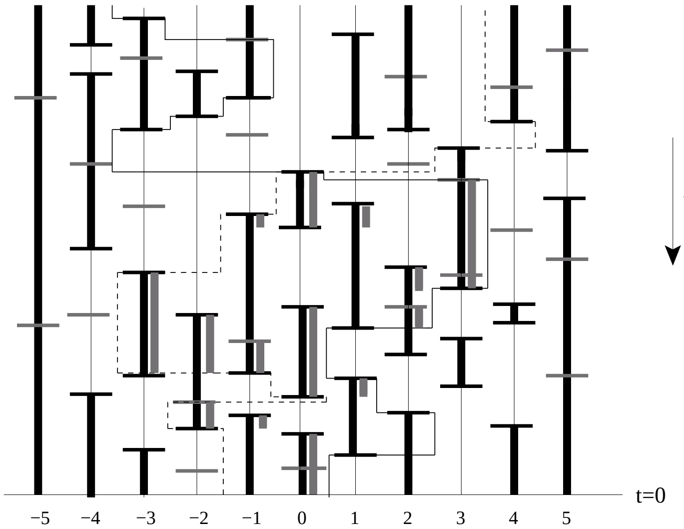
$$Z^i := \left(\zeta_0(j), N_s(j), V_s(j), j \in \{\underline{L}^i_{\rho^i} - 1/2, ..., \overline{R}^i_{\rho^i} + 1/2\}, s \in [0, \rho^i]\right). \tag{5.4}$$

The function Φ^i can not easily be made explicit, see however Step 2 of the algorithm described in the Appendix A.
It clearly suffices to treat the case $i = 0$. We consider the *box* delimited by L^0_t and R^0_t until they meet, i.e. until $t = \rho^0$. Consider an avalanche process starting at some time $T < -\rho^0$ with a given initial condition $\varphi \in E_{\tilde{\zeta}_T}$. We wish to rebuild its value at time 0, thus the time goes now down on Figure 2 (see also Figure 4 below).
Observe, having a look at Figure 2, that on the right and left of this box, the Bernoulli process $\tilde{\zeta}_t$ is vacant (see Lemma 4.3), so that due to our coupling, the avalanche process is also vacant, since it is always smaller (see Proposition 2.4). Hence no interaction can go inside this box (from its left and right sides), since vacant sites *cut* the interaction: indeed, flocks falling outside this box can not make die flocks inside the box.
Next, notice that the horizontal segments delimiting this box on the top side contain sites of the following type:
(a) either vacant sites of the Bernoulli process, so that the avalanche process is also vacant at this site at this time (since it is always smaller);
(b) either sites where the Bernoulli process becomes occupied because of a black mark, so that the avalanche process also becomes occupied at this time (because when the Bernoulli process is vacant, then the avalanche and Bernoulli processes both become occupied when they encounter a black mark);
(c) either a grey mark (in the middle of an occupied zone of the Bernoulli process), so that at this site and at this time, the avalanche process is (or becomes) occupied.
As a conclusion, the avalanche process is vacant on the left and right of the box, and the value of the avalanche process on the top horizontal segments of this box are determined, independently of its starting time $T < -\rho^0$ and initial condition $\varphi \in E_{\tilde{\zeta}_T}$. We thus may rebuild the avalanche process $(\eta^{T,\varphi}_t)_{t \in [\rho^0, 0]}$, and the obtained value $\eta^{T,\varphi}_0(0)$ does not depend on $T < -\rho^0$ nor on $\varphi \in E_{\tilde{\zeta}_T}$. We thus deduce that $\tau_0 \geq -\rho^0$ and that $\eta_0(0) = \eta^{T,\varphi}_0(0)$. This value $\eta_0(0)$ clearly depends only on the values of $N, V, \tilde{\zeta}_t$ in this box, and the fact that the Bernoulli process is vacant on the outside boundary of this box. We thus can say that $\eta_0(0)$ is a (deterministic) function of Z^0.

Step 2. We also observe that for $i \in \mathbb{Z}$, $k \leq i \leq l$, for $T > 0$,

$$\Omega_i(k, l, T) := \left\{\rho^i < T, \underline{L}^i_T \geq k + 1/2, \overline{R}^i_T \leq l - 1/2\right\} \in \mathcal{G}_{k,l,T} \tag{5.5}$$

Figure 4: Reconstruction of η_0 (Step 1).



We explain here graphically how to obtain the values of $\eta_0(i)$ for $i \in \{-2, .., 1\}$. The avalanche process is represented on the right of each site, in grey.
We start from the top of the box, that is at site 0. A flock appears here, so that 0 becomes occupied (independently of its starting time $T < -\rho^0$ and initial condition $\varphi \in E_{\tilde{\zeta}_T}$).
Next, a grey mark appears at the site 3, which thus remains occupied (if it was already) or becomes occupied (if it was not), so that in any case, the site 3 is occupied at this time.
At the same time, the sites 1 and 2 are vacant, since they are vacant for the Bernoulli process.
Next, 1 and then −1 become occupied due to black marks. But then the flock at 0 dies, and makes $-1, 0, 1$ become vacant. And so on...
This way, we see that the site −1 is finally vacant at time 0, while the site 0 is finally occupied.
On the other hand, it is immediate that −2 and 1 are vacant, since the avalanche process is smaller than the Bernoulli process with our coupling.
We could also see that $\eta_0(2) = 1$ and $\eta_0(3) = 0$ here, but it is not possible to decide if $\eta_0(-3) = 1$, because it could be killed by a flock dying at some site $i \leq -6$.

and that $Z^i \mathbb{1}_{\{\Omega_i(k,l,T)\}}$ is $\mathcal{G}_{k,l,T}$-measurable. This is clear from Figure 2.

