STABILITY OF THE STOCHASTIC HEAT EQUATION IN $L^1([0,1])$

NICOLAS FOURNIER AND JACQUES PRINTEMS

ABSTRACT. We consider the white-noise driven stochastic heat equation on $[0,\infty)\times[0,1]$ with Lipschitz-continuous drift and diffusion coefficients. We derive an inequality for the $L^1([0,1])$-norm of the difference between two solutions. Using some martingale arguments, we show that this inequality provides some estimates which allow us to study the stability of the solution with respect the initial condition, the uniqueness of the possible invariant distribution and the asymptotic confluence of solutions.

1. Introduction and results

1.1. **The equation.** Consider the stochastic heat equation with Neumann boundary conditions:

$$\left\{\begin{array}{rcl} \partial_t u(t,x) &=& \partial_{xx}u(t,x)+b(u(t,x))+\sigma(u(t,x))\dot W(t,x), \quad t\geq 0,\ x\in[0,1], \\ u(0,x) &=& u_0(x), \quad x\in[0,1], \\ \partial_x u(t,0) &=& \partial_x u(t,1)=0, \quad t>0. \end{array}\right. \tag{1}$$

Here $b,\sigma:\mathbb{R}\mapsto\mathbb{R}$ are the drift and diffusion coefficients and $u_0:[0,1]\mapsto\mathbb{R}$ is the initial condition. We write formally $W(dt,dx)=\dot W(t,x)dtdx$, for $W(dt,dx)$ a white noise on $[0,\infty)\times[0,1]$ based on $dtdx$, see Walsh [14]. We assume in this paper that b,σ are Lipschitz-continuous: for some constant $C_{b,\sigma}$,

$$\text{for all } r,z\in\mathbb{R},\quad |b(r)-b(z)|+|\sigma(r)-\sigma(z)|\leq C_{b,\sigma}|r-z|. \tag{$\mathcal{H}$}$$

Our goals in this paper are the following:

- prove some new stability estimates of the solution with respect to the initial condition $u_0\in L^1([0,1])$;
- study the uniqueness of invariant measures and the asymptotic confluence of solutions.

Let us mention that our results extend without difficulty to the case of Dirichlet boundary conditions and to the case of the unbounded domain $\mathbb{R}$ (with $u_0\in L^1(\mathbb{R})$).

This equation has been much investigated, in particular since the work of Walsh [14]. In [14], one can find definitions of weak solutions, existence and uniqueness results, as well as proofs that solutions are Hölder-continuous, enjoy a Markov property, etc. Let us mention for example the works of Bally-Gyongy-Pardoux [1] (existence of solutions when the drift is only measurable), Gatarek-Goldys [7] (existence of solutions in law), Donati-Pardoux (comparison results and reflection problems), Bally-Pardoux (smoothness of the law of the solution), Bally-Millet-Sanz [3] (support theorem), etc. Sowers [13], Mueller [10] and Cerrai [4] have obtained some results on the invariant distributions and convergence to equilibrium.

We denote by $L^p([0,1])$ the set of all measurable functions $f:[0,1]\mapsto\mathbb{R}$ such that $||f||_{L^p([0,1])}=(\int_0^1|f(x)|^p dx)^{1/p}<\infty$.

1.2. **Mild solutions.** We now define the classical notion of weak solutions we will use, see Walsh [14]. When we refer to predictability, this is with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W, that is $\mathcal{F}_t=\sigma(W(A),A\in\mathcal{B}([0,t]\times[0,1]))$. We denote by $G_t(x,y)$ the Green kernel associated with the heat

2000 *Mathematics Subject Classification.* 60H15, 35B35.

Key words and phrases. Stochastic partial differential equations, White noise, Invariant distributions, Asymptotic confluence, Stability, Long time behavior, Existence.

equation $\partial_t u = \partial_{xx} u$ on $\mathbb{R}_+ \times [0,1]$ with Neumann boundary conditions, whose explicit form can be found in Walsh [14]. Here we will only use that for some constant C_T, for all $x, y \in [0,1]$, all $t \in (0,T]$, see [14],

$$0 \le G_t(x,y) = G_t(y,x) \le \frac{C_T}{\sqrt{t}} e^{-|x-y|^2/4t}, \quad \int_0^1 G_t(x,y)dx = 1, \quad \int_0^1 G_t^2(x,y)dx \le \frac{C_T}{\sqrt{t}}. \tag{2}$$

Definition 1. *Assume ($\mathcal{H}$) and consider $u_0 \in L^1([0,1])$. A predictable process $u = (u(t,x))_{t\ge 0, x\in[0,1]}$ with values in $\mathbb{R}$ is a mild solution to (1) starting from u_0 if*

$$\text{for all } T > 0, \quad \mathbb{E}\left[\int_0^T \int_0^1 u^2(t,x)dxdt\right] < \infty \tag{3}$$

and if a.s., for a.e. $t \ge 0$, $x \in [0,1]$,

$$u(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)[\sigma(u(s,y))W(ds,dy) + b(u(s,y))dyds]. \tag{4}$$

Da Prato-Zabczyk [5] have checked that for any $u_0 \in L^2([0,1])$, there exists a unique mild solution u to (1). Using a similar approach, together with the estimate

$$\int_0^T \int_0^1 \left[\int_0^1 G_t(x,y)u_0(y)dy\right]^2 dxdt \quad \le \quad C_T ||u_0||^2_{L^1([0,1])}, \tag{5}$$

one easily checks that for any $u_0 \in L^1([0,1])$, there exists a unique mild solution to (1). Furthermore, for any pair u, v of mild solutions to (1) starting from $u_0, v_0 \in L^1([0,1])$, for any $T > 0$,

$$\mathbb{E}\left[\int_0^T \int_0^1 (u(t,x) - v(t,x))^2 dxdt\right] \le C_{T,\sigma,b} ||u_0 - v_0||^2_{L^1([0,1])}, \tag{6}$$

where $C_{T,\sigma,b}$ depends only on T and on the Lipschitz constants of σ and b. We will prove (5) and (6) for the sake of completeness, which implies uniqueness. A similar approach allows one to prove existence.

1.3. Stability in $L^1([0,1])$.

Our first goal is to prove the following stability estimates in $L^1([0,1])$.

Theorem 2. *Assume ($\mathcal{H}$). For $u_0, v_0 \in L^1([0,1])$, consider the two mild solutions u and v to (1) starting from u_0 and v_0.*
(i) For all $\gamma \in (0,1)$, all $T \ge 0$, we have

$$\mathbb{E}\left[\sup_{[0,T]} ||u(t) - v(t)||^\gamma_{L^1([0,1])}\right] \le C_{b,\gamma,T} ||u_0 - v_0||^\gamma_{L^1([0,1])},$$

where $C_{b,\gamma,T}$ depends only on b, γ, T.
(ii) Assume now that b is non-increasing. For all $\gamma \in (0,1)$, we have

$$\mathbb{E}\Big[\sup_{[0,\infty)} ||u(t) - v(t)||^\gamma_{L^1([0,1])} + \left(\int_0^\infty ||b(u(t)) - b(v(t))||_{L^1([0,1])}dt\right)^\gamma + \left(\int_0^\infty ||\sigma(u(t)) - \sigma(v(t))||^2_{L^2([0,1])}dt\right)^{\gamma/2}\Big] \quad \le \quad C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])},$$

where C_γ depends only on γ.

