COLLISIONS OF THE SUPERCRITICAL KELLER-SEGEL PARTICLE SYSTEM

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Abstract. We study a particle system naturally associated to the 2-dimensional Keller-Segel equation. It consists of N Brownian particles in the plane, interacting through a binary attraction in $\theta/(Nr)$, where r stands for the distance between two particles. When the intensity θ of this attraction is greater than 2, this particle system explodes in finite time. We assume that $N > 3\theta$ and study in details what happens near explosion. There are two slightly different scenarios, depending on the values of N and θ, here is one: at explosion, a cluster consisting of precisely k_0 particles emerges, for some deterministic $k_0 \geq 7$ depending on N and θ. Just before explosion, there are infinitely many $(k_0 - 1)$-ary collisions. There are also infinitely many $(k_0 - 2)$-ary collisions before each $(k_0 - 1)$-ary collision. And there are infinitely many binary collisions before each $(k_0 - 2)$-ary collision. Finally, collisions of subsets of $3, \dots, k_0 - 3$ particles never occur. The other scenario is similar except that there are no $(k_0 - 2)$-ary collisions.

1. Introduction and main results

1.1. **Informal definition of the model.** We consider some scalar parameter $\theta > 0$ and a number $N \geq 2$ of particles with positions $X_t = (X_t^1, \dots, X_t^N) \in (\mathbb{R}^2)^N$ at time $t \geq 0$. Informally, we assume that the dynamics of these particles are given by the system of S.D.E.s

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i - \frac{\theta}{N} \sum_{j \neq i} \frac{X_t^i - X_t^j}{\|X_t^i - X_t^j\|^2} \mathrm{d}t, \qquad i \in [\![1, N]\!], \tag{1}$$

where the 2-dimensional Brownian motions $((B_t^i)_{t \geq 0})_{i \in [\![1,N]\!]}$ are independent. In other words, we have N Brownian particles in the plane interacting through an attraction in $1/r$, which is Coulombian in dimension 2. Actually, this S.D.E. does not clearly make sense, due to the singularity of the drift, and we will use, as suggested by Cattiaux-Pédèches [4], the theory of Dirichlet spaces, see Fukushima-Oshima-Takeda [11].

1.2. **Brief motivation and informal presentation of the main results.** This particle system is very natural from a physical point of view, because, as we will see, there is a tight competition between the Brownian excitation and the Coulombian attraction. It can also be seen as an approximation of the famous Keller-Segel equation [16], see also Patlak [20]. This nonlinear P.D.E. has been introduced to model the collective motion of cells, which are attracted by a chemical substance that they emit. It is well-known that a phase transition occurs: if the intensity of the attraction is small, then there exist global solutions, while if the attraction is large, the solution explodes in finite time.

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We will show that this phase transition already occurs at the level of the particle system (1): there exist global (very weak) solutions if $\theta \in (0,2)$ (subcritical case, see Proposition 3 below), but solutions must explode in finite time if $\theta \geq 2$ (supercritical case).

To our knowledge, the supercritical case has not been studied in details, and we aim to describe precisely the explosion phenomenon. Informally, we will show the following (see Theorem 5 below). We assume that $\theta \geq 2$ and $N > 3\theta$, we set $k_0 = \lceil 2N/\theta \rceil \in [\![7, N]\!]$. There exists a (very weak) solution $(X_t)_{t\in[0,\zeta)}$ to (1), with $\zeta < \infty$ a.s. and such that $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists. Moreover, there is a cluster containing precisely k_0 particles in the configuration $X_{\zeta-}$, and no cluster containing strictly more than k_0 particles. Such a cluster containing k_0 particles is inseparable, so that (1) is meaningless (even in a very weak sense) after ζ. Just before explosion, there are infinitely many k_1-ary collisions, where $k_1 = k_0 - 1$. If $(k_0 - 3)(2 - (k_0 - 2)\theta/N) < 2$, we set $k_2 = k_1 - 2$ and just before each k_1-ary collision, there are infinitely many k_2-collisions. Else, we set $k_2 = k_1$. In any case, there are infinitely many binary collisions just before each k_2-ary collision. During the whole time interval $[0, \zeta)$, there are no k-ary collisions, for any $k \in [\![3, k_2 - 1]\!]$.

This phenomenon seems surprising and original, in particular because of the gap between binary and k_2-ary collisions.

1.3. **Sets of configurations.** We introduce, for all $K \subset [\![1, N]\!]$ and all $x = (x^1, \dots, x^N) \in (\mathbb{R}^2)^N$,

$$S_K(x) = \frac{1}{|K|}\sum_{i\in K} x^i \in \mathbb{R}^2 \quad \text{and} \quad R_K(x) = \sum_{i\in K}\|x^i - S_K(x)\|^2 = \frac{1}{2|K|}\sum_{i,j\in K}\|x^i - x^j\|^2 \geq 0.$$

Here $|K|$ is the cardinal of K and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^2$. Observe that $R_K(x) = 0$ if and only if all the particles indexed in K are at the same place. We also set, for $k \geq 2$,

$$E_k = \Big\{x \in (\mathbb{R}^2)^N : \forall K \subset [\![1, N]\!] \text{ with cardinal } |K| = k,\ R_K(x) > 0\Big\},$$

which represents the set of configurations with no cluster of k (or more) particles. Observe that $E_k = (\mathbb{R}^2)^N$ for all $k > N$.

1.4. **Bessel processes.** We recall that a squared Bessel process $(Z_t)_{t\geq 0}$ of dimension $\delta \in \mathbb{R}$ is a nonnegative solution, killed when it reaches 0 if $\delta \leq 0$, of the equation

$$Z_t = Z_0 + 2\int_0^t \sqrt{Z_s}\mathrm{d}W_s + \delta t,$$

where $(W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion. We then say that $(\sqrt{Z_t})_{t\geq 0}$ is a Bessel process of dimension δ. This process has the following property, see Revuz-Yor [21, Chapter XI]:

- if $\delta \geq 2$, then a.s., for all $t > 0$, $Z_t > 0$;
- if $\delta \in (0, 2)$, then a.s., Z is reflected infinitely often at 0;
- if $\delta \leq 0$, then Z a.s. hits 0 and is then killed.

Applying informally the Itô formula, one finds that $Y_t = \sqrt{Z_t}$ should solve

$$Y_t = Y_0 + W_t + \frac{\delta - 1}{2}\int_0^t \frac{\mathrm{d}s}{Y_s},$$

which resembles (1) in that we have a Brownian excitation in competition with an attraction by 0, or a repulsion by 0, depending on the value of δ, proportional to $1/r$. This formula rigorously holds true only when $\delta > 1$, see [21, Chapter XI].

1.5. **Some important quantities.** Consider a (possibly very weak) solution $(X_t)_{t\geq 0}$ to (1). As we will see, when fixing a subset $K \subset [\![1, N]\!]$ and when neglecting the interactions between the particles indexed in K and the other ones, one finds that the process $(R_K(X_t))_{t\geq 0}$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|K|)$, where

$$d_{\theta,N}(k) = (k-1)\Big(2 - \frac{k\theta}{N}\Big). \tag{2}$$

Similar computations already appear in Haškovec-Schmeiser [12], see also [8]. A little study, see Appendix A, see also Figure 1.5 and Subsection 1.8 for numerical examples, shows the following facts. For $r \in \mathbb{R}_+$, we set $\lceil r \rceil = \min\{n \in \mathbb{N} : n \geq r\}$.

Lemma 1. *Fix $\theta > 0$ and $N \geq 2$ such that $N > \theta$. For $k_0 = \lceil \frac{2N}{\theta} \rceil \geq 3$, we have*

$$d_{\theta,N}(k) > 0 \quad \textit{if} \quad k \in [\![2, k_0 - 1]\!] \quad \textit{and} \quad d_{\theta,N}(k) \leq 0 \quad \textit{if} \quad k \geq k_0. \tag{3}$$

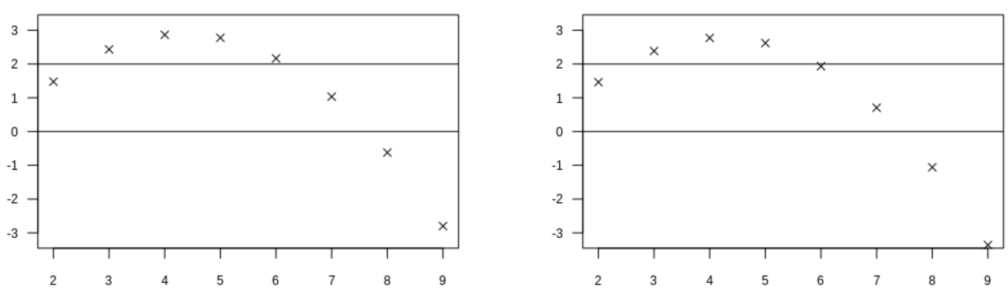
We also define $k_1 = k_0 - 1$, and

$$k_2 = \begin{cases} k_0 - 2 & \textit{if} \quad d_{\theta,N}(k_0 - 2) < 2, \\ k_0 - 1 & \textit{if} \quad d_{\theta,N}(k_0 - 2) \geq 2. \end{cases}$$

If $\theta \geq 2$ and $N > 3\theta$, then $k_0 \in [\![7, N]\!]$ and it holds that

- *$d_{\theta,N}(2) \in (0, 2)$;*
- *$d_{\theta,N}(k) \geq 2$ if $k \in [\![3, k_2 - 1]\!]$;*
- *$d_{\theta,N}(k) \in (0, 2)$ if $k \in \{k_2, k_1\}$;*
- *$d_{\theta,N}(k) \leq 0$ if $k \geq k_0$.*

FIGURE 1. Plot of $d_{\theta,N}(k)$ as a function of $k \in [\![2, N]\!]$ with $N = 9$ and with $\theta = 2.35$ (left) and $\theta = 2.42$ (right).



$k_0 = 8$, $k_1 = 7$, $k_2 = 7$ $\qquad$ $k_0 = 8$, $k_1 = 7$, $k_2 = 6$

We thus expect that there may be some non sticky k-ary collisions for $k \in \{2, k_2, k_1\}$, some sticky k-ary collisions when $k \geq k_0$, but no k-ary collision for $k \in [\![3, k_2 - 1]\!]$.

1.6. **Generator and invariant measure.** As we will see, it will happen that $R_K(X)$ vanishes at some time τ strictly before explosion and for some subset $K \subset [\![1, N]\!]$ of cardinal k_1, and $R_K(X)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(k_1)$ around this instant τ. Since $d_{\theta,N}(k_1)$ can belong to $(0, 1]$ for some values of N and θ, this implies in this case that $\int_{\tau-a}^{\tau+a}[R_K(X_s)]^{-1/2}\mathrm{d}s = \infty$ a.s. for all $a > 0$, which can be shown by comparison with the 1-dimensional Brownian motion. Since now $\|X_s^i - X_s^j\| \leq [R_K(X_s)]^{1/2}$ for all $i, j \in K$, the S.D.E.

(1) cannot have a solution in the classical sense, because the drift term cannot be integrable in time. We will thus define a solution through the theory of the Dirichlet spaces.

For $x = (x^1, \dots, x^N) \in (\mathbb{R}^2)^N$ and for $\mathrm{d}x$ the Lebesgue measure on $(\mathbb{R}^2)^N$, we set

$$\mathbf{m}(x) = \prod_{1 \le i \ne j \le N} \|x^i - x^j\|^{-\theta/N} \quad \text{and} \quad \mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x. \tag{4}$$

Informally, the generator of the solution to (1) is given by $\mathcal{L}^X$, where for $\varphi \in C^2((\mathbb{R}^2)^N)$,

$$\mathcal{L}^X \varphi(x) = \frac{1}{2}\Delta\varphi(x) - \frac{\theta}{N} \sum_{1 \le i \ne j \le N} \frac{x^i - x^j}{\|x^i - x^j\|^2} \cdot \nabla_{x^i}\varphi(x) = \frac{1}{2\mathbf{m}(x)} \mathrm{div}[\mathbf{m}(x)\nabla\varphi(x)]. \tag{5}$$

It is well-defined for all $x \in E_2$ and μ-symmetric. Indeed, an integration by parts shows that

$$\forall\, \varphi, \psi \in C^2_c(E_2), \quad \int_{(\mathbb{R}^2)^N} \varphi \mathcal{L}^X \psi \,\mathrm{d}\mu = -\frac{1}{2} \int_{(\mathbb{R}^2)^N} \nabla\varphi \cdot \nabla\psi \,\mathrm{d}\mu = \int_{(\mathbb{R}^2)^N} \psi \mathcal{L}^X \varphi \,\mathrm{d}\mu. \tag{6}$$

As we will see in Proposition A.1, the measure μ is Radon on $(\mathbb{R}^2)^N$ in the subcritical case $\theta \in (0, 2)$, while it is Radon on E_{k_0} (and not on E_{k_0+1}) in the supercritical case $\theta \ge 2$. This will allow us to use some results found in Fukushima-Oshima-Takeda [11] and to obtain the following existence result.

Proposition 2. *We fix $N \ge 2$ and $\theta > 0$ such that $N > \theta$ and recall that $k_0 = \lceil 2N/\theta \rceil$. We set $\mathcal{X} = E_{k_0}$ and $\mathcal{X}_\triangle = \mathcal{X} \cup \{\triangle\}$, where $\triangle$ is a cemetery point. There exists a continuous Hunt process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\ge0}, (\mathbb{P}^X_x)_{x\in\mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, with regular Dirichlet space $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2((\mathbb{R}^2)^N, \mu)$ with core $C^\infty_c(\mathcal{X})$ defined by*

$$\textit{for all} \quad \varphi \in C^\infty_c(\mathcal{X}), \quad \mathcal{E}^X(\varphi, \varphi) = \frac{1}{2} \int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu = -\int_{(\mathbb{R}^2)^N} \varphi \mathcal{L}^X \varphi \,\mathrm{d}\mu$$

and such that for all $x \in E_2$, all $t > 0$, the law of X_t under $\mathbb{P}_x$ has a density with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$. We call such a process a $KS(\theta, N)$-process and denote by $\zeta = \inf\{t \ge 0 : X_t = \triangle\}$ its life-time.

We refer to Subsection B.1 for a quick summary about the notions used in this proposition: Hunt process, link between its generator, semi-group and Dirichlet space, definition of the one-point compactification topology endowing $\mathcal{X}_\triangle$, etc. Let us mention that by definition, $\triangle$ is absorbing, i.e. $X_t = \triangle$ for all $t \ge \zeta$. Also, $t \mapsto X_t$ is *a priori* continuous on $[0, \infty)$ only for the one-point compactification topology on $\mathcal{X}_\triangle$, which precisely means that it is continuous for the usual topology of $(\mathbb{R}^2)^N$ during $[0, \zeta)$, and it holds that $\zeta = \lim_{n\to\infty} \inf\{t \ge 0 : X_t \notin \mathcal{K}_n\}$ for any increasing sequence of compact subsets $(\mathcal{K}_n)_{n\ge1}$ of E_{k_0} such that $\cup_{n\ge1}\mathcal{K}_n = E_{k_0}$.

As we will see in Remark 28, for all $x \in E_2$, under $\mathbb{P}^X_x$, X_t solves (1) during $[0, \sigma)$, where $\sigma = \inf\{t \ge 0 : X_t \notin E_2\}$. By the Markov property, this implies X_t solves (1) during any open time-interval on which it does not visit $\mathcal{X} \setminus E_2$.

When $\theta < 2$, we have $k_0 > N$ and thus $E_{k_0} = (\mathbb{R}^2)^N$. We will easily prove the following non-explosion result, which is almost contained in Cattiaux-Pédèches [4], who treat the case where $\theta \in (0, 2(N-2)/(N-1))$.

Proposition 3. *Fix $\theta \in (0, 2)$ and $N \ge 2$. Consider the $KS(\theta, N)$-process $\mathbb{X}$ introduced in Proposition 2. For all $x \in E_2$, we have $\mathbb{P}_x(\zeta = \infty) = 1$.*

When $\theta \geq 2$, we will see that there is explosion. Note that any collision of a set of $k \geq k_0$ particles makes the process leave E_{k_0} and thus explode. However, it is not clear at all at this point that explosion is due to a precise collision: the process could explode because it tends to infinity (which is not hard to exclude) or to the boundary of E_{k_0} with possibly many oscillations.

1.7. **Main result.** To avoid any confusion, let us define precisely what we call a collision.

Definition 4. *(i) For $K \subset [\![1, N]\!]$, we say that there is a K-collision in the configuration $x \in (\mathbb{R}^2)^N$ if $R_K(x) = 0$ and if $R_{K\cup\{i\}}(x) > 0$ for all $i \in [\![1, N]\!] \setminus K$.*

(ii) For a $(\mathbb{R}^2)^N$-valued process $(X_t)_{t\in[0,\zeta)}$, we say that there is a K-collision at time $s \in [0, \zeta)$ if there is a K-collision in the configuration X_s.

The main result of this paper is the following description of the explosion phenomenon.

Theorem 5. *Assume that $\theta \geq 2$, that $N > 3\theta$ and recall that $k_0 \in [\![7, N]\!]$, $k_1 = k_0 - 1$ and $k_2 \in \{k_0 - 1, k_0 - 2\}$ were defined in Lemma 1. Consider the $KS(\theta, N)$-process $\mathbb{X}$ introduced in Proposition 2. For all $x \in E_2$, we $\mathbb{P}_x$-a.s. have the following properties:*

(i) ζ is finite and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists for the usual topology of $(\mathbb{R}^2)^N$;

(ii) there is $K_0 \subset [\![1, N]\!]$ with cardinal $|K_0| = k_0$ such that there is a K_0-collision in the configuration $X_{\zeta-}$, and for all $K \subset [\![1, N]\!]$ such that $|K| > k_0$, there is no K-collision in the configuration $X_{\zeta-}$;

(iii) for all $t \in [0, \zeta)$ and all $K \subset K_0$ with cardinal $|K| = k_1$, there is an infinite number of K-collisions during (t, ζ) and none of these instants of K-collision is isolated;

(iv) if $k_2 = k_0 - 2$, then for all $L \subset K \subset K_0$ such that $|L| = k_2$ and $|K| = k_1$, for all instant $t \in (0, \zeta)$ of K-collision and all $s \in [0, t)$, there is an infinite number of L-collisions during (s, t) and none of these instants of L-collision is isolated;

(v) for all $K \subset [\![1, N]\!]$ with cardinal $|K| \in [\![3, k_2 - 1]\!]$, there is no K-collision during $[0, \zeta)$;

(vi) for all $L \subset K \subset K_0$ such that $|L| = 2$ and $|K| = k_2$, for all instant $t \in (0, \zeta)$ of K-collision and all $s \in [0, t)$, there is an infinite number of L-collisions during (s, t) and none of these instants of L-collision is isolated.

The condition $\theta \geq 2$ is crucial to guarantee that $k_0 \leq N$. On the contrary, we impose $N > 3\theta$ for simplicity, because Lemma 1 does not hold true without this assumption. The other cases may also be studied, but we believe this is not very restrictive: N is thought as very large when compared to θ, at least as far as the approximation of the Keller-Segel equation is concerned.

1.8. **Comments.** Let us mention that the very precise values of N and θ influence the value k_2.

(a) If $N = 200$ and $\theta = 4.04$, we have $k_0 = 100$, $k_1 = 99$ and $k_2 = 98$.

(b) If $N = 200$ and $\theta = 4.015$, we have $k_0 = 100$ and $k_1 = k_2 = 99$.

Let us describe informally, in the chronological order, what happens e.g. in case (b) above. We start with 200 particles at 200 different places. During the whole story, there is no k-ary collision for $k = 3, \dots, 98$. Here and there, two particles meet, they collide an infinite number of times, but manage to separate. Then at some times, we have 98 particles close to each other and there are many binary collisions. Then, if a 99-th particle arrives in the same zone (and this eventually occurs), there are infinitely many 99-ary collisions, with infinitely many binary collisions of all possible pairs before each. These 99 particles may manage to separate forever, or for a large time, but if a 100-th particle arrives in the zone (and this situation eventually occurs), then there are infinitely many 99-ary collisions of all the possible subsets and, finally, a 100-ary collision producing explosion, and the story is finished. Informally, the resulting cluster is not able to separate, because the attraction dominates the Brownian excitation, since a Bessel process of dimension $d_{\theta,N}(100) \leq 0$ is absorbed when it reaches 0. We hope to be able, in a future work, to propose and justify a model describing what happens after explosion.

1.9. **References.** In many papers about the Keller-Segel equation, the parameter $\chi = 4\pi\theta$ is used, so that the transition at $\theta = 2$ corresponds to the transition at $\chi = 8\pi$. As already mentioned, this nonlinear P.D.E. has been introduced to model the collective motion of cells, which are attracted by a chemical substance that they emit. It describes the density $f_t(x)$ of particles (cells) with position $x \in \mathbb{R}^2$ at time $t \geq 0$ and writes, in the so-called parabolic-elliptic case,

$$\partial_t f_t(x) + \theta \mathrm{div}_x((K \star f_t)(x) f_t(x)) = \frac{1}{2}\Delta_x f_t(x), \quad \text{where} \quad K(x) = -\frac{x}{|x|^2}. \tag{7}$$

Informally, this solution should be the mean-field limit of the particle system (1) as $N \to \infty$.

We refer to the recent review paper on (7) by Arumugam-Tyagi [1]. The best existence of a global solution to (7), including all the subcritical parameters $\theta \in (0, 2)$, is due to Blanchet-Dolbeault-Perthame [2]. The blow-up of solutions to (7), in the supercritical case $\theta > 2$, has been studied e.g. by Fatkullin [7] and Velasquez [24, 25]. More close to our study, Suzuki [23] has shown, still in the supercritical case, the appearance of a Dirac mass with a precise (critical) weight, at explosion. This is the equivalent, in the limit $N \to \infty$, to the fact that $\lim_{t\to\zeta-} X_t$ exists and corresponds to a K-collision, for some $K \subset [\![1, N]\!]$ with precise cardinal k_0. Let us finally mention Dolbeault-Schmeiser [6], who propose a post-explosion model in the supercritical case.

Concerning particle systems associated with (7), let us mention Stevens [22], who studies a physically more complete particle system with two types of particles, for cells and chemo-attractant particles, with a regularized attraction kernel. Haškovec and Schmeiser [12, 13] study a particle system closer to (1), but with, again, a regularized attraction kernel.

Cattiaux-Pédèches [4], as well as [8], study the system (1) without regularization in the subcritical case: existence of a global solution to (1) has been shown in [8] when $\theta \in (0, 2(N-2)/(N-1))$, and uniqueness of this solution has been established in [4]. Also, the theory of Dirichlet spaces has been used in [4] to build a solution to (1). Finally, the limit as $N \to \infty$ to a solution of (7) is proved in [8] in the very subcritical case where $\theta \in (0, 1/2)$, up to extraction of a subsequence. This last result has been improved by Bresch-Jabin-Wang [3], who remove the necessity of extracting a subsequence and consider the (still very subcritical) case where $\theta \in (0, 1)$. Olivera-Richard-Tomasevic [18] have recently established the $N \to \infty$ convergence of a smoothed version of (1), for all the subcritical cases $\theta \in (0, 2)$. Informally, in view of the mean distance between particles, the regularization used in [18] is not far from being physically reasonable. There is also a related paper of Jabir-Talay-Tomasevic [14] about a one-dimensional but more complicated parabolic-parabolic model.

Let us finally mention the seminal paper of Osada [19], see also [9] for a more recent study, which concerns the vortex model: this is very close to (1), but the attraction $-x/|x|^2$ is replaced by a rotating interaction $x^\perp/|x|^2$, so that particles never encounter.

1.10. **Originality and difficulties.** To our knowledge, this is the first study of the supercritical Keller-Segel particle system near explosion. We hope that this model, which makes compete diffusion and Coulomb interactions, is very natural from a physical point of view, beyond the Keller-Segel community. The phenomenon we discovered seems surprising and original, in particular because of the gap between binary and k_2-ary collisions. We are not aware of other works, possibly dealing with other models, showing such a behavior.

In Section 3, we give the main arguments of the proofs, with quite a high level of precision, but ignoring the technical issues. While it is rather clear, intuitively, that the process explodes in finite time when $\theta \geq 2$ and that no K-collisions may occur for $|K| \in [\![3, k_2 - 1]\!]$, the continuity at explosion is tedious, and some rather deep arguments are required to show that that each k_2-ary collision is preceded by many binary collisions, that each k_1-ary collision is preceded by many k_2-ary collisions, that explosion is preceded by many k_1-ary collisions, and that explosion is due to the emergence of a cluster with precise size k_0 (which more or less says that a possible (k_0+1)-ary collision would necessarily be preceded by a k_0-collision).

Actually, the rigorous proofs are made technically much more involved than those presented in Section 3, because we have to use the theory of Dirichlet spaces. Due to the singularity of the interactions and to the occurrence of many collisions near explosion, we can unfortunately not, as already mentioned, deal at the rigorous level directly with the S.D.E. (1). We thus have to use some suitable heavy versions of some usual tools such as Itô's formula, Girsanov's theorem, time-change, etc.

1.11. **Plan of the paper.** In Section 2, we introduce some notation of constant use. In Section 3, we explain the main ideas of the proofs, with quite a high level of precision, but without speaking of the heavy technical issues related to the use of the theory of Dirichlet spaces. Section 4 is devoted to the existence of a first version of the Keller-Segel process, namely without the property that $\mathbb{P}_x^X \circ X_t^{-1}$ has a density, and we introduce a spherical Keller-Segel process. In Section 5, we show that the Keller-Segel process enjoys a crucial and noticeable decomposition in terms of a 2-dimensional Brownian motion, a squared Bessel process and a spherical process. Section 6 consists in building some smooth approximations of some indicator functions that behave well under the action of the generator $\mathcal{L}^X$. In Section 7, we make use of the Girsanov theorem to prove that when two sets of particles of a KS-process are not too close from each other, they behave as two independent smaller KS-processes. In Section 8, we study explosion and continuity (in the usual sense) at the explosion time. Section 9 is devoted to establish some parts of Theorem 5 for some particular ranges of values of N and θ. Using the results of Section 7, we reduce the general study to the special cases of Section 9 and we prove, in Section 10, that the conclusions of Theorem 5 hold true for quasi all $x \in \mathcal{X}$. Finally, in Section 11, we remove the restriction *for quasi all* $x \in \mathcal{X}$ and conclude the proofs of Propositions 2 and 3 and of Theorem 5.

Appendix A contains a few elementary computations: proof of Lemma 1, proof that μ is Radon on E_{k_0}, and study of a similar measure on a sphere. We end the paper with Appendix B, that summarizes all the notions and results about Dirichlet spaces and Hunt processes we shall use.

2. Notation

We introduce the spaces

$$H = \Big\{x \in (\mathbb{R}^2)^N : S_{[\![1,N]\!]}(x) = 0\Big\}, \quad S = \Big\{x \in (\mathbb{R}^2)^N : \sum_{i=1}^N \|x^i\|^2 = 1\Big\} \quad \text{and} \quad \mathbb{S} = H \cap S.$$

For $u \in \mathbb{S}$, we have $S_{[\![1,N]\!]}(u) = 0$ and $R_{[\![1,N]\!]}(u) = 1$. We consider the (unnormalized) Lebesgue measure σ on $\mathbb{S}$, as well as, recall (4),

$$\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u). \tag{8}$$

We define $\gamma : \mathbb{R}^2 \to (\mathbb{R}^2)^N$ by $\gamma(z) = (z, \dots, z)$ and $\Psi : \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S} \to E_N \subset (\mathbb{R}^2)^N$ by

$$\Psi(z, r, u) = \gamma(z) + \sqrt{r}\, u, \qquad \text{i.e.} \qquad (\Psi(z, r, u))^i = z - \sqrt{r} u^i \quad \text{for} \quad i \in [\![1, N]\!]. \tag{9}$$

We have $S_{[\![1,N]\!]}(\Psi(z, r, u)) = z$ and $R_{[\![1,N]\!]}(\Psi(z, r, u)) = r$.

The orthogonal projection $\pi_H : (\mathbb{R}^2)^N \to H$ is given by

$$\pi_H(x) = x - \gamma(S_{[\![1,N]\!]}(x)), \qquad \text{i.e.} \qquad (\pi_H(x))^i = x^i - S_{[\![1,N]\!]}(x) \quad \text{for} \quad i \in [\![1, N]\!]$$

and we introduce $\Phi_\mathbb{S} : E_N \to \mathbb{S}$ defined by

$$\Phi_\mathbb{S}(x) = \frac{\pi_H x}{||\pi_H x||}, \qquad \text{i.e.} \qquad (\Phi_\mathbb{S}(x))^i = \frac{x^i - S_{[\![1,N]\!]}(x)}{\sqrt{R_{[\![1,N]\!]}(x)}} \quad \text{for} \quad i \in [\![1, N]\!]. \tag{10}$$

For $x \in (\mathbb{R}^2)^N \setminus \{0\}$, the projections $\pi_{x^\perp} : (\mathbb{R}^2)^N \to x^\perp$ and $\pi_x : (\mathbb{R}^2)^N \to \mathrm{span}(x)$ are given by

$$\pi_{x^\perp}(y) = y - \frac{x \cdot y}{||x||^2} x \qquad \text{and} \qquad \pi_x(y) = \frac{x \cdot y}{||x||^2} x,$$

where $x \cdot y = \sum_{i=1}^N x^i \cdot y^i$.

We denote by $b : E_2 \to (\mathbb{R}^2)^N$ the drift coefficient of (1): for $x = (x^1, \dots, x^N) \in E_2$,

$$b(x) = \frac{\nabla \mathbf{m}(x)}{2\mathbf{m}(x)} \in (\mathbb{R}^2)^N, \qquad \text{i.e.} \qquad b^i(x) = -\frac{\theta}{N} \sum_{j \neq i} \frac{x^i - x^j}{\|x^i - x^j\|^2} \in \mathbb{R}^2 \quad \text{for} \quad i \in [\![1, N]\!]. \tag{11}$$

Finally, we introduce the natural operators defined for $\varphi \in C^1(\mathbb{S})$ and $u \in \mathbb{S}$ by

$$\nabla_\mathbb{S} \varphi(u) = \nabla[\varphi \circ \Phi_\mathbb{S}](u) \in (\mathbb{R}^2)^N \quad \text{and} \quad \Delta_\mathbb{S} \varphi(u) = \Delta[\varphi \circ \Phi_\mathbb{S}](u) \in \mathbb{R}, \tag{12}$$

where ∇ and Δ stand for the usual gradient and Laplacian in $(\mathbb{R}^2)^N$. Since $\mathbb{S} \subset E_N \subset (\mathbb{R}^2)^N$, with E_N open, and since $\Phi_\mathbb{S}$ is smooth on E_N, we can indeed define $\nabla[\varphi \circ \Phi_\mathbb{S}](u)$ and $\Delta[\varphi \circ \Phi_\mathbb{S}](u)$ for all $u \in \mathbb{S}$. Similarly, for $\varphi \in C^1(\mathbb{S}, (\mathbb{R}^2)^N)$ and $u \in \mathbb{S}$, we set

$$\mathrm{div}_\mathbb{S} \varphi(u) = \mathrm{div}[\varphi \circ \Phi_\mathbb{S}](u) \in \mathbb{R}. \tag{13}$$

To conclude this subsection, we note that for all $\varphi \in C^\infty((\mathbb{R}^2)^N)$, for all $u \in \mathbb{S}$,

$$\nabla_\mathbb{S}(\varphi|_\mathbb{S})(u) = \pi_H(\pi_{u^\perp}(\nabla \varphi(u))). \tag{14}$$

Indeed, it suffices to observe that setting $G(x) = x/||x||$ for all $x \in (\mathbb{R}^2)^N \setminus \{0\}$, we have $\Phi_\mathbb{S} = G \circ \pi_H$, $\mathrm{d}_x G = \pi_{x^\perp}/||x||$ and $\mathrm{d}_x \pi_H = \pi_H$ and that for $u \in \mathbb{S}$, we have $\pi_H(u) = u$ and $||\pi_H(u)|| = 1$.

3. Main ideas of the proofs

Here we explain the main ideas of the proofs of Proposition 3 and Theorem 5. The arguments below are completely informal. In particular, we do as if our $KS(\theta, N)$-process $(X_t)_{t \in [0,\zeta)}$ was a true solution to (1) until explosion and we apply Itô's formula without care. We always assume at least that $N \geq 2$, $\theta > 0$ and $N > \theta$, which implies that $k_0 = \lceil 2N/\theta \rceil \geq 3$.

3.1. **Existence.** The existence of the $KS(\theta, N)$-process $(X_t)_{t\in[0,\zeta)}$, with values in E_{k_0}, is an easy application of Fukushima-Oshima-Takeda [11, Theorem 7.2.1]. The only difficulty is to show that the invariant measure μ is a Radon on E_{k_0}, see Proposition A.1. The process may explode, i.e. get out of any compact subset of E_{k_0} in finite time. Observe that a typical compact subset of E_{k_0} is of the form, for $\varepsilon > 0$,

$$\mathcal{K}_\varepsilon = \{x \in (\mathbb{R}^2)^N : ||x|| \le 1/\varepsilon \text{ and for all } K \subset [\![1, N]\!] \text{ such that } |K| = k_0,\ R_K(x) \ge \varepsilon\}.$$

3.2. **Center of mass and dispersion process.** One can verify, using Itô's formula, that the center of mass $S_{[\![1,N]\!]}(X)$ is a 2-dimensional Brownian motion with diffusion constant $N^{-1/2}$, that the dispersion process $R_{[\![1,N]\!]}(X)$ is a squared Bessel process with dimension $d_{\theta,N}(N)$, recall (2), and that these two processes are independent.

Consequently, if $\zeta < \infty$, the limits $\lim_{t\to\zeta-} S_{[\![1,N]\!]}(X_t)$ and $\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t)$ a.s. exist, and this implies that $\limsup_{t\to\zeta-} ||X_t|| < \infty$: the process cannot explode to infinity, it can only explode because it tends to the boundary of E_{k_0}. If moreover $k_0 > N$ (i.e. if $\theta < 2$), this is sufficient to show that $\zeta = \infty$, since then $E_{k_0} = (\mathbb{R}^2)^N$.

3.3. **Behavior of distant subsets of particles.** Consider a partition $K_1, \dots, K_p$ of $[\![1, N]\!]$. If we neglect interactions between particles of which the indexes are not in the same subset, we have, for each $\ell \in [\![1, p]\!]$, setting $\tilde\theta_\ell = \theta|K_\ell|/N$,

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i - \frac{\tilde\theta_\ell}{|K_\ell|} \sum_{j\in K_\ell\setminus\{i\}} \frac{X_t^i - X_t^j}{\|X_t^i - X_t^j\|^2}\mathrm{d}t, \qquad i \in K_\ell$$

and we recognize a $KS(\tilde\theta_\ell, |K_\ell|)$-process.

During time intervals where particles indexed in different subsets are far enough from each other, we can indeed bound the interaction between those particles, so that the Girsanov theorem tells us that $(X_t^i)_{i\in K_1}, \dots, (X_t^i)_{i\in K_p}$ behave similarly, in the sense of trajectories, as independent $KS(\tilde\theta_1, |K_1|)$, ..., $KS(\tilde\theta_p, |K_p|)$-processes.

3.4. **Bessel behavior of isolated subsets of particles.** Consider $K \subset [\![1, N]\!]$. As seen just above, during time intervals where the particles indexed in K are far from all the other ones, the system $(X_t^i)_{i\in K}$ behaves, in the sense of trajectories, like a $KS(\theta|K|/N, |K|)$-process. Hence, as seen in Subsection 3.2, $R_K(X_t)$ behaves like a squared Bessel process of dimension $d_{\theta|K|/N,|K|}(|K|)$, which equals $d_{\theta,N}(|K|)$, recall (2).

3.5. **Continuity at explosion.** Here we assume that $N > \theta \ge 2$, so that $k_0 \in [\![2, N]\!]$ and we explain why a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists, in the usual sense of $(\mathbb{R}^2)^N$.

(a) We first show that $\zeta < \infty$ a.s. On the event where $\zeta = \infty$, the squared Bessel process $R_{[\![1,N]\!]}(X)$ is defined for all times and hits zero in finite time, since $d_{\theta,N}(N) \le 0$ (because $\theta \ge 2$). This implies that X has a $[\![1, N]\!]$-collision, and thus leaves E_{k_0} (since $k_0 \le N$) in finite time, which contradicts the fact that $\zeta = \infty$.

(b) We next show by reverse induction that a.s. for all $K \subset [\![1, N]\!]$ with $|K| \ge 2$, we have

$$\text{either} \quad \lim_{t\to\zeta-} R_K(X_t) = 0 \quad \text{or} \quad \liminf_{t\to\zeta-} R_K(X_t) > 0. \tag{15}$$

If $K = [\![1, N]\!]$, $\lim_{t\to\zeta-} R_K(X_t)$ exists by continuity of the (true) squared Bessel process $R_K(X_t)$ and this implies the result. We now fix $n \in [\![3, N]\!]$ and assume that (15) holds true for all K such that $|K| \ge n$. We consider $K \subset [\![1, N]\!]$ with $|K| = n - 1$: by induction assumption, either there is $i \notin K$ such that $\lim_{t\to\zeta-} R_{K\cup\{i\}}(X_t) = 0$ and then $\lim_{t\to\zeta-} R_K(X_t) = 0$, or for all $i \in [\![1,N]\!] \setminus K$, $\liminf_{t\to\zeta-} R_{K\cup\{i\}}(X_t) > 0$. In this last case, and when $\limsup_{t\to\zeta-} R_K(X_t) > 0$ and $\liminf_{t\to\zeta-} R_K(X_t) = 0$ (which is the negation of (15)), there are $\alpha > 0$ and $\varepsilon > 0$ such that (i) $R_K(X_t)$ upcrosses $[\varepsilon/2, \varepsilon]$ infinitely often during $[\zeta-\alpha, \zeta)$ and (ii) for all $t \in [\zeta-\alpha,\zeta)$ such that $R_K(X_t) < \varepsilon$, the particles indexed in K are far from all the other ones (because then $R_K(X_t)$ is small and $R_{K\cup\{i\}}(X_t)$ is large for all $i \notin K$), so that $R_K(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|K|)$, see Subsection 3.4. Points (i) and (ii) are in contradiction, since a squared Bessel process is continuous and thus cannot upcross $[\varepsilon/2,\varepsilon]$ infinitely often during a finite time interval.

(c) We can now show by induction that for all $n \in \mathbb{N}$,

$$\mathcal{P}(n): \begin{cases} \text{for all } \theta \geq 2 \text{ and all } N > \theta \text{ such that } k_0(\theta, N) = N - n, \\ \text{if } (X_t)_{t\in[0,\zeta)} \text{ is a } KS(\theta,N)\text{ -process, then } \lim_{t\to\zeta-} X_t \text{ exists a.s.} \end{cases}$$

Here, and only here, we indicate the dependence of $k_0 = \lceil 2N/\theta \rceil$ in N and θ. Observe that $N > \theta$ implies that $k_0(\theta, N) \geq 3$ and thus that $N \geq n+3$.

We first show $\mathcal{P}(0)$. For $N > \theta \geq 2$ such that $k_0 = N$ and for $(X_t)_{t\in[0,\zeta)}$ a $KS(\theta,N)$-process, we have $\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t) = 0$. Indeed, $\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t)$ exists because $R_{[\![1,N]\!]}(X_t)$ is a (true) squared Bessel process and is thus continuous, and if this limit was positive, the process would remain in a compact of $E_{k_0} = E_N$ and would not explode. Since furthermore $S_{[\![1,N]\!]}(X_t)$ is a (true) Brownian motion on $[0,\zeta)$ and so a.s. has a limit $S \in \mathbb{R}^2$ as $t \to \zeta-$, we conclude that for all $i \in [\![1,N]\!]$, $\lim_{t\to\zeta-} X^i_t = S$.

We next fix $n \in \mathbb{N}$, assume that $\mathcal{P}(0), \dots, \mathcal{P}(n)$ are true and we prove $\mathcal{P}(n+1)$. We fix $N > \theta \geq 2$ such that $k_0(\theta,N) = N-(n+1)$ and consider $(X_t)_{t\in[0,\zeta)}$ a $KS(\theta,N)$-process. By point (b) above, we can find $K \subset [\![1,N]\!]$ such that $\lim_{t\to\zeta-} R_K(X_t) = 0$ and $\liminf_{t\to\zeta-} R_{K\cup\{i\}}(X_t) > 0$ for all $i \notin K$. Consequently, the particles indexed in K are far, just before explosion, from the particles indexed in K^c. Thus by Subsection 3.3, $(X^i_t)_{i\in K}$ and $(X^i_t)_{i\in K^c}$ behave, just before explosion, like two independent $KS(\theta|K|/N, |K|)$ and $KS(\theta|K^c|/N, |K^c|)$-processes, with life-times $\zeta_1, \zeta_2 \geq \zeta$. These two processes are continuous at ζ by induction assumption, let us e.g. detail what happens for the second one:

(i) if $\zeta_2 > \zeta$, this is obvious;

(ii) if $\zeta_2 = \zeta$, then necessarily $\theta|K^c|/N \geq 2$, because else $(X^i_t)_{i\in K^c}$ would not explode, see Subsection 3.2. We also have $|K^c| > \theta|K^c|/N$ because $N > \theta$, and we finally have $k_0(\theta|K^c|/N, |K^c|) = k_0(\theta,N) = N-(n+1) \geq |K^c| - n$. Thus $(X^i_t)_{t\in[0,\zeta_2), i\in K^c}$ has a limit at $\zeta- = \zeta_2-$ by $\mathcal{P}(n)$.

(d) Fix $N > \theta \geq 2$. Then $k_0 \leq N$ and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists by $\mathcal{P}(N-k_0)$.

3.6. **A spherical process.** We recall that $\mathbb{S}$, π_H, $\pi_{u^\perp}$ and b were introduced in Section 2 and introduce the possibly exploding (with life-time ξ) process $(U_t)_{t\in[0,\xi)}$ with values in $\mathbb{S} \cap E_{k_0}$, informally solving (we will also use here the theory of Dirichlet spaces), for some given $U_0 \in \mathbb{S} \cap E_{k_0}$ and some $(\mathbb{R}^2)^N$-valued Brownian motion $(B_t)_{t\geq 0}$,

$$U_t = U_0 + \int_0^t \pi_{U_s^\perp} \pi_H \mathrm{d}B_s + \int_0^t \pi_{U_s^\perp} \pi_H b(U_s) \mathrm{d}s - \frac{2N-3}{2} \int_0^t U_s \mathrm{d}s.$$

We call such a process a $SKS(\theta,N)$-process.

One can check that this process is β-symmetric, where β is defined in (8), and that β is Radon on $\mathbb{S} \cap E_{k_0}$, see Proposition A.3. And we will see that if $k_0 \geq N$, then $\beta(\mathbb{S}) < \infty$, so that the process $(U_t)_{t\geq 0}$ is non-exploding and positive recurrent.

3.7. **Decomposition of the process.** We assume that $N \geq 2$ and $\theta > 0$ are such $d_{\theta,N}(N) < 2$ and, as usual, $N > \theta$. We consider a 2-dimensional Brownian $(M_t)_{t\geq 0}$ with diffusion constant $N^{-1/2}$, a squared Bessel process $(D_t)_{t\in[0,\tau_D)}$ with dimension $d_{\theta,N}(N)$ killed when it hits 0, with life-time τ_D, and a $SKS(\theta, N)$-process $(U_t)_{t\in[0,\xi)}$, these three processes being independent. We introduce the time-change

$$A_t = \int_0^t \frac{\mathrm{d}s}{D_s}, \quad t \in [0, \tau_D).$$

Since $\tau_D < \infty$ (because $d_{\theta,N}(N) < 2$), since $D_{\tau_D} = 0$ and since, roughly, the paths of $(\sqrt{D_t})_{t\in[0,\tau_D)}$ are 1/2-Hölder continuous, it holds that $A_{\tau_D} = \infty$ a.s. We introduce the inverse function $\rho : [0,\infty) \to [0, \tau_D)$ of $A : [0, \tau_D) \to [0,\infty)$.

