EXISTENCE OF DENSITIES FOR STABLE-LIKE DRIVEN SDE'S WITH HÖLDER CONTINUOUS COEFFICIENTS.

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ABSTRACT. Consider a multidimensional stochastic differential equation driven by a stable-like Lévy process. We prove that the law of the solution immediately has a density in some Besov space, under some non-degeneracy condition on the driving Lévy process and some very light Hölder-continuity assumptions on the drift and diffusion coefficients.

1. INTRODUCTION AND RESULTS

In [8], a very simple way to prove the existence of a density for the time-marginals of many stochastic processes has been introduced. The main advantage of this method is to apply when no Malliavin calculus can be used: it allows for instance to study SDE's whose coefficients have low regularity. However, it is based on the use of the Fourier transform and Plancherel identity. It is therefore automatically leading to square integrable densities. It is not possible to prove less "regularity" than L^2-densities. Due to this limitation and to the roughness of the method, only one-dimensional processes can be treated.

In [5], this method has been refined. Using in particular Besov spaces, existence of densities for finite-dimensional projections of the solution to the Navier-Stokes equation perturbed by some Gaussian noise has been proved. In particular, the method is not restricted to one-dimensional processes anymore.

The aim of the present paper is to investigate what this refined method gives for multidimensional stochastic differential equations of jump type. We study some stable-like driven SDE's when the drift and diffusion coefficients have a very low regularity. Let us mention that the present method has also been applied successfully in [7] to prove some regularization property of the homogeneous Boltzmann equation.

This work can be seen as a probabilistic approach to the theory of regularity of solutions to non-local partial differential equations. Indeed, the density of the solution of a stochastic equation satisfies a Fokker-Planck equation which is, in the jump case, non-local. There is a lot of research in this field in the PDE community. In particular, some results are available in the case of coefficients with low regularity. The typical results show that when the initial condition is continuous, the viscosity solution is immediately Hölder continuous. See Barles-Chasseigne-Imbert [2] and the references therein. Here we prove, under similar assumptions on the coefficients, that when the initial condition is a Dirac mass (or any probability measure by linearity), the solution becomes a

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function lying in some (low-index) Besov space. Thus we assume much less on the initial condition, but we get much less regularity.

1.1. **Functional spaces.** We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of all nonnegative finite measures on $\mathbb{R}^d$. When $f \in \mathcal{M}(\mathbb{R}^d)$ has a density, we also denote by f its density. For $\theta \in (0,1)$, we denote by $C^\theta(\mathbb{R}^d)$ the set of all functions on $\mathbb{R}^d$ such that

$$||g||_{C^\theta} := \sup_{x\in\mathbb{R}^d} |g(x)| + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\theta} < \infty.$$

Besov spaces are usually defined by means of Littlewood-Paley's decomposition. Here we use the equivalent definition (see Triebel [14, Theorem 2.5.12] or [15, Theorem 2.6.1]) in terms of differences. Define, for $f : \mathbb{R}^d \mapsto \mathbb{R}$, for $x, h \in \mathbb{R}^d$ and $n \geq 1$,

$$(\Delta_h^1 f)(x) = f(x+h) - f(x),$$
$$(\Delta_h^n f)(x) = \Delta_h^1(\Delta_h^{n-1} f)(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(x+jh).$$

For $s > 0$, introduce

$$\|f\|_{B^s_{1,\infty}} = \|f\|_{L^1} + \sup_{\{|h|\leq 1\}} |h|^{-s} \|\Delta_h^n f\|_{L^1}.$$

Here n is an integer such that $n > s$ and the obtained norm does not depend on $n > s$ (more precisely, the norms obtained with $n > s$ and $n' > s$ are equivalent). Moreover $B^s_{1,\infty}(\mathbb{R}^d)$ can be defined as the set of $L^1(\mathbb{R}^d)$-functions with $\|f\|_{B^s_{1,\infty}} < \infty$.

It is well-known, see e.g. [14, Formula 2.2.2 (18)], that $B^s_{1,\infty}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \in [1, d/(d-s))$. We will also use the estimate: for $f \in C^n(\mathbb{R}^d)$, for all $h \in \mathbb{R}^d$,

$$\|\Delta_h^n f\|_{L^1} \leq C_n |h|^n \|D^n f\|_{L^1}. \tag{1.1}$$

This easily checked: first show recursively that $\Delta_h^n f(x) = \int_0^n K_n(u) D^n f(x+uh).(h,\dots,h)du$ for some bounded function K_n (not depending on f) and use a straightforward computation.

1.2. **Main result.** We consider, on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$, a pure jump d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Lévy process $(Z_t)_{t\geq 0}$ with Lévy measure m and a stochastic differential equation driven by $(Z_t)_{t\geq 0}$:

$$X_t = x + \int_0^t \sigma(X_{s-}) dZ_s + \int_0^t b(X_s) ds. \tag{1.2}$$

Precise assumptions on $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ will be stated below. Denote by f_{Z_t} the law of Z_t. Up to a change of the drift b, we may consider (see Sato [12]) that the characteristic function of Z_t satisfies, for all $\xi \in \mathbb{R}^d$,

$$\begin{cases} \widehat{f_{Z_t}}(\xi) := \mathbb{E}[\exp(i\langle \xi, Z_t\rangle)] = \exp(-t\Psi(\xi)), \\ \text{where} \quad \Psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z\rangle} + i\langle \xi, z\rangle \mathbb{I}_{\{|z|\leq 1\}}\right) m(dz). \end{cases} \tag{1.3}$$

We assume that $(Z_t)_{t\geq 0}$ behaves like some α-stable process for some $\alpha \in (0,2)$:

$$(H_\alpha) \qquad \begin{cases} \text{(i)} \quad \forall\, \beta \in [0,\alpha),\ \int_{\{|z|\geq 1\}} |z|^\beta m(dz) < \infty, \\ \text{(ii)} \quad \exists\, C>0,\ \forall\, a \in (0,1],\ \int_{\{|z|\leq a\}} |z|^2 m(dz) \leq C a^{2-\alpha}, \\ \text{(iii)} \quad \exists\, c>0,\ \exists\, r>0,\ \forall\, \xi \in \mathbb{R}^d \text{ with } |\xi| \geq r,\ \int_{\mathbb{R}^d} (1-\cos(\langle \xi, z\rangle)) m(dz) \geq c|\xi|^\alpha. \end{cases}$$

We prove the following result.

Theorem 1.1. *Let $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be measurable and bounded and let $x \in \mathbb{R}^d$. Consider a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution $(X_t)_{t\geq 0}$ to* (1.2)*, $(Z_t)_{t\geq 0}$ being a Lévy process with Lévy measure m satisfying (H_α) for some $\alpha \in (0,2)$.*

(i) If $\alpha \in [1,2)$, assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$, that b is measurable (then set $\theta_2 = 0$) or that $b \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (0,1)$ and that $\kappa = \min\{1+\theta_1, \alpha+\theta_2\} > 1$.

(ii) If $\alpha \in (0,1)$, assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$, that $\tilde{b} \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (1-\alpha,1)$, where $\tilde{b}(x) := b(x) - \sigma(x)\int_{\{|z|\leq 1\}} z m(dz)$ is the true drift coefficient and set $\kappa = \min\{1+\alpha\theta_1, \alpha+\theta_2, \alpha/(1-\theta_2)\} > 1$.

Then for all $t>0$, the law f_{X_t} of X_t has a density on the set $\{y \in \mathbb{R}^d : \sigma(y)$ invertible$\}$. Furthermore, for all $0 < \gamma < (\kappa-1)\min\{\alpha,1\}$, there exists $n \in \mathbb{N}$ such that $|\sigma^{-1}|^{-n} f_{X_t} \in B^\gamma_{1,\infty}(\mathbb{R}^d)$.

Observe that $\tilde{b}$ is well-defined when $\alpha \in (0,1)$, since $\int_{\{|z|\leq 1\}} |z| m(dz) < \infty$ by Lemma 5.1-(iii) with $a=1$ and $\gamma=1$. We of course take the convention that $|\sigma^{-1}(x)|^{-1} = 0$ if $\sigma(x)$ is not invertible. The existence (and uniqueness) of solutions to (1.2) is not completely clear under such weak assumptions. However, we believe that weak existence holds as soon as b and σ are continuous. We refer to Jacod [9] for many existence and uniqueness results. Clearly, we can not use Malliavin calculus, for example as in Bichteler-Jacod [4], Bichteler-Gravereaux-Jacod [3], Picard [11], Denis [6], Kulik [10], Bally-Clément [1] etc., due to the lack of regularity of the coefficients. To our knowledge, the only regularization result concerning jumping SDE's with Hölder coefficients is that of [8], where there are many restrictions: it works only in dimension $d=1$, for $\alpha \in (1/2,2)$, and more regularity is assumed on the coefficients. Let us mention that when $\alpha > 1$, we obtain the existence of a density when the drift b is only measurable. In the case of Brownian SDE's, this is easily done thanks to the Girsanov theorem, but such a trick cannot work for jumping SDE's.

Note finally that, under the same assumption as in [8] and in particular if $d=1$, we find a density which is in $B^\gamma_{1,\infty}(\mathbb{R})$ with $\gamma > 1/2$. Hence by Sobolev embedding it is in $L^2(\mathbb{R})$ so that we recover the result of [8].

Assumption (H_α)-(i) concerns large jumps and can be removed (loosing the Besov regularity).

Corollary 1.2. *Let $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be measurable and bounded and let $x \in \mathbb{R}^d$. Consider a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution $(X_t)_{t\geq 0}$ to (1.2), where $(Z_t)_{t\geq 0}$ is a Lévy process as in (1.3) with Lévy measure m satisfying (H_α)-(ii)-(iii) for some $\alpha \in (0,2)$ and $\int_{\{|z|\geq 1\}} m(dz) < \infty$.*

(i) If $\alpha \in (1,2)$, assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$.

