# A PURE JUMP MARKOV PROCESS WITH A RANDOM SINGULARITY SPECTRUM

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Abstract. We construct a non-decreasing pure jump Markov process, whose jump measure heavily depends on the values taken by the process. We determine the singularity spectrum of this process, which turns out to be random and to depend locally on the values taken by the process. The result relies on fine properties of the distribution of Poisson point processes and on ubiquity theorems.

## 1. Introduction

Up to the mid-70s, the study of the Hölder regularity of the sample paths of stochastic processes was focused on two main issues: the determination of their uniform modulus of continuity, and the existence of an almost-everywhere pointwise modulus of continuity. However, the first indications that their pointwise regularity could vary from point to point in a subtle way, appeared in the works of Orey-Taylor [18] and Perkins [19], who showed that the fast and slow points of Brownian motion are located on random fractal sets. Furthermore, they determined the Hausdorff dimensions of these sets. Brownian motion, however, only displays very slight changes in its modulus of continuity (which is modified only by logarithmic corrections). This is in sharp contrast with other types of processes, such as Lévy processes for instance, whose modulus of continuity changes completely from point to point. Let us recall the relevant definitions related with pointwise Hölder regularity, in this context.

**Definition 1.** *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a locally bounded function, $t_0\in\mathbb{R}_+$ and let $\alpha>0$. The function $f$ belongs to $C^\alpha(t_0)$ if there exist $C>0$ and a polynomial $P_{t_0}$ of degree less than $\alpha$ such that for all $t$ in a neighborhood of $t_0$,*

$$|f(t)-P_{t_0}(t)|\le C|t-t_0|^\alpha.$$

*The Hölder exponent of $f$ at $t_0$ is (here $\sup\emptyset=0$)*

$$h_f(t_0)=\sup\{\alpha>0:\ \ f\in C^\alpha(t_0)\}.$$

The level sets of the pointwise exponent of the Hölder exponent are called the *iso-Hölder sets* of $f$, and defined, for $h\ge 0$, by

$$E_f(h)=\{t\ge 0:\ \ h_f(t)=h\}.$$

The corresponding notion of "multifractal function" was put into light by Frisch and Parisi [13], who introduced the definition of the *spectrum of singularities* of a function $f$.

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**Definition 2.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded function. The spectrum of singularities of $f$ is the function $D_f$ defined, for $h \geq 0$, by*

$$D_f(h) = \dim_H(E_f(h)).$$

*We also define, for any open set $A \subset \mathbb{R}_+$, $D_f(A, h) = \dim_H(E_f(h) \cap A)$.*

The definition of the Hausdorff dimension can be found in Falconer [12] for instance (by convention $\dim_H \emptyset = -\infty$). The singularity spectrum of $f$ describes the geometric repartition of its Hölder singularities, and encapsulates a geometric information which is usually more meaningful than the Hölder exponent.

Following the way opened by Frisch and Parisi, the spectrum of singularities of large classes of stochastic processes (or random measures, in which case an appropriate notion of Hölder pointwise regularity for measures is used) have been determined. Most examples of stochastic processes $f$ which have been studied display the following remarkable features.

- Though the iso-Hölder sets are random, the spectrum of singularities is deterministic: for some deterministic function $\Delta$, a.s., for all $h \geq 0$, $D_f(h) = \Delta(h)$.
- The spectrum of singularities of $f$ is *homogeneous*: a.s., for any nonempty open subset $A \subset \mathbb{R}_+$, for all $h \geq 0$, $D_f(A, h) = \Delta(h)$.

Though it is easy to construct artificial *ad hoc* processes that do not satisfy these properties, it is remarkable that many "natural" processes of very different kind follow this rule: Lévy processes [16], Lévy processes in multifractal time [5], fractional Brownian motions, random self-similar measures and random Gibbs measures [9], Mandelbrot cascades [2], Poisson cascades [3], among many other examples. See however [10] where Durand constructed a counterexample whose wavelet coefficients are defined using Markov trees.

In this paper we will investigate the regularity properties of some Markov processes. Our purpose at this stage is not to obtain results in the most general form, but rather to consider some specific examples, and check that such processes indeed display a random spectrum, which is not homogeneous. Note that, until now, the only Markov processes which have been analyzed from the multifractal standpoint are the Lévy processes.

We now introduce a new notion, the *local spectrum*, which is tailored to the study of functions with non-homogeneous spectra.

**Definition 3.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded function, $t_0 \in \mathbb{R}_+$ and let $(V_n)_{n\geq 1}$ be a basis of neighborhoods of $t_0$. The local spectrum of $f$ at $t_0$ is the function*

$$\text{for all } h \geq 0,\ D_f(t_0, h) = \lim_{n\to\infty} D_f(V_n, h).$$

By monotonicity (if $A \subset B$, then $D_f(A, h) \leq D_f(B, h)$), the limit exists and it is independent of the particular basis chosen. Clearly, a function has a homogeneous spectrum if and only if for all $h \geq 0$, $D_f(t, h)$ is independent of $t \geq 0$. The local spectrum allows one to recover the spectrum of all possible restrictions of $f$ on an open interval.

**Lemma 4.** *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be a locally bounded function. Then for any open interval $I = (a, b) \subset \mathbb{R}_+$, for any $h \geq 0$, we have $D_f(I, h) = \sup_{t\in I} D_f(t, h)$.*

*Proof.* Let thus $h \geq 0$ be fixed. First, it is obvious that for any $t \in I$, $D_f(t, h) \leq D_f(I, h)$, since for $(V_n)_{n\geq 1}$ a basis of neighborhoods of $t$, $V_n \subset I$ for $n$ large enough. Next, set $\delta = D_f(I, h)$, and consider $\varepsilon > 0$. We want to find $t \in I$ such that for all neighborhood $V$

of $t$, $D_f(V,h) \geq \delta - \varepsilon$. Assume by contradiction that for any $t \in I$, there is a neighborhood $V_t$ of $t$ such that $D_f(V_t,h) < \delta - \varepsilon$. One easily deduces that for any compact $K \subset I$, $D_f(K,h) \leq \delta - \varepsilon$ (use a finite covering of $K$ by the $V_t$'s). This would of course imply that $D_f(I,h) \leq \delta - \varepsilon < \delta$. □

Let us now recall the multifractal nature of Lévy processes without Brownian component. In that case, the spectrum of singularities only depends on one parameter, the *lower index* of Blumenthal and Getoor, which quantifies the "density" of small jumps, and is defined, for any non-negative measure $\nu$ on $\mathbb{R}$ satisfying $\int_{-1}^{1} u^2 \nu(du) < \infty$, by

$$\beta_\nu := \inf \left\{ \alpha \geq 0 : \int_{-1}^{1} |u|^\alpha \nu(du) < \infty \right\}.$$

Note that the integrability condition implies that $\beta_\nu \in [0,2]$. The following result of [16] yields the spectrum of singularities of such Lévy processes.

**Theorem 5.** *Let $(X_t)_{t\geq 0}$ be a Lévy process without Brownian component, with Lévy measure $\nu$. If $\beta_\nu \in (0,2)$, then the singularity spectrum of $X$ is homogeneous and deterministic: a.s., for all $t \geq 0$, for all $h \geq 0$,*

$$D_X(t,h) = D_X(h) = \begin{cases} h \cdot \beta_\nu & \text{if } \ 0 \leq h \leq 1/\beta_\nu, \\ -\infty & \text{if } \ h > 1/\beta_\nu. \end{cases}$$

Let us make a few observations. It is not stated explicitly in [16] that the spectrum of a Lévy process is homogeneous, but it is a direct consequence of the proof. Indeed, the spectrum is computed on a arbitrary interval, and the stationarity of the increments implies that it does not depend on the particular location of this interval. Although a Lévy process is random, its spectrum is almost surely deterministic. As explained above, this is the situation usually met when performing the multifractal analysis of random processes or random measures possessing either stationarity or scaling invariance properties.

The purpose of this paper is to investigate how these results are modified when the stationarity assumption is dropped. We will give examples of Markov processes whose singularity spectra are non-homogeneous and random.

## 2. Statement of the main result

A quite general class of one-dimensional Markov processes consists of stochastic differential equations (S.D.E.) with jumps, see Ethier-Kurtz [11], Ikeda-Watanabe [14]. Since the Brownian motion is mono-fractal, the Brownian part of such a process will not be very relevant. Thus in order to avoid technical difficulties, we consider a jumping S.D.E. without Brownian and drift part, starting e.g. from 0, and with jump measure $\nu(y,du)$ (meaning that when located at $y$, the process jumps to $y+u$ at rate $\nu(y,du)$). Again to make the study as simple as possible, we will assume that the process has finite variations, and even that it is increasing (that is, $\nu(y,(-\infty,0)) = 0$ for all $y \in \mathbb{R}$). Classically, a necessary condition for the process to be well-defined is that $\int_0^\infty u\, \nu(y,du) < \infty$.

