

PARTICLE APPROXIMATION OF SOME LANDAU EQUATIONS

NICOLAS FOURNIER¹

ABSTRACT. We consider a class of nonlinear partial-differential equations, including the spatially homogeneous Fokker-Planck-Landau equation for Maxwell (or pseudo-Maxwell) molecules. Continuing the work of [6, 7, 4], we propose a probabilistic interpretation of such a P.D.E. in terms of a nonlinear stochastic differential equation driven by a standard Brownian motion. We derive a numerical scheme, based on a system of n particles driven by n Brownian motions, and study its rate of convergence. We finally deal with the possible extension of our numerical scheme to the case of the Landau equation for soft potentials, and give some numerical results.

Mathematics Subject Classification (2000): 82C40, 60K35.

Keywords: Fokker-Planck-Landau equation, Plasma physics, Stochastic particle systems.

1. INTRODUCTION AND MAIN RESULTS

1.1. **The equation.** Let S_d be the set of symmetric $d \times d$ matrices with real entries, and S_d^+ its subset of nonnegative matrices. For $a : \mathbb{R}^d \mapsto S_d^+$, we consider the partial differential equation

$$(1) \quad \partial_t f_t(x) = \frac{1}{2} \sum_{i,j=1}^d \partial_i \left\{ \int_{\mathbb{R}^d} a_{ij}(x-y) \left[f_t(y) \partial_j f_t(x) - f_t(x) \partial_j f_t(y) \right] dy \right\},$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ and where the unknown $(f_t)_{t \geq 0}$ is a family of probability density functions $(f_t)_{t \geq 0}$ on \mathbb{R}^d . The spatially homogeneous Landau (or Fokker-Planck-Landau) equation corresponds, in dimension $d \geq 2$, to the case where for some $\kappa : \mathbb{R}_+ \mapsto \mathbb{R}_+$,

$$(2) \quad a_{ij}(z) = \kappa(|z|^2)(|z|^2 \delta_{ij} - z_i z_j).$$

Physically, one assumes that $\kappa(r) = r^{\gamma/2}$, for some $\gamma \in [-3, 1]$. One talks of soft potentials when $\gamma < 0$, Maxwell molecules when $\gamma = 0$, and hard potentials when $\gamma > 0$. We consider in this paper the case of Maxwell molecules, or of pseudo-Maxwell molecules, where κ is supposed to be smooth and bounded.

This equation arises as a limit of the Boltzmann equation when all the collisions become grazing. We refer to Villani [10, 11, 12] and the many references therein for physical and mathematical details on this topic. See Cordier-Mancini [2] and Buet-Cordier-Filbet [1] for a review on deterministic numerical methods to solve (1).

1.2. **Notation.** Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on \mathbb{R}^d , and $\mathcal{P}_k = \{\mu \in \mathcal{P}, m_k(\mu) < \infty\}$, where $m_k(\mu) = \int |x|^k \mu(dx)$.

For $x, y \in \mathbb{R}^d$, we set $|x| = (\sum_1^d x_i^2)^{\frac{1}{2}}$, and $(x, y) = x^*y = \sum_1^d x_i y_i$. We consider the norm $|M| = \sup\{|(Mx, x)|, |x| = 1\} = \max\{|\lambda|, \lambda \text{ eigenvalue of } M\}$ on S_d . Recall that for $A \in S_d^+$, $\inf\{(Ax, x), |x| = 1\} = 1/|A^{-1}|$. All $A \in S_d^+$ admits a unique square root $A^{\frac{1}{2}} \in S_d^+$, and we have $|A^{\frac{1}{2}}| = |A|^{\frac{1}{2}}$.

¹LAMA UMR 8050, Faculté de Sciences et Technologies, Université Paris Est, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France, nicolas.fournier@univ-paris12.fr

Definition 1. Consider $a : \mathbb{R}^d \mapsto S_d^+$. Let $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be defined by $b_i(x) = \sum_{j=1}^d \partial_j a_{ij}(x)$. Assume that $|a(x)| + |b(x)| \leq C(1 + |x|^2)$ (which is the case when a is defined by (2) with $\kappa \in C_b^1$). A measurable family $(P_t)_{t \geq 0} \subset \mathcal{P}_2$ is said to be a weak solution to (1) if for all $t \geq 0$, $\sup_{[0,t]} m_2(P_s) < \infty$ and for all $\varphi \in C_b^2(\mathbb{R}^d)$,

$$(3) \quad \int_{\mathbb{R}^d} \varphi(x) P_t(dx) = \int_{\mathbb{R}^d} \varphi(x) P_0(dx) + \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_s(dx) P_s(dy) L\varphi(x, y),$$

where $L\varphi(x, y) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x-y) \partial_{ij}^2 \varphi(x) + \sum_{i=1}^d b_i(x-y) \partial_i \varphi(x)$.

All the terms make sense due to our conditions on a, b, P_t . See Villani [11] for a similar formulation.

1.3. Known results. To our knowledge, the first (and only) paper proving a rate of convergence for a numerical scheme to solve (1) is that of Fontbona-Guérin-Méléard [4]. Their method relies on a stochastic particle system. The aim of this paper is to go further in this direction.

Let us thus recall briefly the method of [4], relying on the probabilistic interpretation of (1) developed by Funaki [6], Guérin [7].

Let $\sigma : \mathbb{R}^d \mapsto S_d^+$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be Lipschitz continuous functions, and let $P_0 \in \mathcal{P}_2$. A \mathbb{R}^d -valued process $(X_t)_{t \geq 0}$ is said to solve $E_0(P_0, \sigma, b)$ if $\mathcal{L}(X_0) = P_0$, and if for all $t \geq 0$, setting $P_t = \mathcal{L}(X_t)$,

$$(4) \quad X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s - x) W_P(dx, ds) + \int_0^t \int_{\mathbb{R}^d} b(X_s - x) P_s(dx) ds.$$

Here $W_P(dx, dt)$ is a \mathbb{R}^d -valued white noise on $[0, \infty) \times \mathbb{R}^d$, independent of X_0 , with independent coordinates, each of which having covariance measure $P_t(dx) dt$ (see Walsh [14]).

Existence and uniqueness in law for $E_0(P_0, \sigma, b)$ have been proved in Guérin [7]. If furthermore $\sigma(x)\sigma^*(x) = a(x)$ and $b_i(x) = \sum_{j=1}^d \partial_j a_{ij}(x)$, then $(P_t)_{t \geq 0}$ is a weak solution to (1). The condition that σ and b are Lipschitz continuous is satisfied in the case of the Landau equation for Maxwell or pseudo-Maxwell molecules.

In [4], one considers an exchangeable stochastic particle system $(X_t^{i,n})_{t \geq 0, i=1, \dots, n}$, satisfying a S.D.E. driven by n^2 Brownian motions. It is then shown that one may find a coupling between a solution $(X_t^1)_{t \geq 0}$ to $E_0(P_0, \sigma, b)$ and such a particle system in such a way that

$$\mathbb{E} \left[\sup_{[0,T]} |X_t^{1,n} - X_t^1|^2 \right] \leq C_T n^{-2/(d+4)},$$

under the condition that P_0 has a finite moment of order $d+5$. The proof relies on a clever coupling between the the white noise and n Brownian motions. In particular, one has to assume that P_t has a density for all $t > 0$, in order to guarantee the uniqueness of some optimal couplings.

