STOCHASTIC PARTICLE APPROXIMATION OF THE KELLER-SEGEL EQUATION AND TWO-DIMENSIONAL GENERALIZATION OF BESSEL PROCESSES

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Abstract. The Keller-Segel partial differential equation is a two-dimensional model for chemotaxis (and for Newtonian gravitational interaction). When the total mass of the initial density is one, it is known to exhibit blow-up in finite time as soon as the sensitivity χ of bacteria to the chemo-attractant is larger than 8π. We investigate its approximation by a system of N two-dimensional Brownian particles interacting through a singular attractive kernel in the drift term.

In the very subcritical case $\chi < 2\pi$, the diffusion strongly dominates this singular drift: we obtain existence for the particle system and prove that its flow of empirical measures converges, as $N \to \infty$ and up to extraction of a subsequence, to a weak solution of the Keller-Segel equation.

We also show that for any $N \geq 2$ and any value of $\chi > 0$, pairs of particles do collide with positive probability: the singularity of the drift is indeed visited. Nevertheless, when $\chi < 2\pi N$, it is possible to control the drift and obtain existence of the particle system until the first time when at least three particles collide. We check that this time is a.s. infinite, so that global existence holds for the particle system, if and only if $\chi \leq 8\pi(N-2)/(N-1)$.

Finally, we remark that in the system with $N = 2$ particles, the difference between the two positions provides a natural two-dimensional generalization of Bessel processes, which we study in details.

1. Introduction and results

1.1. **The model.** The Keller-Segel equation, introduced by Patlak [34] and Keller and Segel [24], is a model for chemotaxis. It describes the collective motion of cells which are attracted by a chemical substance and are able to emit it. In its simplest form it is a conservative drift/diffusion equation for the density $f_t(x) \geq 0$ of cells (particles) with position $x \in \mathbb{R}^2$ at time $t \geq 0$ coupled with an elliptic equation for the chemo-attractant concentration. By making the chemo-attractant concentration explicit in terms of the cell density, one obtains the following closed equation:

$$\partial_t f_t(x) + \chi \mathrm{div}_x((K \star f_t)(x) f_t(x)) = \Delta_x f_t(x), \tag{1}$$

where $\chi > 0$ is the sensitivity of cells to the chemo-attractant and where

$$K(x) = \frac{-x}{2\pi |x|^2}. \tag{2}$$

In the whole paper, we adopt the convention that $K(0) = 0$. Let us mention that (1) also describes the time-evolution of the particles density in an infinite system of planar Brownian particles subject

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to Newtonian gravitational interaction. Then χ is interpreted as the intensity of the interaction (or, if the latter is fixed, as the inverse of the diffusion coefficient).

This equation preserves mass and $f_t(x)/\int_{\mathbb{R}^2} f_0(y)dy$ solves the same equation with χ replaced by $\chi\int_{\mathbb{R}^2} f_0(y)dy$. We thus may assume without loss of generality that $\int_{\mathbb{R}^2} f_0(x)dx = 1$.

As is well-known, we have formally $\frac{d}{dt}\int_{\mathbb{R}^2} xf_t(x)dx = 0$ and $\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 f_t(x)dx = 4 - \chi/(2\pi)$. Consequently, introducing $V_t := \int_{\mathbb{R}^2} |x - \int_{\mathbb{R}^2} yf_t(y)dy|^2 f_t(x)dx$, it holds that $\frac{d}{dt}V_t = 4 - \chi/(2\pi)$. Since V_t is nonnegative, some kind of blow-up necessarily occurs before time $2\pi V_0/(\chi - 8\pi)$ when χ is larger than the critical value 8π.

Concerning the well-posedness theory, let us mention Jäger and Luckhaus [19], Blanchet, Dolbeault and Perthame [1], Dolbeault and Schmeiser [6], Carrillo, Lisini and Mainini [3] and Egaña and Mischler [7]. In particular, the existence of solutions is verified in [19] (for sufficiently smooth initial conditions), these solutions being local (in time) if $\chi > 0$ is large and global if $\chi > 0$ is small. The uniqueness of bounded solutions was proved in [3] using the quadratic Wasserstein distance. The existence of a unique *strong* (in some precise sense) solution when $\chi < 8\pi$ is shown in [1] (existence) and [7] (uniqueness), still for reasonable initial conditions. The main tool is the *free energy* and its relation with its time derivative. By passing to the limit in a sequence of regularized Keller-Segel equations where the kernel K is replaced by a bounded kernel and by introducing defect measures to take into account blow-up, the existence of generalized weak solutions to (1) is checked in [6], even when $\chi \geq 8\pi$. The blow-up phenomenon has been investigated by Herrero and Velazquez [16, 41, 42]. We refer to Horstmann [17, 18] and Perthame [35] for review papers on this model.

1.2. **Weak solutions.** We denote by $\mathcal{P}(\mathbb{R}^2)$ the set of probability measures on $\mathbb{R}^2$ and we set $\mathcal{P}_1(\mathbb{R}^2) = \{f \in \mathcal{P}(\mathbb{R}^2) \; : \; m_1(f) < \infty\}$, where $m_1(f) = \int_{\mathbb{R}^2}|x|f(dx)$. We will use the following notion of weak solutions.

Definition 1. *Let $\chi > 0$ and $T \in (0, \infty]$ be fixed. We say that a measurable family $(f_t)_{t\in[0,T)}$ of probability measures on $\mathbb{R}^2$ is a weak solution to (1) on $[0, T)$ if the following conditions hold true:*

(a) for all $t \in [0, T)$, $\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x - y|^{-1} f_s(dy)f_s(dx)ds < \infty$;

(b) for all $\phi \in C_b^2(\mathbb{R}^2)$, all $t \in [0, T)$,

$$\int_{\mathbb{R}^2} \phi(x)f_t(dx) = \int_{\mathbb{R}^2} \phi(x)f_0(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta\phi(x)f_s(dx)ds + \chi \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x - y) \cdot \nabla\phi(x)f_s(dy)f_s(dx)ds.$$

Of course, (a) implies that everything makes sense in (b). Performing a symmetrization in the last term leads to another weak formulation of (1) which requires less stringent integrability conditions, but which is not suitable in view of the following probabilistic interpretation.

1.3. **The associated trajectories.** We now introduce a natural probabilistic interpretation of the Keller-Segel equation.

Definition 2. *Let $\chi > 0$ and $T \in (0, \infty]$ be fixed. We say that a $\mathbb{R}^2$-valued continuous process $(X_t)_{t\in[0,T)}$ adapted to some filtration $(\mathcal{F}_t)_{t\in[0,T)}$ solves the nonlinear SDE (3) on $[0, T)$ if, for $f_t := \mathcal{L}(X_t)$, it holds that*

(a) $\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x - y|^{-1} f_s(dy)f_s(dx)ds < \infty$ for all $t \in [0, T)$;

(b) there is a 2-dimensional $(\mathcal{F}_t)_{t\in[0,T)}$-Brownian motion $(B_t)_{t\in[0,T)}$ such that for all $t \in [0,T)$

$$X_t = X_0 + \sqrt{2}B_t + \chi \int_0^t (K \star f_s)(X_s)ds. \tag{3}$$

The main idea is that $(X_t)_{t\in[0,T)}$ represents the time-evolution of the position of a *typical cell*, in an infinite system of cells undergoing the dynamics prescribed by the Keller-Segel equation. The following remark immediately follows from the Itô formula.

Remark 3. *Let $\chi > 0$ be fixed. For $(X_t)_{t\in[0,T)}$ solving the nonlinear SDE (3), the family $(f_t = \mathcal{L}(X_t))_{t\in[0,T)}$ is a weak solution to the Keller-Segel equation (1).*

1.4. **The particle system.** We next consider a natural discretization of the nonlinear SDE: we consider $N \geq 2$ particles (cells) with positions $X_t^{1,N}, \dots, X_t^{N,N}$ solving (recall that $K(0) = 0$)

$$X_t^{i,N} = X_0^{i,N} + \sqrt{2}B_t^i + \frac{\chi}{N}\sum_{j=1}^N \int_0^t K(X_s^{i,N} - X_s^{j,N})ds. \tag{4}$$

More precisely, a solution on $[0,T)$ is a continuous $(\mathbb{R}^2)^N$-valued process $(X_t^{i,N})_{i=1,\dots,N,t\in[0,T)}$ adapted to some filtration $(\mathcal{F}_t)_{t\in[0,T)}$ if $\int_0^t \sum_{i=1}^N |\sum_{j=1}^N K(X_s^{i,N} - X_s^{j,N})|ds < \infty$ a.s. for all $t \in [0,T)$ and if there is a $2N$-dimensional $(\mathcal{F}_t)_{t\in[0,T)}$-Brownian motion $(B_t^1, \dots, B_t^N)_{t\geq 0}$ such that (4) holds true for all $t \in [0,T)$ and all $i = 1, \dots, N$. In the whole paper, we assume that the initial condition $(X_0^{i,N})_{i=1,\dots,N}$ is exchangeable. This is not a limitation since for a fixed value of N, one can always reduce to this case by using an independent uniform random permutation and exchangeability is needed when considering limits as $N \to \infty$.

Of course, such a particle system is not clearly well-defined, due to the singularity of K. Moreover, the singularity *is visited*, as shown by the following statement.

Proposition 4. *For any $N \geq 2$, any $\chi > 0$, any exchangeable initial condition $(X_0^{i,N})_{i=1,\dots,N}$, any $t_0 > 0$ and any solution (if it exists) $(X_t^{i,N})_{i=1,\dots,N,t\in[0,t_0]}$ to (4),*

$$\mathbb{P}\Big(\exists\, s \in [0,t_0],\ \exists\, 1 \leq i < j \leq N \ :\ X_s^{i,N} = X_s^{j,N}\Big) > 0.$$

However, we expect that if independence initially holds, then particles are almost independent (for N large) and look like N copies of the solution to the nonlinear SDE, at least in the subcritical case $\chi \in (0, 8\pi)$ or locally in time in the supercritical case $\chi \geq 8\pi$. This problem seems important, both from a physical point of view, as a step to the rigorous derivation of the Keller-Segel equation, and from a numerical point of view.

We will not use it in the present paper, but let us mention that the formal invariant measure of (4) has a density μ_N with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$ given by

$$\mu_N(x_1, \dots, x_N) = \prod_{1\leq k<\ell\leq N} |x_k - x_\ell|^{-\chi/(2\pi N)}.$$

For any fixed $N \geq 2$, this invariant measure is locally finite if and only if $\chi < 8\pi$.

1.5. **Main results.** We denote by $\mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ the set of exchangeable probability measures on $(\mathbb{R}^2)^N$ with a finite moment of order 1. In other words, the law F_N of a $(\mathbb{R}^2)^N$-valued random variable $(X^1, \dots, X^N)$ belongs to $\mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ if the family $(X^1, \dots, X^N)$ is exchangeable and if $\mathbb{E}[|X^1|] < \infty$.

We first check that the particle system (4) exists when χ is (very) subcritical.

Theorem 5. *Let $N \geq 2$, $\chi \in (0, 2\pi N/(N-1))$ and $F_0^N \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$. There exists a solution $(X_t^{i,N})_{t\in[0,\infty),i=1,\dots,N}$ to (4) such that $\mathcal{L}((X_0^{i,N})_{i=1,\dots,N}) = F_0^N$. Furthermore, the family $\{(X_t^{i,N})_{t\in[0,\infty)},\ i = 1,\dots,N\}$ is exchangeable and for any $\alpha \in ((N-1)\chi/(2\pi N), 1)$, any $T > 0$,*

$$\mathbb{E}\Big[\int_0^T |X_s^{1,N} - X_s^{2,N}|^{\alpha-2} ds\Big] \leq \frac{(2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha}{\alpha(2\alpha - (N-1)\chi/(\pi N))}. \tag{5}$$

We could prove a similar (but slightly different) formula with $\alpha \geq 1$. However, since our goal is to control $\mathbb{E}[|K(X_s^{1,N} - X_s^{2,N})|]$ (with some margin), only the case where $\alpha - 2 < -1$ will be interesting.

As already mentioned, such a result is not obvious, since K is singular and since its singularity is visited. The main point is to observe that (5) *a priori* holds true for some $\alpha < 1$. This will imply that that $\mathbb{E}[|K(X_s^{1,N} - X_s^{2,N})|]$ should be controlled (with some margin since $\alpha - 2 < -1$). This will be sufficient to prove existence by compactness. The formal computation is as follows: by the Itô formula, for $\alpha \in (0,1)$,

$$\begin{aligned} d|X_t^{1,N} - X_t^{2,N}|^\alpha =& \sqrt{2}\alpha|X_t^{1,N} - X_t^{2,N}|^{\alpha-2}(X_t^{1,N} - X_t^{2,N}) \cdot (dB_t^1 - dB_t^2) \\ &+ 2\alpha^2|X_t^{1,N} - X_t^{2,N}|^{\alpha-2}dt - \frac{\alpha\chi}{\pi N}|X_t^{1,N} - X_t^{2,N}|^{\alpha-2}dt \\ &+ \frac{\alpha\chi}{2\pi N}|X_t^{1,N} - X_t^{2,N}|^{\alpha-2}(X_t^{1,N} - X_t^{2,N}) \cdot \sum_{i=3}^N \Big(\frac{X_t^{i,N} - X_t^{1,N}}{|X_t^{i,N} - X_t^{1,N}|^2} + \frac{X_t^{2,N} - X_t^{i,N}}{|X_t^{2,N} - X_t^{i,N}|^2}\Big)dt. \end{aligned} \tag{6}$$

The second term in the right-hand side is the Itô correction due to diffusion, the third term is the contribution of the interaction between the particles 1 and 2 and the last term is the contribution of the interactions with between particles 1, 2 and the rest of the system. By exchangeability and Hölder's inequality, the expectation of the last term in the right-hand side is greater than $-[\alpha(N-2)\chi/(\pi N)]\mathbb{E}[|X_t^{1,N} - X_t^{2,N}|^{\alpha-2}]$. The assumption $\chi < 2\pi N/(N-1)$ ensures us that the Itô correction dominates the drift contribution. More precisely choosing $\alpha \in (\chi(N-1)/(2\pi N), 1)$, integrating in time and taking expectations, one obtains

$$\alpha\Big(2\alpha - \frac{\chi(N-1)}{\pi N}\Big)\int_0^t \mathbb{E}[|X_s^{1,N} - X_s^{2,N}|^{\alpha-2}]ds \leq \mathbb{E}[|X_t^{1,N} - X_t^{2,N}|^\alpha].$$

The right-hand side is easily bounded, uniformly in N, using the oddness of K, whence (5). A similar computation was performed by Osada in [33, Lemma 3.2] for systems of stochastic vortices.

Next, we show some tightness/consistency as $N \to \infty$ in the (very) subcritical case $\chi < 2\pi$. Such a result follows quite easily from the the bound (5), which is uniform in N (when $\chi < 2\pi$). We endow $C([0,\infty), \mathbb{R}^2)$ with the topology of uniform convergence on compact time intervals, and $\mathcal{P}(C([0,\infty), \mathbb{R}^2))$ with the associated weak convergence topology. Finally, we endow $C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ with the topology of uniform convergence on compact time intervals associated with the weak convergence topology in $\mathcal{P}(\mathbb{R}^2)$.

Theorem 6. *Let $\chi \in (0, 2\pi)$ be fixed. For each $N \geq 2$, consider $F_0^N \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ and the particle system $(X_t^{i,N})_{t\in[0,\infty),i=1,\dots,N}$ with initial law F_0^N built in Theorem 5, as well as the empirical measure $\mu^N = N^{-1}\sum_1^N \delta_{(X_t^{i,N})_{t\in[0,\infty)}}$, which a.s. belongs to $\mathcal{P}(C([0,\infty), \mathbb{R}^2))$. For each $t \geq 0$, we also set $\mu_t^N = N^{-1}\sum_1^N \delta_{X_t^{i,N}}$, which a.s. belongs to $\mathcal{P}(\mathbb{R}^2)$. We assume that $\sup_{N\geq 2}\mathbb{E}[|X_0^{1,N}|] < \infty$ and that μ_0^N goes weakly to $f_0 \in \mathcal{P}_1(\mathbb{R}^2)$ in probability as $N \to \infty$.*

(i) The sequence $\{\mu^N,\ N \geq 2\}$ is tight in $\mathcal{P}(C([0,\infty), \mathbb{R}^2))$.

(ii) Any (possibly random) weak limit point μ of $(\mu^N)_{N\geq 2}$ is a.s. the law of a solution to the nonlinear SDE (3) *with initial law f_0.*

(iii) In particular, we can find a subsequence N_k such that $(\mu_t^{N_k})_{t\geq 0}$ goes in law, as $k\to\infty$, in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$, to some $(\mu_t)_{t\geq 0}$, which is a.s. a weak solution to (1) *starting from $\mu_0 = f_0$.*

In particular, for $f_0 \in \mathcal{P}_1(\mathbb{R}^2)$, $F_0^N = f_0^{\otimes N}$ satisfies all the assumptions of this theorem. We are quite satisfied, since this result seems to be the first result concerning the convergence of the *true* particle system (without cutoff) to the Keller-Segel equation. However, there are two main limitations. First, this result should more or less always hold true in the subcritical case $\chi \in (0, 8\pi)$. Second, we are not able to prove the convergence, we have only compactness/consistency. This is due to the fact that we are not able to prove that our limit point $(\mu_t)_{t\geq 0}$ a.s. belongs to the class of weak solutions in which uniqueness is known to hold true. Thanks to Egaña and Mischler [7], it would suffice to show that $(\mu_t)_{t\geq 0}$ satisfies the *free energy dissipation inequality*, which is slightly stronger than the requirement $(\mu_t)_{t\geq 0} \in \cap_{p\geq 1} L^1_{loc}([0,\infty), L^p(\mathbb{R}^2))$ a.s. We believe this is a very difficult problem.

We next prove that, when $\chi < 2\pi N$, the particle system always exists until 3 particles encounter. In view of (6), this is not surprising. Indeed, the assumption $\chi < 2\pi N$ ensures us that the Itô correction still dominates the contribution of the interaction between the particles 1 and 2. Moreover, it is not very hard to control the last term of (6) until a 3-particle collision occurs.

Theorem 7. *Let $\chi > 0$, $N > \max\{2, \chi/(2\pi)\}$ be fixed, as well as $F_0^N \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ such that*

$$F_0^N(\{(x_1,\dots,x_N) \in (\mathbb{R}^2)^N \;:\; x_i \neq x_j,\; \forall\, i \neq j\}) = 1. \tag{7}$$

There exists a solution $(X_t^{i,N})_{t\in[0,\tau_N), i=1,\dots,N}$ to (4)*, with initial law F_0^N, where*

$$\tau_N = \sup_{\ell\geq 1} \inf \Big\{ t \geq 0 \;:\; \exists\, i,j,k \;\; \textit{pairwise different such that} \\ |X_t^{i,N} - X_t^{j,N}| + |X_t^{j,N} - X_t^{k,N}| + |X_t^{k,N} - X_t^{i,N}| \leq 1/\ell \Big\}.$$

The family $\{(X_t^{i,N})_{t\in[0,\tau_N)},\; i = 1,\dots,N\}$ is exchangeable and for any $\alpha \in (\chi/(2\pi N), 1)$,

$$\textit{a.s., for all } t \in [0,\tau_N), \quad \int_0^t |X_s^{1,N} - X_s^{2,N}|^{\alpha-2} ds < \infty. \tag{8}$$

Finally, it holds that (i) $\tau_N = \infty$ a.s. if $\chi \leq 8\pi(N-2)/(N-1)$ and (ii) $\tau_N < \infty$ a.s. if $\chi > 8\pi(N-2)/(N-1)$.

This result thus in particular shows the global existence for the particle system in the subcritical case $\chi < 8\pi$ for all N large enough. This result seems to be new, as well as our method to check it, which is quite specific to the model, and may be considered as the main result of the paper and the most difficult.

As we will see in Lemma 16, for any $I \subset \{1,\dots,N\}$, the process $R_t^I = 2^{-1}\sum_{i\in I} |X_t^{i,N} - \bar{X}_t^I|^2$, where $\bar{X}_t^I = |I|^{-1}\sum_{i\in I} X_t^{i,N}$, behaves like the square of a Bessel process of dimension $(|I|-1)(2 - (\chi|I|)/(4\pi N))$, when neglecting the contribution of the interaction with the other particles. Similar computations for $I = \{1,\dots,N\}$ were performed by Haškovec and Schmeiser [13, Page 139] and Fatkullin [8, Page 89]. The condition $\chi \leq 8\pi(N-2)/(N-1)$ implies that for all $|I| = 3,\dots,N$, the dimension $(|I|-1)(2-(\chi|I|)/(4\pi N))$ is greater than 2, so that R_t^I never reaches 0: there are no collisions involving more than two particles. Of course, the situation is actually much more complicated, since we have to justify that for all I, we can indeed neglect the contribution of the interaction with the other particles.

Remark 8. *When $\chi \in (0, 8\pi)$, we thus show that, for N large enough, triplets of particles a.s. never encounter. To extend the tightness/consistency result of Theorem 6 to some $\chi \in [2\pi, 8\pi)$, we believe that a uniform (in N) version of this fact might be sufficient. For example, one would have to check that $\tau_N^\ell = \inf\{t > 0 \ : \ |X_t^{1,N} - X_t^{2,N}| + |X_t^{2,N} - X_t^{3,N}| + |X_t^{1,N} - X_t^{3,N}| \le 1/\ell\}$ goes a.s. to infinity as $\ell \to \infty$, uniformly in N.*

Finally, we study the case $N = 2$ of two particles. The difference $D_t = X_t^{1,2} - X_t^{2,2}$ formally solves the two-dimensional SDE

$$D_t = D_0 + 2W_t - \frac{\chi}{2\pi} \int_0^t \frac{D_s}{|D_s|^2} ds. \tag{9}$$

This can be seen as a natural two-dimensional generalization of a Bessel process of dimension $(2 - \chi/(4\pi))$. Indeed, one can check that $(|D_t|/2)_{t\ge 0}$ is a Bessel process of dimension $(2 - \chi/(4\pi))$ and the dynamics is radially symmetric. We study in details (9) in Section 6. To summarize, we prove the following.

(i) If $\chi \in (0, 4\pi)$, then (9) has a unique (in law) solution, which a.s. reaches the origin but then instantaneously escapes from this point.

(ii) When $\chi \ge 4\pi$, the formulation (9) becomes meaningless because $\tau = \inf\{t \ge 0 \ : \ D_t = 0\}$ is a.s. finite and $\int_\tau^{\tau+h} |D_s|^{-1} ds = \infty$ a.s. The situation is similar to classical Bessel processes with dimension $\delta \in (0, 1]$: they do not satisfy a classical SDE but their square does. Here also, we introduce an alternative rigorous formulation.

(ii)-(a) For $\chi \in [4\pi, 8\pi)$, we prove the existence of a unique (in law) solution to this new formulation. This solution also a.s. reaches the origin but then instantaneously escapes this point.

(ii)-(b) If finally $\chi \ge 8\pi$, we prove the existence of a pathwise unique solution to this new formulation, frozen when it reaches the origin (and it a.s. does).

We also prove, in all these cases, that the obtained process is the limit in law of the solution to a regularized version of (9), as the regularization parameter tends to 0.

1.6. **References.** Concerning our existence (and uniqueness when $N = 2$) results for the particle system (4), let us mention that classical results about SDEs with singular drift do not apply because the drift under study is too singular and since collisions do occur. This is not surprising, since general results would presumably apply to all values of χ, whereas we observe a phase transition. Existence by a Girsanov transformation would require (at least) that $\int_0^t |B_s^1 - B_s^2|^{-2} ds < \infty$ a.s., which is not the case. The methods of Röckner and Krylov [26] require some integrability of the drift coefficient that (4) never satisfies. Let us however mention that (9) is borderline.

In a very recent paper [4], Cattiaux and Pédèches used Dirichlet forms (see Fukushima [11]) to prove the weak existence and uniqueness for (4) when $N \ge 4$ and $\chi < 8\pi(N-2)/(N-1)$. However, when applying [11], one obtains *abstract* results, and some work is required to get the weak existence and uniqueness for (4). The arguments of [4] rely on some results of the present paper, namely the absence of triple collisions (when $N \ge 4$ and $\chi < 8\pi(N-2)/(N-1)$) and the weak uniqueness when $N = 2$. Cattiaux and Pédèches also prove and use that two binary collisions never occur at the same time.

Let us mention that we of course have tried to prove the pathwise uniqueness for the particle system (4), in particular when $N = 2$, without success. We have no intuition on its plausibility.

Approximating a large particle system by a partial differential equation (for deriving the PDE) or a partial differential equation by a large particle system (to compute numerically the solution of the PDE) is now a classical topic, called *propagation of chaos*. This notion was introduced by Kac [22] as a step to the rigorous justification of the Boltzmann equation. When the interaction is regular, the situation is now well-understood, some important contributions are due to McKean [28], Sznitman [40], Méléard [29], Mischler and Mouhot [30], etc. The main idea is that one can generally prove true quantified convergence when the interaction is Lipschitz continuous and tightness/consistency (and true unquantified convergence if the PDE is known to have a unique solution) when the coefficients are only continuous. Of course, each PDE is specific and these are only formal rules.

The case of singular interactions is much more complicated. In dimension one, let us mention the works of Bossy-Talay [2] and Jourdain [20] which concern the viscous Burgers equation and more general scalar conservation laws (where particles interact through the Heaviside function) and of Cepa-Lépingle [5] on the very singular Dyson model.