If $\nu$ is chosen so that the index $\beta_{\nu(y,.)}$ is constant with respect to $y$, then we expect that $D_M(t,h)$ will be deterministic and independent of $t$. We thus impose that the index of the jump measure depends on the value $y$ of the process. The most natural example of such a situation consists in choosing

$$\nu_\gamma(y,du) := \gamma(y) u^{-1-\gamma(y)} \mathbf{1}_{[0,1]}(u) du,$$

for some function $\gamma : \mathbb{R} \mapsto (0,1)$. The lower exponent of this family of measures is

$$\forall\, y \geq 0, \qquad \beta_{\nu_\gamma(y,.)} = \gamma(y).$$

We will make the following assumption

$$(\mathcal{H}) \qquad \begin{cases} \text{There exists } \varepsilon > 0 \text{ such that } \gamma : [0,\infty) \longmapsto [\varepsilon, 1-\varepsilon] \\ \text{ is a Lipschitz-continuous strictly increasing function.} \end{cases}$$

We impose a monotonicity condition for simplicity. The global Lipschitz condition could be slightly relaxed, as well as the uniform bounds.

**Proposition 6.** *Assume that $(\mathcal{H})$ holds. There exists a strong Markov process $M = (M_t)_{t\geq 0}$ starting from 0, increasing and càdlàg, and with generator $\mathcal{L}$ defined for all $y \in [0,\infty)$ and for any function $\phi : [0,\infty) \mapsto \mathbb{R}$ Lipschitz-continuous by*

$$\mathcal{L}\phi(y) = \int_0^1 [\phi(y+u) - \phi(y)] \nu_\gamma(y,du). \tag{1}$$

*Almost surely, this process is continuous except on a countable number of jump times. We denote by $\mathcal{J}$ the set of its jump times, that is $\mathcal{J} = \{t > 0 : \Delta M(t) \neq 0\}$. Finally, $\mathcal{J}$ is dense in $[0,\infty)$.*

Here and below, $\Delta M_t = M_t - M_{t-}$, where $M_{t-} = \lim_{s\to t,\, s<t} M_s$. Proposition 6 will be checked in Section 3, by using a Poisson S.D.E. This representation of $M$ will be useful for its local regularity analysis in the next sections.

The following theorem summarizes multifractal features of $M$.

**Theorem 7.** *Assume $(\mathcal{H})$ and consider the process $M$ constructed in Proposition 6. Then, the following properties hold almost surely.*

(i) *For every $t \in (0,\infty) \backslash \mathcal{J}$, the local spectrum of $M$ at $t$ is given by*

$$D_M(t,h) = \begin{cases} h \cdot \gamma(M_t) & \text{if } 0 \leq h \leq 1/\gamma(M_t), \\ -\infty & \text{if } h > 1/\gamma(M_t), \end{cases} \tag{2}$$

*while for $t \in \mathcal{J}$,*

$$D_M(t,h) = \begin{cases} h \cdot \gamma(M_t) & \text{if } 0 \leq h < 1/\gamma(M_t), \\ h \cdot \gamma(M_{t-}) & \text{if } h \in [1/\gamma(M_t), 1/\gamma(M_{t-})], \\ -\infty & \text{if } h > 1/\gamma(M_{t-}). \end{cases} \tag{3}$$

(ii) *The spectrum of $M$ on any interval $I = (a,b) \subset (0,+\infty)$ is*

$$\forall h \geq 0, \quad D_M(I,h) = \sup\Big\{ h \cdot \gamma(M_t) : \, t \in I, \; h \cdot \gamma(M_t) < 1 \Big\} \tag{4}$$

$$= \sup\Big\{ h \cdot \gamma(M_{s-}) : \, s \in \mathcal{J} \cap I, \; h \cdot \gamma(M_{s-}) < 1 \Big\}. \tag{5}$$

*In (4) and (5), we adopt the convention that $\sup \emptyset = -\infty$.*

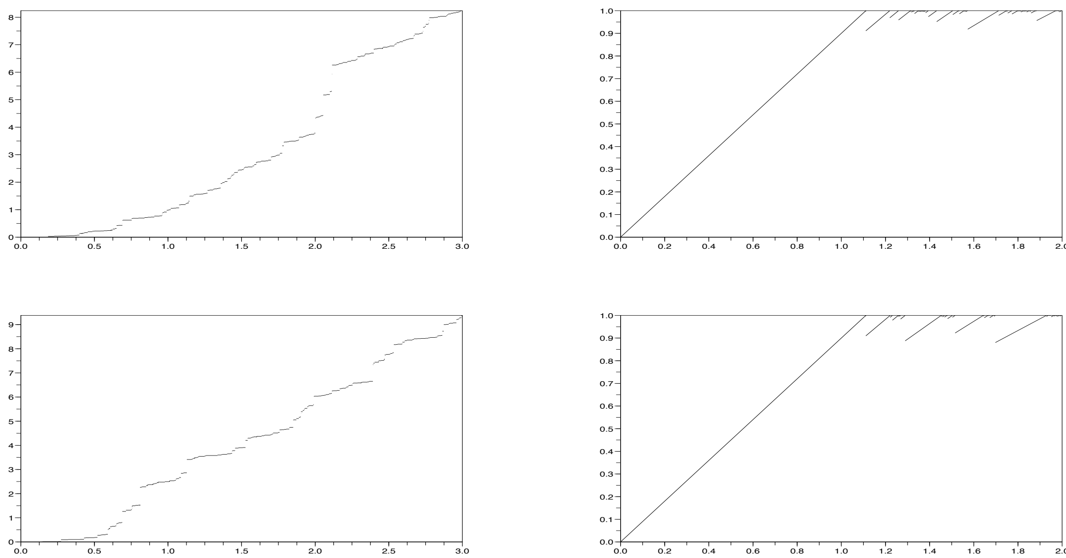

FIGURE 1. Two sample paths of the stochastic process $M$ built using the function $\gamma(y) := \min(1/2 + y/4, 0.9)$. On the right hand-side are plotted the theoretical spectra $D_M([0,3], .)$.

**Remark 8.** *To prove Theorem 7, it is enough to show* (5).

*Proof.* The equality between (4) and (5) follows from the fact that the $\mathcal{J}$ is dense in $\mathbb{R}_+$, and that $t \mapsto \gamma(M_t)$ is càdlàg on $I$.

Next, point (i) follows from (4) applied to $I_n = (t - 1/n, t + 1/n)$ by taking the limit $n \to \infty$ (recall Definition 3). Let us for example assume that $t \in \mathcal{J}$.

• If $h > 1/\gamma(M_{t-})$, then for $n$ large enough, $h \cdot \gamma(M_s) \geq h \cdot \gamma(M_{t-1/n}) \geq 1$ for all $s \in I_n$, whence $D_M(I_n, h) = -\infty$ (we use here that $s \mapsto \gamma(M_{s-})$ is non-decreasing). Thus $D_M(t, h) = -\infty$.

• If $h < 1/\gamma(M_t)$, we get from (4) that $h \cdot \gamma(M_t) \leq D_M(I_n, h) \leq h \cdot \sup_{[t-1/n, t+1/n]} \gamma(M_s) = h \cdot \gamma(M_{t+1/n})$. Since $s \mapsto \gamma(M_s)$ is right continuous, $D_M(t,h) = \lim_n D_M(I_n, h) = h \cdot \gamma(M_t)$.

• If $h \in [1/\gamma(M_t), 1/\gamma(M_{t-})]$, then for all $s \geq t$, $h \cdot \gamma(M_s) \geq 1$, while clearly $h \cdot \gamma(M_{t-1/n}) < 1$. Hence we deduce from (4) that $h \cdot \gamma(M_{t-1/n}) \leq D_M(I_n, h) \leq h \cdot \sup_{[t-1/n, t)} \gamma(M_s) = h \cdot \gamma(M_{t-})$. Finally, $D_M(t,h) = \lim_n D_M(I_n, h) = h \cdot \gamma(M_{t-})$. □

Formula (5) is better understood when plotted: for every $s \in I \cap \mathcal{J}$, plot a segment whose endpoints are $(0,0)$ and $(1/\gamma(M_{s-}), 1)$ (open on the right), and take the supremum to get $D_M(I, .)$. Sample paths of the process $M$ and their associated spectra are given in Figure 1.

The formulae giving the local and global spectra are based on the computation of the pointwise Hölder exponents at all times $t$. We will in particular prove (see Theorem 18 and Proposition 22) the following properties: a.s.,

$$
\begin{aligned}
\text{for every } t \geq 0, \qquad & h_M(t) \leq 1/\gamma(M_t), \\
\text{for Lebesgue-almost every } t, \qquad & h_M(t) = 1/\gamma(M_t), \\
\text{for every } \kappa \in (0,1), \qquad & \dim_H \{t \geq 0 : \ h_M(t) = \kappa/\gamma(M_t)\} = \kappa.
\end{aligned}
$$

It is worth emphasizing that, as expected from the construction of the process $M$, the local spectrum (2) at any point $t > 0$ essentially coincides with that of a stable Lévy subordinator of index $\gamma(M_t)$. This local comparison will be strengthened in Section 7, where we prove the existence of tangent processes for $M$ (which are Lévy stable subordinators).