1.4. Another approach. For $a : \mathbb{R}^d \mapsto S_d^+$, $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying $|a(x)| + |b(x)| \leq C(1 + |x|^2)$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we introduce

$$a(x, \mu) = \int_{\mathbb{R}^d} a(x-y) \mu(dy), \quad b(x, \mu) = \int_{\mathbb{R}^d} b(x-y) \mu(dy).$$

For each $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2$, $a(x, \mu)$ is a nonnegative symmetric matrix and thus admits a unique symmetric nonnegative square root $a^{\frac{1}{2}}(x, \mu) := [a(x, \mu)]^{\frac{1}{2}}$.

Denote by \mathbf{W}_d the law of the d -dimensional Brownian motion, consider $P_0 \in \mathcal{P}_2$, and let $(X_0, B) \sim P_0 \otimes \mathbf{W}_d$. We say that a \mathbb{R}^d -valued process $(X_t)_{t \geq 0}$ solves $E_1(P_0, a, b)$ (or $E_1(P_0, a, b, X_0, B)$) when

needed) if $\mathbb{E}[\sup_{[0,T]} |X_t|^2] < \infty$ for all T and if for all $t \geq 0$, setting $P_t = \mathcal{L}(X_t)$,

$$(5) \quad X_t = X_0 + \int_0^t a^{\frac{1}{2}}(X_s, P_s) dB_s + \int_0^t b(X_s, P_s) ds.$$

This equation is nonlinear in the sense that its coefficients involve the law of the solution. Compared to (4), equation (5) is simpler, since it is driven by a finite-dimensional Brownian motion, and since the nonlinearity does not involve the driving process. However, one may check that at least formally, solutions to (4) and (5) have the same law. The link with (1) relies on a simple application of the Itô formula.

Remark 2. Let $(X_t)_{t \geq 0}$ solve $E_1(P_0, a, b)$. Assume that $b_i = \sum_{j=1}^d \partial_j a_{ij}$, and that $|a(x)| + |b(x)| \leq C(1 + |x|^2)$. Then $(P_t)_{t \geq 0} := (\mathcal{L}(X_t))_{t \geq 0}$ is a weak solution to (1).

The natural linearization of (5) consists of considering n particles $(X_t^{i,n})_{t \geq 0, i=1, \dots, n}$ solving

$$(6) \quad X_t^{i,n} = X_0^i + \int_0^t a^{\frac{1}{2}} \left(X_s^{i,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{k,n}} \right) dB_s^i + \int_0^t b \left(X_s^{i,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{k,n}} \right) ds.$$

Here $(X_0^i, B^i)_{i=1, \dots, n}$ are i.i.d. with law $P_0 \otimes \mathbf{W}_d$. We thus use n Brownian motions. When linearizing (4), one needs to use n^2 Brownian motions, since the white noise is infinite dimensional. However, one may check that the solution to (6) and the particle system built in [4] have the same distribution (provided $\sigma \sigma^* = a$ in [4, Equation (4)]).

1.5. Main results. The main result of this paper is the following.

Theorem 3. Assume that b is Lipschitz continuous, that a is of class C^2 , with all its derivatives of order 2 bounded, and that $P_0 \in \mathcal{P}_2$.

(i) There is strong existence and uniqueness for $E_1(P_0, a, b)$: for any $(X_0, B) \sim P_0 \otimes \mathbf{W}_d$, there is an unique solution $(X_t)_{t \geq 0}$ to $E_1(P_0, a, b, X_0, B)$.

(ii) Let $(X_0^i, B^i)_{i=1, \dots, n}$ be i.i.d. with law $P_0 \otimes \mathbf{W}_d$. There is an unique solution $(X_t^{i,n})_{t \geq 0, i=1, \dots, n}$ to (6). Assume that $P_0 \in \mathcal{P}_4$, and consider the unique solution $(X_t^1)_{t \geq 0}$ to $E_1(P_0, a, b, X_0^1, B^1)$. There is a constant C_T depending only on d, P_0, a, b, T such that

$$(7) \quad \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,n} - X_t^1|^2 \right] \leq C_T \int_0^T \min \left(n^{-1/2}, n^{-1} \sup_{x \in \mathbb{R}^d} (1 + |a(x, P_t)^{-1}|) \right) dt \leq C_T n^{-1/2}.$$

In the general case, we thus prove a rate of convergence in $n^{-1/2}$, which is faster than $n^{-2/(d+4)}$. If we have some information on the nondegeneracy of $a(x, P_t)$, then $a^{\frac{1}{2}}(x, \mu)$ is smooth around $\mu \simeq P_s$, and we can get a better rate of convergence.

Assume for example that a is uniformly elliptic (which is unfortunately not the case of (2), since $a(x)x = 0$ for all $x \in \mathbb{R}^d$). Then $\sup_x |a(x, P_t)^{-1}| \leq \sup_y |a(y)^{-1}| < \infty$, and we get a convergence rate in n^{-1} .

In the case of the Landau equation for true Maxwell molecules, we obtain the following result.

Corollary 4. Consider the Landau equation for Maxwell molecules, where a is given by (2) with $\kappa \equiv 1$ and $b_i(x) = \sum_{j=1}^d \partial_j a_{ij}(x) = -(d-1)x_i$. Then a, b satisfy the assumptions of Theorem 3. Let $P_0 \in \mathcal{P}_4$, and adopt the notation of Theorem 3-(ii).

(i) We have $\mathbb{E}[\sup_{[0,T]} |X_t^{1,n} - X_t^1|^2] \leq C_T n^{-1}(1 + \log n)$.

(ii) Set $x_0 = \int x P_0(dx)$. If $a(x_0, P_0)$ is invertible, then $\mathbb{E}[\sup_{[0,T]} |X_t^{1,n} - X_t^1|^2] \leq C_T n^{-1}$.

We finally consider the case of pseudo-Maxwell molecules.

Corollary 5. Consider the Landau equation for pseudo-Maxwell molecules, where a is given by (2) with $\kappa \in C^2(\mathbb{R}_+)$, and $b_i(x) = \sum_{j=1}^d \partial_j a_{ij}(x) = -(d-1)\kappa(|x|^2)x_i$. Assume that κ' has a bounded support. Then a, b satisfy the assumptions of Theorem 3-(ii).

Assume furthermore that $P_0 \in \mathcal{P}_4$ has a density with a finite entropy $\int P_0(x) \log P_0(x) dx < \infty$, and that κ is bounded below by a positive constant. With the notation of Theorem 3, we have $\mathbb{E}[\sup_{[0,T]} |X_t^{1,n} - X_t^1|^2] \leq C_T n^{-1}$.