A model closely related to the one studied in the present paper is the $2d$-vortex model, that approximates the vorticity formulation of the $2d$-incompressible Navier-Stokes equation. The PDE is the same as (1) and the particle system is the same as (4), replacing everywhere the kernel K, see (2), by the Biot and Savart kernel $x^\perp/(2\pi|x|^2)$. This kernel is as singular as K, but the interaction is of course not *attractive*, so that the situation is simpler. In particular, there is no blow-up for the PDE and Osada [31] has shown that particles never collide so that the particle system is well-posed. Osada [32, 33] has also proved the (true but unquantified) convergence of the particle system to the solution of the PDE when χ is sufficiently small (in our notation), and this limitation has been recently removed in [10]. The method developed in [10] relies on a control of the Fisher information of the law of the particle system provided by the dissipation of its entropy. It has been applied to a *subcritical* Keller-Segel equation by Godinho-Quininao [12], where K is replaced by $-x/(2\pi|x|^{1+\alpha})$ with some $\alpha \in (0, 1)$ and to the Landau equation for moderately soft potentials in [9]. Let us finally mention the propagation of chaos results for some particle systems with deterministic dynamics by Marchioro-Pulvirenti [27] (for the $2d$-Euler equation) by Hauray-Jabin [15] (for some singular Vlasov equations) and by Jourdain-Reygner [21] (for diagonal hyperbolic systems).

In the above mentioned works, some true convergence is derived. Here, we obtain only a tightness/consistency result, but the singularity is really strong and attractive. Concerning the Keller-Segel equation, we are not aware of papers dealing with the convergence of the true particle system *without any cutoff*. Stevens [40] studies a physically more convincing particle system with two kinds of particles (for bacteria and chemo-attractant particles). She proves the convergence of this particle system when the kernel K is regularized. In [13], Haškovec and Schmeiser also prove some results for a regularized kernel of the form $K_\varepsilon(x) = -x/[|x|(|x| + \varepsilon)]$. Finally, Godinho-Quininao [12] study the case where K is replaced by $-x/(2\pi|x|^{1+\alpha})$ for some $\alpha \in (0, 1)$.

1.7. **Plan of the paper.** In the next section, we prove (5) for a regularized particle system. This is the main tool for the proofs of Theorem 5 (existence for the particle system when $\chi \in (0, 2\pi)$) and Theorem 6 (tightness/consistency as $N \to \infty$ when $\chi \in (0, 2\pi)$) given in Section 3, as well as for checking Theorem 7 (local or global existence for the particle system in the general case) in Section 4. We establish Proposition 4 (positive probability of collisions) in Section 5. Section 6 is devoted to a detailed study of the case $N = 2$ and in particular to the natural two-dimensional generalization (9) of Bessel processes. Finally, we quickly and formally discuss in Section 7 how to build an relevant N-particle system when $\chi \geq 8\pi(N-2)/(N-1)$ and we explain why it seems to be a difficult problem.

2. A regularized particle system

Let $\chi > 0$, $N \geq 2$ and an exchangeable initial condition $(X_0^{i,N})_{i=1,\dots,N}$ such that $\mathbb{E}[|X_0^{1,N}|] < \infty$ be fixed. We consider a family $(B_t^i)_{t\geq 0}$, $i = 1,\dots,N$ of independent 2-dimensional Brownian motions, independent of the initial condition. For $\varepsilon \in (0,1)$, we define the regularized version K_ε of K as

$$K_\varepsilon(x) = \frac{-x}{2\pi(|x|^2+\varepsilon^2)}. \tag{10}$$

This kernel is globally Lipschitz continuous, so that the particle system

$$X_t^{i,N,\varepsilon} = X_0^{i,N} + \sqrt{2}B_t^i + \frac{\chi}{N}\sum_{j=1}^N \int_0^t K_\varepsilon(X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon})ds \tag{11}$$

is strongly and uniquely well-defined. These particles are furthermore clearly exchangeable. The following estimates are crucial for our study.

Proposition 9. *(i) For all $t \geq 0$, all $\varepsilon \in (0,1)$, $\mathbb{E}[(1+|X_t^{1,N,\varepsilon}|^2)^{1/2}] \leq \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 2t$.*

(ii) For all $\alpha \in (0,1)$, all $T > 0$, all $\varepsilon \in (0,1)$, all $\eta \in (0,\varepsilon]$,

$$\Big(2\alpha - \frac{\chi}{\pi N}\Big)\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2 + \eta^2)^{\alpha/2-1}ds\Big] \leq \frac{(2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha}{\alpha}$$
$$+ \frac{(N-2)\chi}{\pi N}\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2 + \eta^2)^{(\alpha-1)/2}(|X_s^{1,N,\varepsilon} - X_s^{3,N,\varepsilon}|^2 + \varepsilon^2)^{-1/2}ds\Big].$$

Proof. We start with point (i). Using the Itô formula (with $\phi(x) = (1+|x|^2)^{1/2}$ whence $\nabla\phi(x) = (1+|x|^2)^{-1/2}x$ and $\Delta\phi(x) = (1+|x|^2)^{-3/2}(2+|x|^2)$ and taking expectations, we find

$$\mathbb{E}[(1+|X_t^{1,N,\varepsilon}|^2)^{1/2}] = \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + \mathbb{E}\Big[\int_0^t \frac{2+|X_s^{1,N,\varepsilon}|^2}{(1+|X_s^{1,N,\varepsilon}|^2)^{3/2}}ds\Big]$$
$$+ \frac{\chi}{N}\sum_{j\neq 1}\mathbb{E}\Big[\int_0^t \frac{X_s^{1,N,\varepsilon}}{(1+|X_s^{1,N,\varepsilon}|^2)^{1/2}} \cdot K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{j,N,\varepsilon})ds\Big].$$

By exchangeability and oddness of K_ε, for $j \in \{2,\dots,N\}$,

$$\mathbb{E}\Big[\int_0^t \frac{X_s^{1,N,\varepsilon}}{(1+|X_s^{1,N,\varepsilon}|^2)^{1/2}} \cdot K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{j,N,\varepsilon})ds\Big]$$
$$= \frac{1}{2}\mathbb{E}\Big[\int_0^t \Big(\frac{X_s^{1,N,\varepsilon}}{(1+|X_s^{1,N,\varepsilon}|^2)^{1/2}} - \frac{X_s^{j,N,\varepsilon}}{(1+|X_s^{j,N,\varepsilon}|^2)^{1/2}}\Big) \cdot K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{j,N,\varepsilon})ds\Big]$$

This last expectation is non-positive since for $x, y \in \mathbb{R}^2$, the inequality $|x|^4 + |y|^4 \geq 2|x|^2|y|^2$ implies $(|x|^2(1+|y|^2)^{1/2} + |y|^2(1+|x|^2)^{1/2})^2 \geq (|x||y|((1+|y|^2)^{1/2} + (1+|x|^2)^{1/2}))^2$, whence $(x(1+|y|^2)^{1/2} - y(1+|x|^2)^{1/2}) \cdot (x-y) \geq 0$ and thus $(x(1+|x|^2)^{-1/2} - y(1+|y|^2)^{-1/2}) \cdot (x-y) \geq 0$. Hence

$$\mathbb{E}[(1+|X_t^{1,N,\varepsilon}|^2)^{1/2}] \leq \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + \mathbb{E}\Big[\int_0^t \frac{2+|X_s^{1,N,\varepsilon}|^2}{(1+|X_s^{1,N,\varepsilon}|^2)^{3/2}}ds\Big] \leq \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 2t,$$

as desired. To prove point (ii), we fix $\alpha \in (0,1)$ and start from

$$X_t^{1,N,\varepsilon} - X_t^{2,N,\varepsilon} = X_0^{1,N} - X_0^{2,N} + \sqrt{2}(B_t^1 - B_t^2) + \chi R_t^{12} + \chi S_t^{12},$$

where $R_t^{12} = N^{-1}\sum_{j=3}^N \int_0^t [K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{j,N,\varepsilon}) - K_\varepsilon(X_s^{2,N,\varepsilon} - X_s^{j,N,\varepsilon})]ds$ and where $S_t^{12} = 2N^{-1}\int_0^t K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})ds$. We next fix $\eta \in (0,\varepsilon]$, introduce $\phi_\eta(x) = (|x|^2+\eta^2)^{\alpha/2}$ and use the Itô formula to write

$$\begin{aligned}\mathbb{E}[\phi_\eta(X_T^{1,N,\varepsilon} - X_T^{2,N,\varepsilon})] =& \mathbb{E}[\phi_\eta(X_0^{1,N} - X_0^{2,N})] + \mathbb{E}\Big[\int_0^T 2\Delta\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})ds\Big]\\ &+ \chi\mathbb{E}\Big[\int_0^T \nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})\cdot(dR_s^{12} + dS_s^{12})\Big].\end{aligned}$$

Since $\eta \in (0,1)$, we have $\phi_\eta(x-y) \le [\sqrt{2}((1+|x|^2)^{1/2} + (1+|y|^2)^{1/2}))]^\alpha$, whence $\mathbb{E}[\phi_\eta(X_T^{1,N,\varepsilon} - X_T^{2,N,\varepsilon})] \le (2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha$ by (i). Since furthermore $\mathbb{E}[\phi_\eta(X_0^{1,N} - X_0^{2,N})] \ge 0$,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T 2\Delta\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})ds\Big] \qquad (12)\\ \le& (2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha - \chi\mathbb{E}\Big[\int_0^T \nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})\cdot(dR_s^{12} + dS_s^{12})\Big].\end{aligned}$$

Using exchangeability and recalling the definition of R_t^{12},

$$\begin{aligned}&-\mathbb{E}\Big[\int_0^T \nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})\cdot dR_s^{12}\Big]\\ \le& \frac{2(N-2)}{N}\mathbb{E}\Big[\int_0^T |\nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})||K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{3,N,\varepsilon})|ds\Big].\end{aligned}$$

But $\nabla\phi_\eta(x) = \alpha(|x|^2+\eta^2)^{\alpha/2-1}x$, whence $|\nabla\phi_\eta(x)| \le \alpha(|x|^2+\eta^2)^{\alpha/2-1/2}$. Furthermore, $|K_\varepsilon(x)| \le (|x|^2+\varepsilon^2)^{-1/2}/(2\pi)$. Hence

$$\begin{aligned}&-\mathbb{E}\Big[\int_0^T \nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})\cdot dR_s^{12}\Big] \qquad (13)\\ \le& \frac{(N-2)\alpha}{\pi N}\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2+\eta^2)^{\alpha/2-1/2}(|X_s^{1,N,\varepsilon} - X_s^{3,N,\varepsilon}|^2+\varepsilon^2)^{-1/2}ds\Big].\end{aligned}$$

Recalling the definition of S_t^{12} and using that $|K_\varepsilon(x)| \le (|x|^2+\eta^2)^{-1/2}/(2\pi)$,

$$-\mathbb{E}\Big[\int_0^T \nabla\phi_\eta(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon})\cdot dS_s^{12}\Big] \le \frac{\alpha}{\pi N}\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2+\eta^2)^{\alpha/2-1}ds\Big]. \qquad (14)$$

Finally, we observe that $\Delta\phi_\eta(x) = \alpha(|x|^2+\eta^2)^{\alpha/2-2}(\alpha|x|^2+2\eta^2) \ge \alpha^2(|x|^2+\eta^2)^{\alpha/2-1}$. Inserting this into (12) and using (13) and (14), we find

$$\begin{aligned}&2\alpha^2\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2+\eta^2)^{\alpha/2-1}ds\Big] \qquad (15)\\ \le& (2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha + \frac{\alpha\chi}{\pi N}\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2+\eta^2)^{\alpha/2-1}ds\Big]\\ &+ \frac{(N-2)\alpha\chi}{\pi N}\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2+\eta^2)^{(\alpha-1)/2}(|X_s^{1,N,\varepsilon} - X_s^{3,N,\varepsilon}|^2+\varepsilon^2)^{-1/2}ds\Big].\end{aligned}$$

The conclusion immediately follows. ☐

3. Tightness and consistency in the (very) subcritical case

The aim of this section is to prove Theorems 5 and 6. First, we deduce from Proposition 9 an estimate saying that in some sense, particles do not meet too much, uniformly in $N \geq 2$ and $\varepsilon \in (0,1)$ when $\chi < 2\pi$.

Corollary 10. *For each $N \geq 2$, each $\chi \in (0, 2\pi N/(N-1))$ and each $\varepsilon \in (0,1)$, consider the unique solution $(X_t^{i,N,\varepsilon})_{t\geq 0, i=1,\dots,N}$ to (11) with some exchangeable initial condition $(X_0^{i,N})_{i=1,\dots,N}$ such that $\mathbb{E}[|X_0^{1,N}|] < \infty$. For all $T > 0$ and all $\alpha \in (\chi(N-1)/(2\pi N), 1)$,*

$$\mathbb{E}\Big[\int_0^T |X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^{\alpha-2} ds\Big] \leq \frac{(2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha}{\alpha(2\alpha - (N-1)\chi/(\pi N))}.$$

Proof. We thus fix $\alpha \in (\chi(N-1)/(2\pi N), 1)$. By Hölder's inequality and exchangeability, we have, for any $\eta \in (0, \varepsilon]$,

$$\mathbb{E}[(|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2 + \eta^2)^{(\alpha-1)/2}(|X_s^{1,N,\varepsilon} - X_s^{3,N,\varepsilon}|^2 + \varepsilon^2)^{-1/2}] \leq \mathbb{E}[(|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2 + \eta^2)^{\alpha/2-1}].$$

Applying Proposition 9-(ii), we thus find

$$\Big(2\alpha - \frac{(N-1)\chi}{\pi N}\Big)\int_0^T \mathbb{E}[(|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^2 + \eta^2)^{\alpha/2-1}] ds \leq \frac{(2\sqrt{2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha}{\alpha}.$$

It suffices to let $\eta \searrow 0$ to complete the proof. ☐

Such an estimate easily implies tightness.

Lemma 11. *For each $N \geq 2$, each $\varepsilon \in (0,1)$, consider the unique solution $(X_t^{i,N,\varepsilon})_{t\in[0,\infty), i=1,\dots,N}$ to (11) with some exchangeable initial condition $(X_0^{i,N})_{i=1,\dots,N}$ such that $\mathbb{E}[|X_0^{1,N}|] < \infty$.*

(i) For $N \geq 2$ fixed, if $\chi < 2\pi N/(N-1)$, the family $\{(X_t^{1,N,\varepsilon})_{t\geq 0},\ \varepsilon \in (0,1)\}$ is tight in $C([0,\infty), \mathbb{R}^2)$.

(ii) If $\chi < 2\pi$ and if $\sup_{N\geq 2} \mathbb{E}[|X_0^{1,N}|] < \infty$, the family $\{(X_t^{1,N,\varepsilon})_{t\geq 0},\ N \geq 2,\ \varepsilon \in (0,1)\}$ is tight in $C([0,\infty), \mathbb{R}^2)$.

Proof. We first prove (ii) and thus suppose that $\chi < 2\pi$. Since $C([0,\infty), \mathbb{R}^2)$ is endowed with the topology of the uniform convergence on compact time intervals, it suffices to prove that for all $T > 0$, $\{(X_t^{1,N,\varepsilon})_{t\in[0,T]},\ N \geq 2,\ \varepsilon \in (0,1)\}$ is tight in $C([0,T], \mathbb{R}^2)$. Let thus $T > 0$ be fixed and recall that $X_t^{1,N,\varepsilon} = X_0^{1,N} + \sqrt{2}B_t^1 + J_t^{1,N,\varepsilon}$, where

$$J_t^{1,N,\varepsilon} := \frac{\chi}{N}\sum_{j=2}^N \int_0^t K_\varepsilon(X_s^{1,N,\varepsilon} - X_s^{j,N,\varepsilon}) ds.$$

Since $\{X_0^{1,N}, N \geq 2\}$ is tight by assumption and since the law of $(B_t^1)_{t\in[0,T]}$ does not depend on $N \geq 2$ nor on $\varepsilon > 0$, it suffices to prove that the family $\{(J_t^{1,N,\varepsilon})_{t\in[0,T]},\ N \geq 2,\ \varepsilon \in (0,1)\}$, is tight in $C([0,T], \mathbb{R}^2)$. To do so, we fix $\alpha \in (\chi/(2\pi), 1)$, and we use Hölder's inequality to write, for

$0 \le s < t \le T$,

$$|J_t^{1,N,\varepsilon} - J_s^{1,N,\varepsilon}| \le \frac{\chi}{2\pi N}\sum_{j=2}^N \int_s^t |X_u^{1,N,\varepsilon} - X_u^{j,N,\varepsilon}|^{-1} du$$
$$\le |t-s|^{(1-\alpha)/(2-\alpha)} \frac{\chi}{2\pi N}\sum_{j=2}^N \Big(\int_s^t |X_u^{1,N,\varepsilon} - X_u^{j,N,\varepsilon}|^{\alpha-2} du\Big)^{1/(2-\alpha)}$$
$$\le Z_T^{N,\varepsilon}|t-s|^\beta,$$

where $\beta = (1-\alpha)/(2-\alpha) > 0$ and where $Z_T^{N,\varepsilon} := (\chi/(2\pi N))\sum_{j=2}^N [1+\int_0^T |X_u^{1,N,\varepsilon} - X_u^{j,N,\varepsilon}|^{\alpha-2}du]$. Indeed, $x^{1/(2-\alpha)} \le 1+x$ because $\alpha \in (0,1)$. But we immediately deduce from Corollary 10 and exchangeability that $\sup_{\varepsilon\in(0,1),N\ge 2} \mathbb{E}[Z_T^{N,\varepsilon}] < \infty$, so that there is a constant C_T, not depending on $\varepsilon \in (0,1)$ nor on $N \ge 2$ such that for all $A > 0$, $\mathbb{P}(Z_T^{N,\varepsilon} > A) \le C_T/A$. Since $J_0^{1,N,\varepsilon} = 0$ a.s., we conclude that for all $A > 0$, for all $N \ge 2$, all $\varepsilon \in (0,1)$, $\mathbb{P}[(J_t^{1,N,\varepsilon})_{t\in[0,T]} \notin \mathcal{K}_A] \le C_T/A$, where $\mathcal{K}_A$ is the set of all functions $\gamma : [0,T] \mapsto \mathbb{R}^2$ such that $\gamma(0) = 0$ and for all $0 \le s < t \le T$, $|\gamma(t)-\gamma(s)| \le A|t-s|^\beta$. The Ascoli theorem ensures us that $\mathcal{K}_A$ is a compact subset of $C([0,T],\mathbb{R}^2)$ for all $A > 0$. Since $\lim_{A\to\infty}\sup_{N\ge2,\varepsilon\in(0,1)} \mathbb{P}[(J_t^{1,N,\varepsilon})_{t\in[0,T]} \notin \mathcal{K}_A] = 0$, the proof of (ii) is complete.

The proof of (i) is exactly the same: the only difference is that N is fixed so that we can choose $\alpha \in (\chi(N-1)/(2\pi N), 1)$. □

We now prove the existence of the particle system without cutoff in the very subcritical case.

Proof of Theorem 5. We divide the proof in three steps. Recall that $N \ge 2$ is fixed, as well as $F_0^N \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$, and that $\chi < 2\pi N/(N-1)$.

Step 1. For each $\varepsilon \in (0,1)$, we consider the unique solution $(X_t^{i,N,\varepsilon})_{t\in[0,\infty),i=1,\dots,N}$ to (11) (with initial law F_0^N). By Lemma 11-(i), we know that the family $\{(X_t^{1,N,\varepsilon})_{t\ge0},\ \varepsilon\in(0,1)\}$ is tight in $C([0,\infty),\mathbb{R}^2)$. By exchangeability, we of course deduce that $\{(X_t^{1,N,\varepsilon},\dots,X_t^{N,N,\varepsilon})_{t\ge0},\ \varepsilon\in(0,1)\}$ is tight in $C([0,\infty),(\mathbb{R}^2)^N)$ and consequently that $\{((X_t^{1,N,\varepsilon},B_t^1),\dots,(X_t^{N,N,\varepsilon},B_t^N))_{t\ge0},\ \varepsilon\in(0,1)\}$ is tight in $C([0,\infty),(\mathbb{R}^2\times\mathbb{R}^2)^N)$ (this last assertion only uses that the law of $(B_t^1,\dots,B_t^N)_{t\ge0}$ does not depend on ε). It is thus possible to find a decreasing sequence $\varepsilon_k \searrow 0$ such that the family $((X_t^{1,N,\varepsilon_k},B_t^1),\dots,(X_t^{N,N,\varepsilon_k},B_t^N))_{t\ge0}$ converges in law in $C([0,\infty),(\mathbb{R}^2\times\mathbb{R}^2)^N)$ as $k\to\infty$. By the Skorokhod representation theorem, we can realize this convergence almost surely. All this shows that we can find, for each $k\ge1$, a solution $(\tilde X_t^{1,N,\varepsilon_k},\dots,\tilde X_t^{N,N,\varepsilon_k})_{t\ge0}$ to (11), associated to some Brownian motions $(\tilde B_t^{1,N,\varepsilon_k},\dots,\tilde B_t^{N,N,\varepsilon_k})_{t\ge0}$, in such a way that the sequence $((\tilde X_t^{1,N,\varepsilon_k},\tilde B_t^{1,N,\varepsilon_k}),\dots,(\tilde X_t^{N,N,\varepsilon_k},\tilde B_t^{N,N,\varepsilon_k}))_{t\ge0}$ a.s. goes to some $((X_t^{1,N},B_t^1),\dots,(X_t^{N,N},B_t^N))_{t\ge0}$ in $C([0,\infty),(\mathbb{R}^2\times\mathbb{R}^2)^N)$ as $k\to\infty$. Let us observe at once that the family $\{(X_t^{i,N})_{t\ge0},\ i=1,\dots,N\}$ is exchangeable and that, by Corollary 10 for all $T>0$ and $\alpha\in(\chi(N-1)/(2\pi N),1)$,

$$\sup_k \mathbb{E}\Big[\int_0^T |\tilde X_s^{1,N,\varepsilon_k} - \tilde X_s^{2,N,\varepsilon_k}|^{\alpha-2}ds\Big] \le \frac{(2\sqrt2\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}]+4\sqrt2T)^\alpha}{\alpha(2\alpha-(N-1)\chi/(\pi N))}. \tag{16}$$

By the Fatou Lemma, we deduce that

$$\mathbb{E}\Big[\int_0^T |X_s^{1,N} - X_s^{2,N}|^{\alpha-2}ds\Big] \le \frac{(2\sqrt2\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}]+4\sqrt2T)^\alpha}{\alpha(2\alpha-(N-1)\chi/(\pi N))}. \tag{17}$$

Step 2. We introduce $\mathcal{F}_t = \sigma((X_s^{i,N}, B_s^i)_{i=1,\dots,N,s\in[0,t]})$. Of course, $(X_t^{i,N})_{i=1,\dots,N,t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. The family $(X_0^{i,N})_{i=1,\dots,N}$ is F_0^N-distributed because this is the case of $(X_0^{i,N,\varepsilon_k})_{i=1,\dots,N}$ for all $k \geq 1$ and $(B_s^i)_{i=1,\dots,N,s\in[0,t]}$ is obviously a $2N$-dimensional Brownian motion (because this is the case of $(B_s^{i,N,\varepsilon_k})_{i=1,\dots,N,s\geq 0}$ for all $k \geq 1$). We now show that $(B_s^i)_{i=1,\dots,N,s\in[0,t]}$ is a $2N$-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Let thus $t > 0$ and consider $\phi : C([0,\infty), (\mathbb{R}^2)^N) \mapsto \mathbb{R}$ and $\psi : C([0,t], (\mathbb{R}^2 \times \mathbb{R}^2)^N) \mapsto \mathbb{R}$, both continuous and bounded. We have to check that

$$\begin{aligned}\mathbb{E}[\psi((X_s^{i,N}, B_s^i)_{i=1,\dots,N,s\in[0,t]})\phi((B_{t+s}^i - B_t^i)_{i=1,\dots,N,s\geq 0})]\\ = \mathbb{E}[\psi((X_s^{i,N}, B_s^{i,N})_{i=1,\dots,N,s\in[0,t]})]\mathbb{E}[\phi((B_{t+s}^i - B_t^i)_{i=1,\dots,N,s\geq 0})].\end{aligned}$$

This immediately follows from the fact that for all $k \geq 1$,

$$\begin{aligned}\mathbb{E}[\psi((\tilde{X}_s^{i,N,\varepsilon_k}, \tilde{B}_s^{i,N,\varepsilon_k})_{i=1,\dots,N,s\in[0,t]})\phi((\tilde{B}_{t+s}^{i,N,\varepsilon_k} - \tilde{B}_t^{i,N,\varepsilon_k})_{i=1,\dots,N,s\geq 0})]\\ = \mathbb{E}[\psi((\tilde{X}_s^{i,N,\varepsilon_k}, \tilde{B}_s^{i,N,\varepsilon_k})_{i=1,\dots,N,s\in[0,t]})]\mathbb{E}[\phi((\tilde{B}_{t+s}^{i,N,\varepsilon_k} - \tilde{B}_t^{i,N,\varepsilon_k})_{i=1,\dots,N,s\geq 0})],\end{aligned}$$

which holds true because $(\tilde{X}_t^{i,N,\varepsilon_k})_{i=1,\dots,N,t\geq 0}$ is a strong solution to (11) and is thus adapted to the filtration $\mathcal{F}_t^k = \sigma((X_0^i, B_s^{i,N,\varepsilon_k})_{i=1,\dots,N,s\geq 0})$.