Point (i), which shows some stability with respect to the initial condition, is complementary to (6): we can consider a supremum in time, but we can only deal with the L^1-norm. Point (ii) is quite strong, since it provides some estimates on the time interval $[0,\infty)$. This will allow us to study the large time behavior of solutions when b is non-increasing. To prove these estimates, we rather use the *weak* form of (1). It seems difficult to obtain such results working directly with the *mild* equation.

1.4. **Large time behavior.** We now wish to study the uniqueness of invariant measures.

Definition 3. *A probability measure Q on $L^1([0,1])$ is said to be an invariant distribution for (1) if, for u_0 a $L^1([0,1])$-valued random variable with law Q independent of W, for u the mild solution to (1) starting from u_0, $\mathcal{L}(u(t)) = Q$ for all $t \geq 0$.*

Theorem 4. *Assume $(\mathcal{H})$, that b is non-increasing and that $(\sigma, b) : \mathbb{R} \mapsto \mathbb{R}^2$ is injective. Then (1) admits at most one invariant distribution.*

To prove the asymptotic confluence of solutions, we need to strengthen the injectivity assumption.

$$(\mathcal{I}) \qquad \begin{cases} \text{There is a strictly increasing convex function } \rho : \mathbb{R}_+ \mapsto \mathbb{R}_+ \text{ with } \rho(0) = 0 \text{ such that} \\ \text{for all } r, z \in \mathbb{R}, \quad |b(r) - b(z)| + |\sigma(r) - \sigma(z)|^2 \geq \rho(|r - z|). \end{cases}$$

Theorem 5. *Assume $(\mathcal{H})$, that b is non-increasing and $(\mathcal{I})$.*
(i) The following asymptotic confluence property holds: for $u_0, v_0 \in L^1([0,1])$, for u, v the mild solutions to (1) starting from u_0 and v_0,

$$a.s., \qquad \lim_{t \to \infty} ||u(t) - v(t)||_{L^1([0,1])} = 0.$$

(ii) Assume additionally that (1) admits an invariant distribution Q. Then for any $u_0 \in L^1([0,1])$, for u the mild solution to (1) starting from u_0, $u(t)$ goes in law to Q as $t \to \infty$.

Clearly, $(\mathcal{I})$ holds if b is C^1 with $b' \leq -\epsilon < 0$ (choose $\rho(z) = \epsilon z$) or if σ is C^1 with $|\sigma'| \geq \epsilon > 0$ (choose $\rho(z) = (\epsilon z)^2$). One may also combine conditions on b and σ.
But $(\mathcal{I})$ also holds if b is C^1 and if $b' \leq 0$ vanishes reasonably. For example if $b(z) = -\text{sg}(z) \min(|z|, |z|^p)$ for some $p \geq 1$, choose $\rho = \epsilon \rho_p$ with ϵ small enough and $\rho_p(z) = z^p$ for $z \in [0,1]$ and $\rho_p(z) = pz - p + 1$ for $z \geq 1$. If $b(z) = -z - \sin z$, choose $\rho = \epsilon \rho_3$ with ϵ small enough.
One may also consider the case where σ is monotonous with σ' vanishing reasonably.

Observe that the asymptotic confluence of solutions may hold even if there is no invariant distribution for (1). Consider e.g. the case where $b(x) \geq c > 0$ is non-increasing and $\sigma(x) = x$. Then $(\mathcal{I})$ holds, while clearly, solutions should tend to infinity (due to the uniformly positive drift term).

Let us now compare Theorems 4 and 5 with known results. The works cited below sometimes concern different boundary conditions, but we believe this is not important.

• Sowers [13] has proved the existence of an invariant distribution supported by $C([0,1])$, assuming $(\mathcal{H})$, that σ is bounded and that b is of the form $b(z) = -\alpha z + f(z)$, for some bounded f and some $\alpha > 0$. He obtained uniqueness of this invariant distribution when σ is sufficiently small and bounded from below.

• Mueller [10] has obtained some surprising coupling results, implying in particular the uniqueness of an invariant distribution as well as a the trend to equilibrium. He assumes $(\mathcal{H})$, that σ is bounded from above and from below and that b is non-increasing, with $|b(z) - b(r)| \geq \alpha|z - r|$ for some $\alpha > 0$.

• Cerrai [4] assumed that σ is strictly monotonous (it may vanish, but only at one point).
(i) She obtained an asymptotic confluence result which we do not recall here and concerns, roughly, the case $b(z) \simeq -\text{sg}(z)|z|^m$ as $z \to \pm\infty$, for some $m > 1$.
(ii) Assuming $(\mathcal{H})$, she proved uniqueness of the invariant distribution as well as an asymptotic confluence property, under the conditions that for all $r \leq z$, $b(z) - b(r) \leq \lambda(z - r)$, and $|\sigma(z) - \sigma(r)| \geq \mu|z - r|$, for some $\mu > 0$ and some $\lambda < \mu^2/2$ (if b is non-increasing, choose $\lambda = 0$).

Thus the main advantages of the present paper are that the uniqueness of the invariant measure requires very few conditions, and we allow σ to vanish (it may be compactly supported).

Example 1. Assume $(\mathcal{H})$ and that b strictly decreasing. Then there exists at most one invariant distribution. If $b(z) = -z$ or $b(z) = -z - \sin z$ or $b(z) = -\text{sg}(z) \min(|z|, |z|^p)$ for some $p > 1$, then we have asymptotic confluence of solutions. Here to apply [13, 10] one needs to assume additionally that σ is bounded from above and from below, while to apply [4], one has to suppose that σ is strictly monotonous.

Example 2. Assume $(\mathcal{H})$, that b is non-increasing and that σ is strictly monotonous. Then there exists at most one invariant distribution.
If furthermore σ is C^1 with $0 < c < \sigma' < C$, then we get asymptotic confluence of solutions using [4] or Theorem 5 (here [13, 10] cannot apply, since σ vanishes). But now if $\sigma' \geq 0$ reasonably vanishes then Theorem 5 applies, which is not the case of [4]: take e.g. $\sigma(z) = \text{sg}(z)\min(|z|, |z|^p)$ for some $p > 1$, or $\sigma(z) = z + \sin z$.

Example 3. Consider the compactly supported coefficient $\sigma(z) = (1 - z^2)\mathbf{1}_{\{|z|\leq 1\}}$. Assume that b is C^1, non-increasing, with $b'(z) \leq -\epsilon < 0$ for $z \in (-\infty, -1) \cup \{0\} \cup (1, +\infty)$. Then Theorems 4 and 5 apply, while [13, 10, 4] do not.
Observe here that if $b(z_0) = 0$ for some $z_0 \notin (-1, 1)$, then $u(t) \equiv z_0$ is the (unique) stationary solution. If now $b(-1) > 0$ and $b(1) < 0$, then the invariant measure Q (that exists due to Sowers [13]) is unique and one may show, using the comparison Theorem of Donati-Pardoux [6], that Q is supported by $[-1, 1]$-valued continuous functions on $[0, 1]$.