1.6. Time discretization. To get a simulable particle system, it remains to discretize time in (6). Let $N \geq 1$, and consider $\rho_N(s) = \sum_{k \geq 0} \frac{k}{N} \mathbf{1}_{s \in [k/N, (k+1)/N)}$. Consider the simulable particle system $(X_t^{i,n,N})_{t \geq 0, i=1, \dots, n}$ defined by

$$(8) \quad X_t^{i,n,N} = X_0^i + \int_0^t a^{\frac{1}{2}} \left(X_{\rho_N(s)}^{i,n,N}, \frac{1}{n} \sum_1^n \delta_{X_{\rho_N(s)}^{k,n,N}} \right) dB_s^i + \int_0^t b \left(X_{\rho_N(s)}^{i,n,N}, \frac{1}{n} \sum_1^n \delta_{X_{\rho_N(s)}^{k,n,N}} \right) ds.$$

Theorem 6. Assume that b is Lipschitz continuous, that a is of class C^2 , with all its derivatives of order 2 bounded, and that $P_0 \in \mathcal{P}_2$. Let $(X_0^i, B^i)_{i=1, \dots, n}$ be i.i.d. with law $P_0 \otimes \mathbf{W}_d$. Consider the unique solutions $(X_t^{i,n})_{t \geq 0, i=1, \dots, n}$ to (6) and $(X_t^{i,n,N})_{t \geq 0, i=1, \dots, n}$ to (8). Then there is a constant C_T depending only on d, P_0, a, b, T such that

$$(9) \quad \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,n} - X_t^{1,n,N}|^2 \right] \leq C_T N^{-1}.$$

1.7. Conclusion. Choosing for example a, b , and P_0 as in Corollary 4-(ii) or as in Corollary 5, denoting by $(P_t)_{t \geq 0} = (\mathcal{L}(X_t^1))_{t \geq 0}$ the weak solution to the corresponding Landau equation, we obtain for any $\varphi \in C_b^1$, by exchangeability,

$$\sup_{[0,T]} \mathbb{E} \left[\left| \frac{1}{n} \sum_1^n \varphi(X_t^{i,n,N}) - \int_{\mathbb{R}^d} \varphi(x) P_t(dx) \right| \right] \leq C_T \|\varphi'\|_\infty \sqrt{n^{-1} + N^{-1}}.$$

Thus if one simulates the discretized particle system (8), and if one computes $\frac{1}{n} \sum_1^n \varphi(X_t^{i,n,N})$, we get an approximation of $\int \varphi(x) P_t(dx)$, with a reasonable error.

1.8. Plan of the paper. In Section 2, we give the proofs of Theorems 3 and 6. Section 3 is devoted to the proofs of Corollaries 4 and 5.

In Section 4, we briefly deal with the case of soft potentials, but our theoretical results do not extend well. Numerical results are given in Section 5. Finally an appendix lies at the end of the paper.

2. GENERAL PROOFS

In the whole section, we assume that $P_0 \in \mathcal{P}_2$, that $a : \mathbb{R}^d \mapsto S_d^+$ is of class C^2 , with bounded derivatives of order two, and that $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ is Lipschitz continuous. We denote by C (resp. $C_T, C_{T,p}$) a constant which depend only on a, b, d, P_0 (resp. additionally on T , on T, p) and whose value may change from line to line.

For $\mu, \nu \in \mathcal{P}_2$, we set $\mathcal{W}_2^2(\mu, \nu) = \min \{ \mathbb{E} [|X - Y|^2] ; \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu \}$. See Villani [13] for many informations on the Wasserstein distance \mathcal{W}_2 .

2.1. Preliminaries. Our results are mainly based on the two following Lemmas.

Lemma 7. For all $\mu, \nu \in \mathcal{P}_2$, all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} |a^{\frac{1}{2}}(x, \mu) - a^{\frac{1}{2}}(y, \nu)|^2 + |b(x, \mu) - b(y, \nu)|^2 &\leq C(|x - y|^2 + \mathcal{W}_2^2(\mu, \nu)), \\ |a^{\frac{1}{2}}(x, \mu)|^2 + |b(x, \mu)|^2 &\leq C(1 + m_2(\mu) + |x|^2). \end{aligned}$$

Proof. Step 1. For $\mu \in \mathcal{P}_2$ fixed, we consider the map $A : \mathbb{R}^d \mapsto S_d^+$ defined by $A(x) = a(x, \mu)$. Then $D^2A(x) = \int_{\mathbb{R}^d} D^2a(x - y)\mu(dy)$, is clearly uniformly bounded. Lemma 10 ensures us that $\|D(A^{\frac{1}{2}})\|_\infty$ is uniformly bounded, so that $|a^{\frac{1}{2}}(x, \mu) - a^{\frac{1}{2}}(y, \mu)| = |A^{\frac{1}{2}}(x) - A^{\frac{1}{2}}(y)| \leq C|x - y|$.

Step 2. We now fix $x \in \mathbb{R}^d$, and consider $\mu, \nu \in \mathcal{P}_2$. We introduce a couple (X, Y) of random variables such that $\mathcal{L}(X) = \mu$, $\mathcal{L}(Y) = \nu$, and $\mathcal{W}_2^2(\mu, \nu) = \mathbb{E}[|X - Y|^2]$. We define $A : \mathbb{R} \mapsto S_d^+$ by $A(t) = \mathbb{E}[a(x - [tX + (1 - t)Y])]$. Then $A(0) = \mathbb{E}[a(x - Y)] = a(x, \nu)$ while $A(1) = \mathbb{E}[a(x - X)] = a(x, \mu)$. Furthermore,

$$|D^2A(t)| = |\mathbb{E}[|X - Y|^2 D^2a(x - [tX + (1 - t)Y])]| \leq \|D^2a\|_\infty \mathbb{E}[|X - Y|^2] = C\mathcal{W}_2^2(\mu, \nu).$$

Lemma 10 ensures us that $\|(A^{\frac{1}{2}})'\|_\infty \leq C\mathcal{W}_2(\mu, \nu)$, so that $|a^{\frac{1}{2}}(x, \mu) - a^{\frac{1}{2}}(x, \nu)| = |A^{\frac{1}{2}}(1) - A^{\frac{1}{2}}(0)| \leq C\mathcal{W}_2(\mu, \nu)$.

Step 3. The growth estimate (for a) follows from the Lipschitz estimate, since $|a^{\frac{1}{2}}(0, \delta_0)|^2 = |a^{\frac{1}{2}}(0)|^2 < \infty$, and since $\mathcal{W}_2^2(\mu, \delta_0) = m_2(\mu)$.

Step 4. The case of b is much simpler. For $\mu, \nu \in \mathcal{P}_2$, we introduce X, Y as in Step 2. Then $|b(x, \mu) - b(y, \nu)|^2 = |\mathbb{E}[b(x - X) - b(y - Y)]|^2 \leq C(|x - y|^2 + \mathbb{E}[|X - Y|^2]) \leq C(|x - y|^2 + \mathcal{W}_2^2(\mu, \nu))$. The growth estimate follows from the Lipschitz estimate, since $|b(0, \delta_0)|^2 = |b(0)|^2 < \infty$. \square

Lemma 8. Let Y_i be i.i.d. \mathbb{R}^d -valued random variables with common law $\mu \in \mathcal{P}_4$. Then

$$\mathbb{E} \left[\left| a(Y_1, \mu) - a \left(Y_1, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \right) \right|^2 + \left| b(Y_1, \mu) - b \left(Y_1, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \right) \right|^2 \right] \leq C \frac{1 + m_4(\mu)}{n}.$$

