

# ON EXPONENTIAL MOMENTS OF THE HOMOGENEOUS BOLTZMANN EQUATION FOR HARD POTENTIALS WITHOUT CUTOFF

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ABSTRACT. We consider the spatially homogeneous Boltzmann equation for hard potentials without cutoff. We prove that an exponential moment of order  $\rho = \min\{2\gamma/(2-\nu), 2\}$ , with the usual notation, is immediately created. This is stronger than what happens in the case with cutoff. We also show that exponential moments of order  $\rho \in (0, 2]$  are propagated.

## 1. INTRODUCTION AND RESULTS

**1.1. The Boltzmann equation.** We consider a spatially homogeneous gas modeled by the Boltzmann equation: the density  $f_t(v)$  of particles with velocity  $v \in \mathbb{R}^3$  at time  $t \geq 0$  solves

$$(1) \quad \partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \cos \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)],$$

where

$$(2) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma \quad \text{and} \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

We refer to the book of Cercignani [7] and to the long review papers of Villani [19] and Alexandre [1] for some detailed and complete accounts of what is known, both from the physical and mathematical points of view, about this equation. One may assume without loss of generality that the initial condition satisfies

$$\int_{\mathbb{R}^3} f_0(v) dv = 1, \quad \int_{\mathbb{R}^3} v f_0(v) dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_0(v) dv = 1,$$

and these quantities, namely the mass, momentum and kinetic energy, are constant, at least informally, as time evolves.

**1.2. Assumptions.** We will suppose that for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ ,

$$(H_1(\gamma)) \quad B(|v - v_*|, \cos \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta) \quad \text{for some measurable } \beta : (0, \pi] \rightarrow \mathbb{R}_+,$$

$$(H_2(\nu)) \quad \exists \kappa_1, \kappa_2 \in (0, \infty), \forall \theta \in (0, \pi], \quad \kappa_1 \theta^{-\nu-1} \leq \beta(\theta) \leq \kappa_2 \theta^{-\nu-1}.$$

We consider any possible values of  $\gamma \in (0, 1]$  and  $\nu \in (0, 2)$ , but the only physically relevant situations are the following. As explained in [7, 19, 1], when particles interact through a repulsive force in  $1/r^s$  for some  $s > 2$ , we have  $(H_1(\gamma))$  and  $(H_2(\nu))$  with  $\gamma = (s-5)/(s-1)$  and  $\nu = 2/(s-1)$ . When  $\gamma \in (0, 1)$  (i.e.  $s > 5$ ), one speaks of hard potentials.

One speaks of hard potentials with cutoff when we have  $(H_1(\gamma))$  for some  $\gamma \in (0, 1]$  and when  $(H_2(\nu))$  is replaced by the condition  $\int_0^\pi \beta(\theta) d\theta \in (0, \infty)$ .

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2010 *Mathematics Subject Classification.* 82C40.

*Key words and phrases.* Kinetic equations, Exponential moments, Strong localization.

**1.3. Weak solutions.** First, we parameterize (2) as in [9]. For every  $X \in \mathbb{R}^3 \setminus \{0\}$ , we introduce  $I(X), J(X) \in \mathbb{R}^3$  such that  $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$  is an orthonormal basis of  $\mathbb{R}^3$ . We also put  $I(0) = J(0) = 0$ . For  $X, v, v_* \in \mathbb{R}^3$ ,  $\theta \in (0, \pi]$  and  $\varphi \in [0, 2\pi)$ , we set

$$(3) \quad \begin{cases} \Gamma(X, \varphi) = (\cos \varphi)I(X) + (\sin \varphi)J(X), \\ v' = v - \frac{1 - \cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ v'_* = v_* + \frac{1 - \cos \theta}{2}(v - v_*) - \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi). \end{cases}$$

We denote by  $\mathcal{P}(\mathbb{R}^3)$  the set of probability measures on  $\mathbb{R}^3$ . For  $p \in \mathbb{R}_+$  and  $f \in \mathcal{P}(\mathbb{R}^3)$ , we introduce the moment of order  $p$  of  $f$ :

$$m_p(f) = \int_{\mathbb{R}^3} |v|^p f(dv).$$

We use the following classical notion of weak solutions.

**Definition 1.** Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . A weakly continuous family  $(f_t)_{t \geq 0}$  of probability measures on  $\mathbb{R}^3$  is a weak solution to (1) if for all  $t \geq 0$ ,

$$\int_{\mathbb{R}^3} v f_t(dv) = 0 \quad \text{and} \quad m_2(f_t) = 1$$

and if for any  $\phi \in C_b^2(\mathbb{R}^3)$  and any  $t \geq 0$ , using the parameterization (3),

$$(4) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\pi \int_0^{2\pi} [\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)] |v - v_*|^\gamma d\varphi d\beta(\theta) d\theta f_t(dv_*) f_t(dv).$$

As shown by Lu-Mouhot in [13], see also Villani [19], weak solutions exist starting from any given initial condition  $f_0 \in \mathcal{P}(\mathbb{R}^3)$  such that  $\int_{\mathbb{R}^3} v f_0(dv) = 0$  and  $m_2(f_0) = 1$ , and they satisfy

$$(5) \quad \text{for all } p \geq 0, \text{ all } t_0 > 0, \quad \sup_{t \geq t_0} m_p(f_t) < \infty.$$

Let us mention the recent uniqueness result of Heydecker [12], which concerns the case where  $\nu \in (0, 1)$ , assuming only that  $m_p(f_0) < \infty$  for some sufficiently large  $p$ .

**1.4. Main result.** Here is our main result.

**Theorem 2.** Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . Consider any weak solution  $(f_t)_{t \geq 0}$  to (1).

(i) Put  $\rho = \min\{2\gamma/(2 - \nu), 2\}$ . There are some constants  $T > 0$  and  $\sigma > 0$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} \exp[\sigma t^{\rho/\gamma} |v|^\rho] f_t(dv) \leq 4.$$

(ii) For any  $A > 0$ , any  $\sigma_0 > 0$ , any  $\rho \in (0, 2]$ , there is a constant  $\sigma > 0$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, \rho, \sigma_0, A$ , such that

$$\int_{\mathbb{R}^3} \exp[\sigma_0 |v|^\rho] f_0(dv) \leq A \quad \implies \quad \sup_{t \geq 0} \int_{\mathbb{R}^3} \exp[\sigma |v|^\rho] f_t(dv) \leq 6.$$

Since  $\min\{2\gamma/(2 - \nu), 2\} > \gamma$ , point (i) is stronger than in the cutoff case where, as we will see in the next subsection, only exponential moments of order  $\rho = \gamma$  are created.

In (ii), we have a possible deterioration of the constant  $\sigma$ , as in all the references below.

By (ii), (i) can be extended to: there is  $\sigma > 0$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} \exp[\sigma(t \wedge 1)^{\rho/\gamma} |v|^\rho] f_t(dv) \leq 6 \quad \text{with} \quad \rho = \min \left\{ \frac{2\gamma}{2-\nu}, 2 \right\}.$$

**Remark 3.** *In the physically relevant case where particles interact through a repulsive force in  $1/r^s$  for some  $s > 5$ , we have  $(H_1(\gamma))$  and  $(H_2(\nu))$  with  $\gamma = (s-5)/(s-1) \in (0, 1)$  and  $\nu = 2/(s-1) \in (0, 1/2)$ . We thus can apply Theorem 2: there is  $\sigma > 0$  depending only on  $s, \kappa_1, \kappa_2$  such that for any weak solution  $(f_t)_{t \geq 0}$  to (1),*

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} \exp[\sigma(t \wedge 1)^{(s-1)/(s-2)} |v|^{(s-5)/(s-2)}] f_t(dv) \leq 6.$$

Observe that  $\frac{s-1}{s-2} = \frac{4}{3+\gamma}$  and  $\frac{s-5}{s-2} = \frac{4\gamma}{3+\gamma}$ .

By Bobylev-Gamba [4], point (ii) of Theorem 2 also holds true for Maxwell molecules, i.e. when  $\gamma = 0$  and  $\nu = 1/2$ . Actually, Maxwell molecules with cutoff are studied in [4], but the constants do not depend on the cutoff parameter, as noted in [4, Remark 3.2]. Point (i) is irrelevant in this case, since  $\rho = 0$ .

**1.5. References.** There is a large literature on the subject, because exponential moments can be used for different purposes, such as estimating the rate of convergence to equilibrium, see Mouhot [15], or uniqueness, see [10].

Using the famous Povzner inequality [17], Wennberg [20] discovered that polynomial moments are immediately created by the homogeneous Boltzmann equation for cutoff hard potentials (CHP in short), i.e. when the angular cross section  $\beta$  is assumed to be integrable on  $[0, \pi]$ . This really requires that  $\gamma > 0$  and the main intuition is that particles with large velocities are quickly slowed down, because they collide at large rate (since  $\gamma > 0$ ) with slow particles. Studying a linearized Boltzmann equation, Mouhot-Strain [16] showed that this property is stronger for non cutoff hard potentials (NCHP in short), in the sense that the rate of decay of polynomial moments is higher for NCHP than for CHP.

In his seminal paper [5], Bobylev proved that Gaussian moments ( $\rho = 2$ ) are propagated in the case of hard spheres, i.e. when  $\gamma = 1$  and  $\beta(\theta) = \sin \theta$ . This relies on very tight computations involving a recursive ODE argument to estimate the polynomial moments, which are then summed to estimate Gaussian moments. This was generalized to all CHP by Gamba-Panferov-Villani [11], who also proved some local estimates that we will mention later. Let us also cite Bobylev-Gamba-Panferov [6] who studied inelastic collisions. Following the ideas of [5], Mischler, Mouhot, and Rodriguez Ricard [14] and Mouhot [15] managed to create exponential moments of order  $\rho = \gamma/2$ , still for CHP. Following the same approach, Lu-Mouhot [13] were able to create exponential moments of order  $\rho = \gamma$  for CHP and NCHP. Alonso-Cañizo-Gamba-Mouhot [2] found a much simpler method to create exponential moments of order  $\rho = \gamma$  and propagate exponential moments of order  $\rho \in (0, 2]$ , for CHP. Let us finally quote Alonso-Gamba-Tasković [3], who studied some much stronger Lebesgue and Sobolev norms of  $f_t$  with exponential weights for CHP.

