A SIMPLE PROOF OF NON-EXPLOSION FOR MEASURE SOLUTIONS OF THE KELLER-SEGEL EQUATION

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Abstract. We give a simple proof, relying on a *two-particles* moment computation, that there exists a global weak solution to the 2-dimensional parabolic-elliptic Keller-Segel equation when starting from any initial measure f_0 such that $f_0(\mathbb{R}^2) < 8\pi$.

1. Introduction

1.1. **The model.** We consider the classical parabolic-elliptic Keller-Segel model, also called Patlak-Keller-Segel, of chemotaxis in $\mathbb{R}^2$, which writes

$$\partial_t f + \nabla \cdot (f \nabla c) = \Delta f \qquad \text{and} \qquad \Delta c + f = 0. \tag{1}$$

The unknown (f, c) is composed of two nonnegative functions $f_t(x)$ and $c_t(x)$ of $t \geq 0$ and $x \in \mathbb{R}^2$, and the initial condition f_0 is given.

This equation models the collective motion of a population of bacteria which emit a chemical substance that attracts them. The quantity $f_t(x)$ represents the density of bacteria at position $x \in \mathbb{R}^2$ at time $t \geq 0$, while $c_t(x)$ represents the concentration of chemical substance at position $x \in \mathbb{R}^2$ at time $t \geq 0$. Note that in this model, the speed of diffusion of the chemo-attractant is supposed to be infinite. This equation has been introduced by Keller and Segel [11], see also Patlak [13]. We refer to the recent book of Biler [3] and to the review paper of Arumugam and Tyagi [1] for some complete descriptions of what is known about this model.

We classically observe, see e.g. Blanchet-Dolbeault-Perthame [7, Page 4], that necessarily $\nabla c_t = K * f_t$ for each $t \geq 0$, where

$$K(x) = -\frac{x}{2\pi \|x\|^2} \quad \text{for } x \in \mathbb{R}^2 \setminus \{0\} \quad \text{and (arbitrarily)} \quad K(0) = 0.$$

Hence (1) may be rewritten as

$$\partial_t f + \nabla \cdot [f \ (K * f)] = \Delta f. \tag{2}$$

1.2. **Weak solutions.** We will deal with weak measure solutions. For each $M > 0$, we set

$$\mathcal{M}_M(\mathbb{R}^2) = \Big\{ \mu \text{ nonnegative measure on } \mathbb{R}^2 \text{ such that } \mu(\mathbb{R}^2) = M \Big\}$$

and we endow $\mathcal{M}_M(\mathbb{R}^2)$ with the weak convergence topology, i.e. taking $C_b(\mathbb{R}^d)$, the set of continuous and bounded functions, as set of test functions. We also denote by $C_b^2(\mathbb{R}^d)$ the set of C^2-functions, bounded together with all their derivatives. The following notion of weak solutions is classical, see e.g. Blanchet-Dolbeault-Perthame [7, Page 5].

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Definition 1. *Fix $M > 0$. We say that $f \in C([0,\infty), \mathcal{M}_M(\mathbb{R}^2))$ is a weak solution of (2) if for all $\varphi \in C_b^2(\mathbb{R}^2)$, all $t \geq 0$,*

$$\int_{\mathbb{R}^2} \varphi(x) f_t(\mathrm{d}x) = \int_{\mathbb{R}^2} \varphi(x) f_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) f_s(\mathrm{d}x)\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot(\nabla\varphi(x) - \nabla\varphi(y)) f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s.$$

All the terms in this equality are well-defined. In particular concerning the last term, it holds that $|K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))| \leq \|\nabla^2\varphi\|_\infty/2\pi$. However, $K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))$, which equals 0 when $x = y$ because we (arbitrarily) imposed that $K(0) = 0$, is not continuous near $x = y$. Hence a *good* weak solution has to verify that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(\mathrm{d}x) f_s(\mathrm{d}y) = 0$ for a.e. $s \geq 0$.

1.3. **Main result.** Our goal is to give a simple proof of the following global existence result.

Theorem 2. *Fix $M \in (0, 8\pi)$ and assume that $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. There exists a global weak solution f to (2) with initial condition f_0. Moreover, for all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_{M,\gamma} > 0$ depending only on M and γ such that for all $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s \leq A_{M,\gamma}(1+T). \tag{3}$$

These solutions indeed satisfy that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(\mathrm{d}x) f_s(\mathrm{d}y) = 0$ for a.e. $s \geq 0$.