The proof of Theorem 7 requires a so-called *ubiquity theorem*. Ubiquity theory deals with the search for lower bounds for the Hausdorff dimensions of limsup sets, and is classically required when performing the multifractal analysis of stochastic processes or (random or deterministic) measures with jumps [16, 17, 5, 6]. For our Markov process $M$, the ubiquity theorem needed here is the "localized ubiquity" theorem recently proved in [7]. In order to apply this result, we need to establish fine properties on the distribution of Poisson point processes (see Section 6).

**Remark 9.** *It follows from Theorem 7 that for all $s \in \mathcal{J}$, all $h \in (1/\gamma(M_s), 1/\gamma(M_{s-})]$, $D_M(h) = h \cdot \gamma(M_{s-})$. Thus the spectrum $D_M$ is a straight line on all segments of the form $(1/\gamma(M_s), 1/\gamma(M_{s-})]$, $s \in \mathcal{J}$. Nonetheless, observe that the spectrum we obtain, when viewed as a map from $\mathbb{R}_+$ to $\mathbb{R}_+$, is very irregular, and certainly multifractal itself. This is in sharp contrast with the spectra usually obtained, which are most of the time concave or (piecewise) real-analytic.*

**Remark 10.** *Random processes with random singularity exponents of the most general form have already been constructed in* [1]*, but there is no direct relationship with having a random singularity spectrum. An example of stochastic process $X$ (built using wavelet coefficients) with a random singularity spectrum is provided by* [10]*, but there $D_X(h)$ is random for at most two values of $h$.*

**Remark 11.** *Of course we hope that Theorem 7, which concerns a specific and simple process, might have extensions to more general Markov processes $M = (M_t)_{t\in[0,1]}$. In particular, this is certainly the case if we keep the same measures $\nu_\gamma$ and if we drop the monotonicity assumption on the Lipschitz-continuous function $\gamma$.*

The organization of the paper is the following. In Section 3, Proposition 6 is proved. We also introduce a suitable coupling of $M$ with a family of Lévy processes. In Section 4, we introduce a family of (random) subsets of $[0, 1]$ on which we control the regularity of $M$. We conclude the proof of Theorem 7 in Sections 5 and 6. Finally, in Section 7, we show the existence of tangent processes for $M$.

**In the whole paper, we assume that $(\mathcal{H})$ is satisfied. We will restrict our study to the time interval $[0, 1]$, which of course suffices.**

## 3. Poisson representation of the process

First of all, we observe that using the substitution $u = (1+z)^{-1/\gamma(y)}$ in (1) (for each fixed $y$), we may rewrite, for any $y \in [0,\infty)$, for any $\phi : [0,\infty) \mapsto \mathbb{R}$ Lipschitz-continuous,

$$\mathcal{L}\phi(y) = \int_0^\infty [\phi(y + G(\gamma(y), z)) - \phi(y)]dz \tag{6}$$

where

$$G(\beta, z) := (1+z)^{-1/\beta}.$$

We recall the following representation of the Poisson measures we are going to use.

**Remark 12.** *Let $(Y_n)_{n\geq 1}$ be a sequence of independent exponential random variables with parameter 1. Let $(T_n)_{n\geq 1}$ be a sequence of independent $[0,1]$-valued uniformly-distributed random variables, and assume that $(Y_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ are independent. For each $n \geq 1$, set $Z_n = Y_1 + \cdots + Y_n$. Then the random measure*

$$N(dt, dz) = \sum_{n\geq 1} \delta_{(T_n, Z_n)}(dt, dz)$$

*is a Poisson measure on $[0,1] \times [0,\infty)$ with intensity measure $dtdz$. We denote by $\mathcal{F}_t := \sigma(\{N(A), A \in \mathcal{B}([0,t] \times [0,\infty))\})$ the associated filtration.*

*The law of large numbers ensures us that a.s., $\lim_n Z_n/n = 1$.*

This leads us to the following representation of the process $M$.

**Proposition 13.** *Let $N$ be as in Remark 12. Then there exists an unique càdlàg $(\mathcal{F}_t)_{t\in[0,1]}$-adapted process $M = (M_t)_{t\in[0,1]}$ solution to*

$$M_t = \int_0^t \int_0^\infty G(\gamma(M_{s-}), z) N(ds, dz). \tag{7}$$

*Furthermore, $M$ is a strong Markov process with generator $\mathcal{L}$ (see (1) or (6)), and is a.s. increasing. Finally, $\mathcal{J} = \{t \in [0,1] : \Delta M_t \neq 0\} = \{T_n : n \geq 1\}$.*

Proposition 6 is a consequence of this result.

*Proof.* Owing to classical arguments (see e.g. Ikeda-Watanabe [14]), the (pathwise) existence and uniqueness of the solution to (7), follow from the two following properties, which are easily checked under $(\mathcal{H})$:

- for all $y \in [0,\infty)$, $z \in [0,\infty)$, $G(\gamma(y), z) \leq (1+z)^{-1/(1-\varepsilon)} \in L^1([0,\infty), dz)$,
- for all $x, y \in [0,\infty)$, $\int_0^\infty |G(\gamma(x), z) - G(\gamma(y), z)|dz = |\frac{\gamma(x)}{1-\gamma(x)} - \frac{\gamma(y)}{1-\gamma(y)}| \leq C|x-y|$ (here we use that $G(\beta, z)$ is nondecreasing as a function of $\beta$).

Indeed, it suffices to use the Gronwall Lemma and a Picard iteration (for the norm $||X - Y|| = \mathbb{E}[\sup_{[0,1]} |X_t - Y_t|]$). The strong Markov property follows from the pathwise uniqueness, and the monotonicity of $M$ is obvious since $G$ is non-negative. Finally, for

$\phi : [0,\infty) \mapsto \mathbb{R}$ Lipschitz-continuous, we have

$$\begin{aligned}\phi(M_t) &= \phi(0) + \sum_{s\le t}[\phi(M_{s-}+\Delta M_s) - \phi(M_{s-})]\\ &= \phi(0) + \int_0^t\int_0^\infty [\phi(M_{s-}+G(\gamma(M_{s-}),z)) - \phi(M_{s-})]N(ds,dz).\end{aligned}$$

Taking expectations and using (6), we get $\mathbb{E}[\phi(M_t)] = \phi(0) + \int_0^t \mathbb{E}[\mathcal{L}\phi(M_s)]ds$, so that the generator of $M$ is indeed $\mathcal{L}$. □

We also introduce a one-parameter family of Lévy processes, and check some regularity comparisons with $M$.

**Proposition 14.** *Consider the Poisson measure $N$ and the process $M$ introduced in Remark 12 and Proposition 13. For each fixed $\alpha \in (0,1)$, we define*

$$X_t^\alpha = \int_0^t\int_0^\infty G(\alpha,z)N(ds,dz). \tag{8}$$

*Then $(X_t^\alpha)_{t\in[0,1]}$ is a pure-jump $(\mathcal{F}_t)_{t\in[0,1]}$-adapted Lévy process. Its Lévy measure is $\nu^\alpha(du) = \alpha u^{-1-\alpha}\mathbf{1}_{[0,1]}(u)du$, for which $\beta_{\nu^\alpha} = \alpha$. Almost surely,*

(i) *for all $0 < \alpha < \alpha' < 1$, for all $0 \le s \le t \le 1$,*

$$0 \le X_t^\alpha - X_s^\alpha \le X_t^{\alpha'} - X_s^{\alpha'};$$

(ii) *for all $0 \le s \le t \le 1$,*

$$0 \le X_t^{\gamma(M_s)} - X_s^{\gamma(M_s)} \le M_t - M_s \le X_t^{\gamma(M_{t-})} - X_s^{\gamma(M_{t-})}.$$

*Proof.* For each $\alpha \in (0,1)$, $X^\alpha$ is classically a Lévy process, and its Lévy measure is the image measure of $dz$ by $G(\alpha,.)$, which is nothing but $\nu^\alpha$. Next, point (i) is not hard since for $0 < \alpha < \alpha'$, we have $G(\alpha,z) < G(\alpha',z)$ for all $z \in (0,\infty)$. Point (ii) is also immediate: since $u \mapsto \gamma(M_{u-})$ is nondecreasing, and since $G(.,z)$ is nondecreasing for all $z \in (0,\infty)$, we have a.s., for $0 \le s \le t \le 1$,

$$\begin{aligned}M_t - M_s &= \int_{(s,t]} G(\gamma(M_{u-}),z)N(du,dz)\\ &\le \int_{(s,t]} G(\gamma(M_{t-}),z)N(du,dz) = X_t^{\gamma(M_{t-})} - X_s^{\gamma(M_{t-})}.\end{aligned}$$

Similarly, we obtain $M_t - M_s \ge X_t^{\gamma(M_s)} - X_s^{\gamma(M_s)}$, which ends the proof. □

## 4. Local regularity

We consider a Poisson measure $N$ as in Remark 12, and the associated processes $M$, $X^\alpha$ as in Propositions 13 and 14. We start with a simple observation.