(ii) If $\alpha = 1$, assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$, $b \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_1, \theta_2 \in (0,1)$.

(iii) If $\alpha \in (0,1)$, assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$ and that $\tilde{b} \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (1-\alpha,1)$, where $\tilde{b}(x) := b(x) - \sigma(x)\int_{\{|z|\leq 1\}} z m(dz)$.

Then for all $t>0$, the law f_{X_t} of X_t has a density on the set $\{y \in \mathbb{R}^d : \sigma(y)$ invertible$\}$.

We now state an alternative version of (H_α)-(iii), which resembles the condition of Picard [11] and might be useful for applications. We will prove it in the appendix.

Lemma 1.3. *Let $\alpha \in (0,2)$ and let m satisfy (H_α)-(ii) and $\int_{\{|z|\geq 1\}} m(dz) < \infty$. The two following conditions are equivalent.*
(a) m satisfies (H_α)-(iii);
(b) $\exists\, c>0,\ \forall a \in (0,1],\ \forall \zeta \in \mathbb{R}^d$ with $|\zeta|=1$, $\int_{\{|z|\leq a\}} \langle \zeta, z\rangle^2 m(dz) \geq c a^{2-\alpha}$.

Finally, let us mention the main inconvenient of our results: we have to assume (H_α)-(ii) and (H_α)-(iii) with the same value of α. As shown below, this is satisfied if m is the Lévy measure of some non-degenerated α-stable process. But this is not satisfied, for example, if $d=2$ and if $Z=(Z^1,Z^2)$, where Z^1 and Z^2 are two non-degenerated one-dimensional stable processes with different indexes $\alpha_1 \in (0,2)$ and $\alpha_2 \in (0,2)$. We could also study such a case, but we would obtain much less optimal results.

1.3. **Examples.** The basic example of application is the following.

Example 1.4. *Assume that m is the Lévy measure of some α-stable process. Then, see Sato* [12, Part 14]*, there is a nonnegative finite measure λ on the sphere $\mathbb{S}^{d-1}$ such that for all Borel subset A of $\mathbb{R}^d$, $m(A)=\int_0^\infty r^{-\alpha-1}dr \int_{\mathbb{S}^{d-1}} \mathbb{I}_A(r\sigma)\lambda(d\sigma)$. One immediately checks that m satisfies (H_α)-(i)-(ii). If furthermore m is non-degenerated in the sense that supp λ contains a basis of $\mathbb{R}^d$, then m also satisfies (H_α)-(iii).*

Remark 1.5. *In the case of SDE's driven by true α-stable processes, a little care is needed, because the characteristic function of a true α-stable process is not of the form* (1.3)*, except when $\alpha=1$. Let thus $(Y_t)_{t\geq 0}$ be a true α-stable process with non-degenerated Lévy measure m.*
(a) If $\alpha\in(1,2)$ then $Z_t := Y_t - t\int_{\{|z|>1\}} z m(dz)$ satisfies all our assumptions. The S.D.E. $dX_t = \sigma(X_{t-})dY_t + c(X_t)dt$ rewrites $dX_t=\sigma(X_{t-})dZ_t + b(X_t)dt$, with $b(x):=c(x)+\sigma(x)\int_{\{|z|>1\}} z m(dz)$.
(b) If $\alpha\in(0,1)$, then $Z_t = Y_t - t\int_{\{|z|\leq 1\}} z m(dz)$ satisfies all our assumptions. The S.D.E. $dX_t = \sigma(X_{t-})dY_t + c(X_t)dt$ rewrites $dX_t=\sigma(X_{t-})dZ_t + b(X_t)dt$, with $b(x):=c(x)+\sigma(x)\int_{\{|z|\leq 1\}} z m(dz)$ (whence $\tilde b = c$).

The next example shows that m may be very singular. Roughly, the Lévy measures considered by Bichteler-Gravereaux-Jacod [3] have to be smooth, so that our result is more in the spirit of those of Picard [11].

Example 1.6. *If m is of the form $m(A)=\int_0^\infty \mu(dr)\int_{\mathbb{S}^{d-1}} \mathbb{I}_A(r\sigma)\lambda(d\sigma)$, for some nonnegative finite measure λ on the sphere $\mathbb{S}^{d-1}$ of which the support contains a basis of $\mathbb{R}^d$ and if $\mu = \sum_{n\geq 1} n^{\alpha-1}\delta_{1/n}$, then m satisfies (H_α). Here the support of m may be countable.*

1.4. **Plan of the paper.** In the next section, we recall the absolute continuity criterion of [5] and say a word about our strategy. We prove Theorem 1.1 in Section 3 and deduce Corollary 1.2 in Section 4. An appendix with technical results about our assumptions, estimates of stochastic integrals against Lévy processes and a proof of the absolute continuity criterion of [5] lies at the end of the paper.

1.5. **Convention.** We write C for a (large) finite constant and c for a (small) positive constant, whose values may change from line to line, and which depend only on m, d, σ, b, α. We write in index all the additional dependence of constants. For example, C_β is a finite constant whose value depends on m, d, σ, b, α and β.

2. Main ideas of the proof

Our study relies on the following lemma due to [5]. An alternative proof is given in the appendix.

Lemma 2.1. *Let $g \in \mathcal{M}(\mathbb{R}^d)$. Assume that there are $0 < \eta < a < 1$, $n \geq 1$ and a constant K such that for all $\phi \in C^\eta(\mathbb{R}^d)$, all $h \in \mathbb{R}^d$ with $|h| \leq 1$,*

$$\left| \int_{\mathbb{R}^d} \Delta_h^n \phi(x) g(dx) \right| \leq K ||\phi||_{C^\eta} |h|^a. \tag{2.4}$$

Then g has a density in $B^{a-\eta}_{1,\infty}(\mathbb{R}^d)$ and $||g||_{B^{a-\eta}_{1,\infty}} \leq g(\mathbb{R}^d) + C_{d,a,\eta,\gamma,n} K$.

Let us explain briefly our strategy to apply Lemma 2.1. Recall that $dX_t = \sigma(X_{t_-}) dZ_t + b(X_t)dt$ and assume that σ is uniformly elliptic for simplicity.

• For $\epsilon \in (0,t)$, consider $X_t^\epsilon = X_{t-\epsilon} + \epsilon b(X_{t-\epsilon}) + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon})$ (we will need to do something slightly more tricky when $\alpha \in (0,1)$).

• Study the error $\mathbb{E}[|X_t - X_t^\epsilon|^\eta]$ for $\eta > 0$. We get something like $\mathbb{E}[|X_t - X_t^\epsilon|^\eta] \leq C\epsilon^\gamma$, with γ depending on η, α, and on the Hölder regularity of the coefficients σ and b.

• Conditionally on $X_{t-\epsilon}$, X_t^ϵ has an infinitely divisible distribution, for which many known results are available. In particular, we can bound the norm of any derivative of the density $f_{X_t^\epsilon}$ in $L^1(\mathbb{R}^d)$ by slightly generalizing a result of Schilling-Sztonyk-Wang [13]. Of course, the bound explodes when $\epsilon \to 0$ but the rate of growth is controlled precisely: we will obtain that $\|D^n f_{X_t^\epsilon}\|_{L^1} \leq \epsilon^{-n/\alpha}$.

• Use the discrete integration by part:

$$\mathbb{E}(\Delta_h^n \phi(X_t^\epsilon)) = \int_{\mathbb{R}^d} \Delta_h^n \phi(x) f_{X_t^\epsilon}(x) dx = \int_{\mathbb{R}^d} \phi(x) \Delta_{-h}^n f_{X_t^\epsilon}(x) dx$$

to obtain, using (1.1),

$$|E(\Delta_h^n \phi(X_t^\epsilon))| \leq \|\phi\|_{L^\infty} \|D^n f_{X_t^\epsilon}\|_{L^1} |h|^n \leq C \|\phi\|_{L^\infty} \epsilon^{-n/\alpha} |h|^n.$$

• Finally, write

$$\begin{aligned} |\mathbb{E}[\Delta_h^n \phi(X_t)]| &\leq |\mathbb{E}(\Delta_h^n \phi(X_t^\epsilon))| + |\mathbb{E}(\Delta_h^n \phi(X_t) - \Delta_h^n \phi(X_t^\epsilon))| \\ &\leq C\|\phi\|_{L^\infty} \epsilon^{-n/\alpha} |h|^n + C\|\phi\|_{C^\eta} \mathbb{E}[|X_t - X_t^\epsilon|^\eta]. \end{aligned}$$

For each h, choose ϵ judiciously to end with something like

$$\left| \int_{\mathbb{R}^d} \Delta_h^n \phi(x) f_{X_t}(dx) \right| = |\mathbb{E}[\Delta_h^n \phi(X_t)]| \leq C ||\phi||_{C^\eta} |h|^\delta,$$

for some δ depending on α, on the Hölder regularity of the coefficients σ and b, and on n, η. Try to find η and n such that Lemma 2.1 applies.

3. Proof of the main result

In the whole section, we consider a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Lévy process $(Z_t)_{t\geq 0}$ with Lévy measure m satisfying (H_α) for some $\alpha \in (0,2)$, with characteristic function as in (1.3), as well as some bounded and measurable coefficients $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$. We fix $x \in \mathbb{R}^d$ and consider a $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg solution $(X_t)_{t\geq 0}$ to (1.2).

For $A \in M_{d\times d}(\mathbb{R})$, we put $|A| = \sup_{x\in\mathbb{R}^d,|x|=1} |Ax|$, where $|\,.\,|$ is the Euclidean norm on $\mathbb{R}^d$. One easily checks that $1/|A^{-1}| = \inf_{x\in\mathbb{R}^d,|x|=1} |Ax|$, from which we deduce that for all $A, B \in M_{d\times d}(\mathbb{R})$,

$$\frac{1}{|A^{-1}|} \le |A| \quad \text{and} \quad \left|\frac{1}{|A^{-1}|} - \frac{1}{|B^{-1}|}\right| \le |A - B|. \tag{3.1}$$

3.1. **Approximation.** We first state an approximation lemma. Unfortunately, we have to separate the two cases $\alpha \in (0, 1)$ and $\alpha \in [1, 2)$.