Concerning NCHP, there is the work of Lu-Mouhot [13] already mentioned. The proof of Lemma 4.1 in Fournier-Mouhot [10], which concerns exponential moments for NCHP, is unfortunately false, there is a major gap (the function  $\delta(\eta)$  in (4.6) actually depends on  $p$ ). What was required there for the uniqueness criterion for NCHP to imply a well-posedness result, was the propagation of exponential moments of order  $\rho = \gamma$ . Tasković-Alonso-Gamba-Pavlović [18] have shown, for NCHP,

creation of exponential moments of order  $\rho = \gamma$  and propagation of exponential moments of order  $\rho \in (0, 4/(2 + \nu)]$  (which contains  $\rho = \gamma \in (0, 1]$  and thus fixes the issue in [10]).

It thus seems that concerning NCHP, [18] contains the best available results, and Theorem 2 is stronger both for creation and propagation. In particular, we show that NCHP create more exponential moments than CHP.

The homogeneous Landau equation for hard potentials, which often behaves in a similar way as the Boltzmann equation, but which is considerably simpler in many points, immediately creates Gaussian moments ( $\rho = 2$ ) for any value of  $\gamma \in (0, 1]$ , see [8]. Once this is observed, it is natural to wonder if NCHP create more exponential moments than CHP. The answer is not intuitively clear, because the (considerably many) additional collisions caused by the singularity of  $\beta$  near 0 involve some (considerably) small values of  $\theta$  and do not much slow down particles with high velocity. Actually, the effect is strong enough to modify the behavior of the solutions: exponential moments of order  $\rho = \min\{2\gamma/(2 - \nu), 2\} > \gamma$  are created by NCHP, while only exponential moments of order  $\rho = \gamma$  are created by CHP.

By the way, Theorem 2-(i) implies that for any  $\gamma \in (0, 1]$ , Gaussian moments ( $\rho = 2$ ) are created by NCHP if the angular cross section is singular enough, namely if  $\nu \geq 2 - \gamma$ , but this does never happen in the physical situations described in Remark 3.

Let us finally mention some closely related works: Gamba-Panferov-Villani [11] showed that for CHP, if there are  $A, a > 0$  such that  $f_0(v) \leq Ae^{-a|v|^2}$  for all  $v \in \mathbb{R}^3$ , then there are  $B, b > 0$  such that  $f(t, v) \leq Be^{-b|v|^2}$  for all  $t \geq 0$  and all  $v \in \mathbb{R}^3$ . Bobylev-Gamba [4] established the same result for Maxwell molecules with cutoff. To our knowledge, such a local Gaussian estimate is open in all the other situations.

**1.6. Strategy.** Our strategy is the same as that of [18]: we adapt the ideas of [2] to NCHP, taking advantage of the simplicity of the method. The present paper resembles [18] in several points. In particular, some *Mittag-Leffler moments* of the form

$$\sum_{n \geq 0} \frac{a^n m_{2n}(f)}{\Gamma(\alpha n + 1)},$$

with  $a > 0$  and  $\alpha \geq 1$ , are used in [18], while we are led to use some series of the form

$$\sum_{n \geq 0} \frac{a^n m_{2n}(f)}{(n!)^\alpha},$$

with  $a > 0$  and  $\alpha \geq 1$ . This is almost the same thing and in both cases, this is enough to control some exponential moments of the form  $\int_{\mathbb{R}^3} \exp(b|v|^\rho) f(dv)$ , with  $\rho = 2/\alpha$ , see Lemma 11. The main advantage of using such series is that it then suffices to study integer moments  $m_{2n}(f)$ , which leads to much more explicit computations than if using non-integer moments, as is done e.g. in [2], where  $\int_{\mathbb{R}^3} \exp(b|v|^\rho) f(dv)$  is more naturally studied through  $\sum_{n \geq 0} (n!)^{-1} b^n m_{\rho n}(f)$ .

However, we try to really take advantage of the singularity of the cross section to establish a stronger Povzner inequality than in the cutoff case, see Lemma 4 and the paragraph below. We then have to adapt suitably the proof of [2], on the one hand because we can only deal with integer moments, and on the other hand because we have to exploit the new Povzner inequality.

**1.7. About optimality.** Since the solutions to (1) converge to some Maxwell (Gaussian) distributions, which are stationary solutions, we cannot expect to create or propagate exponential

moments of order  $\rho > 2$ . The propagation result thus seems optimal. Concerning creation, one may get convinced, following the proofs of Lemmas 4 and 7, that for some constant  $c > 0$ , setting  $m'_{2n}(f_t) = \frac{d}{dt}m_{2n}(f_t)$ ,

$$\forall n \geq 2, \quad m'_{2n}(f_t) \geq -cn^{\nu/2}m_{2n+\gamma}(f_t).$$

Admitting, *and this is not so clear*, that the Hölder inequality is sharp enough so that we have  $m_{2n+\gamma}(f_t) \simeq [m_{2n}(f_t)]^{1+\gamma/2n}$ , we end with  $m'_{2n}(f_t) \gtrsim -n^{\nu/2}[m_{2n}(f_t)]^{1+\gamma/2n}$ , from which we easily conclude, if  $m_{2n}(0) = \infty$ , that  $m_{2n}(f_t) \gtrsim [n^{1-\nu/2}/t]^{2n/\gamma}$ . Still informally, this should tell us that  $m_{\rho n}(f_t) \gtrsim [n^{1-\nu/2}/t]^{\rho n/\gamma}$ , so that

$$\int_{\mathbb{R}^3} \exp[\sigma|v|^\rho] f_t(dv) = \sum_{n \geq 0} \frac{\sigma^n m_{\rho n}(f_t)}{n!} \gtrsim \sum_{n \geq 0} \frac{\sigma^n n^{\rho(1-\nu/2)n/\gamma}}{t^{\rho n/\gamma} n!}.$$

By Stirling's formula  $n! \sim \sqrt{2\pi n}(n/e)^n$ , this series is divergent, for any value of  $\sigma > 0$ , when  $\rho(1-\nu/2)/\gamma > 1$ , i.e. when  $\rho > 2\gamma/(2-\nu)$ , which is coherent with Theorem 2-(i).

**1.8. About uniqueness.** Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 1)$ . Using Theorem 2 and [10, Theorem 2.2], it seems possible to prove, in a few pages, the well-posedness of (1) assuming that the initial condition satisfies  $\int_{\mathbb{R}^3} \exp(|v|^\delta) f_0(dv) < \infty$  for some  $\delta > 0$ . This is stronger than [10], where we assumed that  $\int_{\mathbb{R}^3} \exp(|v|^\gamma) f_0(dv) < \infty$ , but weaker than the recent result of Heydecker [12], who only assumes that  $m_p(f_0) < \infty$  for some large explicitable  $p$ .

**1.9. Plan.** The paper is technical and we are guided by computations rather than intuition.

In Section 2, we establish a Povzner lemma, which is stronger than what is known in the cutoff case. We handle the whole computation as explicitly as possible, not relying on any previous Povzner estimate, because this is required if we really want to show that the singular part of the cross section accelerates the slowing down of particles.

In Section 3, we derive some differential inequalities for the even integer moment from the Povzner inequality, and we prove some first rough estimates about these moments.

In Section 4, we quickly study how to control exponential moments by even integer moments and *vice-versa*.

Finally, we adapt the proofs of [2] to show Theorem 2-(i) in Section 5 and Theorem 2-(ii) in Section 6. This requires some work, because we can only use integer moments, and because we start from a different Povzner estimate.

**1.10. Notation.** We use the convention that  $\mathbb{N} = \{0, 1, \dots\}$ . For  $a, b \in \mathbb{N}$  with  $a \leq b$ , we set  $\llbracket a, b \rrbracket = \{a, a+1, \dots, b\}$ . In the whole paper,  $(f_t)_{t \geq 0}$  is a given weak solution satisfying  $m_0(f_t) = m_2(f_t) = 1$  and  $\int_{\mathbb{R}^3} v f_t(dv) = 0$  for all  $t \geq 0$ . For  $p \in \mathbb{R}_+$  and  $t \geq 0$ , we set  $m_p(t) = m_p(f_t)$ .

## 2. A NON-CUTOFF POVZNER LEMMA

The goal of this section is to establish the following Povzner inequality.

**Lemma 4.** *Assume  $(H_2(\nu))$  for some  $\nu \in (0, 2)$ . There are some constants  $\lambda_1, \lambda_2 \in (0, \infty)$ , depending only on  $\nu, \kappa_1, \kappa_2$ , such that for all integer  $n \geq 2$ , all  $v, v_* \in \mathbb{R}^3$ ,*

$$\begin{aligned} D_n(v, v_*) &:= \int_0^\pi \int_0^{2\pi} [|v'|^{2n} + |v'_*|^{2n} - |v|^{2n} - |v_*|^{2n}] d\varphi \beta(\theta) d\theta \\ &\leq -\lambda_1 n^{\nu/2} (|v|^{2n} + |v_*|^{2n}) \\ &\quad + \lambda_2 \sum_{a=1}^{n-1} \binom{n}{a} \left( \frac{n^{\nu/2}}{(n-a)^{\nu/2+1}} + \frac{1}{a} \right) [|v|^{2a} |v_*|^{2(n-a)} + |v|^{2(n-a)} |v_*|^{2a}]. \end{aligned}$$

In the case with cutoff, see e.g. [2], one gets (roughly) something like

$$D_n(v, v_*) \leq -(|v|^{2n} + |v_*|^{2n}) + \epsilon_n \sum_{a=1}^{n-1} \binom{n}{a} [|v|^{2a} |v_*|^{2(n-a)} + |v|^{2(n-a)} |v_*|^{2a}],$$

with  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Here the negative term is reinforced by the factor  $n^{\nu/2}$ , and this is the main fact we will have to exploit. We will also have to play tightly with the positive term, showing that despite the fact it is not clearly multiplied by a small factor, it can be absorbed, in some sense, by the negative term.