Step 3. It only remains to check that for each $i \in \{1,\dots,N\}$, each $t \geq 0$, $X_t^{i,N} = X_0^{i,N} + \sqrt{2}B_t^i + (\chi/N)\sum_{j=1}^N \int_0^t K(X_s^{i,N} - X_s^{j,N})ds$. We of course start from the identity $\tilde{X}_t^{i,N,\varepsilon_k} = \tilde{X}_0^{i,N,\varepsilon_k} + \sqrt{2}\tilde{B}_t^{i,N,\varepsilon_k} + (\chi/N)\sum_{j=1}^N \int_0^t K_{\varepsilon_k}(\tilde{X}_s^{i,N,\varepsilon_k} - \tilde{X}_s^{j,N,\varepsilon_k})ds$ and pass to the limit as $k \to \infty$, e.g. in probability. The only difficulty is to prove that $J_k^{ij}(t)$ tends to $J^{ij}(t)$, where

$$J_k^{ij}(t) = \int_0^t K_{\varepsilon_k}(\tilde{X}_s^{i,N,\varepsilon_k} - \tilde{X}_s^{j,N,\varepsilon_k})ds \quad \text{and} \quad J^{ij}(t) = \int_0^t K(X_s^{i,N} - X_s^{j,N})ds.$$

Observe that $J^{ij}(t)$ is well-defined by (17). We introduce, for $\eta \in (0,1)$,

$$J_{k,\eta}^{ij}(t) = \int_0^t K_\eta(\tilde{X}_s^{i,N,\varepsilon_k} - \tilde{X}_s^{j,N,\varepsilon_k})ds \quad \text{and} \quad J_\eta^{ij}(t) = \int_0^t K_\eta(X_s^{i,N} - X_s^{j,N})ds.$$

For $\alpha \in (0,1)$ and k sufficiently large so that $\varepsilon_k < \eta$, we have

$$|K_\eta(x) - K_{\varepsilon_k}(x)| + |K_\eta(x) - K(x)| \leq \frac{\eta^2}{\pi|x|(|x|^2+\eta^2)} \leq \frac{\eta^{1-\alpha}|x|^{\alpha-2}}{\pi}. \tag{18}$$

We thus deduce from (16)-(17) that, for $\alpha \in (\chi(N-1)/(2\pi N), 1)$, there exists $C_{\alpha,t} < +\infty$ such that

$$\mathbb{E}[|J^{ij}(t) - J_\eta^{ij}(t)|] + \limsup_k \mathbb{E}[|J_k^{ij}(t) - J_{k,\eta}^{ij}(t)|] \leq C_{\alpha,t}\eta^{1-\alpha}.$$

Next, since K_η is continuous and bounded and since $(\tilde{X}_s^{i,N,\varepsilon_k})_{s\geq 0}$ goes a.s. to $(X_s^{i,N})_{s\geq 0}$, it holds that $J_{k,\eta}^{ij}(t) \to J_\eta^{ij}(t)$ a.s. and in L^1 for each $\eta > 0$. Writing

$$\mathbb{E}[|J^{ij}(t) - J_k^{ij}(t)|] \leq \mathbb{E}[|J^{ij}(t) - J_\eta^{ij}(t)|] + \mathbb{E}[|J_\eta^{ij}(t) - J_{k,\eta}^{ij}(t)|] + \mathbb{E}[|J_{k,\eta}^{ij}(t) - J_k^{ij}(t)|],$$

we conclude that $\limsup_{k\to\infty} \mathbb{E}[|J^{ij}(t) - J_k^{ij}(t)|] \leq C_{\alpha,t}\eta^{1-\alpha}$. Since $\eta \in (0,1)$ can be chosen arbitrarily small, we deduce that indeed, $J_k^{ij}(t)$ tends to $J^{ij}(t)$ in L^1 as $k \to \infty$. □

Following some ideas of [10, Proposition 6.1], we now give the

Proof of Theorem 6. For each $N \geq 2$, we consider the particle system $(X_t^{i,N})_{t\in[0,\infty),i=1,\dots,N}$ built in Theorem 5, with initial condition F_0^N. We set $\mu^N = N^{-1}\sum_1^N \delta_{(X_t^{i,N})_{t\in[0,\infty)}}$, which a.s. belongs to $\mathcal{P}(C([0,\infty),\mathbb{R}^2))$. For each $t \geq 0$, we also set $\mu_t^N = N^{-1}\sum_1^N \delta_{X_t^{i,N}}$, which a.s. belongs to $\mathcal{P}(\mathbb{R}^2)$.

Recall that we assume that $\sup_{N\geq 2}\mathbb{E}[|X_0^{1,N}|]<\infty$ and that μ_0^N goes weakly to $f_0\in\mathcal{P}_1(\mathbb{R}^2)$ in probability as $N\to\infty$.

Step 1. For each $N\geq 2$, $(X_t^{i,N})_{t\in[0,\infty),i=1,\dots,N}$ has been obtained as a limit point (in law), of $(X_t^{i,N,\varepsilon})_{t\in[0,\infty),i=1,\dots,N}$ as $\varepsilon\to 0$. By Lemma 11-(ii), the family $\{(X_t^{1,N})_{t\geq 0},\ N\geq 2\}$ is thus tight in $C([0,\infty),\mathbb{R}^2)$. As is well-known, see Sznitman [40, Proposition 2.2], this implies that the family $\{\mu^N,\ N\geq 2\}$ is tight in $\mathcal{P}(C([0,\infty),\mathbb{R}^2))$ (because for each $N\geq 2$, the system is exchangeable). This proves point (i).

Step 2. We now consider a (not relabelled for notational simplicity) subsequence of μ^N going in law to some μ and show that μ a.s. belongs to $\mathcal{S}:=\{\mathcal{L}((X_t)_{t\geq 0})\ :\ (X_t)_{t\geq 0}$ solution to the nonlinear SDE (3) with initial law $f_0\}$, recall Definition 2. This will prove point (ii).

Step 2.1. Consider the identity map $\gamma=(\gamma_t)_{t\geq 0}:C([0,\infty),\mathbb{R}^2)\mapsto C([0,\infty),\mathbb{R}^2)$. Using the classical theory of martingale problems, we realize that $Q\in\mathcal{P}(C([0,\infty),\mathbb{R}^2)))$ belongs to $\mathcal{S}$ as soon as, setting $Q_t=Q\circ\gamma_t^{-1}\in\mathcal{P}(\mathbb{R}^2)$ for each $t\geq 0$,

(a) $Q_0=f_0$;

(b) $\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{-1}Q_s(dy)Q_s(dx)ds<\infty$ for all $T>0$;

(c) for all $0<t_1<\dots<t_k<s<t$, all $\varphi_1,\dots,\varphi_k\in C_b(\mathbb{R}^2)$, all $\varphi\in C_b^2(\mathbb{R}^2)$,

$$\mathcal{F}(Q):=\iint Q(dz)Q(d\tilde z)\varphi_1(z_{t_1})\dots\varphi_k(z_{t_k})\Big[\varphi(z_t)-\varphi(z_s)-\chi\int_s^t K(z_u-\tilde z_u)\cdot\nabla\varphi(z_u)du-\int_s^t\Delta\varphi(z_u)du\Big]=0.$$

Indeed, let $(X_t)_{t\geq 0}$ be Q-distributed, so that $\mathcal{L}(X_t)=Q_t$ for all $t\geq 0$. Then (a) says that X_0 is f_0-distributed, (b) is nothing but the requirement (a) of Definition 2, and (c) tells us that for all $\varphi\in C_b^2(\mathbb{R}^2)$,

$$\varphi(X_t)-\varphi(X_s)-\chi\int_0^t\int K(X_s-\tilde z_s)\cdot\nabla\varphi(X_s)Q(d\tilde z)ds-\int_0^t\Delta\varphi(X_s)ds$$

is a martingale in the (completed) filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by $(X_t)_{t\geq 0}$. This classically implies the existence of a 2-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t)_{t\geq 0}$ such that $X_t=X_0+\sqrt{2}B_t+\chi\int_0^t\int K(X_s-\tilde z_s)Q_s(d\tilde z)ds=X_0+\sqrt{2}B_t+\chi\int_0^t(K\star Q_s)(X_s)ds$ for all $t\geq 0$. See Stroock-Varadhan [39, Section 4.5] for much general statements, but unfortunately assuming that the drift is bounded. However, this is not a tedious problem and the condition (b) is actually sufficient: one easily checks (using some C_b^2 approximating functions) that we can apply (c) to the functions $\varphi(x)=x_i$ and $\varphi(x)=x_ix_j$, with $i,j\in\{1,2\}$, from which one easily checks that $B_t=[X_t-X_0-\chi\int_0^t(K\star Q_s)(X_s)ds]/\sqrt{2}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingale with quadratic variation matrix I_2t and thus a Brownian motion by the Lévy theorem.

We now prove that μ a.s. satisfies these three points. For each $t\geq 0$, we set $\mu_t=\mu\circ\gamma_t^{-1}$.

Step 2.2. Since μ_0^N goes to f_0 by assumption, we have $\mu_0=f_0$ a.s., i.e. μ a.s. satisfies (a).

Step 2.3. Using Corollary 10 and exchangeability, we see that for any $\alpha \in (\chi/(2\pi), 1)$, any $T > 0$, there is a finite constant $C_{\alpha,T}$ such that for all $m > 0$, all $N \geq 2$,

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (m \wedge |x-y|^{\alpha-2})\mu_s^N(dy)\mu_s^N(dx)ds\Big] \leq \frac{mT}{N} + \frac{1}{N^2}\sum_{i\neq j}\mathbb{E}\Big[\int_0^T |X_s^{i,N} - X_s^{j,N}|^{\alpha-2}ds\Big]$$
$$\leq \frac{mT}{N} + C_{\alpha,T}.$$

Since μ^N goes in law to μ, the LHS converges to $\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(m \wedge |x-y|^{\alpha-2})\mu_s(dy)\mu_s(dx)ds\Big]$ as $N \to \infty$. Letting m increase to infinity and using the monotone convergence theorem, we find that

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-y|^{\alpha-2}\mu_s(dy)\mu_s(dx)ds\Big] \leq C_{\alpha,T}.$$

Since $\alpha < 1$, this of course implies that μ a.s. satisfies (b).

Step 2.4. From now on, we consider some fixed $\mathcal{F} : \mathcal{P}(C([0,\infty),\mathbb{R}^2)) \mapsto \mathbb{R}$ as in point (c) and we check that $\mathcal{F}(\mu) = 0$ a.s.

Step 2.4.1. Here we prove that for all $N \geq 2$,

$$\mathbb{E}\Big[(\mathcal{F}(\mu^N))^2\Big] \leq \frac{C_\mathcal{F}}{N}. \tag{19}$$

To this end, we recall that $\varphi \in C_b^2(\mathbb{R}^2)$ is fixed and we apply the Itô formula to (4):

$$O_i^N(t) := \varphi(X_t^{i,N}) - \frac{\chi}{N}\sum_{j=1}^N \int_0^t \nabla\varphi(X_s^{i,N}) \cdot K(X_s^{i,N} - X_s^{j,N})ds - \int_0^t \Delta\varphi(X_s^{i,N})ds$$
$$= \varphi(X_0^{i,N}) + \sqrt{2}\int_0^t \nabla\varphi(X_s^{i,N}) \cdot dB_s^i.$$

By definition of $\mathcal{F}$ (recall that $K(0) = 0$ by convention),

$$\mathcal{F}(\mu^N) = \frac{1}{N}\sum_{i=1}^N \varphi_1(X_{t_1}^{i,N})\dots\varphi_k(X_{t_k}^{i,N})[O_i^N(t) - O_i^N(s)]$$
$$= \frac{\sqrt{2}}{N}\sum_{i=1}^N \varphi_1(X_{t_1}^{i,N})\dots\varphi_k(X_{t_k}^{i,N})\int_s^t \nabla\varphi(X_u^{i,N}) \cdot dB_u^i.$$

Then (19) follows from some classical stochastic calculus argument, using that $0 < t_1 < \dots < t_k < s < t$, that $\varphi_1, \dots, \varphi_k, \nabla\varphi$ are bounded and that the Brownian motions $B^1, \dots, B^N$ are independent.

Step 2.4.2. Next we introduce, for $\eta \in (0,1)$, $\mathcal{F}_\eta$ defined as $\mathcal{F}$ with K replaced by the smooth and bounded kernel K_η, recall (10). Then one easily checks that $Q \mapsto \mathcal{F}_\eta(Q)$ is continuous and bounded from $\mathcal{P}(C([0,\infty),\mathbb{R}^2))$ to $\mathbb{R}$. Since μ^N goes in law to μ, we deduce that for any $\eta \in (0,1)$,

$$\mathbb{E}[|\mathcal{F}_\eta(\mu)|] = \lim_N \mathbb{E}[|\mathcal{F}_\eta(\mu^N)|].$$

Step 2.4.3. We now prove that for all $N \geq 2$, all $\eta \in (0,1)$, all $\alpha \in (\chi/(2\pi), 1)$,

$$\mathbb{E}[|\mathcal{F}(\mu) - \mathcal{F}_\eta(\mu)|] + \sup_{N\geq 2}\mathbb{E}[|\mathcal{F}(\mu^N) - \mathcal{F}_\eta(\mu^N)|] \leq C_{\alpha,\mathcal{F}}\,\eta^{1-\alpha}.$$

Using that all the functions (including the derivatives) involved in $\mathcal{F}$ are bounded and that we have $|K_\eta(x) - K(x)| \le \eta^{1-\alpha}|x|^{\alpha-2}\mathbf{1}_{\{x\neq 0\}}/\pi$ by (18), we get the existence of a finite constant $C_\mathcal{F}$ such that

$$\begin{aligned}|\mathcal{F}(Q) - \mathcal{F}_\eta(Q)| \le & C_\mathcal{F}\, \eta^{1-\alpha}\int\int\int_0^t |z_u - \tilde{z}_u|^{\alpha-2}\mathbf{1}_{\{z_u\neq\tilde{z}_u\}}du\, Q(d\tilde{z})Q(dz)\\ =& C_\mathcal{F}\, \eta^{1-\alpha}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{\alpha-2}\mathbf{1}_{\{x\neq y\}}Q_u(dy)Q_u(dx)du.\end{aligned}$$

The conclusion then follows from Step 2.3. combined with the estimate

$$\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{\alpha-2}\mathbf{1}_{\{x\neq y\}}\mu_s^N(dy)\mu_s^N(dx)ds\Big] \le \frac{1}{N^2}\sum_{i\neq j}\mathbb{E}\Big[\int_0^T |X_s^{i,N} - X_s^{j,N}|^{\alpha-2}ds\Big] \le C_{\alpha,T}$$

deduced from Corollary 10 and exchangeability.

Step 2.4.4. For any $\eta\in(0,1)$, we write

$$\begin{aligned}\mathbb{E}[|\mathcal{F}(\mu)|] \le& \mathbb{E}[|\mathcal{F}(\mu) - \mathcal{F}_\eta(\mu)|] + \limsup_N |\mathbb{E}[|\mathcal{F}_\eta(\mu)|] - \mathbb{E}[|\mathcal{F}_\eta(\mu^N)|]|\\ &+ \limsup_N \mathbb{E}[|\mathcal{F}_\eta(\mu^N) - \mathcal{F}(\mu^N)|] + \limsup_N \mathbb{E}[|\mathcal{F}(\mu^N)|].\end{aligned}$$

By Steps 2.4.1 and 2.4.2, the fourth and second terms on the right-hand side are zero. We thus deduce from Step 2.4.3 that $\mathbb{E}[|\mathcal{F}(\mu)|] \le C_{\alpha,\mathcal{F}}\,\eta^{1-\alpha}$. Since $\eta\in(0,1)$ can be chosen arbitrarily small, we conclude that $\mathbb{E}[|\mathcal{F}(\mu)|] = 0$, whence $\mathcal{F}(\mu) = 0$ a.s. as desired.

Step 3. It only remains to check point (iii). Consider the (not relabelled) subsequence μ^N going to μ in $\mathcal{P}(C([0,\infty),\mathbb{R}^2))$ as in Step 2. This implies that $(\mu_t^N)_{t\ge 0}$ goes to $(\mu_t)_{t\ge0}$ in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$. By Step 2, μ is a.s. the law of a solution to the nonlinear SDE (3). As seen in Remark 3, this implies that a.s., $(\mu_t)_{t\ge0}$ is a weak solution to the Keller-Segel equation (1). □

4. (Local) existence for the particle system in the general case

The aim of this section is to prove Theorem 7. We thus fix $\chi > 0$. Although the goal of the section is to prove some results for N fixed, we give uniform (in N) results as often as possible. We introduce the domain, for $N\ge 2$ and $\ell\ge 1$,

$D_\ell^N := \{(x_1,\dots,x_N)\in(\mathbb{R}^2)^N\ :\ |x_i-x_j|+|x_j-x_k|+|x_k-x_i| > 1/\ell$ for all i,j,k pairwise different$\}$,

and we consider the Lipschitz continuous function $\Phi_\ell^N : (\mathbb{R}^2)^N \mapsto [0,1]$ defined by

$$\Phi_\ell^N(x_1,\dots,x_N) = 0\vee\Big(2\ell\min_{i,j,k\text{ distinct}}\{|x_i-x_j|+|x_j-x_k|+|x_k-x_i|\} - 1\Big)\wedge 1,$$

which satisfies $\mathbf{1}_{D_\ell^N}\le\Phi_\ell^N\le\mathbf{1}_{D_{2\ell}^N}$. As usual, the initial condition $(X_0^{i,N})_{i=1,\dots,N}$ is exchangeable, satisfies $\mathbb{E}[|X_0^{1,N}|]<\infty$ and is independent of the i.i.d. 2-dimensional Brownian motions $(B_t^i)_{t\ge0}$, $i=1,\dots,N$. For $\varepsilon\in(0,1)$ and $\ell\ge1$, the particle system

$$X_t^{i,N,\varepsilon,\ell} = X_0^{i,N} + \sqrt{2}B_t^i + \frac{\chi}{N}\sum_{j=1}^N\int_0^t K_\varepsilon(X_s^{i,N,\varepsilon,\ell} - X_s^{j,N,\varepsilon,\ell})\Phi_\ell^N((X_s^{k,N,\varepsilon,\ell})_{k=1,\dots,N})ds \tag{20}$$

is strongly well-posed, since K_ε and Φ_ℓ^N are bounded and Lipschitz continuous. For a fixed $\ell\ge1$, we can show as in Corollary 10 that particles do not meet too often.

Lemma 12. *Fix $\chi > 0$ and consider, for each $N \geq 2$, $\varepsilon \in (0,1)$ and $\ell \geq 1$, the unique solution $(X_t^{i,N,\varepsilon,\ell})_{t\geq 0, i=1,\dots,N}$ to (20) with some exchangeable initial condition $(X_0^{i,N})_{i=1,\dots,N}$ such that $\sup_{N\geq 2} \mathbb{E}[|X_0^{1,N}|] < \infty$.*

(i) For all $t \geq 0$, all $\ell > 0$, all $\varepsilon \in (0,1)$, $\mathbb{E}[(1+|X_t^{1,N,\varepsilon,\ell}|^2)^{1/2}] \leq \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 2t$.

(ii) For all $T > 0$, all $\alpha \in (0,1)$, all $\ell > 0$, there is a constant $C_{T,\alpha,\ell}$ (depending also on χ and on $\sup_{N\geq 2} \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}]$) such that for all $\varepsilon \in (0,1)$, all $N > \chi/(2\alpha\pi)$,

$$\mathbb{E}\Big[\int_0^T |X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^{\alpha-2} ds\Big] \leq 1 + C_{T,\alpha,\ell}\Big(2\alpha - \frac{\chi}{\pi N}\Big)^{(\alpha-2)/(1-\alpha)}.$$

Proof. First, (i) can be checked exactly as Proposition 9-(i), using only that Φ_ℓ^N is nonnegative and does break the exchangeability. We now prove (ii) and thus fix $\alpha \in (0,1)$. Proceeding exactly as in the proof of Proposition 9-(ii), see (15), we find that for all $\eta \in (0,\varepsilon]$,

$$2\alpha^2 I_{\eta,\alpha,T}^{N,\varepsilon,\ell} \leq A_{\alpha,T} + \frac{\chi\alpha}{\pi N} J_{\eta,\alpha,T}^{N,\varepsilon,\ell} + \frac{(N-2)\chi\alpha}{\pi N} K_{\eta,\alpha,T}^{N,\varepsilon,\ell},$$

where $A_{\alpha,T} = (2\sqrt{2}\sup_{N\geq 2}\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}] + 4\sqrt{2}T)^\alpha$ and where

$$\begin{aligned}
I_{\eta,\alpha,T}^{N,\varepsilon,\ell} =& \mathbb{E}[\int_0^T (|X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^2 + \eta^2)^{\alpha/2-1} ds],\\
J_{\eta,\alpha,T}^{N,\varepsilon,\ell} =& \mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^2 + \eta^2)^{\alpha/2-1} \Phi_\ell^N((X_s^{k,N,\varepsilon,\ell})_{k=1,\dots,N}) ds\Big],\\
K_{\eta,\alpha,T}^{N,\varepsilon,\ell} =& \mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^2 + \eta^2)^{(\alpha-1)/2} (|X_s^{1,N,\varepsilon,\ell} - X_s^{3,N,\varepsilon,\ell}|^2 + \varepsilon^2)^{-1/2}\\
&\qquad \Phi_\ell^N((X_s^{k,N,\varepsilon,\ell})_{k=1,\dots,N}) ds\Big].
\end{aligned}$$

Since $\Phi_\ell^N \leq 1$, we obviously have $J_{\eta,\alpha,T}^{N,\varepsilon,\ell} \leq I_{\eta,\alpha,T}^{N,\varepsilon,\ell}$. We next note that for $u, v > 0$, $u^{(\alpha-1)/2} v^{-1/2} \leq (1+u^{-1/2})(1+v^{-1/2}) \leq (1+\max\{u,v\}^{-1/2})(1+u^{-1/2}+v^{-1/2})$ and that, for $x = (x_1,\dots,x_N) \in (\mathbb{R}^2)^N$, $\Phi_\ell^N(x) > 0$ implies that $x \in D_{2\ell}^N$, whence $|x_1 - x_2| + |x_1 - x_3| + |x_2 - x_3| \geq 1/(2\ell)$ and thus $\max\{|x_1 - x_2|, |x_1 - x_3|\} \geq 1/(8\ell)$. Consequently, since $\eta \in (0,\varepsilon]$,

$$\begin{aligned}
&(|x_1-x_2|^2+\eta^2)^{(\alpha-1)/2}(|x_1-x_3|^2+\varepsilon^2)^{-1/2}\Phi_\ell^N(x)\\
\leq& [1+\max\{\eta^2+|x_1-x_2|^2, \eta^2+|x_1-x_3|^2\}^{-1/2}]\\
&\qquad \times [1+(\eta^2+|x_1-x_2|^2)^{-1}+(\eta^2+|x_1-x_3|^2)^{-1}]\mathbf{1}_{\{x\in D_{2\ell}^N\}}\\
\leq& (1+8\ell)[1+(|x_1-x_2|^2+\eta^2)^{-1/2}+(|x_1-x_3|^2+\eta^2)^{-1/2}].
\end{aligned}$$

This implies that

$$\begin{aligned}
K_{\eta,\alpha,T}^{N,\varepsilon,\ell} \leq& (1+8\ell)\mathbb{E}\Big[\int_0^T \Big(1+(|X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^2+\eta^2)^{-1/2} +\\
&\qquad (|X_s^{1,N,\varepsilon,\ell} - X_s^{3,N,\varepsilon,\ell}|^2+\eta^2)^{-1/2}\Big) ds\Big]\\
\leq& (1+8\ell)T + 2(1+8\ell)\mathbb{E}\Big[\int_0^T (|X_s^{1,N,\varepsilon,\ell} - X_s^{2,N,\varepsilon,\ell}|^2+\eta^2)^{-1/2} ds\Big]\\
\leq& (1+8\ell)T + 2(1+8\ell)T^{(1-\alpha)/(2-\alpha)}[I_{\eta,\alpha,T}^{N,\varepsilon,\ell}]^{1/(2-\alpha)}
\end{aligned}$$

by the Hölder inequality. All in all, we have checked that

$$\Big(2\alpha - \frac{\chi}{\pi N}\Big) I^{N,\varepsilon,\ell}_{\eta,\alpha,T} \le B_{\alpha,T,\ell} + C_{\alpha,T,\ell}[I^{N,\varepsilon,\ell}_{\eta,\alpha,T}]^{1/(2-\alpha)},$$

where $B_{\alpha,T,\ell} = A_{\alpha,T}/\alpha + (1+8\ell)T\chi/\pi$ and $C_{\alpha,T,\ell} = 2(1+8\ell)T^{(1-\alpha)/(2-\alpha)}\chi/\pi$.

Separating the cases $I^{N,\varepsilon,\ell}_{\eta,\alpha,T} \le 1$ and $I^{N,\varepsilon,\ell}_{\eta,\alpha,T} > 1$, we easily conclude that

$$I^{N,\varepsilon,\ell}_{\eta,\alpha,T} \le 1 + \Big(B_{\alpha,T,\ell} + C_{\alpha,T,\ell}\Big)^{(2-\alpha)/(1-\alpha)} \Big(2\alpha - \frac{\chi}{\pi N}\Big)^{(\alpha-2)/(1-\alpha)}.$$

It finally suffices to let $\eta \searrow 0$ to conclude the proof. □

We now deduce some compactness, still for ℓ fixed.