Lemma 3.1. *Assume that $\alpha \in [1, 2)$ and that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ for some $\theta_1 \in (0, 1)$ and either that b is measurable (then we set $\theta_2 = 0$) or that $b \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_2 \in (0, 1)$.*

(i) For all $0 < s < t < s + 1$, all $\beta \in [0, \alpha)$,

$$\mathbb{E}[|X_t - X_s|^\beta] \le C_\beta(t - s)^{\beta/\alpha}.$$

(ii) Define $X^\epsilon_t = X_{t-\epsilon} + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon}) + \epsilon b(X_{t-\epsilon})$. For all $\beta \in (0, \alpha)$, all $0 < \epsilon < t \wedge 1$,

$$\mathbb{E}[|X_t - X^\epsilon_t|^\beta] \le C_\beta[\epsilon^{\beta(1+\theta_1)/\alpha} + \epsilon^{\beta(1+\theta_2/\alpha)}].$$

Proof. We start with (i). Applying Lemma 5.2-(i) (with e.g. $\gamma = 2$) and using that σ, b are bounded, we get

$$\begin{aligned}\mathbb{E}[|X_t - X_s|^\beta] \le& C_\beta \mathbb{E}\left[\left|\int_s^t \sigma(X_{u-})dZ_u\right|^\beta\right] + C_\beta \mathbb{E}\left[\left|\int_s^t b(X_u)du\right|^\beta\right]\\ \le& C_\beta(t - s)^{\beta/\alpha} \sup_{[s,t]} \mathbb{E}[|\sigma(X_u)|^2]^{\beta/2} + C_\beta(t - s)^\beta\\ \le& C_\beta(t - s)^{\beta/\alpha},\end{aligned}$$

since $\beta/\alpha \le \beta$ and $t - s \le 1$.

Next we check (ii). We have

$$X_t - X^\epsilon_t = \int_{t-\epsilon}^t [\sigma(X_{s-}) - \sigma(X_{t-\epsilon})]dZ_s + \int_{t-\epsilon}^t [b(X_s) - b(X_{t-\epsilon})]ds =: I_{t,\epsilon} + J_{t,\epsilon}.$$

First, using Lemma 5.2-(i) with some $\gamma \in (\alpha, \alpha/\theta_1)$, $\gamma \le 2$, and that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$, we find

$$\mathbb{E}[|I_{t,\epsilon}|^\beta] \le C_\beta \epsilon^{\beta/\alpha} \sup_{[t-\epsilon,t]} \mathbb{E}\left[|X_s - X_{t-\epsilon}|^{\gamma\theta_1}\right]^{\beta/\gamma} \le C_\beta \epsilon^{\beta(1+\theta_1)/\alpha},$$

where we used (i) and that $\gamma\theta_1 < \alpha$. It only remains to prove that $E[|J_{t,\epsilon}|^\beta] \le C_\beta \epsilon^{\beta(1+\theta_2/\alpha)}$. If $\beta \le 1$, the Hölder inequality (for the expectation), the fact that $b \in C^{\theta_2}(\mathbb{R}^d)$ and point (i) (which is valid because $\theta_2 < 1 \le \alpha$) yield

$$E[|J_{t,\epsilon}|^\beta] \le \mathbb{E}\left[\int_{t-\epsilon}^t |X_s - X_{t-\epsilon}|^{\theta_2} ds\right]^\beta = \left[\int_{t-\epsilon}^t \mathbb{E}(|X_s - X_{t-\epsilon}|^{\theta_2})ds\right]^\beta \le C_\beta \epsilon^\beta \epsilon^{\beta\theta_2/\alpha}$$

as desired. If finally $\beta > 1$, the Hölder inequality (for the integral in time), the fact that $b \in C^{\theta_2}(\mathbb{R}^d)$ and point (i) (which is valid because $\theta_2\beta < \beta < \alpha$) give

$$E[|J_{t,\epsilon}|^\beta] \le \epsilon^{\beta-1} \mathbb{E}\left[\int_{t-\epsilon}^t |X_s - X_{t-\epsilon}|^{\theta_2\beta} ds\right] = \epsilon^{\beta-1} \int_{t-\epsilon}^t \mathbb{E}(|X_s - X_{t-\epsilon}|^{\theta_2\beta})ds \le C_\beta \epsilon^\beta \epsilon^{\beta\theta_2/\alpha},$$

which ends the proof. □

The case where $\alpha \in (0, 1)$ is slightly more delicate. Indeed, the approximation X^ϵ_t proposed above is too rough.

Lemma 3.2. *Assume that* $\alpha \in (0,1)$, *that* $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ *and* $\tilde{b} \in C^{\theta_2}(\mathbb{R}^d)$ *for some* $\theta_1, \theta_2 \in (0,1)$. *Recall here that* $\tilde{b}(x) := b(x) - \sigma(x)\tau$ *with* $\tau := \int_{\{|z|\leq 1\}} zm(dz)$.

(i) For all $0 < s < t < s+1$, *all* $\beta \in [0,\alpha)$,

$$\mathbb{E}[|X_t - X_s|^\beta] \leq C_\beta (t-s)^\beta.$$

(ii) For all $0 < s < t < s+1$, *all* $\beta \in [0,1)$,

$$\mathbb{E}[|X_t - X_s|^\beta \wedge 1] \leq C_\beta (t-s)^\beta.$$

(iii) For $0 < \epsilon < t \wedge 1$, *we can find a* $\mathcal{F}_{t-\epsilon}$*-measurable random variable* U_t^ϵ *such that, setting* $X_t^\epsilon = U_t^\epsilon + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon})$, *for all* $\beta \in (0,\alpha)$,

$$\mathbb{E}[|X_t - X_t^\epsilon|^\beta] \leq C_\beta[\epsilon^{\beta(1/\alpha+\theta_1)} + \epsilon^{\beta(1+\theta_2/\alpha)} + \epsilon^{\beta/(1-\theta_2)}].$$

Proof. We set $Y_t = Z_t + \tau t$ and rewrite (1.2) as $X_t = x + \int_0^t \sigma(X_{s-})dY_s + \int_0^t \tilde{b}(X_s)ds$. Using Lemma 5.2-(ii) with e.g. $\gamma = 1$ and that σ is bounded, we get $\mathbb{E}[|\int_s^t \sigma(X_{u-})dY_u|^\beta] \leq C_\beta (t-s)^{\beta/\alpha} \leq C_\beta (t-s)^\beta$ for all $\beta \in (0,\alpha)$, since $0 < t-s < 1$. Next, we obviously have $\mathbb{E}[|\int_s^t \tilde{b}(X_u)du|^\beta] \leq C(t-s)^\beta$ for any $\beta > 0$ because $\tilde{b}$ is bounded. This proves (i).

If $\beta \in (0,\alpha)$, (ii) follows from (i). We thus may assume that $\beta \in [\alpha, 1)$. We write $|X_t - X_s|^\beta \wedge 1 \leq |\int_s^t \sigma(X_{u-})dY_u|^\beta \wedge 1 + |\int_s^t \tilde{b}(X_u)du|^\beta \wedge 1 \leq |\int_s^t \sigma(X_{u-})dY_u|^{\beta\alpha} + C(t-s)^\beta$, since $\tilde{b}$ is bounded and since $\alpha \in (0,1)$. It only remains to check that $\mathbb{E}[|\int_s^t \sigma(X_{u-})dY_u|^{\beta\alpha}] \leq C_\beta (t-s)^\beta$, which directly follows from Lemma 5.2-(ii) (with e.g. $\gamma = 1$), since $\beta\alpha \in (0,\alpha)$ and since σ is bounded.

To check (iii), the main idea is to set $U_t^\epsilon = V_t^\epsilon + \epsilon\sigma(X_{t-\epsilon})\tau$, where V_t^ϵ is the value at time t of a solution to $y' = \tilde{b}(y)$, $y(t-\epsilon) = X_{t-\epsilon}$ (there may be several solutions, since $\tilde{b}$ is only Hölder-continuous). The difficulty is then to choose this solution in such a way that V_t^ϵ is a measurable function of $X_{t-\epsilon}$ (so that V_t^ϵ is $\mathcal{F}_{t-\epsilon}$-measurable). We rather consider an approximate solution, which allows us to overcome this difficulty easily.

Let thus $\delta = \epsilon^{1/(1-\theta_2)}$ and define, for $s \in [t-\epsilon, t]$, $s_\delta = t - \epsilon + \delta\lfloor (s-(t-\epsilon))/\delta \rfloor$. Consider the solution, for $u \in [t-\epsilon, t]$, to

$$V_u^\epsilon = X_{t-\epsilon} + \int_{t-\epsilon}^u \tilde{b}(V_{s_\delta}^\epsilon)ds.$$

Then V_t^ϵ is obviously well-defined and $\mathcal{F}_{t-\epsilon}$-measurable, because it is a deterministic continuous function of $X_{t-\epsilon}$: write that $V_u^\epsilon = X_{t-\epsilon} + (u-(t-\epsilon))\tilde{b}(X_{t-\epsilon})$ for $u \in [t-\epsilon, t-\epsilon+\delta)$, that $V_u^\epsilon = X_{t-\epsilon} + \delta\tilde{b}(X_{t-\epsilon}) + (u-(t-\epsilon)-\delta)\tilde{b}(X_{t-\epsilon} + \delta\tilde{b}(X_{t-\epsilon}))$ for $u \in [t-\epsilon+\delta, t-\epsilon+2\delta)$, etc. Observe also that $|V_s^\epsilon - V_{s_\delta}^\epsilon| \leq C\delta$ for all $s \in [t-\epsilon, t]$ because $\tilde{b}$ is bounded. We finally set $X_t^\epsilon = V_t^\epsilon + \sigma(X_{t-\epsilon})(Y_t - Y_{t-\epsilon}) = U_t^\epsilon + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon})$, where $U_t^\epsilon := V_t^\epsilon + \epsilon\sigma(X_{t-\epsilon})\tau$ is also $\mathcal{F}_{t-\epsilon}$-measurable.