We start with an explicit computation of the  $\varphi$ -average.

**Lemma 5.** *For any integer  $n \geq 2$ , any  $v, v_* \in \mathbb{R}^3$ , any  $\theta \in (0, \pi]$ , we have*

$$\Theta_n(v, v_*, \theta) := \frac{1}{2\pi} \int_0^{2\pi} |v'|^{2n} d\varphi = \left( \frac{1 + \cos \theta}{2} \right)^n |v|^{2n} + \left( \frac{1 - \cos \theta}{2} \right)^n |v_*|^{2n} + \Lambda_n(v, v_*, \theta),$$

where, setting  $\mathcal{A}_n = \{(i, j, k) \in \mathbb{N}^2 : i + j + k = n, i \leq n-1, j \leq n-1, k \in 2\mathbb{N}\}$ ,

$$\Lambda_n(v, v_*, \theta) = \sum_{(i, j, k) \in \mathcal{A}_n} \frac{n!}{i! j! [(k/2)!]^2} \left( \frac{1 + \cos \theta}{2} \right)^i \left( \frac{1 - \cos \theta}{2} \right)^j \left( \frac{\sin \theta}{2} \right)^k |v|^{2i} |v_*|^{2j} (|v|^2 |v_*|^2 - (v \cdot v_*)^2)^{k/2}.$$

*Proof.* We fix  $n \geq 2$  and divide the proof into 3 steps.

*Step 1.* Recalling from (3) that  $v' = v - \frac{1 - \cos \theta}{2} (v - v_*) + \frac{\sin \theta}{2} \Gamma(v - v_*, \varphi)$ , that  $|\Gamma(v - v_*, \varphi)| = |v - v_*|$  and that  $(v - v_*) \cdot \Gamma(v - v_*, \varphi) = 0$ , we find

$$\begin{aligned} |v'|^2 &= |v|^2 + \left( \frac{1 - \cos \theta}{2} \right)^2 |v - v_*|^2 + \left( \frac{\sin \theta}{2} \right)^2 |v - v_*|^2 - (1 - \cos \theta) v \cdot (v - v_*) + (\sin \theta) v \cdot \Gamma(v - v_*, \varphi) \\ &= \frac{1 + \cos \theta}{2} |v|^2 + \frac{1 - \cos \theta}{2} |v_*|^2 + (\sin \theta) v \cdot \Gamma(v - v_*, \varphi). \end{aligned}$$

Hence, by Newton's trinomial expansion, setting  $\mathcal{B}_n = \{(i, j, k) \in \mathbb{N}^3 : i + j + k = n\}$ ,

$$|v'|^{2n} = \sum_{(i, j, k) \in \mathcal{B}_n} \frac{n!}{i! j! k!} \left( \frac{1 + \cos \theta}{2} \right)^i \left( \frac{1 - \cos \theta}{2} \right)^j (\sin \theta)^k |v|^{2i} |v_*|^{2j} (v \cdot \Gamma(v - v_*, \varphi))^k.$$

*Step 2.* We now prove that for  $k \in \mathbb{N}$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} (v \cdot \Gamma(v - v_*, \varphi))^k d\varphi = \mathbb{1}_{\{k \in 2\mathbb{N}\}} \frac{k!}{2^k [(k/2)!]^2} (|v|^2 |v_*|^2 - (v \cdot v_*)^2)^{k/2}.$$

We have  $v \cdot \Gamma(v - v_*, \varphi) = a \cos \varphi + b \sin \varphi$ , where  $a = v \cdot I(v - v_*)$  and  $b = v \cdot J(v - v_*)$ , whence

$$v \cdot \Gamma(v - v_*, \varphi) = \sqrt{a^2 + b^2} \sin(\varphi + \varphi_0),$$

for  $\varphi_0$  such that  $\frac{a}{\sqrt{a^2+b^2}} = \sin \varphi_0$  and  $\frac{b}{\sqrt{a^2+b^2}} = \cos \varphi_0$ . We thus recognize a Wallis integral:

$$\frac{1}{2\pi} \int_0^{2\pi} (v \cdot \Gamma(v-v_*, \varphi))^k d\varphi = \frac{2}{\pi} (a^2+b^2)^{k/2} \mathbb{1}_{\{k \in 2\mathbb{N}\}} \int_0^{\pi/2} \sin^k \varphi d\varphi = \mathbb{1}_{\{k \in 2\mathbb{N}\}} \frac{k!}{2^k [(k/2)!]^2} (a^2+b^2)^{k/2}.$$

To complete the step, we recall that  $(\frac{v-v_*}{|v-v_*|}, \frac{I(v-v_*)}{|v-v_*|}, \frac{J(v-v_*)}{|v-v_*|})$  is an orthonormal basis, whence

$$|v|^2 = \frac{[v \cdot (v-v_*)]^2 + [v \cdot I(v-v_*)]^2 + [v \cdot J(v-v_*)]^2}{|v-v_*|^2} = \frac{[v \cdot (v-v_*)]^2 + a^2 + b^2}{|v-v_*|^2}$$

and thus  $a^2 + b^2 = |v|^2 |v-v_*|^2 - [v \cdot (v-v_*)]^2 = |v|^2 |v_*|^2 - (v \cdot v_*)^2$ .

*Step 3.* Gathering Steps 1 and 2 and setting  $\mathcal{C}_n = \{(i, j, k) \in \mathbb{N}^3 : i + j + k = n, k \in 2\mathbb{N}\}$ ,

$$\Theta_n(v, v_*, \theta) = \sum_{(i,j,k) \in \mathcal{C}_n} \frac{n!}{i!j![(k/2)!]^2} \left(\frac{1+\cos\theta}{2}\right)^i \left(\frac{1-\cos\theta}{2}\right)^j \left(\frac{\sin\theta}{2}\right)^k |v|^{2i} |v_*|^{2j} \left(|v|^2 |v_*|^2 - (v \cdot v_*)^2\right)^{k/2}.$$

It then suffices to isolate the two extreme terms  $(i, j, k) = (n, 0, 0)$  and  $(i, j, k) = (0, n, 0)$ .  $\square$

We next estimate, sharply, some integrals in  $\theta$ .

**Lemma 6.** *Assume  $(H_2(\nu))$ . There are  $\zeta_1, \zeta_2 \in (0, \infty)$ , depending only on  $\nu, \kappa_1, \kappa_2$ , such that:*

(i) for all integer  $n \geq 2$ ,

$$a_n := \int_0^\pi \left(1 - \left[\frac{1+\cos\theta}{2}\right]^n - \left[\frac{1-\cos\theta}{2}\right]^n\right) \beta(\theta) d\theta \geq \zeta_1 n^{\nu/2};$$

(ii) for all integers  $a, n$  such that  $1 \leq a \leq n-1$ , setting

$$J_{n,a} := \int_0^\pi \left(\frac{1+\cos\theta}{2}\right)^a \left(\frac{1-\cos\theta}{2}\right)^{n-a} \beta(\theta) d\theta, \quad \text{we have} \quad \binom{n}{a} J_{n,a} \leq \zeta_2 \left[\frac{n^{\nu/2}}{(n-a)^{\nu/2+1}} + \frac{1}{a}\right].$$

*Proof.* We start with (i): the integrand in  $a_n$  is nonnegative, because for  $x = (1 + \cos \theta)/2 \in [0, 1]$ , we have  $x^n + (1-x)^n \leq 1$ . Hence recalling  $(H_2(\nu))$ ,

$$a_n \geq \kappa_1 \int_0^{n^{-1/2}} \left(1 - \left[\frac{1+\cos\theta}{2}\right]^n - \left[\frac{1-\cos\theta}{2}\right]^n\right) \theta^{-\nu-1} d\theta.$$

For all  $\theta \in [0, n^{-1/2}]$ , we have  $(1 + \cos \theta)/2 \leq 1 - \theta^2/5$ , whence  $[(1 + \cos \theta)/2]^n \leq (1 - \theta^2/5)^n \leq \exp(-n\theta^2/5)$  and thus  $1 - [(1 + \cos \theta)/2]^n \geq n\theta^2/10$ . Next, still for  $\theta \in [0, n^{-1/2}]$ , we have  $(1 - \cos \theta)/2 \leq \theta^2/4$ , whence, since  $n \geq 2$ ,

$$1 - \left[\frac{1+\cos\theta}{2}\right]^n - \left[\frac{1-\cos\theta}{2}\right]^n \geq \frac{n\theta^2}{10} - \frac{\theta^{2n}}{4^n} \geq \left(\frac{n}{10} - \frac{1}{16}\right)\theta^2 \geq \left(\frac{n}{10} - \frac{n}{32}\right)\theta^2 \geq \frac{n}{20}\theta^2.$$

Consequently,

$$a_n \geq \kappa_1 \frac{n}{20} \int_0^{n^{-1/2}} \theta^{1-\nu} d\theta = \frac{\kappa_1}{20(2-\nu)} n^{\nu/2}.$$

For (ii), we first use  $(H_2(\nu))$  to write  $J_{n,a} \leq \kappa_2 K_{\nu,n,a} + \kappa_2 L_{\nu,n,a}$ , where

$$K_{\nu,n,a} = \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2}\right)^a \left(\frac{1-\cos\theta}{2}\right)^{n-a} \theta^{-\nu-1} d\theta,$$

$$L_{\nu,n,a} = \left(\frac{2}{\pi}\right)^{\nu+1} \int_{\pi/2}^\pi \left(\frac{1+\cos\theta}{2}\right)^a \left(\frac{1-\cos\theta}{2}\right)^{n-a} d\theta.$$

Using the substitution  $\theta \rightarrow \pi - \theta$ , we see that

$$\begin{aligned} L_{\nu,n,a} &= \left(\frac{2}{\pi}\right)^{\nu+1} \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2}\right)^{n-a} \left(\frac{1-\cos\theta}{2}\right)^a d\theta \\ &\leq \left(\frac{2}{\pi}\right)^\nu \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2}\right)^{n-a} \left(\frac{1-\cos\theta}{2}\right)^a \theta^{-1} d\theta = \left(\frac{2}{\pi}\right)^\nu K_{0,n,n-a}. \end{aligned}$$

We will prove that for  $\nu \in [0, 2)$ , there is a constant  $A_\nu \in (0, \infty)$  such that for all  $1 \leq a \leq n-1$ ,

$$(6) \quad \binom{n}{a} K_{\nu,n,a} \leq A_\nu \frac{n^{\nu/2}}{(n-a)^{1+\nu/2}}.$$

We will deduce that  $\binom{n}{a} L_{\nu,n,a} \leq [2/\pi]^\nu \binom{n}{a} K_{0,n,n-a} \leq [2/\pi]^\nu A_0 a^{-1}$  and this will end the proof.