Proof. We denote by \mathbb{E}_1 the expectation concerning only Y_1 , and by $\mathbb{E}_{2,n}$ the expectation concerning only Y_2, \dots, Y_n . We observe that for all $i = 2, \dots, n$, we have $a(Y_1, \mu) = \mathbb{E}_{2,n}[a(Y_1 - Y_i)]$, whence $a(Y_1, \mu) = \mathbb{E}_{2,n}[\frac{1}{n-1} \sum_{i=2}^n a(Y_1 - Y_i)]$. We also have $a(Y_1, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i}) = \frac{1}{n} \sum_{i=1}^n a(Y_1 - Y_i)$. As a consequence,

$$\begin{aligned} \mathbb{E} \left[\left| a \left(Y_1, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \right) - a(Y_1, \mu) \right|^2 \right] &\leq 2\mathbb{E} \left[\left| \frac{1}{n} \sum_{i=1}^n a(Y_1 - Y_i) - \frac{1}{n-1} \sum_{i=2}^n a(Y_1 - Y_i) \right|^2 \right] \\ &+ 2\mathbb{E}_1 \left\{ \mathbb{E}_{2,n} \left[\left| \frac{1}{n-1} \sum_{i=2}^n a(Y_1 - Y_i) - \mathbb{E}_{2,n} \left[\frac{1}{n-1} \sum_{i=2}^n a(Y_1 - Y_i) \right] \right|^2 \right] \right\} =: 2I_n + 2J_n. \end{aligned}$$

An immediate computation, using that $|a(x)| \leq C(1 + |x|^2)$, shows that $I_n \leq C(1 + m_4(\mu))/n^2$. On the other hand, since the random variables $Y_1 - Y_i$ are i.i.d. under $\mathbb{E}_{2,n}$,

$$\begin{aligned} J_n &\leq \mathbb{E}_1 \left\{ \sum_{k,l=1}^d \text{Var}_{2,n} \left(\frac{1}{n-1} \sum_{i=2}^n a_{kl}(Y_1 - Y_i) \right) \right\} \leq \frac{1}{n-1} \mathbb{E}_1 \left\{ \sum_{k,l=1}^d \text{Var}_{2,n} a_{kl}(Y_1 - Y_2) \right\} \\ &\leq \frac{C}{n-1} \mathbb{E}_1 \{ \mathbb{E}_{2,n} [|a(Y_1 - Y_2)|^2] \} \leq \frac{C}{n} \mathbb{E} [|a(Y_1 - Y_2)|^2] \leq \frac{C}{n} (1 + m_4(\mu)), \end{aligned}$$

again since $|a(x)| \leq C(1 + |x|^2)$. The same computation holds for b , replacing everywhere $m_4(\mu)$ by $m_2(\mu)$, since $|b(x)| \leq C(1 + |x|)$. \square

2.2. Convergence proofs. We start this subsection with some moment estimates.

Lemma 9. (i) Let $(X_t)_{t \geq 0}$ solve $E_1(P_0, a, b)$. Assume that $m_p(P_0) < \infty$ for some $p \geq 2$. Then $\mathbb{E}[\sup_{[0, T]} |X_t|^p] < \infty$ for all $T > 0$.

(ii) Let $(X_t^{i, n})_{t \geq 0, i=1, \dots, n}$ solve (6). For all $0 \leq s \leq t \leq T$, $\mathbb{E}[|X_t^{1, n} - X_s^{1, n}|^2] \leq C_T |t - s|$.

Proof. Point (i). Set $P_t = \mathcal{L}(X_t)$. Using the Burkholder-Davies-Gundy inequality for the Brownian part, and the Hölder inequality for the drift part, we obtain, for all $0 \leq t \leq T$,

$$\mathbb{E} \left[\sup_{[0, t]} |X_s|^p \right] \leq C_p \mathbb{E}[|X_0|^p] + C_p \int_0^t ds \mathbb{E} \left[|a^{\frac{1}{2}}(X_s, P_s)|^p \right] + C_{p, T} \int_0^t ds \mathbb{E} [|b(X_s, P_s)|^p].$$

But Lemma 7 implies that $\mathbb{E}[|a^{\frac{1}{2}}(X_s, P_s)|^p + |b(X_s, P_s)|^p] \leq C_p \mathbb{E}[1 + |X_s|^p + m_2(P_s)^{p/2}]$. Furthermore, since $P_s = \mathcal{L}(X_s)$ and $p \geq 2$, we deduce that $m_2(P_s)^{p/2} \leq \mathbb{E}[|X_s|^p]$. As a conclusion, $\mathbb{E}[\sup_{[0, t]} |X_s|^p] \leq C_p \mathbb{E}[|X_0|^p] + C_{p, T} \int_0^t ds \mathbb{E}[1 + |X_s|^p]$, whence the result by the Gronwall Lemma.

Point (ii). Using the Cauchy-Schwarz and Doob inequalities, we see that for $0 \leq s \leq t \leq T$,

$$\begin{aligned} E \left[|X_t^{1, n} - X_s^{1, n}|^2 \right] &\leq C \int_s^t du \mathbb{E} \left[|a^{\frac{1}{2}}(X_u^{1, n}, \frac{1}{n} \sum_1^n \delta_{X_u^{i, n}})|^2 \right] + C_T \int_s^t du \mathbb{E} \left[|b(X_u^{1, n}, \frac{1}{n} \sum_1^n \delta_{X_u^{i, n}})|^2 \right] \\ (10) \quad &\leq C_T \int_s^t du \mathbb{E} \left[1 + |X_u^{1, n}|^2 + m_2 \left(\frac{1}{n} \sum_1^n \delta_{X_u^{i, n}} \right) \right] \leq C_T \int_s^t du \mathbb{E} [1 + |X_u^{1, n}|^2]. \end{aligned}$$

We used Lemma 7 and that $\mathbb{E}[m_2(\frac{1}{n} \sum_1^n \delta_{X_u^{i, n}})] = \frac{1}{n} \sum_1^n \mathbb{E}[|X_u^{i, n}|^2] = \mathbb{E}[|X_u^{1, n}|^2]$ by exchangeability. Applying (10) with $s = 0$, we get $E[|X_t^{1, n}|^2] \leq C \mathbb{E}[|X_0^{1, n}|^2] + C_T \int_0^t du [1 + \mathbb{E}[|X_u^{1, n}|^2]] du$. The Gronwall Lemma allows us to conclude that $\sup_{[0, T]} E[|X_t^{1, n}|^2] \leq C_T$. Applying a second time (10), we deduce that $E[|X_t^{1, n} - X_s^{1, n}|^2] \leq C_T |t - s|$. \square

Proof of Theorem 3. We consider $P_0 \in \mathcal{P}_2$ fixed.

Point (i). Let $(X_0, B) \sim P_0 \otimes \mathbf{W}_d$.

Uniqueness. Assume that we have two solutions X, Y to $E_1(P_0, a, b, X_0, B)$, and set $P_t = \mathcal{L}(X_t)$, $Q_t = \mathcal{L}(Y_t)$. Using the Cauchy-Schwarz and Doob inequalities, we obtain, for $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E} \left[\sup_{[0, t]} |X_s - Y_s|^2 \right] &\leq C_T \int_0^t \mathbb{E} [|a^{\frac{1}{2}}(X_s, P_s) - a^{\frac{1}{2}}(Y_s, Q_s)|^2 + |b(X_s, P_s) - b(Y_s, Q_s)|^2] ds \\ (11) \quad &\leq C_T \int_0^t \mathbb{E} [|X_s - Y_s|^2 + \mathcal{W}_2^2(P_s, Q_s)] ds \leq C_T \int_0^t \mathbb{E} [|X_s - Y_s|^2] ds. \end{aligned}$$

We used Lemma 7 and the obvious inequality $\mathcal{W}_2^2(P_s, Q_s) \leq \mathbb{E}[|X_s - Y_s|^2]$. The Gronwall Lemma allows us to conclude that $X = Y$.