1.4. **References.** Let us immediately mention that a stronger result is already known: gathering the results of Bedrossian-Masmoudi [2] and Wei [15], there exists a unique global *mild* solution for any $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ with $M < 8\pi$ and for any $f_0 \in \mathcal{M}_{8\pi}(\mathbb{R}^2)$ satisfying $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$. The goal of the present paper is to provide a simple and robust non explosion proof, even if the solution we build is weaker. We could also treat the critical case with more work.

This model was first introduced by Patlak [13] and Keller-Segel [11], as a model for chemotaxis. For an exhaustive summary of the knowledge about this equation and related models, we refer the reader to the review paper Arumugam-Tyagi [1] and to the book of Biler [3]. The main difficulty of this model lies in the tight competition between diffusion and attraction. Therefore it is not clear that a solution exists because a blow-up could occur due to the emergence of a cluster, i.e. a Dirac mass. Thus, the whole problem is about determining if the solution ends-up by being concentrated in finite time or not.

As shown in Jäger-Luckhaus [10], this depends on the initial mass of the solution $M = \int_{\mathbb{R}^2} f_0(\mathrm{d}x)$, the solution globally exists if M is small enough and explodes in the other case. The fact that solutions must explode in finite time if $M > 8\pi$ is rather easy to show. But the fact that 8π is indeed the correct threshold was much more difficult.

Biler-Karch-Laurençot-Nadzieja [4, 5] proved the global existence of a weak solution in the subcritical case for every initial data which is a radially symmetric measure such that $f_0(\{0\}) = 0$ and $f_0(\mathbb{R}^2) = M \leq 8\pi$, with a few other anodyne technical conditions.

At the same time, Blanchet-Dolbeault-Perthame [7] proved the existence of a global weak *free energy* solution for initial data $f_0 \in L^1_+(\mathbb{R}^2)$ with mass $M < 8\pi$, a finite moment of order 2 and a finite entropy. The core of the argument lies in the use of the logarithmic Hardy-Littlewood-Sobolev inequality applied on a well chosen free-energy quantity. Something noticeable is that the authors use this inequality with its optimal constant to get the correct thresold 8π.

Senba-Suzuki [14] showed the *local in time* existence of weak solutions when starting from any nonnegative measure f_0 on $\mathbb{R}^2$ such that $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$. They also showed that this condition is sharp. A simpler proof of the same result was elaborated by Biler-Zienkiewicz [6]. Under the same condition, Bedrossian-Masmoudi [2] have shown the (local) existence and uniqueness of a *mild* solution, which is stronger. The proofs of [2] are conceptually and technically more complicated. Combining the results of [14, 6, 2] with the regularization effect for each $t > 0$ (see e.g. [6]) and the classical result of Blanchet-Dolbeault-Perthame [7], the global existence follows when $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$, with $M < 8\pi$, has a moment of order 2. Actually, using the result of Wei [15], of which the proof is quite involved, instead of the one of [7], this moment condition can be relaxed and the critical case can be treated: one gets from [2, 15] the existence of a unique global *mild* solution for any $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ with $M < 8\pi$ and for any $f_0 \in \mathcal{M}_{8\pi}(\mathbb{R}^2)$ such that $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$.

Let us finally mention [9], where global weak solutions were built for any measure initial condition f_0 such that $f_0(\mathbb{R}^2) < 2\pi$, with a light additional moment condition. This work was inspired by the work of Osada [12] on vortices. The present paper consists in refining this approach, and surprisingly, this allows us to treat the whole subcritical case.

1.5. **Motivation.** Our main goal is to present a simple proof of non explosion. This proof relies on a two-particles moment computation: roughly, we show that for $\gamma \in (0, 2)$ and for $(f_t)_{t\geq 0}$ a solution to (2), it *a priori* holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(||x-y||^\gamma \wedge 1\Big) f_t(\mathrm{d}x)f_t(\mathrm{d}y) \geq c_{\gamma,M}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} ||x-y||^{\gamma-2}\mathbb{1}_{\{||x-y||\leq 1\}} f_t(\mathrm{d}x)f_t(\mathrm{d}y),$$

with $c_{\gamma,M} > 0$ as soon as $\gamma \in (4M/\pi, 2)$. By integration, this implies (3) and such an *a priori* estimate is sufficient to build a global solution.