Step 3. Next, we notice that point (a) (existence and uniqueness of an invariant distribution Π for the avalanche process) follows immediately from Proposition 3.1, provided we know that $\tau_i > -\infty$ a.s. for all $i \in \mathbb{Z}$. But we know from Step 1 that $\tau_i \geq -\rho^i$. Lemma 4.5-(a) implies that for all $i \in \mathbb{Z}$, ρ^i and ρ^0 are identically distributed. As a consequence, it suffices to show that $\rho^0 < \infty$ a.s., which follows from Proposition 4.4-(a).
Hence, $\Pi = \mathcal{L}(\eta_0)$ is the unique invariant distribution of the avalanche process.

Step 4. Point (c) (existence of a perfect simulation algorithm for $(\eta_0(i))_{i=-l,...,l}$) is also immediate, see Appendix for an explicit simulation algorithm. Let $l \geq 0$ be fixed. Due to Step 3, we know that $\eta_0 \sim \Pi$, it thus suffices to simulate (perfectly) $(\eta_0(j))_{j\in\{-l,...,l\}}$. This can be done by simulating first $Z^{-l}, ..., Z^l$. This can be done due to Step 2, which says that Z^i depends on ζ_0, N, and V on an a.s. finite number of sites and on an a.s. finite time intervall. Next, it suffices to compute $\eta_0(i) = \Phi_i(Z^i)$ for all $i \in \{-l, ..., l\}$, which can be done following the rules explained in Step 1, see also Figure 4.

Step 5. We now check the mixing property anounced in point (d). Let thus $n \geq 1$ and $k \in \mathbb{Z}$ be fixed. We consider the events (here $[x]$ stands for the integer part of $x \in \mathbb{R}$)

$$\Omega_n^1 = \left\{ \overline{R}_\infty^k \leq k + [n/3] - 1/2 \right\} \text{ and } \Omega_n^2 = \left\{ \underline{L}_\infty^{k+n} \geq k + [2n/3] + 1/2 \right\}. \quad (5.6)$$

We deduce from Lemma 4.5-(c) that on Ω_n^1, $\overline{R}_\infty^i \leq k + [n/3] - 1/2$ for all $i \leq k$. We thus deduce from Step 2 that the family $(Z^i \mathbb{1}_{\Omega_n^1})_{i \leq k}$ is $\mathcal{G}_{-\infty, k+[n/3], \infty}$-measurable. Hence $(\eta_0(i) \mathbb{1}_{\Omega_n^1})_{i \leq k}$ is $\mathcal{G}_{-\infty, k+[n/3], \infty}$-measurable. By the same way, $(\eta_0(i) \mathbb{1}_{\Omega_n^2})_{i \geq k+n}$ is $\mathcal{G}_{k+[2n/3], \infty, \infty}$-measurable.
But of course, the two σ-fields $\mathcal{G}_{-\infty, k+[n/3], \infty}$ and $\mathcal{G}_{k+[2n/3], \infty, \infty}$ are independent. Hence, for any $A \subset \{0,1\}^{(-\infty, k]}$, $B \subset \{0,1\}^{[k+n, \infty]}$,

$$\begin{aligned} \Big| P\left[(\eta_0(i))_{i \leq k} \in A, (\eta_0(i))_{i \geq k+n} \in B \right] \\ - P\left[(\eta_0(i))_{i \leq k} \in A \right] P\left[(\eta_0(i))_{i \geq k+n} \in B \right] \Big| \\ \leq 2P[(\Omega_n^1)^c] + 2P[(\Omega_n^2)^c], \end{aligned} \quad (5.7)$$

so that

$$\left| \Pi_{(-\infty, k] \cup [k+n, \infty)} - \Pi_{(-\infty, k]} \otimes \Pi_{[k+n, \infty)} \right| \leq 2P((\Omega_n^1)^c) + 2P((\Omega_n^2)^c). \quad (5.8)$$

Using Lemma 4.5-(a), we deduce that $P[(\Omega_n^1)^c] = P[\overline{R}_\infty^k \geq k + [n/3] + 1/2] = P[\overline{R}_\infty^0 \geq [n/3] + 1/2]$ and $P[(\Omega_n^2)^c] = P[\underline{L}_\infty^{k+n} \leq k + [2n/3] - 1/2] = P[\overline{R}_\infty^0 \geq n - [2n/3] + 1/2] \leq P[\overline{R}_\infty^0 \geq [n/3] + 1/2]$, since $n - [2n/3] \geq [n/3]$. Using finally Proposition 4.4-(b), obtain that for some constants $\gamma > 0$, $K > 0$,

$$\begin{aligned} P[(\Omega_n^1)^c] + [(\Omega_n^2)^c] &\leq 2P[\overline{R}_\infty^0 \geq [n/3] + 1/2] \\ &\leq 2e^{-\gamma/2 - \gamma[n/3]} E[e^{\gamma \overline{R}_\infty^0}] \leq K e^{-\gamma n/3}. \end{aligned} \quad (5.9)$$

We thus obtain (1.4), setting $q = e^{-\gamma/3}$ and $C = 2K$.