Lemma 13. *Fix $\chi > 0$ and consider, for each $N \ge 2$, $\varepsilon \in (0,1)$ and $\ell \ge 1$, the unique solution $(X^{i,N,\varepsilon,\ell}_t)_{t\ge0,i=1,\dots,N}$ to (20) with some exchangeable initial condition $(X^{i,N}_0)_{i=1,\dots,N}$ such that $\sup_{N\ge2}\mathbb{E}[|X^{1,N}_0|] < \infty$. For all $\ell \ge 1$, the family $\{(X^{1,N,\varepsilon,\ell}_t)_{t\ge0},\ N > \max\{2,\chi/(2\pi)\},\ \varepsilon\in(0,1)\}$ is tight in $C([0,\infty),\mathbb{R}^2)$.*

Proof. We fix $\ell \ge 1$ and $T > 0$. As in the proof of Lemma 11, the only difficulty is to prove that the family $\{(J^{1,N,\varepsilon,\ell}_t)_{t\in[0,T],i=1,\dots,N},\ N \ge N_0,\ \varepsilon\in(0,1) > 0\}$ is tight in $C([0,T],\mathbb{R}^2)$, where $N_0 = \lfloor \max\{2,\chi/(2\pi)\rfloor + 1$ and

$$J^{1,N,\varepsilon,\ell}_t = \frac{\chi}{N}\sum_{j=1}^N \int_0^t K_\varepsilon(X^{i,N,\varepsilon,\ell}_s - X^{j,N,\varepsilon,\ell}_s)\Phi^N_\ell((X^{k,N,\varepsilon,\ell}_s)_{k=1,\dots,N})ds.$$

We consider $\alpha\in(0,1)$ such that $2\alpha - \chi/(\pi N_0) > 0$, so that, by Lemma 12,

$$\sup_{N\ge N_0,\varepsilon\in(0,1)} \mathbb{E}\Big[\int_0^T |X^{1,N,\varepsilon,\ell}_s - X^{2,N,\varepsilon,\ell}_s|^{\alpha-2}ds\Big] < \infty. \tag{21}$$

Using that $|\Phi^N_\ell| \le 1$, we check as in the proof of Lemma 11 that for all $0 \le s < t \le T$, we have $|J^{1,N,\varepsilon,\ell}_t - J^{1,N,\varepsilon,\ell}_s| \le Z^{N,\varepsilon,\ell}_T |t-s|^\beta$, where $\beta = (1-\alpha)/(2-\alpha)$ and where

$$Z^{N,\varepsilon,\ell}_T = \frac{\chi}{2\pi N}\sum_{j=2}^N \Big[1 + \int_0^T |X^{1,N,\varepsilon,\ell}_s - X^{j,N,\varepsilon,\ell}_s|^{\alpha-2}ds\Big].$$

But (21) and exchangeability imply that $\sup_{N\ge N_0,\varepsilon\in(0,1)}\mathbb{E}[Z^{N,\varepsilon,\ell}_T] < \infty$. We conclude exactly as in the proof of Lemma 11. □

We now make ε tend to 0 in the particle system (20), simultaneously for all $\ell \ge 1$.

Lemma 14. *Let $\chi > 0$, $N > \max\{2,\chi/(2\pi)\}$ and $F^N_0 \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ be fixed. There exists, on some probability space endowed with some filtration $(\mathcal{F}_t)_{t\ge0}$, a F^N_0-distributed $\mathcal{F}_0$-measurable random variable $(X^{i,N}_0)_{i=1,\dots,N}$, a $2N$-dimensional $(\mathcal{F}_t)_{t\ge0}$-Brownian motion $(B^i_t)_{i=1,\dots,N,t\ge0}$ and, for each $\ell \ge 1$, an $(\mathcal{F}_t)_{t\ge0}$-adapted solution to*

$$X^{i,N,\ell}_t = X^{i,N}_0 + \sqrt{2}B^i_t + \frac{\chi}{N}\sum_{j=1}^N \int_0^t K(X^{i,N,\ell}_s - X^{j,N,\ell}_s)\Phi^N_\ell((X^{k,N,\ell}_s)_{k=1,\dots,N})ds. \tag{22}$$

The family $\{(X_t^{i,N,\ell})_{t\geq 0,\ell\geq 1}\ i=1,\dots,N\}$ *is furthermore exchangeable. Moreover, for all* $\ell\geq 1$, *all* $t>0$, *we have* $\mathbb{E}[(1+|X_t^{1,N,\ell}|^2)^{1/2}]\leq \mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}]+2t$ *and, for all* $\alpha\in(\chi/(2\pi N),1)$,

$$\mathbb{E}\Big[\int_0^t |X_s^{1,N,\ell}-X_s^{2,N,\ell}|^{\alpha-2}ds\Big]<\infty.$$

Finally, we have the following compatibility property: for all $\ell'\geq\ell\geq 1$, *a.s.*, $(X_t^{i,N,\ell})_{i=1,\dots,N}=(X_t^{i,N,\ell'})_{i=1,\dots,N}$ *for all* $t\in[0,\tau_N^\ell)$, *where*

$$\tau_N^\ell=\inf\{t\geq 0\ :\ (X_t^{i,N,\ell})_{i=1,\dots,N}\notin D_\ell^N\}.$$

Proof. We thus fix $\chi>0$, $N>\max\{2,\chi/(2\pi)\}$ and $F_0^N\in\mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ and divide the proof in several steps.

Step 1. We know from Lemma 13 that for each $\ell\geq 1$, the family $\{(X_t^{1,N,\varepsilon,\ell})_{t\geq 0},\ \varepsilon\in(0,1)\}$ (all built with the same initial law F_0^N) is tight in $C([0,\infty),\mathbb{R}^2)$. By exchangeability, $\{(X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N},\ \varepsilon\in(0,1)\}$ is thus tight in $C([0,\infty),(\mathbb{R}^2)^N)$, still for each $\ell\geq 1$. Hence for all $\eta>0$, we can find a compact subset $\mathcal{K}_\eta^\ell$ of $C([0,\infty),(\mathbb{R}^2)^N))$ such that

$$\sup_{\varepsilon\in(0,1)}\mathbb{P}((X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N}\notin\mathcal{K}_\eta^\ell)\leq\eta 2^{-\ell}.$$

We now introduce $\mathcal{K}_\eta:=\prod_{\ell\geq 1}\mathcal{K}_\eta^\ell$, which is a compact subset of $[C([0,\infty),(\mathbb{R}^2)^N)]^{\mathbb{N}}$ (endowed with the product topology) by Tychonoff's theorem. It holds that

$$\sup_{\varepsilon\in(0,1)}\mathbb{P}(((X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N})_{\ell\geq 1}\notin\mathcal{K}_\eta)\leq\sum_{\ell\geq 1}\sup_{\varepsilon\in(0,1)}\mathbb{P}((X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N}\notin\mathcal{K}_\eta^\ell)\leq\eta.$$

Consequently, the family $\{((X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N})_{\ell\geq 1},\ \varepsilon\in(0,1)\}$ is tight in $[C([0,\infty),(\mathbb{R}^2)^N)]^{\mathbb{N}}$. Finally, we conclude that the family

$$\{(((X_t^{i,N,\varepsilon,\ell})_{t\geq 0,i=1,\dots,N})_{\ell\geq 1},(B_t^i)_{t\geq 0,i=1,\dots,N}),\ \varepsilon\in(0,1)\}$$

is tight in $[C([0,\infty),(\mathbb{R}^2)^N)]^{\mathbb{N}}\times C([0,\infty),(\mathbb{R}^2)^N)$.

Step 2. We now use the Skorokhod representation theorem: we can find a sequence $\varepsilon_k\searrow 0$ and a sequence $(((\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,i=1,\dots,N})_{\ell\geq 1},(\tilde{B}_t^{i,k})_{t\geq 0,i=1,\dots,N})$ going a.s. in $[C([0,\infty),(\mathbb{R}^2)^N)]^{\mathbb{N}}\times C([0,\infty,(\mathbb{R}^2)^N)$ to some $(((X_t^{i,N,\ell})_{t\geq 0,i=1,\dots,N})_{\ell\geq 1},(B_t^i)_{t\geq 0,i=1,\dots,N})$ and such that, for each $\ell\geq 1$, each $k\geq 1$, $(\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,i=1,\dots,N}$ solves (20) with the Brownian motions $(\tilde{B}_t^{i,k})_{t\geq 0,i=1,\dots,N}$ and some F_0^N-distributed initial condition $(\tilde{X}_0^{i,N,\varepsilon_k})_{i=1,\dots,N}$ (not depending on $\ell\geq 1$). The exchangeability of $\{(X_t^{i,N,\ell})_{t\geq 0,\ell\geq 1},\ i=1,\dots,N\}$ is inherited from that of $\{(\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,\ell\geq 1},\ i=1,\dots,N\}$. Next, Lemma 12 and the Fatou Lemma imply that for all $t\geq 0$, all $\ell\geq 1$,

$$\max\Big\{\mathbb{E}[(1+|X_t^{1,N,\ell}|^2)^{1/2}],\sup_{k\geq 1}\mathbb{E}[(1+|X_t^{1,N,\varepsilon_k,\ell}|^2)^{1/2}]\Big\}\leq\mathbb{E}[(1+|X_0^{1,N}|^2)^{1/2}]+2t$$

and that, for all $\alpha\in(\chi/(2\pi N),1)$, all $T>0$, all $\ell\geq 1$,

$$\mathbb{E}\Big[\int_0^T|X_s^{1,N,\ell}-X_s^{2,N,\ell}|^{\alpha-2}ds\Big]+\sup_{k\geq 1}\mathbb{E}\Big[\int_0^T|X_s^{1,N,\varepsilon_k,\ell}-X_s^{2,N,\varepsilon_k,\ell}|^{\alpha-2}ds\Big]<\infty.$$

Step 3. We introduce $\mathcal{F}_t=\sigma((X_s^{i,N,\ell},B_s^i)_{i=1,\dots,N,s\in[0,t]})$, to which $(X_t^{i,N,\ell})_{t\geq 0,i=1,\dots,N}$ is of course adapted for each $\ell\geq 1$. We clearly have $X_0^{i,N,\ell}=X_0^{i,N,\ell'}$ for all $i=1,\dots,N$ and all $\ell,\ell'\geq 1$ (because $\tilde{X}_0^{i,N,\varepsilon_k,\ell}=\tilde{X}_0^{i,N,\varepsilon_k,\ell'}$ for all $k\geq 1$, all $i=1,\dots,N$ and all $\ell,\ell'\geq 1$). We thus may define $X_0^{i,N}:=X_0^{i,N,\ell}$ for all $i=1,\dots,N$, for any value of ℓ. The family $(X_0^{i,N})_{i=1,\dots,N}$ is of course F_0^N-distributed. Finally, one checks as in the proof of Theorem 5-Step 2 $(B_t^i)_{t\geq 0, i=1,\dots,N}$ is $2N$-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Step 4. It is checked exactly as in the proof of Theorem 5-Step 3 that for each $\ell \geq 1$, $(X_t^{i,N,\ell})_{t\geq 0,i=1,\dots,N}$ solves (22): it suffices to pass to the limit in probability as $k \to \infty$ in the equation satisfied by $(\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,i=1,\dots,N}$, using the estimates proved in Step 2 and that Φ_ℓ^N is continuous.

Step 5. It only remains to prove the compatibility property. We introduce, for $\ell \geq 1$ and $k \geq 1$,

$$\tau_N^{\ell,k} := \inf\{t \geq 0 \ : \ (\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{i=1,\dots,N} \notin D_\ell^N\} \quad \text{and} \quad \tau_N^\ell := \inf\{t \geq 0 \ : \ (X_t^{i,N,\ell})_{i=1,\dots,N} \notin D_\ell^N\}.$$

Since $(\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,i=1,\dots,N}$ goes a.s. to $(X_t^{i,N,\varepsilon_k,\ell})_{t\geq 0,i=1,\dots,N}$ in $C([0,\infty),(\mathbb{R}^2)^N)$ and since $(D_\ell^N)^c$ is a closed subset of $(\mathbb{R}^2)^N$, we deduce that $\tau_N^\ell \leq \liminf_{k\to\infty} \tau_N^{\ell,k}$. But for all $\ell' \geq \ell \geq 1$, we have $(\tilde{X}_t^{i,N,\varepsilon_k,\ell})_{i=1,\dots,N} = (\tilde{X}_t^{i,N,\varepsilon_k,\ell'})_{i=1,\dots,N}$ on the time interval $[0,\tau_N^{\ell,k}]$ for any $k \geq 1$: this follows from the pathwise uniqueness for (20) and from the fact that $\Phi_\ell^N = \Phi_{\ell'}^N = 1$ on D_ℓ^N. Using finally that $(\tilde{X}_t^{i,N,\varepsilon_k,\ell}, \tilde{X}_t^{i,N,\varepsilon_k,\ell'})_{t\geq 0,i=1,\dots,N}$ goes a.s. to $(X_t^{i,N,\ell}, X_t^{i,N,\ell'})_{t\geq 0,i=1,\dots,N}$ in $C([0,\infty),(\mathbb{R}^2)^N \times (\mathbb{R}^2)^N)$, we conclude that indeed, $(X_t^{i,N,\ell})_{i=1,\dots,N} = (X_t^{i,N,\ell'})_{i=1,\dots,N}$ on $[0,\tau_N^\ell]$. ☐

Finally, we let ℓ increase to infinity.

Proof of Theorem 7. We fix $\chi > 0$, $N > \max\{2, \chi/(2\pi)\}$ and $F_0^N \in \mathcal{P}_{sym,1}((\mathbb{R}^2)^N)$ such that (7) holds true. We consider the objects built in Lemma 14: the filtration $(\mathcal{F}_t)_{t\geq 0}$, the $2N$-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t^i)_{i=1,\dots,N,t\geq 0}$, the $(\mathcal{F}_t)_{t\geq 0}$-adapted solution $(X_t^{i,N,\ell})_{t\geq 0,i=1,\dots,N}$, for each $\ell \geq 1$, to (22), and the associated stopping times τ_N^ℓ. Using the compatibility property, we deduce that τ_N^ℓ is a.s. increasing (as a function of ℓ) and we define $\tau_N = \sup_{\ell\geq 1} \tau_N^\ell$. Still using the compatibility property, we deduce that for all $t \in [0,\tau_N)$, all ℓ such that $\tau_N^\ell \geq t$, all $\ell' \geq \ell$, $(X_t^{i,N,\ell})_{i=1,\dots,N} = (X_t^{i,N,\ell'})_{i=1,\dots,N}$. Hence for $t \in [0,\tau_N)$, we can define $(X_t^{i,N})_{i=1,\dots,N}$ as $(X_t^{i,N,\ell})_{i=1,\dots,N}$ for any choice of ℓ such that $\tau_N^\ell \geq t$. Since $\Phi_\ell^N((X_t^{i,N,\ell})_{i=1,\dots,N}) = 1$ for $t \in [0,\tau_N^\ell]$, by the definitions of Φ_N^ℓ and of τ_N^ℓ, we conclude that indeed, $(X_t^{i,N})_{t\in[0,\tau_N),i=1,\dots,N}$ solves (4) with the Brownian motions $(B_t^i)_{i=1,\dots,N,t\geq 0}$, and that $\tau_N^\ell = \inf\{t \geq 0 \ : \ (X_t^{i,N})_{i=1,\dots,N} \notin D_\ell^N\}$, so that

$$\tau_N = \sup_{\ell\geq 1} \inf \{t \geq 0 \ : \ (X_t^{i,N})_{i=1,\dots,N} \notin D_\ell^N\}$$

as in the statement. The exchangeability and $(\mathcal{F}_t)_{t\geq 0}$-adaptation of the family $\{(X_t^{i,N})_{t\in[0,\tau_N)}, \ i = 1,\dots,N\}$ is of course inherited from $\{(X_t^{i,N,\ell})_{t\geq 0,\ell\geq 1}, \ i = 1,\dots,N\}$. We also have a.s., for all $t \in [0,\tau_N)$, all $\alpha \in (\chi/(2\pi N), 1)$,

$$\int_0^t |X_s^{1,N} - X_s^{2,N}|^{\alpha-2} ds = \int_0^t |X_s^{1,N,\ell} - X_s^{2,N,\ell}|^{\alpha-2} ds$$

as soon as ℓ is large enough so that $\tau_N^\ell \geq t$. This last quantity is a.s. finite by Lemma 14 again.

It remains to decide whether τ_N is finite or infinite. For $I \subset \{1,\dots,N\}$ with cardinality $|I| \geq 2$ and $t \in [0,\tau_N)$, let $\bar{X}_t^I = |I|^{-1} \sum_{i\in I} X_t^{i,N}$ and $R_t^I = 2^{-1} \sum_{i\in I} |X_t^{i,N} - \bar{X}_t^I|^2$.

First assume that $\chi > 8\pi(N-2)/(N-1)$. Consider $I_N = \{1,\dots,N\}$. As shown in Lemma 16 below, $(R_t^{I_N})_{t\in[0,\tau_N)}$ is a squared Bessel process with dimension $(N-1)(2-\chi/4\pi) < 2$, restricted to $[0,\tau_N)$. But a squared Bessel process with dimension smaller than 2 a.s. reaches zero in finite time, see [36, page 442]. We conclude that on the event $\{\tau_N = \infty\}$, R^{I_N} reaches zero in finite time, which of course implies that $\tau_N < \infty$. Thus $\mathbb{P}(\tau_N = \infty) = 0$ as desired.

Assume next that $\chi \le 8\pi(N-2)/(N-1)$. Observe that for $(x_1,x_2,x_3)\in(\mathbb{R}^2)^3$ and $\bar x = (x_1+x_2+x_3)/3$,

$$\begin{aligned}|x_1-x_2|+|x_2-x_3|+|x_3-x_1| &\ge (|x_1-x_2|^2+|x_2-x_3|^2+|x_3-x_1|^2)^{1/2}\\ &= \sqrt{3}(|x_1-\bar x|^2+|x_2-\bar x|^2+|x_3-\bar x|^2)^{1/2}.\end{aligned}$$

Consequently, for $\ell \ge 1$,

$$\begin{aligned}\mathbb{P}(\tau_N<\infty) &= \mathbb{P}(\tau_N<\infty, \tau_N^\ell \le \tau_N)\\ &= \mathbb{P}\Big(\tau_N<+\infty, \min_{i,j,k \text{ distinct}} \inf_{t\in[0,\tau_N)} (|X^i_t-X^j_t|+|X^j_t-X^k_t|+|X^k_t-X^i_t|)\le \frac{1}{\ell}\Big)\\ &\le \mathbb{P}\Big(\tau_N<+\infty, \min_{I\,:\,|I|=3} \inf_{t\in[0,\tau_N)} R^I_t \le \frac{1}{6\ell^2}\Big).\end{aligned}$$

This last quantity tends to 0 as $\ell\to\infty$ thanks to the following lemma, whence $\mathbb{P}(\tau_N<\infty)=0$. □

Lemma 15. *Let $N\ge 3$ and $\chi\in(0,8\pi(N-2)/(N-1)]$. Consider $(X^{i,N}_t)_{t\in[0,\tau_N),i=1,\dots,N}$ built in the previous proof. For $I\subset\{1,\dots,N\}$ with cardinality $|I|\ge 2$ and $t\in[0,\tau_N)$, let $\bar X^I_t = |I|^{-1}\sum_{i\in I} X^{i,N}_t$ and $R^I_t = 2^{-1}\sum_{i\in I}|X^{i,N}_t-\bar X^I_t|^2$. For all $I\subset\{1,\dots,N\}$ with $|I|\ge 3$,*

$$\mathbb{P}\Big(\tau_N<\infty, \inf_{t\in[0,\tau_N)} R^I_t = 0\Big) = 0.$$

This lemma requires some preparation.

Lemma 16. *Adopt the notation and assumptions of Lemma 15. Let $I\subset\{1,\dots,N\}$ with $|I|\ge 3$. A.s., $R^I_0>0$ and there exists a one-dimensional Brownian motion $(\beta^I_t)_{t\ge 0}$ such that for $t\in[0,\tau_N)$,*

$$dR^I_t = 2\sqrt{R^I_t}\,d\beta^I_t + (|I|-1)\Big(2-\frac{\chi|I|}{4\pi N}\Big)dt + \frac{\chi}{N}\sum_{i\in I}\sum_{j\notin I}(X^{i,N}_t-\bar X^I_t)\cdot K(X^{i,N}_t-X^{j,N}_t)dt. \tag{23}$$

In particular, with $I_N=\{1,\dots,N\}$, $dR^{I_N}_t = 2\sqrt{R^{I_N}_t}\,d\beta^I_t + (N-1)\Big(2-\frac{\chi}{4\pi}\Big)dt$ and $(R^{I_N}_t)_{t\in[0,\tau_N)}$ is a squared Bessel process with dimension $(N-1)(2-\chi/4\pi)\ge 2$, restricted to $[0,\tau_N)$.

Proof. Fix I as in the statement. By assumption, recall (7), we clearly have $R^I_0>0$ a.s. Also, by definition of τ_N, we have a.s. $R^I_t>0$ for all $t\in[0,\tau_N)$. For all $t\in[0,\tau_N)$, let

$$\beta^I_t = \int_0^t \frac{1}{\sqrt{2R^I_s}}\sum_{i\in I}(X^{i,N}_s-\bar X^I_s)\cdot dB^i_s.$$

This process can easily be extended into a one-dimensional Brownian motion $(\beta^I_t)_{t\ge0}$. We now check (23). We work on $[0,\tau_N)$. Starting from (4) and setting $\bar B^I_t = |I|^{-1}\sum_{i\in I} B^i_t$,

$$d(X^{i,N}_t-\bar X^I_t) = \sqrt{2}d(B^i-\bar B^I)_t + \frac{\chi}{N}\Big[\sum_{j\ne i}K(X^{i,N}_t-X^{j,N}_t) - |I|^{-1}Z^I_t\Big]dt,$$

where $Z^I_t = \sum_{k\in I}\sum_{j\ne k}K(X^{k,N}_t-X^{j,N}_t)$. Using the Itô formula, we thus find

$$\begin{aligned}d|X^{i,N}_t-\bar X^I_t|^2 =& 2\sqrt{2}(X^{i,N}_t-\bar X^I_t)\cdot(dB^i_t-d\bar B^I_t) + 4\frac{|I|-1}{|I|}dt\\ &+\frac{2\chi}{N}(X^{i,N}_t-\bar X^I_t)\cdot\Big[\sum_{j\ne i}K(X^{i,N}_t-X^{j,N}_t)-|I|^{-1}Z^I_t\Big]dt\end{aligned}$$

and thus

$$\begin{aligned} dR_t^I =& \sqrt{2}\sum_{i\in I}(X_t^{i,N}-\bar X_t^I)\cdot(dB_t^i-d\bar B_t^I)+2(|I|-1)dt \\ &+\frac{\chi}{N}\sum_{i\in I}(X_t^{i,N}-\bar X_t^I)\cdot\Big[\sum_{j\neq i}K(X_t^{i,N}-X_t^{j,N})-|I|^{-1}Z_t^I\Big]dt \end{aligned}$$

We now observe that $\sum_{i\in I}(X_t^{i,N}-\bar X_t^I)\cdot(dB_t^i-d\bar B_t^I)=\sum_{i\in I}(X_t^{i,N}-\bar X_t^I)\cdot dB_t^i=\sqrt{2R_t^I}d\beta_t^I$ and that $\sum_{i\in I}(X_t^{i,N}-\bar X_t^I)Z_t^I=0$, so that

$$dR_t^I=2\sqrt{R_t^I}d\beta_t^I+2(|I|-1)dt+\frac{\chi}{N}\sum_{i\in I}\sum_{j\neq i}(X_t^{i,N}-\bar X_t^I)\cdot K(X_t^{i,N}-X_t^{j,N})dt$$

It now suffices to note that $\sum_{i,j\in I,j\neq i}\bar X_t^I\cdot K(X_t^{i,N}-X_t^{j,N})=0$ and that

$$\sum_{i,j\in I,j\neq i}X_t^{i,N}\cdot K(X_t^{i,N}-X_t^{j,N})=\frac12\sum_{i,j\in I,j\neq i}(X_t^{i,N}-X_t^{j,N})\cdot K(X_t^{i,N}-X_t^{j,N})=-\frac{|I|(|I|-1)}{4\pi}$$

to conclude the proof. □

The following remark is a key observation.

Remark 17. *We see in* (23) *that, up to the third-term in the right-hand side, for all* $I\subset\{1,\dots,N\}$ *with* $|I|\geq 3$*, the process* R^I *evolves like the square of a Bessel process of dimension* $(|I|-1)(2-\chi|I|/(4\pi N))$*. The condition* $\chi\in(0,8\pi(N-2)/(N-1)]$ *implies that*

$$\min_{n=3,\dots,N}(n-1)(2-\chi n/(4\pi N))\geq 2. \tag{24}$$

Indeed, observe that $\phi(x)=(x-1)(2-\chi x/(4\pi N))$ *is concave, so that we only have to verify that* $\phi(3)\geq 2$ *and* $\phi(N)\geq 2$*. First,* $\phi(N)\geq 2$ *is equivalent to our condition that* $\chi\leq 8\pi(N-2)/(N-1)$*. Next,* $\phi(3)\geq 2$ *is equivalent to* $\chi\leq 4\pi N/3$*. Finally, it is not hard to verify that,* $N\geq 3$ *being an integer, we always have* $8\pi(N-2)/(N-1)\leq 4\pi N/3$*.*

Since by [36, page 442] a squared Bessel process of dimension $\delta\geq 2$ a.s. never reaches zero, we expect that indeed, for any $|I|\geq 3$, R^I a.s. never reaches zero.

Proof of Lemma 15. We prove by backward induction that for all $n=3,\dots,N$,

$$\forall\, I\subset\{1,\dots,N\}\text{ with }|I|=n,\ \ \mathbb{P}\Big(\tau_N<\infty,\inf_{t\in[0,\tau_N)}R_t^I=0\Big)=0. \tag{25}$$

We first observe that (25) is clear when $n=N$. Indeed, we know from Lemma 16 that for $I_N=\{1,\dots,N\}$, $(R_t^I)_{t\in[0,\tau_N)}$ is a squared Bessel process with dimension $(N-1)(2-\chi/(4\pi))\geq 2$ restricted to $[0,\tau_N)$. Hence $\inf_{[0,\tau_N)}R_t^I>0$ a.s. on the event $\{\tau_N<\infty\}$.

We now assume that (25) holds for some $n\in\{4,\dots,N\}$ and check that it also holds for $n-1$. We thus consider some fixed $I\subset\{1,\dots,N\}$ with cardinality $n-1$. We have to prove that a.s. on $\{\tau_N<\infty\}$, $\inf_{[0,\tau_N)}R_t^I>0$. For each $j\in\{1,\dots,N\}\setminus I$, we introduce $I_j=I\cup\{j\}$.