This computation seems simple and robust. Although they build a more regular weak solution, Blanchet-Dolbeault-Perthame [7] use some optimal Hardy-Littlewood-Sobolev inequality. Moreover, they have some little restrictions on the initial conditions (finite entropy and moment of order 2). The proof of Bedrossian-Masmoudi [2] and Wei [15] is much longer and relies on a fine study of what happens near each possible atom of the initial condition. Let us say again that they build a much stronger solution.

In particular, due to its robustness, we hope to be able to apply such a method to study the convergence of the empirical measure of some stochastic particle system, as the number of particles tends to infinity, to the solution of (2). To establish such a convergence, one needs to show the non explosion of the particle system, uniformly in N in some sense. It seems that the present method works very well and we hope to be able to treat the whole subcritical case $M \in (0, 8\pi)$ and even the critical case $M = 8\pi$, with measure initial conditions. To our knowledge, the best available result concerning particles is that of Bresch-Jabin-Wang [8] and concerns the subcritical case $M \in (0, 8\pi)$, with regular initial conditions and when $\mathbb{R}^2$ is replaced by the torus.

We do not treat the criticial case $M = 8\pi$ in the present paper for the sake of conciseness.

2. Proof

We fix $M > 0$ and $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. For $\varepsilon \in (0, 1]$, we introduce the following regularized versions

$$K_\varepsilon(x) = -\frac{x}{2\pi(\|x\|^2+\varepsilon)} \quad \text{and} \quad f_0^\varepsilon(x) = \frac{1}{2\pi\varepsilon}\int_{\mathbb{R}^2} e^{-\|x-y\|^2/(2\varepsilon)} f_0(\mathrm{d}y).$$

of K and f_0. Since K_ε and f_0^ε are smooth, the equation

$$\partial_t f^\varepsilon + \nabla \cdot [f^\varepsilon (K_\varepsilon * f^\varepsilon)] = \Delta f^\varepsilon \tag{4}$$

starting from f_0^ε has a unique classical solution $(f_t^\varepsilon(x))_{t\geq 0, x\in\mathbb{R}^2}$. This solution preserves mass, i.e

$$\int_{\mathbb{R}^2} f_t^\varepsilon(\mathrm{d}x) = \int_{\mathbb{R}^2} f_0^\varepsilon(\mathrm{d}x) = \int_{\mathbb{R}^2} f_0(\mathrm{d}x) = M \qquad \text{for all } t\geq 0, \tag{5}$$

where we write $f_t^\varepsilon(\mathrm{d}x) = f_t^\varepsilon(x)\mathrm{d}x$. Multiplying (4) by $\varphi \in C_b^2(\mathbb{R}^2)$, integrating on $[0,t]\times\mathbb{R}^2$, proceeding to some integrations by parts and using a symmetry argument, we classically find that

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x) f_t^\varepsilon(\mathrm{d}x) = & \int_{\mathbb{R}^2} \varphi(x) f_0^\varepsilon(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) f_s^\varepsilon(\mathrm{d}x)\mathrm{d}s \\ & + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_\varepsilon(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)] f_s^\varepsilon(\mathrm{d}x) f_s^\varepsilon(\mathrm{d}y)\mathrm{d}s.\end{aligned} \tag{6}$$

We now prove some compactness result.

Proposition 3. *Fix $M>0$, $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ and consider the corresponding family $(f^\varepsilon)_{\varepsilon\in(0,1]}$. The family $(f^\varepsilon)_{\varepsilon\in(0,1]}$ is relatively compact in $C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology.*

Proof. We first prove that for each $t\geq 0$, the family $(f_t^\varepsilon)_{\varepsilon\in(0,1]}$ is tight in $\mathcal{P}(\mathbb{R}^2)$. Since the family $(f_0^\varepsilon)_{\varepsilon\in(0,1]}$ is clearly tight, by the de la Vallée Poussin theorem, there exists $\psi:\mathbb{R}^2\to\mathbb{R}_+$ such that $\lim_{|x|\to\infty}\psi(x)=\infty$ and $A=\sup_{\varepsilon\in(0,1]}\int_{\mathbb{R}^2}\psi(x)f_0^\varepsilon(\mathrm{d}x)<\infty$. Moreover, we can choose ψ smooth and such that $||\nabla^2\psi||$ is bounded by some constant C. It then immediately follows from (6), since $\|z\|\,\|K_\varepsilon(z)\|\leq 1/(2\pi)$, that for all $\varepsilon\in(0,1]$, all $t\geq 0$,

$$\int_{\mathbb{R}^2}\psi(x)f_t^\varepsilon(\mathrm{d}x)\leq \int_{\mathbb{R}^2}\psi(x)f_0^\varepsilon(\mathrm{d}x) + C\Big(M+\frac{M^2}{4\pi}\Big)t \leq A + C\Big(M+\frac{M^2}{4\pi}\Big)t.$$