**Lemma 15.** *Almost surely, for all $\alpha \in (0,1)$,*

$$\mathcal{J} = \{t\in[0,1] : \Delta M_t \ne 0\} = \{t \in [0,1] : \Delta X_t^\alpha \ne 0\} = \bigcup_{n\ge1}\{T_n\}, \tag{9}$$

*and for all $n \geq 1$, all $\alpha \in (0,1)$,*

$$(\Delta M_{T_n})^{\gamma(M_{T_n-})} = (\Delta X^{\alpha}_{T_n})^{\alpha} = (1+Z_n)^{-1}.$$

*Proof.* First, (9) follows from (7) and (8). Next, for $n \geq 1$, $\Delta M_{T_n} = G(\gamma(M_{T_n-}), Z_n) = (1+Z_n)^{-1/\gamma(M_{T_n-})}$ and $\Delta X^{\alpha}_{T_n} = G(\alpha, Z_n) = (1+Z_n)^{-1/\alpha}$. □

Next, we introduce some (random) sets of times, more or less well-approximated by the times of jumps of our process $M$. The main idea is that at times well-approximated by the jump times of $M$, the pointwise regularity of $M$ will be poor, while at times which are far from the jump times of $M$, $M$ will have greater pointwise exponents.

We thus set, for all $\delta \geq 1$,

$$A_\delta = \bigcap_{p\geq 1} \bigcup_{n\geq p} B(T_n, (\Delta M_{T_n})^{\delta\gamma(M_{T_n-})}) = \bigcap_{p\geq 1} \bigcup_{n\geq p} B(T_n, (\Delta X^{\alpha}_{T_n})^{\delta\alpha}) \tag{10}$$

$$= \bigcap_{p\geq 1} \bigcup_{n\geq p} B(T_n, (1+Z_n)^{-\delta}). \tag{11}$$

Here, $B(t,r) = (t-r, t+r)$. Let us observe at once that

$$\text{for all } \delta_1 \leq \delta_2, \quad A_{\delta_2} \subset A_{\delta_1}.$$

**Proposition 16.** *Almost surely, $A_1 \supset [0,1]$.*

*Proof.* Observe that $\sum_{n\geq 1} \delta_{(T_n,(1+Z_n)^{-1})}$ is a Poisson measure on $[0,1]\times(0,1)$ with intensity measure $dt\mu(du)$ where $\mu(du) = du/u^2$ (because $\mu(du)$ is the image measure of $\mathbf{1}_{\{z>0\}}dz$ by the application $(1+z)^{-1}$). Due to Shepp's Theorem [20] (we use here the version used in the papers of Bertoin [8] and Jaffard [16, Lemma 3]), it suffices to prove that

$$S = \int_0^1 \exp\left(2\int_t^1 \mu((u,1))du\right) dt = \infty.$$

But $\mu((u,1)) = u^{-1} - 1$, so that $S = \int_0^1 e^{2(t-1-\log t)}dt \geq e^{-2}\int_0^1 dt/t^2 = \infty$. □

In order to characterize the pointwise exponent of $M$ at every time $t$, we need to introduce the notion of approximation rate by a Poisson process.

**Definition 17.** *Recall that a.s., $\delta \mapsto A_\delta$ is non-increasing and $A_1 = [0,1]$. We introduce, for any $t \in [0,1]$, the (random) index of approximation of $t$*

$$\delta_t := \sup\{\delta \geq 1 : \ t \in A_\delta\} \geq 1. \tag{12}$$

We now are able to give the value of $h_M(t)$.

**Theorem 18.** *Almost surely, for all $t \in [0,1]\backslash\mathcal{J}$,*

$$h_M(t) = \frac{1}{\delta_t \cdot \gamma(M_t)}. \tag{13}$$

*In particular, this implies that for every $t \in [0,1]$, $h_M(t) \leq 1/\gamma(M_t)$.*

We introduce, for $f : \mathbb{R}_+ \mapsto \mathbb{R}$ a locally bounded function and $t_0 \in \mathbb{R}_+$,

$$\tilde{h}_f(t_0) := \sup\{\alpha > 0 : \exists C,\ |f(t) - f(t_0)| \leq C|t - t_0|^\alpha \text{ in a neighborhood of } t_0\}.$$

This notion of Hölder exponent of $f$ at $t_0$ is slightly different of that introduced in Definition 1 (which may involve a polynomial). Note that we always have $\tilde{h}_f(t_0) \leq h_f(t_0)$. In the case where $f$ is purely discontinuous and increasing, one might expect that $h_f(t_0) = \tilde{h}_f(t_0)$ in many cases. This is the case when $f$ is a Lévy subordinator without drift. Indeed, from [16, Lemma 4 and Proposition 2], we have a.s.

$$\text{for all } t \in [0,1]\backslash\mathcal{J}, \text{ for all } \alpha \in (0,1),\ \tilde{h}_{X^\alpha}(t) = h_{X^\alpha}(t) = (\delta_t \cdot \alpha)^{-1}$$

(here we use that $X^\alpha$ is a pure jump Lévy process without drift with Lévy measure $\nu^\alpha$ satisfying $\beta_{\nu^\alpha} = \alpha$).

*Proof of Theorem 18: lowerbound.* Let $t \in [0,1]\backslash\mathcal{J}$ and $\varepsilon > 0$ small enough be fixed. By construction, $M$ is continuous at $t$. Since $\gamma$ is also continuous, there exists $\eta > 0$ such that for all $s \in (t-\eta, t+\eta)$, $\gamma(M_s) \in (\gamma(M_t) - \varepsilon, \gamma(M_t) + \varepsilon)$. We deduce from Proposition 14-(ii) that for all $s \in (t - \eta, t)$,

$$0 \leq M_t - M_s \leq X_t^{\alpha_\varepsilon^+} - X_s^{\alpha_\varepsilon^+},$$

where $\alpha_\varepsilon^+ := \gamma(M_t) + \varepsilon$. Similarly, when $s \in (t, t + \eta)$,

$$0 \leq M_s - M_t \leq X_s^{\alpha_\varepsilon^+} - X_t^{\alpha_\varepsilon^+}.$$

Thus applying the definition of $\tilde{h}$, we conclude that $h_M(t) \geq \tilde{h}_M(t) \geq \tilde{h}_{X^{\alpha_\varepsilon^+}}(t) = (\delta_t.\alpha_\varepsilon^+)^{-1}$. Letting $\varepsilon$ go to zero, we deduce that $h_M(t) \geq (\delta_t.\gamma(M_t))^{-1}$. □

To prove an upper bound for $h_M(t)$, we use the following result of [15, Lemma 1].

**Lemma 19.** *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function discontinuous on a dense set of points, and let $t \in \mathbb{R}$. Let $(t_n)_{n\geq1}$ be a real sequence converging to $t$ such that at each $t_n$, $f$ has right and left limits at $t_n$ and $|f(t_n+) - f(t_n-)| = s_n$. Then*

$$h_f(t) \leq \liminf_{n\to+\infty} \frac{\log s_n}{\log|t_n - t|}.$$

*Proof of Theorem 18: upperbound.* By (10) and (12), for every $\varepsilon > 0$, $t \in A_{\delta_t - \varepsilon}$, so that there exists a sequence of jump instants $(T_{n_k})_{k\geq1}$ converging to $t$ such that $|t - T_{n_k}| \leq (\Delta M_{T_{n_k}})^{(\delta_t-\varepsilon)\gamma(M_{T_{n_k}-})}$. Hence, by Lemma 19, we get

$$\begin{aligned} h_M(t) &\leq \liminf_{k\to+\infty} \frac{\log \Delta M_{T_{n_k}}}{\log|T_{n_k} - t|} \leq \liminf_{k\to+\infty} \frac{\log \Delta M_{T_{n_k}}}{\log(\Delta M_{T_{n_k}})^{(\delta_t-\varepsilon)\gamma(M_{T_{n_k}-})}} \\ &\leq \liminf_{k\to+\infty} \frac{1}{(\delta_t - \varepsilon)\gamma(M_{T_{n_k}-})} = \frac{1}{(\delta_t - \varepsilon)\gamma(M_t)}. \end{aligned}$$

The last point comes from the fact that $M$ is continuous at $t$. Letting $\varepsilon$ tend to zero yields that $h_M(t) \leq (\delta_t.\gamma(M_t))^{-1}$.