For  $\theta \in (0, \pi/2]$ , we have  $\theta \leq 2 \sin \theta$  and  $\theta^{-1} \leq [(1 - \cos \theta)/2]^{-1/2}$ , so that

$$\theta^{-\nu-1} \leq 2\theta^{-\nu-2} \sin \theta \leq 2 \left(\frac{1-\cos\theta}{2}\right)^{-\nu/2-1} \sin \theta$$

and thus

$$K_{\nu,n,a} \leq 2 \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2}\right)^a \left(\frac{1-\cos\theta}{2}\right)^{n-a-\nu/2-1} \sin \theta d\theta = 4 \int_{1/2}^1 x^a (1-x)^{n-a-\nu/2-1} dx,$$

using the change of variables  $x = (1 + \cos \theta)/2$ . Hence

$$K_{\nu,n,a} \leq 4 \int_0^1 x^a (1-x)^{n-a-\nu/2-1} dx = 4 \frac{\Gamma(a+1)\Gamma(n-a-\nu/2)}{\Gamma(n+1-\nu/2)},$$

where  $\Gamma$  is Euler's Gamma function. Using that  $\Gamma(k+1) = k!$ , that  $(n-a-\nu/2)\Gamma(n-a-\nu/2) = \Gamma(n-a+1-\nu/2)$  and setting  $u_{\nu,k} = \Gamma(k+1)/\Gamma(k+1-\nu/2)$ , we realize that

$$\binom{n}{a} K_{\nu,n,a} \leq 4 \frac{n! \Gamma(n-a-\nu/2)}{\Gamma(n+1-\nu/2)(n-a)!} = 4 \frac{u_{\nu,n}}{(n-a-\nu/2)u_{\nu,n-a}}.$$

But using Stirling's formula  $\Gamma(x+1) \sim \sqrt{2\pi x}(x/e)^x$  as  $x \rightarrow \infty$ , one can verify that  $u_{\nu,k} \sim k^{\nu/2}$  as  $k \rightarrow \infty$  so that there is a constant  $A_\nu \in (1, \infty)$  such that for all  $k \geq 1$ ,

$$A_\nu^{-1} k^{\nu/2} \leq u_{\nu,k} \leq A_\nu k^{\nu/2}.$$

We conclude that for all  $1 \leq a \leq n-1$ ,

$$\binom{n}{a} K_{\nu,n,a} \leq 4A_\nu^2 \frac{n^{\nu/2}}{(n-a)^{\nu/2}(n-a-\nu/2)} \leq \frac{4A_\nu^2}{1-\nu/2} \times \frac{n^{\nu/2}}{(n-a)^{\nu/2+1}}$$

because  $n-a \geq 1$  implies that  $n-a-\nu/2 \geq (n-a)(1-\nu/2)$ . We have checked (6) and the proof is complete.  $\square$

We can now handle the

*Proof of Lemma 4.* We fix  $n \geq 2$ . Using (3), we realize that, with the notation of Lemma 5,

$$\frac{1}{2\pi} \int_0^{2\pi} [|v'|^{2n} + |v_*'|^{2n} - |v|^{2n} - |v_*|^{2n}] d\varphi = \Theta_n(v, v_*, \theta) + \Theta_n(v_*, v, -\theta) - |v|^{2n} - |v_*|^{2n}.$$



We deduce from Lemma 5 that

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} [|v'|^{2n} + |v'_*|^{2n} - |v|^{2n} - |v_*|^{2n}] d\varphi \\ &= - \left( 1 - \left[ \frac{1 + \cos \theta}{2} \right]^n - \left[ \frac{1 - \cos \theta}{2} \right]^n \right) \times (|v|^{2n} + |v_*|^{2n}) + \Lambda_n(v, v_*, \theta) + \Lambda_n(v_*, v, -\theta). \end{aligned}$$

Hence  $D_n(v, v_*) = -D_{n,1}(v, v_*) + D_{n,2}(v, v_*)$ , where

$$\begin{aligned} D_{n,1}(v, v_*) &= 2\pi \int_0^\pi \left( 1 - \left[ \frac{1 + \cos \theta}{2} \right]^n - \left[ \frac{1 - \cos \theta}{2} \right]^n \right) \beta(\theta) d\theta \times (|v|^{2n} + |v_*|^{2n}), \\ D_{n,2}(v, v_*) &= 2\pi \int_0^\pi [\Lambda_n(v, v_*, \theta) + \Lambda_n(v_*, v, -\theta)] \beta(\theta) d\theta. \end{aligned}$$

We now divide the proof into 5 steps.

*Step 1.* By Lemma 6-(i), we have  $D_{n,1}(v, v_*) \geq 2\pi \zeta_1 n^{\nu/2} (|v|^{2n} + |v_*|^{2n})$ .

*Step 2.* We next roughly bound  $|v|^2 |v_*|^2 - (v \cdot v_*)^2$  by  $|v|^2 |v_*|^2$  in the expression of  $\Lambda_n$  to find

$$D_{n,2}(v, v_*) \leq 2\pi \sum_{(i,j,k) \in \mathcal{A}_n} \frac{n!}{i!j![(k/2)!]^2} I_{i,j,k} [|v|^{2i+k} |v_*|^{2j+k} + |v|^{2j+k} |v_*|^{2i+k}],$$

where

$$I_{i,j,k} = \int_0^\pi \left( \frac{1 + \cos \theta}{2} \right)^i \left( \frac{1 - \cos \theta}{2} \right)^j \left( \frac{\sin \theta}{2} \right)^k \beta(\theta) d\theta.$$

This can be rewritten as

$$D_{n,2}(v, v_*) \leq 2\pi \sum_{a=0}^n K_{n,a} [|v|^{2a} |v_*|^{2(n-a)} + |v|^{2(n-a)} |v_*|^{2a}]$$

where, setting  $\mathcal{A}_{n,a} = \{(i, j, k) \in \mathcal{A}_n : i + k/2 = a \text{ (whence } j + k/2 = n - a)\}$ ,

$$K_{n,a} = \sum_{(i,j,k) \in \mathcal{A}_{n,a}} \frac{n!}{i!j![(k/2)!]^2} I_{i,j,k}.$$

*Step 3.* We have  $K_{n,n} = K_{n,0} = 0$ . Indeed, we e.g. have  $\mathcal{A}_{n,n} = \emptyset$  because for  $(i, j, k) \in \mathcal{A}_{n,n}$ , we have  $i + j + k = n$  and  $j + k/2 = 0$ , whence  $i = n$ , which is forbidden since  $(i, j, k) \in \mathcal{A}_n$ .

*Step 4.* We now fix  $a \in \llbracket 1, n-1 \rrbracket$ . Then  $(i, j, k) \in \mathcal{A}_{n,a}$  if and only if there is  $\ell \in \mathbb{N}$  such that  $k = 2\ell \in \llbracket 0, n \rrbracket$ ,  $i = a - \ell \in \llbracket 0, n-1 \rrbracket$  and  $j = (n-a) - \ell \in \llbracket 0, n-1 \rrbracket$ , so that

$$K_{n,a} = \sum_{\ell=0}^{a \wedge (n-a)} \frac{n!}{(a-\ell)!(n-a-\ell)!(\ell!)^2} I_{a-\ell, n-a-\ell, 2\ell}.$$

But for any  $\ell \in \llbracket 0, a \wedge (n-a) \rrbracket$ , it holds that

$$I_{a-\ell, n-a-\ell, 2\ell} = \int_0^\pi \left( \frac{1 + \cos \theta}{2} \right)^{a-\ell} \left( \frac{1 - \cos \theta}{2} \right)^{n-a-\ell} \left( \frac{\sin \theta}{2} \right)^{2\ell} \beta(\theta) d\theta = J_{n,a},$$

where

$$J_{n,a} = \int_0^\pi \left( \frac{1 + \cos \theta}{2} \right)^a \left( \frac{1 - \cos \theta}{2} \right)^{n-a} \beta(\theta) d\theta,$$

because  $(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$ . Thus

$$K_{n,a} = J_{n,a} \sum_{\ell=0}^{a \wedge (n-a)} \frac{n!}{(a-\ell)!(n-a-\ell)!(\ell!)^2} = J_{n,a} \binom{n}{a} \sum_{\ell=0}^{a \wedge (n-a)} \binom{a}{\ell} \binom{n-a}{\ell} = J_{n,a} \left[ \binom{n}{a} \right]^2.$$

We finally used the well-known identity  $\sum_{\ell=0}^{a \wedge b} \binom{a}{\ell} \binom{b}{\ell} = \binom{a+b}{a}$ , which can be shown noting that  $\binom{a+b}{a}$  is the coefficient in front of  $X^a$  of  $(1+X)^{a+b}$ , while  $\sum_{\ell=0}^{a \wedge b} \binom{a}{\ell} \binom{b}{\ell} = \sum_{\ell=0}^{a \wedge b} \binom{a}{\ell} \binom{b}{b-\ell}$  is the coefficient in front of  $X^a$  of  $(1+X)^a(1+X)^b$ .