However, there are some cases where [13, 4] provide some better results than ours.

Example 4. If $\sigma(z) = \mu z$ and $b(z) = \lambda z$, then $u(t) \equiv 0$ is an obvious stationary solution. Theorems 4 and 5 apply if $\lambda \leq 0$ and $|\lambda| + |\mu| > 0$. Cerrai [4] was able to treat the case $\lambda > 0$ provided $\mu^2/2 > \lambda$.

Example 5. If σ is small enough and bounded from below and if $b(z) = -\alpha z + h(z)$, with $\alpha > 0$ and h bounded, then Sowers [13] obtains the uniqueness of the invariant distribution even if b is not non-increasing.

1.5. **Plan of the paper.** In the next section, we briefly prove (5) and (6). Section 3 is devoted to the proof of Theorem 2. Theorems 4 and 5 are checked in Section 4. We briefly discuss the multi-dimensional equation in Section 5 and conclude the paper with an appendix containing technical results.

2. Mild solutions

In the whole section, $T > 0$ is fixed. For $u_0 \in L^1([0,1])$ and $t \in [0, T]$, we easily get, using the Cauchy-Schwarz inequality in y and integrating in x using (2):

$$(7)\quad \int_0^1 \left(\int_0^1 G_t(x,y)u_0(y)dy \right)^2 dx \quad \leq \quad ||u_0||_{L^1([0,1])} \int_0^1 \int_0^1 G_t^2(x,y)|u_0(y)|dydx \leq \frac{C_T}{\sqrt{t}} ||u_0||^2_{L^1([0,1])}.$$

This implies (5). Consider now two mild solutions u, v to (1) starting from $u_0, v_0 \in L^1([0,1])$. Put $\delta(t,x) = \mathbb{E}[|u(t,x) - v(t,x)|^2]$, $d(t) = \int_0^1 \delta(t,x)dx$ and $D(t) = \int_0^t d(s)ds$. A straightforward computation using (4) and $(\mathcal{H})$ shows that for all $x \in [0,1]$, all $t \in [0,T]$,

$$\delta(t,x) \leq 2 \left(\int_0^1 G_t(x,y)|u_0(y) - v_0(y)|dy \right)^2 + C_{T,\sigma,b} \int_0^t \int_0^1 G^2_{t-s}(x,y)\delta(s,y)dyds.$$

Integrating in x using (7) and (2), we deduce that for all $t \in [0,T]$,

$$d(t) \leq \frac{C_T}{\sqrt{t}} ||u_0 - v_0||^2_{L^1([0,1])} + C_{T,\sigma,b} \int_0^t \frac{d(s)}{\sqrt{t-s}} ds.$$

Iterating once this formula and using that for all $0 \leq r \leq t$, $\int_r^t \frac{ds}{\sqrt{t-s}\sqrt{s-r}} = \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} < \infty$, we get

$$\begin{aligned} d(t) \quad &\leq \quad \frac{C_T}{\sqrt{t}} ||u_0 - v_0||^2_{L^1([0,1])} + C_{T,\sigma,b} \int_0^t \frac{||u_0 - v_0||^2_{L^1([0,1])}}{\sqrt{s}\sqrt{t-s}} ds + C_{T,\sigma,b} \int_0^t \int_0^s \frac{d(r)}{\sqrt{t-s}\sqrt{s-r}} drds \\ &\leq \quad \frac{C_{T,\sigma,b}}{\sqrt{t}} ||u_0 - v_0||^2_{L^1([0,1])} + C_{T,\sigma,b} \int_0^t d(r)dr. \end{aligned}$$

Finally, integrating in t yields $D(t) \leq C_{T,\sigma,b} ||u_0 - v_0||^2_{L^1([0,1])} + C_{T,\sigma,b} \int_0^t D(s)ds$. The Gronwall Lemma implies that $D(T) \leq C_{T,\sigma,b} ||u_0 - v_0||^2_{L^1([0,1])}$. We have checked (6).

3. On the $L^1([0,1])$-norm of the difference between two mild solutions

The aim of this section is to prove Theorem 2. We first consider continuous initial conditions, and then extend the estimates to general $L^1([0,1])$ initial conditions. All our study is based on the following result. We set $\mathrm{sg}(z)=1$ for $z\geq 0$ and $\mathrm{sg}(z)=-1$ for $z<0$.

Proposition 6. *Assume $(\mathcal{H})$. For two continuous initial conditions u_0, v_0, let u, v be the corresponding mild solutions to (1). Then, enlarging the probability space if necessary, there is a Brownian motion $(B_t)_{t\geq 0}$ such that a.s., for all $t\geq 0$,*

$$\begin{aligned} ||u(t)-v(t)||_{L^1([0,1])} \quad &\leq \quad ||u_0-v_0||_{L^1([0,1])} + \int_0^t ||\sigma(u(s))-\sigma(v(s))||_{L^2([0,1])} dB_s \\ &\quad + \int_0^t \int_0^1 \mathrm{sg}(u(s,x)-v(s,x))(b(u(s,x))-b(v(s,x)))dxds. \end{aligned} \tag{8}$$

Proof. We divide the proof into several steps, following closely the ideas of Donati-Pardoux [6, Theorem 2.1], to which we refer for technical details.

Step 1. Consider an orthonormal basis $(e_k)_{k\geq 1}$ of $L^2([0,1])$. For $k\geq 1$, we set $B^k_t = \int_0^t\int_0^1 e_k(x)W(ds,dx)$. Then $(B^k)_{k\geq 1}$ is a family of independent Brownian motions. For $n\geq 1$, consider the unique adapted solution $u^n\in L^2(\Omega\times[0,T],V)$, where $V=\{f\in H^1([0,1]), f'(0)=f'(1)=0\}$, to

$$u^n(t,x) = u_0(x) + \int_0^t \left[\partial_{xx}u^n(s,x)ds + b(u^n(s,x))\right]ds + \sum_{k=1}^n \int_0^t \sigma(u^n(s,x))e_k(x)dB^k_s.$$

We refer to Pardoux [11] for existence, uniqueness and properties of this solution. We also consider the solution v^n to the same equation starting from v_0. Then, as shown in [6],

$$\lim_n \sup_{[0,T]\times[0,1]} \mathbb{E}[|u^n(t,x)-u(t,x)|^2 + |v^n(t,x)-v(t,x)|^2] = 0. \tag{9}$$