We also set $\zeta' = \rho_\xi$ and observe that $\zeta' \leq \tau_D$, since ρ is $[0, \tau_D)$-valued, and that $\zeta' < \tau_D$ if and only if $\xi < \infty$. A tedious but straightforward computation shows that, recalling (9),

$$X_t = \Psi(M_t, D_t, U_{A_t}), \qquad \text{i.e.} \quad X_t^i = M_t + \sqrt{D_t}U^i_{A_t}, \qquad i \in [\![1, N]\!],$$

which is well-defined during $[0, \zeta')$, solves (1).

This decomposition of the $KS(\theta, N)$-process, which is noticeable in that U satisfies an autonomous S.D.E. and thus is Markov, is at the basis of our analysis.

In other words, $(X_t)_{t\in[0,\zeta')}$ is the restriction to the time interval $[0, \zeta')$ of a $KS(\theta, N)$-process $(X_t)_{t\in[0,\zeta)}$. Moreover, we have $\zeta' = \zeta \wedge \tau_D$: if ξ is finite, then U gets out of $\mathbb{S} \cap E_{k_0}$ at time ξ, so that X gets out of E_{k_0} at time $\zeta' = \rho_\xi < \tau_D$, whence $\zeta = \zeta' = \zeta \wedge \tau_D$; if next $\xi = \infty$, then $\zeta' = \tau_D$ and U remains in E_{k_0} for all times, so that X remains in E_{k_0} during $[0, \tau_D)$, whence $\zeta \geq \tau_D$.

We have $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta \wedge \tau_D)$, because U is $\mathbb{S}$-valued. By definition of $\mathbb{S}$, the process U cannot have any $[\![1, N]\!]$-collision. But for any $K \subset [\![1, N]\!]$ with cardinal at most $N - 1$,

(16) $\qquad U$ has a K-collision at $t \in [0, \xi)$ if and only if X has a K-collision at $\rho_t \in [0, \zeta \wedge \tau_D)$.

Moreover, as seen a few lines above, $\xi < \infty$ is equivalent to $\zeta < \tau_D$. In other words, since $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta \wedge \tau_D)$ and since $\tau_D = \inf\{t > 0 : D_t = 0\}$, we have

$$\xi < \infty \qquad \text{if and only if} \qquad \inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0. \tag{17}$$

3.8. **Some special cases.** Using the Girsanov theorem, see Subsection 3.4, we will manage to reduce a large part of the study to the special cases that we examine in the present subsection. Here we explain the following facts, for $N \geq 2$ and $\theta > 0$ with $N > \theta$:

(a) if $d_{\theta,N}(N - 1) \in (0, 2)$, then a.s., $\tau_D = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) = 0\} \leq \zeta$ and for all $r \in [0, \tau_D)$, all $K \subset [\![1, N]\!]$ with $|K| = N - 1$, $(X_t)_{t\in[0,\zeta)}$ has infinitely many K-collisions during $[r, \tau_D)$;

(b) if $d_{\theta,N}(N - 1) \leq 0$ (whence $k_0 \leq N - 1$), then a.s., $\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0$.

We keep the same notation as in the previous subsection.

(i) We first verify that in (a), $\tau_D \leq \zeta$. Since $d_{\theta,N}(N - 1) \in (0, 2)$, it holds that $k_0 \geq N$. If first $k_0 > N$, then $\zeta = \infty$ by Subsection 3.2 and we are done. If next $k_0 = N$, then $\zeta < \infty$ and $X_{\zeta-}$

exists by Subsection 3.5. Moreover $X_{\zeta-}$ cannot belong to $E_{k_0} = E_N$ by definition of ζ and thus has its N particles at the same place, i.e. $R_{[\![1,N]\!]}(X_{\zeta-}) = 0$: we have $\zeta = \tau_D$.

(ii) In (b), $\zeta < \infty$ by Subsection 3.5 because $d_{\theta,N}(N-1) \leq 0$ implies that $\theta \geq 2$.

(iii) We consider, in any case, the spherical process $(U_t)_{t\in[0,\xi)}$ and assume that $\xi = \infty$. A tedious Itô computation shows that for $K \subset [\![1, N]\!]$, for some 1-dimensional Brownian motion $(W_t)_{t\geq 0}$,

$$\begin{aligned} \mathrm{d}R_K(U_t) =& 2\sqrt{R_K(U_t)(1-R_K(U_t))}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t - d_{\theta,N}(N)R_K(U_t)\mathrm{d}t \\ &- \frac{2\theta}{N}\sum_{i\in K, j\notin K} \frac{U_t^i - U_t^j}{||U_t^i - U_t^j||^2}\cdot (U_t^i - S_K(U_t))\mathrm{d}t. \end{aligned}$$

We fix $\varepsilon > 0$ to be chosen later. During time intervals where $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\| \geq \varepsilon$, we thus have, for some constant C_ε,

$$\mathrm{d}R_K(U_t) \leq 2\sqrt{R_K(U_t)(1-R_K(U_t))}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t + C_\varepsilon\sqrt{R_K(U_t)}\mathrm{d}t, \tag{18}$$

where we used the Cauchy-Schwarz inequality and that $R_K(U_t)$ is uniformly bounded (because U is $\mathbb{S}$-valued). Hence, still during time intervals where $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\| \geq \varepsilon$, by comparison, $R_K(U_t)$ is smaller than S_t, the solution to

$$\mathrm{d}S_t = 2\sqrt{S_t(1-S_t)}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t + C_\varepsilon\sqrt{S_t}\mathrm{d}t. \tag{19}$$

And a little study involving scale functions/speed measures shows that this process hits zero in finite time if and only if $d_{\theta,N}(|K|) < 2$, exactly as a squared Bessel process with dimension $d_{\theta,N}(|K|)$.

(iv) We end the proof of (a). In this case, $k_0 \geq N$, so that U is non-exploding, as seen in Subsection 3.6. Hence $\xi = \infty$ and we can use (iii). Moreover, U is recurrent, still by Subsection 3.6. We fix K with $|K| = N-1$ and we choose $\varepsilon > 0$ small enough so that we have

$$\beta\Big(\Big\{u \in \mathbb{S} : \min_{i\in K, j\notin K} \|u^i - u^j\| \geq \varepsilon\Big\}\Big) > 0,$$

where β is the invariant measure (8) of U. Hence the process $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\|$ visits the zone (ε, ∞) infinitely often and each time, $R_K(U)$ has a (uniformly) positive probability to hit 0 by (iii) and since $d_{\theta,N}(|K|) = d_{\theta,N}(N-1) < 2$. Consequently, for any $s > 0$, $(U_t)_{t\geq 0}$ has infinitely many K-collisions during $[s, \infty)$. Recalling (16) and that $\zeta \wedge \tau_D = \tau_D$ by (i), we conclude that for any $r \in [0, \tau_D)$, $(X_t)_{t\in[0,\zeta)}$ has infinitely many K-collisions during $[r, \tau_D)$.

(v) We finally complete the proof of (b). By (17), it is sufficient to show that $\xi < \infty$ a.s.

Assume that U is recurrent (and thus non-exploding). Then we take $K = [\![2, N]\!]$ and apply the same reasoning as in (iv): since $d_{\theta,N}(|K|) \leq 0 < 2$, $R_K(U)$ hits zero in finite time and this makes U get out of E_{N-1} and thus explode, since U is $(E_{k_0} \cap \mathbb{S})$-valued and since $k_0 \leq N-1$. We thus have a contradiction.

Hence U is transient and it eventually gets out of the compact of $E_{k_0} \cap \mathbb{S}$

$$\mathcal{K} = \{u \in \mathbb{S} : \forall K \subset [\![1, N]\!] \text{ such that } |K| = k_0, \text{ we have } R_K(u) \geq \varepsilon\},$$

for any fixed $\varepsilon > 0$. Hence on the event where $\xi = \infty$, $\lim_{t\to\infty} \min_{|K|=k_0} R_K(U_t) = 0$ a.s. Recalling now that $k_0 \leq N-1$ and that U is $\mathbb{S}$-valued (whence $R_{[\![1,N]\!]}(U_t) = 1$) we can a.s. find K with $|K| \in [\![k_0, N-1]\!]$ such that $\liminf_{t\to\infty} R_K(U_t) = 0$ but $\liminf_{t\to\infty} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 0$. It is then not too hard to find $\alpha > 0$ and $\varepsilon > 0$ such that each time $R_K(U_t) < \alpha$ (which often happens), all the particles indexed in K are far from all the other ones with a distance greater than $\varepsilon > 0$.

We conclude from (iii), since $d_{\theta,N}(|K|) \leq 0$ (because $|K| \geq k_0$) that each time $R_K(U_t) < \alpha$, it has a (uniformly) positive probability to hit zero. On the event $\xi = \infty$, this will eventually happen, so that the process U will have a K-collision and thus will leave E_{k_0} in finite time. Hence U will explode, so that $\xi < \infty$.

3.9. **Size of the cluster.** We assume that $N > 3\theta \geq 6$. Hence $\zeta < \infty$ and $X_{\zeta-}$ exists, by Subsection 3.5. Moreover, by definition of ζ, we know that $X_{\zeta-} \notin E_{k_0}$. We want now to show that $X_{\zeta-} \in E_{k_0+1}$, i.e. that the cluster causing explosion is precisely composed of k_0 particles. If $k_0 = N$, there is nothing to do, since then $E_{k_0+1} = (\mathbb{R}^2)^N$. Now if $k_0 \leq N-1$, we assume by contradiction, that there is $K \subset [\![1, N]\!]$ with $|K| \geq k_0 + 1$ such that $R_K(X_{\zeta-}) = 0$ and $\min_{i \notin K} R_{K\cup\{i\}}(X_{\zeta-}) > 0$. Then there is $\alpha > 0$ such that during $[\zeta - \alpha, \zeta)$, the particles indexed in K are far from the other ones, so that $(X^i_t)_{t\in[0,\zeta), i\in K}$ behaves like a $KS(\theta|K|/N, |K|)$-process by Subsection 3.3. Observe now that $d_{\theta|K|/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) \leq 0$ because $|K|-1 \geq k_0$ and $|K| > \theta|K|/N$ because $N > \theta$. We thus know from the special case (b) of Subsection 3.8 that $\inf_{t\in[\zeta-\alpha,\zeta)} R_K(X_t) > 0$, which contradicts the fact that $R_K(X_{\zeta-}) = 0$.

3.10. **Collisions before explosion.** We fix again $N > 3\theta \geq 6$. We recall that $k_1 = k_0 - 1$ and we show that there are infinitely many k_1-ary collisions just before explosion. We know from the previous subsection that there exists $K_0 \subset [\![1, N]\!]$ such that $|K_0| = k_0$ and $R_{K_0}(X_{\zeta-}) = 0$ and $\min_{i\notin K_0} R_{K_0\cup\{i\}}(X_{\zeta-}) > 0$. Then there is $\alpha > 0$ such that during $[\zeta - \alpha, \zeta)$, the particles indexed in K_0 are far from the other ones, so that $(X^i_t)_{i\in K_0}$ behaves like a $KS(\theta k_0/N, k_0)$-process by Subsection 3.3. Observe now that $d_{\theta k_0/N,k_0}(k_0-1) = d_{\theta,N}(k_0-1) \in (0, 2)$ thanks to Lemma 1 and that $k_0 > \theta k_0/N$ because $N > \theta$. We thus know from the special case (a) of Subsection 3.8 that $(X^i_t)_{i\in K_0}$ has infinitely many $(K_0 \setminus \{i\})$-collisions just before ζ, for all $i \in K_0$.

When $k_2 = k_1 - 1$, one can show in the very same way that for all K with $|K| = k_1$, for all $i \in K$, there are infinitely many $(K \setminus \{i\})$-collisions just before each K-collision. We may also use Subsection 3.8-(a), since $d_{\theta k_1/N,k_1}(k_1 - 1) = d_{\theta,N}(k_2) \in (0, 2)$, see Lemma 1.

3.11. **Absence of other collisions.** We want to show that when $N > 3\theta \geq 6$, for $K \subset [\![1, N]\!]$ with $|K| \in [\![3, k_2 - 1]\!]$, there is no K-collision during $(0, \zeta)$. Suppose by contradiction that there is $K \subset [\![1, N]\!]$ with $|K| \in [\![3, k_2-1]\!]$ and $t \in (0, \zeta)$ such that $R_K(X_t) = 0$ and for all $i \notin K$, $R_{K\cup\{i\}}(X_t) > 0$. Then there is $\alpha > 0$ such that during $[t - \alpha, t]$, the particles indexed in K are far from the other ones, so that $R_K(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta|K|/N,|K|}(|K|)$, see Subsection 3.4. Since $d_{\theta|K|/N,|K|}(|K|) = d_{\theta,N}(|K|) \geq 2$ because $|K| \in [\![3, k_2 - 1]\!]$, see Lemma 1, such a Bessel process cannot hit zero, whence a contradiction.

3.12. **Binary collisions.** We still assume that $N > 3\theta \geq 6$, we suppose that there is a K-collision for some $K \subset [\![1, N]\!]$ such that $|K| = k_2$ at some time $t \in (0, \zeta)$ and we want to show that there are infinitely many binary collisions just before t. There is $\alpha > 0$ such that the particles indexed in K are far from all the other ones during $[t-\alpha, t]$, so that Subsection 3.3 tells us that $(X^i_t)_{i\in K}$ behaves like a $KS(\theta k_2/N, k_2)$-process. We observe that $k_2 \geq 5$, that $d_{\theta k_2/N,k_2}(k_2 - 1) = d_{\theta,N}(k_2 - 1) \geq 2$ and that $d_{\theta k_2/N,k_2}(k_2) = d_{\theta,N}(k_2) \in (0, 2)$ by Lemma 1.

We are reduced to show that a $KS(\theta, N)$-process, that we still denote by $(X^i_t)_{i\in[\![1,N]\!],t\geq 0}$, such that $N \geq 5$, $d_{\theta,N}(N-1) \geq 2$ and $d_{\theta,N}(N) \in (0, 2)$, a.s. has infinitely many binary collisions before the first instant τ_D of $[\![1, N]\!]$-collision. Such a process does not explode, because $k_0 > N$ (since $d_{\theta,N}(N) > 0$), see Subsection 3.2. Hence using (16) (which is licit since $d_{\theta,N}(N) < 2$), we only have to show that e.g. U^1 collides infinitely often with U^2 during $[0, \infty)$.

First, one easily gets convinced that the probability that e.g. X^1 collides with X^2 before τ_D is positive, because the probability that all the particles are pairwise far from each other, except X^1 and X^2, during the time interval $[0,1]$, is positive. On this kind of event, by Subsection 3.4, $R_{\{1,2\}}(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(2) \in (0,2)$ and thus hits zero during $[0,1]$ (and thus before τ_D) with positive probability.

Using again (16), we conclude that the probability that U^1 collides with U^2 in finite time is positive. Since now U is positive recurrent, recall Subsection 3.6 and that $k_0 > N$ (because $d_{\theta,N}(N) > 0$), we conclude that U^1 collides infinitely often with U^2 during $[0,\infty)$ as desired.

4. Construction of the Keller-Segel particle system

The aim of this section is to build a first version of the Keller-Segel particle system using the book of Fukushima-Oshima-Takeda [11]. We also build a $\mathbb{S}$-valued process for later use.

Proposition 6. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$, recall that $k_0 = \lceil 2N/\theta \rceil$ and that μ and β were defined in* (4) *and* (8). *We set $\mathcal{X} = E_{k_0}$ and $\mathcal{X}_\triangle = \mathcal{X} \cup \{\triangle\}$, as well as $\mathcal{U} = \mathbb{S} \cap E_{k_0}$ and $\mathcal{U}_\triangle = \mathcal{U} \cup \{\triangle\}$, where $\triangle$ is a cemetery point.*

(i) There exists a unique continuous Hunt process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}^X_x)_{x\in\mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, with regular Dirichlet space $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2((\mathbb{R}^2)^N, \mu)$ with core $C^\infty_c(\mathcal{X})$ defined by

$$\text{for all} \quad \varphi \in C^\infty_c(\mathcal{X}), \quad \mathcal{E}^X(\varphi,\varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu.$$

We call such a process a $QKS(\theta, N)$-process and denote by $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$ its life-time.

(ii) There exists a unique continuous Hunt process $\mathbb{U} = (\Omega^U, \mathcal{M}^U, (U_t)_{t\geq 0}, (\mathbb{P}^U_u)_{u\in\mathcal{U}_\triangle})$ with values in $\mathcal{U}_\triangle$, which is β-symmetric, with regular Dirichlet space $(\mathcal{E}^U, \mathcal{F}^U)$ on $L^2(\mathbb{S}, \beta)$ with core $C^\infty_c(\mathcal{U})$ defined by

$$\text{for all} \quad \varphi \in C^\infty_c(\mathcal{U}), \quad \mathcal{E}^U(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{S}} \|\nabla_{\mathbb{S}}\varphi\|^2 \mathrm{d}\beta.$$

We call such a process a $QSKS(\theta, N)$ -process and denote by $\xi = \inf\{t \geq 0 : U_t = \triangle\}$ its life-time.

The proof that we can build a $KS(\theta, N)$-process, i.e. a $QKS(\theta, N)$-process such that $\mathbb{P}^X_x \circ X_t^{-1}$ has density for all $x \in E_2$ and all $t > 0$ will be handled in Section 11.

We refer to Subsection B.1 for some explanations about the notions used in this proposition: link between a Hunt process, its generator, semi-group and its Dirichlet space, definition of the one-point compactification topology, i.e. the topology endowing $\mathcal{X}_\triangle$ and $\mathcal{U}_\triangle$, and about the *quasi* notion. The state $\triangle$ is absorbing, i.e. $X_t = \triangle$ for all $t \geq \zeta$ and $U_t = \triangle$ for all $t \geq \xi$.

Remark 7. *By definition of the one-point compactification topology, for any increasing sequence of compact subsets $(\mathcal{K}_n)_{n\geq 1}$ of $\mathcal{X}$ such that $\cup_{n\geq 1}\mathcal{K}_n = \mathcal{X}$, $\zeta = \lim_{n\to\infty} \inf\{t \geq 0 : X_t \notin \mathcal{K}_n\}$.*

Similarly, for any increasing sequence of compact subsets $(\mathcal{L}_n)_{n\geq 1}$ of $\mathcal{U}$ such that $\cup_{n\geq 1}\mathcal{L}_n = \mathcal{U}$, $\xi = \lim_{n\to\infty} \inf\{t \geq 0 : U_t \notin \mathcal{L}_n\}$.

The uniqueness stated e.g. in Proposition 6-(i) has to be understood in the following sense, see [11, Theorem 4.2.8 p 167]: if we have another Hunt process $\mathbb{Y} = (\Omega^Y, \mathcal{M}^Y, (Y_t)_{t\geq 0}, (\mathbb{P}^Y_x)_{x\in\mathcal{X}})$ enjoying the same properties, then for quasi-all $x \in \mathcal{X}$, the law of $(Y_t)_{t\geq 0}$ under $\mathbb{P}^Y_x$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_x$. The *quasi* notion depends on the Hunt process under consideration but, as recalled in Subsection B.1, two Hunt processes with the same Dirichlet space share the same *quasi* notion.

Proof of Proposition 6. We start with (i). We consider the bilinear form $\mathcal{E}^X$ on $C_c^\infty(\mathcal{X})$ defined by $\mathcal{E}^X(\varphi,\varphi) = \int_{(\mathbb{R}^2)^N} ||\nabla\varphi||^2 \mathrm{d}\mu$. It is well-defined, since μ is Radon on $\mathcal{X} = E_{k_0}$ by Proposition A.1.

We first show that it is closable, see [11, page 2], i.e. that if $(\varphi_n)_{n\geq 1} \subset C_c^\infty(\mathcal{X})$ is such that $\lim_n \varphi_n = 0$ in $L^2((\mathbb{R}^2)^N,\mu)$ and $\lim_{n,m} \mathcal{E}^X(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0$, then $\lim_n \mathcal{E}^X(\varphi_n,\varphi_n) = 0$: since $\nabla\varphi_n$ is a Cauchy sequence in $L^2((\mathbb{R}^2)^N,\mu)$, it converges to a limit g and it suffices to prove that $g = 0$ a.e. For $\psi \in C_c^\infty(E_2,(\mathbb{R}^2)^N)$, we have $\int_{(\mathbb{R}^2)^N} g\cdot\psi \mathrm{d}\mu = \lim_n \int_{(\mathbb{R}^2)^N} \nabla\varphi_n\cdot\psi\mathrm{d}\mu$. But, recalling (4),

$$\int_{(\mathbb{R}^2)^N} \nabla\varphi_n\cdot\psi\mathrm{d}\mu = \int_{(\mathbb{R}^2)^N} \nabla\varphi_n(x)\cdot\psi(x)\mathbf{m}(x)\mathrm{d}x = -\int_{(\mathbb{R}^2)^N} \varphi_n(x)\mathrm{div}(\mathbf{m}(x)\psi(x))\mathrm{d}x.$$

Thus by the Cauchy-Schwarz inequality,

$$\Big|\int_{(\mathbb{R}^2)^N} \nabla\varphi_n\cdot\psi\mathrm{d}\mu\Big| \leq \Big(\int_{(\mathbb{R}^2)^N} \varphi_n^2\mathrm{d}\mu\Big)^{1/2}\Big(\int_{(\mathbb{R}^2)^N} \frac{|\mathrm{div}(\mathbf{m}(x)\psi(x))|^2}{\mathbf{m}(x)}\mathrm{d}x\Big)^{1/2},$$

which tends to 0 since $\lim_n \varphi_n = 0$ in $L^2((\mathbb{R}^2)^N,\mu)$, since $\psi \in C_c^\infty(E_2,(\mathbb{R}^2)^N)$ and since $\mathbf{m}$ is smooth and positive on E_2. Thus $\int_{(\mathbb{R}^2)^N} g\cdot\psi\mathrm{d}\mu = 0$ for all $\psi \in C_c^\infty(E_2,(\mathbb{R}^2)^N)$, so that $g = 0$ a.e.

We can thus consider the extension of $\mathcal{E}^X$ to $\mathcal{F}^X = \overline{C_c^\infty(\mathcal{X})}^{\mathcal{E}_1^X}$, where we have set $\mathcal{E}_1^X(\varphi,\varphi) = \int_{(\mathbb{R}^2)^N}(\varphi^2 + \frac{1}{2}||\nabla\varphi||^2)\mathrm{d}\mu$ for $\varphi \in C_c^\infty(\mathcal{X})$.

Next, $(\mathcal{E}^X,\mathcal{F}^X)$ is obviously regular with core $C_c^\infty(\mathcal{X})$, see [11, page 6], because $C_c^\infty(\mathcal{X})$ is dense in $\mathcal{F}^X$ for the norm associated to $\mathcal{E}_1^X$ by definition of $\mathcal{F}^X$ and $C_c^\infty(\mathcal{X})$ is dense, for the uniform norm, in $C_c(\mathcal{X})$. It is also strongly local, see [11, page 6], i.e. $\mathcal{E}^X(\varphi,\psi) = \int_{(\mathbb{R}^2)^N} \nabla\varphi\cdot\nabla\psi\mathrm{d}\mu = 0$ if $\varphi,\psi \in C_c^\infty(\mathcal{X})$ and if φ is constant on a neighborhood of Supp ψ.

Then [11, Theorems 7.2.2 page 380 and 4.2.8 page 167] imply the existence and uniqueness of a Hunt process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, of which the Dirichlet space is $(\mathcal{E}^X,\mathcal{F}^X)$, and such that $t\mapsto X_t$ is $\mathbb{P}_x^X$-a.s. continuous on $[0,\zeta)$ for all $x \in \mathcal{X}$, where $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$.

Furthermore, since $\mathcal{E}^X$ is strongly local, we know from [11, Theorem 4.5.3 page 186] that we can choose $\mathbb{X}$ (modifying $\mathbb{P}_x^X$ only for a quasi-null set of values of x) such that $\mathbb{P}_x(\zeta < \infty, X_{\zeta-} = \triangle) = 1$ for all $x \in \mathcal{X}$. This implies that for all $x \in \mathcal{X}$, $\mathbb{P}_x$-a.s., the map $t \mapsto X_t$ is continuous from $[0,\infty)$ to $\mathcal{X}_\triangle$, endowed with the one-point compactification topology on $\mathcal{X}_\triangle$ recalled in Subsection B.1.

For (ii), the very same strategy applies. The only difference is the integration by parts to be used for the closability: for $\varphi \in C_c^1(\mathcal{U})$ and $\psi \in C_c^1(\mathbb{S}\cap E_2,(\mathbb{R}^2)^N)$, it classically holds that

$$\int_{\mathbb{S}}(\nabla_{\mathbb{S}}\varphi)\cdot\psi\mathrm{d}\beta = \int_{\mathbb{S}}(\nabla_{\mathbb{S}}\varphi(u))\cdot\psi(u)\mathbf{m}(u)\sigma(\mathrm{d}u) = -\int_{\mathbb{S}}\varphi(u)\mathrm{div}_{\mathbb{S}}(\mathbf{m}(u)\psi(u))\sigma(\mathrm{d}u). \tag{20}$$

This can be shown naively using Lemma A.2. □

We now make explicit the generators of $\mathbb{X}$ and $\mathbb{U}$ when applied to some functions enjoying a few properties. See Subsection B.1 for a precise definition of the generator of a Hunt process. We have to introduce a few notation.

For $\varphi \in C^\infty((\mathbb{R}^2)^N)$, $\alpha \in (0,1]$ and $x \in (\mathbb{R}^2)^N$, we set

$$\mathcal{L}^X_\alpha \varphi(x) = \frac{1}{2}\Delta\varphi(x) - \frac{\theta}{N}\sum_{1\le i\ne j\le N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot(\nabla\varphi(x))^i = \frac{1}{2\mathbf{m}_\alpha(x)}\mathrm{div}[\mathbf{m}_\alpha(x)\nabla\varphi(x)], \tag{21}$$

where

$$\mathbf{m}_\alpha(x) = \prod_{1\le i\ne j\le N} (\|x^i - x^j\|^2 + \alpha)^{-\theta/(2N)}.$$

Such a formula makes sense for $x \in E_2$ when $\alpha = 0$ (with $\mathbf{m}_\alpha$ replaced by $\mathbf{m}$) and we recall that for $\varphi \in C^\infty((\mathbb{R}^2)^N)$ and $x \in E_2$, $\mathcal{L}^X\varphi(x)$ was defined in (5) by $\mathcal{L}^X\varphi(x) = \mathcal{L}^X_0\varphi(x)$.

For $\varphi \in C^\infty(\mathbb{S})$, $\alpha \in (0,1]$ and $u \in \mathbb{S}$, we set

$$\mathcal{L}^U_\alpha \varphi(u) = \frac{1}{2}\Delta_{\mathbb{S}}\varphi(u) - \frac{\theta}{N}\sum_{1\le i\ne j\le N} \frac{u^i - u^j}{\|u^i - u^j\|^2 + \alpha}\cdot(\nabla_{\mathbb{S}}\varphi(u))^i = \frac{1}{2\mathbf{m}_\alpha(u)}\mathrm{div}_{\mathbb{S}}[\mathbf{m}_\alpha(u)\nabla_{\mathbb{S}}\varphi(u)]. \tag{22}$$

This formula makes sense for $u \in \mathbb{S}\cap E_2$ when $\alpha = 0$ (with $\mathbf{m}_\alpha$ replaced by $\mathbf{m}$) and we set, for $\varphi \in C^\infty(\mathbb{S})$ and $u \in \mathbb{S}\cap E_2$, $\mathcal{L}^U\varphi(u) = \mathcal{L}^U_0\varphi(u)$.

Remark 8. *(i) Denote by $(\mathcal{A}^X, \mathcal{D}_{A^X})$ the generator of the process $\mathbb{X}$ of Proposition 6-(i). If $\varphi \in C^\infty_c(\mathcal{X})$ satisfies $\sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha\varphi(x)| < \infty$, then $\varphi \in \mathcal{D}_{A^X}$ and $\mathcal{A}^X\varphi = \mathcal{L}^X\varphi$.*

(ii) Denote by $(\mathcal{A}^U, \mathcal{D}_{A^U})$ the generator of the process $\mathbb{U}$ of Proposition 6-(ii). If $\varphi \in C^\infty_c(\mathcal{U})$ satisfies $\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}} |\mathcal{L}^U_\alpha\varphi(u)| < \infty$, then $\varphi \in \mathcal{D}_{A^U}$ and $\mathcal{A}^U\varphi = \mathcal{L}^U\varphi$.

Proof. To check (i), it suffices by (B.1) to verify that (a) $\varphi \in \mathcal{F}^X$, (b) $\mathcal{L}^X\varphi \in L^2(\mathcal{X},\mu)$ and (c) for all $\psi \in \mathcal{F}^X$, we have $\mathcal{E}^X(\varphi,\psi) = -\int_{\mathcal{X}}(\mathcal{L}^X\varphi)\psi\mathrm{d}\mu$.

Point (a) is clear, since $\varphi \in C^\infty_c(\mathcal{X})$. Point (b) follows from the facts that μ is Radon on $\mathcal{X}$, that φ is compactly supported in $\mathcal{X}$ and that $\mathcal{L}^X\varphi \in L^\infty((\mathbb{R}^2)^N, \mathrm{d}x)$, because for all $x \in E_2$, $\mathcal{L}^X\varphi(x) = \lim_{\alpha\to 0}\mathcal{L}^X_\alpha\varphi(x)$. Concerning (c) it suffices, by definition of $(\mathcal{E}^X, \mathcal{F}^X)$ and since $\mathcal{L}^X\varphi \in L^2(\mathcal{X},\mu)$, to show that for all $\psi \in C^\infty_c(\mathcal{X})$, we have $\frac{1}{2}\int_{(\mathbb{R}^2)^N}\nabla\varphi\cdot\nabla\psi\mathrm{d}\mu = -\int_{(\mathbb{R}^2)^N}(\mathcal{L}^X\varphi)\psi\mathrm{d}\mu$. But for $\alpha \in (0,1]$, by a standard integration by parts, since φ, ψ and $\mathbf{m}_\alpha$ are smooth,

$$\begin{aligned}\frac{1}{2}\int_{(\mathbb{R}^2)^N}\nabla\varphi(x)\cdot\nabla\psi(x)\mathbf{m}_\alpha(x)\mathrm{d}x &= -\frac{1}{2}\int_{(\mathbb{R}^2)^N}\mathrm{div}(\mathbf{m}_\alpha(x)\nabla\varphi(x))\psi(x)\mathrm{d}x\\ &= -\int_{(\mathbb{R}^2)^N}[\mathcal{L}^X_\alpha\varphi(x)]\psi(x)\mathbf{m}_\alpha(x)\mathrm{d}x.\end{aligned}$$

We conclude letting $\alpha \to 0$ by dominated convergence, since $\mathbf{m}_\alpha \to \mathbf{m}$ and $\mathcal{L}^X_\alpha\varphi \to \mathcal{L}^X\varphi$ a.e., since by assumption, $|\nabla\varphi(x)\cdot\nabla\psi(x)\mathbf{m}_\alpha(x)| + |[\mathcal{L}^X_\alpha\varphi(x)]\psi(x)\mathbf{m}_\alpha(x)| \le C\mathbb{1}_{\{x\in\mathcal{K}\}}\mathbf{m}(x)$ for some constant C and for $\mathcal{K}$ =Supp ψ which is compact in $\mathcal{X}$, and since $\mu(\mathcal{K}) = \int_{\mathcal{K}}\mathbf{m}(x)\mathrm{d}x < \infty$.

The proof of (ii) is exactly the same, using that if $\varphi, \psi \in C^\infty(\mathbb{S})$, it holds that

$$\frac{1}{2}\int_{\mathbb{S}}\nabla_{\mathbb{S}}\varphi\cdot\nabla_{\mathbb{S}}\psi\ \mathbf{m}_\alpha\mathrm{d}\sigma = -\frac{1}{2}\int_{\mathbb{S}}\mathrm{div}_{\mathbb{S}}(\mathbf{m}_\alpha\nabla_{\mathbb{S}}\varphi)\psi\mathrm{d}\sigma = -\int_{\mathbb{S}}[\mathcal{L}^U_\alpha\varphi]\psi\mathbf{m}_\alpha\mathrm{d}\sigma,$$

which can be shown naively using the projection $\Phi_{\mathbb{S}}$, see (10), and Lemma A.2. □

We end the section with a quick irreducibility/recurrence/transience study of the spherical process, see Subsection B.1 again for definitions.

Lemma 9. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and consider the process $\mathbb{U}$ and its Dirichlet space $(\mathcal{E}^U, \mathcal{F}^U)$ as in Proposition 6-(ii).*

(i) $(\mathcal{E}^U, \mathcal{F}^U)$ is irreducible and we have the alternative:

• *either $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent and in particular it is non-exploding and for all measurable $A \subset \mathcal{U}$ such that $\beta(A) > 0$, for quasi all $u \in \mathcal{U}$, $\mathbb{P}^U_u(\limsup_{t\to\infty}\{U_t \in A\}) = 1$;*

• *or $(\mathcal{E}^U, \mathcal{F}^U)$ is transient and in particular for all compact set $\mathcal{K}$ of $\mathcal{U}$, for quasi all $u \in \mathcal{U}$, we have $\mathbb{P}^U_u(\liminf_{t\to\infty}\{U_t \in \mathcal{K}\}) = 0$.*

(ii) If $d_{\theta,N}(N-1) > 0$, then $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent.

In the transient case, one might also prove that $\mathbb{P}^U_u(\limsup_{t\to\infty}\{U_t \in \mathcal{K}\}) = 0$, but this would be useless for our purpose.

Proof. We start with (i). We first show that in any case, $(\mathcal{E}^U, \mathcal{F}^U)$ is irreducible. By [11, Corollary 4.6.4 page 195] and since $\mathcal{E}^U(\varphi,\varphi) = \int_{\mathbb{S}} \|\nabla_{\mathbb{S}}\varphi\|^2 \mathbf{m} d\sigma$ with $\mathbf{m}$ bounded from below by a constant (on $\mathbb{S}$), it suffices to prove that the σ-symmetric Hunt process with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{U}, \sigma)$ with core $C^\infty_c(\mathcal{U})$ such that for all $\varphi \in C^\infty_c(\mathcal{U})$, $\mathcal{E}(\varphi,\varphi) = \int_{\mathbb{S}} \|\nabla_{\mathbb{S}}\varphi\|^2 d\sigma$ is irreducible. But this Hunt process is nothing but a $\mathbb{S}$-valued Brownian motion. This Brownian motion is *a priori* killed when it gets out of $\mathcal{U}$, but this does a.s. never occur since such a Brownian motion never has two (bi-dimensional) coordinates equal. This $\mathbb{S}$-valued Brownian motion is of course irreducible. We conclude from [11, Lemma 1.6.4 page 55] that $(\mathcal{E}^U, \mathcal{F}^U)$ is either recurrent or transient.

• When $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent, [11, Theorem 4.7.1-(iii) page 202] gives us the result.

• When $(\mathcal{E}^U, \mathcal{F}^U)$ is transient, we fix a compact set $\mathcal{K}$ of $\mathcal{U}$ and we know from Lemma A.3 that $\beta(\mathcal{K}) < \infty$, so that by definition of transience, for β-a.e $u \in \mathcal{U}$, $\mathbb{E}^U_u[\int_0^\infty \mathbb{1}_{\mathcal{K}}(U_s)ds] < \infty$. Setting $\tau_{\mathcal{K}^c} = \inf\{t \geq 0 : U_t \notin \mathcal{K}\}$, we get in particular that for β-a.e $u \in \mathcal{U}$, $\mathbb{P}^U_u(\tau_{\mathcal{K}^c} < \infty) = 1$. But, by [11, (4.1.9) page 155], $u \mapsto \mathbb{P}^U_u(\tau_{\mathcal{K}^c} < \infty)$ is finely continuous. Using [11, Lemma 4.1.5 page 155], we deduce that for quasi all $u \in \mathcal{U}$, $\mathbb{P}^U_u(\tau_{\mathcal{K}^c} < \infty) = 1$. The Markov property allows us to conclude.

Concerning (ii), we recall from Proposition A.3 that $\beta(\mathbb{S}) < \infty$, because $d_{\theta,N}(N-1) > 0$ implies that $k_0 \geq N$, see Lemma 1. Moreover, $k_0 \geq N$ implies that $E_{k_0} \supset E_N \supset \mathbb{S}$, whence $\mathcal{U} = E_{k_0} \cap \mathbb{S} = \mathbb{S}$ is compact: the process cannot explode, i.e. $\xi = \infty$. Consequently, $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent, since $\varphi \equiv 1$ belongs to $L^1(\mathcal{U}, \beta)$ and since $\mathbb{E}^U_u[\int_0^\infty \varphi(U_s)ds] = \mathbb{E}^U_u[\xi] = \infty$. Indeed, as recalled Subsection B.1, if $(\mathcal{E}^U, \mathcal{F}^U)$ was transient, we would have $\mathbb{E}^U_u[\int_0^\infty \varphi(U_s)ds] < \infty$ for all $\varphi \in L^1(\mathcal{U}, \beta)$, with the convention that $\varphi(\triangle) = 0$. □

5. Decomposition

The goal of this section is to prove the following decomposition of the Keller-Segel particle system defined in Proposition 6-(i). This decomposition is noticeable and crucial for our purpose.

Proposition 10. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$, and we recall that $k_0 = \lceil 2N/\theta \rceil$, that $\mathcal{X} = E_{k_0}$ and that $\mathcal{U} = \mathbb{S} \cap E_{k_0}$.*

For $x \in E_N$, we set $r = R_{[\![1,N]\!]}(x) > 0$, $z = S_{[\![1,N]\!]}(x) \in \mathbb{R}^2$ and $u = (x - \gamma(z))/\sqrt{r} \in \mathbb{S}$ and we consider three independent processes:

• *$(M_t)_{t\geq 0}$, a 2-dimensional Brownian motion with diffusion constant $N^{-1/2}$ starting from z,*

• *$(D_t)_{t\geq 0}$ a squared Bessel process with dimension $d_{\theta,N}(N)$ starting from r and killed when it gets out of $(0,\infty)$, with life-time $\tau_D = \inf\{t \geq 0 : D_t = \triangle\}$,*

• $(U_t)_{t\geq 0}$, *a QSKS*(θ, N) *-process starting from* u, *with life-time* $\xi = \inf\{t \geq 0 : U_t = \triangle\}$.

We introduce $A_t = \int_0^{t\wedge\tau_D} D_s^{-1}\mathrm{d}s$, *and its generalized inverse* $\rho_t = \inf\{s > 0 : A_s > t\}$. *We define* $Y_t = \Psi(M_t, D_t, U_{A_t})$, *where we recall from* (9) *that* $\Psi(z, r, u) = \gamma(z) + \sqrt{r}u \in E_N$ *when* $(z, r, u) \in \mathbb{R}^2 \times (0,\infty) \times \mathbb{S}$ *and where we set* $\Psi(z, r, u) = \triangle$ *when* $r = \triangle$ *or* $u = \triangle$. *Observe that the life-time of* Y *equals* $\zeta' = \rho_\xi \wedge \tau_D$.

Consider also a QKS(θ, N)*-process* $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$, *with life-time* ζ, *and* $\mathbb{X}^* = (\Omega^X, \mathcal{M}^X, (X_t^*)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in(\mathcal{X}\cap E_N)\cup\{\triangle\}})$, *where* $X_t^* = X_t \mathbb{1}_{\{t<\tau\}} + \triangle \mathbb{1}_{\{t\geq\tau\}}$ *and where* $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0,\infty)\}$. *In other words,* $\mathbb{X}^*$ *is the version of* $\mathbb{X}$ *killed when it gets out of* E_N. *The life-time of* $\mathbb{X}^*$ *is* τ.

For quasi all $x \in \mathcal{X} \cap E_N$, *the law of* $(Y_t)_{t\geq 0}$ *is the same as that of* $(X_t^*)_{t\geq 0}$ *under* $\mathbb{P}_x^X$.

We take the convention that $R_{[\![1,N]\!]}(\triangle) = 0$, so that $\tau \in [0, \zeta]$. Since $R_{[\![1,N]\!]}(Y_t) = D_t$ and $S_{[\![1,N]\!]}(Y_t) = M_t$ for all $t \in [0, \zeta')$, Proposition 10 in particular implies that $(R_{[\![1,N]\!]}(X_t))_{t\geq 0}$ and $(S_{[\![1,N]\!]}(X_t))_{t\geq 0}$ are some independent squared Bessel process and Brownian motion until the first time $(R_{[\![1,N]\!]}(X_t))_{t\geq 0}$ vanishes. This actually holds true until explosion, as shown in Lemma 11 below. The *quasi* notion refers to the Hunt process $\mathbb{X}$. Observe that when $\theta \geq 2$, we have $k_0 \leq N$, so that $\mathcal{X} \cap E_N = \mathcal{X}$ and $\mathbb{X} = \mathbb{X}^*$.

Proof. We slice the proof in several steps. First, we determine the Dirichlet spaces of the three processes $(M_t)_{t\geq 0}$, $(D_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ involved in the construction of $(Y_t)_{t\geq 0}$. Then we compute the Dirichlet space of $(D_{\rho_t})_{t\geq 0}$. We next identify the Dirichlet space of $(D_{\rho_t}, U_t)_{t\geq 0}$, which allows us to find the one of $(D_t, U_{A_t})_{t\geq 0}$ by a second time-change. By concatenation, we deduce the Dirichlet space of $(M_t, D_t, U_{A_t})_{t\geq 0}$. Finally, and here lies the main computation, we find the Dirichlet space of $(Y_t)_{t\geq 0}$, which allows us to conclude by uniqueness.

Step 1. First, take $\mathbb{U} = (\Omega^U, \mathcal{M}^U, (U_t)_{t\geq 0}, (\mathbb{P}_u^U)_{u\in\mathcal{U}_\triangle})$ as in Proposition 6-(ii).

Second, consider a 2-dimensional Brownian motion $\mathbb{M} = (\Omega^M, \mathcal{M}^M, (M_t)_{t\geq 0}, (\mathbb{P}_z^M)_{z\in\mathbb{R}^2})$ with diffusion constant $N^{-1/2}$. We know from [11, Example 4.2.1 page 167] that $\mathbb{M}$ is a $\mathrm{d}z$-symmetric (here $\mathrm{d}z$ is the Lebesgue measure on $\mathbb{R}^2$) continuous Hunt process with regular Dirichlet space $(\mathcal{E}^M, \mathcal{F}^M)$ on $L^2(\mathbb{R}^2, \mathrm{d}z)$ with core $C_c^\infty(\mathbb{R}^2)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\mathcal{E}^M(\varphi,\varphi) = \frac{1}{2N}\int_{\mathbb{R}^2} \|\nabla_z\varphi(z)\|^2 \mathrm{d}z. \tag{23}$$

Finally, let $\mathbb{D} = (\Omega^D, \mathcal{M}^D, (D_t)_{t\geq 0}, (\mathbb{P}_r^D)_{r\in\mathbb{R}_+^*\cup\{\triangle\}})$ be a squared Bessel process of dimension $d_{\theta,N}(N)$ killed when it gets out of $\mathbb{R}_+^* = (0,\infty)$ and set $\nu = d_{\theta,N}(N)/2 - 1$, see Revuz-Yor [21, page 443]. Fukushima [10, Theorem 3.3] tells us that $\mathbb{D}$ is a continuous $r^\nu\mathrm{d}r$-symmetric Hunt process (here $\mathrm{d}r$ is the Lebesgue measure on $\mathbb{R}_+^*$) with regular Dirichlet space $(\mathcal{E}^D, \mathcal{F}^D)$ on $L^2(\mathbb{R}_+, r^\nu\mathrm{d}r)$ with core $C_c^\infty(\mathbb{R}_+^*)$ where for all $\varphi \in C_c^\infty(\mathbb{R}_+^*)$,

$$\mathcal{E}^D(\varphi,\varphi) = 2\int_{\mathbb{R}_+} |\varphi'(r)|^2 r^{\nu+1}\mathrm{d}r. \tag{24}$$

Together with [10, Theorem 3.3], this uses that the scale function and the speed measure of $(D_t)_{t\geq 0}$ are respectively $r \mapsto r^{-\nu}$ and $-[r^\nu/(2\nu)]\mathrm{d}r$. Actually, we don't take the speed measure as reference measure but $r^\nu\mathrm{d}r$ which is the same up to a constant.