Define $R_{t,\epsilon} = \sum_{s\in[t-\epsilon,t]} |\Delta Y_s|$. Writing

$$V_u^\epsilon = X_{t-\epsilon} + \int_{t-\epsilon}^u \tilde{b}(V_s^\epsilon)ds + \int_{t-\epsilon}^u (\tilde{b}(V_{s_\delta}^\epsilon) - \tilde{b}(V_s^\epsilon))ds.$$

we see that for $u \in [t-\epsilon, t]$,

$$|X_u - V_u^\epsilon| \leq ||\sigma||_{L^\infty} R_{t,\epsilon} + \int_{t-\epsilon}^u |\tilde{b}(X_s) - \tilde{b}(V_s^\epsilon)|ds + \int_{t-\epsilon}^u |\tilde{b}(V_{s_\delta}^\epsilon) - \tilde{b}(V_s^\epsilon)|ds.$$

Setting $S_{t,\epsilon} = \sup_{s\in[t-\epsilon,t]} |X_s - V_s^\epsilon|$ and using that $\tilde b \in C^{\theta_2}(\mathbb{R}^d)$ and that $|V_s^\epsilon - V_{s_\delta}^\epsilon| \le C\delta$, we see that

$$S_{t,\epsilon} \le K(R_{t,\epsilon} + \epsilon S_{t,\epsilon}^{\theta_2} + \epsilon\delta^{\theta_2}) = K(R_{t,\epsilon} + \epsilon S_{t,\epsilon}^{\theta_2} + \epsilon^{1/(1-\theta_2)}).$$

Using the Young inequality $xy \le (1-\theta_2)x^{1/(1-\theta_2)} + \theta_2 y^{1/\theta_2}$ with $x = K\epsilon$ and $y = S_{t,\epsilon}^{\theta_2}$, we get $K\epsilon S_{t,\epsilon}^{\theta_2} \le (1-\theta_2)(K\epsilon)^{1/(1-\theta_2)} + \theta_2 S_{t,\epsilon}$. Consequently $S_{t,\epsilon} \le CR_{t,\epsilon} + C\epsilon^{1/(1-\theta_2)} + \theta_2 S_{t,\epsilon}$, whence, since $\theta_2 < 1$,

$$S_{t,\epsilon} \le CR_{t,\epsilon} + C\epsilon^{1/(1-\theta_2)}. \tag{3.2}$$

We finally recall that $X_t^\epsilon = V_t^\epsilon + \sigma(X_{t-\epsilon})(Y_t - Y_{t-\epsilon}) = X_{t-\epsilon} + \int_{t-\epsilon}^t \tilde b(V_s^\epsilon)ds + \int_{t-\epsilon}^t \sigma(X_{t-\epsilon})dY_s + \int_{t-\epsilon}^t (\tilde b(V_{s_\delta}^\epsilon) - \tilde b(V_s^\epsilon))ds$ so that

$$\begin{aligned}|X_t - X_t^\epsilon| \le & \int_{t-\epsilon}^t |\tilde b(X_s) - \tilde b(V_s^\epsilon)|ds + \left|\int_{t-\epsilon}^t [\sigma(X_{s-}) - \sigma(X_{t-\epsilon})]dY_s\right| + \int_{t-\epsilon}^t |\tilde b(V_{s_\delta}^\epsilon) - \tilde b(V_s^\epsilon)|ds \\ =: & I_\epsilon + J_\epsilon + K_\epsilon.\end{aligned}$$

First, $I_\epsilon \le C\int_{t-\epsilon}^t |X_s - V_s^\epsilon|^{\theta_2} ds \le C\epsilon |S_{t,\epsilon}|^{\theta_2} \le C\epsilon(R_{t,\epsilon}^{\theta_2} + \epsilon^{\theta_2/(1-\theta_2)})$ by (3.2). Consequently, $\mathbb{E}[|I_\epsilon|^\beta] \le C_\beta \epsilon^\beta[\epsilon^{\beta\theta_2/(1-\theta_2)} + \mathbb{E}(R_{t,\epsilon}^{\beta\theta_2})] \le C_\beta[\epsilon^{\beta/(1-\theta_2)} + \epsilon^{\beta(1+\theta_2/\alpha)}]$ thanks to Lemma 5.2-(ii) (note that $0 < \beta\theta_2 < \alpha$). Next, Lemma 5.2-(ii) with some $\gamma \in (\alpha, \alpha/\theta_1 \wedge 1)$ together with the Hölder condition on σ lead us to $\mathbb{E}[|J_\epsilon|^\beta] \le C_\beta \epsilon^{\beta/\alpha} \sup_{[t-\epsilon,t]} \mathbb{E}\left[|X_s - X_{t-\epsilon}|^{\gamma\theta_1}\right]^{\beta/\gamma}$. Using (i) and that $\gamma\theta_1 < \alpha$, we obtain $\mathbb{E}[J_\epsilon^\beta] \le C_\beta \epsilon^{\beta(1/\alpha+\theta_1)}$. Finally, since $\tilde b \in C^{\theta_2}(\mathbb{R}^d)$ and since $|V_s^\epsilon - V_{s_\delta}^\epsilon| \le C\delta$, we have $K_\epsilon \le C\epsilon\delta^{\theta_2} = C\epsilon^{1/(1-\theta_2)}$ a.s., whence $\mathbb{E}[K_\epsilon^\beta] \le C_\beta \epsilon^{\beta/(1-\theta_2)}$. This ends the proof. ☐

3.2. **Small-time density estimate for the Lévy process.** We now use the ideas of Schilling-Sztonyk-Wang [13] to give some regularity estimates on the density of the Lévy process. In [13, Theorem 1.2], it is proved that Z_t has a C^∞-density as soon as $t > 0$ and a sharp estimate on the L^1-norm of the gradient is given. We slightly generalize this result to give estimates on the L^1-norm of higher derivatives.

Lemma 3.3. *Let $(Z_t)_{t\ge 0}$ be a Lévy process with characteristic function Ψ as in (1.3) with a Lévy measure m satisfying (H_α) for some $\alpha \in (0,2)$. Then for all $t > 0$, f_{Z_t} has a C^∞-density and for all $n \ge 0$ there is a constant C_n such that for all $t > 0$ $\|D^n f_{Z_t}\|_{L^1} \le C_n(t\wedge 1)^{-n/\alpha}$.*

Proof. We first show that there are $0 < c < C$ such that for all $\xi \in \mathbb{R}^d$ with $|\xi|$ large enough,

$$c|\xi|^\alpha \le \Re\, \Psi(\xi) \le C|\xi|^\alpha. \tag{3.3}$$

We have $\Re\, \Psi(\xi) = \int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)m(dz)$. Thus the lower-bound is exactly (H_α)-(iii). Next, recalling that $(1 - \cos x) \le 2(x^2 \wedge 1)$ for all $x \in \mathbb{R}$, we get (if $|\xi| \ge 1$)

$$\Re\, \Psi(\xi) \le 2\int_{\{|z|\le 1/|\xi|\}} |z|^2|\xi|^2 m(dz) + 2\int_{\{|z|\ge 1/|\xi|\}} m(dz).$$

Using (H_α)-(ii) for the first term and Lemma 5.1-(i) (with $\beta = 0$) for the second term, we obtain

$$\Re\, \Psi(\xi) \le C|\xi|^2(1/|\xi|)^{2-\alpha} + C(1/|\xi|)^{-\alpha} \le C|\xi|^\alpha.$$

Since $|\widehat{f}_{Z_t}(\xi)| = \exp(-t\Re\, \Psi(\xi)) \le \exp(-ct|\xi|^\alpha)$ (for $|\xi|$ large enough), we classically deduce that f_{Z_t} has a smooth density for all $t > 0$.

Define, for $\rho > 0$ and $t > 0$,

$$\phi(\rho) = \sup_{|\xi|\le\rho} \Re\Psi(\xi) \quad \text{and} \quad h(t) = \frac{1}{\phi^{-1}(1/t)}.$$

We have $c\,\rho^\alpha \le \phi(\rho) \le C\,\rho^\alpha$ by (3.3) whence $c\,t^{1/\alpha} \le h(t) \le C\,t^{1/\alpha}$. It is easy to deduce from (3.3) that for $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} \exp\left(-t\Re\Psi(\xi)\right)|\xi|^n d\xi \le C_n t^{-(n+d)/\alpha} \le C_n(\phi(1/t))^{-(n+d)}.$$

We now write $Z_t = Y_t + \tilde{Y}_t$, for two independent infinitely divisible random variables with laws characterized by $\hat{f}_{Y_t}(\xi) = \exp(-t\Psi_t(\xi))$ and $\hat{f}_{\tilde{Y}_t}(\xi) = \exp(-t(\Psi(\xi) - \Psi_t(\xi)))$ with

$$\Psi_t(\xi) = \int_{\{|z|\le h(t)\}} \left(1 - e^{i\langle \xi, z\rangle} + i\,\langle \xi, z\rangle\right) m(dz).$$

Then by [13, Proposition 2.3] (with the choice $n = d+1$), there exists $t_0 > 0$ such that for $t \le t_0$, Y_t has a C^∞ density f_{Y_t} such that for every $n \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$, with $\sum_{i=1}^d \beta_i = n$, for all $t \in (0, t_0)$,

$$|\partial^\beta f_{Y_t}(y)| \le C_n t^{-(d+n)/\alpha}\left(1 + t^{-1/\alpha}|y|\right)^{-(d+1)},\ y \in \mathbb{R}^d.$$

We immediately deduce that for all $t \in (0, t_0)$,

$$\|\partial^\beta f_{Y_t}\|_{L^1} \le C_n t^{-n/\alpha}.$$

Writing $f_{Z_t} = f_{Y_t} \star f_{\tilde{Y}_t}$, we immediately deduce that $\|\partial^\beta f_{Z_t}\|_{L^1} \le \|\partial^\beta f_{Y_t}\|_{L^1} \le C_n t^{-n/\alpha}$ for all $t \in (0, t_0)$. Finally, for $t \ge t_0$, we write $f_{Z_t} = f_{Z_{t_0}} \star f_{Z_{t-t_0}}$ to get $\|\partial^\beta f_{Z_t}\|_{L^1} \le \|\partial^\beta f_{Z_{t_0}}\|_{L^1}$. Thus $\|\partial^\beta f_{Z_t}\|_{L^1} \le C_n(t \wedge t_0)^{-n/\alpha}$ for all $t > 0$, which ends the proof. □

3.3. **A last preliminary computation.** Let us prepare the application of Lemma 2.1.