To finish the proof of Theorem 18, it is clear from (13) and (12) that for all $t \in [0,1]\backslash\mathcal{J}$, $h_M(t) \leq 1/\gamma(M_t)$. Finally, $h_M(t) = 0$ for all $t \in \mathcal{J}$, since $M$ jumps at $t$. □

## 5. Computation of the spectrum: a localized ubiquity theorem

We are now in a position to prove Theorem 7. Recall that the Poisson measure $N = \sum_n \delta_{(T_n,Z_n)}$ has been introduced in Remark 12, that the process $M$ has been built in Proposition 13, and that $\delta_t$ has been introduced in Definition 17.

We will use here the *localized Jarnik-Besicovitch theorem* of [7], that we explain now. We introduced in (10) and (12) the approximation rate of any real number $t \in [0,1]$ by our Poisson point process. In fact, such an approximation rate can be defined for any system of points.

**Definition 20.** *(i) A* system of points $\mathcal{S} = \{(t_n, l_n)\}_{n\geq 1}$ *is a* $[0,1]\times(0,\infty)$*-valued sequence such that* $l_n$ *decreases to* 0 *when* $n$ *tends to infinity.*
*(ii)* $\mathcal{S} = \{(t_n, l_n)\}_{n\geq 1}$ *is said to satisfy the covering property if*

$$\bigcap_{p\geq 1} \bigcup_{n\geq p} B(t_n, l_n) \supset [0,1]. \tag{14}$$

*(ii) For* $t \in [0,1]$*, the approximation rate of* $t$ *by* $\mathcal{S}$ *is defined as*

$$\delta_t = \sup\{\delta \geq 1 : t \textit{ belongs to an infinite number of balls } B(t_n, l_n^\delta)\}. \tag{15}$$

Set $\lambda_n := (1 + Z_n)^{-1}$. Then $\mathcal{P} = \{(T_n, \lambda_n)\}_{n\geq 1}$ is a system of points. Of course formula (15) coincides with formula (12) when the system of points is $\mathcal{P}$. This system $\mathcal{P}$ is a Poisson point process with intensity measure

$$\Lambda(s, \lambda) = \mathbf{1}_{[0,1]\times(0,1)}(s, \lambda) ds \frac{d\lambda}{\lambda^2}. \tag{16}$$

Let us state the result of [7, Theorem 1.3]. The definitions of a *weakly redundant system* and the *condition* **(C)** are recalled in next section. There, the Poisson system $\mathcal{P}$ is shown to enjoy these properties almost surely.

**Theorem 21.** *Consider a weakly redundant system* $\mathcal{S}$ *satisfying the covering property (14) and condition* **(C)**. *Let* $I = (a, b) \subset [0,1]$ *and* $f : I \to [1, +\infty)$ *be continuous at every* $t \in I\backslash\mathcal{Z}$*, for some countable* $\mathcal{Z} \subset [0,1]$*. Consider*

$$S(I, f) = \{t \in I : \ \delta_t \geq f(t)\} \quad \textit{and} \quad \widetilde{S}(I, f) = \{t \in I : \ \delta_t = f(t)\}.$$

*Then*

$$\dim_H S(I, f) = \dim_H \widetilde{S}(I, f) = \sup\{1/f(t) : t \in I\backslash\mathcal{Z}\}.$$

Observe that $\mathcal{P}$ satisfies the covering property due to Proposition 16. We assume for a while that a.s., the Poisson system $\mathcal{P}$ is weakly redundant and fulfills **(C)**.

**Proposition 22.** *Consider the process* $M$ *built in Proposition 13. Almost surely,*

(i) *for all* $I = (a, b) \subset [0,1]$*, all* $\kappa \in (0,1)$*,*

$$\dim_H\{t \in I : \ h_M(t) = \kappa/\gamma(M_t)\} = \dim_H\{t \in I : \ h_M(t) \leq \kappa/\gamma(M_t)\} = \kappa;$$

(ii) *for Lebesgue-almost every* $t \in [0,1]$*,* $h_M(t) = 1/\gamma(M_t)$*.*

*Proof.* By Theorem 18 and since $\mathcal{J}$ is countable, we observe that for $I=(a,b)\subset[0,1]$,

$$\dim_H\{t\in I:\ h_M(t)=\kappa/\gamma(M_t)\}=\dim_H\{t\in I:\ \delta_t=1/\kappa\}.$$

Let $\kappa\in(0,1)$. Since the Poisson system $\mathcal{P}=\{(T_n,\lambda_n)\}_{n\geq 1}$ satisfies all the required conditions, we may apply Theorem 21 with the constant function $f\equiv 1/\kappa$ and get $\dim_H\{t\in I:\ h_M(t)=\kappa/\gamma(M_t)\}=\kappa$. The same arguments hold for $\dim_H\{t\in I:\ h_M(t)\leq\kappa/\gamma(M_t)\}$, which concludes the proof of (i). Next, we write, for $I=(a,b)\subset[0,1]$,

$$I=\Big\{t\in I: h_M(t)=1/\gamma(M(t))\Big\}\bigcup\left(\cup_{n\geq 1}S_n\right), \tag{17}$$

where $S_n:=\Big\{t\in I: h_M(t)\leq(1-1/n)/\gamma(M(t))\Big\}$. By (i), for every $n\geq 1$, the Lebesgue measure of the set $S_n$ is zero since it is of Hausdorff dimension strictly less than 1. We deduce from (17) that for Lebesgue-a.e. $t\in I$, $h_M(t)=1/\gamma(M(t))$. Since this holds for any $I=(a,b)\subset[0,1]$, the conclusion follows. □

*Proof of Theorem 7.* By Remark 8, it suffices to prove (5). Let $I=(a,b)\subset[0,1]$. By Theorem 18, for all $h\geq 0$,

$$D_M(I,h)=\dim_H\{t\in(a,b):\ h_M(t)=h\}=\dim_H\{t\in(a,b):\ \delta_t=(h\cdot\gamma(M_t))^{-1}\}.$$

The jump times $\mathcal{J}$ are countable and of exponents zero for $M$, so they do not interfere in the computation of Hausdorff dimensions.

For $s\in\mathcal{J}\cap(a,b)$ a jump time of $M$ and $h\in(0,1/\gamma(M_{s-}))$, consider the function $f_s$ defined on the interval $I_s=(a,s)$ by $f_s(t)=(h.\gamma(M_t))^{-1}$. This function is continuous on $I_s\backslash\mathcal{J}$, and satisfies, for every $t\in I_s$, $f_s(t)\geq(h\cdot\gamma(M_{s-}))^{-1}\geq 1$. Applying Theorem 21 to the Poisson system $\mathcal{P}=\{(T_n,\lambda_n)\}_{n\geq 1}$ (which satisfies all the required conditions), we obtain

$$\dim_H\{t\in I_s:\ \delta_t=(h\cdot\gamma(M_t))^{-1}\}=\sup\{h\cdot\gamma(M_t): t\in I_s\backslash\mathcal{J}\}=h\cdot\gamma(M_{s-}),$$

since $\gamma(M_t)$ increases to $\gamma(M_{s-})$ as $t$ increases to $s$. Hence, for every $s\in\mathcal{J}\cap(a,b)$, for every $h$ such that $0<h\leq 1/\gamma(M_{s-})$, we have

$$\dim_H\Big\{E_M(h)\cap I_s\Big\}=h\cdot\gamma(M_{s-}).$$

Furthermore, for $s\in\mathcal{J}\cap(a,b)$, when $h\geq 1/\gamma(M_{s-})$, we have $E_M(h)\cap[s,b]=\emptyset$. Indeed, by Theorem 18, for $t\geq s$, $h_M(t)\leq 1/\gamma(M_t)\leq 1/\gamma(M_s)<1/\gamma(M_{s-})$.

Let now $h\geq 0$ be fixed. Then using the density of $\mathcal{J}$,

$$\begin{aligned}E_M(h)\cap I \ &= \left(\bigcup_{s\in\mathcal{J}\cap(a,b),\gamma(M_{s-})<1/h}(E_M(h)\cap(a,s))\right)\\ &\quad\bigcup\left(\bigcup_{s\in\mathcal{J}\cap(a,b),\gamma(M_{s-})\geq 1/h}(E_M(h)\cap[s,b))\right)\end{aligned}$$

As noted previously, the second term of the right hand side is empty. Thus we get, since $\mathcal{J}$ is countable,

$$\begin{aligned} D_M(I,h) &= \sup\{\dim_H(E_M(h)\cap I_s):\ s\in\mathcal{J}\cap I,\ \gamma(M_{s-})<1/h\} \\ &= \sup\{h\cdot\gamma(M_{s-}) : s\in\mathcal{J}\cap I \text{ and } h\cdot\gamma(M_{s-})<1\}, \end{aligned}$$

which was our aim. Observe that if $h\geq 1/\gamma(M_a)$, this formula gives $D_M(I,h)=-\infty$. □

## 6. Study of the distribution of the Poisson point process

To conclude the proof, we only have to check that $\mathcal{P}$ is a weakly redundant system satisfying **(C)**. Recall that $\mathcal{P}=\{(T_n,\lambda_n)\}_{n\geq 1}$ is a Poisson point process with intensity measure (16).