Step 2. For $\epsilon>0$, we introduce a nonnegative C^2 function ϕ_ϵ such that $\phi_\epsilon(z)=|z|$ for $|z|\geq\epsilon$, with $|\phi'_\epsilon(z)|\leq 1$ and $0\leq\phi''_\epsilon(z)\leq 2\epsilon^{-1}\mathbf{1}_{\{|z|<\epsilon\}}$. When applying the Itô formula (see [6] for details), we get

$$\begin{aligned} \int_0^1 \phi_\epsilon(u^n(t,x)-v^n(t,x))dx &= \int_0^1 \phi_\epsilon(u_0(x)-v_0(x))dx \\ &+ \int_0^t\int_0^1 \phi'_\epsilon(u^n(s,x)-v^n(s,x))\partial_{xx}[u^n(s,x)-v^n(s,x)]dxds \\ &+ \int_0^t\int_0^1 \phi'_\epsilon(u^n(s,x)-v^n(s,x))[b(u^n(s,x))-b(v^n(s,x))]dxds \\ &+ \sum_{k=1}^n\int_0^t\int_0^1 \phi'_\epsilon(u^n(s,x)-v^n(s,x))[\sigma(u^n(s,x))-\sigma(v^n(s,x))]e_k(x)dxdB^k_s \\ &+ \frac{1}{2}\sum_{k=1}^n\int_0^t\int_0^1 \phi''_\epsilon(u^n(s,x)-v^n(s,x))[\sigma(u^n(s,x))-\sigma(v^n(s,x))]^2e_k^2(x)dxds \\ &=: I^1_\epsilon + I^2_\epsilon(t) + I^3_\epsilon(t) + I^4_\epsilon(t) + I^5_\epsilon(t). \end{aligned} \tag{10}$$

Since $|z|\leq\phi_\epsilon(z)\leq|z|+\epsilon$ for all z, we easily get, a.s.,

$$\lim_{\epsilon\to 0}\int_0^1 \phi_\epsilon(u^n(t,x)-v^n(t,x))dx = ||u^n(t)-v^n(t)||_{L^1([0,1])} \quad \text{and} \quad \lim_{\epsilon\to 0} I^1_\epsilon = ||u_0-v_0||_{L^1([0,1])}.$$

An integration by parts, using that $\partial_x[u^n(t,0)-v^n(t,0)]=\partial_x[u^n(t,1)-v^n(t,1)]=0$ shows that

$$I^2_\epsilon(t) = -\int_0^t\int_0^1 \phi''_\epsilon(u^n(s,x)-v^n(s,x))[\partial_x(u^n(s,x)-v^n(s,x))]^2 \leq 0.$$

Since $\phi''_\epsilon(z-r)(\sigma(z)-\sigma(r))^2 \leq C\epsilon^{-1}\mathbf{1}_{\{|z-r|\leq\epsilon\}}|z-r|^2 \leq C\epsilon$ by $(\mathcal{H})$, we have $I^5_\epsilon(t) \leq Cnt\epsilon$, whence

$$\lim_{\epsilon\to 0} I^5_\epsilon(t) = 0 \text{ a.s.}$$

Using that $|\phi'_\epsilon(z) - \text{sg}(z)| \leq \mathbf{1}_{\{|z|\leq\epsilon\}}$ and $(\mathcal{H})$, one obtains a.s.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|I^3_\epsilon(t) - \int_0^t\int_0^1 \text{sg}(u^n(s,x)-v^n(s,x))(b(u^n(s,x))-b(v^n(s,x)))dxds\right| \\
\leq\ & \lim_{\epsilon\to 0}\int_0^t\int_0^1 \mathbf{1}_{\{|u^n(s,x)-v^n(s,x)|\leq\epsilon\}}|b(u^n(s,x))-b(v^n(s,x))|dxds \leq \lim_{\epsilon\to 0} Ct\epsilon = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\mathbb{E}\left[\left(I^4_\epsilon(t) - \sum_{k=1}^n\int_0^t\int_0^1 \text{sg}(u^n(s,x)-v^n(s,x))[\sigma(u^n(s,x))-\sigma(v^n(s,x))]e_k(x)dxdB^k_s\right)^2\right] \\
\leq\ & \lim_{\epsilon\to 0}\mathbb{E}\left[\sum_{k=1}^n\int_0^t\int_0^1 \mathbf{1}_{\{|u^n(s,x)-v^n(s,x)|\leq\epsilon\}}[\sigma(u^n(s,x))-\sigma(v^n(s,x))]^2e_k^2(x)dxds\right] \\
\leq\ & \lim_{\epsilon\to 0} Ctn\epsilon^2 = 0.
\end{aligned}$$

Thus we can pass to the limit as $\epsilon\to 0$ in (10) and get, a.s.,

$$\begin{aligned}
||u^n(t)-v^n(t)||_{L^1([0,1])} \leq\ & ||u_0-v_0||_{L^1([0,1])} \\
&+\int_0^t\int_0^1 \text{sg}(u^n(s,x)-v^n(s,x))[b(u^n(s,x))-b(v^n(s,x))]dxds \\
&+\sum_{k=1}^n\int_0^t\int_0^1 \text{sg}(u^n(s,x)-v^n(s,x))[\sigma(u^n(s,x))-\sigma(v^n(s,x))]e_k(x)dxdB^k_s. \qquad (11)
\end{aligned}$$

Step 3. Using $(\mathcal{H})$, there holds, for all r_1, z_1, r_2, z_2 in $\mathbb{R}$,

$$\big|\text{sg}(r_1-z_1)[\sigma(r_1)-\sigma(z_1)] - \text{sg}(r_2-z_2)[\sigma(r_2)-\sigma(z_2)]\big| \leq C(|r_1-r_2|+|z_1-z_2|), \qquad (12)$$

$$\big|\text{sg}(r_1-z_1)[b(r_1)-b(z_1)] - \text{sg}(r_2-z_2)[b(r_2)-b(z_2)]\big| \leq C(|r_1-r_2|+|z_1-z_2|). \qquad (13)$$

Indeed, it suffices, by symmetry, to check that $\big|\text{sg}(r_1-z_1)[\sigma(r_1)-\sigma(z_1)] - \text{sg}(r_2-z_1)[\sigma(r_2)-\sigma(z_1)]\big| \leq C|r_1-r_2|$. If $\text{sg}(r_1-z_1) = \text{sg}(r_2-z_2)$, this is obvious. If now $r_1\leq z_1\leq r_2$ (or $r_1\geq z_1\geq r_2$) we get the upper-bound $|\sigma(r_1)+\sigma(r_2)-2\sigma(z_1)| \leq C(|r_1-z_1|+|r_2-z_1|) = C|r_1-r_2|$.
Using (9), it is thus routine to make n tend to infinity in (11) and to obtain, a.s.,

$$\begin{aligned}
||u(t)-v(t)||_{L^1([0,1])} \leq\ & ||u_0-v_0||_{L^1([0,1])} + \int_0^t\int_0^1 \text{sg}(u(s,x)-v(s,x))[b(u(s,x))-b(v(s,x))]dxds \\
&+\sum_{k=1}^\infty\int_0^t\int_0^1 \text{sg}(u(s,x)-v(s,x))[\sigma(u(s,x))-\sigma(v(s,x))]e_k(x)dxdB^k_s. \qquad (14)
\end{aligned}$$