As $\lim_{|x|\to\infty}\psi(x)=\infty$, we conclude that indeed, $(f_t^\varepsilon)_{\varepsilon\in(0,1]}$ is tight for each $t\geq 0$.

By the Arzelà-Ascoli theorem, it is enough to prove that f^ε is uniformly Lipschitz continuous in time, in that there exists a constant $C>0$ such that for all $\varepsilon\in(0,1]$, all $t\geq s\geq 0$, $\delta(f_t^\varepsilon,f_s^\varepsilon)\leq C|t-s|$, where δ metrizes the weak convergence topology on $\mathcal{M}_M(\mathbb{R}^2)$. As is well-known, we may find a family $(\varphi_n)_{n\geq 0}$ of elements of $C_b^2(\mathbb{R}^2)$ satisfying

$$\|\varphi_n\|_\infty + \|\nabla\varphi_n\|_\infty + \|\nabla^2\varphi_n\|_\infty \leq 1 \qquad \text{for all } n\geq 0$$

and such that the distance δ on $\mathcal{M}_M(\mathbb{R}^2)$ defined through

$$\delta(f,g)=\sum_{n\geq 0}2^{-n}\Big|\int_{\mathbb{R}^2}\varphi_n(x)f(\mathrm{d}x)-\int_{\mathbb{R}^2}\varphi_n(x)g(\mathrm{d}x)\Big|$$

is suitable. But using (6), for all $n\geq 0$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\varphi_n(x)f_t^\varepsilon(\mathrm{d}x)-\int_{\mathbb{R}^2}\varphi_n(x)f_s^\varepsilon(\mathrm{d}x)\Big| \\ =&\Big|\int_s^t\int_{\mathbb{R}^2}\Delta\varphi_n(x)f_u^\varepsilon(\mathrm{d}x)\mathrm{d}u+\frac{1}{2}\int_s^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K_\varepsilon(x-y)\cdot[\nabla\varphi_n(x)-\nabla\varphi_n(y)]f_u^\varepsilon(\mathrm{d}x)f_u^\varepsilon(\mathrm{d}y)\mathrm{d}u\Big| \\ \leq&(M+M^2/(4\pi))(t-s),\end{aligned}$$

by (5), since $\|\nabla^2\varphi_n\|_\infty \le 1$ and since $\|z\|\,\|K_\varepsilon(z)\| \le 1/(2\pi)$. We conclude that

$$\delta(f_t^\varepsilon, f_s^\varepsilon) \le \sum_{n\ge 0} 2^{-n}(M + M^2/(4\pi))(t-s) = 2(M + M^2/(4\pi))(t-s)$$

as desired. □

The following simple geometrical observation is crucial for our purpose.

Lemma 4. *For all pair of nonincreasing functions $\varphi, \psi : (0,\infty) \to (0,\infty)$, for all $X, Y, Z \in \mathbb{R}^2$ such that $X + Y + Z = 0$, we have*

$$\Delta = [\varphi(\|X\|)X + \varphi(\|Y\|)Y + \varphi(\|Z\|)Z] \cdot [\psi(\|X\|)X + \psi(\|Y\|)Y + \psi(\|Z\|)Z] \ge 0.$$

Proof. We may study only the case where $\|X\| \le \|Y\| \le \|Z\|$. Since $Y = -X - Z$,

$$\varphi(\|X\|)X + \varphi(\|Y\|)Y + \varphi(\|Z\|)Z = \lambda X - \mu Z,$$
$$\psi(\|X\|)X + \psi(\|Y\|)Y + \psi(\|Z\|)Z = \lambda' X - \mu' Z,$$

where $\lambda = \varphi(\|X\|) - \varphi(\|Y\|) \ge 0$, $\mu = \varphi(\|Y\|) - \varphi(\|Z\|) \ge 0$, $\lambda' = \psi(\|X\|) - \psi(\|Y\|) \ge 0$ and $\mu' = \psi(\|Y\|) - \psi(\|Z\|) \ge 0$. Therefore,

$$\Delta = \lambda\lambda'\|X\|^2 + \mu\mu'\|Z\|^2 - (\lambda\mu' + \lambda'\mu)X \cdot Z \ge 0$$

as desired, because $X \cdot Z \le 0$. Indeed, if $X \cdot Z > 0$, then $\|Y\|^2 = \|Z+X\|^2 = \|Z\|^2 + \|X\|^2 + 2X\cdot Z > \|Z\|^2 \ge \|Y\|^2$, which is absurd. □

The following computation is the core of the paper.