### 6.1. **Weak redundancy and condition (C).**

The weak redundancy property asserts that the balls $B(t_n,l_n)$ naturally associated with a system of points $\mathcal{S}$ do not overlap excessively. The precise definition is the following.

**Definition 23.** *Let $\mathcal{S}=\big\{(t_n,l_n)\big\}_{n\geq 1}$ be a system of points. For any integer $j\geq 0$ we set*

$$\mathcal{T}_j=\big\{n:\quad 2^{-(j+1)}<l_n\leq 2^{-j}\big\}.$$

*Then $\mathcal{S}$ is said to be* weakly redundant *if $t_n\neq t_{n'}$ for all $n\neq n'$ and if there exists a non-decreasing sequence of positive integers $(N_j)_{j\geq 0}$ such that*
*(i)* $\lim_{j\to\infty}(\log_2 N_j)/j=0$.
*(ii) for every $j\geq 1$, $\mathcal{T}_j$ can be decomposed into $N_j$ pairwise disjoint subsets (denoted $\mathcal{T}_{j,1},\dots,\mathcal{T}_{j,N_j}$) such that for each $1\leq i\leq N_j$, the balls $B(t_n,l_n)$, $n\in\mathcal{T}_{j,i}$, are pairwise disjoint.*

In other words, for every $x\in[0,1]$, $x$ cannot belong to more than $N_j$ balls $B(t_n,l_n)$ with $2^{-j-1}<l_n\leq 2^j$.

As shown in [4], Proposition 6.2, $\mathcal{P}$ is weakly redundant.

A weak redundant system do not necessarily satisfy the conclusion of Theorem 21. This is the reason why condition **(C)**, which imposes finer properties on the distribution of the system $\mathcal{S}$, has to be introduced.

We denote by $\Phi$ the set of functions $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that

- $\varphi$ is increasing, continuous and $\varphi(0)=0$,
- $r^{-\varphi(r)}$ is decreasing and tends to $\infty$ as $r$ tends to 0,
- $r^{\alpha-\beta\varphi(r)}$ is increasing near 0 for all $\alpha,\beta>0$.

For example, the function $\varphi(r)=\frac{\log|\log(r)|}{|\log r|}$, defined for $r\geq 0$ close enough to 0, has the required behavior around 0.

**Definition 24.** *Suppose that a system of points $\mathcal{S}=\{(t_n,l_n)\}_{n\geq 1}$ is weakly redundant and adopt the notation of Definition 23. For every $\varphi\in\Phi$ and for any $j\geq 1$, we define*

$$\psi(j,\varphi) \;=\; \max\left\{m\in\mathbb{N} : N_m\cdot 2^m\leq 2^{j(1-\varphi(2^{-j}))}\right\}.$$

Of course, the sequence of integers $(N_j)_{j\geq 1}$ is the one appearing in the Definition 24 of the weak redundancy.

Obviously $\psi(j,\varphi)\leq j$, for every $\varphi$, $(N_j)$ and $j$.

For any dyadic interval $U=[k2^{-j},(k+1)2^{-j})$, we set $g(U)=j$, i.e. $g(U)$ is the dyadic generation of $U$. We denote by $\mathcal{G}_j$ the set of all dyadic intervals of generation $j$. Finally, we denote by $\mathcal{G}_* := \bigcup_{j\geq 0}\mathcal{G}_j$ the set of all dyadic intervals.

**Definition 25.** *Suppose that a system of points $\mathcal{S}=\{(t_n,l_n)\}_{n\geq 1}$ is weakly redundant and adopt the notation of Definitions 23 and 24. Let $\varphi\in\Phi$. For every dyadic interval $V\in\mathcal{G}_j$ and every $\delta>1$, the property $\mathcal{P}(V,\delta)$ is satisfied when there exists an integer $n\in\mathcal{T}_j$ such that $t_n\in V$ and*

$$B\left(t_n,(l_n)^\delta\right) \bigcap \left\{t_p : \begin{array}{l} p\in\mathcal{T}_k, \quad \text{where } k \text{ satisfies} \\ \psi(j,\varphi)\ \leq\ k\ \leq\ -\log_2(l_n)^\delta+4 \end{array}\right\} = \{t_n\}.$$

Let us try to give an intuition of the meaning of $\mathcal{P}(V,\delta)$ for a dyadic interval $V$. $\mathcal{P}(V,\delta)$ holds true when, except $t_n$, the family of points $(t_p)_{p\geq 1}$ "avoids" the ball $B\left(t_n,(l_n)^\delta\right)$ when $p$ describes all the sets $\mathcal{T}_k$, for $k$ ranging between $g(V)$ and $-\log_2(l_n)^\delta+4$, i.e. between the dyadic generations of $B\left(t_n,l_n\right)$ and $B\left(t_n,(l_n)^\delta\right)$.

This condition seems to be reasonable, maybe not for all dyadic intervals, but at least for a large number among them. Condition **(C)** is meant to ensure the validity of $\mathcal{P}(V,\delta)$ for a sufficient set of intervals $V$ and approximation degrees $\delta$.

**Definition 26.** *Suppose that a system of points $\mathcal{S}=\{(t_n,l_n)\}_{n\geq 1}$ is weakly redundant and adopt the notation of Definitions 23 and 24. The system $\mathcal{S}$ is said to satisfy condition* **(C)** *if there exist:*

- *a function $\varphi\in\Phi$,*
- *a continuous function $\kappa:(1,+\infty)\to(0,1]$,*
- *a dense subset $\Delta$ of $(1,\infty)$,*

*such that for every $\delta\in\Delta$, for every dyadic interval $U\in\mathcal{G}_*$, there are infinitely many integers $j\geq g(U)$ satisfying*

$$\#\mathcal{Q}(U,j,\delta)\ \geq\ \kappa(\delta)\cdot 2^{d(j-g(U))},$$

*where*

$$\mathcal{Q}(U,j,\delta)=\left\{V\in\mathcal{G}_j:\ V\subset U \quad \text{and} \quad \mathcal{P}\big(V,\delta\big) \text{ holds}\right\}.$$

6.2. **Proof of (C) for the Poisson process $\mathcal{P}$.** We only need to find a function $\varphi\in\Phi$ and a continuous function $\kappa:(1,+\infty)\to\mathbb{R}_+^*$ such that for every $\delta>1$, with probability 1, for every $U\in\mathcal{G}_*$, there are infinitely many integers $j\geq g(U)$ satisfying $\#\mathcal{Q}(U,j,\delta)\geq\kappa(\delta)\cdot 2^{j-g(U)}$. Then, for any countable and dense subset $\Delta$ of $(1,\infty)$, with probability 1, for every $\delta\in\Delta$, for every $U\in\mathcal{G}_*$, there are infinitely many integers $j\geq g(U)$ satisfying $\#\mathcal{Q}(U,j,\delta)\geq\kappa(\delta)\cdot 2^{j-g(U)}$.

In fact, any $\varphi\in\Phi$ is suitable.

Let $\varphi \in \Phi$ and $\delta > 1$. For $U \in \mathcal{G}_*$ and $V \subset U$ such that $V \in \bigcup_{j>g(U)} \mathcal{G}_j$, let us introduce the event

$$\mathcal{A}(U,V,\delta) = \left\{ \begin{array}{l} \exists\ n \in \mathcal{T}_{g(V)} \text{ such that } \ T_n \in V \text{ and} \\ B(T_n,(\lambda_n)^\delta) \bigcap \left( \bigcup_{\psi(g(V),\varphi) \le k \le h(V)} \mathcal{T}_j \right) = \{T_n\} \end{array} \right\}$$

where $h(V) = \left[\delta(g(V)+1)\right]+4$. Recall that $n \in \mathcal{T}_{g(V)}$ means that $2^{-g(V)-1} < \lambda_n \le 2^{-g(V)}$. By construction, we have the inclusion $\mathcal{A}(U,V,\delta) \subset \{\mathcal{P}(V,\delta) \text{ holds}\}$.

For every $j \ge 1$, let $\widetilde{\mathcal{G}}_j = \left\{[2k \cdot 2^{-j}, (2k+1)\cdot 2^{-j}] : 0 \le k \le 2^j - 1\right\}$. The restrictions of the Poisson point process to the strips $V \times (0,1)$, where $V$ describes $\widetilde{\mathcal{G}}_j$, are independent. Consequently, the events $\mathcal{A}(U,V,\delta)$, when $V \in \widetilde{\mathcal{G}}_j$ and $V \subset U$, are independent (we must separate the intervals in $\widetilde{\mathcal{G}}_j$ because if $V \in \mathcal{G}_j$, $T_n \in V$ and $\lambda_n \le 2^{-j}$, then $B(T_n,(\lambda_n)^\delta)$ may overlap with the neighbors of $V$).