Step 2. We apply Lemma B.3 to $\mathbb{D}$ with $g(r) = 1/r$, i.e. with $A_t = \int_0^t D_s^{-1}\mathrm{d}s = \int_0^{t\wedge\tau_D} D_s^{-1}\mathrm{d}s$ thanks to the convention $\triangle^{-1} = 0$ and recall that ρ is its generalized inverse: we find that setting

$D_{\rho_t} = D_{\rho_t} \mathbb{1}_{\{\rho_t<\infty\}} + \triangle \mathbb{1}_{\{\rho_t=\infty\}}$,

$$\mathbb{D}_\rho = (\Omega^D, \mathcal{M}^D, (D_{\rho_t})_{t\geq 0}, (\mathbb{P}^D_r)_{r\in\mathbb{R}_+^*})$$

is a continuous $r^{\nu-1}\mathrm{d}r$-symmetric $(\mathbb{R}_+^* \cup \{\triangle\})$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^{D_\rho}, \mathcal{F}^{D_\rho})$ on $L^2(\mathbb{R}_+, r^{\nu-1}\mathrm{d}r)$ with core $C_c^\infty(\mathbb{R}_+^*)$ such that for all $\varphi \in C_c^\infty(\mathbb{R}_+^*)$,

$$\mathcal{E}^{D_\rho}(\varphi,\varphi) = \mathcal{E}^D(\varphi,\varphi) = 2\int_{\mathbb{R}_+} |\varphi'(r)|^2 r^{\nu+1}\mathrm{d}r = 2\int_{\mathbb{R}_+} |r\varphi'(r)|^2 r^{\nu-1}\mathrm{d}r. \tag{25}$$

We use Lemma B.5 and the notation therein: recalling that $\mathcal{M}^{(D,U)} = \sigma((D_{\rho_t}, U_t) : t \geq 0)$, with the convention that $(r,\triangle) = (\triangle, u) = (\triangle,\triangle) = \triangle$, and that $\mathbb{P}^{(D,U)}_{(r,u)} = \mathbb{P}^D_r \otimes \mathbb{P}^U_u$ if $(r,u) \in \mathbb{R}_+^* \times \mathcal{U}$ and $\mathbb{P}^{(D,U)}_\triangle = \mathbb{P}^D_\triangle \otimes \mathbb{P}^U_\triangle$, it holds that

$$(\mathbb{D}_\rho, \mathbb{U}) = \Big(\Omega^D \times \Omega^U, \mathcal{M}^{(D,U)}, (D_{\rho_t}, U_t)_{t\geq 0}, (\mathbb{P}^{(D,U)}_{(r,u)})_{(r,u)\in(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a continuous $r^{\nu-1}\mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}_+^* \times \mathcal{U}) \cup \{\triangle\}$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^{(D_\rho,U)}, \mathcal{F}^{(D_\rho,U)})$ on $L^2(\mathbb{R}_+ \times \mathbb{S}, r^{\nu-1}\mathrm{d}r\beta(\mathrm{d}u))$ with core $C_c^\infty(\mathbb{R}_+^* \times \mathcal{U})$, and for all $\varphi \in C_c^\infty(\mathbb{R}_+^* \times \mathcal{U})$,

$$\mathcal{E}^{(D_\rho,U)}(\varphi,\varphi) = \int_{\mathbb{R}_+} \mathcal{E}^U(\varphi(r,\cdot),\varphi(r,\cdot)) r^{\nu-1}\mathrm{d}r + \int_{\mathbb{S}} \mathcal{E}^{D_\rho}(\varphi(\cdot,u),\varphi(\cdot,u))\beta(\mathrm{d}u).$$

We now apply Lemma B.3 to $(\mathbb{D}_\rho, \mathbb{U})$ with $g(r,u) = r$ for all $r \in \mathbb{R}_+^*$ and all $u \in \mathcal{U}$. We consider the time-change $\alpha_t = \int_0^t g(D_{\rho_s}, U_s)\mathrm{d}s$, with the convention that $g(r,u) = 0$ as soon as $(r,u) = \triangle$. We also set $B_t = \inf\{s > 0 : \alpha_s > t\}$. As we will see in a few lines, it holds that

$$(D_{\rho_{B_t}}, U_{B_t}) = (D_t, U_{A_t}) \qquad \text{for all } t \geq 0. \tag{26}$$

Hence Lemma B.3 tells us that

$$(\mathbb{D}, \mathbb{U}_A) = \Big(\Omega^D \times \Omega^U, \mathcal{M}^{(D,U)}, (D_t, U_{A_t})_{t\geq 0}, (\mathbb{P}^{(D,U)}_{(r,u)})_{(r,u)\in(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a continuous $r^\nu\mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}_+^* \times \mathcal{U}) \cup \{\triangle\}$-valued Hunt process process, and that its Dirichlet space $(\mathcal{E}^{(D,U_A)}, \mathcal{F}^{(D,U_A)})$ on $L^2(\mathbb{R}_+ \times \mathbb{S}, r^\nu\mathrm{d}r\beta(\mathrm{d}u))$ is regular with core $C_c^\infty(\mathbb{R}_+^* \times \mathcal{U})$ and for all $\varphi \in C_c^\infty(\mathbb{R}_+^* \times \mathcal{U})$,

$$\mathcal{E}^{(D,U_A)}(\varphi,\varphi) = \mathcal{E}^{(D_\rho,U)}(\varphi,\varphi) = \int_{\mathbb{R}_+} \mathcal{E}^U(\varphi(r,\cdot),\varphi(r,\cdot)) r^{\nu-1}\mathrm{d}r + \int_{\mathbb{S}} \mathcal{E}^{D_\rho}(\varphi(\cdot,u),\varphi(\cdot,u))\beta(\mathrm{d}u). \tag{27}$$

We now check the claim (26). Recall that D explodes at time τ_D, that $A_t = \int_0^{t\wedge\tau_D} D_s^{-1}\mathrm{d}s$ and that ρ is the generalized inverse of A. Hence $(\rho_t)_{t\in[0,A_{\tau_D})}$ is the true inverse of $(A_t)_{t\in[0,\tau_D)}$ and we have $\rho'_t = D_{\rho_t}$, whence $\rho_t = \int_0^t D_{\rho_s}\mathrm{d}s$ for $t \in [0, A_{\tau_D})$. We also have $\rho_t = \infty$ for $t \geq A_{\tau_D}$. Next, $\alpha_t = \int_0^t D_{\rho_s}\mathrm{d}s = \rho_t$ for $t \in [0, A_{\tau_D} \wedge \xi)$, because $g(D_{\rho_s}, U_s) = D_{\rho_s}$ if $(D_{\rho_s}, U_s) \neq \triangle$, i.e. if $s < A_{\tau_D} \wedge \xi$. Hence B, the generalized inverse of α, equals A during $[0, \tau_D \wedge \rho_\xi)$, thus in particular $\rho_{B_t} = t$ for $t \in [0, A_{\tau_D} \wedge \xi)$. As conclusion, (26) holds true for $t \in [0, A_{\tau_D} \wedge \xi)$. If now $t \geq \tau_D \wedge \rho_\xi$, then $B_t = \infty$, because B is the generalized inverse of α and because for all $t \geq 0$,

$$\alpha_t \leq \alpha_{A_{\tau_D}\wedge\xi} = \rho_{A_{\tau_D}\wedge\xi} = \tau_D \wedge \rho_\xi.$$

Hence, still if $t \geq \tau_D \wedge \rho_\xi$, we have $(D_{\rho_{B_t}}, U_{B_t}) = \triangle$, while $(D_t, U_{A_t}) = \triangle$ because either $t \geq \tau_D$ and thus $D_t = \triangle$ or $t \geq \rho_\xi$ and thus $A_t \geq \xi$ so that $U_{A_t} = \triangle$. We have proved (26).

We finally conclude, thanks to Lemma B.5 again, setting $\mathcal{M}^{(M,D,U)} = \sigma((M_t, D_t, U_{A_t}) : t \geq 0)$ with the convention that $(z, \triangle) = \triangle$ and setting $\mathbb{P}^{(M,D,U)}_{(z,r,u)} = \mathbb{P}^M_z \otimes \mathbb{P}^{(D,U)}_{(r,u)}$ in the case where $(r, z, u) \in \mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U}$ and $\mathbb{P}^{(M,D,U)}_\triangle = \mathbb{P}^M_\triangle \otimes \mathbb{P}^{(D,U)}_\triangle$, that

$$(\mathbb{M}, \mathbb{D}, \mathbb{U}_\mathbb{A}) = \Big(\Omega^M \times \Omega^D \times \Omega^U, \mathcal{M}^{(M,D,U)}, (M_t, D_t, U_{A_t})_{t\geq 0}, (\mathbb{P}^{(M,D,U)}_{(z,r,u)})_{(z,r,u)\in(\mathbb{R}^2\times\mathbb{R}^*_+\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a continuous $\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U}) \cup \{\triangle\}$-valued Hunt process with Dirichlet space $(\mathcal{E}^{(M,D,U_A)}, \mathcal{F}^{(M,D,U_A)})$ on $L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}, \mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u))$, regular with core $C^\infty_c(\mathbb{R}^2\times\mathbb{R}^*_+\times\mathcal{U})$. Moreover, for all $\varphi \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U})$,

$$\begin{aligned}
\mathcal{E}^{(M,D,U_A)}(\varphi,\varphi) &= \int_{\mathbb{R}_+\times\mathbb{S}} \mathcal{E}^M(\varphi(\cdot, r, u), \varphi(\cdot, r, u)) r^\nu \mathrm{d}r\beta(\mathrm{d}u) + \int_{\mathbb{R}^2} \mathcal{E}^{(D,U_A)}(\varphi(z, \cdot, \cdot), \varphi(z, \cdot, \cdot))\mathrm{d}z \\
&= \int_{\mathbb{R}_+\times\mathbb{S}} \mathcal{E}^M(\varphi(\cdot, r, u), \varphi(\cdot, r, u)) r^\nu \mathrm{d}r\beta(\mathrm{d}u) + \int_{\mathbb{R}^2\times\mathbb{S}} \mathcal{E}^{D_\rho}(\varphi(z, \cdot, u), \varphi(z, \cdot, u))\mathrm{d}z\beta(\mathrm{d}u) \\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathcal{E}^U(\varphi(z, r, \cdot), \varphi(z, r, \cdot))\mathrm{d}z r^{\nu-1}\mathrm{d}r \\
&= \int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \Big[\frac{1}{2N}||\nabla_z\varphi(z, r, u)||^2 + 2r|\partial_r\varphi(z, r, u)|^2 + \frac{1}{2r}||\nabla_\mathbb{S}\varphi(z, r, u)||^2\Big]\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u). \qquad (28)
\end{aligned}$$

For the second line, we used (27). For the last line, we used (23), (25) and the expression of $\mathcal{E}^U$, see Proposition 6-(ii).

Step 3. We recall that $Y_t = \Psi(M_t, D_t, U_{A_t})$, where $\Psi(z, r, u) = \gamma(z) + \sqrt{r}u$ for $(z, r, u) \in \mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U}$ and $\Psi(z, r, u) = \triangle$ for $(z, r, u) = \triangle$. One easily checks that Ψ is a bijection from $(\mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U}) \cup \{\triangle\}$ to $(\mathcal{X} \cap E_N) \cup \{\triangle\}$, recall that $\mathcal{X} = E_{k_0}$ and $\mathcal{U} = E_{k_0} \cap \mathbb{S}$.

We now study

$$\mathbb{Y} = (\Omega^Y, \mathcal{M}^Y, (Y_t)_{t\geq 0}, (\mathbb{P}^Y_y)_{y\in(\mathcal{X}\cap E_N)\cup\{\triangle\}}),$$

where $\Omega^Y = \Omega^M \times \Omega^D \times \Omega^U$, $\mathcal{M}^Y = \mathcal{M}^{(M,D,U)}$ and $\mathbb{P}^Y_y = \mathbb{P}^{(M,D,U)}_{(z,r,u)}$ for $(z, r, u) = \Psi^{-1}(y)$.

First, $\mathbb{Y}$ is a continuous $(\mathcal{X} \cap E_N) \cup \{\triangle\}$-valued Hunt process, because the bijection Ψ from $(\mathbb{R}^2 \times \mathbb{R}^*_+ \times \mathcal{U}) \cup \{\triangle\}$ to $(\mathcal{X} \cap E_N) \cup \{\triangle\}$ is continuous, both sets being endowed with the one-point compactification topology, see Subsection B.1.

Next, we prove that $\mathbb{Y}$ is μ-symmetric: if φ, ψ are nonnegative measurable functions on $\mathcal{X} \cap E_N$ and $t \geq 0$, we have, thanks to Lemma A.2 (recall that $\nu = d_{\theta,N}(N)/2 - 1$),

$$\int_{(\mathbb{R}^2)^N} [P^Y_t \varphi(y)]\psi(y)\mu(\mathrm{d}y) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} [(P^Y_t\varphi)(\Psi(z, r, u))]\psi(\Psi(z, r, u))r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

But $(P^Y_t\varphi)(\Psi(z, r, u)) = \mathbb{E}_{(z,r,u)}[\varphi(\Psi(M_t, D_t, U_{A_t}))] = P^{(M,D,U_A)}_t(\varphi \circ \Psi)(z, r, u)$, so that

$$\int_{(\mathbb{R}^2)^N} [P^Y_t\varphi(y)]\psi(y)\mu(\mathrm{d}y) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} [P^{(M,D,U_A)}_t(\varphi \circ \Psi)(z, r, u)][(\psi \circ \Psi)(z, r, u)]r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Using that $(\mathbb{M}, \mathbb{D}, \mathbb{U}_\mathbb{A})$ is $\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u)$-symmetric and then the same computation in reverse order, one concludes that $\int_{(\mathbb{R}^2)^N}[P^Y_t\varphi]\psi\mathrm{d}\mu = \int_{(\mathbb{R}^2)^N} \varphi[P^Y_t\psi]\mathrm{d}\mu$ as desired.

Thus $\mathbb{Y}$ has a Dirichlet space $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2((\mathbb{R}^2)^N, \mu)$ that we now determine. For $\varphi \in L^2((\mathbb{R}^2)^N, \mu)$, using as above Lemma A.2 and that $(P_t^Y \varphi)(\Psi(z,r,u)) = P_t^{(M,D,U_A)}(\varphi \circ \Psi)(z,r,u)$,

$$\frac{1}{t}\int_{(\mathbb{R}^2)^N} (P_t^Y \varphi - \varphi)\varphi \mathrm{d}\mu$$
$$= \frac{1}{2t}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}} [P_t^{(M,D,U_A)}(\varphi\circ\Psi)(z,r,u) - (\varphi\circ\Psi)(z,r,u)][\varphi\circ\Psi(z,r,u)]r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Since Ψ is bijective, we deduce, see [11, Lemma 1.3.4 page 23], that

$$\mathcal{F}^Y = \Big\{\varphi \in L^2((\mathbb{R}^2)^N, \mu) : \varphi\circ\Psi \in \mathcal{F}^{(M,D,U_A)}\Big\} \tag{29}$$

$$\text{and for } \varphi \in \mathcal{F}^Y, \quad \mathcal{E}^Y(\varphi,\varphi) = \frac{1}{2}\mathcal{E}^{(M,D,U_A)}(\varphi\circ\Psi, \varphi\circ\Psi). \tag{30}$$

Step 4. We now compute $\mathcal{E}^Y(\varphi,\varphi)$ for $\varphi \in C_c^\infty(\mathcal{X}\cap E_N)$, so that $\varphi\circ\Psi \in C_c^\infty(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})$. Thanks to (28) and (30), we have

$$\mathcal{E}^Y(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} I(z,r,u)\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u), \tag{31}$$

where

$$I(z,r,u) = \frac{1}{2N}||\nabla_z(\varphi\circ\Psi)(z,r,u)||^2 + 2r|\partial_r(\varphi\circ\Psi)(z,r,u)|^2 + \frac{1}{2r}||\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u)||^2.$$

We recall that for $\varphi : (\mathbb{R}^2)^N \to \mathbb{R}$, we call $\nabla\varphi(x) = ((\nabla\varphi(x))^1, \dots, (\nabla\varphi(x))^N) \in (\mathbb{R}^2)^N$ the total gradient of φ at $x \in (\mathbb{R}^2)^N$, and we have $(\nabla\varphi(x))^i \in \mathbb{R}^2$ for each $i \in [\![1,N]\!]$. And for $\phi : O \to \mathbb{R}^p$, where O is open in $\mathbb{R}^n$, we denote by $\mathrm{d}_z\phi$ the differential of ϕ at $z \in O$.

We start with the study of $\Psi(z,r,u) = \gamma(z) + \sqrt{r}u$, where we recall that γ was introduced in Section 2 and that $\Phi_{\mathbb{S}}(x) = \pi_H x/||\pi_H x||$ is defined on a neighborhood of $\mathbb{S}$ in $(\mathbb{R}^2)^N$, see (10). It holds that for all $(z,r,u) \in \mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}$ and all $h \in \mathbb{R}^2$, $k \in \mathbb{R}$ and $\ell \in (\mathbb{R}^2)^N$,

$$\mathrm{d}_z\Psi(\cdot,r,u)(h) = \gamma(h), \qquad \mathrm{d}_r\Psi(z,\cdot,u)(k) = \frac{k}{2\sqrt{r}}u, \qquad \mathrm{d}_u[\Psi(z,r,\Phi_{\mathbb{S}}(\cdot))](\ell) = \sqrt{r}\pi_{u^\perp}(\pi_H(\ell)),$$

For the first equality, it suffices to use that γ is linear, so that $\mathrm{d}_z\Psi(\cdot,r,u)(h) = \mathrm{d}_z\gamma(h) = \gamma(h)$. The second equality is obvious. For the third equality, which is the differential at $u \in \mathbb{S}$ of the function $F(x) = \gamma(z) + \sqrt{r}\Phi_{\mathbb{S}}(x)$ defined for $x \in E_N$ (which is open in $(\mathbb{R}^2)^N$ and contains $\mathbb{S}$), we write $\mathrm{d}_uF = \sqrt{r}\mathrm{d}_u\Phi_{\mathbb{S}}$. But $\Phi_S = G\circ\pi_H$, where $G(x) = x/||x||$, and we have $\mathrm{d}_u\pi_H = \pi_H$ and $\mathrm{d}_{\pi_H(u)}G = \mathrm{d}_uG = \pi_{u^\perp}$ for $u \in \mathbb{S}$. All in all, $\mathrm{d}_uF = \sqrt{r}\pi_{u^\perp}\circ\pi_H$.

First, we have $\nabla_z(\varphi\circ\Psi)(z,r,u) = \sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i$. Indeed, for all $h \in \mathbb{R}^2$, it holds that

$$\mathrm{d}_z(\varphi\circ\Psi(\cdot,r,u))(h) = (\mathrm{d}_{\Psi(z,r,u)}\varphi)[(\mathrm{d}_z\Psi(\cdot,r,u))(h)] = (\mathrm{d}_{\Psi(z,r,u)}\varphi)(\gamma(h)) = \nabla\varphi(\Psi(z,r,u))\cdot\gamma(h),$$

which, by definition of γ, equals $h\cdot\sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i$.

This implies that

$$\frac{1}{2N}\|\nabla_z(\varphi\circ\Psi(z,r,u))\|^2 = \frac{1}{2N}\Big\|\sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i\Big\|^2 = \frac{1}{2}\|\pi_{H^\perp}(\nabla\varphi(\Psi(z,r,u)))\|^2. \tag{32}$$

Indeed, recalling the expression of π_H, see Section 2, it suffices to note that for all $x \in (\mathbb{R}^2)^N$, $\|\pi_{H^\perp}(x)\|^2 = \|\gamma(S_{[\![1,N]\!]}(x))\|^2 = N\|S_{[\![1,N]\!]}(x)\|^2 = N^{-1}\|\sum_{i=1}^N x^i\|^2$.

Next, $\partial_r(\varphi\circ\Psi)(z,r,u)=(\nabla\varphi)(\Psi(z,r,u))\cdot u/(2\sqrt{r})$. Indeed, for $k\in\mathbb{R}$,

$$\mathrm{d}_r(\varphi\circ\Psi(z,\cdot,u))(k)=(\mathrm{d}_{\Psi(z,r,u)}\varphi)[(\mathrm{d}_r\Psi(z,\cdot,u))(k)]=(\mathrm{d}_{\Psi(z,r,u)}\varphi)(u)\times\frac{k}{2\sqrt{r}},$$

which is nothing but $(\nabla\varphi)(\Psi(z,r,u))\cdot u\times k/(2\sqrt{r})$.

This implies, recalling that π_u is the orthogonal projection on $\mathrm{Span}(u)\subset(\mathbb{R}^2)^N$, that

$$2r|\partial_r(\varphi\circ\Psi)(z,r,u)|^2=\frac{1}{2}\|\pi_u((\nabla\varphi)(\Psi(z,r,u)))\|^2=\frac{1}{2}\|\pi_H(\pi_u((\nabla\varphi)(\Psi(z,r,u))))\|^2 \tag{33}$$

since $u\in\mathbb{S}$, so that $||u||=1$ and $u\in H$.

Finally, $\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u)=\sqrt{r}\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u))))$. Indeed, for all $\ell\in(\mathbb{R}^2)^N$,

$$\begin{aligned}d_u((\varphi\circ\Psi)(z,r,\Phi_{\mathbb{S}}(\cdot)))(\ell)=&(d_{\Psi(z,r,u)}\varphi)(d_u[\Psi(z,r,\Phi_{\mathbb{S}}(\cdot))](\ell))\\=&\sqrt{r}(d_{\Psi(z,r,u)}\varphi)(\pi_{u^\perp}(\pi_H(\ell)))\\=&\sqrt{r}\nabla\varphi(\Psi(z,r,u))\cdot\pi_{u^\perp}(\pi_H(\ell))\\=&\sqrt{r}\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u))))\cdot\ell,\end{aligned}$$

and we conclude since $\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u)=\nabla_x((\varphi\circ\Psi)(z,r,\Phi_{\mathbb{S}}(\cdot)))(u)$ by definition of $\nabla_{\mathbb{S}}$, see (12).

This implies that

$$\frac{1}{2r}||\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u)||^2=\frac{1}{2}\|\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u))))\|^2. \tag{34}$$

Gathering (32)-(33)-(34), we find that $I(z,r,u)=\frac{1}{2}\|\nabla\varphi(\Psi(z,r,u))\|^2$, since for all $x\in(\mathbb{R}^2)^N$,

$$\|\pi_{H^\perp}(x)\|^2+\|\pi_H(\pi_u(x))\|^2+\|\pi_H(\pi_{u^\perp}(x))\|^2=\|x\|^2$$

because $u\in\mathbb{S}\subset H$.

Injecting the value of I in (31) and using Lemma A.2, we obtain

$$\mathcal{E}^Y(\varphi,\varphi)=\frac{1}{4}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}}\|\nabla\varphi(\Psi(z,r,u))\|^2\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u)=\frac{1}{2}\int_{(\mathbb{R}^2)^N}\|\nabla\varphi\|^2\mathrm{d}\mu.$$

Step 5. As a last technical step, we verify that $(\mathcal{E}^Y,\mathcal{F}^Y)$ is a regular Dirichlet space on $L^2((\mathbb{R}^2)^N,\mu)$ with core $C_c^\infty(\mathcal{X}\cap E_N)$, i.e. that for all $\varphi\in\mathcal{F}^Y$, there is $\varphi_n\in C_c^\infty(\mathcal{X}\cap E_N)$ such that $\lim_n||\varphi_n-\varphi||_{L^2((\mathbb{R}^2)^N,\mu)}+\mathcal{E}^Y(\varphi_n-\varphi,\varphi_n-\varphi)=0$.

Recalling (29) and using that $(\mathcal{E}^{(M,D,U_A)},\mathcal{F}^{(M,D,U_A)})$ on $L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u))$ is regular with core $C_c^\infty(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})$, there is $g_n\in C_c^\infty(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})$ such that

$$||g_n-\varphi\circ\Psi||_{L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u))}+\mathcal{E}^{(M,D,U_A)}(g_n-\varphi\circ\Psi,g_n-\varphi\circ\Psi)\to0.$$

Setting $\varphi_n=g_n\circ\Psi^{-1}$, it holds that $\varphi_n\in C_c^\infty(\mathcal{X}\cap E_N)$ and we have, by (30),

$$\mathcal{E}^Y(\varphi_n-\varphi,\varphi_n-\varphi)=\frac{1}{2}\mathcal{E}^{(M,D,U_A)}(g_n-\varphi\circ\Psi,g_n-\varphi\circ\Psi)\to0,$$

as well as, by Lemma A.2,

$$||\varphi_n-\varphi||_{L^2((\mathbb{R}^2)^N,\mu)}=\frac{1}{2}||g_n-\varphi\circ\Psi||_{L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u))}\to0.$$

Step 6. By Steps 3, 4 and 5, we know that $\mathbb{Y}$ is a continuous μ-symmetric $(\mathcal{X}\cap E_N)\cup\{\triangle\}$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^Y,\mathcal{F}^Y)$ with core $C_c^\infty(\mathcal{X}\cap E_N)$ and with $\mathcal{E}^Y(\varphi,\varphi)=\frac{1}{2}\int_{(\mathbb{R}^2)^N}||\nabla\varphi||^2\mathrm{d}\mu$ for $\varphi\in C_c^\infty(\mathcal{X}\cap E_N)$.

Now, applying Lemma B.6 to $\mathbb{X}$ defined in Proposition 6-(i) with the open set $\mathcal{X} \cap E_N$, we see that $\mathbb{X}^*$, i.e. $\mathbb{X}$ killed when getting outside $\mathcal{X} \cap E_N$, is a continuous μ-symmetric $(\mathcal{X} \cap E_N) \cup \{\triangle\}$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^{X^*}, \mathcal{F}^{X^*})$ with core $C_c^\infty(\mathcal{X} \cap E_N)$ and with $\mathcal{E}^{X^*}(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} ||\nabla\varphi||^2 \mathrm{d}\mu$ for $\varphi \in C_c^\infty(\mathcal{X} \cap E_N)$.

This implies, as recalled in Subsection B.1, that $(\mathcal{E}^{X^*}, \mathcal{F}^{X^*}) = (\mathcal{E}^Y, \mathcal{F}^Y)$. The conclusion follows by uniqueness, see [11, Theorem 4.2.8 p 167]. $\square$

Actually, $(R_{[\![1,N]\!]}(X_t))_{t\geq 0}$ and $(S_{[\![1,N]\!]}(X_t))_{t\geq 0}$ are some independent squared Bessel process and Brownian motion *until explosion* (and not only until the first time where $R_{[\![1,N]\!]}(X_t) = 0$, as shown in Proposition 10), a fact that we shall often use.

Lemma 11. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and we consider a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$. For quasi all $x \in \mathcal{X}$, there are a 2D-Brownian motion $(M_t)_{t\geq 0}$ with diffusion constant $N^{-1/2}$ issued from $S_{[\![1,N]\!]}(x)$ and a squared Bessel process $(D_t)_{t\geq 0}$ with dimension $d_{\theta,N}(N)$ issued from $R_{[\![1,N]\!]}(x)$ (killed when it gets out of $(0,\infty)$ if $d_{\theta,N}(N) \leq 0$) independent of $(M_t)_{t\geq 0}$ such that $\mathbb{P}_x^X$-a.s., $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta)$.*

Proof. If $\theta \geq 2$, this follows from Proposition 10: setting $\tau = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) \notin (0,\infty)\}$, we have $\tau = \zeta$. Indeed, on $\{\tau < \zeta\}$, we have $X_\tau \notin E_N$, whence $X_\tau \notin \mathcal{X}$ since $\mathcal{X} = E_{k_0}$ with $k_0 \leq N$ (because $\theta \geq 2$), which contradicts the fact that $\tau < \zeta$.

We now suppose that $\theta < 2$, so that $k_0 > N$ and thus $\mathcal{X} = (\mathbb{R}^2)^N$. We introduce the shortened notation $R(x) = R_{[\![1,N]\!]}(x)$, $S(x) = (S_1(x), S_2(x)) = S_{[\![1,N]\!]}(x)$ and split the proof in three parts.

Step 1. For $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$, we set $\psi(x) = \varphi(S_1(x), S_2(x), R(x))$ and show that $\psi \in \mathcal{A}^X$ and

$$\begin{aligned}\mathcal{A}^X\psi(x) =& \frac{1}{2N}\Big(\partial_{11}\varphi(S_1(x), S_2(x), R(x)) + \partial_{22}\varphi(S_1(x), S_2(x), R(x))\Big)\\ &+ 2R(x)\partial_{33}\varphi(S_1(x), S_2(x), R(x)) + d_{\theta,N}(N)\partial_3\varphi(S_1(x), S_2(x), R(x)).\end{aligned}$$

To this end, we apply Remark 8. Since $\psi \in C_c^\infty((\mathbb{R}^2)^N)$ and since $\mathcal{X} = (\mathbb{R}^2)^N$, we have to show that $\sup_{\alpha\in(0,1)}\sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}_\alpha^X\psi(x)| < \infty$, and we will deduce that $\mathcal{A}^X\psi = \mathcal{L}^X\psi$. Using that $\nabla_{x^i}S_1(x) = (N^{-1}, 0)$, $\nabla_{x^i}S_2(x) = (0, N^{-1})$ and $\nabla_{x^i}R(x) = 2(x^i - S(x))$, we find

$$\begin{aligned}\nabla_{x^i}\psi(x) =& \frac{1}{N}\Big(\partial_1\varphi(S_1(x), S_2(x), R(x)), \partial_2\varphi(S_1(x), S_2(x), R(x))\Big)\\ &+ 2(x^i - S(x))\partial_3\varphi(S_1(x), S_2(x), R(x)).\end{aligned}$$

Hence by symmetry,

$$\begin{aligned}\frac{\theta}{N}\sum_{1\leq i\neq j\leq N}\frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot\nabla_{x^i}\psi(x) =& \frac{2\theta}{N}\partial_3\varphi(S_1(x), S_2(x), R(x))\sum_{1\leq i\neq j\leq N}\frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot x^i\\ =& \frac{\theta}{N}\partial_3\varphi(S_1(x), S_2(x), R(x))\sum_{1\leq i\neq j\leq N}\frac{\|x^i - x^j\|^2}{\|x^i - x^j\|^2 + \alpha}.\end{aligned} \tag{35}$$

Besides,

$$\Delta_{x^i}\psi(x) =\frac{1}{N^2}\Big(\partial_{11}\varphi(S_1(x),S_2(x),R(x))+\partial_{22}\varphi(S_1(x),S_2(x),R(x))\Big)$$
$$+\frac{4(x_1^i-S_1(x))}{N}\partial_{13}\varphi(S_1(x),S_2(x),R(x))+\frac{4(x_2^i-S_2(x))}{N}\partial_{23}\varphi(S_1(x),S_2(x),R(x))$$
$$+4\Big(1-\frac{1}{N}\Big)\partial_3\varphi(S_1(x),S_2(x),R(x))+4\|x^i-S(x)\|^2\partial_{33}\varphi(S_1(x),S_2(x),R(x)),$$

where $x^i=(x_1^i,x_2^i)\in\mathbb{R}^2$. By symmetry again,

$$\Delta\psi(x) =\frac{1}{N}\Big(\partial_{11}\varphi(S_1(x),S_2(x),R(x))+\partial_{22}\varphi(S_1(x),S_2(x),R(x))\Big)$$
$$+4(N-1)\partial_3\varphi(S_1(x),S_2(x),R(x))+4R(x)\partial_{33}\varphi(S_1(x),S_2(x),R(x)). \tag{36}$$

We conclude by combining (35) and (36) that

$$\mathcal{L}^X_\alpha\psi(x) =\frac{1}{2N}\Big(\partial_{11}\varphi(S_1(x),S_2(x),R(x))+\partial_{22}\varphi(S_1(x),S_2(x),R(x))\Big)$$
$$+2R(x)\partial_{33}\varphi(S_1(x),S_2(x),R(x))$$
$$+\Big(2(N-1)-\frac{\theta}{N}\sum_{1\le i\neq j\le N}\frac{\|x^i-x^j\|^2}{\|x^i-x^j\|^2+\alpha}\Big)\partial_3\varphi(S_1(x),S_2(x),R(x)).$$

We immediately deduce, since φ is compactly supported, that $\sup_{\alpha\in(0,1)}\sup_{x\in(\mathbb{R}^2)^N}|\mathcal{L}^X_\alpha\psi(x)|<\infty$. Hence $\mathcal{A}^X\psi=\mathcal{L}^X\psi$, and since $\mathcal{L}^X\psi=\mathcal{L}^X_\alpha\psi$ with $\alpha=0$, the conclusion follows, noting that $2(N-1)-\theta(N-1)=d_{\theta,N}(N)$.

Step 2. We fix $n\ge 1$ and introduce the stopping time $\tau_n:=\inf\{t>0:X_t\notin A_n\}$, where $A_n=\{x\in(\mathbb{R}^2)^N : S_1^2(x)+S_2^2(x)+R^2(x)\le n\}$. One easily checks that A_n is compact in $(\mathbb{R}^2)^N=\mathcal{X}$ and that $\cup_{n\ge1}A_n=(\mathbb{R}^2)^N$, so that $\lim_n\tau_n=\zeta$ a.s. by Remark 7. We prove here that for quasi all $x\in\mathcal{X}$, there are some $\mathbb{P}^X_x$-martingales $(M_t^{k,n})_{t\ge0}$ and $(M_t^{k,\ell,n})_{t\ge0}$, for $k,\ell\in\{1,2,3\}$, all issued from 0, such that for all $t\in[0,\tau_n]$,

$$S_k(X_t) =S_k(x)+M_t^{k,n}, \qquad k\in\{1,2\}, \tag{37}$$

$$S_k(X_t)S_\ell(X_t) =S_k(x)S_\ell(x)+M_t^{k,\ell,n}+\mathbb{1}_{\{k=\ell\}}N^{-1}t, \qquad k,\ell\in\{1,2\}, \tag{38}$$

$$R(X_t) =R(x)+M_t^{3,n}+d_{\theta,N}(N)t, \tag{39}$$

$$R^2(X_t) =R^2(x)+M_t^{3,3,n}+\int_0^t(4+2d_{\theta,N}(N))R(X_s)\mathrm{d}s, \tag{40}$$

$$R(X_t)S_k(X_t) =R(x)S_k(x)+M_t^{k,3,n}+d_{\theta,N}(N)\int_0^t S_k(X_s)\mathrm{d}s, \qquad k\in\{1,2\}. \tag{41}$$

Applying Lemma B.2 to some $\psi_n\in C_c^\infty((\mathbb{R}^2)^N)\cap\mathcal{A}^X$ such that $\psi_n=S_1$ on A_n, we conclude that for quasi all $x\in(\mathbb{R}^2)^N$, there is a $\mathbb{P}^X_x$-martingale $M^{1,n}$ such that $\psi_n(X_t)=\psi_n(x)+M_t^{1,n}+\int_0^t\mathcal{A}^X\psi_n(X_s)\mathrm{d}s$ for all $t\ge0$. But applying Step 1 with (a truncated version of) $\varphi(s_1,s_2,r)=s_1$, we see that $\mathcal{A}^X\psi_n=0$ on A_n. Hence during $[0,\tau_n]$, we have $S_1(X_t)=S_1(x)+M_t^{1,n}$ and this show (37) with $k=1$. Of course, (37) with $k=2$ is shown similarly. The other equalities are checked similarly, using that by Step 1, with $k,\ell\in\{1,2\}$,

- if $\psi_n=S_kS_\ell$ on A_n, then $\mathcal{A}^X\psi_n(x)=N^{-1}\mathbb{1}_{\{k=\ell\}}$ for all $x\in A_n$ (take $\varphi(s_1,s_2,r)=s_ks_\ell$),
- if $\psi_n=R$ on A_n, then $\mathcal{A}^X\psi_n=d_{\theta,N}(N)$ for all $x\in A_n$ (take $\varphi(s_1,s_2,r)=r$),

- if $\psi_n = R^2$ on A_n, then $\mathcal{A}^X\psi_n(x) = (4+2d_{\theta,N}(N))R(x)$ for all $x \in A_n$ (take $\varphi(s_1, s_2, r) = r^2$),
- if $\psi_n = RS_k$ on A_n, then $\mathcal{A}^X\psi_n(x) = d_{\theta,N}(N)S_k(x)$ for all $x \in A_n$ (take $\varphi(s_1, s_2, r) = rs_k$).

Step 3. We deduce from (37)-(38) that $\langle M^{k,n}, M^{\ell,n}\rangle_t = N^{-1}\mathbb{1}_{\{k=\ell\}}t$ during $[0, \tau_n]$. By the Lévy theorem, $S(X_t) = (S_1(X_t), S_2(X_t))$ is thus a 2-dimensional Brownian motion with diffusion constant N^{-1} issued from $S(x)$, during $[0, \tau_n]$. Next, we deduce from (39)-(40) that $\langle M^{3,n}\rangle_t = 4\int_0^t R(X_s)\mathrm{d}s$, so that there is a Brownian motion $(W_t)_{t\geq 0}$ such that $M_t^{3,n} = \int_0^t 2\sqrt{R(X_s)}\mathrm{d}W_s$ for all $t \in [0, \tau_n]$, whence

$$R(X_t) = R(x) + \int_0^t 2\sqrt{R(X_s)}\mathrm{d}W_s + d_{\theta,N}(N)t. \tag{42}$$

Hence $(R(X_t))_{t\geq 0}$ is a squared Bessel process with dimension $d_{\theta,N}(N)$ during $[0, \tau_n]$, see Revuz-Yor [21, Chapter XI]. Finally, using (37)-(39)-(41), we find $\langle S_k(X), M^{3,n}\rangle_t = 0$, so that we also have $\int_0^t \sqrt{R(X_s)}\mathrm{d}\langle W, S_k(X)\rangle_s = 0$ during $[0, \tau_n]$, for $k \in \{1, 2\}$. Since $\int_0^t \mathbb{1}_{\{R(X_s)=0\}}\mathrm{d}s = 0$ (because $(R(X_t))_{t\geq 0}$ is a squared Bessel process dimension $d_{\theta,N}(N) > 0$ since $\theta \in (0, 2)$), we conclude that $\langle W, S_k(X)\rangle_t = 0$ during $[0, \tau_n]$, for $k \in \{1, 2\}$. By the Lévy theorem, the three Brownian motions $(S_1(X_t))_{t\geq 0}, (S_2(X_t))_{t\geq 0}, (W_t)_{t\geq 0}$ are independent during $[0, \tau_n]$. Since now the S.D.E. (42) has a pathwise unique solution, see Revuz-Yor [21, Theorem 3.5 page 390], this solution is *strong*, i.e. it is measurable in the filtration of $(W_t)_{t\geq 0}$. As a conclusion $(R(X_t))_{t\geq 0}$ is independent of $(S(X_t))_{t\geq 0} = (S_1(X_t), S_2(X_t))_{t\geq 0}$ during $[0, \tau_n]$. We have shown the announced result on $[0, \tau_n]$, and this is sufficient since $\lim_n \tau_n = \zeta$. □

6. Some cutoff functions

We will need several times to approximate some indicator functions by some smooth functions, on which the generator $\mathcal{L}^X$ (or $\mathcal{L}^U$) is bounded. This does not seem obvious, due to the singularity of $\mathcal{L}^X$. We recall that $\mathcal{L}^X_\alpha$ and $\mathcal{L}^U_\alpha$ were defined in (21) and (22).

Lemma 12. *Fix $N \geq 2$, $\theta > 0$, a partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1, N]\!]$, fix $k \in [\![1, N]\!]$ and set*

$$G_k^{\mathbf{K}} = \Big\{x \in E_k : \min_{1\leq p\neq q\leq \ell} \min_{i\in K_p, j\in K_q} ||x^i - x^j|| > 0\Big\}.$$

(i) There exists a family of compact subsets $G_{k,n}^{\mathbf{K}}$ of $G_k^{\mathbf{K}}$ such that

$$\cup_{n\geq 1} G_{k,n}^{\mathbf{K}} = G_k^{\mathbf{K}} \qquad \text{and for each } n \geq 1,\ G_{k,n}^{\mathbf{K}} \text{ is compact in } \mathring{G}_{k,n+1}^{\mathbf{K}}$$

and a family of functions $\Gamma_{k,n}^{\mathbf{K}} \in C_c^\infty((\mathbb{R}^2)^N)$ such that for all $n \geq 1$,

$$\operatorname{Supp}\Gamma_{k,n}^{\mathbf{K}} \subset G_{k,2n}^{\mathbf{K}}, \quad \Gamma_{k,n}^{\mathbf{K}} = 1 \text{ on } G_{k,n}^{\mathbf{K}} \quad \text{and} \quad \sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} \Big|\mathcal{L}_\alpha^X \Gamma_{k,n}^{\mathbf{K}}(x)\Big| < \infty.$$

(ii) With the same sets $G_{k,n}^{\mathbf{K}}$ as in (i), there is a family $\Gamma_{k,n}^{\mathbf{K},\mathbb{S}} \in C^\infty(\mathbb{S})$ such that for all $n \geq 1$,

$$\operatorname{Supp}\Gamma_{k,n}^{\mathbf{K},\mathbb{S}} \subset G_{k,2n}^{\mathbf{K}} \cap \mathbb{S}, \quad \Gamma_{k,n}^{\mathbf{K},\mathbb{S}} = 1 \text{ on } G_{k,n}^{\mathbf{K}} \cap \mathbb{S} \quad \text{and} \quad \sup_{\alpha\in(0,1]} \sup_{u\in\mathbb{S}} \Big|\mathcal{L}_\alpha^U \Gamma_{k,n}^{\mathbf{K},\mathbb{S}}(u)\Big| < \infty.$$

The section is devoted to the proof of this lemma. We start with the following technical result.

Lemma 13. *We define the family $(c_\ell)_{\ell\in[\![1,N]\!]}$ by $c_0 = 1$ and for all $\ell \in [\![1, N-1]\!]$, $c_{\ell+1} = (2+8l)c_\ell$. For all $K \subsetneq [\![1, N]\!]$, all $\varepsilon > 0$, all $x \in (\mathbb{R}^2)^N$ such that*

$$R_K(x) \leq 2c_{|K|}\varepsilon \quad \text{and} \quad \min_{j\notin K} R_{K\cup\{j\}}(x) \geq c_{|K|+1}\varepsilon,$$

it holds that $\|x^i - x^j\|^2 \geq c_{|K|}\varepsilon$ for all $i \in K$, all $j \notin K$.

Proof. We fix $K \subsetneq [\![1, N]\!]$ and $\varepsilon > 0$ and assume by contradiction that there are $i_0 \in K$, $j_0 \notin K$ such that $\|x^{i_0} - x^{j_0}\|^2 < c_{|K|}\varepsilon$. Then for all $i \in K$,

$$\|x^{j_0} - x^i\|^2 \le 2\|x^{i_0} - x^{j_0}\|^2 + 2\|x^{i_0} - x^i\|^2 < 2\|x^{i_0} - x^{j_0}\|^2 + R_K(x) < 4c_{|K|}\varepsilon.$$

This implies that

$$R_{K\cup\{j_0\}}(x) = R_K(x) + 2\sum_{i\in K} \|x^{j_0} - x^i\|^2 < 2c_{|K|}\varepsilon + 8|K|c_{|K|}\varepsilon = c_{|K|+1}\varepsilon,$$

which is a contradiction. □

We are now ready to give the

Proof of Lemma 12. We introduce some nondecreasing C^∞ function $\varrho : \mathbb{R}_+ \to [0, 1]$ such that $\varrho = 0$ on $[0, 1/2]$ and $\varrho = 1$ on $[1, \infty)$. For $a > 0$, we set $\varrho_a(x) = \varrho(ax)$, which equals 0 if $x \in [0, 1/(2a)]$ and 1 if $x \ge 1/a$. We divide the proof in three steps.