The main ideas are the following :

• *If* R^I *was reaching* 0*, then around the time it does, the last term in* (23) *would be reasonable (not too large). Indeed,* R^I *reaches zero when a collision between* $(n-1)$ *particles occur, so that the other particles are not too close since we already know there are no collisions of* n *(or more) particles.* These facts are quantified in Steps 1 and 2.

• *But if the last term in* (23) *is not too large, then* R^I *really behaves as a Bessel process with dimension* $(n-2)(2-\chi(n-1)/(4\pi N)) \geq 2$ *and thus can not reach* 0. This is checked in Steps 3 and 4.

Step 1. We claim that for each $j \in \{1, \dots, N\} \setminus I$, each $(x_1, \dots, x_N) \in (\mathbb{R}^2)^N$, setting $\bar{x}^I = (n-1)^{-1} \sum_{i\in I} x_i$ and $\bar{x}^{I_j} = n^{-1} \sum_{i \in I^j} x_i$,

$$(2n-3) \min_{k\in I} |x_k - x_j|^2 \geq n \sum_{i\in I_j} |x_i - \bar{x}^{I_j}|^2 - 3(n-1) \sum_{i\in I} |x_i - \bar{x}^I|^2.$$

We fix $k \in I$ and start from $|x_k - x_j|^2 = \sum_{i\in I} |x_i - x_j|^2 - \sum_{i\in I, i\neq k} |x_i - x_j|^2$ whence, since $\sum_{i\in I, i\neq k} |x_i - x_j|^2 \leq 2(n-2)|x_k - x_j|^2 + 2\sum_{i\in I, i\neq k} |x_i - x_k|^2$,

$$(2n-3)|x_k - x_j|^2 \geq \sum_{i\in I} |x_i - x_j|^2 - 2 \sum_{i\in I, i\neq k} |x_i - x_k|^2.$$

But one easily checks that $2\max_{k\in I} \sum_{i\in I, i\neq k} |x_i - x_k|^2 \leq \sum_{i,k\in I} |x_i - x_k|^2$, whence

$$(2n-3) \min_{k\in I} |x_k - x_j|^2 \geq \sum_{i\in I} |x_i - x_j|^2 - \sum_{i,k\in I} |x_i - x_k|^2 = \frac{1}{2} \sum_{i,k\in I_j} |x_i - x_k|^2 - \frac{3}{2} \sum_{i,k\in I} |x_i - x_k|^2.$$

The claim then follows from the facts that $\sum_{i,k\in I_j} |x_i - x_k|^2 = 2n \sum_{i\in I^j} |x_i - \bar{x}^{I_j}|^2$ and that $\sum_{i,k\in I} |x_i - x_k|^2 = 2(n-1)\sum_{i\in I} |x_i - \bar{x}^I|^2$.

Step 2. We now fix $a > 0$ and $b = a/3$. Step 1 implies that when $\min_{j\notin I} R_t^{I_j} \geq a$ and $R_t^I \leq b$, we have

$$\min_{k\in I, j\notin I} |X_t^{k,N} - X_t^{j,N}|^2 \geq \frac{2an}{2n-3} - \frac{6(n-1)b}{2n-3} = \frac{2a}{2n-3}, \tag{26}$$

whence

$$\max_{k\in I, j\notin I} |K(X_t^{k,N} - X_t^{j,N})| \leq \frac{\sqrt{2n-3}}{2\pi\sqrt{2a}}.$$

Hence one may bound the third term in the right-hand side of (23) from below:

$$\begin{aligned} \mathbf{1}_{\{\min_{j\notin I} R_t^{I_j} \geq a, R_t^I \leq b\}} \frac{\chi}{N} \sum_{i\in I} \sum_{j\notin I} (X_t^{i,N} - \bar{X}_t^I) \cdot K(X_t^{i,N} - X_t^{j,N}) \\ \geq -\frac{\chi\sqrt{2n-3}}{2\pi N\sqrt{2a}} \sum_{i\in I} \sum_{j\notin I} |X_t^{i,N} - \bar{X}_t^I| \geq -c\sqrt{R_t^I}, \end{aligned} \tag{27}$$

with $c := (N+1-n)\chi\sqrt{(2n-3)(n-1)}/(2\pi N\sqrt{a})$. Let us now define the stopping time

$$\sigma_a = \inf \left\{ t \in [0, \tau_N) : \min_{j\notin I} R_t^{I_j} < a \right\}$$

with convention $\inf \emptyset = \tau_N$ and introduce the process $(R_t^{I,a})_{t\in[0,\tau_N)}$ defined by $R_t^{I,a} = R_t^I$ for $t \in [0, \sigma_a)$ and, when $\sigma_a < \tau_N$, by being the unique solution, for $t \in [\sigma_a, \tau_N)$, to

$$R_t^{I,a} = R_{\sigma_a}^I + 2\int_{\sigma_a}^t \sqrt{R_s^{I,a}} d\beta_s^I + (|I|-1)\Big(2 - \frac{\chi|I|}{4\pi N}\Big)(t - \sigma_a).$$

The existence of a pathwise unique solution to this equation follows from [36, Theorem 3.5 p 390]. We deduce from (23) that this process satisfies, for all $t \in [0, \tau_N)$,

$$R_t^{I,a} = R_0^I + 2\int_0^t \sqrt{R_s^{I,a}}d\beta_s^I + (|I|-1)\Big(2 - \frac{\chi|I|}{4\pi N}\Big)t + \frac{\chi}{N}\int_0^t \mathbf{1}_{\{s<\sigma_a\}}\sum_{i\in I}\sum_{j\notin I}(X_s^{i,N} - \bar{X}_s^I)\cdot K(X_s^{i,N} - X_s^{j,N})ds.$$

Step 3. Recall that $a > 0$ and $b = a/3$ are fixed and that $c > 0$ has been defined in Step 2. The existence of a solution $(\underline{R}_t^{I,b})_{t\ge 0}$ such that $\mathbb{P}(\forall t \ge 0,\ \underline{R}_t^{I,b} \in (0, b]) = 1$ to the SDE reflected at the level b

$$\text{(28)}\qquad \begin{cases} \underline{R}_t^{I,b} = R_0^I \wedge b + 2\int_0^t \sqrt{\underline{R}_s^{I,b}}d\beta_s^I + (|I|-1)\Big(2 - \frac{\chi|I|}{4\pi N}\Big)t - c\int_0^t \sqrt{\underline{R}_s^{I,b}}ds - L_t \\ (L_s)_{s\ge 0} \text{ is an adapted increasing process such that } L_0 = 0 \text{ and } \int_0^t (b - \underline{R}_s^{I,b})dL_s = 0 \end{cases}$$

will be checked at the end of the proof using that $|I| \ge 3$. We take this for granted and show that a.s., for all $t \in [0, \tau_N)$, $R_t^{I,a} \ge \underline{R}_t^{I,b}$.

By [36, Lemma 3.3 p 389] with the choice $\rho(u) = |u|$, the local time at 0 of the continuous semimartingale $S_t = \underline{R}_t^{I,b} - R_t^{I,a}$ vanishes. Indeed, it suffices that a.s., $\int_0^t (\rho(S_s))^{-1}d\langle S, S\rangle_s < \infty$, which follows from the fact that $d\langle S, S\rangle_s = 4(\sqrt{\underline{R}_s^{I,b}} - \sqrt{R_s^{I,a}})^2 ds \le 4|\underline{R}_s^{I,b} - R_s^{I,a}|ds = 4\rho(S_s)ds$.

Hence, setting $x^+ = \max(x, 0)$, one has, by Tanaka's formula, for all $t \in [0, \tau_N)$,

$$(\underline{R}_t^{I,b} - R_t^{I,a})^+ = (\underline{R}_0^{I,b} - R_0^{I,a})^+ + \int_0^t \mathbf{1}_{\{\underline{R}_s^{I,b} > R_s^{I,a}\}}d(\underline{R}_s^{I,b} - R_s^{I,a}).$$

Since $\underline{R}_0^{I,b} - R_0^{I,a} = R_0^I \wedge b - R_0^I \le 0$, we find

$$(\underline{R}_t^{I,b} - R_t^{I,a})^+ \le 2\int_0^t \mathbf{1}_{\{\underline{R}_s^{I,b} > R_s^{I,a}\}}\Big(\sqrt{\underline{R}_s^{I,b}} - \sqrt{R_s^{I,a}}\Big)d\beta_s^I - \int_0^t \mathbf{1}_{\{\underline{R}_s^{I,b} > R_s^{I,a}\}}dL_s + \int_0^t \mathbf{1}_{\{\underline{R}_s^{I,b} > R_s^{I,a}\}}\Big(-c\sqrt{\underline{R}_s^{I,b}} - \frac{\chi}{N}\mathbf{1}_{\{s<\sigma_a\}}\sum_{i\in I}\sum_{j\notin I}(X_s^{i,N} - \bar{X}_s^I)\cdot K(X_s^{i,N} - X_s^{j,N})\Big)ds.$$

Since L is an increasing process, the second term on the right-hand side is nonpositive. The third term on the right-hand side is also nonpositive, because $s < \sigma_a$ implies that $R_s^{I,a} = R_s^I$, so that $\underline{R}_s^{I,b} > R_s^{I,a}$ implies that $R_s^I \le b$, whence, using (27) and the definition of σ_a, for all $s \in [0, \tau_N)$ such that $\underline{R}_s^{I,b} > R_s^{I,a}$,

$$-\frac{\chi}{N}\mathbf{1}_{\{s<\sigma_a\}}\sum_{i\in I}\sum_{j\notin I}(X_s^{i,N} - \bar{X}_s^I)\cdot K(X_s^{i,N} - X_s^{j,N}) \le c\sqrt{R_s^I} = c\sqrt{R_s^{I,a}} < c\sqrt{\underline{R}_s^{I,b}}.$$

We conclude that a.s., for all $t \in [0, \tau_N)$,

$$\text{(29)}\qquad (\underline{R}_t^{I,b} - R_t^{I,a})^+ \le 2\int_0^t \mathbf{1}_{\{\underline{R}_s^{I,b} > R_s^{I,a}\}}\Big(\sqrt{\underline{R}_s^{I,b}} - \sqrt{R_s^{I,a}}\Big)d\beta_s^I.$$

We next introduce $M_t := \int_0^t \mathbf{1}_{\{s<\tau_N, \underline{R}_s^{I,b} > R_s^{I,a}\}}(\sqrt{\underline{R}_s^{I,b}} - \sqrt{R_s^{I,a}})d\beta_s^I$, which is a true martingale (because the integrand is clearly bounded by $\sqrt{b}$), which is a.s. nonnegative for all times by (29) and which starts from 0: we classically conclude that a.s., M_t vanishes for all $t \ge 0$. Coming back to (29), we deduce that $(\underline{R}_t^{I,b} - R_t^{I,a})^+ \le 2M_t = 0$ a.s. for all $t \in [0, \tau_N)$, which ends the step.

Step 4. We now conclude the induction. For any $a>0$ and $b=a/3$, using that $(R^I_t)_{t\in[0,\sigma_a)}=(R^{I,a}_t)_{t\in[0,\sigma_a)}$ and the definition of σ_a,

$$\begin{aligned}\mathbb{P}\Big(\tau_N<\infty,\inf_{t\in[0,\tau_N)}R^I_t=0\Big)&\le\mathbb{P}\Big(\tau_N<\infty,\sigma_a=\tau_N,\inf_{t\in[0,\tau_N)}R^I_t=0\Big)+\mathbb{P}\Big(\tau_N<\infty,\sigma_a<\tau_N\Big)\\&=\mathbb{P}\Big(\tau_N<\infty,\sigma_a=\tau_N,\inf_{t\in[0,\tau_N)}R^{I,a}_t=0\Big)+\mathbb{P}\Big(\tau_N<\infty,\min_{j\notin I}\inf_{t\in[0,\tau_N)}R^{I_j}_t\le a\Big)\\&\le\mathbb{P}\Big(\tau_N<\infty,\inf_{t\in[0,\tau_N)}\underline{R}^{I,b}_t=0\Big)+\mathbb{P}\Big(\tau_N<\infty,\min_{j\notin I}\inf_{t\in[0,\tau_N)}R^{I_j}_t\le a\Big).\end{aligned}$$

Since the continuous process $(\underline{R}^{I,b}_t)_{t\ge0}$ does not reach 0, the first term in the right-hand side is 0. We thus can let a tend to 0 to get

$$\mathbb{P}\Big(\tau_N<\infty,\inf_{t\in[0,\tau_N)}R^I_t=0\Big)\le\mathbb{P}\Big(\tau_N<\infty,\min_{j\notin I}\inf_{t\in[0,\tau_N)}R^{I_j}_t=0\Big).$$

This last quantity vanishes by our induction assumption.

Existence for (28). To conclude the proof, we still have to check the existence of a solution $(\underline{R}^{I,b}_t)_{t\ge0}$ such $\mathbb{P}(\forall t\ge0,\ \underline{R}^{I,b}_t\in(0,b])=1$ to (28). For $\ell\ge1/b$, according to Skorokhod [37], existence and trajectorial uniqueness hold for the reflected (at b) stochastic differential equation with Lipschitz drift and diffusion coefficients

$$\begin{cases}\underline{R}^{I,b,\ell}_t=R^I_0\wedge b+2\int_0^t\sqrt{\ell^{-1}\vee\underline{R}^{I,b,\ell}_s}d\beta^I_s+(|I|-1)\Big(2-\frac{\chi|I|}{4\pi N}\Big)t-c\int_0^t\sqrt{\ell^{-1}\vee\underline{R}^{I,b,\ell}_s}ds-L^\ell_t\\\forall t\ge0,\ \underline{R}^{I,b,\ell}_t\le b\\(L^\ell_s)_{s\ge0}\text{ is an adapted increasing process such that }L^\ell_0=0\text{ and }\int_0^t(b-\underline{R}^{I,b,\ell}_s)dL^\ell_s=0.\end{cases}.$$

Denoting by $\nu_\ell=\inf\{t\ge0:\underline{R}^{I,b,\ell}_t\le1/\ell\}$, we deduce from pathwise uniqueness that for $\ell'\ge\ell$, $(\underline{R}^{I,b,\ell'}_t,L^{\ell'}_t)_{t\in[0,\nu_\ell]}$ and $(\underline{R}^{I,b,\ell}_t,L^\ell_t)_{t\in[0,\nu_\ell]}$ coincide and thus that $\ell\mapsto\nu_\ell$ is a.s. increasing. Setting $\nu_\infty=\sup_{\ell\to\infty}\nu_\ell$, we easily deduce the existence of a solution $(\underline{R}^{I,b}_t,L_t)_{t\in[0,\nu_\infty)}$ to (28) satisfying $\sup_{t\in[0,\nu_\infty)}\underline{R}^{I,b}_t\le b$ and $\underline{R}^{I,b}_t>0$ for all $t\in[0,\nu_\infty)$. More precisely, $\underline{R}^{I,b}_t=\underline{R}^{I,b,\ell}_t\ge1/\ell$ for all ℓ and all $t\in[0,\nu_\ell)$. It thus only remains to prove that $\nu_\infty=\infty$ a.s.

By the Girsanov theorem, under the probability measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\sigma(R^I_0,(\beta^I_s)_{s\in[0,t]})}=\exp(c\beta^I_t/2-c^2t/8)$ (which is of course a true martingale), the process $W_t=\beta^I_t-ct/2$ is a one-dimensional Brownian motion. We introduce the equation, satisfied by $(\underline{R}^{I,b}_t,L^l_t)$ on the time-interval $[0,\nu_\infty)$, for a squared Bessel process $(\rho_t,\Lambda_t)_{t\ge0}$ of dimension $(|I|-1)(2-\chi|I|/(4\pi N))$ driven by W and reflected at the level b,

$$\begin{cases}\rho_t=R^I_0\wedge b+2\int_0^t\sqrt{\rho_s}dW_s+(|I|-1)\Big(2-\frac{\chi|I|}{4\pi N}\Big)t-\Lambda_t\\\forall t\ge0,\ \rho_t\le b\\(\Lambda_s)_{s\ge0}\text{ is an adapted increasing process s.t. }\Lambda_0=0\text{ and }\int_0^t(b-\rho_s)d\Lambda_s=0.\end{cases}.\tag{30}$$

To check global existence for this equation, we set $\underline{\eta}_0=0$ and define, inductively on $k\ge0$, ρ_t to be equal to

- the squared Bessel process

$$R_t=\mathbf{1}_{\{k=0\}}R^I_0\wedge b+\mathbf{1}_{\{k\ge1\}}\frac{b}{3}+2\int_{\underline{\eta}_k}^t\sqrt{R_s}dW_s+(|I|-1)\Big(2-\frac{\chi|I|}{4\pi N}\Big)(t-\underline{\eta}_k)$$

on the time interval $[\underline{\eta}_k, \bar{\eta}_{k+1}]$ where $\bar{\eta}_{k+1} = \inf\{t \geq \underline{\eta}_k : R_t \geq 2b/3\}$,

- the solution to the stochastic differential equation with Lipschitz coefficients

$$R^b_t = \frac{2b}{3} + 2\int_{\bar{\eta}_{k+1}}^t \sqrt{\frac{b}{3} \vee R^b_s}\, dW_s + (|I|-1)\Big(2 - \frac{\chi|I|}{4\pi N}\Big)(t - \bar{\eta}_{k+1}) - \Lambda^b_t,$$

  reflected at b on the time interval $[\bar{\eta}_{k+1}, \underline{\eta}_{k+1}]$ where $\underline{\eta}_{k+1} = \inf\{t \geq \bar{\eta}_{k+1} : R^b_t \leq b/3\}$.

Since, under $\mathbb{Q}$, the delays $(\bar{\eta}_{k+1} - \bar{\eta}_k)_{k\geq 1}$ are i.i.d. and positive, $\mathbb{Q}$-a.s., $\bar{\eta}_k$ goes to ∞ with k by the law of large numbers and ρ_t is defined for $t \in [0, +\infty)$. It is easily checked that the process $(\Lambda)_{t\geq 0}$ defined by the first equality in (30) also satisfies the last one.

Reasoning like in the comparison between $R^{I,a}$ and $\underline{R}^{I,b}$ performed in Step 3.3, we check that the first component of any of two solutions to (30) is above the other one so that the first components coincide.

We deduce that $\underline{R}^{I,b}_t$ and ρ_t coincide for $t \in [0, \nu_\infty)$. With the definition of ν_∞ and the continuity of ρ, this implies that $\{\nu_\infty \leq t\} \subset \{\exists s \in [0,t] : \rho_s = 0\}$. Since $(\rho_t)_{t\geq 0}$ always evolves as a squared Bessel process of dimension $(|I|-1)(2 - \chi|I|/(4\pi N)) \geq 2$ under the level $b/3$, by [36, p 442], $\mathbb{Q}(\exists s \in [0, +\infty) : \rho_s = 0)$. For each $t \in [0, \infty)$, we deduce that $0 = \mathbb{P}(\exists s \in [0,t] : \rho_s = 0) \geq \mathbb{P}(\nu_\infty \leq t)$ by equivalence of $\mathbb{P}$ and $\mathbb{Q}$ on $\sigma(R^I_0, (\beta^I_s)_{s\in[0,t]})$. Letting $t \to \infty$, we conclude that $\mathbb{P}(\nu_\infty < \infty) = 0$. □

5. Positive probability of collisions

The goal of this section is to establish that in the N-particle system (4), pairs of particles do collide during $[0, t_0]$ with positive probability, for any $t_0 > 0$. Assume by contradiction that particles do a.s. not collide before t_0. Then, with e.g. $I = \{1, 2\}$, the third term in the right-hand side of (23) cannot be very large (since it explodes only when there is a collision). Consequently, R^I behaves as a squared Bessel process with dimension $(2 - \chi/(2\pi N)) < 2$. But such a process reaches 0 before t_0 with positive probability, whence a contradiction.

This simple strategy is not so easy to write down. We first reduce to the case where the initial condition is deterministic and equal to $(x^1, \dots, x^N) \in (\mathbb{R}^2)^N$ such that $|x^1 - x^2| = \min_{1\leq i<j\leq N} |x^i - x^j| > 0$.

In Step 1, we introduce a stopping time τ before which particles $3, \dots, N$ are not too close to particles $1, 2$ and particles 1 and 2 do not move too much.

We use Girsanov's theorem in Step 2 to introduce a new probability $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (thanks to the use of τ).

In Step 3, we show that under $\tilde{\mathbb{P}}$, $Y = |X^{1,N} - X^{2,N}|^2/4$ is a Bessel process, $\gamma = (X^{1,N} + X^{2,N} - X^{1,N}_0 - X^{2,N}_0)/2$ is a Brownian motion, and the triple $Y, X^{1,N} + X^{2,N}, (X^{i,N})_{i=3,\dots,N}$ is independent. But all this holds only on $[0, \tau)$, so we extend these processes on $[0, t_0]$ (if $\tau < t_0$). More precisely, we introduce $\tilde{Y}$, γ and $(\tilde{X}^{i,N})_{i=3,\dots,N}$, satisfying the above properties on $[0, t_0]$ and equal to Y, γ and $(X^{i,N})_{i=3,\dots,N}$ on $[0, \tau \wedge t_0)$.

In Step 4, we fix $s_0 \in (0, t_0)$ and we introduce an event Ω_0, involving $\tilde{Y}$, γ and $(\tilde{X}^{i,N})_{i=3,\dots,N}$, on which we have $\tau > s_0$ and $\min_{[0,s_0]} \tilde{Y}_s = 0$, all this implying that $\min_{[0,s_0]} Y_s = 0$.

It only remains to prove that $\tilde{\mathbb{P}}(\Omega_0) > 0$ (which implies that $\mathbb{P}(\Omega_0) > 0$), which we do in Step 5.

Proof of Proposition 4. We thus consider any fixed $N \geq 2$, $\chi > 0$ and any solution (if it exists) $(X_t^{i,N})_{i=1,\dots,N,t\in[0,t_0]}$ to (4) with $(X_0^{i,N})_{i=1,\dots,N}$ exchangeable (with law F_0^N). We work by contradiction and assume that a.s., $X_s^{i,N} \neq X_s^{j,N}$ for all $s \in [0,t_0]$ and all $i \neq j$. Then the singularity of K is a.s. not visited and the particle system (4) is classically strongly well-posed on $[0,t_0]$. Thus for F_0^N-a.e. $(x^1,\dots,x^N) \in (\mathbb{R}^2)^N$, there is a unique strong solution $(X_t^{i,N})_{i=1,\dots,N,t\in[0,t_0]}$ to (4) such that a.s., $X_0^{i,N} = x^i$ for all i and $X_s^{i,N} \neq X_s^{j,N}$ for all $s \in [0,t_0]$ and all $i \neq j$. We fix for the rest of the proof an initial condition $(x^1,\dots,x^N) \in (\mathbb{R}^2)^N$ enjoying these properties. All the processes below are defined on the finite time interval $[0,t_0]$.

Step 1. By construction, $d = \min_{i\neq j} |x^i - x^j| > 0$ and we may of course assume that $d = |x^1 - x^2|$. We introduce $\bar{x} := (x^1 + x^2)/2$ and note that $\min_{3\leq j\leq N} |x^j - \bar{x}| \geq \sqrt{3}d/2$. Fix $1/2 < a < b < \sqrt{3}/2$ and consider the stopping time $\tau = \min\{\tau_1,\tau_2,\tau_3\}$, where

$$\tau_1 = \inf\Big\{t \in [0,t_0] \;:\; |X_t^{1,N} - X_t^{2,N}| \geq \frac{2a+1}{2}d\Big\},$$
$$\tau_2 = \inf\Big\{t \in [0,t_0] :\; |X_t^{1,N} + X_t^{2,N} - 2\bar{x}| \geq \frac{2a-1}{2}d\Big\},$$
$$\tau_3 = \inf\Big\{t \in [0,t_0] :\; \min_{j=3,\dots,N} |X_t^{j,N} - \bar{x}| \leq bd\Big\},$$

with the convention that $\inf\, \emptyset = t_0$. We will use that a.s., for all $t \in [0,\tau]$,

$$\min_{i=1,2,\; j=3,\dots,N} |X_t^{i,N} - X_t^{j,N}| \geq (b-a)d.$$

Indeed, consider e.g. the case $i=1$ and $j=3$, write $|X_t^{1,N} - X_t^{3,N}| \geq |X_t^{3,N} - \bar{x}| - |X_t^{1,N} - \bar{x}|$ and use that $|X_t^{3,N} - \bar{x}| \geq bd$ and that $|X_t^{1,N} - \bar{x}| \leq |X_t^{1,N} - X_t^{2,N}|/2 + |X_t^{1,N} + X_t^{2,N} - 2\bar{x}|/2 \leq (2a+1)d/4 + (2a-1)d/4 = ad$.