Existence. We consider the following Picard iteration: set $X_t^0 = X_0$, and define, for $n \geq 0$, $t \geq 0$,

$$(12) \quad X_t^{n+1} = X_0 + \int_0^t a^{\frac{1}{2}}(X_s^n, \mathcal{L}(X_s^n)) dB_s + \int_0^t b(X_s^n, \mathcal{L}(X_s^n)) ds.$$

We get as in (11), for $0 \leq t \leq T$, $\mathbb{E}[\sup_{[0, t]} |X_s^{n+1} - X_s^n|^2] \leq C_T \int_0^t \mathbb{E}[|X_s^n - X_s^{n-1}|^2] ds$. Thus there classically exists $(X_t)_{t \geq 0}$ such that $\lim_n \mathbb{E}[\sup_{[0, T]} |X_t^n - X_t|^2] = 0$ for all T , which implies

that $\lim_n \sup_{[0,T]} \mathcal{W}_2^2(\mathcal{L}(X_t^n), \mathcal{L}(X_t)) = 0$. Passing to the limit in (12), we see that X solves $E_1(P_0, a, b, X_0, B)$.

Point (ii). First of all, the strong existence and uniqueness for (6) follows from standard theory (see e.g. Stroock-Varadhan [8]), since for each i , the maps $(x_1, \dots, x_n) \mapsto b(x^i, \frac{1}{n} \sum_1^n \delta_{x^k})$ and $(x_1, \dots, x_n) \mapsto a^{\frac{1}{2}}(x^i, \frac{1}{n} \sum_1^n \delta_{x^k})$ are Lipschitz continuous (use Lemmas 7 and 12).

We now consider (X_0^i, B^i) i.i.d. with law $P_0 \otimes \mathbf{W}_d$, the solution $(X_t^{i,n})_{t \geq 0, i=1, \dots, n}$ to (6), and for each $i = 1, \dots, n$, the unique solution $(X_t^i)_{t \geq 0}$ to $E_1(P_0, a, b, X_0^i, B^i)$. For each $t \geq 0$, let $P_t = \mathcal{L}(X_t^1) = \dots = \mathcal{L}(X_t^n)$. Due to the Cauchy-Schwarz and Doob inequalities, for $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E} \left[\sup_{[0,t]} |X_s^{1,n} - X_s^1|^2 \right] &\leq C_T \int_0^t ds \mathbb{E} \left[\left| a^{\frac{1}{2}} \left(X_s^{1,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{i,n}} \right) - a^{\frac{1}{2}}(X_s^1, P_s) \right|^2 \right. \\ &\quad \left. + \left| b \left(X_s^{1,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{i,n}} \right) - b(X_s^1, P_s) \right|^2 \right] \\ &\leq C_T \int_0^t ds \left(\mathbb{E} \left[\left| a^{\frac{1}{2}} \left(X_s^{1,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{i,n}} \right) - a^{\frac{1}{2}} \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) \right|^2 \right. \right. \\ &\quad \left. \left. + \left| b \left(X_s^{1,n}, \frac{1}{n} \sum_1^n \delta_{X_s^{i,n}} \right) - b \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) \right|^2 \right] + \Delta_n(s) \right), \end{aligned}$$

where

$$\begin{aligned} \Delta_n(s) &:= \mathbb{E} \left[\left| a^{\frac{1}{2}} \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) - a^{\frac{1}{2}}(X_s^1, P_s) \right|^2 + \left| b \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) - b(X_s^1, P_s) \right|^2 \right] \\ (13) \quad &=: \Delta_n^1(s) + \Delta_n^2(s). \end{aligned}$$

Using Lemmas 7 and 12, we obtain, for $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E} \left[\sup_{[0,t]} |X_s^{1,n} - X_s^1|^2 \right] &\leq C_T \int_0^t ds \left(\mathbb{E} \left[|X_s^{1,n} - X_s^1|^2 + \mathcal{W}_2^2 \left(\frac{1}{n} \sum_1^n \delta_{X_s^{i,n}}, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) \right] + \Delta_n(s) \right) \\ &\leq C_T \int_0^t ds \mathbb{E} \left[|X_s^{1,n} - X_s^1|^2 + \frac{1}{n} \sum_1^n |X_s^{i,n} - X_s^i|^2 \right] + C_T \int_0^t ds \Delta_n(s) \\ &\leq C_T \int_0^t ds \mathbb{E} [|X_s^{1,n} - X_s^1|^2] + C_T \int_0^t ds \Delta_n(s) \end{aligned}$$

by exchangeability. The Gronwall Lemma ensures us that

$$(14) \quad \mathbb{E} \left[\sup_{[0,T]} |X_s^{1,n} - X_s^1|^2 \right] \leq C_T \int_0^T ds \Delta_n(s).$$

It remains to estimate $\Delta_n(s)$. The random variables X_s^1, \dots, X_s^n are i.i.d. with law P_s . Thus Lemma 8 shows that $\Delta_n^2(s) \leq C(1 + m_4(P_s))/n \leq C_T/n$ for $s \leq T$, due to Lemma 9-(i) and since $P_0 \in \mathcal{P}_4$ by assumption. Next, we use Lemma 11-(i), the Cauchy-Schwarz inequality, and then Lemma 8: for $s \leq T$,

$$\Delta_n^1(s) \leq \mathbb{E} \left[\left| a \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) - a(X_s^1, \mathcal{L}(X_s^1)) \right|^2 \right] \leq C \left(\frac{1 + m_4(P_s)}{n} \right)^{\frac{1}{2}} \leq \frac{C_T}{\sqrt{n}}.$$

But one may also use Lemma 11-(ii) instead of Lemma 11-(i), and this gives, for $s \leq T$,

$$\begin{aligned} \Delta_n^1(s) &\leq \mathbb{E} \left[|a(X_s^1, P_s)^{-1}| \left| a \left(X_s^1, \frac{1}{n} \sum_1^n \delta_{X_s^i} \right) - a(X_s^1, \mathcal{L}(X_s^1)) \right|^2 \right] \\ &\leq C \sup_x |a(x, P_s)^{-1}| \left(\frac{1 + m_4(P_s)}{n} \right) \leq \frac{C_T}{n} \sup_x |a(x, P_s)^{-1}|. \end{aligned}$$

Thus $\Delta_n(s) \leq C_T n^{-1} + C_T \min(n^{-1/2}, n^{-1} \sup_x |a(x, P_s)^{-1}|)$. Inserting this into (14), we obtain (7). \square

Proof of Theorem 6. Using Lemmas 7 and 12, we get as usual (see (11)), by exchangeability,

$$\begin{aligned} \mathbb{E} \left[\sup_{[0,t]} |X_s^{1,n} - X_s^{1,n,N}|^2 \right] &\leq C_T \int_0^t \mathbb{E} \left[|X_s^{1,n} - X_{\rho_N(s)}^{1,n,N}|^2 + \mathcal{W}_2^2 \left(\frac{1}{n} \sum_1^n \delta_{X_s^{i,n}}, \frac{1}{n} \sum_1^n \delta_{X_{\rho_N(s)}^{i,n,N}} \right) \right] ds \\ &\leq C_T \int_0^t \mathbb{E} \left[|X_s^{1,n} - X_{\rho_N(s)}^{1,n,N}|^2 + \frac{1}{n} \sum_1^n |X_s^{i,n} - X_{\rho_N(s)}^{i,n,N}|^2 \right] ds \\ &\leq C_T \int_0^t \mathbb{E} [|X_s^{1,n} - X_{\rho_N(s)}^{1,n,N}|^2] ds \\ &\leq C_T \int_0^t \mathbb{E} [|X_s^{1,n} - X_s^{1,n,N}|^2] ds + C_T \int_0^t \mathbb{E} [|X_s^{1,n} - X_{\rho_N(s)}^{1,n}|^2] ds. \end{aligned}$$

Using finally Lemma 9-(ii), and since $|s - \rho_N(s)| \leq 1/N$, we deduce that $\mathbb{E}[|X_s^{1,n} - X_{\rho_N(s)}^{1,n}|^2] \leq C_T/N$. The Gronwall Lemma allows us to conclude. \square

3. ELLIPTICITY ESTIMATES

We start with the

Proof of Corollary 4. Recall here that a is given by (2) with $\kappa \equiv 1$ and $b(z) = -(d-1)z$. Thus b is Lipschitz continuous, and the second derivatives of a are clearly bounded. We consider a weak solution $(P_t)_{t \geq 0}$ to (1).