*Step 5.* Gathering Steps 2-3-4, we have checked that

$$D_{n,2}(v, v_*) \leq 2\pi \sum_{a=1}^{n-1} J_{n,a} \left[ \binom{n}{a} \right]^2 [|v|^{2a} |v_*|^{2(n-a)} + |v|^{2(n-a)} |v_*|^{2a}].$$

The conclusion then follows from Lemma 6-(ii), from which  $\binom{n}{a} J_{n,a} \leq \zeta_2 \left[ \frac{n^{\nu/2}}{(n-a)^{\nu/2+1}} + \frac{1}{a} \right]$ .  $\square$

### 3. EVEN INTEGER MOMENTS

Using the previous Povzner inequality, we can derive the even integer moments.

**Lemma 7.** *Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . For any integer  $n \geq 2$ , any  $t > 0$ ,*

$$m'_{2n}(t) \leq -c_1 n^{\nu/2} m_{2n+\gamma}(t) + c_2 S_n(t) + 2c_1 n^{\nu/2} 2^{2n/\gamma},$$

where  $c_1 = \lambda_1$  and  $c_2 = 16\lambda_2$  (see Lemma 4) depend only on  $\nu, \kappa_1, \kappa_2$  and where

$$S_n(t) = \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n}{a} \frac{n^{\nu/2}}{a^{\nu/2+1}} m_{2a}(t) m_{2(n-a)+\gamma}(t).$$

*Proof.* We fix  $n \geq 2$  and use the weak formulation (4) with  $\phi(v) = |v|^{2n}$ , which is licit thanks to (5), to find, for any  $t > 0$  and with the notation of Lemma 4,

$$m'_{2n}(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} D_n(v, v_*) |v - v_*|^\gamma f_t(dv_*) f_t(dv).$$

Hence using Lemma 4,  $m'_{2n}(t) \leq -A_n(t) + B_n(t)$ , where

$$\begin{aligned} A_n(t) &= \lambda_1 n^{\nu/2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|v|^{2n} + |v_*|^{2n}) |v - v_*|^\gamma f_t(dv_*) f_t(dv), \\ B_n(t) &= \lambda_2 \sum_{a=1}^{n-1} \binom{n}{a} \left( \frac{n^{\nu/2}}{(n-a)^{\nu/2+1}} + \frac{1}{a} \right) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} [|v|^{2a} |v_*|^{2(n-a)} + |v|^{2(n-a)} |v_*|^{2a}] \\ &\quad |v - v_*|^\gamma f_t(dv_*) f_t(dv). \end{aligned}$$

We now divide the proof into 2 steps.

*Step 1.* Using a symmetry argument and that  $|v - v_*|^\gamma \geq |v|^\gamma - |v_*|^\gamma$ , we see that

$$A_n(t) = 2\lambda_1 n^{\nu/2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^{2n} |v - v_*|^\gamma f_t(dv_*) f_t(dv) \geq 2\lambda_1 n^{\nu/2} (m_{2n+\gamma}(t) - m_{2n}(t) m_\gamma(t)).$$

By Hölder's inequality,  $m_\gamma(t) \leq [m_2(t)]^{\gamma/2} = 1$ . Using moreover that  $x^{2n} \leq \frac{1}{2} x^{2n+\gamma} + 2^{2n/\gamma}$  for all  $x \geq 0$  (separate the cases  $x \leq 2^{1/\gamma}$  and  $x \geq 2^{1/\gamma}$ ),

$$m_\gamma(t) m_{2n}(t) \leq m_{2n}(t) = \int_{\mathbb{R}^3} |v|^{2n} f_t(dv) \leq \int_{\mathbb{R}^3} \left( \frac{1}{2} |v|^{2n+\gamma} + 2^{2n/\gamma} \right) f_t(dv) = \frac{1}{2} m_{2n+\gamma}(t) + 2^{2n/\gamma}.$$

All in all,

$$A_n(t) \geq \lambda_1 n^{\nu/2} m_{2n+\gamma}(t) - 2\lambda_1 n^{\nu/2} 2^{2n/\gamma}.$$

*Step 2.* Using a symmetry argument and that  $|v - v_*|^\gamma \leq |v|^\gamma + |v_*|^\gamma$ , we find

$$\begin{aligned} B_n(t) &\leq 2\lambda_2 \sum_{a=1}^{n-1} \binom{n}{a} \left( \frac{n^{\nu/2}}{(n-a)^{\nu/2+1}} + \frac{1}{a} \right) [m_{2a+\gamma}(t)m_{2(n-a)}(t) + m_{2a}(t)m_{2(n-a)+\gamma}(t)] \\ &= 2\lambda_2 \sum_{a=1}^{n-1} \binom{n}{a} \left( \frac{n^{\nu/2}}{a^{\nu/2+1}} + \frac{1}{a} \right) [m_{2a+\gamma}(t)m_{2(n-a)}(t) + m_{2a}(t)m_{2(n-a)+\gamma}(t)] \end{aligned}$$

by symmetry again. Since now  $1/a \leq n^{\nu/2}/a^{\nu/2+1}$  and since  $a \rightarrow n^{\nu/2}/a^{\nu/2+1}$  is decreasing,

$$B_n(t) \leq 8\lambda_2 \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n}{a} \frac{n^{\nu/2}}{a^{\nu/2+1}} [m_{2a+\gamma}(t)m_{2(n-a)}(t) + m_{2a}(t)m_{2(n-a)+\gamma}(t)].$$

But for  $a \in \llbracket 1, \lfloor n/2 \rfloor \rrbracket$ , we have  $a \leq n-a$  whence  $m_{2a+\gamma}(t)m_{2(n-a)}(t) \leq m_{2a}(t)m_{2(n-a)+\gamma}(t)$  by Lemma 8 below, so that

$$B_n(t) \leq 16\lambda_2 \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n}{a} \frac{n^{\nu/2}}{a^{\nu/2+1}} m_{2a}(t)m_{2(n-a)+\gamma}(t) = 16\lambda_2 S_n(t).$$

Recalling that  $m'_{2n}(t) = -A_n(t) + B_n(t)$  and Step 1, we have reached our goal.  $\square$

We now prove a classical lemma that we used a few lines above.

**Lemma 8.** *For any  $b \geq a \geq 0$ , any  $\alpha > 0$ , any  $f \in \mathcal{P}(\mathbb{R}^3)$ ,*

$$m_{a+\alpha}(f)m_b(f) \leq m_{b+\alpha}(f)m_a(f).$$

*Proof.* We fix  $\alpha > 0$  and  $f \in \mathcal{P}(\mathbb{R}^3)$  and have to prove that the function  $u(a) = m_{a+\alpha}(f)/m_a(f)$  is nondecreasing on  $\mathbb{R}_+$ . Observing that  $\frac{d}{da} m_a(f) = \int_{\mathbb{R}^3} (\log |v|) |v|^a f(dv)$ , we find

$$u'(a) = \frac{1}{m_a(f)} \int_{\mathbb{R}^3} (\log |v|) |v|^{a+\alpha} f(dv) - \frac{1}{m_a^2(f)} \left( \int_{\mathbb{R}^3} (\log |v|) |v|^a f(dv) \right) \left( \int_{\mathbb{R}^3} |v|^{a+\alpha} f(dv) \right).$$

Setting  $g_a(dv) = |v|^a f(dv)/m_a(f)$ , which is a probability measure,

$$u'(a) = \frac{1}{\alpha} \int_{\mathbb{R}^3} (\log |v|^\alpha) |v|^\alpha g_a(dv) - \frac{1}{\alpha} \left( \int_{\mathbb{R}^3} (\log |v|^\alpha) g_a(dv) \right) \left( \int_{\mathbb{R}^3} |v|^\alpha g_a(dv) \right).$$

Hence  $u'(a) \geq 0$  by the Jensen inequality.  $\square$

We will also use the following remark.

**Remark 9.** *For  $f \in \mathcal{P}(\mathbb{R}^3)$  such that  $m_2(f) = 1$ , for all  $r \geq s \geq 2$ , by Hölder's inequality,*

$$m_s(f) = \int_{\mathbb{R}^3} |v|^{s-2} |v|^2 f(dv) \leq \left( \int_{\mathbb{R}^3} |v|^{r-2} |v|^2 f(dv) \right)^{(s-2)/(r-2)} = [m_r(f)]^{(s-2)/(r-2)}.$$

We finally quickly prove some moment estimates that are more or less well-known.

**Lemma 10.** *Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ .*

(i) *For all  $r > 0$ , there is  $K_r \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, r$ , such that for all  $t > 0$ ,*

$$m_r(t) \leq K_r (1 + t^{-(r-2)/\gamma}).$$

(ii) For all integer  $n \geq 2$ , all  $A \geq 1$ , there is  $K_{2n,A} \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, n, A$ , such that

$$m_{2n}(0) \leq A \implies \sup_{t \geq 0} m_{2n}(t) \leq K_{2n,A}.$$

*Proof.* We first prove that for any fixed integer  $n \geq 2$ , there is a constant  $A_n$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, n$ , allowed to vary from line to line, such that for all  $t > 0$ ,

$$(7) \quad m'_{2n}(t) \leq -\frac{c_1}{2} [m_{2n}(t)]^{1+\gamma/(2n-2)} + A_n.$$

Using Lemma 7, a rough upperbound, and then Lemma 8, we get

$$m'_{2n}(t) \leq -c_1 m_{2n+\gamma}(t) + A_n \sum_{a=1}^{\lfloor n/2 \rfloor} m_{2a}(t) m_{2(n-a)+\gamma}(t) + A_n \leq -c_1 m_{2n+\gamma}(t) + A_n m_{2(n-1)+\gamma} m_2(t) + A_n.$$

Using now Remark 9 and that  $m_2(t) = 1$ , we find

$$m'_{2n}(t) \leq -c_1 [m_{2n}(t)]^{1+\gamma/(2n-2)} + A_n [m_{2n}(t)]^{1-(2-\gamma)/(2n-2)} + A_n,$$

from which (7) follows.