Step 1. We fix $n \ge 1$ and define the families $(\tilde{E}_{K,n})_{K\subset[\![1,N]\!]:|K|\ge 2}$ and $(\tilde{\Gamma}_{K,n})_{K\subset[\![1,N]\!]:|K|\ge 2}$ by backward induction on the cardinal of $K \subset [\![1, N]\!]$, using the family $(c_\ell)_{\ell\in[\![1,N]\!]}$ of Lemma 13. We first set

$$\tilde{E}_{[\![1,N]\!],n} = \Big\{x \in (\mathbb{R}^2)^N : R_{[\![1,N]\!]}(x) \ge \frac{c_N}{n}\Big\} \quad \text{and} \quad \tilde{\Gamma}_{[\![1,N]\!],n}(x) = \varrho_{n/c_N}(R_{[\![1,N]\!]}(x)).$$

Then, for all $K \subset [\![1, N]\!]$ such that $|K| \in [\![2, N-1]\!]$, we define

$$\tilde{E}_{K,n} = \Big\{x \in \bigcap_{i\notin K} \tilde{E}_{K\cup\{i\},n} : R_K(x) \ge \frac{c_{|K|}}{n}\Big\} \quad \text{and} \quad \tilde{\Gamma}_{K,n}(x) = \varrho_{n/c_{|K|}}(R_K(x)) \prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}(x).$$

One easily checks by induction on the cardinal of K that for all $K \subset [\![1, N]\!]$,

$$\tilde{\Gamma}_{K,n} \in C^\infty((\mathbb{R}^2)^N), \quad \text{Supp}\, \tilde{\Gamma}_{K,n} \subset \tilde{E}_{K,2n}, \quad \text{and} \quad \tilde{\Gamma}_{K,n} = 1 \text{ on } \tilde{E}_{K,n}. \tag{43}$$

It also holds true, since $R_K(x) > 0$ implies that $R_L(x) > 0$ for all $L \supset K$, that

$$\cup_{n\ge 1}\tilde{E}_{K,n} = \tilde{E}_K, \quad \text{where} \quad \tilde{E}_K = \{x \in (\mathbb{R}^2)^N : R_K(x) > 0\}. \tag{44}$$

We now show, and this is the main difficulty of the step, that for all $A > 0$, all $K \subset [\![1, N]\!]$ with $|K| \ge 2$, we have $\sup_{\alpha\in(0,1]} \sup_{x\in B(0,A)} |\mathcal{L}^X_\alpha \tilde{\Gamma}_{K,n}(x)| < \infty$.

To this end, we proceed by backward induction on the cardinal of K and show that

$$\mathcal{P}(k): \text{ for all } K \subset [\![1, N]\!] \text{ such that } |K| = k, \text{ for all } A > 0, \sup_{\alpha\in(0,1]} \sup_{x\in B(0,A)} |\mathcal{L}^X_\alpha \tilde{\Gamma}_{K,n}(x)| < \infty$$

holds true for all $k \in [\![2, N]\!]$.

We will often use that for all $\varphi \in C^\infty((\mathbb{R}^2)^N)$, all $\phi \in C^\infty(\mathbb{R})$, all $x \in (\mathbb{R}^2)^N$, all $\alpha \in (0, 1]$,

$$\mathcal{L}^X_\alpha(\phi \circ \varphi)(x) = \phi'(\varphi(x))\mathcal{L}^X_\alpha \varphi(x) + \frac{1}{2}\phi''(\varphi(x))\|\nabla\varphi(x)\|^2 \tag{45}$$

and that for all $\varphi, \psi \in C^\infty((\mathbb{R}^2)^N)$, all $x \in (\mathbb{R}^2)^N$, all $\alpha \in (0, 1]$,

$$\mathcal{L}^X_\alpha(\varphi\psi)(x) = \varphi(x)\mathcal{L}^X_\alpha \psi(x) + \psi(x)\mathcal{L}^X_\alpha \varphi(x) + \nabla\varphi(x) \cdot \nabla\psi(x). \tag{46}$$

We will also use several times that for all $K \subset [\![1,N]\!]$, all $x \in (\mathbb{R}^2)^N$, all $\alpha \in (0,1]$,

$$I_{K,\alpha}(x) = \sum_{i,j\in K, i\neq j} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \nabla_{x^i} R_K(x) = 2 \sum_{i,j\in K, i\neq j} \frac{(x^i - x^j)\cdot x^i}{\|x^i - x^j\|^2 + \alpha}$$
$$= \sum_{i,j\in K, i\neq j} \frac{\|x^i - x^j\|^2}{\|x^i - x^j\|^2 + \alpha} \leq |K|(|K|-1). \tag{47}$$

Indeed, a computation shows that $\nabla_{x^i} R_K(x) = 2(x^i - S_K(x))$ for $i \in K$, so that the second equality follows by symmetry, as well as the third one.

We now prove $\mathcal{P}(N)$. Since $\tilde{\Gamma}_{[\![1,N]\!],n} = \varrho_{n/c_N} \circ R_{[\![1,N]\!]}$ and using (45),

$$\mathcal{L}^X_\alpha \tilde{\Gamma}_{[\![1,N]\!],n}(x) = \varrho'_{n/c_N}(R_{[\![1,N]\!]}(x)) \mathcal{L}^X_\alpha R_{[\![1,N]\!]}(x) + \frac{1}{2}\varrho''_{n/c_N}(R_{[\![1,N]\!]}(x)) \|\nabla R_{[\![1,N]\!]}(x)\|^2.$$

The only issue is thus to show that $\sup_{\alpha\in(0,1]} \sup_{x\in B(0,A)} |\mathcal{L}^X_\alpha R_{[\![1,N]\!]}(x)| < \infty$ for all $A > 0$. But

$$\mathcal{L}^X_\alpha R_{[\![1,N]\!]}(x) = \frac{1}{2}\Delta R_{[\![1,N]\!]}(x) - \frac{\theta}{N} \sum_{1\leq i\neq j\leq N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \nabla_{x^i} R_{[\![1,N]\!]}(x),$$

and $\Delta R_{[\![1,N]\!]}$ is constant because $R_{[\![1,N]\!]}$ is a polynomial function of degree 2, while the second term is uniformly bounded by (47). Hence $\mathcal{P}(N)$ holds true.

Next, assume that $\mathcal{P}(k)$ is true for some $k \in [\![3,N]\!]$ and we show that $\mathcal{P}(k-1)$ is true. We fix $K \subset [\![1,N]\!]$ such that $|K| = k-1$. By (46), $\mathcal{L}^X_\alpha \tilde{\Gamma}_{K,n} = S_{1,\alpha} + S_{2,\alpha} + S_3$, where

$$S_{1,\alpha}(x) = \Big(\prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}(x)\Big) \mathcal{L}^X_\alpha \Big(\varrho_{n/c_{|K|}} \circ R_K\Big)(x),$$
$$S_{2,\alpha}(x) = \varrho_{n/c_{|K|}}(R_K(x)) \mathcal{L}^X_\alpha \Big(\prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}\Big)(x),$$
$$S_3(x) = \nabla\Big(\varrho_{n/c_{|K|}} \circ R_K\Big)(x) \cdot \nabla\Big(\prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}\Big)(x).$$

It is clear that S_3, which does not depend on α, is locally bounded. Moreover, we deduce from $\mathcal{P}(k)$ and (46) that $\sup_{\alpha\in(0,1]} \sup_{x\in B(0,A)} S_{2,\alpha}(x) < \infty$ for all $A > 0$.

To complete the step, it remains to show that $\sup_{\alpha\in(0,1]} \sup_{x\in B(0,A)} S_{1,\alpha}(x) < \infty$ for all $A > 0$.

If first $R_K(x) > c_{|K|}/n$ then $S_{1,\alpha}(x) = 0$, because $\mathcal{L}^X_\alpha(\varrho_{n/c_{|K|}} \circ R_K)(x) = 0$, due to the fact that $\varrho_{n/c_{|K|}} \circ R_K \equiv 1$ on the set $\{R_K > c_{|K|}/n\}$.

If next $\min_{i\notin K} R_{K\cup\{i\}}(x) < c_{|K|+1}/(2n)$, then $S_{1,\alpha}(x) = 0$, because $\prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}(x) = 0$.

Finally, we show that for each $A > 0$, there is a constant C_A (also depending on n and K) such that for all $x \in B(0,A)$ satisfying $R_K(x) \leq c_{|K|}/n$ and $\min_{i\notin K} R_{K\cup\{i\}}(x) \geq c_{|K|+1}/(2n)$, all $\alpha \in (0,1]$, we have $|S_{1,\alpha}(x)| \leq C_A$. Recalling (45) and that $|\prod_{i\notin K} \tilde{\Gamma}_{K\cup\{i\},n}(x)| \leq 1$, we see that

$$\begin{aligned}|S_{1,\alpha}(x)| \leq& |\mathcal{L}^X_\alpha(\varrho_{n/c_{|K|}} \circ R_K)(x)| \\ =& |\varrho'_{n/c_{|K|}}(R_K(x)) \mathcal{L}^X_\alpha R_K(x) + \frac{1}{2}\varrho''_{n/c_{|K|}}(R_K(x)) \|\nabla R_K(x)\|^2| \\ \leq& C_A(1 + |\mathcal{L}^X_\alpha R_K(x)|),\end{aligned}$$

for some constant C_A allowed to vary from line to line. Using that $\nabla_{x^i} R_K = 0$ if $i \notin K$, we write $|\mathcal{L}^X_\alpha R_K| \le \frac{1}{2}|\Delta R_K| + |A_{1,\alpha}| + |A_{2,\alpha}|$, where

$$A_{1,\alpha}(x) = \Big| \frac{\theta}{N} \sum_{i,j\in K, i\neq j} \frac{x^i - x^j}{\|x^i - x^j\|^2} \cdot \nabla_{x^i} R_K(x) \Big|$$
$$A_{2,\alpha}(x) = \Big| \frac{\theta}{N} \sum_{i\in K, j\notin K} \frac{x^i - x^j}{\|x^i - x^j\|^2} \cdot \nabla_{x^i} R_K(x) \Big|.$$

First, $|\Delta R_K|$ is constant and $A_{1,\alpha} = \frac{\theta}{N}|I_{K,\alpha}(x)|$ is bounded, see (47). Moreover by Lemma 13 and since $R_K(x) \le c_{|K|}/n$ and $R_{K\cup\{i\}}(x) \ge c_{|K|+1}/(2n)$ for all $i \notin K$ by assumption, we deduce that for all $i \in K$, all $j \notin K$, $\|x^i - x^j\|^2 \ge c_{|K|}/(2n)$. Since $||\nabla_{x^i} R_K||$ is locally bounded, we conclude that $A_{2,\alpha}(x) \le C_A$ as desired.

Step 2. We can now prove (i). We fix $k \in [\![1, N]\!]$ and a partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1, N]\!]$. For each $n \ge 1$, we set

$$G^{\mathbf{K}}_{k,n} = \overline{B}(0, n) \cap \Big(\bigcap_{K\subset[\![1,N]\!]:|K|=k} \tilde{E}_{K,n} \Big) \cap \Big(\bigcap_{1\le p\neq q\le \ell} \; \bigcap_{i\in K_p, j\in K_q} \tilde{E}_{\{i,j\},n} \Big),$$
$$\Gamma^{\mathbf{K}}_{k,n}(x) = g_n(x) \Big(\prod_{K\subset[\![1,N]\!]:|K|=k} \tilde{\Gamma}_{K,n}(x) \Big) \Big(\prod_{1\le p\neq q\le \ell} \; \prod_{i\in K_p, j\in K_q} \tilde{\Gamma}_{\{i,j\},n}(x) \Big),$$

where $g_n(x) = \varrho_n(1/\|x\|)$ with the extension $g_n(0) = 1$.

First, $G^{\mathbf{K}}_{k,n}$ is clearly compact in $\mathring{G}^{\mathbf{K}}_{k,n+1}$ and we deduce from (44) that

$$\cup_{n\ge1} G^{\mathbf{K}}_{k,n} = \Big(\bigcap_{K\subset[\![1,N]\!]:|K|=k} \tilde{E}_K \Big) \cap \Big(\bigcap_{1\le p\neq q\le \ell} \; \bigcap_{i\in K_p, j\in K_q} \tilde{E}_{\{i,j\}} \Big) = G^{\mathbf{K}}_k,$$

where $G^{\mathbf{K}}_k$ was defined in the statement. We also deduce from (43) that for each $n \ge 1$, it holds that $\Gamma^{\mathbf{K}}_{k,n} \in C^\infty((\mathbb{R}^2)^N)$, that $\Gamma^{\mathbf{K}}_{k,n} = 1$ on $G^{\mathbf{K}}_{k,n}$ and that Supp $\Gamma^{\mathbf{K}}_{k,n} \subset G^{\mathbf{K}}_{k,2n}$.

It remains to show that $\sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha \Gamma^{\mathbf{K}}_{k,n}(x)| < \infty$. Thanks to (46) and introducing

$$\chi_{\mathbf{K},k,n}(x) = \Big(\prod_{K\subset[\![1,N]\!]:|K|=k} \tilde{\Gamma}_{K,n}(x) \Big) \Big(\prod_{1\le p\neq q\le \ell} \; \prod_{i\in K_p, j\in K_q} \tilde{\Gamma}_{\{i,j\},n}(x) \Big),$$

which belongs to $C^\infty((\mathbb{R}^2)^N)$ by Step 1, we have $\Gamma^{\mathbf{K}}_{k,n} = g_n \chi_{\mathbf{K},k,n}$ and thus by (46)

$$\mathcal{L}^X_\alpha \Gamma^{\mathbf{K}}_{k,n}(x) = g_n(x) \mathcal{L}^X_\alpha \chi_{\mathbf{K},k,n}(x) + \chi_{\mathbf{K},k,n}(x) \mathcal{L}^X_\alpha g_n(x) + \nabla g_n(x) \cdot \nabla \chi_{\mathbf{K},k,n}(x).$$

The first term is uniformly bounded because g_n is bounded and supported in $B(0, 2n)$ and because $\sup_{\alpha\in(0,1]} \sup_{x\in B(0,2n)} |\mathcal{L}^X \chi_{\mathbf{K},k,n}(x)| < \infty$ by Step 1 and (46). The third term is also uniformly bounded, since $\chi_{\mathbf{K},k,n} \in C^\infty((\mathbb{R}^2)^N)$ and since ∇g_n is bounded and supported in $B(0, 2n)$. Finally, the middle term is bounded because $\chi_{\mathbf{K},k,n}$ is bounded by 1 and because $\mathcal{L}^X_\alpha g_n$ is uniformly bounded, as we now show: Δg_n is obviously bounded since $g_n \in C^\infty_c((\mathbb{R}^2)^N)$ and, since $\nabla_{x^i} g_n(x) = -\varrho'_n(1/||x||) x^i/\|x\|^3$,

$$\sum_{1\le i,j\le N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \nabla_{x^i} g_n(x) = -\frac{\varrho'_n(1/||x||)}{\|x\|^3} \sum_{1\le i,j\le N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot x^i$$
$$= -\frac{\varrho'_n(1/||x||)}{2\|x\|^3} \sum_{1\le i,j\le N} \frac{\|x^i - x^j\|^2}{\|x^i - x^j\|^2 + \alpha}$$

by symmetry. This last quantity is uniformly bounded, since ϱ_n' is bounded and vanishes on $[1/n,\infty)$.

Step 3. We now prove (ii), by showing that the restriction $\Gamma^{\mathbf{K},\mathbb{S}}_{k,n}=\Gamma^{\mathbf{K}}_{k,n}|_{\mathbb{S}}$ satisfies the required conditions. We obviously have $\Gamma^{\mathbb{S}}_{k,n}\in C^\infty(\mathbb{S})$, Supp $\Gamma^{\mathbb{S}}_{k,n}\subset E_{k,2n}\cap\mathbb{S}$ and $\Gamma^{\mathbb{S}}_{k,n}=1$ on $E_{k,n}\cap\mathbb{S}$. It remains to show that $\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}}|\mathcal{L}^U_\alpha\Gamma^{\mathbf{K},\mathbb{S}}_{k,n}|<\infty$, recall (22). Since $\Gamma^{\mathbb{S}}_{k,n}\in C^\infty(\mathbb{S})$, $\Delta_{\mathbb{S}}\Gamma^{\mathbb{S}}_{k,n}$ is bounded. We thus only have to verify that $\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}}|T_\alpha(u)|<\infty$, where

$$T_\alpha(u)=-\frac{\theta}{N}\sum_{1\le i,j\le N}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(\nabla_{\mathbb{S}}\Gamma^{\mathbf{K},\mathbb{S}}_{k,n}(u))^i$$

Setting $b^i_\alpha(u)=-\frac{\theta}{N}\sum_{j=1}^N\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}$ and using (14),

$$T_\alpha(u)=b_\alpha(u)\cdot\nabla_{\mathbb{S}}\Gamma^{\mathbf{K},\mathbb{S}}_{k,n}(u)=b_\alpha(u)\cdot\pi_H(\pi_{u^\perp}(\nabla\Gamma^{\mathbf{K}}_{k,n}(u))).$$

Since now $b(u)\in H$ and since π_H and $\pi_{u^\perp}$ are self-adjoint, as every orthogonal projection, we get

$$T_\alpha(u)=\pi_{u^\perp}(b_\alpha(u))\cdot\nabla\Gamma^{\mathbf{K}}_{k,n}(u)=b_\alpha(u)\cdot\nabla\Gamma^{\mathbf{K}}_{k,n}(u)-(b_\alpha(u)\cdot u)(u\cdot\nabla\Gamma^{\mathbf{K}}_{k,n}(u)).$$

But $b_\alpha(u)\cdot\nabla\Gamma^{\mathbf{K}}_{k,n}(u)=\mathcal{L}^X_\alpha\Gamma^{\mathbf{K}}_{k,n}(u)-\frac12\Delta\Gamma^{\mathbf{K}}_{k,n}(u)$ is uniformly bounded by point (i) and since $\Delta\Gamma^{\mathbf{K}}_{k,n}(u)$ is bounded on $\mathbb{S}$. Next,

$$b_\alpha(u)\cdot u=-\frac{\theta}{N}\sum_{1\le i,j\le N}\frac{(u^i-u^j)\cdot u^i}{\|u^i-u^j\|^2+\alpha}=-\frac{\theta}{2N}\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha}$$

by symmetry, and this last quantity is also uniformly bounded. Finally, $u\cdot\nabla\Gamma^{\mathbf{K}}_{k,n}(u)$ is smooth and thus bounded on $\mathbb{S}$. □

7. A Girsanov theorem for the Keller-Segel particle system.

The goal of this section is to provide a rigorous version of the intuitive argument presented in Subsection 3.4.

For $x\in(\mathbb{R}^2)^N$, all $K\subset[\![1,N]\!]$, we denote by $x|_K=(x^i)_{i\in K}$. For $\mathbf{K}=(K_p)_{p\in[\![1,\ell]\!]}$ a partition of $[\![1,N]\!]$, for $y_1\in(\mathbb{R}^2)^{|K_1|},\dots,y_\ell\in(\mathbb{R}^2)^{|K_\ell|}$, we abusively denote by $(y_p)_{p\in[\![1,\ell]\!]}$ the element y of $(\mathbb{R}^2)^N$ such that for all $i\in[\![1,\ell]\!]$, $y|_{K_i}=y_i$.

We adopt the convention that for any $\theta>0$, a $QKS(\theta,1)$-process is a 2-dimensional Brownian motion. This is natural in view of (1).

Proposition 14. *Let $N\ge2$, $\theta>0$ such that $N>\theta$ and set $k_0=\lceil 2N/\theta\rceil$. Fix $\ell\in[\![2,N]\!]$ and some partition $\mathbf{K}=(K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1,N]\!]$. We introduce the state spaces $\mathcal{X}=E_{k_0}$ and, for each $p\in[\![1,\ell]\!]$,*

$$\mathcal{Y}_p=\Big\{y\in(\mathbb{R}^2)^{|K_p|}:\forall K\subset[\![1,|K_p|]\!]\text{ with }|K|\ge k_0,\ \sum_{i,j=1}^{|K_p|}||y^i-y^j||^2>0\Big\}.$$

Consider

- $\mathbb{X}=(\Omega^X,\mathcal{M}^X,(X_t)_{t\ge0},(\mathbb{P}^X_x)_{x\in\mathcal{X}_\triangle})$ *a $QKS(\theta,N)$-process,*
- *For all $p\in[\![1,\ell]\!]$, $\mathbb{Y}^p=(\Omega^p,\mathcal{M}^p,(Y_{p,t})_{t\ge0},(\mathbb{P}^p_y)_{y\in\mathcal{Y}^p_\triangle})$ a $QKS(\theta|K_p|/N,|K_p|)$-process.*

We set $\Omega^Y = \prod_{p=1}^{\ell} \Omega^p$ and $Y_t = (Y_{p,t})_{p\in[\![1,\ell]\!]}$, with the convention that $Y_t = \triangle$ as soon as $Y_{p,t} = \triangle$ for some $p \in [\![1,\ell]\!]$. We also introduce $\mathcal{M}^Y = \sigma(Y_t : t \geq 0)$, as well as $\mathbb{P}^Y_y = \otimes_{p=1}^{\ell} \mathbb{P}^p_{y_p}$ for all $y = (y_p)_{p\in[\![1,\ell]\!]} \in (\mathbb{R}^2)^N$.

We fix $\varepsilon > 0$ and set

$$G_{\mathbf{K},\varepsilon} = \Big\{x \in \mathcal{X} : \min_{1\leq p\neq q\leq \ell} \min_{i\in K_p, j\in K_q} \|x^i - x^j\|^2 > \varepsilon\Big\} \cap B(0, 1/\varepsilon),$$
$$\tau_{\mathbf{K},\varepsilon} = \big\{t \geq 0 : X_t \notin G_{\mathbf{K},\varepsilon}\big\} \quad \text{and} \quad \tilde\tau_{\mathbf{K},\varepsilon} = \big\{t \geq 0 : Y_t \notin G_{\mathbf{K},\varepsilon}\big\}.$$

Fix $T > 0$. For quasi all $x \in G_{\mathbf{K},\varepsilon}$, there is a probability measure $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x$ on $(\Omega^X, \mathcal{M}^X)$, equivalent to $\mathbb{P}^X_x$, such that the law of $(X_{t\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{t\geq 0}$ under $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x$ is the same as that of $(Y_{t\wedge T\wedge\tilde\tau_{\mathbf{K},\varepsilon}})_{t\geq 0}$ on $(\Omega^Y, \mathcal{M}^Y)$ under $\mathbb{P}^Y_x$.

Furthermore, the Radon-Nikodym density $\frac{\mathrm{d}\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x}{\mathrm{d}\mathbb{P}^X_x}$ is $\mathcal{M}^X_{T\wedge\tau_{\mathbf{K},\varepsilon}}$-measurable, where as usual $\mathcal{M}^X_t = \sigma(X_s, s \leq t)$, and there exists a deterministic constant $C_{T,\varepsilon,\mathbf{K}} > 0$ such that for quasi all $x \in G_{\mathbf{K},\varepsilon}$,

$$C^{-1}_{T,\varepsilon,\mathbf{K}} \leq \frac{d\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x}{d\mathbb{P}^X_x} \leq C_{T,\varepsilon,\mathbf{K}}.$$

The quasi notion refers to the process $\mathbb{X}$. Let us mention that for ζ the life-time of $\mathbb{X}$, we have $\tau_{\mathbf{K},\varepsilon} \in [0, \zeta]$ when $\zeta < \infty$ because $\triangle \notin G_{\mathbf{K},\varepsilon}$. However, when $\tau_{\mathbf{K},\varepsilon} = \zeta$, it is generally not true that X_t goes to the boundary of $G_{\mathbf{K},\varepsilon}$ as $t \to \zeta-$.

Proof. We only consider the case where $\ell = 2$. The general case is heavier in terms of notation but contains no additional difficulty. We fix $\mathbf{K} = (K_1, K_2)$ a non-trivial partition of $[\![1, N]\!]$. Applying informally Lemma B.7 to $\mathbb{X}$ with the function $\varrho(x) = \exp(u(x))$, where

$$u(x) = \frac{\theta}{N} \sum_{i\in K_1, j\in K_2} \log(\|x^i - x^j\|),$$

until $\tau_{\mathbf{K},\varepsilon}$ would prove the result, mainly because $\varrho^2\mu = \mu_1 \otimes \mu_2$, with μ defined in (4) and with the measures on $(\mathbb{R}^2)^{|K_1|}$ and $(\mathbb{R}^2)^{|K_2|}$ defined by

$$\mu_1(\mathrm{d}y) = \prod_{i,j\in K_1, i\neq j} ||y^i - y^j||^{-\theta/N}\mathrm{d}y \quad \text{and} \quad \mu_2(\mathrm{d}y) = \prod_{i,j\in K_2, i\neq j} ||y^i - y^j||^{-\theta/N}\mathrm{d}y.$$

Unfortunately, this is not licit, because u does not belong to $\mathcal{F}^X$, mainly because $G_{\mathbf{K},\varepsilon}$ is not relatively compact in $\mathcal{X}$, at least if $|K_1| \geq k_0$ or $|K_2| \geq k_0$. To overcome this difficulty, we introduce, for $n \geq 1$, $\varrho_n(x) = e^{u_n(x)}$, where

$$u_n(x) = u(x)\Gamma^{\mathbf{K}}_{k_0,n}(x),$$

with $\Gamma^{\mathbf{K}}_{k_0,n}$ defined in Lemma 12-(i). Observe that $u_n \in C^\infty_c((\mathbb{R}^2)^N)$ because $\Gamma^{\mathbf{K}}_{k_0,n}$ vanishes in the zone where u is not smooth. We will first apply Lemma B.7 with ϱ_n and then let $n \to \infty$.

Step 1. We first prove, recalling the definition (21) of $\mathcal{L}^X_\alpha$, that

$$\sup_{\alpha\in(0,1]} \sup_{x\in G_{\mathbf{K},\varepsilon}} |\mathcal{L}^X_\alpha \varrho(x)| < \infty \quad \text{and} \quad \forall\, n \geq 1, \quad \sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha \varrho_n(x)| < \infty.$$

By (45),

$$\mathcal{L}^X_\alpha \varrho(x) = e^{u(x)}\mathcal{L}^X_\alpha u(x) + \frac{1}{2}e^{u(x)}\|\nabla u(x)\|^2, \quad \mathcal{L}^X_\alpha \varrho_n(x) = e^{u_n(x)}\mathcal{L}^X_\alpha u_n(x) + \frac{1}{2}e^{u_n(x)}\|\nabla u_n(x)\|^2.$$

Since $u \in C^\infty(\bar{G}_{\mathbf{K},\varepsilon})$ with $\bar{G}_{\mathbf{K},\varepsilon}$ compact in $(\mathbb{R}^2)^N$ and since $u_n \in C_c^\infty((\mathbb{R}^2)^N)$, the only difficulties are to check that

$$\sup_{\alpha\in(0,1]} \sup_{x\in G_{\mathbf{K},\varepsilon}} |\mathcal{L}_\alpha^X u(x)| < \infty \qquad \text{and} \qquad \sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}_\alpha^X u_n(x)| < \infty.$$

By (46),

$$\mathcal{L}_\alpha^X u_n(x) = \Gamma_{k_0,n}^{\mathbf{K}}(x)\mathcal{L}_\alpha^X u(x) + u(x)\mathcal{L}_\alpha^X \Gamma_{k_0,n}^{\mathbf{K}}(x) + \nabla\Gamma_{k_0,n}^{\mathbf{K}}(x)\cdot\nabla u(x).$$

Again, the only difficulty consists of the first term, because $\mathcal{L}_\alpha^X \Gamma_{k_0,n}^{\mathbf{K}}$ is uniformly bounded by Lemma 12. Since Supp $\Gamma_{k_0,n}^{\mathbf{K}} \subset G_{k_0,2n}^{\mathbf{K}}$, see Lemma 12, and since $G_{k_0,2n}^{\mathbf{K}} \subset G_{\mathbf{K},\varepsilon}$ (actually for some other value of $\varepsilon > 0$), we are reduced to show that $\sup_{\alpha\in(0,1]}\sup_{x\in G_{\mathbf{K},\varepsilon}} |\mathcal{L}_\alpha^X u(x)| < \infty$. But

$$\mathcal{L}_\alpha^X u = \frac{1}{2}\Delta u - \frac{\theta}{N}S_\alpha, \quad \text{where} \quad S_\alpha(x) = \sum_{1\le i,j\le N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot\nabla_{x^i}u(x),$$

and we only have to verify that $\sup_{\alpha\in(0,1]}\sup_{x\in G_{\mathbf{K},\varepsilon}} |S_\alpha(x)| < \infty$.

For $k \in K_1$ and $\ell \in K_2$, we have

$$\nabla_{x^k}u(x) = \sum_{j\in K_2}\frac{\theta}{N}\frac{x^k - x^j}{\|x^k - x^j\|^2} \quad \text{and} \quad \nabla_{x^\ell}u(x) = \sum_{i\in K_1}\frac{\theta}{N}\frac{x^\ell - x^i}{\|x^\ell - x^i\|^2}.$$

Hence $S_\alpha = S_{1,\alpha} + S_{2,\alpha} + S_{3,\alpha} + S_{4,\alpha}$, where

$$S_{1,\alpha}(x) = \frac{\theta}{N}\sum_{i,j\in K_1}\frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot\sum_{k\in K_2}\frac{x^i - x^k}{\|x^i - x^k\|^2},$$

$$S_{2,\alpha}(x) = \frac{\theta}{N}\sum_{i\in K_2, j\in K_1}\frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\cdot\sum_{k\in K_1}\frac{x^i - x^k}{\|x^i - x^k\|^2},$$

and $S_{3,\alpha}$ (resp. $S_{4,\alpha}$) is defined as $S_{1,\alpha}$ (resp. $S_{2,\alpha}$) exchanging the roles of K_1 and K_2. First, $S_{2,\alpha}$ (and $S_{4,\alpha}$) is obviously uniformly bounded on $G_{\mathbf{K},\varepsilon}$. Next, by symmetry,

$$S_{1,\alpha}(x) = \frac{\theta}{2N}\sum_{i,j\in K_1}\frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha}\sum_{k\in K_2}\Big(\frac{x^i - x^k}{\|x^i - x^k\|^2} - \frac{x^j - x^k}{\|x^j - x^k\|^2}\Big).$$

Moreover, there is $C_\varepsilon > 0$ such that for all $x \in G_{\mathbf{K},\varepsilon}$, all $i, j \in K_1$ such that $i \neq j$, all $k \in K_2$,

$$\Big\|\frac{x^i - x^k}{\|x^i - x^k\|^2} - \frac{x^j - x^k}{\|x^j - x^k\|^2}\Big\| \le C_\varepsilon\|x^i - x^j\|,$$

so that $S_{1,\alpha}$ (and $S_{3,\alpha}$) is bounded on $G_{\mathbf{K},\varepsilon}$, uniformly in $\alpha \in (0,1]$, as desired.

Step 2. We denote by

$$\mathbb{X}^{*,n} = \Big(\Omega^X, \mathcal{M}^X, (X_t^{*,n})_{t\ge0}, (\mathbb{P}_x^X)_{x\in G_{k_0,3n}^{\mathbf{K}}\cup\{\triangle\}}\Big)$$

the version of $\mathbb{X}$ killed when exiting $G_{k_0,3n}^{\mathbf{K}}$. Lemma B.6 implies that $\mathbb{X}^{*,n}$ is a continuous $\mu|_{G_{k_0,3n}^{\mathbf{K}}}$-symmetric $G_{k_0,3n}^{\mathbf{K}} \cup \{\triangle\}$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^n, \mathcal{F}^n)$ with core $C_c^\infty(G_{k_0,3n}^{\mathbf{K}})$ and for all $\varphi \in C_c^\infty(G_{k_0,3n}^{\mathbf{K}})$,

$$\mathcal{E}^n(\varphi,\varphi) = \frac{1}{2}\int_{G_{k_0,3n}^{\mathbf{K}}} ||\nabla\varphi||^2 \mathrm{d}\mu.$$

Step 3. We want to apply Lemma B.7 to $\mathbb{X}^{*,n}$ with the function u_n and so we first check that the conditions are satisfied. We have $u_n \in C_c^\infty((\mathbb{R}^2)^N)$ with Supp $u_n = G^{\mathbf{K}}_{k_0,2n}$, which is compact in $G^{\mathbf{K}}_{k_0,3n}$, see Lemma 12. We conclude that $u_n \in \mathcal{F}^n$ and that u_n is bounded. We denote by $(\mathcal{A}^n, \mathcal{D}_{\mathcal{A}^n})$ the generator of $\mathbb{X}^{*,n}$. Thanks to Remark 8 and Step 1, we have $\varrho_n \in \mathcal{D}_{\mathcal{A}^n}$ and $\mathcal{A}^n \varrho_n = \mathcal{L}^X \varrho_n$, and $\mathcal{A}^n \varrho_n$ is bounded, again by Step 1. And with the convention that $\varrho_n(\triangle) = 1$, ϱ_n is continuous on $G^{\mathbf{K}}_{k_0,3n} \cup \{\triangle\}$, because u_n is continuous and vanishes on a neighborhood of the boundary of $G^{\mathbf{K}}_{k_0,3n}$ (because it is supported in $G^{\mathbf{K}}_{k_0,2n}$). Hence we can apply Lemma B.7. We introduce

$$L_t^{\varrho_n} = \frac{\varrho_n(X_t^{*,n})}{\varrho_n(X_0^{*,n})} \exp\Big(- \int_0^t \frac{\mathcal{A}^n \varrho_n(X_s^{*,n})}{\varrho_n(X_s^{*,n})} \mathrm{d}s \Big),$$

with the same conventions as in Lemma B.7, i.e. $\varrho_n(\triangle) = 1$ and $\mathcal{A}^n \varrho_n(\triangle) = 0$. By Lemma B.7, there exists a family of probability measures $(\mathbb{Q}_x^n)_{x \in G^{\mathbf{K}}_{k_0,3n} \cup \{\triangle\}}$ such that

$$\mathbb{Q}_x^n = L_t^{\varrho_n} \cdot \mathbb{P}_x^X \quad \text{on} \quad \sigma(\{X_s^{*,n}, s \le t\})$$

for all $t \ge 0$ and for quasi all $x \in G^{\mathbf{K}}_{k_0,3n} \cup \{\triangle\}$, and such that

$$\tilde{\mathbb{X}}^{*,n} = (\Omega^X, \mathcal{M}^X, (X_t^{*,n})_{t\ge 0}, (\mathbb{Q}_x^n)_{x \in G^{\mathbf{K}}_{k_0,3n} \cup \{\triangle\}})$$

is a continuous $\varrho_n^2 \mu|_{G^{\mathbf{K}}_{k_0,3n}}$-symmetric $G^{\mathbf{K}}_{k_0,3n} \cup \{\triangle\}$-valued Hunt process with regular Dirichlet space $(\tilde{\mathcal{E}}^n, \tilde{\mathcal{F}}^n)$ with core $C_c^\infty(G^{\mathbf{K}}_{k_0,3n})$ such that for all $\varphi \in C_c^\infty(G^{\mathbf{K}}_{k_0,3n})$,

$$\tilde{\mathcal{E}}^n(\varphi, \varphi) = \frac{1}{2} \int_{G^{\mathbf{K}}_{k_0,3n}} ||\nabla \varphi||^2 \varrho_n^2 \mathrm{d}\mu.$$

Step 4. Denote by

$$\tilde{\mathbb{X}}^{*,n,\varepsilon} = \Big(\Omega^X, \mathcal{M}^X, (X_t^{*,n})_{t\ge 0}, (\mathbb{Q}_x^n)_{x \in (G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}) \cup \{\triangle\}}\Big)$$

the version of $\tilde{\mathbb{X}}^{*,n}$ killed when exiting the open set $G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}$. By Lemma B.6, $\tilde{\mathbb{X}}^{*,n}$ is a continuous $\varrho_n^2 \mu|_{G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}}$-symmetric $(G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}) \cup \{\triangle\}$-valued Hunt process with regular Dirichlet space $(\tilde{\mathcal{E}}^{n,\varepsilon}, \tilde{\mathcal{F}}^{n,\varepsilon})$ with core $C_c^\infty(G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon})$ such that for all $\varphi \in C_c^\infty(G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon})$,

$$\tilde{\mathcal{E}}^{n,\varepsilon}(\varphi, \varphi) = \frac{1}{2} \int_{G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}} ||\nabla \varphi||^2 \varrho_n^2 \mathrm{d}\mu = \frac{1}{2} \int_{G^{\mathbf{K}}_{k_0,n} \cap G_{\mathbf{K},\varepsilon}} ||\nabla \varphi||^2 \varrho^2 \mathrm{d}\mu.$$

Indeed, $\Gamma^{\mathbf{K}}_{k_0,n} = 1$, whence $\varrho_n = \varrho$, on $G^{\mathbf{K}}_{k_0,n}$, see Lemma 12.

Step 5. Consider the measures

$$\mu_1(\mathrm{d}y) = \prod_{i,j \in K_1, i \ne j} ||y^i - y^j||^{-\theta/N} \mathrm{d}y \quad \text{and} \quad \mu_2(\mathrm{d}y) = \prod_{i,j \in K_2, i \ne j} ||y^i - y^j||^{-\theta/N} \mathrm{d}y$$

on $(\mathbb{R}^2)^{|K_1|}$ and $(\mathbb{R}^2)^{|K_2|}$, with $\mu_i(\mathrm{d}y) = \mathrm{d}y$ if $|K_i| = 1$. Recall that $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$, see (4) and that by definition, $\varrho(x) = \prod_{i \in K_1, j \in K_2} \|x^i - x^j\|^{\theta/N}$: we deduce that

$$\mu_1 \otimes \mu_2 = \varrho^2 \mu.$$

Recall from Proposition 6 that for $p = 1, 2$, $\mathbb{Y}^p$ is a continuous $\mathcal{Y}^p_\triangle$-valued μ_p-symmetric (since $(\theta|K_p|/N)/|K_p| = \theta/N$) Hunt process with regular Dirichlet space $(\mathcal{E}_p, \mathcal{F}_p)$ with core $C_c^\infty(\mathcal{Y}_p)$ and, for $\varphi \in C_c^\infty(\mathcal{Y}_p)$, $\mathcal{E}_p(\varphi, \varphi) = \frac{1}{2} \int_{(\mathbb{R}^2)^{|K_p|}} ||\nabla \varphi||^2 \mathrm{d}\mu_p$. This also holds true if e.g. $|K_1| = 1$, see [11, Example 4.2.1 page 167], since then μ_1 is nothing but the Lebesgue measure on $\mathbb{R}^2$.

Since now $\mu_1 \otimes \mu_2 = \varrho^2\mu$, by Lemma B.5,

$$\mathbb{Y} = \Big(\Omega^Y, \mathcal{M}^Y, (Y_t)_{t\geq 0}, (\mathbb{P}_y^Y)_{y\in(\mathcal{Y}_1\times\mathcal{Y}_2)\cup\{\triangle\}}\Big)$$

is a $\varrho^2\mu$-symmetric continuous $\mathcal{X}_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(\mathcal{Y}_1\times\mathcal{Y}_2, \varrho^2\mathrm{d}\mu)$ with core $C_c^\infty(\mathcal{Y}_1\times\mathcal{Y}_2)$ and, for $\varphi \in C_c^\infty(\mathcal{Y}_1\times\mathcal{Y}_2)$,

$$\begin{aligned}\mathcal{E}^Y(\varphi,\varphi) =& \int_{(\mathbb{R}^2)^{|K_1|}} \mathcal{E}_2(\varphi(y,\cdot),\varphi(y,\cdot))\mu_1(\mathrm{d}y) + \int_{(\mathbb{R}^2)^{|K_2|}} \mathcal{E}_1(\varphi(\cdot,z),\varphi(\cdot,z))\mu_2(\mathrm{d}z)\\ =&\frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2\varrho^2\mathrm{d}\mu.\end{aligned}$$

Finally, we apply Lemma B.6 to $\mathbb{Y}$ with the open set $G^{\mathbf{K}}_{k_0,n}\cap G_{\mathbf{K},\varepsilon} \subset \mathcal{X} \subset \mathcal{Y}_1\times\mathcal{Y}_2$, to find that the resulting killed process

$$\mathbb{Y}^{*,n,\varepsilon} = \Big(\Omega^Y, \mathcal{M}^Y, (Y_t^{*,n,\varepsilon})_{t\geq 0}, (\mathbb{P}_y^Y)_{y\in(G^{\mathbf{K}}_{k_0,n}\cap G_{\mathbf{K},\varepsilon})\cup\{\triangle\}}\Big)$$

is a continuous $\varrho^2\mu|_{G^{\mathbf{K}}_{k_0,n}\cap G_{\mathbf{K},\varepsilon}}$-symmetric $(G^{\mathbf{K}}_{k_0,n}\cap G_{\mathbf{K},\varepsilon})\cup\{\triangle\}$-valued Hunt process with the same regular Dirichlet space as $\tilde{\mathbb{X}}^{*,n,\epsilon}$, see Step 4.

Step 6. By uniqueness, see [11, Theorem 4.2.8 p 167], for quasi all $x \in G^{\mathbf{K}}_{k_0,n}\cap G_{\mathbf{K},\varepsilon}$, the law of $(X_t^{*,n,\varepsilon})_{t\geq 0}$ under $\mathbb{Q}^n_x$ is the same as that of $(Y_t^{*,n,\varepsilon})_{t\geq 0}$ under $\mathbb{P}^Y_x$.

Step 7. Here we fix $T>0$ and we prove that there exist some constant $C_{T,\varepsilon,\mathbf{K}}>0$ and some $\mathcal{M}_{T\wedge\tau_{\mathbf{K},\varepsilon}}$-measurable random variable $J_{T,\varepsilon,\mathbf{K}}$ such that $C^{-1}_{T,\varepsilon,\mathbf{K}} \leq J_{T,\varepsilon,\mathbf{K}} \leq C_{T,\varepsilon,\mathbf{K}}$ and such that for quasi all $x\in G_{\mathbf{K},\varepsilon}$, $\mathbb{E}^X_x[J_{T,\varepsilon,\mathbf{K}}]=1$ and there is $n_x\geq 1$ such that for all $n\geq n_x$,

$$\mathbb{E}^X_x[J_{T,\varepsilon,\mathbf{K}}|\mathcal{M}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}] = L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}},$$

where we have set $\zeta_n = \inf\{t>0 : X_t \notin G^{\mathbf{K}}_{k_0,n}\}$.

Recall that $\cup_{n\geq 1}G^{\mathbf{K}}_{k_0,n} = G^{\mathbf{K}}_{k_0}$, see Lemma 12, and observe that $G^{\mathbf{K}}_{k_0}$ contains $G_{\mathbf{K},\varepsilon}$. Fix $x\in G_{\mathbf{K},\varepsilon}$, and consider $n_x\geq 1$ such that $x\in G^{\mathbf{K}}_{k_0,n_x}$. We know from Lemma B.7 that for all $n\geq n_x$, $(L^{\varrho_n}_t)_{t\geq 0}$ is a $\mathbb{P}^X_x$-martingale. Moreover, for all $n\geq n_x$, we have $\varrho_{n+1}=\varrho=\varrho_n$ and $\mathcal{A}^{n+1}\varrho_{n+1} = \mathcal{L}^X\varrho_{n+1} = \mathcal{L}^X\varrho = \mathcal{L}^X\varrho_n = \mathcal{A}^n\varrho_n$ on $G^{\mathbf{K}}_{k_0,n}$, so that $L^{\varrho_{n+1}}_t = L^{\varrho_n}_t$ on $[0,\zeta_n]$. Hence

$$\mathbb{E}^X_x[L^{\varrho_{n+1}}_{\zeta_{n+1}\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}|\mathcal{M}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}] = L^{\varrho_{n+1}}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}} = L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}.$$

Consequently, $(L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{n\geq n_x}$ is a $(\mathcal{M}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{k\geq n_x}$-martingale under $\mathbb{P}^X_x$. Moreover, recalling the expression of L^{ϱ_n}, that $\varrho_n=\varrho$ and $\mathcal{A}^n\varrho_n = \mathcal{L}^X\varrho$ on $G^{\mathbf{K}}_{k_0,n}$, that ϱ is bounded from above on from below on $G_{\mathbf{K},\varepsilon}$ and that $\mathcal{L}^X\varrho$ is bounded on $G_{\mathbf{K},\varepsilon}$ by Step 1, we conclude that $(L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{n\geq n_x}$ is bounded from above and from below, uniformly in $x\in G_{\mathbf{K},\varepsilon}$ and $n\geq n_x$. Hence $(L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{n\geq n_x}$ is a uniformly integrable martingale and has a $\mathcal{M}_{T\wedge\tau_{\mathbf{K},\varepsilon}}$-measurable limit $J_{T,\varepsilon}$ (a.s. and in L^1), which is also bounded by above and by below, uniformly in $x\in G_{\mathbf{K},\varepsilon}$. And it holds that $\mathbb{E}^X_x[J_{T,\varepsilon}] = 1$ and $\mathbb{E}^X_x[J_{T,\varepsilon}|\mathcal{M}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}] = L^{\varrho_n}_{\zeta_n\wedge T\wedge\tau_{\mathbf{K},\varepsilon}}$ for all $n\geq n_x$ as desired.