Lemma 3.4. *Assume that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$. For any $n \ge 1$, there exists a constant C_n such that for all $\eta \in (0,1)$, $\phi \in C^\eta(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $t > 0$, $\epsilon \in (0, t \wedge 1]$*

$$\left|\mathbb{E}\left(\frac{\Delta_h^n \phi(X_t)}{|\sigma^{-1}(X_t)|^n}\right)\right| \le C_n||\phi||_{C^\eta}\left(|h|^\eta \mathbb{E}\left[|X_t - X_{t-\epsilon}|^{\theta_1} \wedge 1\right] + \mathbb{E}\left[|X_t - X_t^\epsilon|^\eta\right] + \epsilon^{-n/\alpha}|h|^n\right),$$

where X_t^ϵ was introduced in Lemma 3.1 (if $\alpha \in [1,2)$) or Lemma 3.2 (if $\alpha \in (0,1)$).

Proof. Call $I_{t,h}^\phi$ the LHS of the above inequality and write $I_{t,h}^\phi \le I_{t,h,\epsilon}^{\phi,1} + I_{t,h,\epsilon}^{\phi,2} + I_{t,h,\epsilon}^{\phi,3}$, where

$$I_{t,h,\epsilon}^{\phi,1} := \left|\mathbb{E}\left(\Delta_h^n\phi(X_t)\left[\frac{1}{|\sigma^{-1}(X_t)|^n} - \frac{1}{|\sigma^{-1}(X_{t-\epsilon})|^n}\right]\right)\right|,$$
$$I_{t,h,\epsilon}^{\phi,2} := \left|\mathbb{E}\left([\Delta_h^n\phi(X_t) - \Delta_h^n\phi(X_t^\epsilon)]\frac{1}{|\sigma^{-1}(X_{t-\epsilon})|^n}\right)\right|,$$
$$I_{t,h,\epsilon}^{\phi,3} := \left|\mathbb{E}\left(\frac{\Delta_h^n\phi(X_t^\epsilon)}{|\sigma^{-1}(X_{t-\epsilon})|^n}\right)\right|.$$

By (3.1) and since $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ (and thus is bounded), we get $||\sigma^{-1}(X_t)|^{-n} - |\sigma^{-1}(X_{t-\epsilon})|^{-n}| \le C_n(|X_t - X_{t-\epsilon}|^{\theta_1} \wedge 1)$. Furthermore, $\Delta_h^n\phi$ is bounded by $C_n||\phi||_{C^\eta}|h|^\eta$. Consequently, $I_{t,h,\epsilon}^{\phi,1} \le C_n||\phi||_{C^\eta}|h|^\eta\mathbb{E}[(|X_t - X_{t-\epsilon}|^{\theta_1} \wedge 1)]$.

By (3.1), since σ is bounded and since $||\Delta_h^n\phi||_{C^\eta} \le C_n||\phi||_{C^\eta}$, $I_{t,h,\epsilon}^{\phi,2} \le C_n||\phi||_{C^\eta}\mathbb{E}\left[|X_t - X_t^\epsilon|^\eta\right]$.

The last term is slightly more complicated. We call f_{Z_ϵ} the law of Z_ϵ. Observe that in any case, we can write $X_t^\epsilon = U_t^\epsilon + \sigma(X_{t-\epsilon})(Z_t - Z_{t-\epsilon})$ for some $\mathcal{F}_{t-\epsilon}$-measurable random variable U_t^ϵ. Since

$Z_t - Z_{t-\epsilon}$ has the same law as Z_ϵ and is independent of $\mathcal{F}_{t-\epsilon}$, we observe that (f_{Z_ϵ} has a smooth density due to Lemma 3.3)

$$\begin{aligned} I^{\phi,3}_{t,h,\epsilon} =& \left|\mathbb{E}\left[\int_{\mathbb{R}^d} \Delta_h^n \phi(U_t^\epsilon + \sigma(X_{t-\epsilon})x)\frac{f_{Z_\epsilon}(x)}{|\sigma^{-1}(X_{t-\epsilon})|^n}dx\right]\right| \\ =& \left|\mathbb{E}\left[\int_{\mathbb{R}^d} \phi(U_t^\epsilon + \sigma(X_{t-\epsilon})x)\frac{\Delta^n_{-\sigma^{-1}(X_{t-\epsilon})h}f_{Z_\epsilon}(x)}{|\sigma^{-1}(X_{t-\epsilon})|^n}dx\right]\right| \\ \le& ||\phi||_{L^\infty}\mathbb{E}\left[\frac{1}{|\sigma^{-1}(X_{t-\epsilon})|^n}\int_{\mathbb{R}^d}\left|\Delta^n_{-\sigma^{-1}(X_{t-\epsilon})h}f_{Z_\epsilon}(x)\right|dx\right]. \end{aligned}$$

Using (1.1) and then Lemma 3.3, we have a.s.

$$\int_{\mathbb{R}^d}\left|\Delta^n_{-\sigma^{-1}(X_{t-\epsilon})h}f_{Z_\epsilon}(x)\right|dx \le C_n|\sigma^{-1}(X_{t-\epsilon})h|^n\|D^n f_{Z_\epsilon}\|_{L^1} \le C_n|\sigma^{-1}(X_{t-\epsilon})|^n|h|^n\epsilon^{-n/\alpha}.$$

We thus find $I^{\phi,3}_{t,h,\epsilon} \le C_n||\phi||_{L^\infty}|h|^n\epsilon^{-n/\alpha}$, which ends the proof since $||\phi||_{L^\infty} \le ||\phi||_{C^\eta}$. □

3.4. **Optimization and conclusion.** We have all the weapons to conclude the

Proof of Theorem 1.1. We fix $t > 0$. For $n \in \mathbb{N}$ to be chosen later, call $g_t(dx) = f_{X_t}(dx)/|\sigma^{-1}(x)|^n$. This is a nonnegative finite measure on $\mathbb{R}^d$, since by (3.1), $g_t(\mathbb{R}^d) \le ||\sigma||^n_{L^\infty}$. Observe that if g_t has a density, then f_{X_t} has a density on $\{y \in \mathbb{R}^d : \sigma(y) \text{ invertible}\}$. For $h \in \mathbb{R}^d$ with $|h| \le 1$ and $\phi \in C^\eta(\mathbb{R}^d)$, we define

$$I^\phi_{t,h} = \int_{\mathbb{R}^d}\Delta_h^n\phi(x)g_t(dx) = E[\Delta_h^n\phi(X_t)/|\sigma^{-1}(X_t)|^n].$$

We aim to apply Lemma 2.1. By Lemma 3.4, we know that for all $|h| < 1$, $\epsilon \in (0, t\wedge 1)$, $\eta \in (0,1)$,

$$I^\phi_{t,h} \le C_n||\phi||_{C^\eta}\left(|h|^\eta\mathbb{E}\left[|X_t - X_{t-\epsilon}|^{\theta_1}\wedge 1\right] + \mathbb{E}\left[|X_t - X_t^\epsilon|^\eta\right] + \epsilon^{-n/\alpha}|h|^n\right).$$

Case 1: $\alpha \in [1,2)$. Recall that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$, $b \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_1 \in (0,1)$, some $\theta_2 \in [0,1)$ (with the unusual convention that $b \in C^0(\mathbb{R}^d)$ only means that b is measurable). Fix $\eta \in (0,1)$. For any $\epsilon \in (0, t\wedge 1]$, Lemma 3.1-(i) tells us that $\mathbb{E}[|X_t - X_{t-\epsilon}|^{\theta_1}\wedge 1] \le C\epsilon^{\theta_1/\alpha}$ (because $0 < \theta_1 < 1 \le \alpha$) and Lemma 3.1-(ii) implies that $\mathbb{E}[|X_t - X_t^\epsilon|^\eta] \le C_\eta(\epsilon^{\eta(1+\theta_1)/\alpha} + \epsilon^{\eta(1+\theta_2/\alpha)})$ (because $0 < \eta < \alpha$), whence $\mathbb{E}[|X_t - X_t^\epsilon|^\eta] \le C_\eta\epsilon^{\kappa\eta/\alpha}$ where $\kappa = \min\{1+\theta_1, \alpha+\theta_2\}$ as in the statement. We thus have, for $|h| \le 1$, $\eta \in (0,1)$, $\epsilon \in (0, t\wedge 1)$,

$$I^\phi_{t,h} \le C_{\eta,n}||\phi||_{C^\eta}\left[|h|^\eta\epsilon^{\theta_1/\alpha} + \epsilon^{\kappa\eta/\alpha} + |h|^n\epsilon^{-n/\alpha}\right].$$

Choose $\epsilon = (t\wedge 1)|h|^a$, where $a = \alpha n/(n+\kappa\eta)$, which gives

$$I^\phi_{t,h} \le \frac{C_{\eta,n}}{(t\wedge 1)^{n/\alpha}}||\phi||_{C^\eta}\left[|h|^{n\theta_1/(n+\kappa\eta)+\eta} + |h|^{n\kappa\eta/(n+\kappa\eta)}\right].$$

Let $\gamma \in (0, \kappa-1)$ as in the statement and recall that $\theta_1 \ge \kappa - 1$. Choose $n \in \mathbb{N}$ large enough and $\eta \in (0,1)$ close enough to 1 so that

$$n\theta_1/(n+\kappa\eta) + \eta > \gamma + \eta \quad \text{and} \quad n\kappa\eta/(n+\kappa\eta) > \gamma + \eta.$$

Then

$$I^\phi_{t,h} \le \frac{C_{\eta,n,\gamma}}{(t\wedge 1)^{n/\alpha}}||\phi||_{C^\eta}|h|^{\gamma+\eta}.$$

Lemma 2.1 then tells us that g_t has a density belonging to $B^\gamma_{1,\infty}(\mathbb{R}^d)$, which was our goal.