For the last term, we used that, by the Plancherel identity, setting for simplicity

$$\begin{aligned}
\alpha_n(s,x) &= \text{sg}(u^n(s,x)-v^n(s,x))[\sigma(u^n(s,x))-\sigma(v^n(s,x))], \\
\alpha(s,x) &= \text{sg}(u(s,x)-v(s,x))[\sigma(u(s,x))-\sigma(v(s,x))],
\end{aligned}$$

there holds

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{k=1}^n \int_0^t\int_0^1 \alpha_n(s,x)e_k(x)dB_s^k - \sum_{k=1}^\infty \int_0^t\int_0^1 \alpha(s,x)e_k(x)dB_s^k\Big)^2\Big]\\
\le\ & \int_0^t \mathbb{E}\Big[\sum_{k\ge1}\Big(\int_0^1 \Big\{\alpha_n(s,x)-\alpha(s,x)\Big\}e_k(x)dx\Big)^2\Big]ds + \sum_{k\ge n+1}\int_0^t \mathbb{E}\Big[\Big(\int_0^1 \alpha(s,x)e_k(x)dx\Big)^2\Big]ds\\
\le\ & \int_0^t \mathbb{E}\Big[||\alpha_n(s)-\alpha(s)||^2_{L^2([0,1])}\Big]ds + \sum_{k\ge n+1}\int_0^t \mathbb{E}\Big[\Big(\int_0^1 \alpha(s,x)e_k(x)dx\Big)^2\Big]ds =: I_n(t)+J_n(t).
\end{aligned}$$

Using (12) and then (9), $I_n(t) \le C\int_0^t\int_0^1 \mathbb{E}[|u^n(s,x)-u(s,x)|^2 + |v^n(s,x)-v(s,x)|^2]dxds$ tends to 0 as $n\to\infty$. Finally, $J_n(t)$ tends to 0 because $\sum_{k\ge1}\int_0^t \mathbb{E}[(\int_0^1 \alpha(s,x)e_k(x)dx)^2]ds = \int_0^t \mathbb{E}[||\alpha(s)||^2_{L^2([0,1])}]ds \le C\int_0^t\int_0^1 \mathbb{E}(|u(s,x)-v(s,x)|^2)dxds < \infty$.

Step 4. A standard representation argument (see e.g. Revuz-Yor [12, Proposition 3.8 and Theorem 3.9 p 202-203]) concludes the proof, because the last term on the RHS of (14) is a continuous local martingale with bracket

$$\int_0^t \sum_{k=1}^\infty \Big(\int_0^1 \text{sg}(u(s,x)-v(s,x))[\sigma(u(s,x))-\sigma(v(s,x))]e_k(x)dx\Big)^2 ds = \int_0^t ||\sigma(u(s))-\sigma(v(s))||^2_{L^2([0,1])}ds.$$

We used here again that $(e_k)_{k\ge1}$ is an orthonormal basis of $L^2([0,1])$. □

Corollary 7. *Adopt the notation and assumptions of Proposition 6.*
(i) For all $\gamma\in(0,1)$, all $T\ge0$,

$$\mathbb{E}\left[\sup_{[0,T]} ||u(t)-v(t)||^\gamma_{L^1([0,1])}\right] \le C_{b,\gamma,T}||u_0-v_0||^\gamma_{L^1([0,1])},$$

where $C_{b,\gamma,T}$ depends only on b,γ,T.
(ii) Assume that b is non-increasing. Then for all $\gamma\in(0,1)$,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{[0,\infty)} ||u(t)-v(t)||^\gamma_{L^1([0,1])} + \Big(\int_0^\infty ||b(u(t))-b(v(t))||_{L^1([0,1])}dt\Big)^\gamma&\\
+ \Big(\int_0^\infty ||\sigma(u(t))-\sigma(v(t))||^2_{L^2([0,1])}dt\Big)^{\gamma/2}\Big] \quad &\le\quad C_\gamma||u_0-v_0||^\gamma_{L^1([0,1])},
\end{aligned}$$

where C_γ depends only on γ.

Proof. Let us first prove point (i). Let C_b be the Lipschitz constant of b. Denote by L_t the RHS of (8). The Itô formula yields

$$\begin{aligned}
||u(t)-v(t)||_{L^1([0,1])}e^{-C_bt} \quad&\le\quad L_te^{-C_bt}\\
&=\quad ||u_0-v_0||_{L^1([0,1])} - C_b\int_0^t e^{-C_bs}L_sds\\
&\quad\ + \int_0^t ||\sigma(u(s))-\sigma(v(s))||_{L^2([0,1])}e^{-C_bs}dB_s\\
&\quad\ + \int_0^t\int_0^1 e^{-C_bs}\text{sg}(u(s,x)-v(s,x))(b(u(s,x))-b(v(s,x)))dxds.
\end{aligned}$$

But $\int_0^1 \text{sg}(u(s,x)-v(s,x))(b(u(s,x))-b(v(s,x)))dx \le C_b||u(s)-v(s)||_{L^1([0,1])} \le C_bL_s$. Hence

$$||u(t)-v(t)||_{L^1([0,1])}e^{-C_bt} \quad\le\quad ||u_0-v_0||_{L^1([0,1])} + \int_0^t ||\sigma(u(s))-\sigma(v(s))||_{L^2([0,1])}e^{-C_bs}dB_s =: M_t.$$

The process M_t is a nonnegative continuous local martingale starting from $||u_0 - v_0||_{L^1([0,1])}$, whence by Lemma 8, for any $\gamma \in (0,1)$, $\mathbb{E}\left[\sup_{[0,\infty)} M_t^\gamma\right] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}$. We deduce that

$$\mathbb{E}\left[\sup_{[0,\infty)} \left(||u(t)-v(t)||^\gamma_{L^1([0,1])} e^{-C_b\gamma t}\right)\right] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}.$$

Point (i) immediately follows (with $C_{b,\gamma,T} = C_\gamma e^{C_b \gamma T}$).

We now check point (ii). Since b is non-increasing, Proposition 6 yields

$$\begin{aligned} &||u(t)-v(t)||_{L^1([0,1])} + \int_0^t ||b(u(s)) - b(v(s))||_{L^1([0,1])} ds \\ \le\ & ||u_0 - v_0||_{L^1([0,1])} + \int_0^t ||\sigma(u(s)) - \sigma(v(s))||_{L^2([0,1])} dB_s =: N_t, \end{aligned}$$

which is a nonnegative continuous local martingale starting from $||u_0 - v_0||_{L^1([0,1])}$ with bracket $\langle N \rangle_t = \int_0^t ||\sigma(u(s)) - \sigma(v(s))||^2_{L^2([0,1])} ds$. Due to Lemma 8, $\mathbb{E}[\sup_{[0,\infty)} N_t^\gamma + \langle N \rangle_\infty^{\gamma/2}] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}$ holds true for all $\gamma \in (0,1)$. Hence

$$\begin{aligned} \mathbb{E}\left[\sup_{[0,\infty)} ||u(t)-v(t)||^\gamma_{L^1([0,1])}\right] &\le \mathbb{E}\left[\sup_{[0,\infty)} N_t^\gamma\right] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}, \\ \mathbb{E}\left[\left(\int_0^\infty ||b(u(t)) - b(v(t))||_{L^1([0,1])} dt\right)^\gamma\right] &\le \mathbb{E}\left[\sup_{[0,\infty)} N_t^\gamma\right] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}, \\ \mathbb{E}\left[\left(\int_0^\infty ||\sigma(u(t)) - \sigma(v(t))||^2_{L^2([0,1])} dt\right)^{\gamma/2}\right] &= \mathbb{E}\left[\langle N \rangle_\infty^{\gamma/2}\right] \le C_\gamma ||u_0 - v_0||^\gamma_{L^1([0,1])}. \end{aligned}$$

This ends the proof. ☐

Finally, we extend the previous estimates to general initial conditions.