Step 8. We introduce, for each $x\in G_{\mathbf{K},\varepsilon}$, the probability measure $\mathbb{Q}^{T,\varepsilon}_x = J_{T,\varepsilon}\cdot\mathbb{P}^X_x$. It obviously satisfies the last assertion of the statement since $C^{-1}_{T,\varepsilon,\mathbf{K}} \leq J_{T,\varepsilon,\mathbf{K}} \leq C_{T,\varepsilon,\mathbf{K}}$ by Step 7. Recalling the notation of the statement, and since the laws of two continuous Hunt processes coincide as soon as their semi-groups coincide, we have to verify that for any bounded continuous $\varphi : G_{\mathbf{K},\varepsilon} \to \mathbb{R}$, for quasi all $x \in G_{\mathbf{K},\varepsilon}$,

$$\mathbb{E}^{\mathbb{Q}_x^{T,\varepsilon}}[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\}}] = \mathbb{E}_x^Y[\varphi(Y_t)\mathbb{1}_{\{t<\tilde\tau_{\mathbf{K},\varepsilon}\wedge T\}}].$$

Fix $n \ge n_x$, recall that $\zeta_n = \inf\{t > 0 : X_t \notin G^{\mathbf{K}}_{k_0,n}\}$ and set $\tilde\zeta_n = \inf\{t > 0 : Y_t \notin G^{\mathbf{K}}_{k_0,n}\}$. By Step 7 and since $(X_{t\wedge\zeta_n\wedge\tau_{\mathbf{K},\varepsilon}\wedge T})_{t\ge0}$ is $\mathcal{M}_{T\wedge\zeta_n\wedge\tau_{\mathbf{K},\varepsilon}}$-measurable, it holds that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_x^{T,\varepsilon}}[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}] =&\mathbb{E}_x^X[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}J_{T,\varepsilon}]\\ =&\mathbb{E}_x^X[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}L^{\varrho_n}_{T\wedge\zeta_n\wedge\tau_{\mathbf{K},\varepsilon}}].\end{aligned}$$

Since now $(L^{\varrho_n}_t)_{t\ge0}$ is a $(\mathcal{M}_t)_{t\ge0}$-martingale under $\mathbb{P}_x$,

$$\mathbb{E}^{\mathbb{Q}_x^{T,\varepsilon}}[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}] = \mathbb{E}_x^X[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}L^{\varrho_n}_T] = \mathbb{E}^{\mathbb{Q}^n_x}[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}],$$

recall that $\mathbb{Q}^n = L^{\varrho_n}_T \cdot \mathbb{P}^X_x$ was defined in Step 3. By the identity between the laws of the killed processes established at Step 6,

$$\mathbb{E}^{\mathbb{Q}_x^{T,\varepsilon}}[\varphi(X_t)\mathbb{1}_{\{t<T\wedge\tau_{\mathbf{K},\varepsilon}\wedge\zeta_n\}}] = \mathbb{E}_x^Y[\varphi(Y_t)\mathbb{1}_{\{t<T\wedge\tilde\tau_{\mathbf{K},\varepsilon}\wedge\tilde\zeta_n\}}].$$

The conclusion follows, letting $n \to \infty$ by dominated convergence: $\zeta_n\wedge\tau_{\mathbf{K},\varepsilon} \to \tau_{\mathbf{K},\varepsilon}$ and $\tilde\zeta_n\wedge\tilde\tau_{\mathbf{K},\varepsilon} \to \tilde\tau_{\mathbf{K},\varepsilon}$ as $n \to \infty$ because $\cup_{n\ge1}G^{\mathbf{K}}_{k_0,n} = G^{\mathbf{K}}_{k_0}$, see Lemma 12, and because $G^{\mathbf{K}}_{k_0}$ contains $G_{\mathbf{K},\varepsilon}$. ☐

8. Explosion and continuity at explosion

In this section we consider a $QKS(\theta, N)$-process $\mathbb{X}$ with life-time ζ. We show that $\zeta = \infty$ when $\theta \in (0, 2)$ and that $\zeta < \infty$ when $\theta \ge 2$. In the latter case, we also prove that $\lim_{t\to\zeta-} X_t$ a.s. exists, *for the usual topology of* $(\mathbb{R}^2)^N$: the Keller-Segel process is continuous at explosion. This is not clear at all at first sight: we know that $\lim_{t\to\zeta-} X_t = \triangle$ a.s. for the one-point compactification topology, which means that the process escapes from every compact of $\mathcal{X}$, but it could either go to infinity, which is not difficult to exclude, or it could tend to the boundary of $\mathcal{X}$ without converging, e.g. because it could alternate very fast between having its particles labeled in $[\![1, k_0]\!]$ very close and having its particles labeled in $[\![2, k_0 + 1]\!]$ very close. The goal of the section is to prove the following result.

Proposition 15. *Fix $\theta > 0$ and $N \ge 2$ such that $N > \theta$, set $k_0 = \lceil 2N/\theta\rceil$ and $\mathcal{X} = E_{k_0}$ and consider a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\ge0}, (\mathbb{P}^X_x)_{x\in\mathcal{X}\cup\{\triangle\}})$ with life-time ζ.*

(i) If $\theta < 2$, then for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x(\zeta = \infty) = 1$.

(ii) If $\theta \ge 2$, then for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x$-a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta} X_t$ exists for the usual topology of $(\mathbb{R}^2)^N$ and does not belong to E_{k_0}.

We first show that the process does not explode in the subcritical case and cannot go to infinity at explosion in the supercritical case.

Lemma 16. *(i) If $\theta < 2$ and $N \ge 2$, then for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x(\zeta = \infty) = 1$.*

(ii) If $\theta \ge 2$ and $N > \theta$, then for quasi all $x \in \mathcal{X}$,

$$\mathbb{P}^X_x\Big(\zeta < \infty \text{ and } \sup_{[0,\zeta)} \|X_t\| < \infty\Big) = 1.$$

Proof. The arguments below only apply for quasi all $x \in \mathcal{X}$, since we use Proposition 10. In both cases, we have for all $i \in [\![1,N]\!]$ and all $t \in [0,\zeta)$,

$$||X_t||^2 \leq 2\sum_{i=1}^{N}(\|X_t^i - S_{[\![1,N]\!]}(X_t)\|^2 + \|S_{[\![1,N]\!]}(X_t)\|^2) = 2R_{[\![1,N]\!]}(X_t) + 2N\|S_{[\![1,N]\!]}(X_t)\|^2.$$

By Lemma 11, there are a Brownian motion $(M_t)_{t\geq 0}$ and a squared Bessel process $(D_t)_{t\geq 0}$ with dimension $d_{\theta,N}(N)$ (killed when it gets out of $(0,\infty)$ if $d_{\theta,N}(N) \leq 0$), such that $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0,\zeta)$. These processes being locally bounded, we conclude that

$$\text{a.s., for all } T > 0, \quad \sup_{t\in[0,\zeta\wedge T)} \|X_t\| < \infty. \tag{48}$$

(i) When $\theta < 2$ and $N \geq 2$, we have $k_0 = \lceil 2N/\theta \rceil > N$, so that $\mathcal{X} = (\mathbb{R}^2)^N$. Hence on the event $\{\zeta < \infty\}$, we necessarily have $\limsup_{t\to\zeta-} ||X_t|| = \infty$, and this is incompatible with (48) with $T = \zeta$.

(ii) When $N > \theta \geq 2$, we have $d_{\theta,N}(N) \leq 0$, so that $(D_t)_{t\geq 0}$ is killed at some finite time τ. It holds that $\zeta \leq \tau$. Indeed, on the event where $\tau < \zeta$, we have $R_{[\![1,N]\!]}(X_\tau) = \lim_{t\to\tau-} R_{[\![1,N]\!]}(X_t) = \lim_{t\to\tau-} D_t = 0$, so that $X_\tau \notin E_{k_0}$ (since $k_0 \leq N$), which is not possible since $\tau < \zeta$. Hence ζ is also a.s. finite and it holds that $\sup_{[0,\zeta)} \|X_t\| < \infty$ a.s. by (48) with the choice $T = \zeta$. □

To show the continuity at explosion in the supercritical case, we need to prove the following tedious lemma.

Lemma 17. *Assume that $N > \theta \geq 2$. For quasi all $x \in \mathcal{X}$, for all $K \subset [\![1,N]\!]$ with $|K| \geq 2$,*

$$\mathbb{P}_x^X\text{-a.s,} \qquad \lim_{t\to\zeta-} R_K(X_t) = 0 \quad \text{or} \quad \liminf_{t\to\zeta-} R_K(X_t) > 0.$$

Proof. We proceed by reverse induction on the cardinal of K. If first $K = [\![1,N]\!]$, the result is clear because $(R_{[\![1,N]\!]}(X_t))_{t\in[0,\zeta)}$ is a (killed) squared Bessel process on $[0,\zeta)$ by Lemma 11 (and since $\zeta \leq \tau$ exactly as in the proof of Lemma 16-(ii)), hence it has a limit in $\mathbb{R}_+$ as $t \to \zeta$. Then, we assume that the property is proved if $|K| \geq n$ where $n \in [\![3,N]\!]$, we take $K \subset [\![1,N]\!]$ such that $|K| = n-1$ and we show in several steps that a.s., either $\lim_{t\to\zeta-} R_K(X_t) = 0$ or $\liminf_{t\to\zeta-} R_K(X_t) > 0$.

Step 1. We fix $\varepsilon \in (0,1]$ and introduce $\tilde\sigma_0^\varepsilon = 0$ and, for $k \geq 1$,

$$\sigma_k^\varepsilon = \inf\{t \in (\tilde\sigma_{k-1}^\varepsilon,\zeta) : R_K(X_t) \leq \varepsilon\} \quad \text{and} \quad \tilde\sigma_k^\varepsilon = \inf\{t \in (\sigma_k^\varepsilon,\zeta) : R_K(X_t) \geq 2\varepsilon\},$$

with the convention that $\inf\emptyset = \zeta$. We show in this step that for all deterministic $A > 0$, setting $a_\varepsilon = c_{|K|+1}\varepsilon/c_{|K|}$ (recall Lemma 13), there exists a constant $p_{A,\varepsilon} > 0$ such that for all $k \geq 1$, for quasi-all $x \in \mathcal{X}$, on $\{\sigma_k^\varepsilon < \zeta\}$,

$$\mathbb{P}_x^X\Big(\tilde\sigma_k^\varepsilon \geq (\sigma_k^\varepsilon + A)\wedge\zeta \;\text{ or }\; \max_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} ||X_t|| \geq 1/\varepsilon \;\text{ or }\; \min_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \Big| \mathcal{M}^X_{\sigma_k^\varepsilon}\Big) \geq p_{A,\varepsilon},$$

where $\mathcal{M}_t^X = \sigma(X_s : s \in [0,t])$.

By the strong Markov property of $\mathbb{X}$, on $\{\sigma_k^\varepsilon < \zeta\}$,

$$\mathbb{P}_x^X\Big(\tilde\sigma_k^\varepsilon \geq (\sigma_k^\varepsilon + A)\wedge\zeta \;\text{ or }\; \max_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon]} ||X_t|| \geq 1/\varepsilon \;\text{ or }\; \min_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon]} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \Big| \mathcal{M}^X_{\sigma_k^\varepsilon}\Big) = g(X_{\sigma_k^\varepsilon}),$$

where

$$g(y) = \mathbb{P}^X_y\Big(\tilde\sigma_1^\varepsilon \geq A\wedge\zeta \quad \text{or} \quad \max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||\geq 1/\varepsilon \quad \text{or} \quad \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K} R_{K\cup\{i\}}(X_t)\leq a_\varepsilon\Big).$$

We used that $R_K(X_{\sigma_k^\varepsilon})\leq\varepsilon$ on $\{\sigma_k^\varepsilon<\zeta\}$ by definition of σ_k^ε, so that $\sigma_1^\varepsilon=0$ under $\mathbb{P}^X_{X_{\sigma_k^\varepsilon}}$. Using again that $R_K(X_{\sigma_k^\varepsilon})\leq\varepsilon$ on $\{\sigma_k^\varepsilon<\zeta\}$, it suffices to show that there is a constant $p_{A,\varepsilon}>0$ such that $g(y)\geq p_{A,\varepsilon}$ for quasi all $y\in\mathcal{X}$ such that $R_K(y)\leq\varepsilon$.

If first $||y||\geq 1/\varepsilon$ or $\min_{i\notin K}R_{K\cup\{i\}}(y)\leq a_\varepsilon$, then clearly, $g(y)=1$.

Otherwise, $y\in G_{\mathbf{K},\varepsilon}$, where

$$G_{\mathbf{K},\varepsilon}=\{x\in\mathcal{X}:\ \text{for all } i\in K,\ \text{all } j\notin K,\ \|x^i-x^j\|^2>\varepsilon\}\cap B(0,1/\varepsilon)$$

as in Proposition 14 with $\mathbf{K}=(K,K^c)$, because $||y||<1/\varepsilon$ and because $R_K(y)\leq\varepsilon<2\varepsilon$ and $\min_{i\notin K}R_{K\cup\{i\}}(y)>a_\varepsilon=c_{|K|+1}\varepsilon/c_{|K|}$ imply that $||x^i-x^k||^2>\varepsilon$ for all $i\in K$, $j\notin K$ by Lemma 13. Hence, we can apply Proposition 14 with $T=A$ (and ε) and we find that (for quasi all $y\in G_{\mathbf{K},\varepsilon}$)

$$g(y)\geq C^{-1}_{A,\varepsilon,\mathbf{K}}\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y\Big(\tilde\sigma_1^\varepsilon\geq A\wedge\zeta \text{ or } \max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||\geq 1/\varepsilon \text{ or } \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)\leq a_\varepsilon\Big).$$

Using that $\mathbb{P}(A_1\cup A_2\cup A_3)=\mathbb{P}(A_1\cap A_2^c\cap A_3^c)+\mathbb{P}(A_2\cup A_3)$ for three events A_1,A_2,A_3, we deduce that $g(y)\geq g_1(y)+g_2(y)$, where

$$\begin{aligned}
g_1(y)=&C^{-1}_{A,\varepsilon,\mathbf{K}}\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y\Big(\tilde\sigma_1^\varepsilon\geq A\wedge\zeta \text{ and } \max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||<1/\varepsilon \text{ and } \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\Big),\\
g_2(y)=&C^{-1}_{A,\varepsilon,\mathbf{K}}\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y\Big(\max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||\geq 1/\varepsilon \text{ or } \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)\leq a_\varepsilon\Big).
\end{aligned}$$

But we know from Proposition 14 and Lemma 11 that under $\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y$, $(R_K(X_t))_{t\in[0,\tau_{K,\varepsilon}\wedge A]}$ is a squared Bessel process with dimension $d_{\theta|K|/N,|K|}(|K|)=d_{\theta,N}(|K|)$, issued from $R_K(y)\leq\varepsilon$, stopped at time $\tau_{\mathbf{K},\varepsilon}\wedge A$, where $\tau_{\mathbf{K},\varepsilon}=\inf\{t>0: X_t\notin G_{\mathbf{K},\varepsilon}\}$. Hence there exists, under $\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y$, a squared Bessel process $(S_t)_{t\geq0}$ with dimension $d_{\theta,N}(|K|)$ such that $S_t=R_K(X_t)$ for all $t\in[0,\tau_{\mathbf{K},\varepsilon}\wedge A]$. We introduce $\kappa_\varepsilon=\inf\{t>0:S_t\geq 2\varepsilon\}$ and we observe that

$$\begin{aligned}
&\Big\{\kappa_\varepsilon\geq A\wedge\zeta \text{ and } \max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||<1/\varepsilon \text{ and } \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\Big\}\\
=&\Big\{\tilde\sigma_1^\varepsilon\geq A\wedge\zeta \text{ and } \max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||<1/\varepsilon \text{ and } \min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\Big\}.
\end{aligned}$$

Indeed, on the event $\{\max_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||<1/\varepsilon$ and $\min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\}$, it holds that $\tau_{\mathbf{K},\varepsilon}\geq\tilde\sigma_1^\varepsilon$, because during $[0,\tilde\sigma_1^\varepsilon)$, we have $X_t\in G_{\mathbf{K},\varepsilon}$, because $||X_t||<1/\varepsilon$ and because $R_K(X_t)<2\varepsilon$, which, together with $\min_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon=c_{|K|+1}\varepsilon/c_{|K|}$, implies that $||X_t^i-X_t^j||^2\geq\varepsilon$ for all $i\in K$ and $j\notin K$ by Lemma 13. Hence, still on this event, $R_K(X_t)=S_t$ for all $t\in[0,\tilde\sigma_1^\varepsilon\wedge A)$, from which we conclude that $\kappa_\varepsilon\geq A\wedge\zeta$ if and only $\tilde\sigma_1^\varepsilon\geq A\wedge\zeta$.

Hence

$$\begin{aligned}
g_1(y)=&C^{-1}_{A,\varepsilon,\mathbf{K}}\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y\Big(\kappa_\varepsilon\geq A\wedge\zeta \text{ and } \max_{t\in[0,\tilde\sigma_1^\varepsilon]}||X_t||<1/\varepsilon \text{ and } \min_{t\in[0,\tilde\sigma_1^\varepsilon]}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\Big)\\
\geq&C^{-1}_{A,\varepsilon,\mathbf{K}}\mathbb{Q}^{A,\varepsilon,\mathbf{K}}_y\Big(\kappa_\varepsilon\geq A \text{ and } \max_{t\in[0,\tilde\sigma_1^\varepsilon]}||X_t||<1/\varepsilon \text{ and } \min_{t\in[0,\tilde\sigma_1^\varepsilon]}\min_{i\notin K}R_{K\cup\{i\}}(X_t)>a_\varepsilon\Big).
\end{aligned}$$

Using again that $\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1 \cap A_2^c \cap A_3^c) + \mathbb{P}(A_2 \cup A_3)$, we conclude that

$$g(y) = g_1(y) + g_2(y) \geq C_{A,\varepsilon,\mathbf{K}}^{-1} \mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\kappa_\varepsilon \geq A).$$

The step is complete, since $q_{A,\varepsilon}(y) = \mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\kappa_\varepsilon \geq A)$ is the probability that a squared Bessel process with dimension $d_{\theta,N}(|K|)$ issued from $R_K(y) \leq \varepsilon$ remains below 2ε during $[0, A]$ and is thus strictly positive, uniformly in y (such that $y \in G_{\mathbf{K},\varepsilon}$ and $R_K(y) = \varepsilon$).

Step 2. We prove here that for all $\varepsilon \in (0, 1]$, all $A > 0$, for quasi all $x \in \mathcal{X}$,

$$\mathbb{P}_x^X\Big(\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon \quad \text{or} \quad \liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \exists\, k \geq 1,\ \sigma_k^\varepsilon \geq \zeta \wedge A\Big) = 1.$$

All the arguments below only hold for quasi all $x \in \mathcal{X}$, even if we do not mention it explicitly during this step. For $k \geq 1$, we introduce

$$\Omega_{k+1} = \Big\{\sigma_{k+1}^\varepsilon < \zeta \wedge A \quad \text{and} \quad \min_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) > a_\varepsilon \quad \text{and} \quad \sup_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} ||X_t|| < 1/\varepsilon\Big\}$$

and we first show that $\mathbb{P}_x^X(\liminf_k \Omega_k) = 0$. To this end, it suffices to check that for all $\ell \geq 1$, $\mathbb{P}_x^X(\cap_{k=\ell}^\infty \Omega_k) = 0$. Since Ω_k is $\mathcal{M}_{\sigma_k^\varepsilon}$-measurable, for all $m \geq \ell \geq 1$,

$$\mathbb{P}_x^X\Big(\cap_{k=\ell}^{m+1} \Omega_k\Big) = \mathbb{E}_x^X\Big[\mathbb{1}_{\cap_{k=\ell}^m \Omega_k} \mathbb{P}_x^X\Big(\Omega_{m+1}\Big|\mathcal{M}_{\sigma_m^\varepsilon}\Big)\Big].$$

Since moreover $\cap_{k=\ell}^m \Omega_k \subset \{\sigma_m^\varepsilon < \zeta\}$ and since $\sigma_{m+1}^\varepsilon \geq \tilde\sigma_m^\varepsilon \geq \tilde\sigma_m^\varepsilon - \sigma_m^\varepsilon$, we deduce from Step 1 that on $\cap_{k=\ell}^m \Omega_k$,

$$\begin{aligned}
&\mathbb{P}_x^X\Big(\Omega_{m+1}\Big|\mathcal{M}_{\sigma_m^\varepsilon}\Big)\\
=&1 - \mathbb{P}_x^X\Big(\sigma_{m+1}^\varepsilon \geq \zeta \wedge A \quad \text{or} \quad \min_{t\in[\sigma_m^\varepsilon,\tilde\sigma_m^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \sup_{t\in[\sigma_m^\varepsilon,\tilde\sigma_m^\varepsilon)} ||X_t|| \geq 1/\varepsilon\Big|\mathcal{M}_{\sigma_m^\varepsilon}\Big)\\
\leq&1 - \mathbb{P}_x^X\Big(\tilde\sigma_m^\varepsilon \geq (\sigma_m^\varepsilon + A) \wedge \zeta \quad \text{or} \quad \min_{t\in[\sigma_m^\varepsilon,\tilde\sigma_m^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \sup_{t\in[\sigma_m^\varepsilon,\tilde\sigma_m^\varepsilon)} ||X_t|| \geq 1/\varepsilon\Big|\mathcal{M}_{\sigma_m^\varepsilon}\Big)\\
\leq&1 - p_{A,\varepsilon}.
\end{aligned}$$

Hence we conclude that $\mathbb{P}_x^X(\cap_{k=\ell}^{m+1} \Omega_k) \leq (1 - p_{A,\varepsilon})\mathbb{P}_x^X(\cap_{k=\ell}^m \Omega_k)$ for all $m \geq \ell \geq 1$, so that $\mathbb{P}_x^X(\cap_{k=\ell}^\infty \Omega_k) = 0$ as desired.

Hence $\mathbb{P}_x^X(\liminf_k \Omega_k) = 0$, so that a.s., an infinite number of Ω_k^c are realized. Consequently,

- either there is $k \geq 1$ such that $\sigma_k^\varepsilon \geq \zeta \wedge A$;
- or for all $k \geq 1$, $\sigma_k^\varepsilon < \zeta$ and $\min_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon$ for infinitely many k's, which implies that $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon$ because necessarily, $\lim_\infty \sigma_k^\varepsilon = \zeta$ by definition of the sequence $(\sigma_k^\varepsilon)_{k\geq1}$ and by continuity of $t \to R_K(X_t)$ on $[0, \zeta)$;
- or for all $k \geq 1$, $\sigma_k^\varepsilon < \zeta$ and there are infinitely many k's for which $\sup_{t\in[\sigma_k^\varepsilon,\tilde\sigma_k^\varepsilon)} ||X_t|| \geq 1/\varepsilon$ and this implies that $\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon$, because $\lim_\infty \sigma_k^\varepsilon = \zeta$ as previously.

Step 3. We conclude. Applying Step 2 for each $A \in \mathbb{N}$ and each $\varepsilon \in \mathbb{Q} \cap (0, 1]$, we conclude that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., for all $A > 0$ and all $\varepsilon \in (0, 1]$,

$$\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon \quad \text{or} \quad \liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \exists\, k \geq 1,\ \sigma_k^\varepsilon \geq \zeta \wedge A.$$

By Lemma 16 -(ii), we know that $\zeta < \infty$, so that choosing $A = \lceil \zeta \rceil$, we conclude that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., for all $\varepsilon \in (0,1]$

$$\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon \quad \text{or} \quad \liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \exists\, k \geq 1,\ \sigma_k^\varepsilon = \zeta. \tag{49}$$

And by Lemma 16 -(ii) again, $\limsup_{t\to\zeta-} ||X_t|| \leq 1/\varepsilon_0$ for some (random) $\varepsilon_0 \in (0,1]$.

On the event where $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) = 0$, there exists some (random) $i_0 \notin K$ such that $\liminf_{t\to\zeta-} R_{K\cup\{i_0\}}(X_t) = 0$, whence $\lim_{t\to\zeta-} R_{K\cup\{i_0\}}(X_t) = 0$ by induction assumption, and this obviously implies that $\lim_{t\to\zeta-} R_K(X_t) = 0$.

On the complementary event, we fix $\varepsilon_1 \in (0, \varepsilon_0]$ such that $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) > a_{\varepsilon_1}$ and we conclude from (49) and the fact that $\limsup_{t\to\zeta-} ||X_t|| \leq 1/\varepsilon_0$ that for all $\varepsilon \in (0, \varepsilon_1]$, there exists $k_\varepsilon \geq 1$ such that $\sigma_{k_\varepsilon}^\varepsilon = \zeta$. Recalling the definition of $(\sigma_k^\varepsilon)_{k\geq 1}$, we deduce that for all $\varepsilon \in (0, \varepsilon_1]$, $R_K(X_t)$ upcrosses the segment $[\varepsilon, 2\varepsilon]$ a finite number of times during $[0, \zeta)$. Hence for all $\varepsilon \in (0, \varepsilon_1]$, there exists $t_\varepsilon \in [0, \zeta)$ such that either $R_K(X_t) > \varepsilon$ for all $t \in [t_\varepsilon, \zeta)$ or $R_K(X_t) < 2\varepsilon$ for all $t \in [t_\varepsilon, \zeta)$. If there is $\varepsilon \in (0, \varepsilon_1]$ such that $R_K(X_t) > \varepsilon$ for all $t \in [t_\varepsilon, \zeta)$, then $\liminf_{t\to\zeta-} R_K(X_t) \geq \varepsilon > 0$. If next for all $\varepsilon \in (0, \varepsilon_1]$, we have $R_K(X_t) < 2\varepsilon$ for all $t \in [t_\varepsilon, \zeta)$, then $\lim_{t\to\zeta-} R_K(X_t) = 0$.

Hence in any case, we have either $\lim_{t\to\zeta-} R_K(X_t) = 0$ or $\liminf_{t\to\zeta-} R_K(X_t) > 0$. □

We finally give the

Proof of Proposition 15. Point (i), which concerns the subcritical case, has already been checked in Lemma 16-(i). Concerning point (ii), which concerns the supercritical case $\theta \geq 2$, we already know that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(\zeta < \infty) = 1$ by Lemma 16-(ii), and it remains to prove that $\mathbb{P}_x^X$-a.s., $\lim_{t\to\zeta-} X_t$ exists and does not belong to E_{k_0}. We divide the proof in two steps.

Step 1. We show by induction that for all $n \in \mathbb{N}$,

$$\mathcal{P}(n)\ :\ \begin{cases} \text{for all } \theta \geq 2 \text{ and all } N > \theta \text{ such that } k_0(\theta, N) = N - n, \\ \text{if } \mathbb{X} \text{ is a } QKS(\theta, N)\text{, then for quasi all } x \in \mathcal{X},\ \mathbb{P}_x^X\text{-a.s.}, \lim_{t\to\zeta-} X_t \text{ exists.} \end{cases}$$

We indicate here the dependence of $k_0 = \lceil 2N/\theta \rceil$ on θ and N. For N and θ satisfying the conditions of $\mathcal{P}(n)$, we have $k_0(\theta, N) \geq 3$ (because $N > \theta$) and thus $N \geq n + 3$. All the arguments below only hold for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., even if we do not mention it explicitly.

If $n = 0$, we consider $N > \theta \geq 2$ with $k_0(\theta, N) = N$. We thus have $\mathcal{X} = E_N$. But Lemma 16-(ii) tells us that $\limsup_{t\to\zeta-} ||X_t|| < \infty$, so that on the event where $\liminf_{t\to\zeta-} R_{[\![1,N]\!]}(X_t) > 0$, we can find a compact of E_N of the form $\{x \in (\mathbb{R}^2)^N : ||x|| \leq 1/\varepsilon \text{ and } R_{[\![1,N]\!]}(x) \geq \varepsilon\}$ in which $(X_t)_{t\in[0,\zeta)}$ remains and this forbids explosion, which is absurd. Hence $\liminf_{t\to\zeta-} R_{[\![1,N]\!]}(X_t) = 0$ and we conclude from Lemma 17 that $\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t) = 0$. Moreover, according to Lemma 11, there exists $(M_t)_{t\geq 0}$, a 2-dimensional Brownian motion with diffusion constant $N^{-1/2}$, such that for all $t \in [0, \zeta)$, $M_t = S_{[\![1,N]\!]}(X_t)$. Hence $M = \lim_{t\to\zeta-} S_{[\![1,N]\!]}(X_t)$ exists. But for all $i \in [\![1, N]\!]$, all $t \in [0, \zeta)$,

$$\|X_t^i - M\|^2 \leq 2\|X_t^i - S_{[\![1,N]\!]}(X_t)\|^2 + 2\|S_{[\![1,N]\!]}(X_t) - M\|^2 \leq 2R_{[\![1,N]\!]}(X_t) + 2\|S_{[\![1,N]\!]}(X_t) - M\|^2,$$

which tends to 0 as $t \to \zeta-$. Hence $\lim_{t\to\zeta-} X_t^i = M$ for all $i \in [\![1, N]\!]$ and we have proved $\mathcal{P}(0)$.

We next fix $n \in \mathbb{N}$, we assume that $\mathcal{P}(k)$ holds true for all $k \in [\![0, n]\!]$, and we show that $\mathcal{P}(n+1)$ holds true. We consider $N > \theta \geq 2$ such that $k_0(\theta, N) = N - n - 1$, a $QKS(\theta, N)$-process $\mathbb{X}$, we fix $x \in \mathcal{X} = E_{N-n-1}$ and we show that $\mathbb{P}_x^X$-a.s., $\lim_{t\to\zeta-} X_t$ exists in $(\mathbb{R}^2)^N$.

(a) We first prove that for all non-empty $K \subsetneq [\![1,N]\!]$, for all $T>0$, for all $\varepsilon>0$,

$$(X_t^i)_{i\in K} \text{ and } (X_t^i)_{i\notin K} \text{ have a limit in } (\mathbb{R}^2)^{|K|} \text{ and } (\mathbb{R}^2)^{N-|K|} \text{ as } t \text{ goes to } (\tau_{\mathbf{K},\varepsilon}\wedge T)-, \tag{50}$$

where $\tau_{\mathbf{K},\varepsilon} = \inf\{t>0 : X_t \notin G_{\mathbf{K},\varepsilon}\}$ as in Proposition 14 with the partition $\mathbf{K}=(K,K^c)$. Let us recall that since $\triangle \notin G_{\mathbf{K},\varepsilon}$, we have $\tau_{\mathbf{K},\varepsilon}\in[0,\zeta]$ when $\zeta<\infty$. However, when $\tau_{\mathbf{K},\varepsilon}=\zeta$, it is not true in general that X_t goes to the boundary of $G_{\mathbf{K},\varepsilon}$ as $t\to\zeta-$.

If first $x\notin G_{K,\varepsilon}$, (50) is obvious since then $\tau_{\mathbf{K},\varepsilon}=0$.

Else, we use Proposition 14: we know that under $\mathbb{Q}_x^{T,\varepsilon,\mathbf{K}}$, $(X_t^i)_{i\in K, t\in[0,\tau_{\mathbf{K},\varepsilon}]}$ and $(X_t^i)_{i\notin K, t\in[0,\tau_{\mathbf{K},\varepsilon}]}$ are two (stopped) independent $QKS(\theta|K|/N,|K|)$ and $QKS(\theta|K^c|/N,|K^c|)$ processes. Since $\mathbb{Q}_x^{T,\varepsilon,\mathbf{K}}$ is equivalent to $\mathbb{P}_x^X$, it is sufficient to prove that $\mathbb{Q}_x^{T,\varepsilon,\mathbf{K}}$-a.s., the limits $\lim_{t\to\tau_{\mathbf{K},\varepsilon}\wedge T}(X_t^i)_{i\in K}$ and $\lim_{t\to\tau_{\mathbf{K},\varepsilon}\wedge T}(X_t^i)_{i\in K^c}$ exist for the usual Euclidean topology. Let us e.g. consider $(X_t^i)_{i\in K}$.

• If $(X_t^i)_{i\in K}$ does not explode at time ζ, then it obviously has a limit at time $\tau_{\mathbf{K},\varepsilon}\wedge T\in[0,\zeta]$.

• If next the process $(X_t^i)_{i\in K}$ does explode at time ζ, then necessarily, we have $|K|\geq 2$ and $\theta|K|/N\geq 2$ by Lemma 16 -(i). Thus its parameters $\theta|K|/N$ and $|K|$ satisfy $\mathcal{P}(n+1+|K|-N)$. Indeed, $|K|>\theta|K|/N$ (since $N>\theta$) and

$$k_0(\theta|K|/N,|K|) = k_0(\theta,N) = N-n-1 = |K|-(n+1+|K|-N).$$

But $n+1+|K|-N\leq n$ (because $|K|\leq N-1$), so that we can use our induction assumption and conclude that $(X_t^i)_{i\in K}$ has a limit as $t\to\zeta-$. Since moreover $\tau_{\mathbf{K},\varepsilon}\wedge T\in[0,\zeta]$ and since $(X_t^i)_{i\in K}$ is continuous on $[0,\zeta)$, this shows that it has a limit as $t\to(\tau_{\mathbf{K},\varepsilon}\wedge T)-$

(b) For all $\varepsilon\in(0,1]$ and all non empty $K\subsetneq[\![1,N]\!]$, we set $\tilde\eta_0^{K,\varepsilon}=0$ and, for all $k\geq 0$,

$$\eta_{k+1}^{K,\varepsilon} = \inf\{t\geq \tilde\eta_k^{K,\varepsilon} : X_t\in G_{\mathbf{K},2\varepsilon}\} \quad\text{and}\quad \tilde\eta_{k+1}^{K,\varepsilon} = \inf\{t\geq \eta_{k+1}^{K,\varepsilon} : X_t\notin G_{\mathbf{K},\varepsilon}\},$$

with the convention that $\inf\emptyset=\zeta$. Using (50) and the strong Markov property, we conclude that for all non empty $K\subsetneq[\![1,N]\!]$, all $\varepsilon\in(0,1]\cap\mathbb{Q}$, all $k\geq 1$, all $T\in\mathbb{Q}_+$, if $\eta_k^{K,\varepsilon}<\zeta$, then

$$(X_t^i)_{i\in K} \text{ and } (X_t^i)_{i\notin K} \text{ admit a limit in } (\mathbb{R}^2)^{|K|} \text{ and } (\mathbb{R}^2)^{N-|K|} \text{ as } t \text{ goes to } (\tilde\eta_k^{K,\varepsilon}\wedge T)-. \tag{51}$$

(c) On the event $\{\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t)=0\}$, one can check as when proving $\mathcal{P}(0)$ that $\lim_{t\to\zeta-}X_t$ exists: $M_t=S_{[\![1,N]\!]}(X_t)$ is a 2-dimensional Brownian motion during $[0,\zeta)$, it thus has a limit $M\in\mathbb{R}^2$ as $t\to\zeta-$ which, together with $\lim_{t\to\zeta-}R_{[\![1,N]\!]}(X_t)=0$, implies that $\lim_{t\to\zeta-}X_t^i=M$ for all $i\in[\![1,N]\!]$.

(d) On $\{\lim_{t\to\zeta-}R_{[\![1,N]\!]}(X_t)=0\}^c$, we can find some non empty $K_0\subsetneq[\![1,N]\!]$ such that

$$\lim_{t\to\zeta-} R_{K_0}(X_t)=0 \quad\text{and}\quad \min_{i\notin K_0}\liminf_{t\to\zeta-} R_{K_0\cup\{i\}}(X_t)>0.$$

Indeed, $\zeta<\infty$ and $\sup_{t\in[0,\zeta)}||X_t||<\infty$ by Lemma 16-(ii). This implies that the set

$$\mathcal{S}=\Big\{k\in[\![2,N]\!] \text{ such that } \min_{K\subset[\![1,N]\!]:|K|=k}\ \liminf_{t\to\zeta-} R_K(X_t)=0\Big\}$$

is non-empty, because else there would exist $t_0\in[0,\zeta)$ and a compact subset of E_2 of the form

$$\{x\in(\mathbb{R}^2)^N : \|x\|\leq 1/\varepsilon \text{ and for all } i,j\in[\![1,N]\!], \|x^i-x^j\|\geq\varepsilon\}$$

in which X_t would remain for all $t\in[t_0,\zeta)$ and this would contradict the fact that $\zeta<\infty$. On the event $\{\lim_{t\to\zeta-}R_{[\![1,N]\!]}(X_t)=0\}^c$, we know from Lemma 17 that $\liminf_{t\to\zeta-}R_{[\![1,N]\!]}(X_t)>0$, so that $n_0=\max\mathcal{S}<N$. Hence we can find $K_0\subset[\![1,N]\!]$, with cardinal $|K_0|=n_0<N$ such

that $\liminf_{t\to\zeta-} R_{K_0}(X_t) = 0$ and, of course, $\min_{i\notin K_0} \liminf_{t\to\zeta-} R_{K_0\cup\{i\}}(X_t) > 0$. Using again Lemma 17, we also have $\lim_{t\to\zeta-} R_{K_0}(X_t) = 0$

(e) Using moreover that $\lim_{t\to\zeta-} ||X_t|| < \infty$ by Lemma 16-(ii), we conclude that on the event $\{\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t) = 0\}^c$, there exists $\varepsilon > 0$ and $t_0 \in [0,\zeta)$ such that for all $t \in [t_0,\zeta)$,

$$||X_t|| < 1/\varepsilon, \quad R_{K_0}(X_t) < 2\varepsilon \quad \text{and} \quad \min_{i\notin K} R_{K_0\cup\{i\}}(X_t) > \frac{c_{|K_0|+1}}{c_{|K_0|}}\varepsilon,$$

whence $X_t \in G_{\mathbf{K}_0,\varepsilon}$ by Lemma 13, where $\mathbf{K}_0 = (K_0, K_0^c)$. Thus, recalling (b), there exists $k \geq 1$ such that $\eta_k^{K_0,\varepsilon} < \zeta$ and $\tilde\eta_k^{K_0,\varepsilon} = \zeta$. By (51) this implies that $\lim_{t\to(\zeta\wedge T)-}(X_t^i)_{i\in K_0}$ and $\lim_{t\to(\zeta\wedge T)-}(X_t^i)_{i\in K_0^c}$ exist for every $T > 0$, and thus that $\lim_{t\to\zeta-} X_t$ exists.

Step 2. We fix $N > \theta \geq 2$ as in the statement. Then $k_0 \in [\![2,N]\!]$, so that N and θ satisfy all the conditions of $\mathcal{P}(n)$ with $n = N - k_0$. Hence for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists in $(\mathbb{R}^2)^N$. Moreover, $X_{\zeta-}$ cannot belong to $\mathcal{X} = E_{k_0}$, because $\lim_{t\to\zeta-} X_t = \triangle$ when $E_{k_0}\cup\{\triangle\}$ is endowed with the one-point compactification topology, see Subsection B.1. □

9. Some special cases

During a K-collision, the particles labeled in K are isolated from the other ones. Thanks to Proposition 14, it will thus be possible to describe what happens in a neighborhood of the instant of this K-collision, by studying a $QKS(\theta|K|/N, |K|)$-process. In other words, we may assume that $|K| = N$, so that the following special cases, which are the purpose of this section, will be crucial.

Proposition 18. *Let $N \geq 4$ and $\theta > 0$ such that $N > \theta$. Consider a $QKS(\theta,N)$-process $\mathbb{X}$ as in Proposition 6. Recall that $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$ and set $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0,\infty)\}$ with the convention that $R_K(\triangle) = 0$, so that $\tau \in [0,\zeta]$.*

(i) If $d_{\theta,N}(N-1) \leq 0$ and $d_{\theta,N}(N) < 2$, then for quasi all $x \in \mathcal{X}$,

$$\mathbb{P}_x^X\Big(\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0\Big) = 1.$$

(ii) If $d_{\theta,N}(N-1) \in (0,2)$ and $d_{\theta,N}(N) < 2$, then for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s, for all $K \subset [\![1,N]\!]$ with cardinal $|K| = N-1$, there is $t \in [0,\tau)$ such that $R_K(X_t) = 0$.

(iii) If $0 < d_{\theta,N}(N) < 2 \leq d_{\theta,N}(N-1)$, then for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s, for all $K \subset [\![1,N]\!]$ with cardinal $|K| = 2$, there is $t \in [0,\tau)$ such that $R_K(X_t) = 0$.

The proof of this proposition is very long. First, we recall some notation about the decomposition of $\mathbb{X}$ obtained in Proposition 10 and we study the involved time-change. We then derive a formula describing $R_K(U_t)$, valid on certain time intervals, for any $K \subset [\![1,N]\!]$. This formula is of course not closed, but it allows us to compare $R_K(U_t)$, when it is close to 0, to some process resembling a squared Bessel process, of which one easily studies the behavior near 0. Finally, we prove Proposition 18, unifying a little points (i) and (ii) and treating separately point (iii).

9.1. **Notation and preliminaries.** We recall the decomposition of Proposition 10, which holds true for quasi all $x \in \mathcal{X}\cap E_N$. Consider a Brownian motion $(M_t)_{t\geq 0}$ with diffusion coefficient $N^{-1/2}$ starting from $S_{[\![1,N]\!]}(x)$, a squared Bessel process $(D_t)_{t\geq0}$ starting from $R_{[\![1,N]\!]}(x) > 0$ killed when leaving $(0,\infty)$ with life-time $\tau_D = \inf\{t \geq 0 : D_t = \triangle\}$ and a $QSKS(\theta,N)$ -process $(U_t)_{t\geq0}$ starting from $\Phi_{\mathbb{S}}(x)$ with life-time $\xi = \inf\{t \geq 0 : U_t = \triangle\}$, all these processes being independent. For $t \in [0,\tau_D)$, we put $A_t = \int_0^t \frac{\mathrm{d}s}{D_s}$. We also consider the inverse $\rho : [0, A_{\tau_D}) \to [0,\tau_D)$ of A.

Lemma 19. *If $d_{\theta,N}(N) < 2$, then $\tau_D < \infty$ and $A_{\tau_D} = \infty$ a.s.*

Proof. Since $(D_t)_{t\geq 0}$ is a (killed) squared Bessel process with dimension $d_{\theta,N}(N) < 2$, we have $\tau_D < \infty$ a.s according to Revuz-Yor [21, Chapter XI]. Moreover, there is a Brownian motion $(B_t)_{t\geq 0}$ such that $D_t = r + 2\int_0^t \sqrt{D_s}\mathrm{d}B_s + d_{\theta,N}(N)t$ for all $t \in [0, \tau_D)$, where $r = R_{[\![1,N]\!]}(x) > 0$. A simple computation shows the existence of a Brownian motion $(W_t)_{t\geq 0}$ such that for all $t \in [0, A_{\tau_D})$,

$$D_{\rho_t} = r + 2\int_0^t D_{\rho_s}\mathrm{d}W_s + d_{\theta,N}(N)\int_0^t D_{\rho_s}\mathrm{d}s.$$

Hence for all $t \in [0, A_{\tau_D})$, $D_{\rho_t} = r\exp(2W_t + (d_{\theta,N}(N) - 2)t)$. On the event where $A_{\tau_D} < \infty$, we have $0 = D_{\tau_D -} = \lim_{t\to A_{\tau_D}} D_{\rho_t} = \exp(2W_{A_{\tau_D}} + (d_{\theta,N}(N) - 2)A_{\tau_D}) > 0$. Hence $A_{\tau_D} = \infty$ a.s. $\square$

From now on, we assume that $d_{\theta,N}(N) < 2$. Hence $A : [0, \tau_D) \to [0, \infty)$ is an increasing bijection, as well as $\rho : [0, \infty) \to [0, \tau_D)$. By Proposition 10, for quasi all $x \in \mathcal{X} \cap E_N$, we can find a triple $(M_t, D_t, U_t)_{t\geq 0}$ as above such that for $\mathbb{X}$ our $QKS(\theta, N)$ process starting from x, for all $t \in [0, \tau_D \wedge \rho_\xi)$, and actually for all $t \in [0, \rho_\xi)$ because $\rho_\xi \leq \tau_D$ since ρ is $[0, \tau_D)$-valued,

$$X_t = \Psi(M_t, D_t, U_{A_t}), \quad \text{i.e.} \quad M_t = S_{[\![1,N]\!]}(X_t), \quad D_t = R_{[\![1,N]\!]}(X_t) \quad \text{and} \quad U_{A_t} = \Phi_{\mathbb{S}}(X_t).$$

We recall that $\Psi(m, r, u) = \gamma(m) + \sqrt{r}u$ if $(m, r, u) \in \mathbb{R}^2 \times (0, \infty) \times \mathcal{U}$ and $\Psi(m, r, u) = \triangle$ if $(m, r, u) = \triangle$. Observe that $\tau = \tau_D \wedge \rho_\xi = \rho_\xi$, where $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0, \infty)\} \in [0, \zeta]$.