Proposition 5. *Recall that $M \in (0, 8\pi)$, that $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. For all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_{M,\gamma} > 0$ depending only on M and γ such that for all $\varepsilon \in (0,1]$, all $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s^\varepsilon(\mathrm{d}x) f_s^\varepsilon(\mathrm{d}y)\mathrm{d}s \le A_{M,\gamma}(1+T).$$

Proof. For any smooth $\psi : (\mathbb{R}^2)^2 \to \mathbb{R}$ such that $\psi(x,y) = \psi(y,x)$, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(x,y) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) =& 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(x,y)[\Delta f_t^\varepsilon(x) - \nabla\cdot(f_t^\varepsilon(x)(K_\varepsilon * f_t^\varepsilon))(x)] f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y \\ =& 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\Delta_x\psi(x,y) + (K_\varepsilon * f_t^\varepsilon)(x)\cdot\nabla_x\psi(x,y)] f_t^\varepsilon(x) f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We fix $\gamma \in (M/(4\pi), 2)$, introduce $\varphi(r) = r^{\gamma/2}/(1+r^{\gamma/2})$, and set $\psi(x,y) = \varphi(||x-y||^2)$. We have

$$\varphi'(r) = \frac{\gamma}{2}\frac{r^{\gamma/2-1}}{(1+r^{\gamma/2})^2} \qquad \text{and} \qquad \varphi''(r) = \frac{\gamma}{2}\frac{r^{\gamma/2-2}}{(1+r^{\gamma/2})^2}\Big(\frac{\gamma}{2} - 1 - \frac{\gamma r^{\gamma/2}}{1+r^{\gamma/2}}\Big)$$

and

$$\nabla_x\psi(x,y) = 2\varphi'(||x-y||^2)(x-y) = \gamma\frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}(x-y),$$

$$\Delta_x\psi(x,y) = 4\varphi'(||x-y||^2) + 4||x-y||^2\varphi''(||x-y||^2) = \gamma^2\frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}\Big(1 - 2\frac{||x-y||^\gamma}{1+||x-y||^\gamma}\Big).$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(||x-y||^2) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) = J_t^\varepsilon + S_t^\varepsilon, \tag{7}$$

where

$$J_t^\varepsilon = 2\gamma^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}\Big(1 - 2\frac{||x-y||^\gamma}{1+||x-y||^\gamma}\Big) f_t^\varepsilon(x) f_t^\varepsilon(y) \mathrm{d}x\mathrm{d}y,$$

$$S_t^\varepsilon = 2\gamma \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}(x-y)\cdot K_\varepsilon(x-z) f_t^\varepsilon(x) f_t^\varepsilon(y) f_t^\varepsilon(z) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

First, we have

$$J_t^\varepsilon \geq \gamma(\gamma + M/(4\pi)) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} f_t^\varepsilon(x) f_t^\varepsilon(y) \mathrm{d}x\mathrm{d}y - M^2 C_{M,\gamma}, \tag{8}$$

where $C_{M,\gamma} > 0$ is a constant such that for all $a > 0$, recall that $\gamma > M/(4\pi)$,

$$2\gamma^2 \frac{a^{\gamma-2}}{(1+a^\gamma)^2}\Big(1 - 2\frac{a^\gamma}{1+a^\gamma}\Big) \geq 2\gamma\Big(\frac{\gamma + M/(4\pi)}{2}\Big)\frac{a^{\gamma-2}}{(1+a^\gamma)^2} - C_{M,\gamma}.$$