Proof of Theorem 2. We divide the proof into two steps.

Step 1. Let u be the mild solution to (1) starting from $u_0 \in L^1([0,1])$. Consider a sequence of continuous initial conditions $(u_0^n)_{n\ge1}$ such that $||u_0^n - u_0||_{L^1([0,1])} \le 2^{-n}$, and the corresponding sequence $(u^n)_{n\ge1}$ of mild solutions to (1) starting from u_0^n. The aim of this step is to check that for any $T > 0$,

$$a.s., \quad \lim_{n\to\infty} \sup_{[0,T]} ||u^n(t) - u(t)||_{L^1([0,1])} = 0, \tag{15}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T ||u^n(t) - u(t)||^2_{L^2([0,1])} dt\right] = 0. \tag{16}$$

First, (16) follows immediately from (6). Next, we use Corollary 7-(i) (with $\gamma = 1/2$) to deduce that

$$\mathbb{E}\left[\sum_{n\ge1} \sup_{[0,T]} ||u^{n+1}(t) - u^n(t)||^{1/2}_{L^1([0,1])}\right] \le C_{b,\gamma,T} \sum_{n\ge1} ||u_0^{n+1} - u_0^n||^{1/2}_{L^1([0,1])} < \infty.$$

As a consequence, $\sum_{n\ge1} \sup_{[0,T]} ||u^{n+1}(t) - u^n(t)||^{1/2}_{L^1([0,1])} < \infty$ a.s., which finally implies that a.s., $\sum_{n\ge1} \sup_{[0,T]} ||u^{n+1}(t) - u^n(t)||_{L^1([0,1])} < \infty$. Using some completeness arguments, we deduce that there is a process v such that a.s., for all $T > 0$, $\lim_n \sup_{[0,T]} ||u^n(t) - v(t)||_{L^1([0,1])} = 0$. But recalling (16), we necessarily have $v = u$, which ends the step.

Step 2. We now prove the desired estimates. For u_0 and v_0 in $L^1([0,1])$, we consider u_0^n and v_0^n continuous with $||u_0^n - u_0||_{L^1([0,1])} + ||v_0^n - v_0||_{L^1([0,1])} \le 2^{-n}$. We denote by u, v, u^n, v^n the corresponding mild solutions to (1). For each $n \ge 1$, we may apply Corollary 7-(i) to u^n, v^n. Using the Fatou Lemma and

Step 1, we deduce that for all $T>0$, all $\gamma\in(0,1)$,

$$\begin{aligned}\mathbb{E}\left[\sup_{[0,T]}||u(t)-v(t)||^\gamma_{L^1([0,1])}\right] &\leq \liminf_n \mathbb{E}\left[\sup_{[0,T]}||u^n(t)-v^n(t)||^\gamma_{L^1([0,1])}\right]\\ &\leq \liminf_n C_{b,\gamma,T}||u^n_0-v^n_0||^\gamma_{L^1([0,1])} = C_{b,\gamma,T}||u_0-v_0||^\gamma_{L^1([0,1])}.\end{aligned}$$

If furthermore b is non-increasing, we get similarly, using Corollary 7-(ii), for any $T>0$, any $\gamma\in(0,1)$,

$$\mathbb{E}\left[\sup_{[0,T]}||u(t)-v(t)||^\gamma_{L^1([0,1])}\right]\leq C_\gamma||u_0-v_0||^\gamma_{L^1([0,1])}.$$

Since C_γ does not depend on T, we can let T increase to infinity. Next, using again Step 1, Corollary 7-(ii) and $(\mathcal{H})$, we deduce that for any $T>0$, any $\gamma\in(0,1)$,

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T ||b(u(t))-b(v(t))||_{L^1([0,1])}dt\right)^\gamma+\left(\int_0^T ||\sigma(u(t))-\sigma(v(t))||^2_{L^2([0,1])}dt\right)^{\gamma/2}\right]\\ =\;& \lim_n \mathbb{E}\left[\left(\int_0^T ||b(u^n(t))-b(v^n(t))||_{L^1([0,1])}dt\right)^\gamma+\left(\int_0^T ||\sigma(u^n(t))-\sigma(v^n(t))||^2_{L^2([0,1])}dt\right)^{\gamma/2}\right]\\ \leq\;& \lim_n C_\gamma||u^n_0-v^n_0||^\gamma_{L^1([0,1])}=C_\gamma||u_0-v_0||^\gamma_{L^1([0,1])}.\end{aligned}$$

To conclude the proof, it suffices to let T increase to infinity. □

4. Large time behavior

We now prove the uniqueness of the invariant measure.

Proof of Theorem 4. Consider two invariant distributions Q and $\tilde{Q}$ for (1), see Definition 3. Let u_0 be Q-distributed and $\tilde{u}_0$ be $\tilde{Q}$-distributed. Consider the corresponding (stationary) mild solutions $u,\tilde{u}$ to (1). Applying Theorem 2-(ii) and the Cauchy-Schwarz inequality, $\int_0^\infty K_s ds<\infty$ a.s., where

$$K_s:=K(u(s),\tilde{u}(s))=||b(u(s))-b(\tilde{u}(s))||_{L^1([0,1])}+||\sigma(u(s))-\sigma(\tilde{u}(s))||^2_{L^1([0,1])}.$$

Using Lemma 9, there is a (deterministic) sequence $(t_n)_{n\geq1}$ such that K_{t_n} tends to 0 in probability. Consider the function $\phi(r)=r/(1+r)$ on $\mathbb{R}_+$, and define $\Psi: L^1([0,1])\times L^1([0,1])\mapsto[0,1]$ as $\Psi(f,g)=\phi(K(f,g))$. Then $\lim_n\mathbb{E}[\Psi(u(t_n),v(t_n))]=\lim_n\mathbb{E}[\phi(K_{t_n})]=0$.
We now apply Lemma 10. The space $L^1([0,1])$ is Polish and for each $n\geq1$, $\mathcal{L}(u(t_n))=Q$ and $\mathcal{L}(\tilde{u}(t_n))=\tilde{Q}$. The function Ψ is clearly continuous on $L^1([0,1])\times L^1([0,1])$, (because σ,b are Lipschitz-continuous). Finally, $\Psi(f,g)>0$ for all $f\neq g$ (because $\Psi(f,g)=0$ implies that $b\circ f=b\circ g$ and $\sigma\circ f=\sigma\circ g$ a.e., whence $f=g$ a.e. since (σ,b) is injective). Lemma 10 thus yields $Q=\tilde{Q}$. □

Finally, we give the

Proof of Theorem 5. Point (ii) is immediately deduced from point (i). Let thus $u_0,v_0\in L^1([0,1])$ be fixed and let u,v be the corresponding mild solutions to (1). We know from $(\mathcal{I})$, the Jensen inequality and Theorem 2-(ii) that a.s.,

$$\begin{aligned}\int_0^\infty \rho(||u(t)-v(t)||_{L^1([0,1])})dt &\leq \int_0^\infty ||\rho(|u(t)-v(t)|)||_{L^1([0,1])}dt\\ &\leq \int_0^\infty \Big|\Big||b(u(t))-b(v(t))|+|\sigma(u(t))-\sigma(v(t))|^2\Big|\Big|_{L^1([0,1])}dt<\infty.\end{aligned}$$