Simple computations using (3) (with $\varphi(x) = x_i$, $\varphi(x) = |x|^2$) show that $\partial_t \int x P_t(dx) = 0$ and $\partial_t m_2(P_t) = 0$. We classically may assume without loss of generality that $\int x P_t(dx) = \int x P_0(dx) = 0$. We also assume that $m_2(P_t) = m_2(P_0) > 0$ (else $X_t^1 = X_t^{1,n} = 0$ a.s.).

We now bound from below $(a(x, P_t)y, y)$ for $x, y \in \mathbb{R}^d$, $t \geq 0$.

A simple computation, using that $\int x P_t(dx) = 0$, shows that $a(x, P_t) = a(x) + a(0, P_t)$. Thus for all $t \geq 0$, $x, y \in \mathbb{R}^d$, setting $m_2^{ij}(P_t) = \int x_i x_j P_t(dx)$

$$(a(x, P_t)y, y) \geq (a(0, P_t)y, y) = \sum_{i,j} y_i y_j [m_2(P_t) \delta_{ij} - m_2^{ij}(P_t)] = m_2(P_0) |y|^2 - \sum_{i,j} y_i y_j m_2^{ij}(P_t).$$

Using (3) with $\varphi(x) = x_i x_j$, we deduce that

$$\partial_t m_2^{ij}(P_t) = 2m_2(P_t) \delta_{ij} - 2d m_2^{ij}(P_t) = 2m_2(P_0) \delta_{ij} - 2d m_2^{ij}(P_t).$$

We thus obtain

$$\begin{aligned} \partial_t (a(0, P_t)y, y) &= - \sum_{i,j} y_i y_j \partial_t m_2^{ij}(P_t) = 2d \sum_{i,j} y_i y_j m_2^{ij}(P_t) - 2m_2(P_0) |y|^2 \\ &= 2(d-1) m_2(P_0) |y|^2 - 2d (a(0, P_t)y, y). \end{aligned}$$

Set $\lambda_0 = \inf\{(a(0, P_0)y, y), |y| = 1\} \geq 0$ and $\lambda_1 = \frac{d-1}{d}m_2(P_0) > 0$. For all $t \geq 0$, all $x, y \in \mathbb{R}^d$,

$$(15) \quad \begin{aligned} (a(x, P_t)y, y) &\geq (a(0, P_t)y, y) = (a(0, P_0)y, y)e^{-2dt} + \lambda_1|y|^2(1 - e^{-2dt}) \\ &\geq |y|^2[\lambda_0 e^{-2dt} + \lambda_1(1 - e^{-2dt})]. \end{aligned}$$

We now prove point (i). We deduce from (15) that

$$(a(x, P_t)y, y) \geq \lambda_1(1 - e^{-2dt})|y|^2.$$

As a consequence, $|a(x, P_t)^{-1}| \leq 1/[\lambda_1(1 - e^{-2dt})] \leq C/t + C$. Inserting this into (7), we get $\mathbb{E}[\sup_{[0, T]} |X_t^1 - X_t^{1, n}|^2] \leq C_T \int_0^T \min(n^{-1/2}, n^{-1} + (nt)^{-1}) dt \leq C_T n^{-1}(1 + \log n)$.

To get (ii), we use (15) and that by assumption, $\lambda_0 > 0$. We deduce that

$$(a(x, P_t)y, y) \geq |y|^2(\lambda_0 e^{-2dt} + \lambda_1(1 - e^{-2dt})) \geq \min(\lambda_0, \lambda_1)|y|^2/2.$$

As a consequence, $|a(x, P_t)^{-1}| \leq 2/\min(\lambda_0, \lambda_1)$. Inserting this into (7), we get $\mathbb{E}[\sup_{[0, T]} |X_t^1 - X_t^{1, n}|^2] \leq C_T \int_0^T \min(n^{-1/2}, n^{-1}) dt \leq C_T n^{-1}$. \square

It remains to give the

Proof of Corollary 5. Recall here that $a_{ij}(x) = \kappa(|x|^2)(|x|^2\delta_{ij} - x_i x_j)$ and that $b(x) = -(d-1)\kappa(|x|^2)x$, that κ is C^2 and that κ' has a bounded support, so that a has bounded derivatives of order 2, and b is Lipschitz continuous. We consider a weak solution $(P_t)_{t \geq 0}$ to (1). As previously, we classically have $m_2(P_t) = m_2(P_0)$. Furthermore, it is again classical and widely used that the entropy of P_t is non-increasing, so that $\int P_t(x) \log P_t(x) dx \leq \int P_0(x) \log P_0(x) dx = C < \infty$ for all times, see Villani [10, 11, 12].

If we prove that there is $\lambda_0 > 0$ such that for all $t \geq 0$, $x, y \in \mathbb{R}^d$, $(a(x, P_t)y, y) \geq \lambda_0|y|^2$, then we deduce that $|a(x, P_t)^{-1}|$ is uniformly bounded, so that the Corollary follows from (7).

Observe that setting $\alpha_{ij}(x) = |x|^2\delta_{ij} - x_i x_j$, we have $(a(x, P_t)y, y) \geq \lambda_1(\alpha(x, P_t)y, y)$, where $\lambda_1 > 0$ is a lowerbound of κ . But it is shown in Desvillettes-Villani [3, Proposition 4] that for $E_0 \in \mathbb{R}_+$, $H_0 \in \mathbb{R}_+$, there is a constant $c_{E_0, H_0} > 0$ such that for any probability density function f on \mathbb{R}^d such that $m_2(f) \leq E_0$ and $\int f(x) \log f(x) dx \leq H_0$, $(\alpha(x, f)y, y) \geq c_{E_0, H_0}|y|^2$. Actually, they consider the case where $\alpha_{ij}(x) = |x|^\gamma(|x|^2\delta_{ij} - x_i x_j)$ for some $\gamma > 0$, but one can check that their proof works without modification when $\gamma = 0$. We finally obtain $(a(x, P_t)y, y) \geq \lambda_1 c_{E_0, H_0}|y|^2$ for all $t \geq 0$, $x, y \in \mathbb{R}^d$, which concludes the proof. \square

4. ON SOFT POTENTIALS

We consider in this section the spatially homogeneous Landau equation for soft potentials, which writes (1) with $a_{ij}(z) = |z|^\gamma(|z|^2\delta_{ij} - z_i z_j)$ for some $\gamma \in [-3, 0)$, the Coulomb case $\gamma = -3$ being the most interesting from a physical point of view. Then we have $b_i(z) = \sum_1^d \partial_j a_{ij}(z) = -(d-1)|z|^\gamma z_i$.