Step 2. Consider the exponential martingale defined on $[0,t_0]$ by

$$M_t = \exp\Big[\frac{\chi}{\sqrt{2}N}\sum_{i=1}^N \int_0^{t\wedge\tau} \Big(\mathbf{1}_{\{i\leq 2\}}\sum_{j=3}^N K(X_s^{j,N} - X_s^{i,N}) + \mathbf{1}_{\{i\geq 3\}}\sum_{j=1}^2 K(X_s^{j,N} - X_s^{i,N})\Big)\cdot dB_s^i$$
$$- \frac{\chi^2}{4N^2}\sum_{i=1}^N \int_0^{t\wedge\tau} \Big|\mathbf{1}_{\{i\leq 2\}}\sum_{j=3}^N K(X_s^{j,N} - X_s^{i,N}) + \mathbf{1}_{\{i\geq 3\}}\sum_{j=1}^2 K(X_s^{j,N} - X_s^{i,N})\Big|^2 ds\Big].$$

This is indeed a true martingale, because $K(X_s^{j,N} - X_s^{i,N})$ is bounded by $(2\pi(b-a)d)^{-1}$ on $[0,\tau]$ for each $i=1,2$ and $j=3,\dots,N$, see Step 1. Hence $\tilde{\mathbb{P}} := M_{t_0}\cdot\mathbb{P}$ is a probability measure equivalent to $\mathbb{P}$. In particular, it also holds that $\tilde{\mathbb{P}}$-a.s., $X_s^{i,N} \neq X_s^{j,N}$ for all $s \in [0,t_0]$ and all $i \neq j$. The Girsanov theorem tells us that, under $\tilde{\mathbb{P}}$, the processes

$$W_t^i := B_t^i + \frac{\chi}{\sqrt{2}N}\int_0^{t\wedge\tau} \Big(\mathbf{1}_{\{i\leq 2\}}\sum_{j=3}^N K(X_s^{i,N} - X_s^{j,N}) + \mathbf{1}_{\{i\geq 3\}}\sum_{j=1}^2 K(X_s^{i,N} - X_s^{j,N})\Big)ds$$

are independent two-dimensional Brownian motions on $[0,t_0]$. We next introduce

$$\beta_t = \int_0^t \frac{(X_s^{1,N} - X_s^{2,N})}{|X_s^{1,N} - X_s^{2,N}|}\cdot d\Big(\frac{W_s^1 - W_s^2}{\sqrt{2}}\Big) \quad \text{and} \quad \gamma_t = \frac{W_t^1 + W_t^2}{\sqrt{2}}.$$

It is easily seen, computing brackets and using Karatzas and Shreve [23, Theorem 4.13 p 179], that still under $\tilde{\mathbb{P}}$, β is a one-dimensional Brownian motion on $[0,t_0]$, $\gamma, W^3,\dots,W^N$ are two-dimensional Brownian motions on $[0,t_0]$, and all these processes are independent.

Step 3. We have

$$\begin{aligned}X_t^{1,N}-X_t^{2,N}=&x^1-x^2+\sqrt{2}(B_t^1-B_t^2)+\frac{2\chi}{N}\int_0^t K(X_s^{1,N}-X_s^{2,N})ds\\&+\frac{\chi}{N}\sum_{j=3}^N\int_0^t\Big(K(X_s^{1,N}-X_s^{j,N})-K(X_s^{2,N}-X_s^{j,N})\Big)ds\\=&x^1-x^2+\sqrt{2}(W_t^1-W_t^2)+\frac{2\chi}{N}\int_0^t K(X_s^{1,N}-X_s^{2,N})ds\end{aligned}$$

for all $t\in[0,\tau]$. By the Itô formula, $Y_t=|X_t^{1,N}-X_t^{2,N}|^2/4$ thus solves, still for $t\in[0,\tau]$,

$$Y_t:=\frac{d^2}{4}+2\int_0^t\sqrt{Y_s}d\beta_s+\Big(2-\frac{\chi}{2\pi N}\Big)t.$$

We also have, for all $t\in[0,\tau]$

$$\begin{aligned}X_t^{1,N}+X_t^{2,N}=&2\bar{x}+\sqrt{2}(B_t^1+B_t^2)+\frac{\chi}{N}\sum_{j=3}^N\int_0^t\Big(K(X_s^{1,N}-X_s^{j,N})+K(X_s^{2,N}-X_s^{j,N})\Big)ds\\=&2\bar{x}+\sqrt{2}(W_t^1+W_t^2)\\=&2\bar{x}+2\gamma_t,\end{aligned}$$

and, for all $t\in[0,\tau]$ and all $i=3,\dots,N$ (recall that $K(0)=0$),

$$\begin{aligned}X_t^{i,N}=&x^i+\sqrt{2}B_t^i+\frac{\chi}{N}\int_0^t\sum_{j=1}^N K(X_s^{i,N}-X_s^{j,N})ds\\=&x^i+\sqrt{2}W_t^i+\frac{\chi}{N}\int_0^t\sum_{j=3}^N K(X_s^{i,N}-X_s^{j,N})ds.\end{aligned}$$

We introduce $(\tilde{Y}_t)_{t\in[0,t_0]}$ the unique strong solution, see [36, Theorem 3.5 p 390], to

$$\tilde{Y}_t:=\frac{d^2}{4}+2\int_0^t\sqrt{|\tilde{Y}_s|}d\beta_s+\Big(2-\frac{\chi}{2\pi N}\Big)t.$$

We clearly have $(Y_t)_{t\in[0,\tau]}=(\tilde{Y}_t)_{t\in[0,\tau]}$. We next consider the system

$$\tilde{X}_t^{i,N}=x^i+\sqrt{2}W_t^i+\frac{\chi}{N}\int_0^t\sum_{j=3}^N K(\tilde{X}_s^{i,N}-\tilde{X}_s^{j,N})ds,\quad i=3,\dots,N,$$

which classically has a unique strong solution $(\tilde{X}_t^{i,N})_{i=3,\dots,N,t\in[0,\sigma)}$ up to $\sigma=\lim_{\ell\to\infty}\inf\{t\in[0,t_0]\ :\ \min_{3\le i<j\le N}|\tilde{X}_t^{i,N}-\tilde{X}_t^{j,N}|\le 1/\ell\}$ (convention : $\inf\emptyset=t_0$), which is a.s. positive because the initial conditions $x^3,\dots,x^N$ are pairwise different. Clearly, $(X_t^{i,N})_{i=3,\dots,N,t\in[0,\tau\wedge\sigma)}=(\tilde{X}_t^{i,N})_{i=3,\dots,N,t\in[0,\tau\wedge\sigma)}$. We conclude this step mentioning that the processes $(\tilde{Y}_t)_{t\in[0,t_0]}$, $(\gamma_t)_{t\in[0,t_0]}$ and $(\tilde{X}_t^{i,N})_{i=3,\dots,N,t\in[0,\sigma)}$ are independent under $\tilde{\mathbb{P}}$.

Step 4. For any $s_0\in(0,t_0)$, we claim that

$$\Omega_1\cap\Omega_2\cap\Omega_3\subset\Big\{\min_{[0,s_0]}|X_s^{1,N}-X_s^{2,N}|=0\Big\},$$

where

$$\Omega_1 = \Big\{ \min_{[0,s_0]} \tilde{Y}_s = 0,\ \max_{[0,s_0]} \tilde{Y}_s < \frac{(2a+1)^2 d^2}{16} \Big\}, \quad \Omega_2 = \Big\{ \max_{[0,s_0]} |\gamma_s| < \frac{(2a-1)d}{4} \Big\},$$
$$\Omega_3 = \Big\{ \sigma > s_0,\ \min_{s\in[0,s_0], j\geq 3} |\tilde{X}^{j,N}_s - \bar{x}| > bd \Big\}.$$

Indeed, on Ω_1, we have $\max_{[0,s_0]} \tilde{Y}_s < (2a+1)^2 d^2/16$, whence, since $|X^{1,N}_t - X^{2,N}_t|^2 = 4\tilde{Y}_t$ on $[0,\tau]$, $\max_{[0,s_0\wedge\tau]} |X^{1,N}_s - X^{2,N}_s| < (2a+1)d/2$ and thus $\tau_1 > s_0\wedge\tau$. Since $X^{1,N}_t + X^{2,N}_t = 2\bar{x} + 2\gamma_t$ on $[0,\tau]$, we deduce that on Ω_2, $\max_{[0,s_0\wedge\tau]} |X^{1,N}_s + X^{2,N}_s - 2\bar{x}| \leq \sup_{[0,s_0\wedge\tau]} 2|\gamma_s| < (2a-1)d/2$, whence $\tau_2 > s_0 \wedge \tau$. On Ω_3, since $\sigma > s_0$, we have $(X^{i,N}_t)_{i=3,\dots,N,t\in[0,\tau\wedge s_0]} = (\tilde{X}^{i,N}_t)_{i=3,\dots,N,t\in[0,\tau\wedge s_0]}$, and thus $\min_{s\in[0,s_0\wedge\tau],j\geq 3} |X^{j,N}_s - \bar{x}| > bd$, so that $\tau_3 > s_0 \wedge \tau$. As a conclusion, $\tau > s_0 \wedge \tau$ and thus $\tau > s_0$ on $\Omega_1 \cap \Omega_2 \cap \Omega_3$. We deduce that $\Omega_1 \cap \Omega_2 \cap \Omega_3 \subset \Big\{ \tau > s_0,\ \min_{[0,s_0]} \tilde{Y}_s = 0 \Big\} \subset \Big\{ \min_{[0,s_0]} |X^{1,N}_s - X^{2,N}_s| = 0 \Big\}$, because $\tilde{Y}_t = |X^{1,N}_t - X^{2,N}_t|^2/4$ for all $t \in [0,\tau]$.

Step 5. Here we show that we can find $s_0 \in (0,t_0)$ such that $\tilde{\mathbb{P}}(\Omega_1 \cap \Omega_2 \cap \Omega_3) > 0$. As seen at the end of Step 3, the events Ω_1, Ω_2 and Ω_3 are independent (under $\tilde{\mathbb{P}}$). It obviously holds true that $\tilde{\mathbb{P}}(\Omega_2) > 0$ (for any $s_0 > 0$) and that $\tilde{\mathbb{P}}(\Omega_3) > 0$ if $s_0 > 0$ is small enough because $\sigma > 0$ a.s. and by continuity of the sample-paths (at time 0, we have $\min_{j\geq 3} |\tilde{X}^{j,N}_0 - \bar{x}| = \min_{j\geq 3} |x^j - \bar{x}| \geq \sqrt{3}d/2 > bd$). It thus only remains to verify that $\tilde{\mathbb{P}}(\Omega_1)$ for all $s_0 \in (0,t_0)$. Since, by the comparison principle stated in [36, Theorem 3.7 p 394], $\tilde{\mathbb{P}}(\Omega_1)$ is non-decreasing with χ, it is enough to check that $\tilde{\mathbb{P}}(\Omega_1) > 0$ for all $s_0 \in (0,t_0)$ when $\chi < 4\pi N$, which we now do.

It holds that $\tilde{Y}$ is a squared Bessel process of dimension $\delta := 2 - \chi/(2\pi N)$ started at $y = d^2/4$ and restricted to the time-interval $[0,t_0]$. We set $z = (2a+1)^2 d^2/16$ and observe that $z > y$. For $x \geq 0$, we also introduce $\tau_x = \inf\{t \in [0,t_0] \ :\ \tilde{Y}_t = x\}$. Then $\Omega_1 = \{\tau_0 < s_0 \wedge \tau_z\}$.

For $x \geq 0$, we denote by Q_x the law of the squared Bessel process of dimension δ starting from x (on the whole time interval $[0,\infty)$), and by $q_s(x,u)$ the density of its marginal at time $s > 0$, which is a positive function of u on $(0,+\infty)$ according to [36, Corollary 4.1 p441]. For all $u \neq v$, we define τ_u as the first passage time at u and τ_{uv} as the first passage time at v after τ_u. It holds that $\tilde{\mathbb{P}}(\Omega_1) = Q_y(\tau_0 < s_0 \wedge \tau_z)$ and what we have to check is that $Q_y(\tau_0 < s_0 \wedge \tau_z) > 0$ for all $s_0 \in (0,t_0)$.

We first show that $Q_x(\tau_0 < t) > 0$ for all $t > 0$ and all $x > 0$. Since $\delta < 2$, we know from [36, page 442] that $Q_x(\tau_0 < \infty) = 1$ for all $x > 0$. With the Markov property, we deduce that

$$1 = \sum_{n\geq 0} Q_x(\tau_0 \in (nt/2,(n+1)t/2]) \leq Q_x(\tau_0 \leq t/2) + \int_0^{+\infty} Q_u(\tau_0 \leq t/2)\Big(\sum_{n\geq 1} q_{nt/2}(x,u)\Big)du.$$

Since, $u \mapsto q_{t/2}(x,u)$ is positive on $(0,+\infty)$, this ensures the positivity of

$$Q_x(\tau_0 \leq t/2) + 1_{\{Q_x(\tau_0\leq t/2)=0\}} \int_0^{+\infty} Q_u(\tau_0 \leq t/2) q_{t/2}(x,u)du \leq Q_x(\tau_0 \leq t).$$

Using the strong Markov property, that $0 < y < z$ and the monotonicity of $t \mapsto Q_y(\tau_0 \leq t)$,

$$Q_y(\tau_z < \tau_0 \leq t) = Q_y(\tau_{zy} < \tau_0 \leq t) = \int \mathbf{1}_{\{\tau_{zy}<t\}} Q_y(\tau_0 \leq t-s)|_{s=\tau_{zy}} dQ_y \leq Q_y(\tau_{zy} < t) Q_y(\tau_0 \leq t).$$

By continuity of the sample-paths, $\lim_{s\to 0} Q_y(\tau_{zy} < s) = 0$ and we can find $s_1 \in (0, t_0)$ so that for all $s_0 \in (0, s_1]$, $Q_y(\tau_{zy} < s_0) < 1$. We conclude that for all $s_0 \in (0, s_1]$,

$$Q_y(\tau_0 \le s_0 \wedge \tau_z) = Q_y(\tau_0 \le s_0) - Q_y(\tau_z < \tau_0 \le s_0) \ge (1 - Q_y(\tau_{zy} < s_0))Q_y(\tau_0 \le s_0) > 0.$$

If now $s_0 \in [s_1, t_0]$, we obviously have $Q_y(\tau_0 \le s_0 \wedge \tau_z) \ge Q_y(\tau_0 \le s_1 \wedge \tau_z) > 0$. This ends the step.

Step 6. We deduce from Steps 4 and 5 that $\tilde{\mathbb{P}}(\min_{[0,t_0]} |X_s^{1,N} - X_s^{2,N}| = 0) > 0$. But $\mathbb{P}$ and $\tilde{\mathbb{P}}$ being equivalent, this implies that $\mathbb{P}(\min_{[0,t_0]} |X_s^{1,N} - X_s^{2,N}| = 0) > 0$, whence a contradiction. □

6. Two particles system

In this section we consider the particle system (4) with $N = 2$. Assuming that $(X_t^1, X_t^2)_{t\ge 0}$ solves (4) with $N = 2$, we easily find that $S_t = X_t^1 + X_t^2$ and $D_t = X_t^1 - X_t^2$ solve two autonomous equations, namely $S_t = S_0 + 2B_t$ and

$$D_t = D_0 + 2W_t + \chi \int_0^t K(D_s)ds, \tag{31}$$

with the two independent 2-dimensional Brownian motions $B_t = (B_t^1 + B_t^2)/\sqrt{2}$ and $W_t = (B_t^1 - B_t^2)/\sqrt{2}$. The equation satisfied by $(S_t)_{t\ge 0}$ being trivial, only the study of (31) is interesting. This equation can be seen as a natural two-dimensional generalization of a Bessel process of dimension $(2 - \chi/(4\pi))$. Indeed, $(|D_t|/2)_{t\ge 0}$ is a Bessel process of dimension $(2 - \chi/(4\pi))$ and the dynamics of $(D_t)_{t\ge 0}$ is radially symmetric.

During the whole section, the initial condition D_0 is only assumed to be a $\mathbb{R}^2$-random variable independent of $(W_t)_{t\ge 0}$.

Remark 18. *Theorem 5 ensures us existence for* (31) *when* $\chi < 4\pi$ *and* D_0 *is the difference of two i.i.d. integrable random vectors. When* $\chi \ge 4\pi$*, the equation* (31) *has no global (in time) solution in the usual sense. More precisely, assume that it has a global solution* $(D_t)_{t\ge 0}$*. Then* $\tau = \inf\{t \ge 0 \,:\, D_t = 0\}$ *is a.s. finite and a.s.,* $\int_\tau^{\tau+h} |K(D_s)|ds = \infty$ *for all* $h > 0$*.*

Proof. Let thus $\chi \ge 4\pi$ and assume that there is a global solution $(D_t)_{t\ge 0}$ to (31). By a direct application of the Itô formula, this implies that $R_t = |D_t|^2/4$ solves $R_t = R_0 + 2\int_0^t \sqrt{|R_s|}d\beta_s + (2 - \chi/(4\pi))t$, where $\beta_t = \int_0^t \mathbf{1}_{\{D_s \ne 0\}} |D_s|^{-1} D_s \cdot dW_s + \int_0^t \mathbf{1}_{\{D_s = 0\}} d\tilde{\beta}_s$ is a 1-dimensional Brownian motion (here $\tilde{\beta}$ is any one-dimensional Brownian motion independent of (D_0, W)). According to [36, p 442] combined, when $\chi > 8\pi$, with the comparison theorem [36, Theorem 3.7 p 394], $\tau = \inf\{t \ge 0 \,:\, R_t = 0\}$ is a.s. finite. By the strong Markov property (for the process R), the comparison theorem [36, Theorem 3.7 p 394] and since $\chi \ge 4\pi$, $(R_{\tau+t})_{t\ge 0}$ can be bounded from above by a squared 1-dimensional Bessel process starting from 0, process with the same law as $(|\beta_t|^2)_{t\ge 0}$. For $h > 0$, by the occupation times formula [36, Corollary 1.6 p 224], $\int_0^h |\beta_s|^{-1}ds = \int_{\mathbb{R}} |a|^{-1} L_h^a da$. But $L_h^0 > 0$ as soon as $h > 0$ and we know from [36, Corollary 1.8 p 226] that $a \mapsto L_h^a$ is a.s. continuous, so that $\int_0^h |\beta_s|^{-1}ds = \infty$ for all $h > 0$ a.s. Thus $4\pi \int_\tau^{\tau+h} |K(D_s)|ds = \int_\tau^{\tau+h} R_s^{-1/2}ds = \infty$ for all $h > 0$ a.s. □

Hence (31) has no global solution for $\chi \ge 4\pi$, while we expect that in some sense, the dynamics it represents is meaningful at least for all $\chi \in (0, 8\pi)$. We thus would like to refomulate it, in such a way that it is possible to build global solutions. More precisely, we would like to identify, for any value of $\chi > 0$, the limit, as $\varepsilon > 0$, of the smoothed equation

$$D_t^\varepsilon = D_0 + 2W_t + \chi \int_0^t K_\varepsilon(D_s^\varepsilon)ds, \tag{32}$$

where K_ε was defined in (10). The regularized drift coefficient K_ε being Lipschitz, existence and trajectorial uniqueness hold for this SDE. We introduce the equation formally satisfied by $Z_t = |D_t|^2 D_t$ for $(D_t)_{t\geq 0}$ solution to (31):

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)dW_s + \int_0^t b(Z_s)ds, \tag{33}$$

where $\sigma(z) = 2|z|^{-4/3}(|z|^2 I_2 + 2zz^*)$ and $b(z) = (16 - 3\chi/(2\pi))|z|^{-2/3}z$. Here and below, I_2 is the identity matrix and z^* is the transpose of z.

It might seem more natural to rather consider $|D_t|D_t$ (since this resembles more a squared Bessel process), but this unfortunately leads to discontinuous (although bounded) diffusion and drift coefficients. Here is the main result of this section.

Theorem 19. *Set $Z_0 = |D_0|^2 D_0$.*

(i) If $\chi \in (0, 8\pi)$, (33) *has a unique (in law) solution $(Z_t)_{t\geq 0}$ such a.s., $\int_0^\infty \mathbf{1}_{\{Z_t=0\}}dt = 0$. Moreover, if $\chi \in (0, 4\pi)$,* (31) *has a unique (in law) solution.*

(ii) If $\chi \geq 8\pi$, (33) *has a pathwise unique solution frozen when it reaches* 0 *(and it a.s. reaches* 0*).*

(iii) In any case, the solution $(D_t^\varepsilon)_{t\geq 0}$ to (32) *goes in law, as $\varepsilon \to 0$, to $(D_t)_{t\geq 0}$ defined by $D_t = |Z_t|^{-2/3} Z_t \mathbf{1}_{\{Z_t\neq 0\}}$ and, when $\chi \in (0, 4\pi)$, this process $(D_t)_{t\geq 0}$ solves* (31) .

In point (i), uniqueness in law cannot hold true without restriction for (33): the time passed at 0 by the solution that we consider is Lebesgue-nul, while it is easy to build a solution by freezing the process when it reaches 0.

The rest of the section is devoted to the proof of this theorem. The following lemma is more or less standard.

Lemma 20. *Let $\chi > 0$ be fixed. For each $\varepsilon \in (0,1)$, we consider the unique solution $(D_t^\varepsilon)_{t\geq 0}$ to* (32) *and we put $Z_t^\varepsilon = |D_t^\varepsilon|^2 D_t^\varepsilon$.*

(i) The family $\{(Z_t^\varepsilon)_{t\geq 0}, \varepsilon \in (0,1)\}$ is tight in $C([0,\infty), \mathbb{R}^2)$.

(ii) Any limit point $(Z_t)_{t\geq 0}$ is a weak solution to (33) *and, setting $R_t = |Z_t|^{2/3}/4$, it holds that*
(a) if $\chi \in (0, 8\pi)$, then $(R_t)_{t\geq 0}$ is a $(2 - \chi/(4\pi))$-dimensional squared Bessel process;
(b) if $\chi \geq 8\pi$, then $(R_t)_{t\geq 0}$ is a $(2 - \chi/(4\pi))$-dimensional squared Bessel process frozen when it reaches 0.

Proof. We divide the proof in several steps.

Step 1. Direct applications of the Itô formula show that

$$Z_t^\varepsilon = Z_0 + \int_0^t \sigma(Z_s^\varepsilon)dW_s + \int_0^t b_\varepsilon(Z_s^\varepsilon)ds,$$

where $b_\varepsilon(z) = 16|z|^{-2/3}z - (3\chi/(2\pi))(|z|^{2/3} + \varepsilon^2)^{-1}z$ and that $R_t^\varepsilon := |D_t^\varepsilon|^2/4$ solves

$$R_t^\varepsilon = R_0 + 2\int_0^t \sqrt{R_s^\varepsilon}d\beta_s^\varepsilon + \int_0^t \Big(2 - \frac{\chi R_s^\varepsilon}{\pi(\varepsilon^2 + 4R_s^\varepsilon)}\Big)ds,$$

where $\beta_t^\varepsilon = \int_0^t \mathbf{1}_{\{D_s^\varepsilon\neq 0\}}|D_s^\varepsilon|^{-1}D_s^\varepsilon \cdot dW_s$. Since $\sup_{r\geq 0}(\chi r)/[2\pi\sqrt{r}(\varepsilon^2 + 4r)] = \chi/(8\pi\varepsilon)$, the Girsanov theorem ensures us that for all $T \in (0, +\infty)$, the law of $(R_t^\varepsilon)_{t\in[0,T]}$ is equivalent to the law of the restriction to the time interval $[0,T]$ of a 2-dimensional squared Bessel process starting from R_0. By

[36, p 442], we deduce that a.s., for all $t>0$, $R_t^\varepsilon>0$. As a consequence $(\beta_t^\varepsilon)_{t\geq 0}$ is a one-dimensional Brownian motion.

Step 2. By trajectorial uniqueness for (32), for $M>0$, on the event $\{|D_0|\leq M\}$, the solution starting from D_0 coincides with the one starting from $D_0\mathbf{1}_{\{|D_0|\leq M\}}$. Therefore, by both implications in the Prokhorov theorem, to check that the family $\{(Z_t^\varepsilon)_{t\geq 0},\varepsilon\in(0,1)\}$ is tight in $C([0,\infty),\mathbb{R}^2)$ it is enough to do so when D_0 is bounded. The tightness property then easily follows from the Kolmogorov criterion, using that $\sup_{\varepsilon\in(0,1)}|b_\varepsilon(z)|$ and $|\sigma(z)|$ both have at most affine growth: one classically verifies successively that for all $\rho\geq 2$ and all $T>0$ there is $C_{T,\rho}$ such that for all $\varepsilon\in(0,1)$, $\sup_{[0,T]}\mathbb{E}[|Z_t^\varepsilon|^\rho]\leq C_{T,\rho}$ and $\mathbb{E}[|Z_t^\varepsilon-Z_s^\varepsilon|^\rho]]\leq C_{T,\rho}|t-s|^{\rho/2}$ for all $0\leq s\leq t\leq T$.

Step 3. Using martingale problems, that b and σ are continuous and that b_ε converges (uniformly) to b, it is checked without difficulty that any limit point $(Z_t)_{t\geq 0}$ (as $\varepsilon\to 0$) of the family $\{(Z_t^\varepsilon)_{t\geq 0},\varepsilon>0\}$ is indead a (weak) solution to (33).

Step 4. Here we assume that $\chi\in(0,8\pi)$ and we prove that $(R_t^\varepsilon)_{t\geq 0}$ goes in law to the squared $(2-\chi/(4\pi))$-dimensional Bessel process. We consider the $(2-\chi/(4\pi))$-dimensional Bessel process $(R_t)_{t\geq 0}$ associated to $(\beta_t^\varepsilon)_{t\geq 0}$, that is $R_t=R_0+2\int_0^t\sqrt{R_s}d\beta_s^\varepsilon+(2-\chi/(4\pi))t$ (its law does of course not depend on ε) and we prove that $\lim_{\varepsilon\to 0}\mathbb{E}[\sup_{[0,T]}|R_t^\varepsilon-R_t|]=0$ for all $T>0$, which clearly suffices.