Point (ii) clearly follows from (7). When  $r = 2n \geq 4$  is an even integer, point (i) also follows from (7). For a general  $r > 2$ , we consider an integer  $n_r \geq 2$  such that  $2n_r \geq r$  and we conclude from Remark 9 that

$$m_r(t) \leq [m_{2n_r}(t)]^{(r-2)/(2n_r-2)} \leq (K_{2n_r} (1 + t^{-(2n_r-2)/\gamma}))^{(r-2)/(2n_r-2)},$$

whence the result. Finally, for  $r \in (0, 2]$ , we obviously have  $m_r(t) \leq [m_2(t)]^{r/2} \leq 1$ .  $\square$

#### 4. CONTROL OF EXPONENTIAL MOMENTS BY EVEN INTEGER MOMENTS

We explain how to control exponential moments from even integer moments and *vice versa*.

**Lemma 11.** (i) For  $f \in \mathcal{P}(\mathbb{R}^3)$ , for  $\sigma_0 \in (0, \infty)$ ,  $\alpha \in [1, \infty)$  and  $K \in [1, \infty)$ ,

$$\sup_{n \geq 0} \frac{\sigma_0^n m_{2n}(f)}{(n!)^\alpha} \leq K \implies \int_{\mathbb{R}^3} \exp(\sigma_0^{1/\alpha} |v|^{2/\alpha}/2) f(dv) \leq 2K^{1/\alpha}.$$

(ii) For  $\rho \in (0, 2]$ ,  $\sigma_0 \in (0, 1]$  and  $K \in (1, \infty)$ , there is  $\sigma_1 \in (0, \infty)$ , depending only on  $\rho, \sigma_0, K$ , such that for all  $f \in \mathcal{P}(\mathbb{R}^3)$ ,

$$\int_{\mathbb{R}^3} \exp(\sigma_0 |v|^\rho) f(dv) \leq K \implies \sup_{n \geq 0} \frac{\sigma_1^n m_{2n}(f)}{(n!)^{2/\rho}} \leq 1.$$

*Proof.* We start with (i). By Hölder's inequality, since  $\alpha \geq 1$ ,

$$m_{2n/\alpha}(f) \leq [m_{2n}(f)]^{1/\alpha} \leq K^{1/\alpha} \sigma_0^{-n/\alpha} n!,$$

whence

$$\int_{\mathbb{R}^3} \exp[\sigma_0^{1/\alpha} |v|^{2/\alpha}/2] f(dv) = \sum_{n \geq 0} \frac{\sigma_0^{n/\alpha} m_{2n/\alpha}(f)}{2^n n!} \leq K^{1/\alpha} \sum_{n \geq 0} 2^{-n} = 2K^{1/\alpha}.$$

We now check (ii). We know that

$$\sup_{n \geq 0} \frac{\sigma_0^n m_{\rho n}(f)}{n!} \leq \sum_{n \geq 0} \frac{\sigma_0^n m_{\rho n}(f)}{n!} = \int_{\mathbb{R}^3} \exp(\sigma_0 |v|^\rho) f(dv) \leq K.$$

For  $n \geq 1$ , we set  $k_n = \lceil 2n/\rho \rceil \in [2n/\rho, 2n/\rho + 1)$  and write, using that  $\rho k_n \geq 2n$ ,

$$m_{2n}(f) \leq 1 + m_{\rho k_n}(f) \leq 1 + \frac{K k_n!}{\sigma_0^{k_n}}, \quad \text{so that} \quad \frac{\sigma_1^n m_{2n}(f)}{(n!)^{2/\rho}} \leq \frac{\sigma_1^n}{(n!)^{2/\rho}} + \frac{\sigma_1^n K k_n!}{\sigma_0^{k_n} (n!)^{2/\rho}}.$$

By Stirling's formula  $n! \sim \sqrt{2\pi n}(n/e)^n$  and since  $k_n = \lceil 2n/\rho \rceil$ , we find that, for some constant  $A \in (0, \infty)$ , depending only on  $\rho$  and allowed to vary, for all  $n \geq 1$ ,

$$\frac{k_n!}{(n!)^{2/\rho}} \leq A \frac{n^{1/2} [(2n/\rho + 1)/e]^{2n/\rho + 1}}{n^{1/\rho} [n/e]^{2n/\rho}} \leq A n^{3/2 - 1/\rho} \frac{[(3n)/(e\rho)]^{2n/\rho}}{[n/e]^{2n/\rho}} \leq A n^{3/2} \left(\frac{3}{\rho}\right)^{2n/\rho} \leq A \left(\frac{4}{\rho}\right)^{2n/\rho}.$$

Observing next that  $\sigma_0^{k_n} \geq \sigma_0^{2n/\rho + 1}$  (since  $\sigma_0 \in (0, 1]$ ), we end with

$$\text{for all } n \geq 1, \quad \frac{\sigma_1^n m_{2n}(f)}{(n!)^{2/\rho}} \leq \frac{\sigma_1^n}{(n!)^{2/\rho}} + \frac{KA}{\sigma_0} \frac{\sigma_1^n}{\sigma_0^{2n/\rho}} \left(\frac{4}{\rho}\right)^{2n/\rho}.$$

This last quantity is bounded by 1 if  $\sigma_1 > 0$  is small enough (depending on  $\rho$ ,  $K$ , and  $\sigma_0$ ).  $\square$

## 5. CREATION OF EXPONENTIAL MOMENTS

The following estimate will allow us to prove the creation result by Lemma 11.

**Lemma 12.** *Assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$  and set  $\alpha = \max\{1, (2 - \nu)/\gamma\}$ . There are  $\sigma \in (0, 1]$  and  $T > 0$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that*

$$\sup_{t \in [0, T]} \sum_{n=0}^{\infty} \frac{(\sigma t)^{2n/\gamma} m_{2n}(t)}{(n!)^\alpha} \leq 2.$$

*Proof.* We recall that by Lemma 7, for any integer  $n \geq 2$ , any  $t > 0$ ,

$$(8) \quad m'_{2n}(t) \leq -c_1 n^{\nu/2} m_{2n+\gamma}(t) + c_2 S_n(t) + 2c_1 n^{\nu/2} 2^{2n/\gamma},$$

where  $S_n(t) = \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n}{a} \frac{n^{\nu/2}}{a^{\nu/2+1}} m_{2a}(t) m_{2(n-a)+\gamma}(t)$ .

*Step 1.* We introduce, for  $\sigma \in (0, 1]$  to be chosen later, for  $p \geq 2$  and  $t \geq 0$ ,

$$E_p(t) = \sum_{n=0}^p \frac{(\sigma t)^{2n/\gamma} m_{2n}(t)}{(n!)^\alpha}.$$

By Lemma 10-(i) and since  $m_0(t) = 1$ , it holds that for some constant  $C_p \in (0, \infty)$ ,

$$1 \leq E_p(t) \leq 1 + C_p \sum_{n=1}^p t^{2n/\gamma} m_{2n}(t) \leq 1 + C_p \sum_{n=1}^p t^{2n/\gamma} [1 + t^{-(2n-2)/\gamma}] = 1 + C_p \sum_{n=1}^p [t^{2n/\gamma} + t^{2/\gamma}],$$

whence  $\lim_{t \rightarrow 0} E_p(t) = 1$ .

*Step 2.* Since  $m'_0(t) = m'_2(t) = 0$ , we deduce from (8) that for all  $p \geq 2$ , all  $t \in [0, 1]$ ,

$$E'_p(t) \leq -c_1 F_p(t) + c_2 G_p(t) + \frac{2\sigma}{\gamma} H_p(t) + C,$$

where

$$F_p(t) = \sum_{n=2}^p n^{\nu/2} \frac{(\sigma t)^{2n/\gamma} m_{2n+\gamma}(t)}{(n!)^\alpha}, \quad G_p(t) = \sum_{n=2}^p \frac{(\sigma t)^{2n/\gamma} S_n(t)}{(n!)^\alpha}, \quad H_p(t) = \sum_{n=1}^p \frac{n(\sigma t)^{2n/\gamma - 1} m_{2n}(t)}{(n!)^\alpha}$$

and where, since  $\sigma t \leq 1$ ,

$$C = 2c_1 \sum_{n=2}^{\infty} \frac{n^{\nu/2} 2^{2n/\gamma}}{(n!)^\alpha} < \infty.$$

*Step 3.* We first prove that for all  $\epsilon \in (0, \infty)$ , there is  $A_\epsilon \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, \epsilon$ , such that for any choice of  $\sigma \in (0, 1]$ , for all  $p \geq 2$ , all  $t \in [0, 1]$ ,

$$G_p(t) \leq \epsilon E_p(t) F_p(t) + \sigma^{2/\gamma} A_\epsilon (F_p(t) + 1).$$

We start from

$$G_p(t) = \sum_{n=2}^p \sum_{a=1}^{\lfloor n/2 \rfloor} \frac{(\sigma t)^{2n/\gamma}}{(n!)^\alpha} \frac{n^{\nu/2}}{a^{\nu/2+1}} \binom{n}{a} m_{2a}(t) m_{2(n-a)+\gamma}(t).$$

Since  $\alpha \geq 1$  and since  $a \leq \lfloor n/2 \rfloor$  implies  $n \leq 2(n-a)$  and thus  $n^{\nu/2} \leq 2^{\nu/2} (n-a)^{\nu/2}$ ,

$$\begin{aligned} G_p(t) &\leq 2^{\nu/2} \sum_{n=2}^p \sum_{a=1}^{\lfloor n/2 \rfloor} \frac{(\sigma t)^{2n/\gamma}}{(n!)^\alpha} \frac{(n-a)^{\nu/2}}{a^{\nu/2+1}} \left[ \binom{n}{a} \right]^\alpha m_{2a}(t) m_{2(n-a)+\gamma}(t) \\ &= 2^{\nu/2} \sum_{n=2}^p \sum_{a=1}^{\lfloor n/2 \rfloor} (n-a)^{\nu/2} \frac{(\sigma t)^{2(n-a)/\gamma} m_{2(n-a)+\gamma}(t)}{((n-a)!)^\alpha} \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha a^{\nu/2+1}} \\ &= 2^{\nu/2} \sum_{a=1}^{\lfloor p/2 \rfloor} \sum_{n=2a}^p (n-a)^{\nu/2} \frac{(\sigma t)^{2(n-a)/\gamma} m_{2(n-a)+\gamma}(t)}{((n-a)!)^\alpha} \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha a^{\nu/2+1}}. \end{aligned}$$