Simulation with cutoff. We restrict our study to the case where $\gamma \in (-2, 0]$. We assume that P_0 has finite moments of all orders, and has a density with a finite entropy $\int P_0(x) \log P_0(x) dx < \infty$. For $\varepsilon > 0$ let $\kappa_\varepsilon : \mathbb{R}_+ \mapsto \mathbb{R}_+$ of class C^2 , nondecreasing, with $\kappa_\varepsilon(z) = z$ for $z \geq \varepsilon$, $\kappa_\varepsilon(z) = \varepsilon/2$ for $z \in [0, \varepsilon/2]$, with $|\kappa'_\varepsilon(z)| + \varepsilon|\kappa''_\varepsilon(z)| \leq C$. Consider then $a_\varepsilon, b_\varepsilon$ be defined as a, b with $|z|^\gamma$ replaced by $[\kappa_\varepsilon(|z|)]^\gamma$. Then a_ε is of class C^2 , with all its derivatives of order 2 bounded by $C\varepsilon^\gamma$, and b_ε is Lipschitz continuous with Lipschitz constant $C\varepsilon^\gamma$.

We thus may apply Corollary 5 and Theorem 6. Denote by $(P_t^\varepsilon)_{t \geq 0} = (\mathcal{L}(X_t^{1, \varepsilon}))_{t \geq 0}$ a weak solution to (1) with a_ε and $P_0^\varepsilon = P_0$. Then we believe that our results, plus some moment and ellipticity estimates (uniform in $\varepsilon \in (0, 1]$), will give something like $\mathbb{E}[\sup_{[0, T]} |X_t^{1, n, N, \varepsilon} - X_t^{1, \varepsilon}|^2] \leq (n^{-1} + N^{-1}) \exp(C_T \varepsilon^{2\gamma})$, where $(X_t^{i, n, N, \varepsilon})_{t \geq 0, i=1, \dots, n}$ solves (8) with $a_\varepsilon, b_\varepsilon$ instead of a, b .

On the other hand, we may apply the techniques introduced in [5] to estimate $\mathcal{W}_2^2(P_t, P_t^\varepsilon)$, where $(P_t)_{t \geq 0} = (\mathcal{L}(X_t^1))_{t \geq 0}$ is a weak solution to (1) with a and P_0 . We believe that, with a convenient coupling, it is possible to obtain something like $\sup_{[0, T]} \mathbb{E}[|X_t^{1, \varepsilon} - X_t^1|^2] \leq C_T \varepsilon^2$.

One would thus get $\sup_{[0, T]} \mathbb{E}[|X_t^{1, n, N, \varepsilon} - X_t^1|^2] \leq C_T \left(\varepsilon^2 + (n^{-1} + N^{-1}) e^{C_T \varepsilon^{2\gamma}} \right)$. This is of course an awful rate of convergence. It does not seem reasonable to handle a rigorous proof.

Simulation without cutoff. However, the particle system (8) is still well-defined and simulable for soft potentials (with $\gamma \in [-3, 0]$), at least if we replace $\frac{1}{n} \sum_k \delta_{X_t^{k, n, N}}$ by $\frac{1}{n} \sum_{k \neq i} \delta_{X_t^{k, n, N}}$ and if P_0 has a density. Based on the well-posedness result of [5], we hope that, at least when $\gamma \in (-2, 0]$, one might obtain the same estimates as in Corollary 5 and Theorem 6 (under additional conditions on P_0). The proof however seems to be quite difficult: we do not know how to get a sufficiently good estimate of quantities like $|X_t^{i, n, N} - X_t^{j, n, N}|^\gamma$.

5. NUMERICS

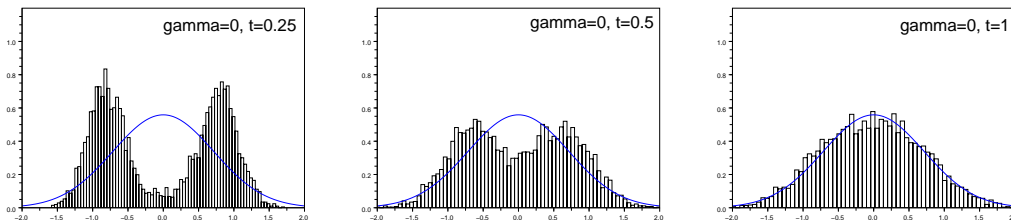
Let us first observe that for the Landau equation 1 where a is given by (2) and $b_i = \sum_j \partial_j a_{ij}$, the simulable particle system (8) is conservative, in the sense that it preserves, in mean, momentum and kinetic energy: for all $i = 1, \dots, n$, all $t \geq 0$, $\mathbb{E}[X_t^{i, n, N}] = \int x P_0(dx)$ and $\mathbb{E}[|X_t^{i, n, N}|^2] = m_2(P_0)$.

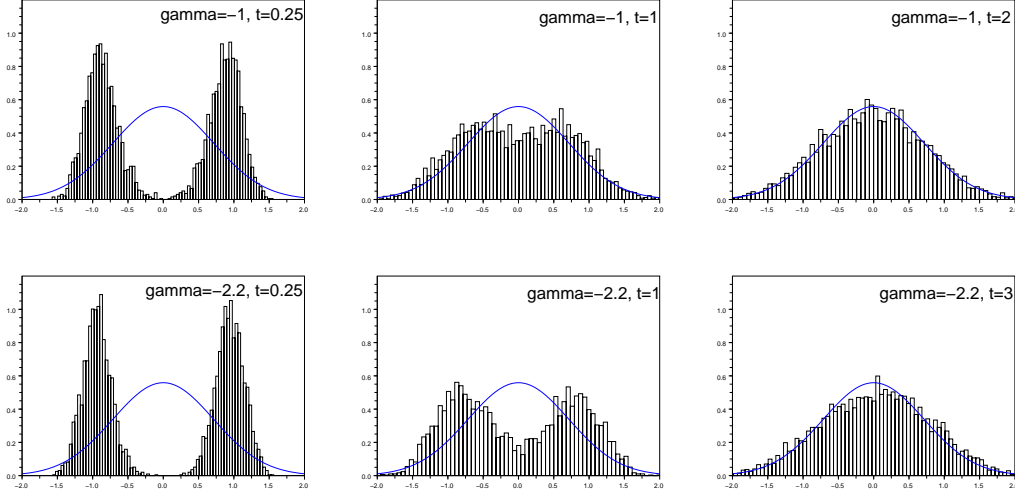
We consider here the Landau equation for soft potentials, for some $\gamma \in [-3, 0]$, described in the previous section, in dimension $d = 2$. We use no cutoff procedure in the case $\gamma < 0$. We consider the initial condition P_0 with density $P_0(x_1, x_2) = f(x_1)g(x_2)$, where f is the Gaussian density with mean 0 and variance 0.1, while $g(x) = (f(x-1) + f(x+1))/2$. The momentum and energy of P_0 are given by (0, 0) and 1.02.

Thus in large time, the solution P_t should converge to the Gaussian distribution with mean (0, 0) and covariance matrix $0.51I_2$, see Villani [12].