Since $\int_{\varepsilon^{3/2}}^\varepsilon x^{-1}dx=\log(1/\varepsilon)/2$, one may construct a family of C^2 nondecreasing convex functions $\varphi_\varepsilon:[0,\infty)\mapsto[0,\infty)$, indexed by $\varepsilon\in(0,1/2)$ such that $\varphi_\varepsilon(x)=0$ for $x\leq\varepsilon^{3/2}$, $\varphi'(x)=1$ for $x\geq\varepsilon$ and $\varphi_\varepsilon''(x)\leq C1_{\{\varepsilon^{3/2}\leq x\leq\varepsilon\}}/[x\log(1/\varepsilon)]$ for some constant $C\in(1,+\infty)$ not depending on ε. Such functions are called Yamada functions in the literature. We then observe that $R_t^\varepsilon\geq R_t$ for all $t\geq 0$ by the comparison theorem stated in [36, Theorem 3.7 p 394]. Computing $R_t^\varepsilon-R_t$ and applying the Itô formula, we obtain that

$$\begin{aligned}\varphi_\varepsilon(R_t^\varepsilon-R_t)=&2\int_0^t\varphi_\varepsilon'(R_s^\varepsilon-R_s)(\sqrt{R_s^\varepsilon}-\sqrt{R_s})d\beta_s+\frac{\chi}{4\pi}\int_0^t\varphi_\varepsilon'(R_s^\varepsilon-R_s)\frac{\varepsilon^2}{\varepsilon^2+4R_s^\varepsilon}ds\\&+2\int_0^t\varphi_\varepsilon''(R_s^\varepsilon-R_s)(\sqrt{R_s^\varepsilon}-\sqrt{R_s})^2ds.\end{aligned}$$

We next remark that

$$\varphi_\varepsilon'(R_s^\varepsilon-R_s)\frac{\varepsilon^2}{\varepsilon^2+4R_s^\varepsilon}\leq\varphi_\varepsilon'(R_s^\varepsilon-R_s)\frac{\varepsilon^2}{\varepsilon^2+4(R_s^\varepsilon-R_s)}\leq\mathbf{1}_{\{R_s^\varepsilon-R_s\geq\varepsilon^{3/2}\}}\frac{\varepsilon^2}{\varepsilon^2+4(R_s^\varepsilon-R_s)}\leq\frac{\sqrt{\varepsilon}}{4}$$

and that

$$\varphi_\varepsilon''(R_s^\varepsilon-R_s)(\sqrt{R_s^\varepsilon}-\sqrt{R_s})^2\leq\varphi_\varepsilon''(R_s^\varepsilon-R_s)(R_s^\varepsilon-R_s)\leq\frac{C}{\log(1/\varepsilon)},$$

whence (the constant C may now change from line to line)

$$\varphi_\varepsilon(R_t^\varepsilon-R_t)\leq 2\int_0^t\varphi_\varepsilon'(R_s^\varepsilon-R_s)(\sqrt{R_s^\varepsilon}-\sqrt{R_s})d\beta_s+\frac{\chi\sqrt{\varepsilon}}{16\pi}t+\frac{C}{\log(1/\varepsilon)}t. \tag{34}$$

Taking expectations, we conclude that $\mathbb{E}[\varphi_\varepsilon(R_t^\varepsilon-R_t)]\leq Ct/\log(1/\varepsilon)$. But since $\varphi_\varepsilon(x)\leq x\leq\varphi_\varepsilon(x)+\varepsilon$, we deduce that $\mathbb{E}[R_t^\varepsilon-R_t]\leq\varepsilon+Ct/\log(1/\varepsilon)$. Coming back to (34), using the Doob inequality and that $0\leq\varphi_\varepsilon'\leq 1$ and $(\sqrt{R_s^\varepsilon}-\sqrt{R_s})^2\leq R_s^\varepsilon-R_s$, we conclude that $\mathbb{E}[\sup_{[0,T]}\varphi_\varepsilon(R_t^\varepsilon-R_t)]\leq CT/\log(1/\varepsilon)+C(\varepsilon T+CT^2/\log(1/\varepsilon))^{1/2}$ and, finally, that $\mathbb{E}[\sup_{[0,T]}(R_t^\varepsilon-R_t)]\leq\varepsilon+CT/\log(1/\varepsilon)+C(\varepsilon T+CT^2/\log(1/\varepsilon))^{1/2}$, from which the conclusion follows.

Step 5. Finally, we assume that $\chi \geq 8\pi$ and we prove that $(R_t^\varepsilon)_{t\geq 0}$ goes in law to the $(2-\chi/(4\pi))$-dimensional squared Bessel process frozen when it reaches 0. We consider the frozen $(2-\chi/(4\pi))$-dimensional squared Bessel process associated to $(\beta_t^\varepsilon)_{t\geq 0}$, that is $R_t = R_0 + 2\int_0^t \sqrt{R_s} d\beta_s^\varepsilon + (2 - \chi/(4\pi))t$ for all $t \in [0, \tau]$, with $\tau = \inf\{t \geq 0 \,:\, R_t = 0\}$ and $R_t = 0$ for all $t \geq \tau$. We will check that for all $\alpha > 0$, all $T > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}(\sup_{[0,T]} |R_t^\varepsilon - R_t| > \alpha) = 0$ and this will complete the proof. We introduce $\tau_k = \inf\{t \geq 0 \,:\, R_t \leq 1/k\}$ and observe that $\tau = \sup_{k\geq 1} \tau_k$.

Step 5.1. For any $\alpha > 0$, $t \geq 0$ and $k \geq 1$, $\lim_{\varepsilon\to 0} \mathbb{P}(\sup_{[0,t\wedge\tau_k)} |R_s^\varepsilon - R_s| \geq \alpha) = 0$. Indeed, using that $R_t^\varepsilon \geq R_t$ for all $t \geq 0$ by the comparison theorem [36, Theorem 3.7 p 394], that $R_t^\varepsilon - R_t = 2\int_0^t (\sqrt{R_s^\varepsilon} - \sqrt{R_s}) d\beta_s^\varepsilon + (\chi/4\pi)\int_0^t [\varepsilon^2/(\varepsilon^2 + 4R_s^\varepsilon)] ds$ for all $t \in [0, \tau_k]$, and that $|\sqrt{x} - \sqrt{y}| \leq k^{1/2}|x-y|/2$ for all $x, y \geq 1/k$, it is easily checked, by the Doob inequality, that

$$\mathbb{E}\Big[\sup_{[0,t\wedge\tau_k)} (R_s^\varepsilon - R_s)^2\Big] \leq Ck\int_0^t \mathbb{E}\Big[\sup_{[0,s\wedge\tau_k)} (R_u^\varepsilon - R_u)^2\Big] ds + C\varepsilon^4 k^2 t^2,$$

whence $\mathbb{E}[\sup_{[0,t\wedge\tau_k)} (R_s^\varepsilon - R_s)^2] \leq C\varepsilon^4 k^2 t^2 \exp(Ckt)$ by the Gronwall lemma.

Step 5.2. We write, for $\alpha > 0$ and $k \geq 1$ fixed,

$$\begin{aligned}\mathbb{P}\Big(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha\Big) \leq& \mathbb{P}\Big(\sup_{[0,T\wedge\tau_k]} (R_t^\varepsilon - R_t) \geq \alpha\Big) + \mathbb{P}\Big(\tau_k < T, R_{\tau_k}^\varepsilon > 2/k\Big) \\ &+ \mathbb{P}\Big(\tau_k < T, R_{\tau_k}^\varepsilon \leq 2/k, \sup_{[\tau_k,T]} R_t^\varepsilon \geq \alpha\Big).\end{aligned}$$

For the last term, we used that $\sup_{[\tau_k,T]} (R_t^\varepsilon - R_t) \geq \alpha$ implies that $\sup_{[\tau_k,T]} R_t^\varepsilon \geq \alpha$ because $0 \leq R_t \leq R_t^\varepsilon$. By Step 5.1, the two first terms tend to 0 as $\varepsilon \to 0$ (recall that $R_{\tau_k} = 1/k$), whence

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha\Big) \leq \limsup_{\varepsilon\to 0} \mathbb{P}\Big(\tau_k < T, R_{\tau_k}^\varepsilon \leq 2/k, \sup_{[\tau_k,T]} R_t^\varepsilon \geq \alpha\Big).$$

Using the strong Markov property for the process R^ε as well as its monotony with respect to its initial condition (by the comparison theorem), we deduce that

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha\Big) \leq \limsup_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{[0,T]} R_t^{2/k,\varepsilon} \geq \alpha\Big),$$

where

$$R_t^{2/k,\varepsilon} = 2/k + 2\int_0^t \sqrt{R_s^{2/k,\varepsilon}} d\beta_s^\varepsilon + \int_0^t \Big(2 - \frac{\chi R_s^{2/k,\varepsilon}}{\pi(\varepsilon^2 + 4R_s^{2/k,\varepsilon})}\Big) ds.$$

We introduce, for $r \in (0,1)$ and $\varepsilon \in (0, 1/2)$, the solution $(S_t^{r,\varepsilon})_{t\geq 0}$ to $S_t^{r,\varepsilon} = r + 2\int_0^t \sqrt{|S_s^{r,\varepsilon}|} d\beta_s^\varepsilon + 2\varepsilon^2 \int_0^t (\varepsilon^2 + 4|S_s^{r,\varepsilon}|)^{-1} ds$. Such a solution is pathwise unique by [36, Theorem 3.5 p 390] and nonnegative by the comparison theorem [36, Theorem 3.7 p 394]. Again by the comparison theorem, and since $\chi \geq 8\pi$, we find that a.s., $R_t^{2/k,\varepsilon} \leq S_t^{2/k,\varepsilon}$ for all $t \geq 0$. Hence

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha\Big) \leq \limsup_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{[0,T]} S_t^{2/k,\varepsilon} \geq \alpha\Big) \leq \limsup_{\varepsilon\to 0} \frac{\mathbb{E}[\sup_{[0,T]} S_t^{2/k,\varepsilon}]}{\alpha}.$$

We will verify in the next step that (if $r \in (0,1]$)

$$\mathbb{E}\Big[\sup_{[0,T]} S_t^{r,\varepsilon}\Big] \leq C(1+T)(r + 1/\log(1/\varepsilon))^{1/2}, \tag{35}$$

so that $\limsup_{\varepsilon\to 0} \mathbb{P}(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha) \leq C(1+T)k^{-1/2}/\alpha$. Letting k tend to infinity, we conclude that, as desired, $\limsup_{\varepsilon\to 0} \mathbb{P}(\sup_{[0,T]} (R_t^\varepsilon - R_t) \geq \alpha) = 0$.

Step 5.3. To show (35), we consider the Yamada function φ_ε built in Step 4. By the Itô formula,

$$\varphi_\varepsilon(S_t^{r,\varepsilon}) = \varphi_\varepsilon(r) + 2\int_0^t \varphi_\varepsilon'(S_s^{r,\varepsilon})\sqrt{S_s^{r,\varepsilon}}d\beta_s^\varepsilon + \int_0^t \varphi_\varepsilon'(S_s^{r,\varepsilon})\frac{2\varepsilon^2}{\varepsilon^2+4S_s^{r,\varepsilon}}ds + 2\int_0^t \varphi_\varepsilon''(S_s^{r,\varepsilon})S_s^{r,\varepsilon}ds.$$

Proceeding as in Step 4, we find that

$$\begin{aligned}\varphi_\varepsilon(S_t^{r,\varepsilon}) \le & r + 2\int_0^t \varphi_\varepsilon'(S_s^{r,\varepsilon})\sqrt{S_s^{r,\varepsilon}}d\beta_s^\varepsilon + \frac{\sqrt{\varepsilon}}{2}t + \frac{C}{\log(1/\varepsilon)}t \\ \le & r + \frac{C}{\log(1/\varepsilon)}t + 2\int_0^t \varphi_\varepsilon'(S_s^{r,\varepsilon})\sqrt{S_s^{r,\varepsilon}}d\beta_s.\end{aligned} \tag{36}$$

Taking expectations, we deduce that $\mathbb{E}[\varphi_\varepsilon(S_t^{r,\varepsilon})] \le r + Ct/\log(1/\varepsilon)$, whence $\mathbb{E}[S_t^{r,\varepsilon}] \le r + \varepsilon + Ct/\log(1/\varepsilon)$. Coming back to (36) and using the Doob inequality and that $0 \le \varphi_\varepsilon' \le 1$, we conclude that

$$\mathbb{E}\Big[\sup_{[0,T]} \varphi_\varepsilon(S_t^{r,\varepsilon})\Big] \le r + \frac{CT}{\log(1/\varepsilon)} + C\Big(rT + \varepsilon T + \frac{T^2}{\log(1/\varepsilon)}\Big)^{1/2} \le C(1+T)\Big(r + \frac{1}{\log(1/\varepsilon)}\Big)^{1/2}$$

because $r \in (0,1]$. Then (35) follows from the fact that $x \le \varepsilon + \varphi_\varepsilon(x)$.

□

This allows us to conclude when $\chi \ge 8\pi$.

Proof of Theorem 19 when $\chi \ge 8\pi$. The existence of a (weak) solution to (33) follows from Lemma 20, and the solution built there is frozen when it reaches 0. The pathwise uniqueness of such a frozen solution follows from the Lipschitz continuity of coefficients σ, b on $\mathbb{R}^2 \setminus \{0\}$ and can easily be verified using the stopping times $\tau_\ell = \inf\{t \ge 0 \ : \ |Z_t| \le 1/\ell\}$ and that $\tau = \inf\{t \ge 0 \ : \ |Z_t| = 0\} = \sup_{\ell \ge 1} \tau_\ell$ (because $t \mapsto Z_t$ is a.s. continuous on $[0,\infty)$). Using Lemma 20, we easily conclude that $(Z_t^\varepsilon)_{t\ge 0}$ goes in law to this $(Z_t)_{t\ge0}$. Since $D_t^\varepsilon = |Z_t^\varepsilon|^{-2/3}Z_t^\varepsilon$ and since the map $z \mapsto |z|^{-2/3}z\mathbf{1}_{\{z\neq 0\}}$ is continuous, we conclude that $(D_t^\varepsilon)_{t\ge0}$ goes in law, as $\varepsilon \to 0$, to $(|Z_t|^{-2/3}Z_t\mathbf{1}_{\{Z_t\neq0\}})_{t\ge0}$. □

To conclude the proof when $\chi \in (0, 8\pi)$, the only issue is to check the the uniqueness in law of the solution. We define $h_{2\pi}(\theta) = \theta - 2\pi\lfloor\theta/(2\pi)\rfloor \in [0, 2\pi)$.

Lemma 21. *Consider $0 \le s_0 < t_0$, a continuous function $r : [0,\infty) \mapsto \mathbb{R}_+$ satisfying that $r_{s_0} = r_{t_0} = 0$ and $r_t > 0$ for all $t \in (s_0,t_0)$ and $\int_{s_0}^t (r_s)^{-1}ds = \infty$ for all $t \in (s_0,t_0)$. There is a law $\Gamma(s_0,t_0,(r_s)_{s\in[s_0,t_0]})$ on $C((s_0,t_0),[0,2\pi))$ (with the torus topology on $[0,2\pi)$) such that for any filtration $(\mathcal{H}_t)_{t\ge0}$ in which we have a 1-dimensional $(\mathcal{H}_t)_{t\ge0}$-Brownian motion $(\gamma_t)_{t\ge0}$ and a $(\mathcal{H}_t)_{t\ge0}$-adapted process $(T_t)_{t\in(s_0,t_0)}$ with $T_t = h_{2\pi}(T_u + \int_u^t (r_s)^{-1/2}d\gamma_s)$ for all $s_0 < u < t < t_0$, $(T_t)_{t\in(s_0,t_0)}$ is independent of $\mathcal{H}_{s_0}$ and is $\Gamma(s_0,t_0,(r_s)_{s\in[s_0,t_0]})$-distributed.*

Proof of Lemma 21. Existence. Let $u_0 \in (s_0,t_0)$ be chosen arbitrarily. We consider a Brownian motion $(\gamma_t)_{t\ge0}$, independent of a random variable Θ, uniformly distributed on $[0,2\pi)$. We put $T_t = h_{2\pi}(\Theta + \int_{u_0}^t (r_s)^{-1/2}d\gamma_s)$ for all $t \in (s_0,t_0)$ (with $\int_{u_0}^t (r_s)^{-1/2}d\gamma_s = -\int_t^{u_0}(r_s)^{-1/2}d\gamma_s$ when $t < u_0$). Then $(T_t)_{t\in(s_0,t_0)}$ is clearly continuous for the torus topology and it holds that $T_t = h_{2\pi}(T_u + \int_u^t (r_s)^{-1/2}d\gamma_s)$ for all $s_0 < u < t < t_0$. Furthermore, for each fixed $t \in (s_0,t_0)$, by independence between Θ and γ, the conditional law of T_t knowing $(\gamma_s)_{s\ge0}$ is the uniform distribution on $[0,2\pi)$, which implies that T_t is independent of $(\gamma_s)_{s\ge0}$. Finally, we have to verify that setting $\mathcal{H}_t = \sigma((T_s),(\gamma_s)_{s\in[0,t]})$, $(\gamma_s)_{s\ge0}$ is a $(\mathcal{H}_t)_{t\ge0}$-Brownian motion. Let thus $t \in (s_0,t_0)$ be fixed. We have to verify that $(\gamma_s - \gamma_t)_{s\ge t}$ is independent of $(T_s,\gamma_s)_{s\in(s_0,t]}$. Since $T_s = h_{2\pi}(T_t - \int_s^t (r_u)^{-1/2}d\gamma_u)$ for all $s \in (s_0, t]$, it holds that $\sigma((T_s, \gamma_s)_{s\in(s_0,t]}) = \sigma(T_t, (\gamma_s)_{s\in(s_0,t]})$ and the conclusion easily follows from the independence between T_t and $(\gamma_s)_{s\in[s_0,t_0]}$.

Uniqueness. We thus consider a filtration $(\mathcal{H}_t)_{t\geq 0}$ in which we have a Brownian motion $(\gamma_t)_{t\geq 0}$ and an adapted process $(T_t)_{t\in(s_0,t_0)}$ satisfying $T_t = h_{2\pi}(T_u + \int_u^t (r_s)^{-1/2} d\gamma_s)$ for all $s_0 < u < t < t_0$. We will show that for any fixed $u_0 \in (s_0, t_0)$, T_{u_0} is uniformly distributed on $[0, 2\pi)$ and independent of $\mathcal{H}_{s_0} \vee \sigma((\gamma_t)_{t\geq 0})$. Since $(T_t)_{t\in[s_0,t_0]}$ is $\sigma(T_{u_0}, (\gamma_t - \gamma_{s_0})_{t\in(s_0,t_0)})$-measurable and since $(\gamma_t)_{t\geq 0}$ is a $(\mathcal{H}_t)_{t\geq 0}$-Brownian motion, we conclude that $(T_t)_{t\in(s_0,t_0)}$ is independent of $\mathcal{H}_{s_0}$. Furthermore, the process $(T_t)_{t\in(s_0,t_0)}$ clearly has the same law as the one built above.

For $0 < \varepsilon < \eta < u_0 - s_0$, we have $T_{u_0} = h_{2\pi}(T_{s_0+\epsilon} + \int_{s_0+\epsilon}^{s_0+\eta} (r_s)^{-1/2} d\gamma_s + \int_{s_0+\eta}^{u_0} (r_s)^{-1/2} d\gamma_s)$. By assumption, the vector $(\int_{s_0+\epsilon}^{s_0+\eta} (r_s)^{-1/2} d\gamma_s, \int_{s_0+\eta}^{u_0} (r_s)^{-1/2} d\gamma_s)$ has independent components and is independent of $\mathcal{H}_{s_0} \vee \sigma(T_{s_0+\epsilon})$. Setting $\sigma_{\varepsilon,\eta} = \int_{s_0+\epsilon}^{s_0+\eta} (r_s)^{-1} ds$, we thus have, for any $\varphi : \mathbb{R} \mapsto [0, \infty)$ continuous and 2π-periodic,

$$
\begin{aligned}
(37) \qquad & \mathbb{E}[\varphi(T_{u_0}) \mid \mathcal{H}_{s_0} \vee \sigma(T_{s_0+\varepsilon}, (\gamma_s - \gamma_{s_0+\eta})_{s\geq s_0+\eta})] \\
&= \int_{\mathbb{R}} \varphi\Big(T_{s_0+\varepsilon} + \int_{s_0+\eta}^{u_0} (r_s)^{-1/2} d\gamma_s + \sigma_{\varepsilon,\eta} x\Big) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \to (2\pi)^{-1} \int_0^{2\pi} \varphi(x) dx
\end{aligned}
$$

a.s. as $\varepsilon \to 0$. This last convergence follows from the facts that $\lim_{\varepsilon\to 0} \sigma_{\varepsilon,\eta} = \infty$ and that, setting $\bar\varphi(x) := \varphi(x) - (2\pi)^{-1} \int_0^{2\pi} \varphi(y) dy$ and $\Phi(x) := \int_0^x \bar\varphi(y) dy$, for all $\theta \in [0, 2\pi)$,

$$
\begin{aligned}
\Big| \int_{\mathbb{R}} \varphi\Big(\theta + \sigma x\Big) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - \frac{1}{2\pi} \int_0^{2\pi} \varphi(x) dx \Big| &= \frac{1}{\sqrt{2\pi}} \Big| \sum_{k\in\mathbb{Z}} \int_0^{2\pi/\sigma} \bar\varphi(\sigma y) e^{-(y-(\theta+2k\pi)/\sigma)^2/2} dy \Big| \\
&= \frac{1}{\sigma\sqrt{2\pi}} \Big| \sum_{k\in\mathbb{Z}} \int_0^{2\pi/\sigma} \Phi(\sigma y) \times (y - (\theta + 2k\pi)/\sigma) e^{-(y-(\theta+2k\pi)/\sigma)^2/2} dy \Big| \\
&\leq \frac{\sqrt{2\pi}}{\sigma} \sup_{x\in[0,2\pi)} |\varphi(x)| \int_{\mathbb{R}} |z| e^{-z^2/2} dz \\
&= \frac{2\sqrt{2\pi}}{\sigma} \sup_{x\in[0,2\pi)} |\varphi(x)|.
\end{aligned}
$$

We used an integration by parts, that $\Phi(0) = \Phi(2\pi) = 0$ and that $|\Phi(y)| \leq 2\pi \sup_{x\in[0,2\pi)} |\varphi(x)|$ for all $y \in [0, 2\pi)$.

We deduce from (37) that T_{u_0} is uniformly distributed on $[0, 2\pi)$ and is independent of $\mathcal{H}_{s_0} \vee \sigma((\gamma_s - \gamma_{s_0+\eta})_{s\geq s_0+\eta})$. Since $\eta > 0$ can be chosen arbitrarily small, we conclude that T_{u_0} is independent of $\mathcal{H}_{s_0} \vee \sigma((\gamma_s - \gamma_{s_0})_{s\geq s_0}) = \mathcal{H}_{s_0} \vee \sigma((\gamma_s)_{s\geq 0})$ as desired. ☐

Lemma 22. *Assume that $\chi \in (0, 8\pi)$. There is uniqueness in law for* (33) *among solutions such that a.s., $\int_0^\infty \mathbf{1}_{\{Z_t=0\}} dt = 0$.*

Proof. As in the proof of Theorem 19 when $\chi \geq 8\pi$, (33) admits a pathwise unique solution until it reaches 0. All the difficulty is thus to prove the uniqueness in law of the solution started at 0. We thus consider, if it exists, a continuous solution $(Z_t)_{t\geq 0}$ to (33) with $Z_0 = 0$, adapted to some filtration $(\mathcal{F}_t)_{t\geq 0}$ in which $(W_t)_{t\geq 0}$ is a 2-dimensional Brownian motion, and such that $\int_0^\infty \mathbf{1}_{\{Z_t=0\}} dt$ vanishes a.s.

Step 1. We define $R_t = |Z_t|^{2/3}/4$ and $\beta_t = \int_0^t \mathbf{1}_{\{Z_s\neq 0\}}|Z_s|^{-1}Z_s \cdot dW_s$, which is clearly a 1-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Here we prove that

$$R_t = 2\int_0^t \sqrt{R_s}d\beta_s + (2-\chi/(4\pi))t. \tag{38}$$

Starting from (33) (with $Z_0 = 0$) and using the Itô formula, we easily find that

$$|Z_t|^2 = 12\int_0^t |Z_s|^{5/3}d\beta_s + (72 - 3\chi/\pi)\int_0^t |Z_s|^{4/3}ds.$$

For $\eta > 0$, using again Itô's formula, we find that

$$\begin{aligned}(|Z_t|^2+\eta)^{1/3} =&\eta^{1/3} + 4\int_0^t |Z_s|^{5/3}(|Z_s|^2+\eta)^{-2/3}d\beta_s + (24-\chi/\pi)\int_0^t |Z_s|^{4/3}(|Z_t|^2+\eta)^{-2/3}ds\\ &- 16\int_0^t |Z_s|^{10/3}(|Z_s|^2+\eta)^{-5/3}ds.\end{aligned}$$

Since $\int_0^t \mathbf{1}_{\{Z_s=0\}}ds = 0$ a.s. by assumption, the Lebesgue theorem ensures us that the sum of the two last terms in the right-hand side converges a.s. to $(8-\chi/\pi)t$ as $\eta \to 0$. The Itô isometry ensures that the second term in the right-hand side converges in L^2 to $4\int_0^t |Z_s|^{1/3}d\beta_s$. All in all, we find that $|Z_t|^{2/3} = 4\int_0^t |Z_s|^{1/3}d\beta_s + (8-\chi/\pi)t$. Dividing by 4 completes the proof of (38).

Step 2. We consider, for each $\eta > 0$, a nondecreasing C^2-function $\psi_\eta : [0,\infty) \mapsto [0,\infty)$ such that $\psi(u) = 0$ for all $u \in [0,\eta/2]$ and $\psi(u) = 1$ for all $u \geq \eta$. Observing that $\psi_\eta(R_t)|Z_t|^{-1}Z_t = \Psi_\eta(Z_t)$ where $\Psi_\eta(z) = \psi_\eta(|z|^{2/3}/4)|z|^{-1}z$ is of class C^2 on $\mathbb{R}^2$, we easily obtain, starting from (33) and applying the Itô formula,

$$\psi_\eta(R_t)\frac{Z_t}{|Z_t|} = \int_0^t \psi_\eta(R_s)\Big(2Z_s^\perp\frac{Z_s^\perp.dW_s}{|Z_s|^{7/3}} - \frac{2Z_s}{|Z_s|^{5/3}}ds\Big) + \int_0^t \frac{Z_s}{|Z_s|}(\psi_\eta'(R_s)dR_s + 2\psi_\eta''(R_s)R_sds) \tag{39}$$

where, for $z \in \mathbb{R}^2$ with respective coordinates z_1 and z_2, $z^\perp$ denotes the element of $\mathbb{R}^2$ with respective coordinates $-z_2$ and z_1.