Step 6. Finally, it remains to study the trend to equilibrium. For $\varphi \in E$, $t \geq 0$, we denote by Π_t^φ the law of the φ-avalanche process at time t. The main difficulty here is to obtain the trend to equilibrium for any initial datum $\varphi \in E$, since our coupling allows us *a priori* to deal only with initial data stochastically smaller than Γ. We thus have to introduce a final coupling, which mixes those of Lemma 2.3 and Proposition 3.1. We keep, however, all the notations introduced in this proof. We fix $\varphi \in E$.
Step 6.1. We consider a third IFPP W, independent of N, V, and ζ_0. We consider, for $T < 0$ using Lemma 2.3, the $(\tilde{\zeta}_T, \varphi, N^T, W^T)$-coupled Bernoulli processes $(\tilde{\zeta}_t, \bar{\zeta}_t^T)_{t \in [T,0]}$. Recall that $(\tilde{\zeta}_t)_{t \in [T,0]}$ is the (φ, N^T)-Bernoulli process,

while $(\bar{\zeta}^T_t)_{t\in[T,0]}$ is the (φ, M_T)-Bernoulli process for some IFPP M_T. Notice that due to Lemma 2.3-(ii)-(iii), we obtain that for any $k \geq 0$, any $t \in [T, 0]$,

$$\begin{aligned} &\text{if } \Omega_1(t,T,k) := \{\bar{\zeta}^T_s(i) = \tilde{\zeta}_s(i); \forall\, i \in [-k,k], s \in [t,0]\} \\ &\qquad \text{then } P[\Omega_1(t,T,k)] \geq 1 - (2k+1)e^{-2(t-T)}. \end{aligned} \tag{5.10}$$

We finally consider the $(\varphi, \varphi, M_T, V^T)$-coupled Bernoulli-avalanche processes $(\bar{\zeta}_t, \bar{\eta}^T_t)_{t\in[T,0]}$.

Step 6.2. We consider the event, for $n \geq 0$, $t > 0$,

$$\Omega_0(n,t) = \left\{ \overline{R}^l_\infty \leq l + n + 1/2, \underline{L}^{-l}_\infty \geq -l - n - 1/2, \max_{i\in\{-l,\dots,l\}} \rho^i < t \right\}. \tag{5.11}$$

We know that on this event, $\eta^{-t,\varphi}_0(i) = \eta_0(i)$ for all $i \in \{-l, ..., l\}$, as soon as $\varphi \in E_{\zeta_t}$. But we easily understand, using Step 6.1 and some arguments as in Step 1, that on $\Omega_1(-t, -2t, l+n+1) \cap \Omega_0(n,t)$, we also have $\bar{\eta}^{-2t}_0(i) = \eta_0(i)$ for all $i \in \{-l, ..., l\}$.

Step 6.3. On the other hand, $\mathcal{L}(\bar{\eta}^{-2t}_0) = \Pi^\varphi_{2t}$ and $\mathcal{L}(\eta_0) = \Pi$, so that we classically deduce that for all $n \geq 0$,

$$\left|(\Pi^\varphi_{2t})_{[-l,l]} - \Pi_{[-l,l]}\right|_{TV} \leq 2P[(\Omega_0(n,t))^c \cup (\Omega_1(-t,-2t,l+n+1))^c]. \tag{5.12}$$

Step 6.4. We obtain, using Proposition 4.4 and Lemma 4.5-(a), that

$$\begin{aligned} P[(\Omega_0(n,t))^c] &\leq P\left[\overline{R}^l_\infty \geq l+n+3/2\right] + P\left[\underline{L}^{-l}_\infty \leq -l-n-3/2\right] \\ &\quad + \sum_{i=-l}^{l} P[\rho^i \geq t] \\ &\leq 2\left[\overline{R}^0_\infty \geq n+3/2\right] + (2l+1)P[\rho^0 \geq t] \\ &\leq 2e^{-\gamma n} E\left[e^{\gamma \overline{R}^0_\infty}\right] + (2l+1)e^{-\beta t} E\left[e^{\beta\rho^0}\right] \\ &\leq A(e^{-\gamma n} + (2l+1)e^{-\beta t}). \end{aligned} \tag{5.13}$$

for some constants $A > 0$ $\beta > 0$, $\gamma > 0$.