Using the change of indices  $n \rightarrow \ell = n - a$

$$\begin{aligned} G_p(t) &\leq 2^{\nu/2} \sum_{a=1}^{\lfloor p/2 \rfloor} \sum_{\ell=a}^{p-a} \ell^{\nu/2} \frac{(\sigma t)^{2\ell/\gamma} m_{2\ell+\gamma}(t)}{(\ell!)^\alpha} \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha a^{\nu/2+1}} \\ &\leq \left( \sum_{\ell=1}^p \ell^{\nu/2} \frac{(\sigma t)^{2\ell/\gamma} m_{2\ell+\gamma}(t)}{(\ell!)^\alpha} \right) \times 2^{\nu/2} \left( \sum_{a=1}^p \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha a^{\nu/2+1}} \right) \\ &= \left( F_p(t) + (\sigma t)^{2/\gamma} m_{2+\gamma}(t) \right) I_p(t), \end{aligned}$$

where

$$I_p(t) = 2^{\nu/2} \sum_{a=1}^p \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha a^{\nu/2+1}}.$$

But setting  $N_\epsilon = \lceil 2^{\nu/2} \epsilon^{-2/(\nu+2)} \rceil$ , it holds that  $a \geq N_\epsilon$  implies  $2^{\nu/2} a^{-\nu/2-1} \leq \epsilon$ , whence

$$I_p(t) \leq \epsilon E_p(t) + J_\epsilon(t), \quad \text{where} \quad J_\epsilon(t) = 2^{\nu/2} \sum_{a=1}^{N_\epsilon} \frac{(\sigma t)^{2a/\gamma} m_{2a}(t)}{(a!)^\alpha}.$$

By Lemma 10-(i), we see that for some constants  $A, A_\epsilon \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, \epsilon$  and allowed to vary, for any choice of  $\sigma \in (0, 1]$ , for all  $t \in [0, 1]$ ,

$$(\sigma t)^{2/\gamma} m_{2+\gamma}(t) \leq A (\sigma t)^{2/\gamma} (1+t^{-1}) \leq A \sigma^{2/\gamma} \quad \text{and} \quad J_\epsilon(t) \leq A_\epsilon \sum_{a=1}^{N_\epsilon} (\sigma t)^{2a/\gamma} [1+t^{-(2a-2)/\gamma}] \leq A_\epsilon \sigma^{2/\gamma}.$$

All in all, we have proved that for any choice of  $\sigma \in (0, 1]$ , for all  $t \in [0, 1]$ ,

$$G_p(t) \leq (F_p(t) + A \sigma^{2/\gamma}) (\epsilon E_p(t) + A_\epsilon \sigma^{2/\gamma}) \leq \epsilon E_p(t) F_p(t) + \sigma^{2/\gamma} [\epsilon A E_p(t) + A_\epsilon F_p(t) + A A_\epsilon].$$

Since  $E_p(t) \leq m_0(t) + \sigma t m_2(t) + F_p(t) \leq 2 + F_p(t)$ , we conclude that, as desired,

$$G_p(t) \leq \epsilon E_p(t) F_p(t) + \sigma^{2/\gamma} [(\epsilon A + A_\epsilon) F_p(t) + A A_\epsilon + 2\epsilon A].$$

*Step 4.* We now verify that there are some constants  $\kappa, B \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that for any choice of  $\sigma \in (0, 1]$ , for all  $p \geq 2$ , all  $t \in [0, 1]$ ,

$$H_p(t) \leq \kappa F_p(t) + B.$$

We first observe that, for  $\kappa > 0$  to be chosen later,

$$\frac{nm_{2n}(t)}{\sigma t} \leq \kappa n^{\nu/2} [m_{2n}(t)]^{1+\gamma/(2n-2)} + \frac{n}{\sigma t} \left( \frac{n^{1-\nu/2}}{\kappa \sigma t} \right)^{(2n-2)/\gamma}.$$

Indeed,  $nm_{2n}(t)/(\sigma t)$  is bounded by the second term if  $m_{2n}(t) \leq [n^{1-\nu/2}/(\kappa \sigma t)]^{(2n-2)/\gamma}$  and by the first term else. Since  $[m_{2n}(t)]^{1+\gamma/(2n-2)} \leq m_{2n+\gamma}(t)$  by Remark 9, we conclude that

$$\begin{aligned} H_p(t) &= \sum_{n=1}^p \frac{nm_{2n}(t)}{\sigma t} \frac{(\sigma t)^{2n/\gamma}}{(n!)^\alpha} \\ &\leq \kappa \sum_{n=1}^p n^{\nu/2} \frac{(\sigma t)^{2n/\gamma} m_{2n+\gamma}(t)}{(n!)^\alpha} + \sum_{n=1}^p \frac{n}{\sigma t} \left( \frac{n^{1-\nu/2}}{\kappa \sigma t} \right)^{(2n-2)/\gamma} \frac{(\sigma t)^{2n/\gamma}}{(n!)^\alpha} \\ &= \kappa F_p(t) + \kappa (\sigma t)^{2/\gamma} m_{2+\gamma}(t) + (\sigma t)^{2/\gamma-1} \sum_{n=1}^p \frac{n^{(2-\nu)n/\gamma-(2-\nu)/\gamma+1}}{(n!)^\alpha \kappa^{(2n-2)/\gamma}}. \end{aligned}$$

But by Lemma 10-(i), there is  $A \in (0, \infty)$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that for all  $t \in [0, 1]$ ,

$$t^{2/\gamma} m_{2+\gamma}(t) \leq A t^{2/\gamma} [1 + t^{-1}] \leq 2A.$$

Moreover, since  $\alpha = \max\{1, (2-\nu)/\gamma\} \geq (2-\nu)/\gamma$ , the series

$$S := \sum_{n=1}^{\infty} \frac{n^{(2-\nu)n/\gamma-(2-\nu)/\gamma+1}}{(n!)^\alpha \kappa^{(2n-2)/\gamma}} \leq \kappa^{2/\gamma} \sum_{n=1}^{\infty} \frac{n^{\alpha n+1}}{(n!)^\alpha \kappa^{2n/\gamma}}$$

is convergent if  $\kappa = 2e^{\alpha\gamma/2}$ , by Stirling's formula  $n! \sim \sqrt{2\pi n}(n/e)^n$ . Hence for any choice of  $\sigma \in (0, 1]$ , for all  $t \in [0, 1]$ ,

$$H_p(t) \leq \kappa F_p(t) + 2\kappa A + S.$$

*Step 5.* By Steps 2-3-4, for any choice of  $\epsilon \in (0, \infty)$  and  $\sigma \in (0, 1]$ , for all  $p \geq 2$ , all  $t \in (0, 1]$ ,

$$\begin{aligned} E_p'(t) &\leq -c_1 F_p(t) + c_2 G_p(t) + \frac{2\sigma}{\gamma} H_p(t) + C \\ &\leq -c_1 F_p(t) + c_2 [\epsilon E_p(t) F_p(t) + \sigma^{2/\gamma} A_\epsilon (F_p(t) + 1)] + \frac{2\sigma}{\gamma} [\kappa F_p(t) + B] + C. \end{aligned}$$

Choosing  $\epsilon = c_1/(4c_2)$  and  $\sigma \in (0, 1]$  small enough so that  $c_2 \sigma^{2/\gamma} A_\epsilon + 2\kappa \sigma/\gamma \leq c_1/2$ , we find

$$E_p'(t) \leq -\frac{c_1}{2} F_p(t) + \frac{c_1}{4} E_p(t) F_p(t) + D,$$

for some constant  $D \in (0, \infty)$ . By Step 1, we know that  $E_p(0) = \lim_{t \rightarrow 0} E_p(t) = 1$ , whence

$$T_p = \sup\{t > 0 : E_p(t) \leq 2\} > 0.$$

For  $p \geq 2$  and  $t \in [0, T_p \wedge 1)$ , we have  $E_p'(t) \leq D$ . Recalling that  $E_p(0) = 1$ , we deduce that  $T_p \geq \min\{1, 1/D\} =: T$ . In other words, with our choice of  $\sigma$ , all  $t \in [0, T]$ ,  $\sup_{p \geq 2} E_p(t) \leq 2$ .  $\square$

We can finally give the

*Proof of Theorem 2-(i).* We assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . We fix  $\rho = \min\{2\gamma/(2-\nu), 2\}$  and observe that  $\alpha := 2/\rho = \max\{1, (2-\nu)/\gamma\}$ . Thus by Lemma 12, there are  $\sigma \in (0, 1]$  and  $T > 0$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2$ , such that

$$\sup_{[0, T]} \sum_{n=0}^{\infty} \frac{(\sigma t)^{2n/\gamma} m_{2n}(t)}{(n!)^\alpha} \leq 2.$$

We deduce from Lemma 11-(i) (with  $\sigma_0 = (\sigma t)^{2/\gamma}$ ) that

$$\sup_{[0, T]} \int_{\mathbb{R}^3} \exp[(\sigma t)^{\rho/\gamma} |v|^\rho / 2] f_t(dv) = \sup_{[0, T]} \int_{\mathbb{R}^3} \exp[(\sigma t)^{2/(\gamma\alpha)} |v|^{2/\alpha} / 2] f_t(dv) \leq 2^{1+1/\alpha} \leq 4.$$

as desired.  $\square$

## 6. PROPAGATION OF EXPONENTIAL MOMENTS

We proceed as in the previous section.