We note that if $\xi < \infty$, then $\rho_\xi < \tau_D$, because ρ is an increasing bijection from $[0, \infty)$ into $[0, \tau_D)$. Hence, still if $\xi < \infty$, then X explodes at time ρ_ξ strictly before τ_D, whence

$$\{\xi < \infty\} \subset \Big\{\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0\Big\}. \tag{52}$$

Finally note that since U is $\mathbb{S}$-valued, it cannot have a $[\![1, N]\!]$-collision. But for any $K \subset [\![1, N]\!]$ with cardinal $|K| \leq N - 1$, it holds that

(53) U has a K-collision at $t \in [0, \xi)$ if and only if X has a K-collision at $\rho_t \in [0, \tau)$,

which follows from the facts that

- for all $(m, r, u) \in \mathbb{R}^2 \times (0, \infty) \times \mathcal{U}$, $R_K(\Psi(m, r, u)) = 0$ if and only if $R_K(u) = 0$;
- ρ is an increasing bijection from $[0, \xi)$ into $[0, \tau)$, because $\rho_\xi = \tau$.

We conclude this subsection with a remark about the quasi notions of $\mathbb{X}$ and $\mathbb{U}$, of course in the case where they are related as above. See Subsection B.1 for a short reminder on this notion.

Remark 20. *Fix $B \in \mathcal{M}^U$ such that $\mathbb{P}^U_u(B) = 1$ for quasi all $u \in \mathcal{U}$ (here quasi refers to the Hunt process $\mathbb{U}$). Then $\mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}(B) = 1$ for quasi all $x \in \mathcal{X}$ (here quasi refers to the Hunt process $\mathbb{X}^*$, which is $\mathbb{X}$ killed when it gets outside E_N).*

Proof. By definition, there exists $\mathcal{N}^U$ a properly exceptional set relative to $\mathbb{U}$ such that for all $u \in \mathcal{U} \setminus \mathcal{N}^U$, $\mathbb{P}^U_u(B) = 1$. Thus for all $x \in \Phi_{\mathbb{S}}^{-1}(\mathcal{U} \setminus \mathcal{N}^U)$, $\mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}(B) = 1$.

By Proposition 10, there exists $\mathcal{N}^X$ a properly exceptional set relative to $\mathbb{X}^*$, such that for all $x \in (\mathcal{X} \cap E_N) \setminus \mathcal{N}^X$, the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_x$ is equal to the the law of $(Y_t = \Psi(M_t, D_t, U_{A_t}))_{t\geq 0}$ under $\mathbb{Q}^Y_x = \mathbb{P}^M_{\pi_{H^\perp}(x)} \otimes \mathbb{P}^D_{\|\pi_H(x)\|^2} \otimes \mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}$, with some obvious notation.

Hence we only have to prove that $\mathcal{N} = \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X$ is properly exceptional for $\mathbb{X}^*$.

• First, we have $\mathbb{P}^X_x(X^*_t \notin \mathcal{N}$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus \mathcal{N}$. Indeed, since $x \in \mathcal{X} \setminus \mathcal{N}$, the law of $(X^*_t)_{t\geq 0}$ under $\mathbb{P}^X_x$ equals the law of $(Y_t)_{t\geq 0}$ under $\mathbb{Q}^Y_x$. Since $\mathbb{P}^U_u(U_t \notin \mathcal{N}^U$ for all $t \geq 0) = 1$ for all $u \in \mathcal{U} \setminus \mathcal{N}^U$ and since $\Phi_{\mathbb{S}}(Y_t) = U_{A_t}$, we have $\mathbb{Q}^Y_x(Y_t \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$ for all $t \geq 0) = 1$ for all

$x \in \mathcal{X} \setminus \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$. Hence $\mathbb{P}_x^X(X_t^* \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus (\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X)$. Finally, $\mathbb{P}_x^X(X_t^* \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus (\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X)$ because $\mathcal{N}^X$ is properly exceptional for $\mathbb{X}^*$.

- We have $\mu(\mathcal{N}) = 0$. Indeed, $\mu(\mathcal{N}^X) = 0$ by definition and, using Lemma A.2,

$$\mu(\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}} \mathbb{1}_{\{\Psi(z,r,u)\in\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)\}} r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^*} \beta(\mathcal{N}^U) r^\nu \mathrm{d}z\mathrm{d}r = 0,$$

because $\beta(\mathcal{N}^U) = 0$. We used that $\Psi(z,r,u) \in \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \Leftrightarrow u \in \mathcal{N}^U$, since $\Phi_{\mathbb{S}}(\Psi(z,r,u)) = u$. $\square$

9.2. **An expression of dispersion processes on the sphere.** We now study the dispersion process $(R_K(U_t))_{t\geq 0}$, for $K \subset [\![1,N]\!]$. The equation below can be informally established if assuming that (1) rigorously holds true, after a time-change and several tedious Itô computations.

Lemma 21. *Fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and recall that $k_0 = \lceil 2N/\theta \rceil$. Consider a $QSKS(\theta, N)$ -process $\mathbb{U}$ with life-time ξ, fix $K \subset [\![1,N]\!]$ such that $|K| \geq 2$, and set $\mathbf{K} = (K, K^c)$. Recall that $G_{k_0}^{\mathbf{K}}$ was introduced in Lemma 12 and observe that, since $\mathbb{S} \cap E_{k_0} = \mathcal{U}$,*

$$G_{k_0}^{\mathbf{K}} \cap \mathbb{S} = \Big\{ u \in \mathcal{U} : \min_{i\in K, j\notin K} ||u^i - u^j|| > 0 \Big\}.$$

For quasi all $u \in G_{k_0}^{\mathbf{K}} \cap \mathbb{S}$, enlarging the filtered probability space $(\Omega^U, \mathcal{M}^U, (\mathcal{M}_t^U)_{t\geq 0}, \mathbb{P}_u^U)$ if necessary, there exists a 1-dimensional $(\mathcal{M}_t^U)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}$ under $\mathbb{P}_u^U$ such that

$$\begin{aligned} R_K(U_t) =& R_K(u) + 2\int_0^t \sqrt{R_K(U_s)(1-R_K(U_s))}\mathrm{d}W_s + d_{\theta,N}(|K|)t \\ &- d_{\theta,N}(N)\int_0^t R_K(U_s)\mathrm{d}s - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\int_0^t \frac{U_s^i - U_s^j}{\|U_s^i - U_s^j\|^2}\cdot(U_s^i - S_K(U_s))\mathrm{d}s \end{aligned} \tag{54}$$

for all $t \in [0, \kappa_K)$, where $\kappa_K = \inf\{t \geq 0 : U_t \notin G_{k_0}^{\mathbf{K}} \cap \mathbb{S}\}$.

As usual, $\kappa_K \leq \xi$ because $\triangle \notin G_{k_0}^{\mathbf{K}} \cap \mathbb{S}$. Note also that if $K = [\![1,N]\!]$, then $R_K(U_t) = 1$ for all $t \in [0,\xi)$, and that the constant process 1 indeed solves (54).

Proof. We divide the proof in several steps. The main idea is to compute $\mathcal{L}^U R_K$ and $\mathcal{L}^U(R_K)^2$ and to use that $R_K(U_t) = R_K(u) + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s + M_t$, for some martingale $(M_t)_{t\geq 0}$ of which we can compute the bracket. However, we need to regularize R_K and to localize space in a zone where the last term of (54) is bounded.

Step 1. We fix $n \geq 1$ and recall $\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}} \in C^\infty(\mathbb{S})$, satisfying Supp $\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}} \subset G_{k_0,2n}^{\mathbf{K}} \subset \mathcal{U}$, was defined in Lemma 12. We want to apply Remark 8 to $R_K\Gamma_{N,n}^{\mathbf{K},\mathbb{S}}$ and $(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2$, for $n \geq 1$. We thus have to show that $R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}}$ and $(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2$ belong to $C_c^\infty(\mathcal{U})$ for all $n \geq 1$, which is clear, and that

$$\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}}\Big(|\mathcal{L}_\alpha^U[R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}}](u)| + |\mathcal{L}_\alpha^U[(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2](u)|\Big) < \infty$$

for all $n \geq 1$. Since

$$\mathcal{L}_\alpha^U(fg) = f\mathcal{L}_\alpha^U g + g\mathcal{L}_\alpha^U f + \nabla_{\mathbb{S}} f\cdot\nabla_{\mathbb{S}} g \tag{55}$$

for all $f, g \in C^\infty(\mathbb{S})$ and recalling that $\sup_{\alpha\in(0,1]} \sup_{u\in\mathbb{S}} |\mathcal{L}_\alpha^U \Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}}(u)| < \infty$ by Lemma 12 and that $\operatorname{Supp} \Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}} \subset G_{k_0,2n}^{\mathbf{K}}$, the only difficulty is to verify that

$$\sup_{\alpha\in(0,1]} \sup_{u\in G_{k_0,2n}^{\mathbf{K}}} |\mathcal{L}_\alpha^U R_K(u)| < \infty. \tag{56}$$

Step 2. Here we prove that

$$\begin{aligned}\mathcal{L}_\alpha^U R_K(u) =& 2(|K|-1) - 2(N-1)R_K(u) + \frac{\theta}{N} R_K(u) \sum_{1\le i,j\le N} \frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} \\ & - \frac{\theta}{N} \sum_{i\in K, j\in K} \frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} - \frac{2\theta}{N} \sum_{i\in K, j\notin K} \frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)),\end{aligned} \tag{57}$$

and this will imply (56): the first four terms are obviously uniformly bounded on $\mathbb{S}$, and the last one is uniformly bounded on $G_{k_0,2n}^{\mathbf{K}}$ by definition of $G_{k_0,2n}^{\mathbf{K}}$, see Lemma 12.

This will also imply, taking $\alpha = 0$ and observing that $2(|K|-1) - \frac{\theta}{N}|K|(|K|-1) = d_{\theta,N}(|K|)$ and $2(N-1) - \frac{\theta}{N}N(N-1) = d_{\theta,N}(N)$, that for all $u \in \mathbb{S}\cap E_2$,

$$\mathcal{L}^U R_K(u) = d_{\theta,N}(|K|) - d_{\theta,N}(N)R_K(u) - \frac{2\theta}{N}\sum_{i\in K, j\notin K} \frac{u^i-u^j}{\|u^i-u^j\|^2}\cdot(u^i - S_K(u)). \tag{58}$$

Step 2.1. We first verify that for all $u \in \mathbb{S}$,

$$(\nabla_{\mathbb{S}} R_K(u))^i = 2(u^i - S_K(u))\mathbb{1}_{\{i\in K\}} - 2R_K(u)u^i, \qquad i \in [\![1,N]\!], \tag{59}$$

$$\Delta_{\mathbb{S}} R_K(u) = 4(|K|-1) - 4(N-1)R_K(u). \tag{60}$$

First, a simple computation shows that for $x \in (\mathbb{R}^2)^N$, for $i \in [\![1,N]\!]$,

$$\nabla_{x^i} R_K(x) = 2(x^i - S_K(x))\mathbb{1}_{\{i\in K\}} \quad \text{and} \quad \Delta_{x^i} R_K(x) = \frac{4(|K|-1)}{|K|}\mathbb{1}_{\{i\in K\}}, \tag{61}$$

so that in particular $\nabla R_K(x) \in H$ and

$$\nabla R_K(x)\cdot x = 2\sum_{i\in K}(x^i - S_K(x))\cdot x^i = 2\sum_{i\in K}(x^i - S_K(x))\cdot(x^i - S_K(x)) = 2R_K(x). \tag{62}$$

Next, proceeding as in (14), we get $\nabla[R_K \circ \Phi_{\mathbb{S}}](x) = ||\pi_H(x)||^{-1}\pi_H(\pi_{(\pi_H(x))^\perp}(\nabla R_K(\Phi_{\mathbb{S}}(x))))$ for all $x \in E_N$, so that

$$\nabla[R_K\circ\Phi_{\mathbb{S}}](x) = \frac{\pi_H\Big(\nabla R_K(\Phi_{\mathbb{S}}(x)) - \frac{\pi_H(x)\cdot\nabla R_K(\Phi_{\mathbb{S}}(x))}{||\pi_H(x)||^2}\pi_H(x)\Big)}{||\pi_H(x)||} = \frac{\nabla R_K(x) - 2R_K(x)\frac{\pi_H(x)}{||\pi_H(x)||^2}}{||\pi_H(x)||^2}.$$

We used that $\nabla R_K(\Phi_{\mathbb{S}}(x)) = \nabla R_K(x)/\|\pi_H(x)\|$ thanks to (61), that $\nabla R_K(x) \in H$ by (61) and that $\pi_H(x)\cdot\nabla R_K(x) = x\cdot\nabla R_K(x) = 2R_K(x)$ by (62).

We first conclude that for $u \in \mathbb{S}$, since $\pi_H(u) = u$ and $||u|| = 1$,

$$\nabla_{\mathbb{S}} R_K(u) = \nabla[R_K\circ\Phi_{\mathbb{S}}](u) = \nabla R_K(u) - 2R_K(u)u, \tag{63}$$

which implies (59) by (61).

Second, we deduce that for $x \in E_N$,

$$\Delta[R_K \circ \Phi_{\mathbb{S}}](x) = \frac{1}{||\pi_H(x)||^2}\Big(\Delta R_K(x) - 2\nabla R_K(x) \cdot \frac{\pi_H(x)}{||\pi_H(x)||^2} - 2R_K(x)\frac{\mathrm{div}\pi_H(x)}{||\pi_H(x)||^2} + \frac{4R_K(x)}{||\pi_H(x)||^2}\Big) - \frac{2\pi_H(x)}{||\pi_H(x)||^4} \cdot \Big(\nabla R_K(x) - 2R_K(x)\frac{\pi_H(x)}{||\pi_H(x)||^2}\Big).$$

Using that $\mathrm{div}\,\pi_H(x) = 2(N-1)$, we conclude that for $u \in \mathbb{S}$, since $\pi_H(u) = u$, $||u|| = 1$ and $u \cdot \nabla R_K(u) = 2R_K(u)$ by (62),

$$\Delta_{\mathbb{S}} R_K(u) = \Delta[R_K \circ \Phi_{\mathbb{S}}](u) = \Delta R_K(u) - 4R_K(u) - 4(N-1)R_K(u) + 4R_K(u).$$

Since finally $\Delta R_K(u) = 4(|K|-1)$ by (61), this leads to (60).

Step 2.2. We fix $u \in \mathbb{S}$ and show that setting $I_\alpha(u) = -\frac{\theta}{N}\sum_{1\le i,j\le N} \frac{u^i - u^j}{\|u^i-u^j\|^2+\alpha} \cdot (\nabla_{\mathbb{S}} R_K(u))^i$, it holds that

$$I_\alpha(u) = -\frac{\theta}{N}\sum_{i\in K, j\in K} \frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} + \frac{\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)). \tag{64}$$

By (59), we may write $I_\alpha = I_{1,\alpha} + I_{2,\alpha}$, where

$$I_{1,\alpha}(u) = -\frac{2\theta}{N}\sum_{i\in K, j\in[\![1,N]\!]}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)),$$
$$I_{2,\alpha}(u) = \frac{2\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot u^i.$$

First, by symmetry,

$$\begin{aligned} I_{1,\alpha}(u) &= -\frac{2\theta}{N}\sum_{i\in K, j\in K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)) - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)) \\ &= -\frac{2\theta}{N}\sum_{i\in K, j\in K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot u^i - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)) \\ &= -\frac{\theta}{N}\sum_{i\in K, j\in K}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)). \end{aligned}$$

Second, it immediately follows from (47) that

$$I_{2,\alpha}(u) = \frac{\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha}.$$

Step 2.3. Since $\mathcal{L}^U_\alpha R_K(u) = \frac{1}{2}\Delta_{\mathbb{S}} R_K(u) + I_\alpha(u)$, (57) follows from (60) and (64).

Step 3. By Steps 1 and 2, we can apply Remark 8 and Lemma B.2: for quasi all $u \in \mathcal{U}$, all $n \geq 1$, there exist two $(\mathcal{M}_t^U)_{t\geq 0}$-martingales $(M_t^{1,n})_{t\geq 0}$ and $(M_t^{2,n})_{t\geq 0}$ under $\mathbb{P}_u^U$, such that

$$(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})(U_t) = (R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})(u) + M_t^{1,n} + \int_0^t \mathcal{L}^U(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})(U_s)\mathrm{d}s,$$
$$(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2(U_t) = (R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2(u) + M_t^{2,n} + \int_0^t \mathcal{L}^U(R_K\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}})^2(U_s)\mathrm{d}s$$

for all $t \geq 0$. We introduce

$$\kappa_{K,n} = \inf\{t \geq 0 : U_t \notin G_{k_0,n}^{\mathbf{K}}\}$$

and observe that, since $\cup_{n\geq 1} G_{k_0,n}^{\mathbf{K}} = G_{k_0}^{\mathbf{K}}$, see Lemma 12, $\kappa_{K,n}$ a.s. increases to κ_K, defined in the statement, as $n \to \infty$. Next, since $\Gamma_{k_0,n}^{\mathbf{K},\mathbb{S}} = 1$ on $G_{k_0,n}^{\mathbf{K}} \cap \mathbb{S}$, we have, for all $t \in [0, \kappa_{K,n}]$,

$$R_K(U_t) = R_K(u) + M_t^{1,n} + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s, \tag{65}$$

$$(R_K(U_t))^2 = (R_K(u))^2 + M_t^{2,n} + \int_0^t \mathcal{L}^U(R_K^2)(U_s)\mathrm{d}s. \tag{66}$$

Applying the Itô formula to compute $(R_K(U_t))^2$ from (65), recalling from (55) that $\mathcal{L}^U(R_K^2) = 2R_K\mathcal{L}^U R_K + ||\nabla_{\mathbb{S}} R_K||^2$ and comparing to (66), we obtain that for $t \in [0, \kappa_{K,n}]$,

$$\langle M^{1,n}\rangle_t = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|^2 \mathrm{d}s.$$

Hence, enlarging the probability space if necessary, we can find a Brownian motion $(W_t)_{t\geq 0}$, which is defined by $W_t = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|^{-1}\mathrm{d}M_s^{1,n}$ for $t \in [0, \kappa_{K,n}]$ and which is then extended to $\mathbb{R}_+$, such that $M_t^{1,n} = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|\mathrm{d}W_s$ during $[0, \kappa_{K,n}]$, whence, still for $t \in [0, \kappa_{K,n}]$,

$$R_K(U_t) = R_K(u) + \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|\mathrm{d}W_s + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s. \tag{67}$$

But $\nabla_{\mathbb{S}} R_K(u) = \nabla R_K(u) - 2R_K(u)u$ by (63), whence

$$\|\nabla_{\mathbb{S}} R_K(u)\|^2 = \|\nabla R_K(u)\|^2 - 4R_K(u)\nabla R_K(u)\cdot u + 4(R_K(u))^2.$$

Since $||\nabla R_K(u)||^2 = 4R_K(u)$ by (61) and $\nabla R_K(u)\cdot u = 2R_K(u)$ by (62),

$$\|\nabla_{\mathbb{S}} R_K(u)\|^2 = 4R_K(u) - 4(R_K(u))^2 = 4R_K(u)(1 - R_K(u)).$$

Inserting this, as well as the expression (58) of $\mathcal{L}^U R_K$, in (67), shows that $R_K(U_t)$ satisfies the desired equation during $[0, \kappa_{K,n}]$. Since finally $\lim_n \kappa_{K,n} = \kappa_K$ a.s., the proof is complete. ☐

9.3. **A squared Bessel-like process.** The equation obtained in the previous lemma will be studied by comparison with the process we now introduce. This process behaves, near 0, like a squared Bessel processes.

Lemma 22. *Fix $\delta \in \mathbb{R}$, $a > 0$ and $b > 0$ such that $\delta + a\sqrt{b} < 2$. For $(W_t)_{t\geq 0}$ a 1-dimensional Brownian motion and for $x \in [0, 1)$, consider the unique solution $(S_t)_{t\geq 0}$ of*

$$S_t = x + \int_0^t 2\sqrt{|S_s(1 - S_s)|}\mathrm{d}W_s + \delta t + a\int_0^t \sqrt{b + |S_s|}\mathrm{d}s. \tag{68}$$

For $z \in \mathbb{R}$, set $\tau_z = \inf\{t > 0 : S_t = z\}$. For all $y \in (x, 1)$, it holds that $\mathbb{P}(\tau_0 < \tau_y) > 0$.

Proof. This equation is classically well-posed, since the diffusion coefficient is 1/2-Hölder continuous and the drift coefficient is Lipschitz continuous, see Revuz-Yor [21, Theorem 3.5 page 390]. As in Karatzas-Shreve [15, (5.42) page 339], we introduce the scale function

$$f(z) = \int_{1/2}^{z} \exp\Big(- \int_{1/2}^{u} \frac{\delta + a\sqrt{b+|v|}}{2|v(1-v)|} \mathrm{d}v \Big) \mathrm{d}u.$$

This function is obviously continuous on $(0,1)$ and one gets convinced, for example approximating $(\delta + a\sqrt{b+|v|})/(2|v(1-v)|)$ by $(\delta + a\sqrt{b})/(2|v|)$, that it is also continuous at 0 because $\delta + a\sqrt{b} < 2$. By [15, (5.61) page 344], we have

$$\mathbb{P}(\tau_0 < \tau_y) = \frac{f(y) - f(x)}{f(y) - f(0)}. \tag{69}$$

for all $y \in (x,1)$. This last quantity is nonzero (which would not be the case if $\delta + a\sqrt{b} \geq 2$, since then $f(0) = -\infty$). ☐

9.4. **Collisions of large clusters.** We are now ready to give the

Proof of Proposition 18-(i)-(ii). We fix $N \geq 4$, $\theta > 0$ such that $N > \theta$. We always assume that $d_{\theta,N}(N) < 2$ and we use the notation of Subsection 9.1.

Step 1. We consider $\varepsilon > 0$ and $K \subset [\![1,N]\!]$ such that $|K| \in [\![2, N-1]\!]$ and $d_{\theta,N}(|K|) < 2$. We introduce the constant $a_K = c_{|K|+1}/(2c_{|K|})$ with $(c_\ell)_{\ell \in [\![1,N]\!]}$ defined in Lemma 13. We prove in this step that there are some constants $p_{K,\varepsilon} > 0$ and $T_{K,\varepsilon} > 0$ such that, setting

$$\tilde{\sigma}^{K,\varepsilon} = \inf\Big\{ t > 0 : R_K(U_t) \geq \varepsilon \;\text{ or }\; \min_{i \notin K} R_{K \cup \{i\}}(U_t) \leq a_K \varepsilon \Big\} \wedge T_{K,\varepsilon},$$

with the convention that $\inf \emptyset = \xi$, it holds that for quasi all $u \in \mathcal{U}$ satisfying $R_K(u) \leq \varepsilon/2$,

$$\mathbb{P}_u^U\Big(\tilde{\sigma}^{K,\varepsilon} = \xi \;\text{ or }\; \inf_{t \in [0, \tilde{\sigma}^{K,\varepsilon})} \min_{i \notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon \text{ or } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon}) \Big) \geq p_{K,\varepsilon}.$$

We note that for all $t \in [0, \tilde{\sigma}^{K,\varepsilon})$, $R_K(U_t) \leq \varepsilon$ and $\min_{t \in [0,\tilde{\sigma}^{K,\varepsilon})} \min_{i \notin K} R_{K\cup\{i\}}(U_t) \geq a_K \varepsilon$ so that $\min_{i \in K, j \notin K} \|U_t^i - U_t^j\| \geq \varepsilon/2$ thanks to the definition of a_K and to Lemma 13. This implies that $\tilde{\sigma}^{K,\varepsilon} \leq \kappa_K$, where we recall that $\kappa_K = \inf\{t \geq 0 : U_t \notin G_{k_0}^{\mathbf{K}} \cap \mathbb{S}\}$ was defined in Lemma 21, with $G_{k_0}^{\mathbf{K}} \cap \mathbb{S} = \{u \in \mathcal{U} : \min_{i \in K, j \notin K} ||u^i - u^j|| > 0\}$.

By the Cauchy-Schwarz inequality, and since R_K is bounded on $\mathcal{U}$, there is a deterministic constant $C_{K,\varepsilon} > 0$, allowed to change from line to line, such that for all $t \in [0, \tilde{\sigma}^{K,\varepsilon})$, we have

$$\begin{aligned}
& - d_{\theta,N}(N) R_K(U_t) - \frac{2\theta}{N} \sum_{i \in K, j \notin K} \frac{U_t^i - U_t^j}{\|U_t^i - U_t^j\|^2} \cdot (U_t^i - S_K(U_t)) \\
\leq & C_{K,\varepsilon} \sqrt{R_K(U_t)} + C_{K,\varepsilon} \Big(\sum_{i \in K} \|U_t^i - S_K(U_t)\|^2 \Big)^{1/2} \\
\leq & C_{K,\varepsilon} \sqrt{R_K(U_t)} \\
\leq & C_{K,\varepsilon} \sqrt{b + R_K(U_t)}
\end{aligned}$$

where $b > 0$ is chosen small enough so that $d_{\theta,N}(|K|) + C_{K,\varepsilon}\sqrt{b} < 2$. Actually, b is only introduced to make the drift coefficient of (68) Lipschitz continuous.

Recalling that $R_K(U_0) \le \varepsilon/2$, the formula describing $R_K(U_t) \in [0,1]$ for $t \in [0, \kappa_K) \supset [0, \tilde\sigma^{K,\varepsilon})$, see Lemma 21, considering the process $(S_t)_{t\ge0}$ solution to (68) with $x = \varepsilon/2$, $\delta = d_{\theta,N}(|K|)$, $a = C_{K,\varepsilon}$ and with b introduced a few lines above, driven by the same Brownian motion $(W_t)_{t\ge0}$, and using the comparison theorem, we conclude that $R_K(U_t) \le S_t$ for all $t \in [0, \tilde\sigma^{K,\varepsilon})$.

Setting $\tau_z = \inf\{t \ge 0 : S_t = z\}$ for $z \in \mathbb{R}$ and recalling the definition of $\tilde\sigma^{K,\varepsilon}$, we conclude that

$$\Big\{ \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon \Big\} \subset \{\tilde\sigma^{K,\varepsilon} \ge \tau_\varepsilon \wedge T_{K,\varepsilon}\}.$$

Indeed, on $\{\inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon\}$, either $\tilde\sigma^{K,\varepsilon} = T_{K,\varepsilon}$, or $(R_K(U_t))_{t\ge0}$ reaches ε at time $\tilde\sigma^{K,\varepsilon}$ and we then have $\tau_\varepsilon \le \tilde\sigma^{K,\varepsilon}$. In both cases, $\tilde\sigma^{K,\varepsilon} \ge \tau_\varepsilon \wedge T_{K,\varepsilon}$.

Hence, using again that $R_K(U_t) \le S_t$ for all $t \in [0, \tilde\sigma^{K,\varepsilon})$,

$$\begin{aligned}&\Big\{\tilde\sigma^{K,\varepsilon} < \xi \text{ and } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon \text{ and } S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon}]\Big\}\\ \subset&\Big\{\tilde\sigma^{K,\varepsilon} < \xi \text{ and } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon \text{ and } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde\sigma^{K,\varepsilon})\Big\}.\end{aligned}$$

Hence we may write

$$\begin{aligned}&\mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} = \xi \text{ or } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \le 2a_K\varepsilon \text{ or } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde\sigma^{K,\varepsilon})\Big)\\ =&\mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} = \xi \text{ or } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \le 2a_K\varepsilon\Big)\\ &+ \mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} < \xi \text{ and } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon \text{ and } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde\sigma^{K,\varepsilon})\Big)\\ \ge&\mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} = \xi \text{ or } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \le 2a_K\varepsilon\Big)\\ &+ \mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} < \xi \text{ and } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon \text{ and } S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon})\Big)\\ =&\mathbb{P}^U_u\Big(\tilde\sigma^{K,\varepsilon} = \xi \text{ or } \inf_{t\in[0,\tilde\sigma^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \le 2a_K\varepsilon \text{ or } S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon})\Big)\\ \ge&\mathbb{P}^U_u\Big(S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon})\Big).\end{aligned}$$

This last quantity equals $\mathbb{P}(\tau_0 < \tau_\varepsilon \wedge T_{K,\varepsilon})$ and does not depend on u such that $R_K(u) \le \varepsilon/2$. Since $\mathbb{P}(\tau_0 < \tau_\varepsilon) > 0$ by Lemma 22 and since $d_{\theta,N}(|K|) + C_{K,\varepsilon}\sqrt{b} < 2$, there exists $T_{K,\varepsilon} > 0$ so that $\mathbb{P}(\tau_0 < \tau_\varepsilon \wedge T_{K,\varepsilon}) > 0$ and this completes the step.

Step 2. We prove (ii), i.e. that when $d_{\theta,N}(N-1) \in (0,2)$, for any $K \subset [\![1,N]\!]$ with cardinal $|K| = N-1$, for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x$-a.s., $R_K(X_t)$ vanishes during $[0,\zeta)$. By (53) and Remark 20, and since $\mathbb{P}^U_u(\xi = \infty) = 1$ for quasi all $u \in \mathcal{U}$ by Lemma 9-(ii), it suffices to check that for quasi all $u \in \mathcal{U}$, $\mathbb{P}^U_u$-a.s., $(R_K(U_t))_{t\ge0}$ vanishes at least once during $[0,\infty)$.

We fix $K \subset [\![1,N]\!]$ with $|K| = N-1$, set $\varepsilon_0 = 1/(4a_K)$ and introduce $\tilde\tau^K_0 = 0$ and for all $k \ge 0$,

$$\begin{aligned}\tau^K_{k+1} &= \inf\{t \ge \tilde\tau^K_k : R_K(U_t) \le \varepsilon_0/2\},\\ \tilde\tau^K_{k+1} &= \inf\{t \ge \tau^K_{k+1} : R_K(U_t) \ge \varepsilon_0\} \wedge (\tau^K_{k+1} + T_{K,\varepsilon_0}).\end{aligned}$$

with T_{K,ε_0} defined in Step 1. All these stopping times are finite since $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent by Lemma 9-(ii). We also put, for $k \geq 1$,

$$\Omega_k^K = \{R_K(U_t) = 0 \text{ for some } t \in [\tau_k^K, \tilde{\tau}_k^K]\}.$$

We now prove that $\mathbb{P}_u^U(\cap_{k\geq 1}(\Omega_k^K)^c) = 0$ for quasi all $u \in \mathcal{U}$.

For $\ell \geq 1$, since $\cap_{k=1}^{\ell}(\Omega_k^K)^c$ is $\mathcal{M}^U_{\tau^K_{\ell+1}}$-measurable, the strong Markov property tells us that

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^K)^c\Big) = \mathbb{E}_u^U\Big[\Big(\prod_{k=1}^{\ell}\mathbb{I}_{(\Omega_k^K)^c}\Big)\mathbb{P}^U_{U_{\tau^K_{\ell+1}}}((\Omega_1^K)^c)\Big].$$

We now prove that $\mathbb{P}_u^U(\Omega_1^K) \geq p_{K,\varepsilon_0}$ for quasi all $u \in \mathcal{U}$ such that $R_K(u) \leq \varepsilon_0/2$. For such a u, we have $\tau_1^K = 0$. Moreover, for all $i \notin K$, we have $R_{K\cup\{i\}}(u) = R_{[\![1,N]\!]}(u) = 1 > 2a_K\varepsilon_0$ thanks to our choice of ε_0. Hence $\tilde{\tau}_1^K = \tilde{\sigma}^{K,\varepsilon_0}$, recall Step 1. Since finally $\tilde{\sigma}^{K,\varepsilon_0} < \infty = \xi$ and since $R_{K\cup\{i\}}(U_t) = R_{[\![1,N]\!]}(U_t) = 1 > 2a_K\varepsilon_0$ for all $t \geq 0$ and all $i \notin K$,

$$\begin{aligned}\Omega_1^K =& \Big\{R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon_0}]\Big\}\\ =& \Big\{\tilde{\sigma}^{K,\varepsilon_0} = \xi \;\text{ or }\; \inf_{t\in[0,\tilde{\sigma}^{K,\varepsilon_0})}\min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon_0 \;\text{ or }\; R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon_0}]\Big\}.\end{aligned}$$

Hence Step 1 tells us that $\mathbb{P}_u^U(\Omega_1^K) \geq p_{K,\varepsilon_0}$ for quasi all $u \in \mathcal{U}$ such that $R_K(u) \leq \varepsilon_0/2$.

Since $R_K(U_{\tau^K_{\ell+1}}) \leq \varepsilon_0/2$, we have proved that for all $\ell \geq 1$,

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^K)^c\Big) \leq (1 - p_{K,\varepsilon_0})\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell}(\Omega_k^K)^c\Big).$$

This allows us to conclude that indeed, $\mathbb{P}_u^U(\cap_{k=1}^{\infty}(\Omega_k^K)^c) = 0$.

Step 3. We prove (i), i.e. that if $d_{\theta,N}(N-1) \leq 0$, then $\mathbb{P}_x^X(\inf_{[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0) = 1$ for quasi all $x \in \mathcal{X}$. By Remark 20 and (52), it suffices to show that for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U(\xi < \infty) = 1$.

For all $K \subset [\![1,N]\!]$, all $\varepsilon > 0$, we introduce $\tilde{\sigma}_0^{K,\varepsilon} = 0$ and for all $k \geq 0$,

$$\begin{aligned}\sigma_{k+1}^{K,\varepsilon} &= \inf\Big\{t \geq \tilde{\sigma}_k^{K,\varepsilon} : R_K(U_t) \leq \varepsilon/2 \;\text{ and }\; \min_{i\notin K} R_{K\cup\{i\}}(U_t) \geq 2a_K\varepsilon\Big\},\\ \tilde{\sigma}_{k+1}^{K,\varepsilon} &= \inf\Big\{t \geq \sigma_{k+1}^{K,\varepsilon} : R_K(U_t) \geq \varepsilon \;\text{ or }\; \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq a_K\varepsilon\Big\} \wedge (\sigma_{k+1}^{K,\varepsilon} + T_{K,\varepsilon}),\end{aligned}$$

with $T_{K,\varepsilon}$ defined in Step 1 and with the convention that $\inf\emptyset = \xi$.

Step 3.1. We fix $\varepsilon > 0$ and assume that $|K| \geq k_0$, so that $d_{\theta,N}(|K|) \leq 0$ by Lemma 1. We prove here that for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U$-a.s., either there is $t \in [0,\xi)$ such that $R_K(U_t) = 0$ or there is $k \geq 1$ such that $\sigma_{k+1}^{K,\varepsilon} = \xi$ or there is $k \geq 1$ such that $\inf_{t\in[\sigma_k^{K,\varepsilon},\tilde{\sigma}_k^{K,\varepsilon})}\min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon$.

It suffices to prove that $\mathbb{P}_u^U(\cap_{k\geq 1}(\Omega_k^{K,\varepsilon})^c) = 0$, where

$$\begin{aligned}\Omega_k^{K,\varepsilon} = \Big\{\sigma_{k+1}^{K,\varepsilon} = \xi \;\text{ or }\; &\inf_{t\in[\sigma_k^{K,\varepsilon},\tilde{\sigma}_k^{K,\varepsilon})}\min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon\\ &\text{or }\; R_K(U_t) = 0 \text{ for some } t \in [\sigma_k^{K,\varepsilon}, \tilde{\sigma}_k^{K,\varepsilon})\Big\}.\end{aligned}$$

But for all $\ell \geq 1$, $\cap_{k=1}^{\ell}(\Omega_k^{K,\varepsilon})^c$ is $\mathcal{M}^U_{\sigma_{\ell+1}^{K,\varepsilon}}$-measurable, whence, by the strong Markov property,

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^{K,\varepsilon})^c\Big) = \mathbb{E}_u^U\Big[\Big(\prod_{k=1}^{\ell}\mathbb{1}_{(\Omega_k^{K,\varepsilon})^c}\Big)\mathbb{P}^U_{U_{\sigma_{\ell+1}^{K,\varepsilon}}}\Big((\Omega_1^{K,\varepsilon})^c\Big)\Big] \leq (1-p_{K,\varepsilon})\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell}(\Omega_k^{K,\varepsilon})^c\Big).$$

We used Step 1, that $R_K(U_{\sigma_{\ell+1}^{K,\varepsilon}}) \leq \varepsilon/2$ on the event $(\Omega_\ell^{K,\varepsilon})^c \subset \{\sigma_{\ell+1}^{K,\varepsilon} < \xi\}$, as well as the inclusion $\{\tilde\sigma_k^{K,\varepsilon} = \xi\} \subset \{\sigma_{k+1}^{K,\varepsilon} = \xi\}$. One easily concludes.

Step 3.2. For all $K \subset [\![1,N]\!]$ such that $|K| \geq k_0$, for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U$-a.s., there is no $t \in [0,\xi)$ such that $R_K(U_t) = 0$. Indeed, on the contrary event, there is $t \in [0,\xi)$ such that $U_t \notin E_{k_0}$, whence $U_t \notin \mathcal{U}$, which contradicts the fact that $t \in [0,\xi)$.

Step 3.3. We show by decreasing induction that

$\mathcal{P}(n)$: for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U$-a.s. on the event $\{\xi = \infty\}$, $b_n = \min_{\{|K|=n\}} \inf_{t\geq 0} R_K(U_t) > 0$

holds true for every $n \in [\![k_0, N]\!]$.

The result is clear when $n = N$, because for all $t \in [0,\xi)$, $R_{[\![1,N]\!]}(U_t) = 1$.

We next assume $\mathcal{P}(n)$ for some $n \in [\![k_0+1, N]\!]$ and we show that $\mathcal{P}(n-1)$ is true. We fix $K \subset [\![1,N]\!]$ with cardinal $|K| = n-1$ and we applying Step 3.1 with K and with some $\varepsilon \in (0, b_n/(4a_K))$ (b_n is random but we may apply Step 3.1 simultaneously for all $\varepsilon \in \mathbb{Q}_+^*$) and Step 3.2, we find that on the event $\{\xi = \infty\}$, there either exists $k \geq 1$ such that $\sigma_{k+1}^{K,\varepsilon} = \infty$ or $k \geq 1$ such that $\inf_{t\in[\sigma_k^{K,\varepsilon}, \tilde\sigma_k^{K,\varepsilon}]} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon$. This second choice is not possible, since by induction assumption, $R_{K\cup\{i\}}(U_t) \geq b_n$ for all $t > 0$ and all $i \notin K$. Hence there is $k \geq 1$ such that $\sigma_{k+1}^{K,\varepsilon} = \infty$.

By definition of $\sigma_{k+1}^{K,\varepsilon}$, this implies that, still on the event where $\xi = \infty$, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, either $R_K(U_t) \geq \varepsilon/2$ or $\min_{i\in K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon$. Using again the induction assumption, we get that the second choice is never possible, so that actually, $R_K(U_t) \geq \varepsilon/2$ for all $t \geq t_0$. Since $(R_K(U_t))_{t\geq 0}$ is continuous and positive on $[0,t_0]$ according to Step 3.2, this completes the step.

Step 3.4. We conclude from Step 3.3 that for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U$-a.s. on the event $\{\xi = \infty\}$, $U_t \in \mathcal{K}$ for all $t \geq 0$, where

$$\mathcal{K} = \{u \in \mathcal{U} : \text{for all } n \in [\![k_0,N]\!], \text{ all } K \subset [\![1,N]\!] \text{ with } |K| = n,\ R_K(u) \geq b_n\}.$$

This (random) set is compact in $\mathcal{U}$, so that Lemma 9-(i) tells us, both in the case where $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent and in the case where $(\mathcal{E}^U, \mathcal{F}^U)$ is transient, that this happens with probability 0. Hence for quasi all $u \in \mathcal{U}$, $\mathbb{P}_u^U(\xi = \infty) = 0$ as desired. □

9.5. **Binary collisions.** We finally give the

Proof of Proposition 18-(iii). We assume that $N \geq 4$, that $0 < d_{\theta,N}(N) < 2 \leq d_{\theta,N}(N-1)$ and observe that $\theta < 2$ and $k_0 > N$, so that $\mathcal{X} = (\mathbb{R}^2)^N$ and $\mathcal{U} = \mathbb{S}$. The $QKS(\theta,N)$-process $\mathbb{X}$ is non-exploding by Proposition 15-(i), and the $QSKS(\theta,N)$-process $\mathbb{U}$ is irreducible recurrent by Lemma 9-(ii). In particular, $\zeta = \xi = \infty$ a.s. We divide the proof in 4 steps. First, we prove that $\mathbb{X}$ may have some binary collisions with positive probability. Then we check that this implies that $\mathbb{U}$ also may have some binary collisions with positive probability. Since $\mathbb{U}$ is recurrent, it will then necessarily be a.s. subjected to (infinitely many) binary collisions. Finally, we conclude using (53).

Step 1. We set $\mathbf{K} = (\{1,2\},\{3\},\dots,\{N\})$ and, for $\varepsilon \in (0,1)$ to be chosen later,

$$\mathcal{K}_\varepsilon = \Big\{x \in B\Big(0, \frac{1}{9\varepsilon}\Big) : \|x^1 - x^2\|^2 < \frac{\varepsilon}{4} \text{ and } \min_{i\in[\![1,N]\!], j\in[\![3,N]\!], i\neq j} \|x^i - x^j\|^2 > 9\varepsilon\Big\}.$$

We show in this step that if $\varepsilon \in (0, 1/(2N))$, then $\mathbb{P}^X_x(A) > 0$ for quasi all $x \in \mathcal{K}_\varepsilon$, where

$$A = \Big\{X^1_t = X^2_t \text{ for some } t \in [0,1] \text{ and } \min_{t\in[0,1]} R_{[\![1,N]\!]}(X_t) > 0\Big\}.$$

To this end, we fix $x \in \mathcal{K}_\varepsilon$ and introduce the sets

$$O = \Big\{y \in (\mathbb{R}^2)^2 : R_{\{1,2\}}(y) < \varepsilon/2\Big\}, \qquad O' = \Big\{y \in (\mathbb{R}^2)^2 : \Big\|\frac{y^1+y^2}{2} - \frac{x^1+x^2}{2}\Big\|^2 < \frac{\varepsilon}{16}\Big\},$$

and $B_i = \{y \in \mathbb{R}^2 : ||y - x^i||^2 < \varepsilon\}$ for $i \in [\![3,N]\!]$. We first claim that

$$L = \Big\{y \in (\mathbb{R}^2)^N : (y^1,y^2) \in O \cap O' \text{ and } y^i \in B_i \text{ for all } i \in [\![3,N]\!]\Big\} \subset G_{\mathbf{K},\varepsilon},$$

where we recall that $G_{\mathbf{K},\varepsilon} = \{y \in B(0,1/\varepsilon) : \forall\, i \in [\![1,N]\!],\ \forall\, j \in [\![3,N]\!] \setminus \{i\},\ ||y^i - y^j||^2 > \varepsilon\}$. First, for $y \in L$, we have $(y^1,y^2) \in O \cap O'$, so that

$$\|y^1 - x^1\| \le \Big\|\frac{y^1+y^2}{2} - \frac{x^1+x^2}{2}\Big\| + \Big\|\frac{y^1-y^2}{2} - \frac{x^1-x^2}{2}\Big\| \le \frac{\sqrt{\varepsilon}}{4} + \frac{\|y^1-y^2\|}{2} + \frac{\|x^1-x^2\|}{2} < \sqrt{\varepsilon},$$

because $\|x^1 - x^2\| < \sqrt{\varepsilon}/2$ and $\|y^1 - y^2\| = \sqrt{2R_{\{1,2\}}(y^1,y^2)} < \sqrt{\varepsilon}$. The same bound applies to $\|x^2 - y^2\|$. Hence for all $i \in [\![1,N]\!]$, all $j \in [\![3,N]\!] \setminus \{i\}$,

$$\|y^i - y^j\| \ge \|x^i - x^j\| - \|y^i - x^i\| - \|y^j - x^j\| \ge \|x^i - x^j\| - 2\sqrt{\varepsilon} > \sqrt{\varepsilon}.$$

Finally, $\|y\|^2 \le 2\sum_{i=1}^N \|y^i - x^i\|^2 + 2\|x\|^2 < 2N\varepsilon + 2/(9\varepsilon) < 1/\varepsilon$ since $\varepsilon < 1/(2N)$.