Step 6.5. Gathering (5.10), (5.12), (5.13), we finally obtain that for any $t \geq 0$, any $n \geq 1$,

$$\begin{aligned} \left|(\Pi^\varphi_{2t})_{[-l,l]} - \Pi_{[-l,l]}\right|_{TV} &\leq 2A(e^{-\gamma n} + (2l+1)e^{-\beta t}) \\ &\quad + 2(2(l+n+1)+1)e^{-2t}. \end{aligned} \tag{5.14}$$

Choosing finally $n = [t]$, we deduce that for $a = \min(1, \gamma, \beta)$, we may find a constant $K > 0$ such that for all $t > 0$, all $l \geq 0$,

$$\left|(\Pi^\varphi_{2t})_{[-l,l]} - \Pi_{[-l,l]}\right|_{TV} \leq K(1+l)e^{-at}. \tag{5.15}$$

This yields (1.3), and concludes the proof. □

6 A related mean-field model

This section, quite independent of the rest of the paper, is devoted to the brief study of a mean-field coagulation-fragmentation model related to the avalanche process.
To obtain a process which preserves the total mass, we will slightly change our point of view: we assume that each edge of $\mathbb{Z}$ has a mass equal to 1.
Consider a possible state $\eta \in E$ of the avalanche process. Two neighbour edges, say $(i-1, i)$ and $(i, i+1)$, are said to belong to the same *particle* if $\eta(i) = 1$: the flock lying at i *glues* the two edges. For example, the edge $(0, 1)$ belongs to a particle with mass 3 if $\eta(-1) = \eta(2) = 0$ and $\eta(0) = \eta(1) = 1$, or if $\eta(0) = \eta(3) = 0$ and $\eta(1) = \eta(2) = 1$. Similarly, $(0, 1)$ belongs to a particle with mass 1 if and only if $\eta(0) = \eta(1) = 0$.

Then we consider, for $\eta \in E$ and for $k \in \mathbb{N}$, if it exists,

$$c_k(\eta) = \lim_{n \to \infty} \frac{\text{number of particles with mass } k \text{ in } [-n, n]}{2n+1}, \tag{6.1}$$

which represents the average number of particles with mass k per unit of length.
Consider an avalanche process $(\eta_t)_{t \geq 0}$. Assume for a moment that for each $t \geq 0$, the successive masses of particles in η_t are independent (which is intuitively far from being exact). Then, using the invariance by translation of the model, one would have, for $k \geq 1$, $t \geq 0$,

$$c_k(t) := c_k(\eta_t) = \frac{1}{k} P\left[(0, 1) \text{ belongs to a particle with mass } k \text{ in } \eta_t\right] \tag{6.2}$$

The family $(c(t))_{t \geq 0} = (c_k(t))_{t \geq 0, k \geq 1}$ would also satisfy $\sum_{k \geq 1} k c_k(t) = 1$ for all $t \geq 0$, and the following infinite system of differential equations:

$$\begin{aligned} \frac{d}{dt} c_1(t) &= -2c_1(t) + \sum_{k \geq 1} (k-1) k c_k(t), \\ \frac{d}{dt} c_k(t) &= -2c_k(t) - (k-1) c_k(t) + \frac{1}{m_0(c(t))} \sum_{i=1}^{k-1} c_i(t) c_{k-i}(t) \quad (k \geq 2), \end{aligned} \tag{6.3}$$

where $m_0(c(t)) = \sum_{k \geq 1} c_k(t)$ is the average number of particles per unit of length. Indeed, the first equation expresses that an isolated edge merges with its two neighbours with rate 1, while each time a flock falls on a particle with mass k, which happens at rate $k-1$ (since a particle with mass k contains k edges and thus $k-1$ sites), an avalanche occurs and k new particles with mass 1 appear. Next, the second equation expresses that particles with mass k become larger at rate 2 (when a flock falls on one of its two extremities), that particles with mass k disappear when they are subject to an avalanche (which happens at rate $k-1$), and that particles with mass k do appear when a flock falls between a particle with mass i and a particle with mass $k-i$. This last event occurs at rate 1, proportionnaly to the number (per unit of length) of pairs of neighbour particles with masses i and $k-i$, which is exactly $c_i(t)c_{k-i}(t)/m_0(c(t))$. We use the abusive independence assumption when computing this last rate.

We refer to Aldous [1, Construction 5] for very similar considerations, without fragmentation, where the independence between neighbours really holds.

The system (6.3) can be seen as a coagulation-fragmentation equation with constant coagulation rate $K(i,j) = 2$, with a splitting rate (from one particle with mass k into k particles with mass 1) $F(k;1,...,1) = k-1$, the change of time $1/m_0(c(t))$ lying in front of the coagulation term. Indeed, we could write, for example when $k \geq 2$,

$$\begin{aligned} \frac{d}{dt}c_k(t) &= -F(k;1,...,1)c_k(t) \quad (6.4)\\ &\quad + \frac{1}{m_0(c(t))}\left[-c_k(t)\sum_{i\geq 1}K(k,i)c_i(t) + \sum_{i=1}^{k-1}K(i,k-i)c_i(t)c_{k-i}(t)\right]. \end{aligned}$$

The term in brackets on the second line is the right-hand side member of the well-known Smoluchowski coagulation equation. See Aldous [1], Laurençot-Mischler [14] for reviews on these types of equation. No result about trend to equilibrium for such a model without detailed balance condition (here the coagulation is binary, which is not the case of fragmentation) seem to be available. See however Fournier-Mischler [11] for partial results about a coagulation-fragmentation without balance condition in the case of binary fragmentation.

We aim here to compute the steady state of this mean-field model, and to show numerically that it approximates closely the invariant distribution of the avalanche process.