**Lemma 13.** *Assume  $(H_1(\gamma))$  and  $(H_2(\gamma))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . For any  $\sigma_0 \in (0, \infty)$  and any  $\alpha \geq 1$ , there is  $\sigma \in (0, \sigma_0]$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, \sigma_0$ , such that*

$$\sup_{n \geq 0} \frac{\sigma_0^n m_{2n}(0)}{(n!)^\alpha} \leq 1 \implies \sup_{t \geq 0} \sum_{n=0}^{\infty} \frac{\sigma^n m_{2n}(t)}{(n!)^\alpha} \leq 3.$$

*Proof.* We fix  $\alpha \geq 1$  and  $\sigma_0 \in (0, \infty)$  and assume that  $\sup_{n \geq 0} (n!)^{-\alpha} \sigma_0^n m_{2n}(0) \leq 1$ .

*Step 1.* We introduce, for  $\sigma \in (0, 1]$  to be chosen later, for  $p \geq 2$  and  $t \geq 0$ ,

$$E_p(t) = \sum_{n=0}^p \frac{\sigma^n m_{2n}(t)}{(n!)^\alpha}.$$

If  $\sigma \in (0, \sigma_0/2]$ ,  $E_p(0) \leq 2$  for all  $p \geq 2$ , because

$$E_p(0) \leq 1 + \sum_{n \geq 1} \frac{(\sigma_0/2)^n m_{2n}(0)}{(n!)^\alpha} \leq 1 + \sum_{n \geq 1} 2^{-n} = 2.$$

*Step 2.* By Lemma 7, since  $m'_0(t) = m'_2(t) = 0$  and since  $\sigma \in (0, 1]$ ,

$$E'_p(t) \leq -c_1 F_p(t) + c_2 G_p(t) + C,$$

where

$$F_p(t) = \sum_{n=2}^p n^{\nu/2} \frac{\sigma^n m_{2n+\gamma}(t)}{(n!)^\alpha}, \quad G_p(t) = \sum_{n=2}^p \frac{\sigma^n S_n(t)}{(n!)^\alpha}, \quad C = 2c_1 \sum_{n=2}^{\infty} \frac{n^{\nu/2} 2^{2n/\gamma}}{(n!)^\alpha}.$$

*Step 3.* We prove that for all  $\epsilon \in (0, \infty)$ , there is a constant  $A_\epsilon \in (0, \infty)$ , depending only on  $\epsilon, \gamma, \nu, \kappa_1, \kappa_2, \alpha, \sigma_0$ , such that for any choice of  $\sigma \in (0, 1]$ , for all  $p \geq 2$ , all  $t \geq 0$ ,

$$G_p(t) \leq \epsilon E_p(t) F_p(t) + \sigma A_\epsilon (F_p(t) + 1).$$

Exactly as in the previous section, Step 3, we have

$$G_p(t) \leq (F_p(t) + \sigma m_{2+\gamma}(t)) (\epsilon E_p(t) + J_\epsilon(t)),$$

where, setting  $N_\epsilon = \lceil 2^{\nu/(\nu+2)} \epsilon^{-2/(\nu+2)} \rceil$ ,

$$J_\epsilon(t) = 2^{\nu/2} \sum_{a=1}^{N_\epsilon} \frac{\sigma^a m_{2a}(t)}{(a!)^\alpha},$$



For all  $n \geq 2$ ,  $m_{2n}(0) \leq \sigma_0^{-n}(n!)^\alpha$ , so that by Lemma 10-(ii), there are some constants  $A, A_\epsilon \in (0, \infty)$ , depending only on  $\epsilon, \gamma, \nu, \kappa_1, \kappa_2, \alpha, \sigma_0$  and allowed to vary, such that for all  $t \geq 0$ ,

$$m_{2+\gamma}(t) \leq [m_4(t)]^{(2+\gamma)/4} \leq A \quad \text{and} \quad J_\epsilon(t) \leq A_\epsilon \sum_{a=1}^{N_\epsilon} \sigma^a \leq A_\epsilon \sigma.$$

All in all, we have proved that for any choice of  $\sigma \in (0, 1]$ , for all  $t \geq 0$ ,

$$G_p(t) \leq (F_p(t) + A\sigma)(\epsilon E_p(t) + A_\epsilon \sigma) \leq \epsilon E_p(t) F_p(t) + \sigma[\epsilon A E_p(t) + A_\epsilon F_p(t) + A A_\epsilon].$$

The conclusion follows since  $E_p(t) \leq m_0(t) + \sigma m_2(t) + F_p(t) \leq 2 + F_p(t)$ .

*Step 4.* We now prove that for all  $p \geq 2$ , all  $t \geq 0$ ,

$$F_p(t) \geq \sigma^{-\gamma/2}(E_p(t) - e).$$

We write

$$F_p(t) = \sum_{n=2}^p n^{\nu/2} \frac{\sigma^n m_{2n+\gamma}(t)}{(n!)^\alpha} \geq \frac{1}{\sigma^{\gamma/2}} \sum_{n=2}^p \frac{\sigma^{n+\gamma/2} m_{2n+\gamma}(t)}{(n!)^\alpha} \geq \frac{1}{\sigma^{\gamma/2}} \sum_{n=2}^p \frac{\sigma^n m_{2n}(t) - 1}{(n!)^\alpha},$$

because

$$\sigma^{n+\gamma/2} m_{2n+\gamma}(t) = \int_{\mathbb{R}^3} (\sigma|v|^2)^{n+\gamma/2} f_t(dv) \geq \int_{\mathbb{R}^3} [(\sigma|v|^2)^n - 1] f_t(dv) = \sigma^n m_{2n}(t) - 1.$$

Hence, since  $m_0(t) = m_2(t) = 1$ , since  $\sigma \in (0, 1]$  and since  $\alpha \geq 1$ ,

$$F_p(t) \geq \frac{1}{\sigma^{\gamma/2}} \left[ E_p(t) - m_0(t) - \sigma m_2(t) - \sum_{n=2}^p \frac{1}{(n!)^\alpha} \right] \geq \frac{1}{\sigma^{\gamma/2}} \left[ E_p(t) - \sum_{n=0}^p \frac{1}{(n!)^\alpha} \right] \geq \frac{1}{\sigma^{\gamma/2}} [E_p(t) - e].$$

*Step 5.* By Steps 2-3-4, for any choice of  $\epsilon \in (0, \infty)$  and  $\sigma \in (0, 1]$ , for all  $p \geq 2$ , all  $t \geq 0$ ,

$$E'_p(t) \leq -c_1 F_p(t) + c_2 G_p(t) + C \leq -c_1 F_p(t) + c_2 [\epsilon E_p(t) F_p(t) + \sigma A_\epsilon (F_p(t) + 1)] + C.$$

Choosing  $\epsilon = c_1/(16c_2)$  and then  $\sigma_1 \in (0, 1]$  small enough so that  $c_2 \sigma_1 A_\epsilon \leq c_1/2$ , we conclude that for some constant  $D \in (0, \infty)$ , for any choice of  $\sigma \in (0, \sigma_1]$ , any  $t \geq 0$ ,

$$E'_p(t) \leq -\frac{c_1}{2} F_p(t) + \frac{c_1}{16} E_p(t) F_p(t) + D.$$

We now recall from Step 1 that for  $\sigma \in (0, \sigma_0/2]$ , we have  $E_p(0) \leq 2$  and by continuity,

$$T_p = \sup\{t \geq 0 : E_p(t) \leq 4\} > 0.$$

But for all  $t \in [0, T_p)$ , if  $\sigma \in (0, (\sigma_0/2) \wedge \sigma_1]$ ,

$$E'_p(t) \leq -\frac{c_1}{4} F_p(t) + D \leq -\frac{c_1}{4\sigma^{\gamma/2}} [E_p(t) - e] + D$$

by Step 4. Since  $E_p(0) \leq 2$ , this implies that if  $\sigma \in (0, (\sigma_0/2) \wedge \sigma_1]$ , for all  $p \geq 2$  and all  $t \in [0, T_p)$ ,

$$E_p(t) \leq e + \frac{4\sigma^{\gamma/2} D}{c_1}.$$

Choosing  $\sigma \in (0, (\sigma_0/2) \wedge \sigma_1]$  small enough so that  $4\sigma^{\gamma/2} D/c_1 \leq 3 - e$ , we conclude that for all  $p \geq 2$ ,  $E_p(t) \leq 3$  for all  $t \in [0, T_p)$ , whence  $T_p = \infty$  by continuity. In other words, for all  $p \geq 2$ , all  $t \geq 0$ ,  $E_p(t) \leq 3$ , which was our goal.  $\square$

We finally can give the

*Proof of Theorem 2-(ii).* We assume  $(H_1(\gamma))$  and  $(H_2(\nu))$  for some  $\gamma \in (0, 1]$  and some  $\nu \in (0, 2)$ . We fix  $A > 1$ ,  $\rho \in (0, 2]$  and  $\sigma_0 > 0$  and assume that  $\int_{\mathbb{R}^3} \exp(\sigma_0|v|^\rho) f_0(dv) \leq A$ .

We set  $\alpha = 2/\rho \geq 1$ . By Lemma 11-(ii), for some  $\sigma_1 \in (0, \infty)$ , depending only on  $\rho, \sigma_0, A$ ,

$$\sup_{n \geq 0} \frac{\sigma_1^n m_{2n}(0)}{(n!)^\alpha} \leq 1.$$

We thus may apply Lemma 13: there is  $\sigma_2 \in (0, \sigma_1]$ , depending only on  $\gamma, \nu, \kappa_1, \kappa_2, \sigma_1$ , such that

$$\sup_{t \geq 0} \sum_{n=0}^{\infty} \frac{\sigma_2^n m_{2n}(t)}{(n!)^\alpha} \leq 3.$$

We deduce from Lemma 11-(i) that, setting  $\sigma_3 = \sigma_2^{1/\alpha}/2$ ,

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} \exp[\sigma_3|v|^\rho] f_t(dv) = \sup_{t \geq 0} \int_{\mathbb{R}^3} \exp[\sigma_2^{1/\alpha}|v|^{2/\alpha}/2] f_t(dv) \leq 2.3^{1/\alpha} \leq 6$$

as desired.  $\square$

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