Let $\gamma_t = \int_0^t \mathbf{1}_{\{Z_s\neq 0\}}|Z_s|^{-1}Z_s^\perp \cdot dW_s$. Since $\langle \beta,\gamma\rangle_t = \int_0^t \mathbf{1}_{\{Z_s\neq 0\}}|Z_s|^{-2}Z_s \cdot Z_s^\perp ds = 0$, the process $(\gamma_t)_{t\geq 0}$ is a 1-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian independent of $(\beta_t)_{t\geq 0}$ and thus also of $(R_s)_{s\geq 0}$ (because $(R_s)_{s\geq 0}$ is $\sigma(R_0,(\beta_s)_{s\geq 0})$-measurable by pathwise uniqueness for the SDE it solves).

For any $0 < u < t$, on the event $\{\inf_{[u,t]} R_s > 0\}$, choosing $\eta \in (0, \inf_{[u,t]} R_s)$ in the difference between (39) and the same equation with t replaced by u, we obtain

$$\frac{Z_t}{|Z_t|} = \frac{Z_u}{|Z_u|} + \int_u^t \Big(2\frac{Z_s^\perp}{|Z_s|}\frac{d\gamma_s}{|Z_s|^{1/3}} - \frac{2Z_s}{|Z_s|^{5/3}}ds\Big) = \frac{Z_u}{|Z_u|} + \int_u^t \Big(\frac{Z_s^\perp}{|Z_s|}\frac{d\gamma_s}{\sqrt{R_s}} - \frac{Z_s}{|Z_s|}\frac{ds}{2R_s}\Big). \tag{40}$$

Step 3. For $s > 0$ such that $R_s > 0$ we define $T_s \in [0,2\pi)$ through the equality $|Z_s|^{-1}Z_s = e^{iT_s}$. For $s \geq 0$ with $R_s = 0$, we simply put $T_s = 0$. We used the natural identification between $\mathbb{R}^2$ and $\mathbb{C}$: for $\theta \in \mathbb{R}$, we denote by $e^{i\theta}$ (resp. $ie^{i\theta}$) the 2-dimensional vector with coordinates $\cos\theta$ and $\sin\theta$ (resp. $-\sin\theta$ and $\cos\theta$). We claim that for all $0 < u < t$, on the event $\{\inf_{[u,t]} R_s > 0\}$, it holds that $T_t = h_{2\pi}(T_u + \int_u^t R_s^{-1/2}d\gamma_s)$.

To check this claim, on the event $\{\inf_{[u,t]} R_s > 0\}$, we introduce $\mathcal{T}_v = T_u + \int_u^v R_s^{-1/2}d\gamma_s$, for all $v \in [u,t]$. Since $(\gamma_v)_{v\geq 0}$ is independent of the event $\{\inf_{[u,t]} R_s > 0\}$, we can apply the Itô formula:

$$\text{for all } v \in [u,t], \quad e^{i\mathcal{T}_v} = e^{iT_u} + \int_u^v \Big(ie^{i\mathcal{T}_s}\frac{d\gamma_s}{\sqrt{R_s}} - e^{i\mathcal{T}_s}\frac{ds}{2R_s}\Big).$$

Recalling (40) and using a uniqueness argument, we deduce that on the event $\{\inf_{[u,t]} R_s > 0\}$, $|Z_v|^{-1}Z_v = e^{i\mathcal{T}_v}$ whence $T_v = h_{2\pi}(\mathcal{T}_v)$ for all $v \in [u,t]$.

Step 4. Here we check that a.s., $\int_t^{t+h} R_s^{-1}ds = \infty$ for all $t \geq 0$ such that $R_t = 0$ and all $h > 0$. This follows from the fact that for all $T > 0$, $\lim_{u \searrow 0} \sup_{t\in[0,T]}[u(1\vee\log(1/u))]^{-1/2}|\sqrt{R_{t+u}} - \sqrt{R_t}| = \sqrt{2}$ a.s., see Khoshnevisan [25, (2.1a) p 1299] and recall that $(R_s)_{s\geq 0}$ is a squared $(2 - \chi/(4\pi))$-dimensional Bessel process starting from 0 by Step 1, with $2 - \chi/(4\pi) > 0$.

Step 5. Here we verify that conditionally on $(R_s)_{s\geq 0}$, for any $\sigma((R_s)_{s\geq 0})$-measurable finite family $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$ such that for all $k = 1,\dots,n$, $R_{s_k} = R_{t_k} = 0$ and $R_s > 0$ on (s_k, t_k), the variables $\{(T_s)_{s\in(s_k,t_k)}, k = 1,\dots,n\}$ are independent and for each $k = 1,\dots,n$, $(T_s)_{s\in(s_k,t_k)}$ is $\Gamma(s_k, t_k, (R_s)_{s\in(s_k,t_k)})$-distributed. The function Γ was introduced in Lemma 21.

Let $(\mathbf{Z}_t, \mathbf{g}_t)_{t\geq 0}$ denote the canonical process on $C([0,\infty), \mathbb{R}^2 \times \mathbb{R})$ endowed with the conditional law of $(Z_t, \gamma_t)_{t\geq 0}$ knowing $(R_t)_{t\geq 0}$. We define $\mathbf{T}_t \in [0, 2\pi)$ by $|\mathbf{Z}_t|^{-1}\mathbf{Z}_t = e^{i\mathbf{T}_t}$ if $\mathbf{Z}_t \neq 0$ and $\mathbf{T}_t = 0$ else. We introduce the filtration $\mathcal{H}_t = \sigma((\mathbf{T}_s, \mathbf{g}_s)_{s\in[0,t]})$. We claim that a.s., $(\mathbf{g}_t)_{t\geq 0}$ is a $(\mathcal{H}_t)_{t\geq 0}$-Brownian motion, because $(\gamma_t)_{t\geq 0}$ is independent of $\sigma((R_s)_{s\geq 0})$ and is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$ to which $(T_t)_{t\geq 0}$ is adapted: for all $t > 0$, all bounded measurable Φ, Ψ,

$$\begin{aligned}
&\mathbb{E}\Big[\Phi((\gamma_{t+s} - \gamma_t)_{s\geq 0})\Psi((\gamma_s, T_s)_{s\in[0,t]})\Big|(R_s)_{s\geq 0}\Big] \\
=&\mathbb{E}\Big[\Psi((\gamma_s, T_s)_{s\in[0,t]})\mathbb{E}\Big[\Phi((\gamma_{t+s} - \gamma_t)_{s\geq 0})\Big|\mathcal{F}_t \vee \sigma((R_s)_{s\geq 0})\Big]\Big|(R_s)_{s\geq 0}\Big] \\
=&\mathbb{E}\Big[\Phi((\gamma_{t+s} - \gamma_t)_{s\geq 0})\Big]\mathbb{E}\Big[\Psi((\gamma_s, T_s)_{s\in[0,t]})\Big|(R_s)_{s\geq 0}\Big].
\end{aligned}$$

Fix now $k \in \{1,\dots,n\}$. It a.s. holds that $\mathbf{T}_t = h_{2\pi}(\mathbf{T}_u + \int_u^t R_s^{-1/2}d\mathbf{g}_s)$ for all $s_k < u < t < t_k$ by Step 3 and that $\int_{s_k}^t R_s^{-1}ds = \infty$ for all $t \in (s_k, t_k)$ by Step 4. Applying Lemma 21, we find that a.s., $(\mathbf{T}_s)_{s\in(s_k,t_k)}$ is independent of $\mathcal{H}_{s_k}$ and is $\Gamma(s_k, t_k, (R_s)_{s\in(s_k,t_k)})$-distributed. Using that $(\mathbf{T}_s)_{s\in(s_k,t_k)}$ is $\mathcal{H}_{t_k}$-measurable for each $k = 1,\dots,n$, the independence easily follows.

Step 6. By Step 1, $(R_t)_{t\geq 0}$ is a $(2 - \chi/(4\pi))$-dimensional Bessel process starting from 0. By Step 5, the conditional law of $(T_t\mathbf{1}_{\{R_t\neq 0\}})_{t\geq 0}$ knowing $(R_t)_{t\geq 0}$ is also determined: conditionally on $(R_s)_{s\geq 0}$, for any $\sigma((R_s)_{s\geq 0})$-measurable finite family $\{(s_k, t_k), k = 1,\dots,n\}$ of excursions of $(R_s)_{s\geq 0}$, we know the law of $(T_s)_{s\in\cup_{k=1}^n(s_k,t_k)}$. Since by construction $Z_t = (4R_t)^{3/2}e^{iT_t}\mathbf{1}_{\{R_t\neq 0\}}$, the law of $(Z_t)_{t\geq 0}$ is thus entirely characterized. □

Finally, we can give the

Proof of Theorem 19 when $\chi \in (0, 8\pi)$. First, the existence of a solution $(Z_t)_{t\geq 0}$ to (33) such that a.s. $\int_0^\infty \mathbf{1}_{\{Z_t=0\}}dt = 0$ follows from Lemma 20: the solution $(Z_t)_{t\geq 0}$ built there satisfies that $|Z_t|^{2/3}/4$ is a $(2 - \chi/(4\pi))$-dimensional Bessel process, whence $\int_0^\infty \mathbf{1}_{\{Z_t=0\}}dt = 0$ a.s. by [36, p 442]. The uniqueness in law of this solution has been checked in Lemma 22. The convergence of $(Z_t^\varepsilon)_{t\geq 0}$ to $(Z_t)_{t\geq 0}$ clearly follows from Lemma 20 and from this uniqueness in law. This implies as in the case $\chi \geq 8\pi$ that $(D_t^\varepsilon)_{t\geq 0}$ goes in law to $(|Z_t|^{-2/3}Z_t\mathbf{1}_{\{Z_t\neq 0\}})_{t\geq 0}$.

It remains to verify that when $\chi \in (0, 4\pi)$, $D_t = |Z_t|^{-2/3}Z_t\mathbf{1}_{\{Z_t\neq 0\}}$ solves (31) and that uniqueness in law holds true for (31).

For $(D_t)_{t\geq 0}$ a solution to (31), one easily checks by Itô's formula that $Z_t = |D_t|^2 D_t$ solves (33) and that $|D_t|^2$ is a $(2-\chi/(4\pi))$-dimensional Bessel process, whence $\int_0^\infty \mathbf{1}_{\{Z_t=0\}}dt = 0$ a.s. by [36, p 442]. The uniqueness in law for (31) then follows from Lemma 22.

For $(Z_t)_{t\geq 0}$ built above, by Itô's formula, for $\eta > 0$,

$$
\begin{aligned}
(|Z_t|^2+\eta)^{-1/3}Z_t &= (|Z_0|^2+\eta)^{-1/3}Z_0 + 2\int_0^t |Z_s|^{2/3}(|Z_s|^2+\eta)^{-1/3}dW_s \\
&\quad + 4\int_0^t \Big(|Z_s|^{-1/3}(|Z_s|^2+\eta)^{-1/3} - |Z_s|^{5/3}(|Z_s|^2+\eta)^{-4/3}\Big) Z_s d\beta_s \\
&\quad + \int_0^t \bigg((16-3\chi/(2\pi))|Z_s|^{-2/3}(|Z_t|^2+\eta)^{-1/3} + (\chi/\pi - 48)|Z_s|^{4/3}(|Z_t|^2+\eta)^{-4/3} \\
&\qquad\qquad + 32|Z_s|^{10/3}(|Z_t|^2+\eta)^{-7/3}\bigg) Z_s ds.
\end{aligned}
$$

By the Itô isometry and the Lebesgue theorem and since a.s. $\int_0^t \mathbf{1}_{\{Z_s=0\}}ds = 0$, the second term on the RHS tends to $2W_t$ in L^2 and the third term on the RHS tends to 0 in L^2. Since $|Z_t|^{2/3}/4$ is a $(2-\chi/(4\pi))$-dimensional squared Bessel process and $2-\chi/(4\pi) > 1$, [36, Exercise 1.26 p 451] ensures that a.s. $\int_0^t |Z_s|^{-1/3}ds < \infty$. Hence the Lebesgue theorem ensures us that the last term on the RHS converges a.s. to $-(\chi/(2\pi))\int_0^t |Z_s|^{-4/3}Z_s ds$. We conclude that $D_t = |Z_t|^{-2/3}Z_t\mathbf{1}_{\{Z_t\neq 0\}}$ solves $D_t = D_0 + 2W_t - (\chi/(2\pi))\int_0^t |D_s|^{-2}D_s ds$, which completes the proof. □

7. On the system with $N \geq 3$ particles

7.1. **Classification of reflecting and sticky collisions.** We have seen in the proof of Lemma 15-Step 2 that very roughly, the empirical variance of the positions of k particles in the system with N particles resembles a squared Bessel process of dimension $\delta_{N,\chi}(k) = (k-1)(2-\chi k/(4\pi N))$. Fix $\chi > 0$ and $N \geq 3$ and consider the regularized particle system (11), which is always well-posed. We now describe formally the expected behavior of its limit as $\varepsilon \to 0$. According to [36, Page 442] and the comparison theorem [36, Theorem 3.7 p 394], the following events should occur:

- if $\delta_{N,\chi}(k) \geq 2$, no collisions of subsystems of k particles,
- if $\delta_{N,\chi}(k) \in (0,2)$, (instantaneously) reflecting collisions of subsystems of k particles,
- if $\delta_{N,\chi}(k) \leq 0$, sticky collisions of subsystems of k particles.

Let us now study the inequality $\delta_{N,\chi}(k) \geq 2$. We have already seen in the proof of Lemma 15-Step 2 that when $\chi \in (0, 8\pi(N-2)/(N-1)]$, $\delta_{N,\chi}(k) \geq 2$ for all $k \in \{3,\dots,N\}$. When $\chi \in (8\pi(N-2)/(N-1), 4\pi N/3]$, $\delta_{N,\chi}(3) \geq 2$ whereas $\delta_{N,\chi}(2) < 2$ and $\delta_{N,\chi}(N) < 2$, hence the two roots $x^{\pm}_{N,\chi} = [1+(8\pi N)/\chi \pm \sqrt{(1+8\pi N/\chi)^2 - 64\pi N/\chi}]/2$ of the second order equation $\delta_{N,\chi}(x) = 2$ are such that $x^-_{N,\chi} \in (2,3]$ and $x^+_{N,\chi} \in [3,N)$, so that $\delta_{N,\chi}(2) < 2$, $\delta_{N,\chi}(k) \geq 2$ for $k \in \{3,\dots,\lfloor x^+_{N,\chi}\rfloor\}$ and $\delta_{N,\chi}(k) < 2$ for $k \in \{\lfloor x^+_{N,\chi}\rfloor + 1,\dots,N\}$. Finally, one easily checks that $x^-_{N,4\pi N/3} = 3$ and $x^+_{N,4\pi N/3} = 4$. By strict monotonicity of the map $\chi \mapsto \delta_{N,\chi}(k)$, we conclude that if $\chi > 4\pi N/3$, then $\delta_{N,\chi}(k) < 2$ for all $k \in \{2,\dots,N\}$.

Let us next study the inequality $\delta_{N,\chi}(k) \leq 0$, which, for $k \in \{2,\dots,N\}$ is equivalent to $k \geq 8\pi N/\chi$. Hence for $\chi \in (0,8\pi)$, $\delta_{N,\chi}(k) > 0$ for all $k \in \{2,\dots,N\}$ whereas for $\chi \in [8\pi, 4\pi N)$, $\delta_{N,\chi}(k) > 0$ for all $k \in \{2,\dots,\lceil 8\pi N/\chi\rceil - 1\}$ and $\delta_{N,\chi}(k) \leq 0$ for all $k \in \{\lceil 8\pi N/\chi\rceil,\dots,N\}$ with the two sets non empty. When $\chi \geq 4\pi N$, $\delta_{N,\chi}(k) \leq 0$ for all $k \in \{2,\dots,N\}$.

When $N \geq 6$, we end up with the following picture.

(a) If $\chi \in (0, 8\pi(N-2)/(N-1)]$, the regularized particle system should tend to the particle system (4) and the latter should have a unique (in law) solution. Indeed, it holds that $\delta_{N,\chi}(k) \geq 2$ for all $k \geq 3$ and that $\delta_{N,\chi}(2) \in (0,2)$, so that only binary reflecting collisions occur. We have already checked a tightness/consistency result in this spirit in Theorem 7. Only the uniqueness in law remains open.

(b) If $\chi \in (8\pi(N-2)/(N-1), 8\pi)$, the regularized particle system should tend to the particle system (4) and the latter should also have a unique (in law) solution. One may check that $k_0 := \lfloor x^+_{N,\chi} \rfloor + 1 \in \{N-1, N\}$ (it suffices to verify that $\delta_{N,\chi}(N-2) \geq \delta_{N,8\pi}(N-2) \geq 2$). In this situation, there should be binary reflecting collisions and also reflecting collisions of subsystems of particles with cardinality in $\{k_0, N\}$. To check the existence (and *a fortiori* uniqueness) of such a process, one has to control the drift term during the collisions with reflection. In the present paper, we are more or less able to contol the drift during a (reflecting) binary collision, but we have not the least idea of what to do during a k-ary reflecting collision with $k \geq 3$.

(c) If $\chi \in [8\pi, 4\pi N/3]$, the regularized particle system should tend to a particle system with sticky collisions that we will describe more precisely in the next subsection. One can check that, for $k_0 := \lfloor x^+_{N,\chi} \rfloor + 1 > 4$ and $k_1 := \lceil 8\pi N/\chi \rceil \leq N$, we have $k_0 \in \{k_1 - 2, k_1 - 1\}$ (just verify that $\delta_{N,\chi}(k_1 - 3) \geq \delta_{N,\chi}(8\pi N/\chi - 2) \geq 2$ and $\delta_{N,\chi}(k_1 - 1) \leq \delta_{N,\chi}(8\pi N/\chi - 1) < 2$). Thus, binary reflecting collisions, as well as k-ary reflecting collisions, for $k \in \{k_0, k_1 - 1\}$, should occur, as well as sticky collisions of subsytem of k-particles, for $k \in \{k_1, \dots, N\}$. Assume e.g. that $k_0 = k_1 - 1$. What might happen is that, at some time, k_0 particles become close to each other, they may collide (with reflection) a few times, then another particle is attracted in the zone, the $k_0 + 1 = k_1$ particles meet and then remain stuck forever. Such a cluster will move with a very small diffusion coefficient and should collide later with other particles (or clusters) in a sticky way. Of course, such a result would be very interesting but it seems very difficult to prove, because to check the existence of such a process, one would have to control the drift term during the collisions with reflection, as mentioned previously. The sticky collisions should be easier to describe.

(d) If $\chi \in (4\pi N/3, 4\pi N)$, the same situation as previously should arise, except that there should be k-ary reflecting collisions for all $k \in \{2, \dots, \lceil 8\pi N/\chi \rceil - 1\}$ and sticky k-ary collisions for all $k \in \{\lceil 8\pi N/\chi \rceil, \dots, N\}$. In addition, when $\chi \geq 2\pi N$, there is a problem of definition of the drift due to binary collisions as in Remark 18: the particle system without cutoff (4) should have solutions only until the first binary collision. It is not clear to us how to rewrite the equation in a way that makes sense

(e) If finally $\chi \geq 4\pi N$, then there should be sticky k-ary collisions for all $k \in \{2, \dots, N\}$.

When $N = 5$, we find the following dichotomy. If $\chi \in (0, 6\pi]$, only binary reflecting collisions. If $\chi \in (6\pi, 20\pi/3]$, only reflecting collisions of subsystems of $k \in \{2, 5\}$ particles. If $\chi \in (20\pi/3, 8\pi)$, only reflecting collisions of subsystems of $k \in \{2, 3, 4, 5\}$ particles. If $\chi \in [8\pi, 20\pi)$, reflecting collisions of subsystems of $k \in \{2, \dots, \lceil 40\pi/\chi \rceil - 1\}$ particles and sticky collisions of subsystems of $k \in \{\lceil 40\pi/\chi \rceil, \dots, 5\}$ particles. If $\chi \geq 20\pi$, k-ary sticky collisions for all $k \in \{2, \dots, 5\}$.

When finally $N \in \{3, 4\}$, $8\pi(N-2)/(N-1) = 4\pi N/3$ and the situation is as follows. If $\chi \in (0, 8\pi(N-2)/(N-1)]$, only binary reflecting collisions. If $\chi \in (4\pi N/3, 8\pi)$, reflecting collisions of subsystems of $k \in \{2, \dots, N\}$ particles. If $\chi \in [8\pi, 4\pi N)$, k-ary reflecting collisions for $k \in \{2, \dots, \lceil 8\pi N/\chi \rceil - 1\}$ and k-ary sticky collisions of subsystems of $k \in \{\lceil 8\pi N/\chi \rceil, \dots, N\}$ particles. If $\chi \geq 4\pi N$, k-ary sticky collisions for all $k \in \{2, \dots, N\}$.

7.2. **A particle system in the supercritical case.** When $\chi \geq 8\pi$, the following dynamics should describe the limit of the regularized particle system as $\varepsilon \to 0$. Particles are characterized by their masses and their positions. Initially, we start with N particles with masses $\nu_0^1, \dots, \nu_0^N$ all equal to $1/N$ and with some given positions $X_0^{1,N}, \dots, X_0^{N,N}$. If now at some time $t \geq 0$, we have N_t particles (N_t will be a.s. nonincreasing) with masses $\nu_t^1, \dots, \nu_t^{N_t}$ (such that $\sum_1^{N_t} \nu_t^i = 1$), we make the positions evolve according to

$$dX_t^{i,N} = \sqrt{\frac{2}{N\nu_t^i}}dB_t^i + \chi \sum_{j=1}^{N_t} \nu_t^j K(X_t^{i,N} - X_t^{j,N})dt, \quad i = 1, \dots, N_t \tag{41}$$

until the next collision between at least two of these N_t particles. If the sum S of the masses of the particles involved in the collision is smaller than $8\pi/\chi$, they should automatically separate instantaneously and we carry on making evolve the system according to (41) (with the same values for the masses and for N_t) until the next collision. If now S exceeds $8\pi/\chi$, the particles involved in the collision are replaced by a single particle with mass S, the number of particles is decreased accordingly, the particles are relabeled, and we make evolve the system according to (41) with these new values for N_t and for the masses until the next collision.

By construction, the masses take values in $\{1/N, 2/N, \dots, N/N\}$ and actually in $\{k/N\ :\ k = 1$ or $8\pi N/\chi \leq k \leq N\}$. A particle of mass k/N with $k \geq 2$ has to be seen as a *cluster* of k elementary particles. The drift term is thus easily understood: a single elementary particle interacts with the other ones proportionaly to $1/N$, so that a cluster consisting of k elementary particles interacts with the other ones proportionally to its mass k/N. The diffusion coefficients are also quite natural: a single particle being subjected to a Brownian excitation with coefficient $\sqrt{2}$, a cluster with mass k/N is excited by the mean of k Brownian motions with coefficient $\sqrt{2}$, that is, by a Brownian motion with coefficient $\sqrt{2/k}$.

If $N_t \geq 2$, setting for $I \subset \{1, \dots, N_t\}$ with cardinality $|I| \geq 2$, $\bar{X}_t^I = \sum_{i \in I} \nu_t^i X_t^{i,N} / \sum_{i \in I} \nu_t^i$ and $R_t^I = (N/2) \sum_{i \in I} \nu_t^i |X_t^{i,N} - \bar{X}_t^I|^2$, a simple computation shows that, when neglecting the interaction with particles with label outside I, R_t^I behaves like a squared Bessel process of dimension $2(|I| - 1) - (\chi N/4\pi) \sum_{i,j \in I, i \neq j} \nu_t^i \nu_t^j \leq (|I| - 1)[2 - S\chi/(4\pi)]$, which is nonpositive as soon as $S = \sum_{i \in I} \nu_t^i \geq 8\pi/\chi$.

Let us mention that once a *cluster* is formed, its mass necessarily exceeds $8\pi/\chi$, so that any collision involving a cluster will be sticky.

The existence of such a process is not clear. Sticky collisions should not be very hard to treat. The main difficulty is to control reflecting collisions. As explained just above, reflecting collisions only concern particles with masses $1/N$, so that the classification given in Subsection 7.1 should still be relevant. Thus we believe that the main difficulty is to build a (necessarily nontrivial) local (in time) solution to (4) when $\chi \geq 8\pi$ and starting from an initial condition where k particles have the same initial positions, for some $k \in \{2, \dots, \lceil 8\pi N/\chi \rceil - 1\}$.

7.3. **Comments.** Observe that this process is different of the one introduced by Haškovec and Schmeiser in [13] where they consider a system of particles with different masses to approximate the singular solution to the Keller-Segel equation. In fact, rather than considering like us the limit $\varepsilon \to 0$ of the regularized particle system (11), they first prove propagation of chaos as $N \to \infty$ for a fixed $\varepsilon > 0$ in [14]. More precisely, they check that for fixed $k \geq 1$, the density of $(X_t^{1,N,\varepsilon}, \dots, X_t^{k,N,\varepsilon})$ solving (11) (with another regularized kernel K_ε) converges as $N \to \infty$ to

$\prod_{i=1}^k f_t^\varepsilon(x_i)$ where $(f_t^\varepsilon)_{t\geq 0}$ solves the regularized Keller-Segel partial differential equation

$$\partial_t f_t^\varepsilon(x) + \chi \mathrm{div}_x((K_\varepsilon \star f_t^\varepsilon)(x) f_t^\varepsilon(x)) = \Delta_x f_t^\varepsilon(x).$$

The limiting behaviour of $(f_t^\varepsilon)_{t\geq 0}$ as $\varepsilon \to 0$ was studied in [6] and involves a defect measure. Then Haškovec and Schmeiser introduce in [13] a particle system associated with this limit, in which there are heavy particles that occupy a positive proportion of the mass, interact with the other particles, but do not undergo any Brownian excitation.

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