Using Lemma 9, one may thus find an increasing (deterministic) sequence $(t_n)_{n\geq1}$ such that $\rho(||u(t_n)-v(t_n)||_{L^1([0,1])})$ tends to 0 in probability, so that $||u(t_n)-v(t_n)||_{L^1([0,1])}$ also tends to 0 in probability (because due to $\mathcal{I}$, ρ is strictly increasing and vanishes only at 0). Next, we use Theorem 2-(ii) with e.g. $\gamma = 1/2$ to get, setting $\Delta_t = \sup_{[t,\infty)} ||u(s) - v(s)||_{L^1([0,1])}$,

$$\mathbb{E}\left[\Delta_{t_n}^{1/2} \middle| \mathcal{F}_{t_n}\right] \leq C||u(t_n) - v(t_n)||_{L^1([0,1])}^{1/2} \to 0 \text{ in probability.}$$

We used here that conditionally on $\mathcal{F}_{t_n}$, $(u(t_n + t, x))_{t\geq 0, x\in[0,1]}$ is a mild solution to (1), starting from $u(t_n)$ (with a translated white noise). Thus for any $\epsilon > 0$, using the Markov inequality

$$P\left[\Delta_{t_n} > \epsilon\right] = \mathbb{E}\left[P\left(\Delta_{t_n} > \epsilon | \mathcal{F}_{t_n}\right)\right] \leq \mathbb{E}\left[\min\left(1, \epsilon^{-1/2}\mathbb{E}\left[\Delta_{t_n}^{1/2} \middle| \mathcal{F}_{t_n}\right]\right)\right],$$

which tends to 0 as $n \to \infty$ by dominated convergence. Consequently, as n tends to infinity,

$$\Delta_{t_n} \text{ tends to 0 in probability.} \tag{17}$$

But a.s. $s \mapsto \Delta_s = \sup_{[s,\infty)} ||u(t) - v(t)||_{L^1([0,1])}$ is non-increasing, and thus admits a limit as $s \to \infty$, which can be only 0 due to (17). □

5. Toward the multi-dimensional case?

Consider now a bounded smooth domain $D \subset \mathbb{R}^d$, for some $d \geq 2$. Consider the (scalar) equation

$$\partial_t u(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t \geq 0,\ x \in D, \tag{18}$$

with some Neumann boundary condition. Here $W(dt, dx) = \dot{W}(t,x)dtdx$ is a white noise on $[0,\infty) \times D$ based on $dtdx$. We assume that $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz-continuous.

It is well known that the mild equation makes no sense in such a case, since even if $\sigma(u)$ is bounded, $G_{t-s}(x,y)\sigma(u(s,y))$ does not belong to $L^2([0,t] \times D)$. The existence of solutions is thus still an open problem. See however Walsh [14] when $\sigma \equiv 1$, $b(u) = \alpha u$ and Nualart-Rozovskii [9] when $\sigma(u) = u$, $b(u) = \alpha u$. In these works, the authors manage to define some *ad-hoc* notion of solutions, using that the equations can be solved more or less explicitly. In the literature, one almost always considers the simpler case where the noise W is colored, see Da Prato-Zabczyk [5].

However the weak form makes sense: a predictable process $u = (u(t,x))_{t\geq 0, x\in D}$ is a weak solution if a.s.,

$$\text{for all } T > 0, \quad \sup_{[0,T]} ||u(t)||_{L^1(D)} + \int_0^T ||\sigma(u(t))||_{L^2(D)}^2 dt < \infty \tag{19}$$

and if for all function $\varphi \in C_b^2(D)$ (with Neumann conditions on ∂D), all $t \geq 0$, a.s.,

$$\int_D u(t,x)\varphi(x)dx = \int_D u_0(x)\varphi(x)dx + \int_0^t\int_D [\{u(s,x)\Delta\varphi(x) + b(u(s,x))\varphi(x)\}dxds + \sigma(u(s,x))\varphi(x)W(ds,dx)].$$

Assume now that $\sigma(0) = b(0) = 0$. Then $v \equiv 0$ is a weak solution. Furthermore, the estimate of Theorem 2-(i) *a priori* holds. Choosing $u_0 \in L^1(D)$ and $v_0 = 0$, this would imply (19). Unfortunately, we are not able to make this *a priori* estimate rigorous.

But following the proof of Proposition 6 and Corollary 7, one can easily check rigorously the following result. For $(e_k)_{k\geq 1}$ an orthonormal basis of $L^2(D)$, set $B_t^k = \int_0^t \int_D e_k(x)W(ds,dx)$. For $u_0 \in L^\infty(D)$ and $n \geq 1$, consider the solution (see Pardoux [11]) to

$$u^n(t,x) = u_0(x) + \int_0^t [\partial_{xx}u^n(s,x) + b(u^n(s,x))]ds + \sum_{k=1}^n \int_0^t \sigma(u^n(s,x))e_k(x)dB_s^k.$$

Then if $\sigma(0) = b(0) = 0$, for any $\gamma \in (0,1)$, any $T > 0$,

$$\mathbb{E}\left[\sup_{[0,T]} ||u^n(t)||_{L^1(D)}^\gamma + \left\{\int_0^T \sum_{k=1}^n \left(\int_{\mathbb{R}^d} \sigma(u^n(t,x))e_k(x)dx\right)^2 ds\right\}^\gamma\right] \leq C_{b,\gamma,T}||u_0||_{L^1(D)}^\gamma, \tag{20}$$

where the constant $C_{b,\gamma,T}$ depends only on γ, T, b (the important fact is that it does not depend on n). Passing to the limit formally in (20) would yield (19). Unfortunately, (20) is not sufficient to ensure that the sequence u^n is compact and tends, up to extraction of a subsequence, to a weak solution u to (18). But this suggests that, when $\sigma(0) = b(0) = 0$, (generalized) weak solutions to (18) do exist and satisfy something like (19), possibly in an extended sense. Recall that when $D = \mathbb{R}^d$, $b \equiv 0$ and $\sigma(z) = \sqrt{z}$, the (generalized) solution is known to exist and is called the *super Brownian motion*, see e.g. Krylov [8]. The super Brownian motion takes its values in the set of finite measures, which is the closure of L^1 in some sense. Such a construction is possible, essentially because in this case, $||u(t)||_{L^1} = ||\sigma(u(t))||^2_{L^2}$.

6. Appendix

First, we recall the following results on continuous local martingales.