Proposition 6.1 *The system of equations (6.3) admits a unique steady state $c = (c_k)_{k\geq 1}$, in the sense that: $c_k \geq 0$ for all $k \geq 1$, $\sum_{k\geq 1} kc_k = 1$, and, with $m_0(c) = \sum_{k\geq 1} c_k$,*

$$\begin{aligned} 2c_1 &= \sum_{k\geq 1}(k-1)kc_k,\\ (k+1)c_k &= \frac{1}{m_0(c)}\sum_{i=1}^{k-1}c_ic_{k-i} \quad (k\geq 2). \quad (6.5) \end{aligned}$$

This steady state is given by $c_k = a_k g^{k-1}2^{-k}$, where $a_1 = 1$ and for $k \geq 2$, $a_k = \frac{1}{k+1}\sum_{i=1}^{k-1} a_j a_{k-j}$, while $g > 0$ is the unique solution of $\sum_{k\geq 1} a_k (g/2)^k = 1$. We also have $c_1 = 1/2$, $\sum_{k\geq 1} k^2 c_k = 2$, and $m_0(c) = 1/g$.

Proof Consider the sequence $(a_k)_{k\geq 1}$ defined in the statement. Remark that for any $x > 0$, any $g > 0$, the sequence $x_1 = x$, $x_k = \frac{g}{k+1}\sum_{i=1}^{k-1} x_j x_{k-j}$ is explicitely given by $x_k = a_k x^k g^{k-1}$.
Thus, $(c_k)_{k\geq 1}$ is a steady state of (6.3) if and only if there exist $x > 0$ and $g > 0$ such that

(i) $\forall\ k \geq 1$, $c_k = a_k x^k g^{k-1}$,
(ii) $g = 1/m_0(c)$,
(iii) $\sum_{k\geq 1} k c_k = 1$,
(iv) $x = \frac{1}{2}(\sum_{k\geq 1} k^2 c_k - 1)$.

Points (i) and (ii) imply that necessarily, $\sum_{k\geq 1} a_k (xg)^k = 1$. Thus $q := xg$ is clearly uniquely defined, and satisfies $0 < q < 1$ (since $a_1 = 1$ and $a_2 = 1/3 > 0$). Next, using (iii), we deduce that $g = \sum_{k\geq 1} k a_k q^k$ is also uniquely defined (and finite, since $q < 1$ and since one easily checks recursively that $a_k \leq 1$ for all $k \geq 1$). Thus $x = q/g$ is also uniquely defined. This shows that there exists at most one steady state. We next have to verify that these values for x and g imply point (iv). Using the definition of $(a_k)_{k\geq 1}$, we obtain that on the one hand,

$$\sum_{k\geq 2} (k+1) a_k q^k = \sum_{k\geq 2} q^k \sum_{j=1}^{k-1} a_j a_{k-j} = \left(\sum_{j\geq 1} a_j q^j \right)^2 = 1, \tag{6.6}$$

while on the other hand, since $a_1 = 1$,

$$\sum_{k\geq 2} (k+1) a_k q^k = \sum_{k\geq 1} k a_k q^k + \sum_{k\geq 1} a_k q^k - 2q = g + 1 - 2q. \tag{6.7}$$

We obtain by this way $g = 2q$, so that $x = 1/2$, and thus $c_1 = 1/2$.
To conclude the proof, it suffices to check that $\sum_{k\geq 1} k^2 c_k = 2$ with the previous values for x and g. But again, we obtain on the one hand that

$$\sum_{k\geq 2} k(k+1) a_k q^k = \sum_{k\geq 1} k^2 a_k q^k + \sum_{k\geq 1} k a_k q^k - 2q = g \sum_{k\geq 1} k^2 c_k + g - 2q, \tag{6.8}$$

while on the other hand,

$$\begin{aligned} \sum_{k\geq 2} k(k+1) a_k q^k &= \sum_{k\geq 2} k q^k \sum_{j=1}^{k-1} a_j a_{k-j} = \sum_{j\geq 1} a_j q^j \sum_{k\geq j+1} k q^{k-j} a_{k-j} \\ &= \sum_{j\geq 1} a_j q^j \sum_{l\geq 1} (j+l) q^l a_l = 2 \left(\sum_{j\geq 1} a_j q^j \right) \left(\sum_{l\geq 1} l q^l a_l \right) = 2g. \end{aligned} \tag{6.9}$$

Hence $g \sum_{k\geq 1} k^2 c_k + g - 2q = 2g$, so that $\sum_{k\geq 1} k^2 c_k = 1 + 2q/g = 2$. $\square$

To conclude this section, let us give some numerical results.
We obtain numerically, computing the values of $a_1, a_2, ..., a_{10000}$, and studying the function $z \mapsto \sum_1^{10000} a_k (z/2)^k$, that $g \simeq 1.4458$, with quite a good precision. We deduce then from Proposition 6.1 that at equilibrium, the mean-field model (6.3) satisfies

$$c_1 = 0.5,\ c_2 \simeq 0.1204,\ c_3 \simeq 0.04354,$$
$$c_4 \simeq 0.01679,\ c_5 \simeq 0.006574,\ c_6 \simeq 0.002582,$$
$$\textstyle\sum_{k\geq 1} c_k \simeq 0.6916,\ \sum_{k\geq 1} k^2 c_k = 2.$$