Case 2: $\alpha \in (0,1)$. Recall that $\sigma \in C^{\theta_1}(\mathbb{R}^d)$ and $\tilde{b} \in C^{\theta_2}(\mathbb{R}^d)$ for some $\theta_1, \theta_2 \in (0,1)$. Fix $\eta \in (0,\alpha)$ and $\epsilon \in (0, t\wedge 1]$. By Lemma 3.2-(ii), we know that $\mathbb{E}[|X_t - X_{t-\epsilon}|^{\theta_1} \wedge 1] \le C\epsilon^{\theta_1}$. Lemma 3.2-(iii) yields, since $0<\eta<\alpha$, $\mathbb{E}[|X_t - X_t^\epsilon|^\eta] \le C_\eta[\epsilon^{\eta(\theta_1+1/\alpha)} + \epsilon^{\eta(1+\theta_2/\alpha)} + \epsilon^{\eta/(1-\theta_2)}] \le C_\eta \epsilon^{\kappa\eta/\alpha}$, where $\kappa = \min\{1+\alpha\theta_1, \alpha+\theta_2, \alpha/(1-\theta_2)\}$ as in the statement. We have checked that for $n\ge 1$, $\eta \in (0,\alpha)$,

$$I_{t,h}^\phi \le C_{\eta,n}||\phi||_{C^\eta}\left[\epsilon^{\theta_1}|h|^\eta + \epsilon^{\kappa\eta/\alpha} + |h|^n\epsilon^{-n/\alpha}\right].$$

Observe that $\kappa>1$ because $\theta_2 > 1-\alpha$ by assumption. Choose $\epsilon = (t\wedge 1)|h|^a \le t\wedge 1$, where $a = \alpha n/(n+\kappa\eta)$, which gives

$$I_{t,h}^\phi \le \frac{C_{\eta,n,\delta}}{(t\wedge 1)^{n/\alpha}}||\phi||_{C^\eta}\left[|h|^{n\alpha\theta_1/(n+\kappa\eta)+\eta} + |h|^{n\kappa\eta/(n+\kappa\eta)}\right].$$

Let $\gamma \in (0, \alpha(\kappa-1))$ as in the statement and recall that $\alpha\theta_1 \ge \kappa - 1$. Choose $n\in\mathbb{N}$ large enough and $\eta \in (0,\alpha)$ close enough to α so that

$$n\alpha\theta_1/(n+\kappa\eta)+\eta > \gamma+\eta \quad \text{and} \quad n\kappa\eta/(n+\kappa\eta) > \gamma+\eta.$$

With these choices,

$$I_{t,h}^\phi \le \frac{C_{\eta,n,\delta}}{(t\wedge 1)^{n/\alpha}}||\phi||_{C^\eta}|h|^{\gamma+\eta}.$$

Thus we deduce from Lemma 2.1 that g_t has a density in $B^\gamma_{1,\infty}(\mathbb{R}^d)$ as desired. □

4. Proof of the corollary

Proof of Corollary 1.2. We thus consider a d-dimensional $(\mathcal{F}_t)_{t\ge 0}$-Lévy process $(Z_t)_{t\ge0}$ with Lévy measure m satisfying only $m(\{|z|\ge 1\})<\infty$ and (H_α)-(ii)-(iii) for some $\alpha\in(0,2)$, with characteristic function as in (1.3), as well as some bounded measurable coefficients $\sigma : \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$ and $b:\mathbb{R}^d\mapsto\mathbb{R}^d$. We fix $x\in\mathbb{R}^d$ and consider a $(\mathcal{F}_t)_{t\ge0}$-adapted càdlàg solution $(X_t)_{t\ge0}$ to (1.2). For $A>1$ (large), we introduce the Lévy process $(Z_t^A)_{t\ge0}$ built from $(Z_t)_{t\ge0}$ by erasing all its jumps with norm greater than A. The Lévy measure of $(Z_t^A)_{t\ge0}$ is given by $m_A(dz) = \mathbb{1}_{\{|z|\le A\}}m(dz)$, so that m_A satisfies (H_α). Introduce, for $T>0$, the event $\Omega_{A,T} := \{(Z_t^A)_{t\in[0,T]} = (Z_t)_{t\in[0,T]}\}$. Then $\Pr[\Omega_{A,T}] = 1-\exp(-m(\{|z|\ge A\})T)$ increases to 1 as $A\to\infty$ because $m(\{|z|\ge1\})<\infty$. Conditionally on $\Omega_{A,T}$, $(X_t)_{t\in[0,T]}$, solves

$$X_t = x + \int_0^t \sigma(X_{s-})dZ_s^A + \int_0^t b(X_s)ds.$$

Since $(Z_t^A)_{t\ge0}$ is independent of $\Omega_{A,T}$, $(Z_t^A)_{t\ge0}$ is still a Lévy process with Lévy measure m_A conditionally on $\Omega_{A,T}$. Thus under the assumptions stated in (i), (ii) or (iii), we may apply Theorem 1.1: conditionally on $\Omega_{A,T}$, f_{X_t} has a density on the set $\{y\in\mathbb{R}^d : \sigma(y) \text{ invertible}\}$ for all $t\in(0,T]$. In other words, for all Lebesgue-null Borel subset B of $\{y\in\mathbb{R}^d:\sigma(y)\text{ invertible}\}$, any $t\in(0,T]$, any $A>1$, it holds that $\Pr[X_t\in B, \Omega_{A,T}]=0$. By the monotone convergence theorem, we have $\Pr[X_t\in B] = \lim_{A\to\infty}\Pr[X_t\in B,\Omega_{A,T}]$. Consequently, $\Pr[X_t\in B]=0$ for all $t>0$ and all Lebesgue-null Borel subset B of $\{y\in\mathbb{R}^d:\sigma(y)\text{ invertible}\}$, which ends the proof. □

5. Appendix

5.1. **About our assumptions.** We first establish some consequences of (H_α).

Lemma 5.1. *Let m be a nonnegative measure on $\mathbb{R}^d$.*

(i) If m satisfies (H_α)-(i)-(ii), then for all $\beta \in [0,\alpha)$, all $a \in (0,1]$,

$$\int_{\{|z|\geq a\}} |z|^\beta m(dz) \leq C_\beta a^{\beta-\alpha}.$$

(ii) If m satisfies (H_α)-(ii) and $\int_{\{|z|\geq 1\}} m(dz) < \infty$, then for all $a \in (0,1]$,

$$\int_{\{|z|\geq a\}} m(dz) \leq C a^{-\alpha}.$$

(iii) If m satisfies (H_α)-(ii), then for all $\gamma \in (\alpha, 2)$, all $a \in (0,1]$,

$$\int_{\{|z|\leq a\}} |z|^\gamma m(dz) \leq C a^{\gamma-\alpha}.$$

Proof. We start with (i). Using (H_α)-(i)-(ii), we see that for $a \in (0,1]$, $\beta \in [0,\alpha)$,

$$\begin{aligned}
\int_{\{|z|\geq a\}} |z|^\beta m(dz) \leq& C_\beta + \int_{\{a\leq |z|\leq 1\}} |z|^\beta m(dz)\\
=& C_\beta + (2-\beta) \int_{\{a\leq |z|\leq 1\}} |z|^2 m(dz) \int_{|z|}^\infty u^{\beta-3} du\\
=& C_\beta + (2-\beta) \int_a^\infty u^{\beta-3} du \int_{\{a\leq |z|\leq u\wedge 1\}} |z|^2 m(dz)\\
\leq& C_\beta + C_\beta \int_a^\infty u^{\beta-3} (u\wedge 1)^{2-\alpha} du\\
\leq& C_\beta (1 + a^{\beta-\alpha}),
\end{aligned}$$

from which we conclude since $a \in (0,1]$. Point (ii) is checked similarly: copy the proof of (i) with $\beta = 0$ and observe that we do not use (H_α)-(i). We finally check (iii), using (H_α)-(ii):

$$\begin{aligned}
\int_{\{|z|\leq a\}} |z|^\gamma m(dz) =& \sum_{n=0}^\infty \int_{\{a2^{-n-1}<|z|\leq a2^{-n}\}} |z|^\gamma m(dz)\\
\leq& \sum_{n=0}^\infty (a2^{-n-1})^{\gamma-2} \int_{\{|z|\leq a2^{-n}\}} |z|^2 m(dz)\\
\leq& C \sum_{n=0}^\infty (a2^{-n-1})^{\gamma-2} (a2^{-n})^{2-\alpha}\\
=& C a^{\gamma-\alpha} \sum_{n=0}^\infty 2^{-n(\gamma-\alpha)},
\end{aligned}$$

from which the conclusion follows, since $\alpha < \gamma$. ∎

Next we give the

Proof of Lemma 1.3. Recall that m is supposed to satisfy (H_α)-(ii) and $\int_{\{|z|\geq 1\}} m(dz) < \infty$.