We denote by $X(U,V,\delta)$ the random variable $\mathbf{1}_{\mathcal{A}(U,V,\delta)}$. For a given generation $j > g(U)$, the random variables $(X(U,V,\delta))_{V\in\widetilde{\mathcal{G}}_j}$ are i.i.d Bernoulli variables, whose common parameter is denoted by $p_j(\delta)$. The following Lemma holds.

**Lemma 27.** *There exists a continuous function $\kappa_1 : (1,+\infty) \to (0,1)$ such that for every $j \ge 1$, $p_j(\delta) \ge \kappa_1(\delta)$.*

Let us assume Lemma 27 for the moment. By definition we have

$$\#\mathcal{Q}(U,j,\delta) \ge \sum_{V\in\widetilde{\mathcal{G}}_j:\ V\subset U} X(U,V,\delta).$$

The right hand side of this inequality is a binomial variable of parameters $(2^{j-g(U)}, p_j(\delta))$, with $p_j(\delta) \ge \kappa_1(\delta) > 0$. Consequently, there exists a constant $\kappa(\delta) > 0$ satisfying

$$\mathbb{P}\Big( \sum_{V\in\mathcal{G}_j,\ V\subset U} X(U,V,\delta) \ge \kappa(\delta)\cdot 2^{j-g(U)} \Big) \ge 1/2 \tag{18}$$

provided that $j$ is large enough. The continuity of $\kappa$ with respect to the parameter $\delta > 1$ follows from the continuity of $\kappa_1$.

Let $(j_n)_{n\ge1}$ be the sequence defined inductively by $j_1 = g(U)+1$ and $j_{n+1} = (j_n+1)\delta+5$. We notice that the events $E_n$ defined for $n \ge 1$ by

$$E_n = \{\#\mathcal{Q}(U,j_n,\delta) \ge \kappa(\delta)\cdot 2^{j_n-g(U)}\}$$

are independent. Moreover, (18) implies that $\sum_{n\ge1}\mathbb{P}(E_n) = +\infty$. The Borel-Cantelli Lemma yields that, with probability 1, there is an infinite number of generations $j_n$ satisfying $\#\mathcal{Q}(U,j_n,\delta) \ge \kappa(\delta)\cdot 2^{j-g(U)}$. This holds true for every $U \in \mathcal{G}_*$ almost surely, hence almost surely for every $U \in \mathcal{G}_*$. Condition **(C)** is proved.

We prove Lemma 27. For every $V \in \mathcal{G}_*$, let us introduce the sets

$$S_V = V \times [2^{-(g(V)+1)}, 2^{-g(V)}] \ \text{ and } \ \widetilde{S}_V = V \times [2^{-h(V)}, 2^{-\psi(g(V),\varphi)}].$$

We denote by $N_V$ and $\widetilde{N}_V$ respectively the cardinality of $\mathcal{P}\cap S_V$ and $\mathcal{P}\cap(\widetilde{S}_V \setminus S_V)$. These random variables $N_V$ and $\widetilde{N}_V$ are independent. We set $l_V = \Lambda(S_V)$ and $\widetilde{l}_V = \Lambda(\widetilde{S}_V)$ ($\Lambda$ is the intensity of the Poisson point process (16)). Due to the form of the intensity $\Lambda$,

$N_V$ and $\widetilde{N}_V$ are Poisson random variables of parameter $l_V = 1$ and $\widetilde{l}_V = 2^{-g(V)}\big(2^{h(V)} - 2^{g(V)+1} + 2^{g(V)} - 2^{\psi(g(V),\varphi)}\big)$ respectively. Observe that $\widetilde{l}_V \leq 2^{h(V)-g(V)}$ since by definition $\psi(g(V),\varphi) \leq g(V)$.

We also consider two sequences of random variables in $\mathbb{R}^2$ $(\xi_p = (X_p, Y_p))_{p\geq 1}$ and $(\widetilde{\xi}_q = (\widetilde{X}_q, \widetilde{Y}_q))_{q\geq 1}$ such that

$$\begin{aligned}\mathcal{P} \cap S_V &= \{\xi_p : \ 1 \leq p \leq N_V\}\\ \mathcal{P} \cap (\widetilde{S}_V \setminus S_V) &= \{\widetilde{\xi}_q : \ 1 \leq q \leq \widetilde{N}_V\}.\end{aligned}$$

The event $\mathcal{A}(U,V,\delta)$ contains the event $\widetilde{\mathcal{A}}(U,V,\delta)$ defined as

$$\Big\{\ N_V = 1 \text{ and } B\big(X_1, Y_1^{\delta)}\big) \bigcap \big\{\widetilde{X}_q : 1 \leq q \leq \widetilde{N}_V\big\} = \{X_1\}\Big\},$$

where $\xi_1 = (X_1, Y_1)$. The difference between $\mathcal{A}(U,V,\delta)$ and $\widetilde{\mathcal{A}}(U,V,\delta)$ is that the latter one imposes that there is one, and only one, Poisson point in $S_V$. We have

$$\begin{aligned}&\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))\\ =\ &\mathbb{P}\Big(\Big\{B\big(X_1, Y_1^{\delta}\big) \bigcap \big\{\widetilde{X}_q : 1 \leq q \leq \widetilde{N}_V\big\} = \emptyset \,\Big|\, \big\{N_V = 1\big\}\Big\}\Big)\\ &\times \mathbb{P}(\{N_V = 1\})\\ =\ &\mathbb{P}\Big(\Big\{\forall\, 1 \leq q \leq \widetilde{N}_V,\ \widetilde{X}_q \notin B\big(X_1, Y_1^{\delta}\big)\Big\} \ \Big|\ \big\{N_V = 1\big\}\Big\}\Big) \times e^{-1}.\end{aligned}$$

where $\mathbb{P}(\{N_V = 1\}) = e^{-1}$ since $N_V$ is a Poisson random variable of parameter 1. The random variables $\widetilde{X}_q$ are independant and uniformly distributed in $V$. Thus,

$$\begin{aligned}&\mathbb{P}\Big(\Big\{\forall\, 1 \leq q \leq \widetilde{N}_V,\ \widetilde{X}_q \notin B\big(X_1, Y_1^{\delta}\big)\Big\} \ \Big|\ \big\{N_V = 1\big\}\Big\}\Big)\\ &\geq \mathbb{E}\Big(\Big[1 - \frac{\ell\big(B(X_1, Y_1^{\delta})\big)}{2^{-g(V)}}\Big]^{\widetilde{N}_V}\Big).\end{aligned}$$

Observe that, since $\delta > 1$, provided that $g(V)$ is large enough, conditionally on $\{N_V \geq 1\}$, $\ell\big(B(X_1, Y_1^{\delta})\big) \leq 2^{-g(V)\delta}$. This implies that

$$\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta)) \geq e^{-1} \times \mathbb{E}\Big(\big[1 - 2^{-g(V)(\delta-1))}\big]^{\widetilde{N}_V}\Big). \tag{19}$$

Let us define $\eta_{g(V)} = 2^{-g(V)(\delta-1)}$. Using that $\widetilde{N}_V$ is a Poisson random variable of parameter $\widetilde{l}_V$, a classical calculus shows that (19) can be rewritten as

$$\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta)) \geq e^{-1} e^{-\widetilde{l}_V \cdot \eta_{g(V)}}.$$

By using the definition of $h(V) = \big[(g(V)+1)\delta\big] + 4$, we can get

$$\widetilde{l}_V \cdot \eta_{g(V)} \ \leq \ 2^{h(V)-g(V))} 2^{-g(V)(\delta-1)} \leq 16 \cdot 2^{\delta}.$$

Thus, $\widetilde{l}_V \eta_{g(V)}$ is bounded from above independently of $V$ by a continuous function of $\delta$. Consequently, $\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))$, and thus $\mathbb{P}(\mathcal{A}(U,V,\delta))$, is bounded from below by some quantity $\kappa_1(\delta)$ which is strictly positive and continuously dependent on $\delta > 1$. Lemma 27 is proved.

## 7. Some tangent stable Lévy processes

Let $t_0 \geq 0$ be fixed. Then one observes (recall Theorems 5 and 7) that the local multifractal spectrum $D_M(t_0,.)$ of our process $M$ essentially coincides with the multifractal spectrum of a stable Lévy process with Lévy measure $\gamma(M_{t_0})u^{-1-\gamma(M_{t_0})}du$. A possible explanation for this is that such a stable Lévy process is *tangent* to our process.