On the other hand, simulating 10^8 times the mass M of the particle containg the edge $(0,1)$ in the avalanche process η_0 at equilibrium (see Appendix A) we obtain the following Monte-Carlo approximations for $c_k(\eta_0) := P[M=k]/k$

$$c_1(\eta_0) \simeq 0.499934,\ c_2(\eta_0) \simeq 0.12312,\ c_3(\eta_0) \simeq 0.0422142,$$
$$c_4(\eta_0) \simeq 0.0161849,\ c_5(\eta_0) \simeq 0.00648257,\ c_6(\eta_0) \simeq 0.00263739,$$
$$\textstyle\sum_{k\geq 1} c_k(\eta_0) \simeq 0.692419,\ \sum_{k\geq 1} k^2 c_k(\eta_0) \simeq 1.99979.$$

It appears clearly that the two sets of values are very similar, even if numerical computations indicate that no equality holds, except maybe concerning c_1 and $\sum_k k^2 c_k$. We have no explanation for this phenomenon. It might indicate that correlations between the masses of successive clusters are nearly insignificant. We have no proof that the mean-field model is the (very fast) limit, in some asymptotic regime, of the avalanche process.

A Appendix

Let us now write down the simulation algorithm, which we deduce from Sections 3 and 4. For $l \geq 0$, the algorithm below simulates a random variable $(\widehat{\eta}_0(i))_{i\in[-l,l]}$ with law $\Pi_{[-l,l]}$, where Π is the unique invariant distribution of the avalanche process. The idea is to simulate N, ζ and V in an a.s. finite random space-time domain and then to reconstruct η following the graphical construction 2.5.

We construct a random process $\widehat{\zeta}_n(k)$ containing the values of ζ at some random times $\hat{T}_n$ (times of jumps of N and V in a finite growing spatial domain $[\ell_n, r_n]$) and an additionnal information: roughly,
$\widehat{\zeta}_n(k) = 0$ if $\zeta_{\hat{T}_n}(k) = 0$;
$\widehat{\zeta}_n(k) = 2$ if $\zeta_{\hat{T}_n}(k) = 1$ and $(k, \hat{T}_n)$ belongs to the box delimited by the contour processes;
$\widehat{\zeta}_n(k) = 1$ if $\zeta_{\hat{T}_n}(k) = 1$ and $(k, \hat{T}_n)$ is outside the box.

We invite the reader to have a look to Figures 2 (for Step 1) and 4 (for Step 2) while reading the simulation algorithm below. We will say *the box* for the box delimited by the contour processes.

Simulation Algorithm for $\Pi_{[-l,l]}$

Step 0: Initialization.
Simulate the initial Bernoulli configuration $\zeta_0(k)$ for $k \in [l_0, r_0]$, where l_0 (resp. r_0) is the first vacant site on the left (resp. right) of $-l$ (resp. l).
If $\zeta_0(k) = 0$ for all $k \in [-l, l]$, set $\widehat{\eta}_0(k) = 0$ for all $k \in [-l, l]$, and stop here.
Else, set $\widehat{\zeta}_0(k) = 2\zeta_0(k)$ for $k \in [l_0, r_0]$, set $n = 0$, and proceed to Step 1.

Initially, all the sites in $[l_0 + 1, r_0 - 1]$ are in the box. We thus assign the value 0 to vacant sites, and the value 2 to occupied sites.

Step 1: Simulation of black/grey marks and contour processes
Set $n = n + 1$.

Choose i_n uniformly in $[\ell_{n-1}, r_{n-1}]$ *representing the involved site.*
Choose $m_n \sim \text{Ber}(1/2)$, *here* $m_n = 0$ *for a black mark,* $m_n = 1$ *for a grey mark.*

- If $m_n = 0$, and $\widehat{\zeta}_{n-1}(i_n) \geq 1$ then we set $\widehat{\zeta}_n(i_n) = 0$.

  If the site i_n *is occupied, it becomes vacant due to a black mark.*

- If $m_n = 1$ and $\widehat{\zeta}_{n-1}(i_n) = 2$;

  set $\widehat{\zeta}_n(i_n) = 1$ if $\widehat{\zeta}_{n-1}(i_n - 1) = \widehat{\zeta}_{n-1}(i_n + 1) = 0$ and if $\forall k \in [\ell_{n-1}, i_n - 1]$, $\widehat{\zeta}_{n-1}(k) \leq 1$ or $\forall k \in [i_n + 1, r_{n-1}]$, $\widehat{\zeta}_{n-1}(k) \leq 1$;

  The site i_n *remains occupied but leaves the box due to a grey mark, because its two neighbors are vacant, and because it is on the boundary of the box.*

  otherwise, set $\widehat{\zeta}_n(i_n) = 2$.

  The site i_n *remains occupied and in the box, because either the site* i_n *is in the strict interior of the box or one of its neighbors is occupied.*

- If $m_n = 0$ and $\widehat{\zeta}_{n-1}(i_n) = 0$ then we consider the connected component I_n of occupied sites (plus i_n) around i_n (at time $n-1$).

  If $\forall k \in I_n \cup [i_n + 1, r_{n-1}]$, $\widehat{\zeta}_{n-1}(k) \leq 1$ or if $\forall k \in I_n \cup [\ell_{n-1}, i_n - 1]$, $\widehat{\zeta}_n(k) \leq 1$, then, for all $k \in I_n$, set $\widehat{\zeta}_n(i_n) = 1$.

  Otherwise, set $\widehat{\zeta}_{n+1}(k) = 2$ for all $k \in I_n$.