Step 1: (b) implies (a). Recall that $1-\cos x \geq x^2/4$ for all $x \in [0,1]$. For $|\xi| \geq 1$, we write

$$\int_{\mathbb{R}^d} (1-\cos\langle \xi, z\rangle) m(dz) \geq \frac{1}{4}\int_{\{|z||\xi|\leq 1\}} \langle \xi, z\rangle^2\, m(dz) = \frac{|\xi|^2}{4}\int_{\{|z|\leq 1/|\xi|\}} \left\langle \frac{\xi}{|\xi|}, z\right\rangle^2 m(dz),$$

which is bounded from below by $c|\xi|^2(1/|\xi|)^{2-\alpha} = c|\xi|^\alpha$ by (b).

Step 2: (a) implies (b). Recall that $r>0$ was defined in (H_α)-(iii). Using (a) (i.e. (H_α)-(iii)) and that $1-\cos x \leq x^2 \wedge 2$, we deduce that for all $a \in (0,1]$, all $\delta > ar$, all $|\zeta| = 1$,

$$c(\delta/a)^\alpha \leq \int_{\mathbb{R}^d} \left(1-\cos\left\langle \frac{\delta\zeta}{a}, z\right\rangle\right) m(dz) \leq \int_{\{|z|\leq a\}} \left\langle \frac{\delta\zeta}{a}, z\right\rangle^2 m(dz) + 2\int_{\{|z|\geq a\}} m(dz).$$

But $\int_{\{|z|\geq a\}} m(dz) \leq Ca^{-\alpha}$ by Lemma 5.1-(ii). Hence for all $|\zeta|=1$, all $a\in(0,1]$, all $\delta > ar$,

$$\int_{\{|z|\leq a\}} \langle \zeta, z\rangle^2\, m(dz) \geq c(\delta/a)^{\alpha-2} - C(\delta/a)^{-2}a^{-\alpha} = \delta^{-2}(c\delta^\alpha - C)a^{2-\alpha}.$$

Choose $\delta = \delta_0 = r + (2C/c)^{1/\alpha}$ (which indeed satisfies $\delta > ar$ for all $a \in (0,1]$): we get, for all $a \in (0,1]$ and all $|\zeta| = 1$,

$$\int_{\{|z|\leq a\}} \langle \zeta, z\rangle^2\, m(dz) \geq \kappa a^{2-\alpha},$$

where $\kappa = \delta_0^{-2}(c\delta_0^\alpha - C) > 0$ as desired. □

5.2. **Stochastic integrals.** We next compute some moments of some stochastic integrals.

Lemma 5.2. *Let $(Z_t)_{t\geq 0}$ be a Lévy process with characteristic function as in (1.3) with a Lévy measure m satisfying (H_α)-(i)-(ii) for some $\alpha \in (0,2)$. Let also $(H_s)_{s\geq 0}$ be a predictable process (with values in $M_{d\times d}(\mathbb{R})$).*

(i) If $\alpha \in [1,2)$, then for all $0<\beta<\alpha<\gamma\leq 2$, all $0\leq s\leq t\leq s+1$,

$$\mathbb{E}\left[\left|\int_s^t H_u dZ_u\right|^\beta\right] \leq C_{\beta,\gamma}(t-s)^{\beta/\alpha} \sup_{[s,t]} \mathbb{E}\left[|H_u|^\gamma\right]^{\beta/\gamma}.$$

(ii) If $\alpha \in (0,1)$, then for all $0<\beta<\alpha<\gamma\leq 1$, all $0\leq s\leq t\leq s+1$,

$$\mathbb{E}\left[\left|\int_s^t H_u dY_u\right|^\beta\right] \leq C_{\beta,\gamma}(t-s)^{\beta/\alpha} \sup_{[s,t]} \mathbb{E}\left[|H_u|^\gamma\right]^{\beta/\gamma} \quad \text{and} \quad \mathbb{E}\left[\left(\sum_{s\leq u\leq t} |\Delta Y_u|\right)^\beta\right] \leq C_\beta (t-s)^{\beta/\alpha},$$

where $Y_t := Z_t + t\int_{\{|z|\leq 1\}} z m(dz)$.

Proof. Let us first treat (i). Write $Z_t = \int_0^t\int_{\{|z|\leq 1\}} z\tilde{N}(ds,dz) + \int_0^t\int_{\{|z|>1\}} zN(ds,dz)$ for some Poisson measure N with intensity measure $dsm(dz)$ and $\int_s^t H_u dZ_u = I^1_{s,t} + I^2_{s,t} + I^3_{s,t}$, where

$$I^1_{s,t} = \int_s^t \int_{\{|z|\leq (t-s)^\alpha\}} H_u z\tilde{N}(du,dz), \quad I^2_{s,t} = \int_s^t \int_{\{(t-s)^\alpha<|z|\leq 1\}} H_u z\tilde{N}(du,dz),$$

$$I^3_{s,t} = \int_s^t \int_{\{|z|>1\}} H_u zN(du,dz).$$

The Burkholder-Davis-Gundy inequality, the sub-additivity of $x \mapsto x^{\gamma/2}$, the Hölder inequality and finally Lemma 5.1-(iii) give

$$
\begin{aligned}
\mathbb{E}\left[\left|I^1_{s,t}\right|^\beta\right] \le & C_\beta \mathbb{E}\left[\left|\int_s^t \int_{\{|z|\le (t-s)^{1/\alpha}\}} |H_u|^2 |z|^2 N(du,dz)\right|^{\beta/2}\right] \\
\le & C_\beta \mathbb{E}\left[\left|\int_s^t \int_{\{|z|\le (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma N(du,dz)\right|^{\beta/\gamma}\right] \\
\le & C_\beta \mathbb{E}\left[\int_s^t \int_{\{|z|\le (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma N(du,dz)\right]^{\beta/\gamma} \\
= & C_\beta \mathbb{E}\left[\int_s^t \int_{\{|z|\le (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma m(dz)du\right]^{\beta/\gamma} \\
\le & C_{\beta,\gamma} \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma} \left[(t-s)\times((t-s)^{1/\alpha})^{\gamma-\alpha}\right]^{\beta/\gamma} \\
= & C_{\beta,\gamma} \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma} (t-s)^{\beta/\alpha}.
\end{aligned}
$$

The Burkholder-Davis-Gundy inequality, the sub-additivity of $x \mapsto x^{\beta/2}$ and Lemma 5.1-(i) yield

$$
\begin{aligned}
\mathbb{E}\left[\left|I^2_{s,t}\right|^\beta\right] \le & C_\beta \mathbb{E}\left[\left|\int_s^t \int_{\{(t-s)^{1/\alpha}<|z|\le 1\}} |H_u|^2 |z|^2 N(du,dz)\right|^{\beta/2}\right] \\
\le & C_\beta \mathbb{E}\left[\int_s^t \int_{\{(t-s)^{1/\alpha}<|z|\le 1\}} |H_u|^\beta |z|^\beta N(du,dz)\right] \\
= & C_\beta \mathbb{E}\left[\int_s^t \int_{\{(t-s)^{1/\alpha}<|z|\le 1\}} |H_u|^\beta |z|^\beta m(dz)du\right] \\
\le & C_\beta \sup_{[s,t]} \mathbb{E}[|H_u|^\beta] \times (t-s)((t-s)^{1/\alpha})^{\beta-\alpha} \\
= & C_\beta \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma} (t-s)^{\beta/\alpha}.
\end{aligned}
$$

To treat $I^3_{s,t}$, we separate two cases. First assume that $\beta \le 1$ and write, using the sub-additivity of $x \mapsto x^\beta$ and (H_α)-(i)

$$
\begin{aligned}
\mathbb{E}\left[\left|I^3_{s,t}\right|^\beta\right] \le & C_\beta \mathbb{E}\left[\int_s^t \int_{\{|z|>1\}} |H_u|^\beta |z|^\beta N(du,dz)\right] \\
= & C_\beta \mathbb{E}\left[\int_s^t \int_{\{|z|>1\}} |H_u|^\beta |z|^\beta m(dz)du\right] \\
\le & C_\beta \sup_{[s,t]} \mathbb{E}[|H_u|^\beta](t-s) \\
\le & C_\beta \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma} (t-s)^{\beta/\alpha}
\end{aligned}
$$

(because $t-s\le 1$ and $\beta/\alpha<1$). If finally $\beta>1$ (whence $\alpha>1$), we put $\tau=\int_{\{|z|>1\}} zm(dz)$ and write, using similar arguments as previously (and that $\gamma>\beta$ and $\beta/\alpha<1<\beta$),

$$\begin{aligned}
\mathbb{E}\left[\left|I^3_{s,t}\right|^\beta\right] \le & C_\beta \mathbb{E}\left[\left|\int_s^t\int_{\{|z|>1\}} H_u z\tilde{N}(du,dz)\right|^\beta\right] + C_\beta\mathbb{E}\left[\left|\tau\int_s^t H_u du\right|^\beta\right]\\
\le & C_\beta \mathbb{E}\left[\left|\int_s^t\int_{\{|z|>1\}} |H_u|^2|z|^2 N(du,dz)\right|^{\beta/2}\right] + C_\beta\mathbb{E}\left[(t-s)^{\beta-1}\int_s^t |H_u|^\beta du\right]\\
\le & C_\beta \mathbb{E}\left[\int_s^t\int_{\{|z|>1\}} |H_u|^\beta|z|^\beta N(du,dz)\right] + C_\beta \sup_{[s,t]}\mathbb{E}[|H_u|^\beta](t-s)^\beta\\
\le & C_\beta \sup_{[s,t]}\mathbb{E}[|H_u|^\beta][(t-s)+(t-s)^\beta]\\
\le & C_\beta \sup_{[s,t]}\mathbb{E}[|H_u|^\gamma]^{\beta/\gamma}(t-s)^{\beta/\alpha}.
\end{aligned}$$