Next, by symmetrization, we have

$$\begin{aligned} S_t^\varepsilon =& \gamma \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}(x-y)\cdot (K_\varepsilon(x-z) - K_\varepsilon(y-z)) f_t^\varepsilon(x) f_t^\varepsilon(y) f_t^\varepsilon(z) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ =& \frac{\gamma}{3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F_\varepsilon(x,y,z) f_t^\varepsilon(x) f_t^\varepsilon(y) f_t^\varepsilon(z) \mathrm{d}x\mathrm{d}y\mathrm{d}z, \end{aligned} \tag{9}$$

where

$$\begin{aligned} F_\varepsilon(x,y,z) =& [K_\varepsilon(x-z) - K_\varepsilon(y-z)]\cdot(x-y)\frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} \\ &+ [K_\varepsilon(y-x) - K_\varepsilon(z-x)]\cdot(y-z)\frac{||y-z||^{\gamma-2}}{(1+||y-z||^\gamma)^2} \\ &+ [K_\varepsilon(z-y) - K_\varepsilon(x-y)]\cdot(z-x)\frac{||z-x||^{\gamma-2}}{(1+||z-x||^\gamma)^2}. \end{aligned}$$

Introducing $X = x-y$, $Y = y-z$ and $Z = z-x$ and recalling that $K_\varepsilon(X) = \frac{-X}{2\pi(\|X\|^2+\varepsilon)}$, we find

$$\begin{aligned} 2\pi F_\varepsilon(x,y,z) =& \Big[\frac{Z}{\|Z\|^2+\varepsilon} + \frac{Y}{\|Y\|^2+\varepsilon}\Big]\cdot X\frac{||X||^{\gamma-2}}{(1+||X||^\gamma)^2} \\ &+ \Big[\frac{X}{\|X\|^2+\varepsilon} + \frac{Z}{\|Z\|^2+\varepsilon}\Big]\cdot Y\frac{||Y||^{\gamma-2}}{(1+||Y||^\gamma)^2} \\ &+ \Big[\frac{Y}{\|Y\|^2+\varepsilon} + \frac{X}{\|X\|^2+\varepsilon}\Big]\cdot Z\frac{||Z||^{\gamma-2}}{(1+||Z||^\gamma)^2}. \end{aligned}$$

We now introduce

$$\begin{aligned} G(x,y,z) =& \frac{\|X\|^{\gamma-2}}{(1+\|X\|^\gamma)^2} + \frac{\|Y\|^{\gamma-2}}{(1+\|Y\|^\gamma)^2} + \frac{\|Z\|^{\gamma-2}}{(1+\|Z\|^\gamma)^2} \\ \geq& X\cdot\frac{X}{\|X\|^2+\varepsilon}\frac{\|X\|^{\gamma-2}}{(1+\|X\|^\gamma)^2} + Y\cdot\frac{Y}{\|Y\|^2+\varepsilon}\frac{\|Y\|^{\gamma-2}}{(1+\|Y\|^\gamma)^2} + Z\cdot\frac{Z}{\|Z\|^2+\varepsilon}\frac{\|Z\|^{\gamma-2}}{(1+\|Z\|^\gamma)^2}. \end{aligned}$$

Hence $G(x,y,z) + 2\pi F_\varepsilon(x,y,z)$ is larger than

$$\Big(\frac{X}{\|X\|^2+\varepsilon} + \frac{Y}{\|Y\|^2+\varepsilon} + \frac{Z}{\|Z\|^2+\varepsilon}\Big)\cdot\Big(X\frac{||X||^{\gamma-2}}{(1+||X||^\gamma)^2} + Y\frac{||Y||^{\gamma-2}}{(1+||Y||^\gamma)^2} + Z\frac{||Z||^{\gamma-2}}{(1+||Z||^\gamma)^2}\Big),$$

which is nonnegative according to Lemma 4, since $r \to 1/(r^2+\varepsilon)$ and $r \to r^{\gamma-2}/(1+r^\gamma)^2$ are both nonincreasing on $(0,\infty)$ and since $X+Y+Z=0$. Thus $F_\varepsilon(x,y,z) \geq -G(x,y,z)/(2\pi)$. Recalling (9), we conclude that

$$S_t^\varepsilon \geq -\frac{\gamma}{6\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G(x,y,z) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) f_t^\varepsilon(\mathrm{d}z)$$
$$= -\frac{\gamma M}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2} f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) \tag{10}$$

by symmetry again.

