ERRATUM

Samir Salem pointed out an error in the proof point (i) of Lemma C.2 of [1]: we applied the Jensen inequality to the Lebesgue measure on $(\mathbb{R}^3)^{N-k}$, which is not a probability measure. And by the way, we did not provide any rigorous argument ensuring that $\int_{(\mathbb{R}^3)^{N-k}} \nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^n)\, d\boldsymbol{v}_{N-k}^N$, which appears in argument of Ψ_r, is well defined almost everywhere. We can justify the use of the Jensen inequality by saying that Ψ_r is actually homogeneous of degree one. That would allow to apply the Jensen inequality with respect to the uniform probability of a ball of given radius, and then obtain the requested inequality by passing to the limit. But passing to the limit requires a proper answer to the second point.

Here is a more direct proof that fixes both issues. Since F is symmetric, we can write

$$I^r(F) = \int_{(\mathbb{R}^3)^N} \frac{|\nabla_{v_1} F(\boldsymbol{v}^N)|^{2r}}{[F(\boldsymbol{v}^N)]^{2r-1}} d\boldsymbol{v}^N \quad \text{and} \quad I_\gamma(F) = \int_{(\mathbb{R}^3)^N} \langle v_1 \rangle^\gamma \frac{|\nabla_{v_1} F(\boldsymbol{v}^N)|^2}{F(\boldsymbol{v}^N)}.$$

Fix $k \in \{1, \dots, N-1\}$ and set $\boldsymbol{v}_k^N = (v_1, \dots, v_k)$ and $\boldsymbol{v}_{N-k}^N = (v_{k+1}, \dots, v_N)$. Since $2r > 1$, we have, by the Hölder inequality,

$$\begin{aligned}
\int_{(\mathbb{R}^3)^{N-k}} \left|\nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)\right| d\boldsymbol{v}_{N-k}^N &= \int_{(\mathbb{R}^3)^{N-k}} \frac{\left|\nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)\right|}{[F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)]^{\frac{2r-1}{2r}}} [F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)]^{\frac{2r-1}{2r}} \, d\boldsymbol{v}_{N-k}^N \\
&\le \left(\int_{(\mathbb{R}^3)^{N-k}} \frac{\left|\nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)\right|^{2r}}{[F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)]^{2r-1}} d\boldsymbol{v}_{N-k}^N \right)^{\frac{1}{2r}} \left(\int_{(\mathbb{R}^3)^{N-k}} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)\, d\boldsymbol{v}_{N-k}^N \right)^{\frac{2r-1}{2r}} \\
&= \left(\int_{(\mathbb{R}^3)^{N-k}} \frac{\left|\nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)\right|^{2r}}{[F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)]^{2r-1}} d\boldsymbol{v}_{N-k}^N \right)^{\frac{1}{2r}} \left(F_k(\boldsymbol{v}_k^N)\right)^{\frac{2r-1}{2r}}.
\end{aligned}$$

The two integrals in the r.h.s. are finite almost everywhere, since F is a probability density and $I^r(F) < \infty$. Consequently,

$$\frac{|\nabla_{v_1} F_k(\boldsymbol{v}_k^N)|^{2r}}{[F_k(\boldsymbol{v}_k^N)]^{2r-1}} \le \int_{(\mathbb{R}^3)^{N-k}} \frac{|\nabla_{v_1} F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)|^{2r}}{[F(\boldsymbol{v}_k^N, \boldsymbol{v}_{N-k}^N)]^{2r-1}} d\boldsymbol{v}_{N-k}^N.$$

Integrating this inequality in $\boldsymbol{v}_k^N$, we obtain $I^r(F_k) \le I^r(F)$. Choosing next $r = 1$, multiplying the inequality by $\langle v_1 \rangle^\gamma$ and integrating in $\boldsymbol{v}_k^N$, we obtain $I_\gamma(F_k) \le I_\gamma(F)$.

References

[1] Fournier, N., Hauray, M. Propagation of chaos for the Landau equation with moderately soft potentials. *Ann. Probab. 44* (2016), no. 6, 3581–3660.