We now prove (ii), writing $Y_t=\int_0^t\int_{\mathbb{R}^d} zN(ds,dz)$ for some Poisson measure N with intensity measure $dsm(dz)$. It holds that $\mathbb{E}[|\int_s^t H_u dY_u|^\beta]\le J^1_{s,t,\beta}+J^2_{s,t,\beta}$, where,

$$\begin{aligned}
J^1_{s,t,\beta} :=& \mathbb{E}\left[\left|\int_s^t\int_{\{|z|>(t-s)^{1/\alpha}\}} |H_u||z|N(du,dz)\right|^\beta\right],\\
J^2_{s,t,\beta} :=& \mathbb{E}\left[\left|\int_s^t\int_{\{|z|\le(t-s)^{1/\alpha}\}} |H_u||z|N(du,dz)\right|^\beta\right].
\end{aligned}$$

Using the sub-additivity of $x\mapsto x^\beta$ and then Lemma 5.1-(i),

$$\begin{aligned}
J^1_{s,t,\beta} \le & \mathbb{E}\left[\int_s^t\int_{\{|z|>(t-s)^{1/\alpha}\}} |H_u|^\beta|z|^\beta N(du,dz)\right]\\
=& \mathbb{E}\left[\int_s^t\int_{\{|z|>(t-s)^{1/\alpha}\}} |H_u|^\beta|z|^\beta m(dz)du\right]\\
\le & C_\beta(t-s)\sup_{[s,t]}\mathbb{E}[|H_u|^\beta][(t-s)^{1/\alpha}]^{\beta-\alpha}\\
\le & C_\beta(t-s)^{\beta/\alpha}\sup_{[s,t]}\mathbb{E}[|H_u|^\gamma]^{\beta/\gamma}.
\end{aligned}$$

The sub-additivity of $x \mapsto x^\gamma$, the fact that $\beta/\gamma < 1$ and Lemma 5.1-(iii)

$$
\begin{aligned}
J^2_{s,t,\beta} \leq & \mathbb{E}\left[\left|\int_s^t \int_{\{|z|\leq (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma N(du,dz)\right|^{\beta/\gamma}\right] \\
\leq & \mathbb{E}\left[\int_s^t \int_{\{|z|\leq (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma N(du,dz)\right]^{\beta/\gamma} \\
= & \mathbb{E}\left[\int_s^t \int_{\{|z|\leq (t-s)^{1/\alpha}\}} |H_u|^\gamma |z|^\gamma m(dz)du\right]^{\beta/\gamma} \\
\leq & C_{\beta,\gamma}(t-s)^{\beta/\gamma} \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma} ([(t-s)^{1/\alpha}]^{\gamma-\alpha})^{\beta/\gamma} \\
= & C_{\beta,\gamma}(t-s)^{\beta/\alpha} \sup_{[s,t]} \mathbb{E}[|H_u|^\gamma]^{\beta/\gamma}
\end{aligned}
$$

as desired. Finally, $\sum_{s\leq u\leq t} |\Delta Y_u| = \int_0^t \int_{\mathbb{R}^d} |z| N(du,dz)$, so that exactly the same computation as previously (with $H_u \equiv 1$ and e.g. $\gamma = 1$) gives $\mathbb{E}\left[\left(\sum_{s\leq u\leq t} |\Delta Y_u|\right)^\beta\right] \leq C_\beta (t-s)^{\beta/\alpha}$. □

5.3. **Proof of the absolute continuity criterion.** A proof using several functional analysis theorems can be found in [5, end of the proof of Theorem 5.1]. For readers familiar with functional analysis, the proof of [5] should be more natural. Here we present an *elementary* proof, extending slightly that of [7] which concerns the case $n = 1$.

Proof of Lemma 2.1. Recall that $n \geq 1$, $K > 0$ and $0 < \eta < a < 1$ are fixed and let $g \in \mathcal{M}(\mathbb{R}^d)$ satisfy (2.4).

Step 1. For $r > 0$, consider the function $\chi_r(x) = (v_d r^d)^{-1} \mathbb{1}_{\{|x|<r\}}$, where v_d is the volume of the unit ball in $\mathbb{R}^d$. We first check that for all $\psi \in L^\infty(\mathbb{R}^d)$, $||\psi \star \chi_r||_{C^\eta} \leq C_d ||\psi||_{L^\infty}(1 + r^{-\eta})$. It obviously holds that $||\psi \star \chi_r||_{L^\infty} \leq ||\psi||_{L^\infty}$. Next, an easy computation shows that for all $x, y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} |\chi_r(x-z) - \chi_r(y-z)|dz \leq C_d \min(1, |x-y|/r)$, whence $|\psi \star \chi_r(x) - \psi \star \chi_r(y)| \leq C_d ||\psi||_{L^\infty} \min(1, |x-y|/r) \leq C_d ||\psi||_{L^\infty} r^{-\eta} |x-y|^\eta$.

We now set $\rho_r = \sum_{j=1}^n (-1)^{j+1} \binom{n}{j} \chi_{jr}$ and deduce from above that: $||\psi \star \rho_r||_{C^\eta} \leq C_{n,d} ||\psi||_{L^\infty}(1 + r^{-\eta}) \leq C_{n,d} ||\psi||_{L^\infty} r^{-\eta}$ for all $r \in (0,1]$.

Step 2. Here we set $g_r = g \star \rho_r$ and check that for all $r \in (0,1]$, all $|h| \leq 1$, $||\Delta_h^n g_r||_{L^1} \leq C_{n,d} K |h|^a r^{-\eta}$. It suffices to prove that for any $\psi \in L^\infty(\mathbb{R}^d)$, $I_r(h,\psi) := |\int_{\mathbb{R}^d} \psi(x) \Delta_h^n g_r(x) dx| \leq C_{n,d} K ||\psi||_{L^\infty} |h|^a r^{-\eta}$. But using (2.4) and Step 1, we get

$$
I_r(h,\psi) = \left|\int_{\mathbb{R}^d} \Delta_{-h}^n [\psi \star \rho_r](y) g(dy)\right| \leq K ||\psi \star \rho_r||_{C^\eta} |h|^a \leq C_{n,d} K ||\psi||_{L^\infty} |h|^a r^{-\eta}.
$$

Step 3. Here we assume additionally that g has a density in $C^1(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} |\nabla g(x)| dx < \infty$, which implies that all the computations below are licit, and we check that

$$
\sup_{|h|\leq 1} |h|^{\eta - a} ||\Delta_h^n g||_{L^1} \leq C_{d,a,\eta,n} K.
$$

To this end, we first write, using a change of variable,

$$g_r(x) = \frac{1}{v_d r^d} \sum_{j=1}^n (-1)^{j+1} \binom{n}{j} \int_{\{|y|<r\}} g(x+jy)dy.$$

Therefore $g(x) - g_r(x) = (-1)^n (v_d r^d)^{-1} \int_{\{|y|<r\}} \Delta_y^n g(x)dy$, so that for $r \in (0,1]$,

$$||g - g_r||_{L^1} \le \frac{1}{v_d r^d} \int_{\{|y|<r\}} dy \|\Delta_y^n g\|_{L^1} \le \frac{1}{v_d r^d} \int_{\{|y|<r\}} dy|y|^{a-\eta} \sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1}$$

and thus

$$||g - g_r||_{L^1} \le r^{a-\eta} \sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1}.$$

Next, one easily checks that $||\Delta_h^n f||_{L^1} \le 2^n ||f||_{L^1}$, whence, using Step 2 and the previous estimate, for all $|h| \le 1$, all $r \in (0,1]$,

$$||\Delta_h^n g||_{L^1} \le ||\Delta_h^n g_r||_{L^1} + 2^n ||g - g_r||_{L^1} \le C_{n,d} K |h|^a r^{-\eta} + 2^n r^{a-\eta} \sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1}.$$

Thus for all $|h| \in (0,1]$, all $r \in (0,1]$,

$$|h|^{\eta-a} ||\Delta_h^n g||_{L^1} \le C_{n,d} K |h|^\eta r^{-\eta} + 2^n |h|^{\eta-a} r^{a-\eta} \sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1}.$$

Choosing $r = 2^{-(n+1)/(a-\eta)}|h|$ and taking the supremum over $|h| \le 1$, we get

$$\sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1} \le 2^{(n+1)\eta/(a-\eta)} C_{n,d} K + \frac{1}{2} \sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1}.$$

This implies $\sup_{|h|\le 1} |h|^{\eta-a} \|\Delta_h^n g\|_{L^1} \le 2^{1+(n+1)\eta/(a-\eta)} C_{n,d} K$ as desired.

Step 4. Consider now g as in the statement. For $\ell \ge 1$, put $g_\ell = g \star G_\ell$, where $G_\ell(x) = (\ell/\pi)^{d/2} e^{-\ell |x|^2}$. Then $g_\ell \in C^1(\mathbb{R}^d)$, $g_\ell(\mathbb{R}^d) = g(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |\nabla g_\ell(x)|dx < \infty$. Furthermore, one easily checks that g_ℓ satisfies (2.4) with the same constant K as g. Thus we can apply Step 3 and deduce that $||g_\ell||_{B^{a-\eta}_{1,\infty}} \le g(\mathbb{R}^d) + C_{d,a,\eta,n} K$.

Recall that $B^{a-\eta}_{1,\infty}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for every $p \in [1, \frac{d}{d-a+\eta})$ (by [14, formula 2.2.2/(18)] and Sobolev's embeddings), it follows that $(g_\ell)_{\ell \in \mathbb{N}}$ is uniformly integrable and since it converges to g in $\mathcal{M}$, the convergence holds in fact in L^p. Moreover, since the bound above is independent of ℓ, we easily conclude that g has a density satisfying $||g||_{B^{a-\eta}_{1,\infty}} \le g(\mathbb{R}^d) + C_{d,a,\alpha} K$. □

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