  The site i_n *becomes occupied due to a black mark. Then its whole connected component of occupied sites joins the box, except if all these sites were outside the box at time* $n - 1$*.*

- Set $\widehat{\zeta}_n(k) = \widehat{\zeta}_{n-1}(k)$ for all sites $\ell_{n-1} \leq k \leq r_{n-1}$ of which the value (at time n) has not been defined yet.

  We update all other sites. Observe that we have not considered the case $m_n = 1, \widehat{\zeta}_{n-1}(i_n) \in \{0, 1\}$*: grey marks have no effect on vacant sites in the Bernoulli process, and cannot increase the number of sites in the box.*

If $\widehat{\zeta}_n(r_{n-1}) \leq 1$, set $r_n = r_{n-1}$. Else, consider $s_n^r \sim \text{Geo}(1/2)$ (i.e. $P[s_n^r = k] = (1/2)^k$ for $k \geq 1$), set $r_n = r_{n-1} + s_n^r$, $\widehat{\zeta}_n(k) = 2$ for $k \in [r_{n-1}, r_n - 1]$, and $\widehat{\zeta}_n(r_n) = 0$.
We extend the box if necessary, i.e. when $\widehat{\zeta}_n(r_{n-1})$ *is occupied and in the box. We thus extend the Bernoulli process to the right until we meet a vacant site, at some site* r_n*. Then the (occupied) sites* $k \in [r_{n-1} + 1, r_n - 1]$ *are in the box.*

If $\widehat{\zeta}_n(\ell_{n-1}) \leq 1$, set $\ell_n = \ell_{n-1}$. Else, consider $s_n^\ell \sim \text{Geo}(1/2)$, set $\ell_n = \ell_{n-1} - s_n^\ell$, $\widehat{\zeta}_n(k) = 2$ for $k \in [\ell_n + 1, \ell_{n-1}]$, and $\widehat{\zeta}_n(\ell_n) = 0$.
Here we use the same arguments on the left of the domain.

Check $\{\ell_n \leq k \leq r_n, \widehat{\zeta}_n(k) = 2\}$. If this set is non empty then repeat Step 1. Otherwise, set $T = n$ and proceed to Step 2.

If all the sites have a value equal to 0 *or* 1*, this means that the contour processes have met.*

Step 2: Deduction of the avalanche invariant realization.
Start with $\widehat{\eta}_T(k) = 0$ for all $\ell_T \leq k \leq r_T$. Then for all $1 \leq n \leq T$, define recursively (for n decreasing from T to 1) $\widehat{\eta}_{n-1}(k)$, for all $\ell_{n-1} \leq k \leq r_{n-1}$, by

- $\widehat{\eta}_{n-1}(k) = 0$ if $m_n = 0$ and if k belongs to the connected component of occupied sites of i_n (in $\widehat{\eta}_n$).

  Black marks kill connected components of occupied sites.

- $\widehat{\eta}_{n-1}(k) = 1$ if $k = i_n$, if $m_n = 1$ and if $\widehat{\zeta}_{n-1}(i_n) \geq 1$.

  Grey marks make appear flocks when the Bernoulli process is occupied.

- $\widehat{\eta}_{n-1}(k) = 1$ if $k = i_n$, if $m_n = 0$ and if $\widehat{\eta}_n(i_n) = \widehat{\zeta}_n(i_n) = 0$.

  Black marks make appear flocks at vacant sites.

- $\widehat{\eta}_{n-1}(k) = \widehat{\eta}_n(k)$ for all sites $k \in [\ell_{n-1}, r_{n-1}]$ for which the value $\widehat{\eta}_{n-1}(k)$ has not been defined yet.

Conclusion.
Then $\{\widehat{\eta}_0(k), k \in [-l, l]\}$ is distributed according to $\Pi_{[-l,l]}$.

Remark that the number of iterations of Step 1 is finite due to our results: when the contour processes L^{-l} and R^l meet, there are no more sites with value 2.

Alternative
We finally propose another version of Step 1. The advantage is that the number of involved sites is much smaller. The idea is to take better advantage of the so-called *grey marks* : we will keep track only of what may really be needed to reconstruct $\{\widehat{\eta}_0(k), k \in [-l, l]\}$.

Step 1'.
Set $n = n + 1$. Choose i_n uniformly in $[\ell_{n-1}, r_{n-1}]$. Choose $m_n \sim \text{Ber}(1/2)$,

- If $m_n = 0$, and $\widehat{\zeta}_{n-1}(i_n) \geq 1$ then we set $\widehat{\zeta}_n(i_n) = 0$.

- If $m_n = 1$ and $\widehat{\zeta}_{n-1}(i_n) = 2$; we set $\widehat{\zeta}_n(i_n) = 1$ as soon as $\widehat{\zeta}_{n-1}(i_n - 1) \leq 1$ or $\widehat{\zeta}_{n-1}(i_n + 1) \leq 1$; else we let $\widehat{\zeta}_n(i_n) = 2$.