**Proposition 28.** *Let $t_0 \geq 0$ be fixed. Conditionally on $\mathcal{F}_{t_0}$, the family of processes $\Big(\frac{M_{t_0+\alpha t}-M_{t_0}}{\alpha^{1/\gamma(M_{t_0})}}\Big)_{t\in[0,1]}$ converges in law, as $\alpha \to 0^+$, to a stable Lévy subordinator with Lévy measure $\gamma(M_{t_0})u^{-1-\gamma(M_{t_0})}du$. Here the Skorokhod space of càdlàg functions on $[0,1]$ is endowed with the uniform convergence topology (which is stronger than the Skorokhod topology).*

One might conjecture that under many restrictive conditions, a result of the following type might hold: if a process $(M_t)_{t\in[0,1]}$ has a tangent process $(Y_t^{t_0})_{t\in[0,1]}$ at time $t_0$, then $D_M(t_0,.)$ coincides with the multifractal spectrum of $Y^{t_0}$. This would allow, for example, to generalize our results to the study of the multifractal spectrum of any reasonable jumping S.D.E. (which is always tangent, in some sense, to a Lévy process).

*Proof.* Using the Markov property, it suffices to treat the case $t_0 = 0$. Let thus $N(ds,dz)$ be a Poisson measure on $[0,1]\times[0,\infty)$ with intensity measure $dsdz$. Recall that

$$M_t = \int_0^t \int_0^\infty (1+z)^{-1/\gamma(M_{s-})} N(ds,dz)$$

and introduce the Lévy processes

$$L_t = \int_0^t \int_0^\infty (1+z)^{-1/\gamma(0)} N(ds,dz), \quad \tilde{L}_t = \int_0^t \int_1^\infty z^{-1/\gamma(0)} N(ds,dz),$$
$$S_t = \int_0^t \int_0^\infty z^{-1/\gamma(0)} N(ds,dz).$$

One immediately checks that $(L_t)_{t\in[0,1]}$ and $(\tilde{L}_t)_{t\in[0,1]}$ have the same law, and that $(S_t)_{t\in[0,1]}$ is a stable Lévy process with Lévy measure $\gamma(0)u^{-1-\gamma(0)}du$. Thus our aim is to prove that $(\alpha^{-1/\gamma(0)}M_{\alpha t})_{t\in[0,1]}$ tends in law to $(S_t)_{t\in[0,1]}$. First, $(\alpha^{-1/\gamma(0)}S_{\alpha t})_{t\in[0,1]}$ has the same law as $(S_t)_{t\in[0,1]}$ for each $\alpha > 0$. Next, it is clear that

$$\mathbb{P}\Big[(\alpha^{-1/\gamma(0)}\tilde{L}_{\alpha t})_{t\in[0,1]} = (\alpha^{-1/\gamma(0)}S_{\alpha t})_{t\in[0,1]}\Big] \geq \mathbb{P}\Big[N([0,\alpha]\times[0,1]) = 0\Big] = e^{-\alpha},$$

which tends to 1 as $\alpha$ tends to 0.

We will now show that $\alpha^{-1/\gamma(0)}\Delta_\alpha$ tends to 0 in probability, where

$$\Delta_t := \sup_{[0,t]} |M_s - L_s| = \int_0^t \int_0^\infty [(1+z)^{-1/\gamma(M_{s-})} - (1+z)^{-1/\gamma(0)}] N(ds,dz),$$

and this will conclude the proof. A first computation, using that $\gamma(y) \leq 1-\varepsilon < 1$ by assumption, shows that for all $t \geq 0$,

$$\mathbb{E}[M_t] = \int_0^t \int_0^\infty \mathbb{E}[(1+z)^{-1/\gamma(M_s)}]dzds \leq \int_0^t \int_0^\infty (1+z)^{-1/(1-\varepsilon)}dzds \leq Ct$$

for some constant $C$. Next, we introduce, for $\eta > 0$, the stopping time $\tau_\eta = \inf\{t \geq 0 : \gamma(M_t) > \gamma(0) + \eta\}$. Denoting by $A$ the Lipschitz constant of $\gamma$, one easily gets

$$\mathbb{P}[\tau_\eta < \alpha] \leq \mathbb{P}[M_\alpha \geq \eta/A] \leq (A/\eta)\mathbb{E}[M_\alpha] \leq CA\alpha/\eta = C_\eta \alpha.$$

Now for $\beta \in (\gamma(0) + \eta, 1]$, by subadditivity, one obtains, for all $t \geq 0$,

$$\begin{aligned}
\mathbb{E}[M^\beta_{t\wedge\tau_\eta}] &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_0^\infty (1+z)^{-\beta/\gamma(M_{s-})} N(ds,dz)\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_0^\infty (1+z)^{-\beta/\gamma(M_s)} dzds\right] \\
&\leq \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_0^\infty (1+z)^{-\beta/(\gamma(0)+\eta)} dzds\right] \leq C_{\eta,\beta} t.
\end{aligned}$$

Let us introduce for $m \geq 0$ the quantity $\kappa(m) := 1/\gamma(0) - 1/\gamma(m) \geq 0$. Still by subadditivity, for $\beta \in (\gamma(0) + \eta, 1]$, for $t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[\Delta^\beta_{t\wedge\tau_\eta}] &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_0^\infty [(1+z)^{-1/\gamma(M_s)} - (1+z)^{-1/\gamma(0)}]^\beta dzds\right] \\
&\leq \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_0^{2^{1/\kappa(M_s)}-1} (1+z)^{-\beta/\gamma(0)}[(1+z)^{\kappa(M_s)} - 1]^\beta dzds\right] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_\eta}\int_{2^{1/\kappa(M_s)}-1}^\infty (1+z)^{-\beta/\gamma(M_s)} dzds\right] =: I^{\beta,\eta}_t + J^{\beta,\eta}_t.
\end{aligned}$$

But for $z \leq 2^{1/\kappa(m)} - 1$, there holds $(1+z)^{\kappa(m)} - 1 \leq C\kappa(m)\log(1+z) \leq Cm\log(1+z)$, the last inequality following from the facts that $\kappa(0) = 0$ and $\kappa$ is Lipschitz-continuous. Hence

$$\begin{aligned}
I^{\beta,\eta}_t &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_\eta} CM^\beta_s \int_0^\infty (1+z)^{-\beta/\gamma(0)}(\log(1+z))^\beta dzds\right] \leq C_\beta \mathbb{E}\left[\int_0^{t\wedge\tau_\eta} M^\beta_s ds\right] \\
&\leq C_\beta \int_0^t \mathbb{E}\left[M^\beta_{s\wedge\tau_\eta}\right] ds \leq C_{\eta,\beta} t^2.
\end{aligned}$$

Next, $\beta/\gamma(M_s) \geq \beta/(\gamma(0) + \eta) > 1$ on $[0, \tau_\eta)$, whence (since $2^{-ax} \leq C_a x$ for all $x > 0$),

$$\begin{aligned}
J^{\beta,\eta}_t &\leq C_{\eta,\beta}\mathbb{E}\left[\int_0^{t\wedge\tau_\eta} 2^{[1-\beta/(\gamma(0)+\eta)]/\kappa(M_s)} ds\right] \leq C_{\eta,\beta}\mathbb{E}\left[\int_0^{t\wedge\tau_\eta} \kappa(M_s) ds\right] \\
&\leq C_{\eta,\beta}\mathbb{E}\left[\int_0^{t\wedge\tau_\eta} M_s ds\right] \leq C_{\eta,\beta}\int_0^t \mathbb{E}\left[M_{s\wedge\tau_\eta}\right] ds \leq C_{\eta,\beta} t^2.
\end{aligned}$$

Again, we used here that $\kappa(0) = 0$ and that $\kappa$ is Lipschitz-continuous. As a conclusion, $\mathbb{E}[\Delta^\beta_{t\wedge\tau_\eta}] \leq C_{\eta,\beta} t^2$.

We may now conclude that for all $\delta > 0$, all $\alpha > 0$, all $\eta > 0$, all $\beta \in (\gamma(0) + \eta, 1]$,

$$\begin{aligned}
\mathbb{P}[\alpha^{-1/\gamma(0)}\Delta_\alpha \geq \delta] &\leq \mathbb{P}[\tau_\eta \geq \alpha] + \mathbb{P}[\Delta_{\alpha\wedge\tau_\eta} \geq \delta\alpha^{1/\gamma(0)}] \\
&\leq C_\eta \alpha + (\delta\alpha^{1/\gamma(0)})^{-\beta} E[\Delta^\beta_{\alpha\wedge\tau_\eta}] \leq C_\eta \alpha + C_{\eta,\delta,\beta}\alpha^{2-\beta/\gamma(0)}.
\end{aligned}$$

Choosing $\eta = \min(\gamma(0), 1-\gamma(0))/2$ and then $\beta \in (\gamma(0)+\eta, 1 \wedge 2\gamma(0))$, we deduce that $2 - \beta/\gamma(0) > 0$, so that $\alpha^{-1/\gamma(0)}\Delta_{\alpha T}$ tends to 0 in probability when $\alpha$ goes to 0, which was our aim. □

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