Since $G_{\mathbf{K},\varepsilon}$ is obviously included in $\{y \in (\mathbb{R}^2)^N : R_{[\![1,N]\!]}(y) > 0\}$, we conclude that

$$\begin{aligned}\mathbb{P}^X_x(A) \ge& \mathbb{P}^X_x\Big(X^1_t = X^2_t \text{ for some } t \in [0,1] \text{ and } X_t \in L \text{ for all } t \in [0,1]\Big)\\ \ge& C^{-1}_{1,\varepsilon,\mathbf{K}} \mathbb{Q}^{1,\varepsilon,\mathbf{K}}_x\Big(X^1_t = X^2_t \text{ for some } t \in [0,1] \text{ and } X_t \in L \text{ for all } t \in [0,1]\Big)\end{aligned}$$

by Proposition 14 with $T = 1$. We now set $\tau_{\mathbf{K},\varepsilon} = \inf\{t > 0 : X_t \notin G_{\mathbf{K},\varepsilon}\}$. Proposition 14 tells us that (for quasi all $x \in \mathcal{K}_\varepsilon \subset G_{\mathbf{K},\varepsilon}$), the law of $(X_t)_{t\in[0,\tau_{\mathbf{K},\varepsilon}]}$ under $\mathbb{Q}^{1,\varepsilon,\mathbf{K}}_x$ equals the law of $Y_t = (Y^1_t,\dots,Y^N_t)_{t\in[0,\tilde\tau_{\mathbf{K},\varepsilon}]}$ where $(Y^1_t,Y^2_t)_{t\ge0}$ is a $QKS(2\theta/N,2)$-process issued from (x^1,x^2), where for all $i \in [\![3,N]\!]$, $(Y^i_t)_{t\ge0}$ is a $QKS(\theta/N,1)$-process, i.e. a 2-dimensional Brownian motion, issued from x^i, and where all these processes are independent. We have set $\tilde\tau_{\mathbf{K},\varepsilon} = \inf\{t > 0 : Y_t \notin G_{\mathbf{K},\varepsilon}\}$.

This, together with the fact that $\{X_t \in L \text{ for all } t \in [0,1]\} \subset \{\tau_{\mathbf{K},\varepsilon} > 1\}$, tells us that

$$\mathbb{P}^X_x(A) \ge C^{-1}_{1,\varepsilon,\mathbf{K}} q_{\varepsilon,1,2} \prod_{i=3}^N q_{\varepsilon,i}$$

for quasi all $x \in \mathcal{K}_\varepsilon$, where

$$q_{\varepsilon,1,2} = \mathbb{P}\Big(\min_{s\in[0,1]} R_{\{1,2\}}((Y^1_s,Y^2_s)) = 0 \text{ and } (Y^1_t,Y^2_t) \in O \cap O' \text{ for all } t \in [0,1]\Big)$$

and, for $i \in [\![3,N]\!]$, $q_{\varepsilon,i} = \mathbb{P}(Y^i_t \in B_i \text{ for all } t \in [0,1])$, which is obviously positive since $(Y^i_t)_{t\ge0}$ is nothing but a Brownian motion issued from x^i. Moreover, we know from Lemma 11 that $(M_t = (Y^1_t + Y^2_t)/2)_{t\ge0}$ is a 2-dimensional Brownian motion with diffusion coefficient $2^{-1/2}$ issued from $m = (x^1 + x^2)/2$, that $(R_t = R_{\{1,2\}}((Y^1_t,Y^2_t)))_{t\ge0}$ is a squared Bessel process of dimension

$d_{2\theta/N,2}(2) = d_{\theta,N}(2)$ issued from $r = ||x^1 - x^2||^2/2 \in (0, \varepsilon/8)$, and that these processes are independent. Hence, recalling the definitions of O and O',

$$q_{\varepsilon,1,2} = \mathbb{P}\Big(\min_{s\in[0,1]} R_s = 0 \text{ and } \sup_{s\in[0,1]} R_s < \frac{\varepsilon}{2}\Big)\mathbb{P}\Big(\sup_{s\in[0,1]} ||M_t - m|| < \frac{\varepsilon}{16}\Big).$$

This last quantity is clearly positive, because a squared Bessel process with dimension $d_{\theta,N}(2) \in (0,2)$, see Lemma 1, does hit zero, see Revuz-Yor [21, Chapter XI].

Step 2. We now deduce from Step 1 that the set $F = \{u \in \mathcal{U} : u^1 = u^2\}$ is not exceptional for $\mathbb{U}$. Indeed, if it was exceptional, we would have $\mathbb{P}^U_u(\exists\, t \geq 0 : U_t \in F) = 0$ for quasi all $u \in \mathcal{U}$. By (53) and Remark 20, this would imply that for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x(\exists\, t \in [0,\tau) : X_t \in G) = 0$, where $G = \{x \in \mathcal{X} : x^1 = x^2\}$ and $\tau = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) = 0\}$. But on the event A defined in Step 1, there is $t \in [0,1]$ such that $X_t \in G$ and it holds that $\tau > 1$. As a conclusion, $\mathbb{P}^X_x(\exists\, t \in [0,\tau) : X_t \in G) > 0$ for quasi all $x \in \mathcal{K}_\varepsilon$, whence a contradiction, since $\mu(\mathcal{K}_\varepsilon) > 0$.

Step 3. Since $(\mathcal{E}^U, \mathcal{F}^U)$ is irreducible-recurrent and since F is not exceptional, we know from Fukushima-Oshima-Takeda [11, Theorem 4.7.1-(iii) page 202] that for quasi all $u \in \mathcal{U}$,

$$\mathbb{P}^U_u(\forall\, r > 0, \exists\, t \geq r \ : \ U_t \in F) = 1.$$

Step 4. Using again (53) and Remark 20 and recalling that $\xi = \infty$ and that ρ is an increasing bijection from $[0,\infty)$ to $[0,\tau)$, we conclude that for quasi all $x \in \mathcal{X}$, $\mathbb{P}^X_x$-a.s., X_t visits F (an infinite number of times) during $[0,\tau)$. Of course, the same arguments apply when replacing $\{1,2\}$ by any subset of $[\![1,N]\!]$ with cardinal 2, and the proof is complete. □

10. Conclusion for quasi all initial conditions

Here we prove that the conclusions of Theorem 5 hold for quasi all $x \in \mathcal{X}$.

Partial proof of Theorem 5. We assume that $\theta \geq 2$ and $N > 3\theta$, so that $k_0 = \lceil 2N/\theta \rceil \in [\![7,N]\!]$, and consider a $\mathcal{X}_\triangle$-valued $QKS(\theta,N)$-process $\mathbb{X}$ with life-time ζ as in Proposition 6, where $\mathcal{X} = E_{k_0}$.

Preliminaries. For $K \subset [\![1,N]\!]$ and $\varepsilon > 0$, we write $\tau_{K,\varepsilon} = \inf\{t > 0 : X_t \notin G_{K,\varepsilon}\} \in [0,\zeta]$ and $G_{K,\varepsilon} = \{x \in \mathcal{X} : \min_{i\in K, j\notin K} ||x^i - x^j||^2 > \varepsilon\} \cap B(0,1/\varepsilon)$ instead of $\tau_{\mathbf{K},\varepsilon}$ and $G_{\mathbf{K},\varepsilon}$ with $\mathbf{K} = (K, K^c)$ as in Proposition 14. We also write $\mathbb{Q}^{T,\varepsilon,K}_x$ instead of $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}$ and recall that it is equivalent to $\mathbb{P}^X_x$ on $\mathcal{M}^X_T = \sigma(X_s : s \in [0,T])$.

Setting $X^K_t = (X^i_t)_{i\in K}$ and $X^{K^c}_t = (X^i_t)_{i\in K^c}$, we know that for quasi all $x \in G_{K,\varepsilon}$, the law of $(X^K_t, X^{K^c}_t)_{t\in[0,\tau_{K,\varepsilon}\wedge T]}$ under $\mathbb{Q}^{T,\varepsilon,K}_x$ is the same as the law of $(Y_t, Z_t)_{t\in[0,\tilde\tau_{K,\varepsilon}\wedge T]}$, where $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process issued from $x|_K$ and $(Z_t)_{t\geq 0}$ is a $QKS(|K^c|\theta/N, |K^c|)$-process issued from $x|_{K^c}$, these two processes being independent, and where $\tilde\tau_{K,\varepsilon} = \inf\{t > 0 : (Y_t, Z_t) \notin G_{K,\varepsilon}\}$. We denote by ζ^Y and ζ^Z the life-times of $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$. The life-time of $(Y_t, Z_t)_{t\geq 0}$ is given by $\zeta' = \zeta^Y \wedge \zeta^Z$ and it holds that $\tilde\tau_{K,\varepsilon} \in [0,\zeta']$.

No isolated points. Here we prove that for all $K \subset [\![1,N]\!]$ with $d_{\theta,N}(|K|) \in (0,2)$, for quasi all $x \in \mathcal{X}$, we have $\mathbb{P}^X_x(A_K) = 0$, where $A_K = \{\mathcal{Z}_K$ has an isolated point$\}$ and

$$\mathcal{Z}_K = \{t \in (0,\zeta) : \text{ there is a } K\text{-collision in the configuration } X_t\}.$$

On A_K, we can find $u, v \in \mathbb{Q}_+$ such that $u < v < \zeta$ and such that there is a unique $t \in (u,v)$ with $R_K(X_t) = 0$ and $\min_{i\notin K} R_{K\cup\{i\}}(X_t) > 0$. By continuity, we deduce that on A_K, there exist $r, s \in \mathbb{Q}_+$ and $\varepsilon \in \mathbb{Q}^*_+$ such that $r < s < \zeta$, $X_t \in G_{K,\varepsilon}$ for all $t \in [r,s]$ and such that

$\{t \in (r,s) : R_K(X_t) = 0\}$ has an isolated point. It thus suffices that for all $r < s$ and all $\varepsilon > 0$, that we all fix from now on, for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(A_{K,r,s,\varepsilon}) = 0$, where

$$A_{K,r,s,\varepsilon} = \Big\{X_t \in G_{K,\varepsilon} \text{ for all } t \in (r,s) \text{ and } \{t \in (r,s) : R_K(X_t) = 0\} \text{ has an isolated point}\Big\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(A_{K,0,s,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{s,\varepsilon,K}(A_{K,0,s,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$. We write, recalling the preliminaries,

$$\begin{aligned}\mathbb{Q}_x^{s,\varepsilon,K}(A_{K,0,s,\varepsilon}) =&\mathbb{Q}_x^{s,\varepsilon,K}\Big(\tau_{K,\varepsilon} \geq s \text{ and } \{t \in (0,s) : R_K(X_t) = 0\} \text{ has an isolated point}\Big)\\ =&\mathbb{P}\Big(\tilde{\tau}_{K,\varepsilon} \geq s \text{ and } \{t \in (0,s) : R_K(Y_t) = 0\} \text{ has an isolated point}\Big)\\ \leq&\mathbb{P}\Big(\ \{t \in (0,s) : R_K(Y_t) = 0\} \text{ has an isolated point}\Big).\end{aligned}$$

But $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process, so that we know from Lemma 11 that $(R_K(Y_t))_{t\geq 0}$ is a squared Bessel process with dimension $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \in (0,2)$. Such a process has no isolated zero, see Revuz-Yor [21, Chapter XI].

Point (i). We have already seen in Proposition 15-(ii) that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists in $(\mathbb{R}^2)^N$ and does not belong to E_{k_0}.

Point (ii). We want to show that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., there is $K_0 \subset [\![1,N]\!]$ with $|K_0| = k_0$ such that there is a K_0-collision and no K-collision with $|K| > k_0$ in the configuration $X_{\zeta-}$. We already know that $X_{\zeta-} \notin E_{k_0}$, so that there is $K \subset [\![1,N]\!]$ with $|K| \geq k_0$ such that there is a K-collision in the configuration $X_{\zeta-}$. Hence the goal is to verify that for quasi all $x \in \mathcal{X}$, for all $K \subset [\![1,N]\!]$ with $|K| > k_0$, $\mathbb{P}_x^X(B_K) = 0$, where

$$B_K = \{\text{There is a } K\text{-collision in the configuration } X_{\zeta-}\}.$$

On B_K, there is $\varepsilon \in \mathbb{Q}_+^*$ such that $X_{\zeta-} \in G_{K,2\varepsilon}$. By continuity, there also exists, still on B_K, some $r \in \mathbb{Q}_+ \cap [0,\zeta)$ such that $X_t \in G_{K,\varepsilon}$ for all $t \in [r,\zeta)$. Hence we only have to prove that for all $\varepsilon \in \mathbb{Q}_+^*$, all $t \in \mathbb{Q}_+$, all $T \in \mathbb{Q}_+$ such that $T > r$, for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(B_{K,r,T,\varepsilon}) = 0$, where

$$B_{K,r,T,\varepsilon} = \{\zeta \in (r,T],\ X_t \in G_{K,\varepsilon} \text{ for all } t \in [r,\zeta) \text{ and } R_K(X_{\zeta-}) = 0\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(B_{K,0,T,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$, for all $\varepsilon \in \mathbb{Q}_+^*$ and all $T \in \mathbb{Q}_+^*$. We now fix $\varepsilon \in \mathbb{Q}_+^*$ and $T \in \mathbb{Q}_+^*$. By equivalence, it suffices to prove that $\mathbb{Q}_x^{T,\varepsilon,K}(B_{K,0,T,\varepsilon}) = 0$. Using the notation introduced in the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(B_{K,0,T,\varepsilon}) =&\mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \leq T,\ \tau_{K,\varepsilon} = \zeta \text{ and } R_K(X_{\zeta-}) = 0\Big)\\ =&\mathbb{P}\Big(\zeta' \leq T,\ \tilde{\tau}_{K,\varepsilon} = \zeta' \text{ and } R_K(Y_{\zeta'-}) = 0\Big)\\ \leq&\mathbb{P}\Big(\inf_{t\in[0,\zeta^Y)} R_K(Y_t) = 0\Big).\end{aligned}$$

But $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process with $|K| > k_0 \geq 7$ and with $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) \leq 0$ by Lemma 1 because $|K|-1 \geq k_0$. We also have $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \leq 0$. Hence Proposition 18-(i) tells us that $\mathbb{P}(\inf_{t\in[0,\zeta^Y)} R_K(Y_t) = 0) = 0$.

Point (iii). We recall that $k_1 = k_0 - 1$ and we fix $L \subset K \subset [\![1,N]\!]$ with $|K| = k_0$ and $|L| = k_1$. We want to prove that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., if $R_K(X_{\zeta-}) = 0$, then for all $t \in [0,\zeta)$, the set $\mathcal{Z}_L \cap (t,\zeta)$ is infinite and has no isolated point. But since $d_{\theta,N}(k_1) \in (0,2)$, see Lemma 1, we already know that $\mathcal{Z}_L$ has no isolated point. It thus suffices to check that for quasi all $x \in \mathcal{X}$, for all $r \in \mathbb{Q}_+$, we have $\mathbb{P}_x^X(C_{K,L,r}) = 0$, where

$$C_{K,L,r} = \{\zeta > r,\ R_K(X_{\zeta-}) = 0,\ \text{and } R_L(X_t) > 0 \text{ for all } t \in (r,\zeta)\}.$$

We used that since $|L| = k_1 = k_0 - 1$, for all $x \in \mathcal{X} = E_{k_0}$, there is a L collision in the configuration x if and only if $R_L(x) = 0$.

On $C_{K,L,r}$, thanks to point (ii) , there are $\varepsilon \in \mathbb{Q}_+^*$, $T \in \mathbb{Q}_+$ and $s \in \mathbb{Q}_+^* \cap [r,\zeta)$ such that $\zeta \in (s,T]$ and $X_t \in G_{K,\varepsilon}$ for all $t \in [s,\zeta)$. Thus it suffices to prove that for all $s < T$ and all $\varepsilon > 0$, that we now fix, for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(C_{K,L,s,T,\varepsilon}) = 0$, where

$$C_{K,L,s,T,\varepsilon} = \{\zeta \in (s,T],\ R_K(X_{\zeta-}) = 0,\ X_t \in G_{K,\varepsilon} \text{ and } R_L(X_t) > 0 \text{ for all } t \in [s,\zeta)\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(C_{K,L,0,T,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) = 0$. Recalling the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) =& \mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \le T,\ R_K(X_{\zeta-}) = 0,\ \tau_{K,\varepsilon} = \zeta \text{ and } R_L(X_t) > 0 \text{ for all } t \in [0,\zeta)\Big)\\ =& \mathbb{P}\Big(\zeta' \le T,\ R_K(Y_{\zeta'-}) = 0,\ \tilde{\tau}_{K,\varepsilon} = \zeta' \text{ and } R_L(Y_t) > 0 \text{ for all } t \in [0,\zeta')\Big).\end{aligned}$$

Setting $\sigma_K = \inf\{t > 0 : R_K(Y_t) = 0\}$, we observe that $\sigma_K = \zeta^Y$. Indeed, $|K| = k_0$ and $(Y_t)_{t\ge 0}$ is a $QKS(|K|\theta/N, |K|)$-process, of which the state space is given by $\mathcal{Y}_\triangle = \mathcal{Y} \cup \{\triangle\}$, where $\mathcal{Y} = \{y \in (\mathbb{R}^2)^{|K|} : R_M(y) > 0 \text{ for all } M \subset [\![1,N]\!] \text{ such that } |M| \ge k_0\}$, because $\lceil 2|K|/(|K|\theta/N)\rceil = \lceil 2N/\theta\rceil = k_0$. Hence $\{R_K(Y_{\zeta'-}) = 0\} \subset \{\zeta' = \sigma_K\}$, so that

$$\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) \le \mathbb{P}(R_L(Y_t) > 0 \text{ for all } t \in [0,\sigma_K)).$$

This last quantity equals zero by Proposition 18-(ii), since $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) = d_{\theta,N}(k_0 - 1) \in (0,2)$ by Lemma 1 and since $|L| = k_1 = |K| - 1$ and since $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) = d_{\theta,N}(k_0) \le 0 < 2$.

Point (iv). We assume that $k_2 = k_0 - 2$, i.e. that $d_{\theta,N}(k_0 - 2) \in (0,2)$. We fix $L \subset K \subset [\![1,N]\!]$ with $|K| = k_1$ and $|L| = k_2$. We want to prove that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., for all $t \in [0,\zeta)$, if there is a K-collision in the configuration X_t, then for all $r \in [0,t)$, the set $\mathcal{Z}_L \cap (r,t)$ is infinite and has no isolated point. We already know that $\mathcal{Z}_L$ has no isolated point. It thus suffices to check that for quasi all $x \in \mathcal{X}$, for all $r \in \mathbb{Q}_+$, we have $\mathbb{P}_x^X(D_{K,L,r}) = 0$, where

$$\begin{aligned}D_{K,L,r} = \{\zeta > r \text{ and there is } t \in (r,\zeta) \text{ such that there is a } K\text{-collision at time } t\\ \text{but no } L\text{-collision during } (r,t)\}.\end{aligned}$$

We set $\sigma_{K,r} = \inf\{t > r : \text{there is a } K\text{-collision in the configuration } X_t\}$. It holds that

$$D_{K,L,r} = \{\zeta > r,\ \sigma_{K,r} < \zeta \text{ and there is no } L\text{-collision during } u \in [r,\sigma_{K,r})\}.$$

On $D_{K,L,r}$, there exists $\varepsilon \in \mathbb{Q}_+^*$ such that $X_{\sigma_{K,r}} \in G_{K,2\varepsilon}$, so that by continuity, there exists $v \in \mathbb{Q}_+ \cap [r,\sigma_{K,r})$ such that $X_u \in G_{K,\varepsilon}$ for all $u \in [v,\sigma_{K,r}]$. Observe that $\sigma_{K,v} = \sigma_{K,r}$ and that for all $t \in [v,\sigma_{K,v})$, there is a L-collision at time t if and only if $R_L(X_t) = 0$, by definition of $\sigma_{K,v}$ and since $X_t \in G_{K,\varepsilon}$. All in all, it suffices to prove that for all $v \in \mathbb{Q}_+$, all $\varepsilon \in \mathbb{Q}_+^*$, all $T \in \mathbb{Q}_+^*$, $\mathbb{P}_x^X(D_{K,L,v,T,\varepsilon}) = 0$ for quasi all $x \in \mathcal{X}$, where

$$D_{K,L,v,T,\varepsilon} = \{\zeta \in (v,T],\ \sigma_{K,v} < \zeta,\ X_u \in G_{K,\varepsilon} \text{ and } R_L(X_u) > 0 \text{ for all } u \in [v,\sigma_{K,v})\}.$$

By the Markov property, it suffices to prove that $\mathbb{P}_x^X(D_{K,L,0,T,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$ and, by equivalence, we may use $\mathbb{Q}_x^{T,\varepsilon,K}$ instead of $\mathbb{P}_x^X$. But recalling the preliminaries,

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(D_{K,L,0,T,\varepsilon}) =&\mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \le T,\ \sigma_{K,0} < \zeta,\ \tau_{K,\varepsilon} \ge \sigma_{K,0} \ \text{ and }\ R_L(X_t) > 0 \text{ for all } t \in [0, \sigma_{K,0})\Big)\\ =&\mathbb{P}\Big(\zeta' \le T,\ \tilde\sigma_{K,0} < \zeta',\ \tilde\tau_{K,\varepsilon} \ge \tilde\sigma_{K,0} \ \text{ and }\ R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})\Big)\\ \le&\mathbb{P}\Big(R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})\Big),\end{aligned}$$

where we have set $\tilde\sigma_{K,0} = \inf\{t > 0 : R_K(Y_t) = 0\}$. Finally, $\mathbb{P}(R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})) = 0$ by Proposition 18-(ii), because $(Y_t)_{t\ge0}$ is a $QKS(|K|\theta/N, |K|)$-process, because $|L| = k_2 = |K|-1$, because $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) = d_{\theta,N}(k_2) \in (0,2)$ and because $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) = d_{\theta,N}(k_1) \in (0,2)$.

Point (v). We fix $K \subset [\![1,N]\!]$ with cardinal $|K| \in [\![3, k_2-1]\!]$, so that $d_{\theta,N}(|K|) \ge 2$. We want to prove that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X$-a.s., for all $t \in [0,\zeta)$, there is no K-collision in the configuration X_t. We introduce $\sigma_K = \inf\{t > 0 : \text{there is a } K\text{-collision in the configuration } X_t\}$, with the convention that $\inf\emptyset = \zeta$, and we have to verify that for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(\sigma_K < \zeta) = 0$.

On the event $\{\sigma_K < \zeta\}$, there exist $\varepsilon \in \mathbb{Q}_+^*$ and $r \in \mathbb{Q}_+^* \cap [0, \sigma_K)$ such that $X_t \in G_{K,\varepsilon}$ for all $t \in [r, \sigma_K]$. Hence it suffices to check that for all $\varepsilon \in \mathbb{Q}_+^*$, all $r \in \mathbb{Q}_+^*$ and all $T \in \mathbb{Q}_+^* \cap (r,\infty)$, which we now fix, for quasi all $x \in \mathcal{X}$, $\mathbb{P}_x^X(F_{K,r,T,\varepsilon}) = 0$, where

$$F_{K,r,T,\varepsilon} = \{\sigma_K \in (r, \zeta\wedge T) \text{ and } X_t \in G_{K,\varepsilon} \text{ for all } t \in [r, \sigma_K]\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(F_{K,0,T,\varepsilon}) = 0$ for quasi all $x \in G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{T,\varepsilon,K}(F_{K,0,T,\varepsilon}) = 0$. Recalling the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(F_{K,0,T,\varepsilon}) =&\mathbb{Q}_x^{T,\varepsilon,K}\Big(\sigma_K \in (0, \zeta\wedge T) \text{ and } \tau_{K,\varepsilon} \ge \sigma_K\Big)\\ =&\mathbb{P}\Big(\tilde\sigma_K \in (0, \zeta'\wedge T) \text{ and } \tilde\tau_{K,\varepsilon} \ge \tilde\sigma_K\Big)\\ \le&\mathbb{P}\Big(\inf_{t\in[0,T]} R_K(Y_t) = 0\Big),\end{aligned}$$

where we have set $\tilde\sigma_K = \inf\{t > 0 : \text{there is a } K\text{-collision in the configuration } (Y_t, Z_t)\}$. Since $(Y_t)_{t\ge0}$ is a $QKS(|K|\theta/N, |K|)$-process, we know from Lemma 11 that $(R_K(Y_t))_{t\ge0}$ is a squared Bessel process with dimension $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \ge 2$. Such a process does a.s. never reach 0.

Point (vi). The proof is exactly the same as that of (iv), replacing everywhere k_1 by k_2 and k_2 by 2, and using Proposition 18-(iii) instead of Proposition 18-(ii), which is licit because $0 < d_{k_2\theta/N,k_2}(k_2) < 2 \le d_{k_2\theta/N,k_2}(k_2-1)$, since $d_{k_2\theta/N,k_2}(k_2) = d_{\theta,N}(k_2)$ and $d_{k_2\theta/N,k_2}(k_2-1) = d_{\theta,N}(k_2-1)$ and by Lemma 1. ☐

11. Extension to all initial conditions in E_2

We first prove Proposition 2: we can build a $KS(\theta,N)$-process, i.e. a $QKS(\theta,N)$-process such that $\mathbb{P}_x^X \circ X_t^{-1}$ is absolutely continuous for all $x \in E_2$ and all $t > 0$. We next conclude the proofs of Proposition 3 and of Theorem 5.

11.1. **Construction of a $KS(\theta, N)$-process.** We fix $\theta > 0$ and $N \geq 2$ such that $N > \theta$ during the whole subsection. For each $n \in \mathbb{N}^*$, we introduce $\phi_n \in C^\infty(\mathbb{R}_+, \mathbb{R}_+^*)$ such that $\phi_n(r) = r$ for all $r \geq 1/n$ and we set, for $x \in (\mathbb{R}^2)^N$,

$$\mathbf{m}_n(x) = \prod_{1\leq i\neq j\leq N} [\phi_n(\|x^i - x^j\|^2)]^{-\theta/N} \qquad \text{and} \qquad \mu_n(\mathrm{d}x) = \mathbf{m}_n(x)\mathrm{d}x.$$

We then consider the $(\mathbb{R}^2)^N$-valued S.D.E

$$X_t^n = x + B_t + \int_0^t \frac{\nabla \mathbf{m}_n(X_s^n)}{2\mathbf{m}_n(X_s^n)}\mathrm{d}s, \tag{70}$$

which is strongly well-posed, for every initial condition, since the drift coefficient is smooth and bounded. We denote by $\mathbb{X}^n = (\Omega^n, \mathcal{M}^n, (X_t^n)_{t\geq 0}, (\mathbb{P}_x^n)_{x\in(\mathbb{R}^2)^N})$ the corresponding Markov process, which is of course a Hunt process.

Lemma 23. *For all $n \geq 1$, $\mathbb{X}^n$ is a μ_n-symmetric $(\mathbb{R}^2)^N$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^n, \mathcal{F}^n)$ with core $C_c^\infty((\mathbb{R}^2)^N)$ such that for all $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$,*

$$\mathcal{E}^n(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu_n.$$

Moreover $\mathbb{P}_x^n \circ (X_t^n)^{-1}$ has a density with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$ for all $t > 0$ and all $x \in (\mathbb{R}^2)^N$.

Proof. Classically, $\mathbb{X}^n$ is a μ_n-symmetric Hunt process and its (strong) generator $\mathcal{L}^n$ satisfies that for all $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$, all $x \in (\mathbb{R}^2)^N$, $\mathcal{L}^n\varphi(x) = \frac{1}{2}\Delta\varphi(x) + \frac{\nabla \mathbf{m}_n(x)}{2\mathbf{m}_n(x)} \cdot \nabla\varphi(x)$. Hence, see Subsection B.1, one easily shows that for $(\mathcal{E}^n, \mathcal{F}^n)$ the Dirichlet space of $\mathbb{X}^n$, we have $C_c^\infty((\mathbb{R}^2)^N) \subset \mathcal{F}^n$ and, for $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$, $\mathcal{E}^n(\varphi,\varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu_n$. Since $(\mathcal{E}^n, \mathcal{F}^n)$ is closed, we deduce that

$$\overline{C_c^\infty((\mathbb{R}^2)^N)}^{\mathcal{E}_1^n} \subset \mathcal{F}^n,$$

where $\mathcal{E}_1^n(\cdot,\cdot) = \mathcal{E}^n(\cdot,\cdot) + \|\cdot\|^2_{L^2((\mathbb{R}^2)^N,\mu_n)}$. But thanks to [11, Lemma 3.3.5 page 136],

$$\mathcal{F}^n \subset \{\varphi \in L^2((\mathbb{R}^2)^N, \mu_n) : \nabla\varphi \in L^2((\mathbb{R}^2)^N, \mu_n)\},$$

where ∇ is understood in the sense of distributions. Since finally

$$\overline{C_c^\infty((\mathbb{R}^2)^N)}^{\mathcal{E}_1^n} = \{\varphi \in L^2((\mathbb{R}^2)^N, \mu_n) : \nabla\varphi \in L^2((\mathbb{R}^2)^N, \mu_n)\},$$

$\mathbb{X}^n$ has the announced Dirichlet space. Finally, the absolute continuity of $\mathbb{P}_x^n \circ (X_t^n)^{-1}$, for $t > 0$ and $x \in (\mathbb{R}^2)^N$, immediately follows from the (standard) Girsanov theorem, since the drift coefficient is bounded. □

For all $x \in E_2$ we set $d_x = \min_{i\neq j} \|x^i - x^j\|^2$. For $n \geq 1$, we introduce the open set

$$E_2^n = \Big\{x \in (\mathbb{R}^2)^N : d_x > \frac{1}{n} \ \text{ and } \ ||x|| < n\Big\}. \tag{71}$$

We also fix a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$ for the whole subsection.

Lemma 24. *There exists an exceptional set $\mathcal{N}_0 \subset E_2$ with respect to $\mathbb{X}$ such that for all $n \geq 1$, for all $x \in E_2^n \setminus \mathcal{N}_0$, the law of $(X_{t\wedge\tau_n}^n)_{t\geq 0}$ under $\mathbb{P}_x^n$ equals the law of $(X_{t\wedge\sigma_n})_{t\geq 0}$ under $\mathbb{P}_x^X$, where*

$$\tau_n = \inf\{t > 0 : X_t^n \notin E_2^n\} \qquad \text{and} \qquad \sigma_n = \inf\{t > 0 : X_t \notin E_2^n\}.$$

Proof. We fix $n \geq 1$. Applying Lemma B.6 to $\mathbb{X}^n$ and $\mathbb{X}$ with the open set E_2^n, using that $\mathbf{m}_n = \mathbf{m}$ on E_2^n and Lemma 23, we find that the processes $\mathbb{X}^n$ and $\mathbb{X}$ killed when leaving E_2^n have the same Dirichlet space. By uniqueness, see [11, Theorem 4.2.8 page 167], there exists an exceptional set $\mathcal{N}_n$ such that for all $x \in E_2^n \setminus \mathcal{N}_n$, the law of $(X_t^n)_{t\geq 0}$ killed when leaving E_2^n under $\mathbb{P}_x^n$ equals the law of $(X_t)_{t\geq 0}$ killed when leaving E_2^n under $\mathbb{P}_x^X$. We conclude setting $\mathcal{N}_0 = \cup_{n\geq 1}\mathcal{N}_n$. □

Lemma 25. *For all exceptional set $\mathcal{N}$ with respect to $\mathbb{X}$, all $n \geq 1$ and all $x \in E_2^n$, we have $\mathbb{P}_x^n(X_{\tau_n}^n \notin \mathcal{N}) = 1$.*

Proof. We fix $\mathcal{N}$ an exceptional set with respect to $\mathbb{X}$, $n \geq 1$ and $x \in E_2^n$. For any $\varepsilon > 0$, we write

$$\mathbb{P}_x^n(X_{\tau_n}^n \in \mathcal{N}) \leq \mathbb{P}_x^n(\tau_n \leq \varepsilon) + \mathbb{P}_x^n(\tau_n > \varepsilon, X_{\tau_n}^n \in \mathcal{N}) = \mathbb{P}_x^n(\tau_n \leq \varepsilon) + \mathbb{E}_x^n[\mathbb{1}_{\{\tau_n > \varepsilon\}}\mathbb{P}^n_{X_\varepsilon^n}(X_{\tau_n}^n \in \mathcal{N})]$$

by the Markov property. But by Lemma 24, for all $y \in E_2^n \setminus \mathcal{N}_0$, the law of $(X_{t\wedge\tau_n}^n)_{t\geq 0}$ under $\mathbb{P}_y^n$ is equal to the law of $(X_{t\wedge\sigma_n})_{t\geq 0}$ under $\mathbb{P}_y^X$. Since $\mathcal{N}_0 \cup \mathcal{N}$ is exceptional for $\mathbb{X}$, we can find $\mathcal{N}' \supset \mathcal{N}_0 \cup \mathcal{N}$ properly exceptional for $\mathbb{X}$ (see Subsection B.1). Hence for all $y \in E_2^n \setminus \mathcal{N}'$,

$$\mathbb{P}_y^n(X_{\tau_n}^n \in \mathcal{N}) \leq \mathbb{P}_y^n(X_{\tau_n}^n \in \mathcal{N}') = \mathbb{P}_y^X(X_{\sigma_n} \in \mathcal{N}') = 0.$$

Since now $\mathbb{P}_x^n \circ (X_\varepsilon^n)^{-1}$ has a density by Lemma 24, we conclude that $\mathbb{P}_x^n(X_\varepsilon^n \in \mathcal{N}') = 0$ and thus that $\mathbb{P}_x^n$-a.s., we have $\mathbb{P}^n_{X_\varepsilon^n}(X_{\tau_n}^n \in \mathcal{N}) = 0$. All in all, we have proved that $\mathbb{P}_x^n(X_{\tau_n}^n \in \mathcal{N}) \leq \mathbb{P}_x^n(\tau_n \leq \varepsilon)$, and it suffices to let $\varepsilon \to 0$, since $\mathbb{P}_x^n(\tau_n > 0) = 1$ by continuity and since $x \in E_2^n$. □

Using Lemmas 24 and 25, it is slightly technical but not difficult to build from $\mathbb{X}$ and the family $(\mathbb{X}^n)_{n\geq 1}$ a continuous $\mathcal{X}_\triangle$-valued Hunt process $\tilde{\mathbb{X}} = (\tilde{\Omega}^X, \tilde{\mathcal{M}}^X, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x^X)_{x\in\mathcal{X}_\triangle})$ such that

- for all $x \in \mathcal{X}_\triangle \setminus \mathcal{N}_0$, the law of $(\tilde{X}_t)_{t\geq 0}$ under $\tilde{\mathbb{P}}_x^X$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}_x^X$,

- for all $x \in \mathcal{N}_0$, setting $n = 1 + \lfloor \max(1/d_x, ||x||)\rfloor$ (so that $x \in E_2^n$), the law of $(\tilde{X}_{t\wedge\tilde{\sigma}_n})_{t\geq 0}$ under $\tilde{\mathbb{P}}_x^X$ is the same as that of $(X_{t\wedge\tau_n}^n)_{t\geq 0}$ under $\mathbb{P}_x^n$ and the law of $(\tilde{X}_{\tilde{\sigma}_n+t})_{t\geq 0}$ under $\tilde{\mathbb{P}}_x^X$ conditionally on $\tilde{\mathcal{M}}_{\tilde{\sigma}_n}^X$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_{\tilde{X}_{\sigma_n}}$. We have used the notation $\tilde{\sigma}_n = \inf\{t > 0 : \tilde{X}_t \notin E_2^n\}$ and $\tilde{\mathcal{M}}_t^X = \sigma(\tilde{X}_s : s \in [0, t])$.

Remark 26. *For all $x \in E_2$, setting $n = 1 + \lfloor \max(1/d_x, ||x||)\rfloor$, the law of $(\tilde{X}_{t\wedge\tilde{\sigma}_n})_{t\geq 0}$ under $\tilde{\mathbb{P}}_x^X$ is the same as that of $(X_{t\wedge\tau_n}^n)_{t\geq 0}$ under $\mathbb{P}_x^n$.*

Proof. This follows from Lemma 24 when $x \in E_2 \setminus \mathcal{N}_0$ and from the definition of $\tilde{\mathbb{X}}$ otherwise. □

We can finally give the

Proof of Proposition 2. We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and we prove that $\tilde{\mathbb{X}}$ defined above is a $KS(\theta, N)$-process. First, it is clear that $\tilde{\mathbb{X}}$ is a $QKS(\theta, N)$-process because $\tilde{\mathbb{X}}$ is a continuous $\mathcal{X}_\triangle$-valued Hunt process and since for all $x \in \mathcal{X}_\triangle \setminus \mathcal{N}_0$, the law of $(\tilde{X}_t)_{t\geq 0}$ under $\tilde{\mathbb{P}}_x^X$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}_x^X$, with $\mathcal{N}_0$ exceptional for $\mathbb{X}$. It remains to prove that for all $x \in E_2$, all $t > 0$ and all Lebesgue-null $A \subset (\mathbb{R}^2)^N$, we have $\tilde{\mathbb{P}}_x^X(\tilde{X}_t \in A) = 0$. We set $n = 1 + \lfloor \max(1/d_x, ||x||)\rfloor$ and write, for any $\varepsilon \in (0, t)$,

$$\tilde{\mathbb{P}}_x^X(\tilde{X}_t \in A) \leq \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n > \varepsilon, \tilde{X}_t \in A) + \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n \leq \varepsilon) = \tilde{\mathbb{E}}_x^X[\mathbb{1}_{\{\tilde{\sigma}_n > \varepsilon\}}\tilde{\mathbb{P}}^X_{\tilde{X}_\varepsilon}(\tilde{X}_{t-\varepsilon} \in A)] + \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n \leq \varepsilon).$$

Since $\tilde{\mathbb{X}}$ is μ-symmetric (because it is a $QKS(\theta, N)$-process), since $\tilde{P}_{t-\varepsilon}1 \leq 1$, where $\tilde{P}_t$ is the semi-group of $\tilde{\mathbb{X}}$ and since A is Lebesgue-null,

$$\int_{(\mathbb{R}^2)^N} \tilde{\mathbb{P}}_y(\tilde{X}_{t-\varepsilon} \in A)\mu(\mathrm{d}y) \leq \mu(A) = 0.$$

Hence there is a Lebesgue-null subset B of $(\mathbb{R}^2)^N$ (depending on $t-\varepsilon$) such that $\tilde{\mathbb{P}}_y(\tilde{X}_{t-\varepsilon} \in A) = 0$ for every $y \in (\mathbb{R}^2)^N \setminus B$. We conclude that

$$\tilde{\mathbb{P}}_x^X(\tilde{X}_t \in A) \le \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n > \varepsilon, \tilde{X}_\varepsilon \in B) + \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n \le \varepsilon) = \mathbb{P}_x^n(\tau_n > \varepsilon, X_\varepsilon^n \in B) + \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n \le \varepsilon),$$

where we finally used Remark 26. Since B is Lebesgue-null, we deduce from Lemma 23 that $\mathbb{P}_x^n(\tau_n > \varepsilon, X_\varepsilon^n \in B) = 0$. Thus $\tilde{\mathbb{P}}_x^X(\tilde{X}_t \in A) \le \tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n \le \varepsilon)$, which tends to 0 as $\varepsilon \to 0$ because $\tilde{\mathbb{P}}_x^X(\tilde{\sigma}_n > 0) = 1$ by continuity. □

11.2. **Final proofs.** We fix $\theta > 0$, $N \ge 2$ such that $N > \theta$ and a $KS(\theta, N)$-process $\mathbb{X}$, which exists thanks to Subsection 11.1. We recall that E_2^n was introduced in (71) and define, for all $n \ge 1$, $\sigma_n = \inf\{t \ge 0 : X_t \notin E_2^n\}$, as well as the σ-field

$$\mathcal{G} = \cap_{n \ge 1} \sigma(X_{\sigma_n + t}, t \ge 0).$$

Lemma 27. *Fix $A \in \mathcal{G}$. If $\mathbb{P}_x^X(A) = 0$ for quasi all $x \in \mathcal{X}$, then $\mathbb{P}_x^X(A) = 0$ for all $x \in E_2$.*

Proof. We fix $A \in \mathcal{G}$ such that $\mathbb{P}_x^X(A) = 0$ for quasi all $x \in \mathcal{X}$. There is an exceptional set $\mathcal{N}$ such that for all $x \in E_2 \setminus \mathcal{N}$, $\mathbb{P}_x^X(A) = 0$. We now fix $x \in E_2$ and set $n = 1 + \lfloor \max(1/d_x, ||x||) \rfloor$. For any $\varepsilon > 0$,

$$\mathbb{P}_x^X(A) \le \mathbb{P}_x^X(\sigma_n \le \varepsilon) + \mathbb{P}_x^X[\sigma_n > \varepsilon, A].$$

By the Markov property and since $A \in \mathcal{G} \subset \sigma(X_{\sigma_n + t}, t \ge 0)$, we get

$$\mathbb{P}_x^X[\sigma_n > \varepsilon, A] = \mathbb{E}_x^X[\mathbb{1}_{\{\sigma_n > \varepsilon\}} \mathbb{P}_{X_\varepsilon}^X(A)].$$

But the law of X_ε under $\mathbb{P}_x^X$ has a density, so that $\mathbb{P}_x^X(X_\varepsilon \in \mathcal{N}) = 0$, whence $\mathbb{P}_x^X(\mathbb{P}_{X_\varepsilon}^X(A) = 0) = 1$. Hence $\mathbb{P}_x^X[\sigma_n > \varepsilon, A] = 0$ and we end with $\mathbb{P}_x^X(A) \le \mathbb{P}_x^X(\tau_n \le \varepsilon)$. As usual, we conclude that $\mathbb{P}_x^X(A) = 0$ by letting $\varepsilon \to 0$. □

We are now ready to give the

Proof of Proposition 3. Let $\theta \in (0, 2)$ and $N \ge 2$. Since our $KS(\theta, N)$-process $\mathbb{X}$ is a $QKS(\theta, N)$-process, we know from Proposition 15-(i) that $\mathbb{P}_x^X(\zeta = \infty) = 1$ for quasi all $x \in \mathcal{X}$. We want to prove that $\mathbb{P}_x^X(\zeta = \infty) = 1$ for all $x \in E_2$. By Lemma 27, it thus suffices to check that $\{\zeta = \infty\}$ belongs to $\mathcal{G}$, which is not hard since for each $n \ge 1$,

$$\{\zeta = \infty\} = \{X_t \in \mathcal{X} \text{ for all } t \ge 0\} = \{X_t \in \mathcal{X} \text{ for all } t \ge \sigma_n\} \in \sigma(X_{\sigma_n + t}, t \ge 0).$$

For the second equality, we used that $X_t \in \bar{E}_2^n \subset \mathcal{X}$ for all $t \in [0, \sigma_n]$ by definition. □

Proof of Theorem 5. Let $\theta \ge 2$ and $N > 3\theta$. Since our $KS(\theta, N)$-process $\mathbb{X}$ is a $QKS(\theta, N)$-process, we know from Section 10 that all the conclusions of Theorem 5 hold for quasi all $x \in \mathcal{X}$. In other words, $\mathbb{P}_x^X(A) = 1$ for quasi all $x \in \mathcal{X}$, where A is the event on which we have $\zeta < \infty$, $X_{\zeta-} = \lim_{t \to \zeta-} X_t \in (\mathbb{R}^2)^N$, there is $K_0 \in [\![1, N]\!]$ with cardinal $|K_0| = k_0$ such that there is a K_0-collision in the configuration $X_{\zeta-}$, etc. We want to prove that $\mathbb{P}_x^X(A) = 1$ for all $x \in E_2$. By Lemma 27, it thus suffices to check that A belongs to $\mathcal{G}$. But for each $n \ge 1$, A indeed belongs to $\sigma(X_{\sigma_n + t}, t \ge 0)$, because no collision (nor explosion) may happen before getting out of E_2^n. □

We end this section with the following remark (that we will not use anywhere).