- If $m_n = 0$ and $\widehat{\zeta}_{n-1}(i_n) = 0$ then we consider

  $$I_n+ = \{k \in [i_n + 1, r_{n-1}], \widehat{\zeta}_{n-1}(k) = 2, \ \forall i_n < j < k, \widehat{\zeta}_{n-1}(j) \geq 1\},$$
  $$I_n- = \{k \in [\ell_{n-1}, i_n - 1], \widehat{\zeta}_{n-1}(k) = 2, \ \forall k < j < i_n, \widehat{\zeta}_{n-1}(j) \geq 1\}.$$

  If $I_n+ = I_n- = \emptyset$, we set $\widehat{\zeta}_n(i_n) = 1$.

  Else, we set $\widehat{\zeta}_n(k) = 2$ for all $k \in [\min(I_n-), \max(I_n+)]$.

- Set $\widehat{\zeta}_n(k) = \widehat{\zeta}_{n-1}(k)$ for all sites $\ell_{n-1} \leq k \leq r_{n-1}$ of which the value (at time n) has not been defined yet.

If there is $k \in [\ell_{n-1}, r_{n-1}]$ such that $\widehat{\zeta}_n(k) = 2$ and $\widehat{\zeta}_n(j) \geq 1$ for all $\ell_{n-1} \leq j \leq k$, consider $s_n^\ell \sim \text{Geo}(1/2)$, set $\ell_n = \ell_{n-1} - s_n^\ell$, $\widehat{\zeta}_n(j) = 1$ for $j \in [\ell_n + 1, \ell_{n-1} - 1]$ and $\widehat{\zeta}_n(\ell_n) = 0$.
Otherwise, set $\ell_n = \ell_{n-1}$.

Act symmetrically on the right of the domain.

Check $\{\ell_n \leq k \leq r_n, \widehat{\zeta}_n(k) = 2\}$. If this set is non empty then repeat Step 1'. Otherwise, set $T = n$ and proceed to Step 2.

Let us emphasize the differences between Step 1 and Step 1'. The domain is extended only if there is k with $\widehat{\zeta}_n(k) = 2$ in the *connected* component touching the boundary. The first and fourth switching rules of $\widehat{\zeta}_n(k)$ are left unchanged. In the second one, it is made *easier* to set $\widehat{\zeta}_n(k)$ from 2 to 1. In the third one, a much smaller set of indices is switched from 1 to 2.

One may understand that changing Step 1 into Step 1' does not change the law of the final values $(\widehat{\eta}_0(k))_{k \in [-l,l]}$. It defines, in some sense, a much more precise contour process (leftmost and rightmost sites k with $\widehat{\zeta}_n(k) = 2$) than the one defined in Section 4: we numerically observe that using Step 1', the algorithm is 15 times faster than when using Step 1. We hope to take advantage of this idea to generalize our methods to more general particle systems.

References

[1] D.J. Aldous, *Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory for Probabilists*, Bernoulli, 5, p 3-48, 1999.

[2] P. Bak, C. Tang, K. Wiesenfeld, *Self-organized criticality : an explanation of $1/f$ noise*, Phys. Rev. Letters 59, 381–384, 1987.

[3] P. Bak, C. Tang, K. Wiesenfeld, *Self-organized criticality*, Phys. Rev. A (3) 38, no. 1, 364–374, 1988.

[4] J. van den Berg, R. Brouwer, *Self-organized forest-fires near the critical time*, Comm. Math. Phys. 267, no. 1, 265–277, 2006.

[5] J. van den Berg, A.A. Jarai, *On the asymptotic density in a one-dimensional self-organized critical forest-fire model*, Comm. Math. Phys. 253, no. 3, 633–644, 2005.

[6] R. Brouwer and J. Pennanen, *The cluster size distribution for a forest-fire process on $\mathbb{Z}$*, EJP, 11, n 43, 1133–1143, 2006.

[7] D. Dhar, *Theoretical studies of self-organized criticality*, Phys. A, no. 1, 29–70, 2006.

[8] B. Drossel, F. Schwabl, *Self-organized critical forest-fire model*, Phys. Rev. Lett. 69, n11, 1629–1632, 1992

[9] M. Dürre, *Existence of multi-dimensional infinite volume self-organized critical*, Electron. J. Probab. 11, no. 21, 513–539, 2006 (electronic).

[10] M. Dürre, *Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models*, Electronic Communications in Probability, 11, 31, 2006.

[11] N. Fournier, S. Mischler, *Exponential trend to equilibrium for discrete coagulation equations with strong fragmentation and without balance condition*, Proceedings of the Royal Soc. of London, series A, Vol. 460, no 2049, 2477-2486, 2004.

[12] P. Grassberger, *Critical Behaviour of the Drossel-Schwabl Forest Fire Model*, New Journal of Physics 4, 17, 2002.

[13] C.L. Henley, *Self-organized percolation: a simpler model*, Bull. Am. Phys. Soc. 34, 838, 1989.

[14] P. Laurençot, S. Mischler, *On coalescence equations and related models,* Modeling and computational methods for kinetic equations, 321–356, Model. Simul. Sci. Eng. Technol., Birkhauser Boston, Boston, MA, 2004.

[15] T.M. Liggett, *Interacting particle systems*, Springer, 1985.

[16] Z. Olami, H. J. S. Feder, K. Christensen, *Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes*, Physical Review Letters 68: 12441247, 1992.

[17] J.G. Propp, D.B. Wilson, *Exact sampling with coupled Markov chains and applications to statistical mechanics,* Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995), Random Structures Algorithms 9, no. 1-2, 223–252, 1996.