Lemma 8. *Let $(M_t)_{t\geq 0}$ be a nonnegative continuous local martingale starting from $m \in (0,\infty)$. For all $\gamma \in (0,1)$, there exists a constant C_γ (depending only on γ) such that*

$$\mathbb{E}\left[\sup_{[0,\infty)} M_t^\gamma + \langle M\rangle_\infty^{\gamma/2}\right] \leq C_\gamma m^\gamma.$$

Proof. Classically (see e.g. Revuz-Yor [12, Theorems 1.6 and 1.7 p 181-182]), enlarging the probability space if necessary, there is a standard Brownian motion β such that $M_t = m + \beta_{\langle M\rangle_t}$. Denote now by $\tau_a = \inf\{t \geq 0;\ \beta_t = a\}$. Since M is nonnegative, we deduce that

$$\langle M\rangle_\infty \leq \tau_{-m} \text{ and } \sup_{[0,\infty)} M_t \leq m + \sup_{[0,\tau_{-m})} \beta_s.$$

Thus we just have to prove that $\mathbb{E}[\tau_{-m}^{\gamma/2}] + \mathbb{E}[S_m^\gamma] \leq C_\gamma m^\gamma$, where $S_m = \sup_{[0,\tau_{-m})} \beta_s$.

First, for $x \geq 0$, $P[S_m \geq x] = P[\tau_x \leq \tau_{-m}] = m/(m+x)$. As a consequence, since $\gamma \in (0,1)$,

$$\mathbb{E}[S_m^\gamma] = \int_0^\infty P[S_m^\gamma \geq x]dx = \int_0^\infty \frac{m}{m + x^{1/\gamma}}dx = m^\gamma \int_0^\infty \frac{1}{1+y^{1/\gamma}}dy = C_\gamma m^\gamma.$$

Next, for $t \geq 0$, $P[\tau_{-m} \geq t] = P[\inf_{[0,t]} \beta_s > -m]$. Recalling that $\inf_{[0,t]} \beta_s$ has the same law as $-\sqrt{t}|\beta_1|$, we get $P[\tau_{-m} \geq t] = P[|\beta_1| < m/\sqrt{t}]$. Hence

$$\mathbb{E}[\tau_{-m}^{\gamma/2}] = \int_0^\infty P[\tau_{-m}^{\gamma/2} \geq t]dt = \int_0^\infty P[|\beta_1| < m/t^{1/\gamma}]dt = \int_0^\infty P[(m/|\beta_1|)^\gamma > t]dt = m^\gamma \mathbb{E}\left[|\beta_1|^{-\gamma}\right].$$

This concludes the proof, since $\mathbb{E}\left[|\beta_1|^{-\gamma}\right] < \infty$ for $\gamma \in (0,1)$. □

Next, we state a technical result on a.s. converging integrals.

Lemma 9. *Let $(K_t)_{t\geq 0}$ be a nonnegative process. Assume that $A_\infty = \int_0^\infty K_t dt < \infty$. Then one may find a deterministic sequence $(t_n)_{n\geq 1}$ increasing to infinity such that K_{t_n} tends to 0 in probability as $n \to \infty$.*

Proof. Consider a strictly increasing continuous concave function $\phi : \mathbb{R}_+ \mapsto [0,1]$ such that $\phi(0) = 0$. Using the Jensen inequality, we deduce that

$$\frac{1}{T}\int_0^T \mathbb{E}[\phi(K_s)]ds = \mathbb{E}\left[\frac{1}{T}\int_0^T \phi(K_s)ds\right] \leq \mathbb{E}\left[\phi\left(\frac{1}{T}\int_0^T K_s ds\right)\right] \leq \mathbb{E}\left[\phi\left(\frac{A_\infty}{T}\right)\right],$$

which tends to 0 as $T \to \infty$ by the dominated convergence Theorem. As a consequence, we may find a sequence $(t_n)_{n\geq 1}$ such that $\lim_n \mathbb{E}[\phi(K_{t_n})] = 0$. The conclusion follows. □

Finally, we prove a technical result on coupling.

Lemma 10. *Consider two probability measures μ, ν on a Polish space $\mathcal{X}$. Let $\Psi : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}_+$ be continuous and assume that $\Psi(x,y) > 0$ for all $x \neq y$. If there is a sequence of $\mathcal{X} \times \mathcal{X}$-valued random variables $(X_n, Y_n)_{n\geq 1}$ such that for all $n \geq 1$, $\mathcal{L}(X_n) = \mu$ and $\mathcal{L}(Y_n) = \nu$ and if $\lim_n \mathbb{E}[\Psi(X_n, Y_n)] = 0$, then $\mu = \nu$.*

Proof. The sequence of probability measures $(\mathcal{L}(X_n, Y_n))_{n\geq 1}$ is obviously tight, so up to extraction of a subsequence, we may assume that (X_n, Y_n) converges in law, to some (X, Y). Of course, $\mathcal{L}(X) = \mu$ and $\mathcal{L}(Y) = \nu$. Since $\Psi\wedge 1$ is continuous and bounded, we deduce that $\mathbb{E}[\Psi(X,Y)\wedge 1] = \lim_n \mathbb{E}[\Psi(X_n, Y_n)\wedge 1] = 0$, whence $\Psi(X, Y) = 0$ a.s. By assumption, this implies that $X = Y$ a.s., so that $\mu = \nu$. □

References

[1] V. Bally, I. Gyongy, E. Pardoux, *White noise driven parabolic SPDEs with measurable drift,* J. Funct. Anal. **120** (1994), 484–510.

[2] V. Bally, E. Pardoux, *Malliavin calculus for white noise driven parabolic SPDEs,* Potential Anal. **9** (1998), 27–64.

[3] V. Bally, A. Millet, M. Sanz-Solé, *Approximation and support in Hölder norm for parabolic stochastic partial differential equations*, Ann. Probab. **23** (1995), 178–222.

[4] S. Cerrai, *Stabilization by noise for a class of stochastic reastion-diffusion equations*, Probab. Theory Related Fields **133** (2005), 190–214.

[5] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions,*, Cambridge University Press, 1992.

[6] C. Donati-Martin, E. Pardoux, *White nise driven SPDEs with reflection*, Probab. Theory Related Fields **95** (1993), 1–24.

[7] D. Gatarek, B. Goldys, *On weak solutions of stochastic equations in Hilbert spaces*, Stochastics Stochastics Rep. **46** (1994), 41–51.

[8] N.V. Krylov, *On SPDEs and superdiffusions*, Ann. Probab. **25** (1997), 1789–1809.

[9] D. Nualart, B. Rozovskii, *Weighted stochastic Sobolev spaces and bilinear SPDEs driven by space-time white noise*, J. Funct. Anal. **149** (1997), 200–225.

[10] C. Mueller, *Coupling and invariant measures for the heat equation with noise*, Ann. Probab. **21** (1993), 2189–2199.

[11] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), 127–167.

[12] D. Revuz, M. Yor, *Continuous martingales and Brownian motion,* third edition, Springer Verlag, 1999.

[13] R. Sowers, *Large deviations for the invariant measure of a reaction-diffusion equation with non-Gaussian perturbations*, Probab. Theory Related Fields **92** (1992), 393–421.

[14] J.B. Walsh, *An introduction to stochastic partial differential equations,* Ecole d'été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math., 1180, Springer, Berlin, 1986.

Université Paris Est, LAMA UMR 8050, Faculté des Sciences et Technologies, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France.

E-mail address: nicolas.fournier@univ-paris12.fr, printems@univ-paris12.fr