Remark 28. *Fix $\theta \ge 0$ and $N \ge 2$ such that $N > \theta$. Consider a $KS(\theta, N)$ process $\mathbb{X}$ and define $\sigma = \inf\{t \ge 0 : X_t \notin E_2\}$. For all $x \in E_2$, there is some $(\mathcal{M}_t^X)_{t \ge 0}$-Brownian motion $((B_t^i)_{t \ge 0})_{i \in [\![1, N]\!]}$ (of dimension $2N$) under $\mathbb{P}_x^X$ such that for all $t \in [0, \sigma)$, all $i \in [\![1, N]\!]$,*

$$X_t^i = x^i + B_t^i - \frac{\theta}{N} \sum_{j \ne i} \int_0^t \frac{X_s^i - X_s^j}{||X_s^i - X_s^j||^2} \mathrm{d}s. \tag{72}$$

Proof. It of course suffices to prove the result during $[0, \sigma_n)$, where $\sigma_n = \inf\{t \geq 0 : X_t \notin E_2^n\}$. For any $x \in E_2^n$ and for a given Brownian motion, the solutions to (72) and (70) classically coincide while they remain E_2^n, because their drift coefficients coincide and are smooth inside E_2^n. Hence, recalling the notation of Subsection 11.1, it suffices to prove that the semi-groups $P_t(x, \cdot)$ and $P_t^n(x, \cdot)$ of the Markov processes $\mathbb{X}$ and $\mathbb{X}^n$ killed when getting out of E_2^n coincide for all $x \in E_2^n$.

By Lemma 24, there is an exceptional set $\mathcal{N}_0$ such that $P_t(x, \cdot) = P_t^n(x, \cdot)$ for all $x \in E_2^n \setminus \mathcal{N}_0$. We next fix $x \in E_2^n$. For any $\varepsilon \in (0, t)$, using that $P_\varepsilon(x, \cdot)$ has a density and that $\mathcal{N}_0$ is Lebesgue-null, we easily deduce that $P_t(x, \cdot) = (P_\varepsilon P_{t-\varepsilon})(x, \cdot) = (P_\varepsilon P_{t-\varepsilon}^n)(x, \cdot)$. It is then not difficult, using that P_t^n is Feller, to let $\varepsilon \to 0$ and conclude that indeed, $P_t(x, \cdot) = P_t^n(x, \cdot)$. □

Appendix A. A few elementary computations

We recall that $d_{\theta,N}(k) = (k-1)(2 - \theta k/N)$ for $k \geq 2$ and give the

Proof of Lemma 1. First, (3), which says that $d_{\theta,N}(k) > 0$ if and only if $k < k_0 = \lceil 2N/\theta \rceil$, is clear. We next fix $N > 3\theta \geq 6$, so that $k_0 \in [\![7, N]\!]$ and $d_{\theta,N}(2) = 2 - 2\theta/N \in (4/3, 2)$. By concavity of $x \to (x-1)(2 - \theta x/N)$, it only remains to check that (i) $d_{\theta,N}(3) \geq 2$, (ii) $d_{\theta,N}(k_0 - 3) \geq 2$, and (iii) $d_{\theta,N}(k_0 - 1) < 2$. We introduce $a = 2N/\theta > 6$ and observe that $d_{\theta,N}(k) = 2a^{-1}(k-1)(a-k)$ and that $k_0 = \lceil a \rceil$.

For (i), we write $d_{\theta,N}(3) = 4a^{-1}(a-3) = 4 - 12a^{-1} > 2$ since $a > 6$.

For (ii), we have $d_{\theta,N}(k_0-3) = 2a^{-1}(\lceil a \rceil - 4)(a - \lceil a \rceil + 3)$ and we need $(\lceil a \rceil - 4)(a - \lceil a \rceil + 3) \geq a$. Writing $a = n + \alpha$ with an integer $n \geq 6$ and $\alpha \in (0, 1]$, we need that $(n-3)(2+\alpha) \geq n + \alpha$, and this holds true because $2(n-3) \geq n$ and $(n-3)\alpha \geq \alpha$.

For (iii), we write $d_{\theta,N}(k_0 - 1) = 2a^{-1}(\lceil a \rceil - 2)(a - \lceil a \rceil + 1) \leq 2a^{-1}(\lceil a \rceil - 2) < 2$. □

We next study the reference measure of the Keller-Segel particle system.

Proposition A.1. *Let $N \geq 2$ and $\theta > 0$ be such that $N > \theta$. Recall that $k_0 = \lceil 2N/\theta \rceil$ and the definition* (4) *of $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$.*

(i) The measure μ is Radon on E_{k_0}.

(ii) If $k_0 \leq N$, then μ is not Radon on E_{k_0+1}.

Proof. (i) To show that μ is radon on E_{k_0}, we have to check that for all $x = (x^1, \dots, x^N) \in E_{k_0}$, which we now fix, there is an open set $O_x \subset E_{k_0}$ such that $x \in O_x$ and $\mu(O_x) < \infty$. We choose $O_x = \prod_{i=1}^N B(x^i, d_x)$, where the balls are subsets of $\mathbb{R}^2$ and where

$$d_x = 1 \wedge \min \Big\{ \frac{\|x^i - x^j\|}{3} : i, j \in [\![1, N]\!] \text{ such that } x^i \neq x^j \Big\} > 0.$$

We consider the partition $K_1, \dots, K_\ell$ of $[\![1, N]\!]$ such that for all $p \neq q$ in $[\![1, \ell]\!]$, for all $i, j \in K_p$ and all $k \in K_q$, $x^i = x^j$ and $x^i \neq x^k$. Since $x \in E_{k_0}$, it holds that $\max_{p \in [\![1,\ell]\!]} |K_p| \leq k_0 - 1$. By definition of O_x and d_x, we see that for all $y \in O_x$, for all $p \neq q$ in $[\![1, \ell]\!]$, for all $i \in K_p$, all $j \in K_q$,

$$\|y^i - y^j\| \geq \|x^i - x^j\| - \|x^i - y^i\| - \|x^j - y^j\| \geq \|x^i - x^j\| - 2d_x \geq d_x.$$

This implies that for some finite constant C depending on x, for all $y \in O_x$,

$$\mathbf{m}(y) = \prod_{1 \leq i \neq j \leq N} ||y^i - y^j||^{-\theta/N} \leq C \prod_{p=1}^{\ell} \Big(\prod_{i,j \in K_p, i \neq j} ||y^i - y^j||^{-\theta/N} \Big).$$

Recall now that $\mu(\mathrm{d}y) = \mathbf{m}(y)\mathrm{d}y$ and that we want to show that $\mu(O_x) < \infty$. Since $x^i = x^j$ for all $i, j \in K_p$ and all $p \in [\![1, \ell]\!]$, since $|K_p| \leq k_0 - 1$, $d_x \leq 1$ and by a translation argument, we are reduced to show that for any $n \in [\![2, k_0 - 1]\!]$, (when $k_0 > N$, one could study only $n \in [\![2, N]\!]$)

$$I_n = \int_{(B(0,1))^n} \Big(\prod_{1 \leq i \neq j \leq n} \|y^i - y^j\|^{-\theta/N} \Big) \mathrm{d}y^1 \dots \mathrm{d}y^n < \infty.$$

We fix $n \in [\![2, k_0 - 1]\!]$ and show that $I_n < \infty$. Since $\|u\|^2 \geq |u_1 u_2|$ for all $u = (u_1, u_2) \in \mathbb{R}^2$, we have $I_n \leq J_n^2$, where

$$J_n = \int_{[-1,1]^n} \Big(\prod_{1 \leq i \neq j \leq n} |t^i - t^j|^{-\theta/(2N)} \Big) \mathrm{d}t^1 \dots \mathrm{d}t^n.$$

But for all $t^1, \dots, t^n \in \mathbb{R}$,

$$\prod_{1 \leq i \neq j \leq n} |t^i - t^j|^{-\theta/(2N)} = \prod_{i=1}^n \Big(\prod_{j=1, j \neq i}^n |t^i - t^j|^{-\theta/(2N)} \Big) \leq \frac{1}{n} \sum_{i=1}^n \prod_{j=1, j \neq i}^n |t^i - t^j|^{-\theta n/(2N)}$$

by the inequality of arithmetic and geometric means. Thus by symmetry,

$$J_n \leq \int_{[-1,1]^n} \Big(\prod_{j=2}^n |t^1 - t^j|^{-\theta n/(2N)} \Big) \mathrm{d}t^1 \dots \mathrm{d}t^n = \int_{-1}^1 \Big(\int_{-1}^1 |t^1 - t^2|^{-\theta n/(2N)} \mathrm{d}t^2 \Big)^{n-1} \mathrm{d}t^1.$$

Consequently,

$$J_n \leq \int_{-1}^1 \Big(\int_{-2}^2 |s|^{-\theta n/(2N)} \mathrm{d}s \Big)^{n-1} \mathrm{d}t^1.$$

Since $n \leq k_0 - 1 = \lceil 2N/\theta \rceil - 1 < 2N/\theta$, we have $\theta n/(2N) < 1$, so that $J_n < \infty$, whence $I_n < \infty$.

(ii) We next assume that $k_0 \in [\![2, N]\!]$. To prove that μ is not radon on E_{k_0+1}, we show that $\mu(K) = \infty$ for the compact subset

$$K = \prod_{i=1}^{k_0} \overline{B}(0,1) \times \prod_{k=k_0+1}^{N} \overline{B}((2k, 0), 1/2)$$

of E_{k_0+1}. All the balls in the previous formula are balls of $\mathbb{R}^2$. For $x = (x^1, \dots, x^N) \in K$, it holds that $x^{k_0+1}, \dots, x^N$ are far from each other and far from $x^1, \dots, x^{k_0}$, which explains that K is indeed compact in E_{k_0+1}. There is a positive constant $c > 0$ such that for all $x \in K$,

$$\mathbf{m}(x) = \prod_{1 \leq i \neq j \leq N} ||x^i - x^j||^{-\theta/N} \geq c \prod_{1 \leq i \neq j \leq k_0} ||x^i - x^j||^{-\theta/N},$$

whence, the value of $c > 0$ being allowed to vary,

$$\mu(K) \geq c \int_{(B(0,1))^{k_0}} \Big(\prod_{1 \leq i \neq j \leq k_0} \|x^i - x^j\|^{-\theta/N} \Big) \mathrm{d}x^1 \dots \mathrm{d}x^{k_0}.$$

We now observe that

$$A = \{x = (x^1, \dots, x^{k_0}) : x^1, x^2 \in B(0, 1/3),\ \forall i \notin \{1, 2\},\ x^i \in B(x^1, \|x^1 - x^2\|)\} \subset (B(0,1))^{k_0}$$

and that for $x \in A$, we have $||x^i - x^j|| \leq ||x^i - x^1|| + ||x^j - x^1|| \leq 2||x^1 - x^2||$ for all $i, j = 1, \dots, k_0$, from which

$$\prod_{1 \leq i \neq j \leq k_0} \|x^i - x^j\|^{-\theta/N} \geq c \|x^1 - x^2\|^{-k_0(k_0-1)\theta/N}.$$

As a conclusion,

$$\begin{aligned}\mu(K) \geq& c\int_{(B(0,1/3))^2} \|x^1-x^2\|^{-k_0(k_0-1)\theta/N}\mathrm{d}x^1\mathrm{d}x^2 \int_{(B(x_1,\|x^1-x^2\|))^{k_0-2}} \mathrm{d}x^3\dots\mathrm{d}x^{k_0}\\ \geq& c\int_{(B(0,1/3))^2} \|x^1-x^2\|^{-k_0(k_0-1)\theta/N+2(k_0-2)}\mathrm{d}x^1\mathrm{d}x^2\\ \geq& c\int_{B(0,1/3)} \|u\|^{-k_0(k_0-1)\theta/N+2(k_0-2)}\mathrm{d}u,\end{aligned}$$

where we finally used the change of variables $u = x^1 - x^2$ and $v = x^1 + x^2$. This last integral diverges, because $-k_0(k_0-1)\theta/N + 2(k_0-2) = d_{\theta,N}(k_0) - 2 \leq -2$, recall that $d_{\theta,N}(k_0) = (k_0-1)(2-k_0\theta/N) \leq 0$ by definition of k_0. □

We need a similar result on the sphere $\mathbb{S}$ defined in Section 2, where $\gamma : \mathbb{R}^2 \to (\mathbb{R}^2)^N$ and $\Psi : \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S} \to E_N \subset (\mathbb{R}^2)^N$ were also introduced. First, we show an explicit link between $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$ and $\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u)$ defined in (4) and (8), that we use several times.

Lemma A.2. *We fix $N \geq 2$, $\theta > 0$ and set $\nu = d_{\theta,N}(N)/2 - 1$. For all Borel $\varphi : (\mathbb{R}^2)^N \to \mathbb{R}_+$,*

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}} \varphi(\Psi(z,r,u))r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Proof. Since $H = \{y = (y^1,\dots,y^N) \in (\mathbb{R}^2)^N : \sum_1^N y^i = 0\}$ and since $\mathbf{m}$ is translation invariant,

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \int_{(\mathbb{R}^2)^N} \varphi(x)\mathbf{m}(x)\mathrm{d}x = \int_{\mathbb{R}^2\times H} \varphi(\gamma(z)+y)\mathbf{m}(y)\mathrm{d}z\mathrm{d}y.$$

We next note that $\mathbb{S}$ is the (true) unit sphere of the $(2N-2)$-dimensional Euclidean space H and proceed to the substitution $(\ell, u) = (\|y\|, y/||y||)$:

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \varphi(\gamma(z)+\ell u)\mathbf{m}(\ell u)\ell^{2N-3}\mathrm{d}z\mathrm{d}\ell\sigma(\mathrm{d}u).$$

We finally substitute $\ell = \sqrt{r}$ and obtain

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \varphi(\gamma(z)+\sqrt{r}u)\mathbf{m}(\sqrt{r}u)r^{N-2}\mathrm{d}z\mathrm{d}r\sigma(\mathrm{d}u).$$

But $\mathbf{m}(\sqrt{r}u)r^{N-2} = r^{N-2-\theta(N-1)/2}\mathbf{m}(u)$ by (4) and $\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u)$, whence

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \varphi(\Psi(z,r,u))r^{N-2-\theta(N-1)/2}\mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Since finally $\nu = d_{\theta,N}(N)/2 - 1 = N - 2 - \theta(N-1)/2$, the conclusion follows. □

We can now study the measure β on $\mathbb{S}$.

Proposition A.3. *Let $N \geq 2$ and $\theta > 0$ such that $N > \theta$. Recall that $k_0 = \lceil 2N/\theta \rceil$.*

(i) The measure β is Radon on $\mathbb{S} \cap E_{k_0}$.

(ii) If $k_0 \geq N$, then $\beta(\mathbb{S}) < \infty$.

Proof. We start with (i). For $\varepsilon > 0$, we introduce

$$\mathcal{K}_\varepsilon = \{x \in (\mathbb{R}^2)^N : \forall K \subset [\![1,N]\!] \text{ such that } |K| \geq k_0, \text{ we have } R_K(x) \geq \varepsilon\} \quad \text{and} \quad \mathcal{L}_\varepsilon = \mathcal{K}_\varepsilon \cap \mathbb{S}.$$

Since $\mathcal{K}_\varepsilon \cap \overline{B}(0,1)$ is compact in E_{k_0}, with here $B(0,1)$ the unit ball of $(\mathbb{R}^2)^N$, we know from Proposition A.1-(i) that $\mu(\mathcal{K}_\varepsilon \cap B(0,1)) < \infty$. Now by Lemma A.2,

$$\mu(\mathcal{K}_\varepsilon \cap B(0,1)) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \mathbb{I}_{\{\gamma(z)+\sqrt{r}u\in\mathcal{K}_\varepsilon\cap B(0,1)\}} r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

But for $(z,r,u) \in \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}$,

$$\gamma(z) + \sqrt{r}u \in \mathcal{K}_\varepsilon \cap B(0,1) \qquad \text{if and only if} \qquad u \in \mathcal{L}_{\varepsilon/r} \quad \text{and} \quad N||z||^2 + r < 1.$$

Indeed, $R_K(\gamma(z) + \sqrt{r}u) = rR_K(u)$ for all $K \subset [\![1,N]\!]$ and $||\gamma(z) + \sqrt{r}u||^2 = \sum_1^N ||z + \sqrt{r}u^i||^2 = N||z||^2 + r$ because $\sum_1^N u^i = 0$ and $\sum_1^N ||u^i||^2 = 1$. Thus

$$\mu(\mathcal{K}_\varepsilon \cap B(0,1)) = \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathbb{I}_{\{N||z||^2+r<1\}} r^\nu \beta(\mathcal{L}_{\varepsilon/r})\mathrm{d}z\mathrm{d}r.$$

All this implies that for all $\varepsilon > 0$, for almost all $r \in (0,1)$, $\beta(\mathcal{L}_{\varepsilon/r}) < \infty$. Since $\varepsilon \to \mathcal{L}_\varepsilon$ is monotone, we conclude that $\beta(\mathcal{L}_\varepsilon) < \infty$ for all $\varepsilon > 0$. Since finally $\cup_{\varepsilon>0}\mathcal{L}_\varepsilon = \mathbb{S} \cap E_{k_0}$ and since $\mathcal{L}_\varepsilon$ is compact in $\mathbb{S} \cap E_{k_0}$ for each $\varepsilon > 0$, we conclude as desired that β is Radon on $\mathbb{S} \cap E_{k_0}$.

We next prove (ii). It holds that $\mathbb{S} \subset E_N$, because for $u \in \mathbb{S}$, we have $R_{[\![1,N]\!]}(u) = 1$. Hence if $k_0 \geq N$, then $\mathbb{S} \subset E_N \subset E_{k_0}$, whence $\mathbb{S} = \mathbb{S} \cap E_{k_0}$ and thus β is Radon on $\mathbb{S}$ by point (i). Since finally $\mathbb{S}$ is compact, we conclude that $\beta(\mathbb{S}) < \infty$. ☐

Appendix B. Markov processes and Dirichlet spaces

In a first subsection, we recall some classical definitions and results about Hunt processes and Dirichlet spaces found in Fukushima-Oshima-Takeda [11]. In a second subsection, we mention a few results about martingales, times-changes, concatenation, killing and Girsanov transformation of Hunt processes found in [11] and elsewhere.

B.1. **Main definitions and properties.** Let E be a locally compact separable metrizable space endowed with a Radon measure α such that Supp $\alpha = E$. We set $E_\triangle = E \cup \{\triangle\}$, where $\triangle$ is a cemetery point. See [11, Section A2] for the definition of a Hunt process $\mathbb{Y} = (\Omega, \mathcal{M}, (Y_t)_{t\geq 0}, (\mathbb{P}_y)_{y\in E_\triangle})$: it is a strong Markov process in its canonical filtration, $\mathbb{P}_y(Y_0 = y) = 1$ for all $y \in E_\triangle$, $\triangle$ is an absorbing state, i.e. $Y_t = \triangle$ for all $t \geq 0$ under $\mathbb{P}_\triangle$, and a few more technical properties are satisfied. The life-time of $\mathbb{Y}$ is defined by $\zeta = \inf\{t \geq 0 : Y_t = \triangle\}$.

Let us denote by $P_t(y, \mathrm{d}z)$ its transition kernel. Our Hunt process is said to be α-symmetric if $\int_E \varphi P_t\psi \mathrm{d}\alpha = \int_E \psi P_t\varphi \mathrm{d}\alpha$ for all measurable $\varphi, \psi : E \to \mathbb{R}_+$ and all $t \geq 0$, see [11, page 30]. The Dirichlet space $(\mathcal{E}, \mathcal{F})$ of our Hunt process on $L^2(E, \alpha)$ is then defined, see [11, page 23], by

$$\mathcal{F} = \Big\{\varphi \in L^2(E,\alpha) : \lim_{t\to 0} \frac{1}{t}\int_E \varphi(P_t\varphi - \varphi)\mathrm{d}\alpha \text{ exists}\Big\},$$

$$\mathcal{E}(\varphi,\psi) = -\lim_{t\to 0}\frac{1}{t}\int_E \varphi(P_t\psi - \psi)\mathrm{d}\alpha \qquad \text{for all } \varphi,\psi \in \mathcal{F}.$$

The generator $(\mathcal{A}, \mathcal{D}_A)$ of $\mathbb{Y}$ is defined as follows:

$$\mathcal{D}_\mathcal{A} = \Big\{\varphi \in L^2(E,\alpha) : \lim_{t\to 0}\frac{1}{t}(P_t\varphi - \varphi) \text{ exists in } L^2(E,\alpha)\Big\},$$

and for $\varphi \in \mathcal{D}_\mathcal{A}$, we denote by $\mathcal{A}\varphi \in L^2(E, \alpha)$ this limit. By [11, Pages 20-21], it holds that

$$\text{(B.1)} \qquad \mathcal{D}_\mathcal{A} = \Big\{\varphi \in \mathcal{F} : \exists\, h \in L^2(E,\alpha) \text{ such that } \forall\, \psi \in \mathcal{F}, \text{ we have } \mathcal{E}(\varphi,\psi) = -\int_E h\psi\mathrm{d}\alpha\Big\}$$

and in such a case $\mathcal{A}\varphi = h$.

The one-point compactification $E_\triangle = E \cup \{\triangle\}$ of E is endowed with the topology consisting of all the open sets of E and of all the sets of the form $K^c \cup \{\triangle\}$ with K compact in E, see page [11, page 69]. Observe that for a $E_\triangle$-valued sequence $(x_n)_{n\geq 0}$, we have $\lim_n x_n = x$ if and only if

- either $x \in E$, $x_n \in E$ for all n large enough, and $\lim_n x_n = x \in E$ in the usual sense;
- or $x = \triangle$ and for all compact subset K of E, there is $n_K \in \mathbb{N}$ such that for all $n \geq n_K$, $x_n \notin K$.

We say that our Hunt process is continuous if $t \to Y_t$ is continuous from $\mathbb{R}_+$ into $E_\triangle$, where $E_\triangle$ is endowed with the one-point compactification topology.

A Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ is said to be regular if it has a core, see [11, page 6], i.e. a subset $\mathcal{C} \subset C_c(E) \cap \mathcal{F}$ which is dense in $\mathcal{F}$ for the norm $||\varphi|| = [\int_E \varphi^2 \mathrm{d}\alpha + \mathcal{E}(\varphi, \varphi)]^{1/2}$ and dense in $C_c(E)$ for the uniform norm.

Observe two regular Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F}')$ such that $\mathcal{E}(\varphi, \varphi) = \mathcal{E}'(\varphi, \varphi)$ for all φ in a common core $\mathcal{C}$ are necessarily equal, i.e. $\mathcal{F} = \mathcal{F}'$ and $\mathcal{E} = \mathcal{E}'$. This follows from the fact that by definition, see [11, page 5], a Dirichlet space is closed.

We say that a Borel set A of E is $(P_t)_{t\geq 0}$-invariant if for all $\varphi \in L^2(E, \alpha)$, all $t > 0$ we have $P_t(\mathbb{I}_A \varphi) = \mathbb{I}_A P_t \varphi$ α-a.e, see [11, page 53]. According to [11, page 55], we say that $(\mathcal{E}, \mathcal{F})$ is irreducible if for all $(P_t)_{t\geq 0}$-invariant set A, we have either $\alpha(A) = 0$ or $\alpha(E \setminus A) = 0$.

We say that $(\mathcal{E}, \mathcal{F})$ is recurrent if for all nonnegative $\varphi \in L^1(E, \alpha)$, for α-a.e. $y \in E$, we have $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] \in \{0, \infty\}$, see [11, page 55].

We finally say that $(\mathcal{E}, \mathcal{F})$ is transient if for all nonnegative $\varphi \in L^1(E, \alpha)$, for α-a.e. $y \in E$, we have $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] < \infty$, with the convention that $\varphi(\triangle) = 0$, see [11, page 55].

By [11, Lemma 1.6.4 page 55], if $(\mathcal{E}, \mathcal{F})$ is irreducible, then it is either recurrent or transient.

A Borel set $\mathcal{N} \subset E$ is properly exceptional if $\alpha(\mathcal{N}) = 0$ and $\mathbb{P}_y(\exists t \geq 0 : Y_t \in \mathcal{N}) = 0$ for all $y \in E \setminus \mathcal{N}$, see [11, page 153]. A property is said to hold true for quasi all $y \in E$ if it holds true outside a properly exceptional set.

Remark B.1. *Two Hunt processes with the same Dirichlet space share the same quasi notion, up to the restriction that the capacity of every compact set is finite, which is always the case in the present work.*

Proof. We fix a Hunt process $\mathbb{Y}$ and explain why its quasi-notion depends only on its Dirichlet space. A set $\mathcal{N} \subset E$ is exceptional, see [11, page 152], if there exists a Borel set $\tilde{\mathcal{N}}$ such that $\mathcal{N} \subset \tilde{\mathcal{N}}$ and $\mathbb{P}_y(\exists t \geq 0 : Y_t \in \tilde{\mathcal{N}}) = 0$ for α-a.e. $y \in E$. A properly exceptional set is clearly exceptional and [11, Theorem 4.1.1 page 155] tells us that any exceptional set is included in a properly exceptional set. Thus, a property is true for quasi all $y \in E$ if and only if it holds true outside an exceptional set. Next, [11, Theorem 4.2.1-(ii) page 161] tells us that a set $\mathcal{N}$ is exceptional if and only if its capacity is 0, where the capacity of $\mathcal{N} \subset E$ is entirely defined from the Dirichlet space. And for [11, Theorem 4.2.1-(ii) page 161] to apply, one needs that the capacity of all compact sets is finite. □

B.2. **Toolbox.** We start with martingales.

Lemma B.2. *Let E be a locally compact separable metrizable space endowed with a Radon measure α such that Supp $\alpha = E$, and $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ a continuous α-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ and generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$. Assume that*

$\varphi : E \mapsto \mathbb{R}$ belongs to $\mathcal{D}_{\mathcal{A}}$ and that both φ and $\mathcal{A}\varphi$ are bounded. Define

$$M_t^\varphi = \varphi(Z_t) - \varphi(Z_0) - \int_0^t \mathcal{A}\varphi(Z_s)\mathrm{d}s,$$

with the convention that $\varphi(\triangle) = \mathcal{A}\varphi(\triangle) = 0$. For quasi all $z \in E$, $(M_t^\varphi)_{t\geq 0}$ is a $\mathbb{P}_z$-martingale in the canonical filtration of $(Z_t)_{t\geq 0}$.

This can be found in [11, page 332]. There the assumption on φ is that there is f bounded and measurable such that $\varphi = R_1 f$, i.e. $\varphi = (I - \mathcal{A})^{-1} f$, which simply means that $\varphi - \mathcal{A}\varphi$ is bounded. Also, the conclusion is that $(M_t^\varphi)_{t\geq 0}$ is a MAF, which indeed implies that $(M_t^\varphi)_{t\geq 0}$ is a martingale, see [11, page 243].

Next, we deal with time-changes.

Lemma B.3. *Let E be a C^∞-manifold, α a Radon measure on E such that $\mathrm{Supp}(\alpha) = E$, and $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ a continuous α-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ with core $C_c^\infty(E)$. We also fix $g : E \to (0, \infty)$ continuous and take the convention that $g(\triangle) = 0$. We consider the time-change $A_t = \int_0^t g(Z_s)\mathrm{d}s$ and its generalized inverse $\rho_t = \inf\{s > 0 : A_s > t\}$. We introduce $Y_t = Z_{\rho_t}\mathbb{1}_{\{\rho_t<\infty\}} + \triangle\mathbb{1}_{\{\rho_t=\infty\}}$. Then $(\Omega, \mathcal{M}, (Y_t)_{t\geq 0}, (\mathbb{P}_y)_{y\in E_\triangle})$ is a continuous $g\alpha$-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E}, \mathcal{F}')$ on $L^2(E, g\alpha)$ with core $C_c^\infty(E)$, i.e. $\mathcal{F}'$ is the closure of $C_c^\infty(E)$ with respect to the norm $[\int_E \varphi^2 g\mathrm{d}\alpha + \mathcal{E}(\varphi, \varphi)]^{1/2}$.*

Remark B.4. *If we apply the preceding result to the simple case where E is an open subset of $\mathbb{R}^d$ and where $\mathcal{E}(\varphi, \varphi) = \int_{\mathbb{R}^d} \|\nabla\varphi\|^2\mathrm{d}\alpha$ for all $\varphi \in C_c^\infty(E)$, then when $\mathcal{E}$ is seen as the Dirichlet form of a $g\alpha$-symmetric process, it may be better understood as $\mathcal{E}(\varphi, \varphi) = \int_{\mathbb{R}^d} \|g^{-1/2}\nabla\varphi\|^2 g\mathrm{d}\alpha$.*

This lemma is nothing but a particular case of [11, Theorem 6.2.1 page 316], see also the few pages before. We only have to check that the Revuz measure in our case is $g\alpha$, i.e., see [11, (5.1.13) page 229], that for all bounded nonnegative measurable functions φ, ψ on E, for all $t > 0$,

$$\int_E \mathbb{E}_x\Big[\int_0^t \varphi(Z_s)g(Z_s)\mathrm{d}s\Big]\psi(x)\alpha(\mathrm{d}x) = \int_0^t \int_E (P_s^Z\psi)\varphi g\mathrm{d}\alpha,$$

where P_t^Z is the semi-group of Z. The left hand side equals $\int_0^t \int_E P_s^Z(\varphi g)\psi\mathrm{d}\alpha$, so that the claim is obvious since Z is α-symmetric.

The following concatenation result can be found in Li-Ying [17, Proposition 3.2].

Lemma B.5. *Let E_V, E_W be two C^∞-manifolds, α_V, α_W be some Radon measures on E_V and E_W such that $\mathrm{Supp}(\alpha_V) = E_V$ and $\mathrm{Supp}(\alpha_W) = E_W$. Let $(\Omega^V, \mathcal{M}^V, (V_t)_{t\geq 0}, (\mathbb{P}_v^V)_{v\in E_V\cup\{\triangle\}})$ be a continuous α_V-symmetric $(E_V \cup \{\triangle\})$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^V, \mathcal{F}^V)$ on $L^2(E_V, \alpha_V)$ with core $C_c^\infty(E_V)$. Let also $(\Omega^W, \mathcal{M}^W, (W_t)_{t\geq 0}, (\mathbb{P}_w^W)_{w\in E_W\cup\{\triangle\}})$ be a continuous α_W-symmetric $(E_W \cup \{\triangle\})$-valued Hunt process with regular Dirichlet space $(\mathcal{E}^W, \mathcal{F}^W)$ on $L^2(E_W, \alpha_W)$ with core $C_c^\infty(E_W)$. Introduce the measure $\alpha = \alpha_V \otimes \alpha_W$ on $E = E_V \times E_W$. We take the convention that $(v, \triangle) = (\triangle, w) = (\triangle, \triangle) = \triangle$ for all $v \in E_V$, all $w \in E_W$. Moreover, we set $\mathcal{M}^{(V,W)} = \sigma(\{(V_t, W_t) : t \geq 0\})$ and we define $\mathbb{P}_{(v,w)}^{(V,W)} = \mathbb{P}_v^V \otimes \mathbb{P}_w^W$ if $(v, w) \in E_V \times E_W$ and $\mathbb{P}_\triangle^{(V,W)} = \mathbb{P}_\triangle^V \otimes \mathbb{P}_\triangle^W$. The process*

$$\Big(\Omega^V \times \Omega^W, \mathcal{M}^{(V,W)}, (V_t, W_t)_{t\geq 0}, (\mathbb{P}_{(v,w)}^{(V,W)})_{(v,w)\in(E_V\times E_W)\cup\{\triangle\}}\Big)$$

is a continuous $E_\triangle$-valued α-symmetric Hunt process, with regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(E,\alpha)$ with core $C_c^\infty(E)$ and, for $\varphi\in C_c^\infty(E)$,

$$\mathcal{E}(\varphi,\varphi)=\int_{E_V}\mathcal{E}^W(\varphi(v,\cdot),\varphi(v,\cdot))\alpha_V(\mathrm{d}v)+\int_{E_W}\mathcal{E}^V(\varphi(\cdot,w),\varphi(\cdot,w))\alpha_W(\mathrm{d}w).$$

Observe that $\mathcal{M}^{(V,W)}$ may be strictly smaller than $\mathcal{M}^V\otimes\mathcal{M}^W$ due to the identification of all the cemetery points. Also, it actually holds true that $\mathbb{P}^V_\triangle\otimes\mathbb{P}^W_w=\mathbb{P}^V_v\otimes\mathbb{P}^W_\triangle=\mathbb{P}^V_\triangle\otimes\mathbb{P}^W_\triangle$ on $\mathcal{M}^{(V,W)}$ so that the choice $\mathbb{P}^{(V,W)}_\triangle=\mathbb{P}^V_\triangle\otimes\mathbb{P}^W_\triangle$ is arbitrary but legitimate.

The following killing result is a summary, adapted to our context, of Theorems 4.4.2 page 173 and 4.4.3-(i) page 174 in [11, Section 4.4].

Lemma B.6. *Let E be a C^∞-manifold, let α be a Radon measure on E such that $\mathrm{Supp}(\alpha)=E$, and let $(\Omega,\mathcal{M},(Z_t)_{t\geq0},(\mathbb{P}_z)_{z\in E_\triangle})$ be a continuous α-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(E,\alpha)$ with core $C_c^\infty(E)$. Let O be an open subset of E and consider $\tau_O=\inf\{t\geq0:X_t\notin O\}$, with the convention that $\inf\emptyset=\infty$. Then, setting*

$$Z_t^O=Z_t\mathbb{1}_{\{t<\tau_O\}}+\triangle\mathbb{1}_{\{t\geq\tau_O\}},$$

$(\Omega,\mathcal{M},(Z_t^O)_{t\geq0},(\mathbb{P}_z)_{z\in O\cup\{\triangle\}})$ is a continuous $\alpha|_O$-symmetric $O\cup\{\triangle\}$-valued Hunt process with regular Dirichlet space $(\mathcal{E}_O,\mathcal{F}_O)$ on $L^2(O,\alpha|_O)$ with core $C_c^\infty(O)$ and for $\varphi\in\mathcal{F}_O$,

$$\mathcal{E}_O(\varphi,\varphi)=\mathcal{E}(\varphi,\varphi).$$

Note that since O is an open subset of the manifold E and since the Hunt process is continuous, the regularity condition (4.4.6) of [11, Theorem 4.4.2 page 173] is obviously satisfied.

We finally give an adaptation of the Girsanov theorem in the context of Dirichlet spaces, which is a particular case of Chen-Zhang [5, Theorem 3.4].

Lemma B.7. *Let E be an open subset of $\mathbb{R}^d$, with $d\geq1$, α be a Radon measure on E such that* $\mathrm{Supp}(\alpha)=E$ *and $(\Omega,\mathcal{M},(Z_t)_{t\geq0},(\mathbb{P}_z)_{z\in E_\triangle})$ be a continuous α-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(E,\alpha)$ with core $C_c^\infty(E)$ such that for all $\varphi\in C_c^\infty(E)$,*

$$\mathcal{E}(\varphi,\varphi)=\int_E\|\nabla\varphi\|^2\mathrm{d}\alpha.$$

Let $(\mathcal{A},\mathcal{D}_\mathcal{A})$ stand for its generator. Let $u\in\mathcal{F}$ be bounded, such that $\varrho=e^u\in\mathcal{D}_\mathcal{A}$ and such that $\mathcal{A}\varrho$ is bounded. Set

$$L_t^\varrho=\frac{\varrho(Z_t)}{\varrho(Z_0)}\exp\Big(-\int_0^t\frac{\mathcal{A}\varrho(Z_s)}{\varrho(Z_s)}\mathrm{d}s\Big),$$

with the conventions that $\varrho(\triangle)=1$ and $\mathcal{A}\varrho(\triangle)=0$.

Assume that ϱ is continuous on $E_\triangle$. Then for quasi all $z\in E$, $(L_t^\varrho)_{t\geq0}$ is a bounded $(\mathcal{M}_t)_{t\geq0}$-martingale under $\mathbb{P}_z$, where we have set $\mathcal{M}_t=\sigma(\{Z_s:s\in[0,t]\})$, and there exists a probability measure $\tilde{\mathbb{P}}_z$ on $(\Omega,\mathcal{M})$, such that for all $t>0$, $\tilde{\mathbb{P}}_z=L_t^\varrho\cdot\mathbb{P}_z$ on $\mathcal{M}_t$.

Moreover, $(\Omega,\mathcal{M},(Z_t)_{t\geq0},(\tilde{\mathbb{P}}_z)_{z\in E_\triangle})$ is a continuous $\varrho^2\alpha$-symmetric $E_\triangle$-valued Hunt process with regular Dirichlet space $(\tilde{\mathcal{E}},\mathcal{F})$ on $L^2(E,\varrho^2\alpha)$ such that for all $\varphi\in\mathcal{F}$,

$$\tilde{\mathcal{E}}(\varphi,\varphi)=\frac{1}{2}\int_E\|\nabla\varphi\|^2\varrho^2\mathrm{d}\alpha.$$

Actually, they speak of *right processes* in [5], but this is not an issue since we only consider continuous Hunt processes. Also, they assume that L^ϱ is bounded from above and from below by some deterministic constants, on each compact time interval, but this is obvious under our assumptions on u and $\mathcal{A}\varrho$. Finally, their expression of L^ϱ is different, see [5, pages 485-486]: first, they define M_t^ϱ as the martingale part of $\varrho(X_t)$. By Lemma B.2, we see that $M_t^\varrho = \varrho(Z_t) - \varrho(Z_0) - \int_0^t \mathcal{A}\varrho(Z_s)\mathrm{d}s$. Then they put $M_t = \int_0^t [\varrho(Z_s)]^{-1}\mathrm{d}M_s^\varrho$ and $L_t^\varrho = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$. But by Itô's formula, $\log\varrho(Z_t) = \log\varrho(Z_0) + \int_0^t[\varrho(Z_s)]^{-1}\mathrm{d}M_s^\varrho + \int_0^t[\varrho(Z_s)]^{-1}\mathcal{A}\varrho(Z_s)\mathrm{d}s - \frac{1}{2}\int_0^t[\varrho(Z_s)]^{-2}\mathrm{d}\langle M^\varrho\rangle_s$, whence $\log\varrho(Z_t) = \log\varrho(Z_0) + M_t + \int_0^t[\varrho(Z_s)]^{-1}\mathcal{A}\varrho(Z_s)\mathrm{d}s - \frac{1}{2}\langle M\rangle_t$, so that $L_t^\varrho = \exp(M_t - \frac{1}{2}\langle M\rangle_t) = [\varrho(Z_0)]^{-1}\varrho(Z_t)\exp(-\int_0^t[\varrho(Z_s)]^{-1}\mathcal{A}\varrho(Z_s)\mathrm{d}s)$ as desired.

References

[1] G. Arumugam, J. Tyagi, Keller-Segel Chemotaxis Models: A Review, *Acta Appl. Math.* **171** (2021), 6.

[2] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, *Electron. J. Differential Equations* **44** (2006), 32 pp.

[3] D. Bresch, P.E. Jabin, Z. Wang On mean-field limits and quantitative estimates with a large class of singular kernels: application to the Patlak-Keller-Segel model, *C. R. Math. Acad. Sci. Paris* **357** (2019), 708–720.

[4] P. Cattiaux, L. Pédèches, The 2-D stochastic Keller-Segel particle model: existence and uniqueness, *ALEA, Lat. Am. J. Probab. Math. Stat.* **13** (2016), 447–463.

[5] Z. Chen, T.S. Zhang, Girsanov and Feynman-Kac type transformations for symmetric Markov processes. *Ann. Inst. H. Poincaré Probab. Statist.* **38** (2002), 475–505.

[6] J. Dolbeault, C. Schmeiser, The two-dimensional Keller-Segel model after blow-up, *Discrete Contin. Dyn. Syst.* **25** (2009), 109–121.

[7] I. Fatkullin, A study of blow-ups in the Keller-Segel model of chemotaxis, *Nonlinearity* **26** (2013), 81–94.

[8] N. Fournier, B. Jourdain, Stochastic particle approximation of the Keller-Segel equation and two-dimensional generalization of Bessel processes, *Ann. Appl. Probab.* **27** (2017), 2807–2861.

[9] N. Fournier, M. Hauray, S. Mischler, Propagation of chaos for the 2D viscous vortex model, *J. Eur. Math. Soc.* **16** (2014), 1423–1466.

[10] M. Fukushima, From one dimensional diffusions to symmetric Markov processes, *Stochastic Process. Appl.* **120** (2010), 590–604.

[11] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet forms and symmetric Markov processes. Second revised and extended edition. Walter de Gruyter, 2011.

[12] J. Haškovec, C. Schmeiser, Stochastic particle approximation for measure valued solutions of the 2D Keller-Segel system, *J. Stat. Phys.* **135**, 133–151.

[13] J. Haškovec, C. Schmeiser, Convergence of a stochastic particle approximation for measure solutions of the 2D Keller-Segel system, *Comm. Partial Differential Equations* **36(6)** (2011), 940–960.

[14] J.F. Jabir, D. Talay, M. Tomasevic, Mean-field limit of a particle approximation of the one-dimensional parabolic-parabolic Keller-Segel model without smoothing, *Electron. Commun. Probab.* **23** (2018), Paper No. 84.

[15] I. Karatzas, S.E. Shreve, Brownian motion and stochastic calculus. Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1991.

[16] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26** (1970), 399–415.

[17] L. Li,, J. Ying, Regular subspaces of Dirichlet forms. Festschrift Masatoshi Fukushima, 397–420, Interdiscip. Math. Sci., 17, World Sci. Publ., Hackensack, NJ, 2015.

[18] C. Olivera, A. Richard, M. Tomasevic, Particle approximation of the 2-d parabolic-elliptic Keller-Segel system in the subcritical regime, arXiv:2011.00537.

[19] H. Osada, Propagation of chaos for the two-dimensional Navier-Stokes equation, *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), 8–11.

[20] C.S. Patlak, Random walk with persistence and external bias, *Bull. Math. Biophys.* **15** (1953), 311–338.

[21] D. Revuz, M. Yor, Continuous martingales and Brownian motion. Third Edition. Springer, 2005.

[22] A. Stevens, The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems, *SIAM J. Appl. Math.* **61** (2000), 183–212.

[23] T. Suzuki Exclusion of boundary blowup for 2D chemotaxis system provided with Dirichlet boundary condition for the Poisson part, *J. Math. Pures Appl.* **100** (2013), 347–367.
[24] J.J.L. Velazquez, Point dynamics in a singular limit of the Keller-Segel model. I. Motion of the concentration regions, *SIAM J. Appl. Math.*, **64** (2004), 1198–1223.
[25] J.J.L. Velazquez, Point dynamics in a singular limit of the Keller-Segel model. II. Formation of the concentration regions, *SIAM J. Appl. Math.*, **64** (2004), 1224–1248.

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