We use the particle system (8) with n particles, and N steps per unit of time. Easy considerations show that the computation of (8) until time T is essentially proportionnal to TNn^2 , and should not depend too much on γ . However, it is consequently faster when $\gamma = 0$ for obvious computational reasons. Let us also remark that the law of (8) does not change when replacing $a^{\frac{1}{2}}$ by any σ such that $\sigma(x, \mu)\sigma(x, \mu)^* = a(x, \mu)$. We thus use a Cholesky decomposition, which is numerically quite fast. Let us give an idea of the time needed to perform one time-step: with $\gamma = 0$, it takes around $7 \cdot 10^{-3}$ seconds ($n = 500$), 0.15 s ($n = 2500$), 3.5 s ($n = 12500$), and 13 s ($n = 25000$). The computations are around 10 times slower when $\gamma < 0$.

Now we always use $n = 5000$ particles, and $N = 200$ steps per unit of time. We draw, for different values of t and γ , the histogram (with 80 sticks) based on the second coordinates of $(X_t^{i, n, N})_{i=1, \dots, n}$. The plain curve is the expected asymptotic Gaussian density, with mean 0 and variance 0.51. The convergence to equilibrium seems to be slower and slower as γ is more and more negative.





For too small values of γ (say $\gamma < -2.5$), the numerical results are not so convincing. This is not surprising, since the coefficients are more and more singular as γ becomes smaller and smaller.

6. APPENDIX

The following Lemma can be found in Stroock-Varadhan (when $p = d$) [8, Theorem 5.2.3], or in Villani [9, Theorem 1] (for a more refined statement including all possible values of p and d).

Lemma 10. *Let $A : \mathbb{R}^p \mapsto S_d^+$, for some $p \geq 1$, $d \geq 1$, be of class C^2 , with all its derivatives of order 2 bounded. Then $\|D(A^{\frac{1}{2}})\|_\infty \leq C_{p,d} \sqrt{\|D^2 A\|_\infty}$, where $C_{p,d}$ depends only on p and d .*

We also need the following estimates, which are probably standard.

Lemma 11. *For $A, B \in S_d^+$,*

- (i) *there holds $|A^{\frac{1}{2}} - B^{\frac{1}{2}}| \leq \sqrt{|A - B|}$*
- (ii) *and $|A^{\frac{1}{2}} - B^{\frac{1}{2}}| \leq \sqrt{\min(|A^{-1}|, |B^{-1}|)} \times |A - B|$.*

Proof. We start with point (i). Let $\sigma = |A^{\frac{1}{2}} - B^{\frac{1}{2}}|$. There is a unit vector $e \in \mathbb{R}^d$ such that $|(A^{\frac{1}{2}} - B^{\frac{1}{2}})e| = \sigma e$, and we may assume that $(A^{\frac{1}{2}} - B^{\frac{1}{2}})e = \sigma e$ (else, change the roles of A, B). Then, using that $B^{\frac{1}{2}}$ is nonnegative,

$$|A - B| \geq (Ae - Be, e) = ((A^{\frac{1}{2}} - B^{\frac{1}{2}})e, (A^{\frac{1}{2}} + B^{\frac{1}{2}})e) = (\sigma e, \sigma e + 2B^{\frac{1}{2}}e) \geq \sigma^2 |e|^2 = \sigma^2.$$

We now prove (ii). First observe that $(A^{\frac{1}{2}}x, x) \geq |x|^2/|A^{-1/2}|$ for all $x \in \mathbb{R}^d$. As previously,

$$\begin{aligned} |A - B| &\geq ((A^{\frac{1}{2}} - B^{\frac{1}{2}})e, (A^{\frac{1}{2}} + B^{\frac{1}{2}})e) = \sigma(A^{\frac{1}{2}}e, e) + \sigma(B^{\frac{1}{2}}e, e) \\ &\geq \sigma|e|^2/|A^{-1/2}| + \sigma|e|^2/|B^{-1/2}| = \sigma/\sqrt{|A^{-1}|} + \sigma/\sqrt{|B^{-1}|} \geq \sigma/\sqrt{\min(|A^{-1}|, |B^{-1}|)}, \end{aligned}$$

whence $|A^{\frac{1}{2}} - B^{\frac{1}{2}}| = \sigma \leq \sqrt{\min(|A^{-1}|, |B^{-1}|)} |A - B|$. \square

We conclude this annex with an elementary fact on the Wasserstein distance.

Lemma 12. *For $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^d$, $\mathcal{W}_2^2(\frac{1}{n} \sum_1^n \delta_{x_i}, \frac{1}{n} \sum_1^n \delta_{y_i}) \leq \frac{1}{n} \sum_1^n |x_i - y_i|^2$.*

Proof. Let U be uniformly distributed on $\{1, \dots, n\}$, set $X = x_U$ and $Y = y_U$. Then $X \sim \frac{1}{n} \sum_1^n \delta_{x_i}$, $Y \sim \frac{1}{n} \sum_1^n \delta_{y_i}$, and $\mathbb{E}[|X - Y|^2] = \frac{1}{n} \sum_1^n |x_i - y_i|^2$. \square

REFERENCES

- [1] C. BUET, S. CORDIER, F. FILBET, *Comparison of numerical schemes for Fokker-Planck-Landau equation.*, CEMRACS 1999 (Orsay), 161-181, ESAIM Proc., 10, Soc. Math. Appl. Indust., Paris, 1999.
- [2] S. CORDIER, S. MANCINI, *A brief review on numerical methods for the collision operators*, ENUMATHS Proceedings 2001, Springer, 2003.
- [3] L. DESVILLETES, C. VILLANI, *On the spatially homogeneous Landau equation for hard potentials Part I: existence, uniqueness, and smoothness*, Comm. Partial differential equations, 25, no 1-2, 175-259, 2000.
- [4] J. FONTBONA, H. GUÉRIN, S. MÉLÉARD, *Measurability of optimal transportation and convergence rate for Landau type interacting particle systems*, to appear in Probab. Theory Related Fields, 2008.
- [5] N. FOURNIER, H. GUÉRIN, *Well-posedness of the spatially homogeneous Landau equation for soft potentials*, Preprint, 2008.
- [6] T. FUNAKI, *The diffusion approximation of the spatially homogeneous Boltzmann equation*, Duke Math. J. 52, 1-23, 1985.
- [7] H. GUÉRIN, *Solving Landau equation for some soft potentials through a probabilistic approach*, Ann. Appl. Probab. 13, no. 2, 515-539, 2003.
- [8] D.W. STROOCK, S.R.S. VARADHAN, *Multidimensional diffusion processes*, Reprint of the 1997 edition. Classics in Mathematics. Springer, 2006.
- [9] P.L. LIONS, C. VILLANI, *Régularité optimale des racines carrées*, C.R. Acad. Sci. 321, 1537-1541, 1995.
- [10] C. VILLANI, *On the spatially homogeneous Landau equation for Maxwellian molecules.*, Math. Models Methods Appl. Sci. 8, no. 6, 957-983, 1998.
- [11] C. VILLANI, *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations*, Arch. Rational Mech. Anal. 143, no. 3, 273-307, 1998.
- [12] C. VILLANI, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam, 2002.
- [13] C. VILLANI, *Topics in Optimal Transportation*, Graduate Studies in Mathematics Vol. 58, AMS 2003.
- [14] J.B. WALSH, *An introduction to stochastic partial differential equations*, École d'été de Probabilités de Saint-Flour XIV, Lect. Notes in Math. 1180, 265-437, 1986.