Gathering (7)-(8)-(10), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(||x-y||^2) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) \geq \gamma\Big(\gamma - \frac{M}{4\pi}\Big)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2} f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) - M^2 C_{M,\gamma}.$$

Integrating on $[0,T]$, using that $\gamma > M/(4\pi)$ and and that φ is $[0,1]$-valued, we end with

$$\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2} f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y)\mathrm{d}t \leq \frac{M^2 + M^2 C_{M,\gamma} T}{\gamma(\gamma - M/(4\pi))}.$$

One easily completes the proof, using that there is $D_\gamma > 0$ such that $a^{\gamma-2} \leq 2a^{\gamma-2}/(1+a^\gamma)^2 + D_\gamma$ for all $a > 0$ ☐

We finally give the

Proof of Theorem 2. Recall that $M \in (0,8\pi)$, that $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$, and that $(f^\varepsilon)_{\varepsilon\in(0,1]}$ is the corresponding family of regularized solutions. By Proposition 3, we can find $(\varepsilon_k)_{k\geq 0}$ and $f \in C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$ such that $\lim_k \varepsilon_k = 0$ and $\lim_k f^{\varepsilon_k} = f$ in $C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology. By definition of f_0^ε, we obviously have $f|_{t=0} = f_0$. By Proposition 5 and the Fatou lemma, for all $\gamma \in (M/(4\pi), 2)$, we have

$$\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s \leq A_{M,\gamma}(1+T). \tag{11}$$

It only remains to check that f is a weak solution to (2). We fix $\varphi \in C_b^2(\mathbb{R}^2)$ and use (6) to write $\int_{\mathbb{R}^2}\varphi(x) f_t^{\varepsilon_k}(\mathrm{d}x) = I_k(t) + J_k(t)$, where

$$I_k(t) = \int_{\mathbb{R}^2}\varphi(x) f_0^{\varepsilon_k}(\mathrm{d}x) + \int_0^t\int_{\mathbb{R}^2}\Delta\varphi(x) f_s^{\varepsilon_k}(\mathrm{d}x)\mathrm{d}s,$$
$$J_k(t) = \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_{\varepsilon_k}(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)] f_s^{\varepsilon_k}(\mathrm{d}x) f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s.$$

Since φ and $\Delta\varphi$ are continuous and bounded, we immediately conclude that

$$\lim_k \int_{\mathbb{R}^2}\varphi(x) f_t^{\varepsilon_k}(\mathrm{d}x) = \int_{\mathbb{R}^2}\varphi(x) f_t(\mathrm{d}x) \quad \text{and} \quad \lim_k I_k(t) = \int_{\mathbb{R}^2}\varphi(x) f_0(\mathrm{d}x) + \int_0^t\int_{\mathbb{R}^2}\Delta\varphi(x) f_s(\mathrm{d}x)\mathrm{d}s,$$

and it only remains to check that for $J(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s(\mathrm{d}x)f_s(\mathrm{d}y)\mathrm{d}s$, we have $\lim_k J_k(t)=J(t)$. To this end, we write $J_k(t)=J_k^1(t)+J_k^2(t)$, where

$$J_k^1(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s,$$
$$J_k^2(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[K_{\varepsilon_k}(x-y)-K(x-y)]\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s.$$

Recalling the expression of K and that $\varphi\in C_b^2(\mathbb{R}^2)$, we see that $g(x,y)=K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]$ is bounded and continuous on the set $\mathbb{R}^2\setminus D$, where $D=\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2: x=y\}$. Since $f_s^{\varepsilon_k}\otimes f_s^{\varepsilon_k}$ goes weakly to $f_s\otimes f_s$ for each $s\geq 0$ and since $(f_s\otimes f_s)(D)=0$ for a.e. $s\geq 0$ by (11), 1, we conclude that $\lim_k\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}g(x,y)f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}g(x,y)f_s(\mathrm{d}x)f_s(\mathrm{d}y)$ for a.e. $s\geq 0$, whence $\lim_k J_k^1(t)=J_k(t)$ by dominated convergence.

We finally have to verify that $\lim_k J_k^2(t)=0$. We fix $\gamma\in(M/(4\pi),2)$ and write

$$\|z\|\,\|K(z)-K_\varepsilon(z)\|=\frac{\varepsilon}{2\pi(\varepsilon+\|z\|^2)}\leq\min(1,\varepsilon\|z\|^{-2})\leq(\varepsilon\|z\|^{-2})^{1-\gamma/2}=\varepsilon^{1-\gamma/2}\|z\|^{\gamma-2}.$$

Thus

$$|J_k^2(t)|\leq\|\nabla^2\varphi\|_\infty\varepsilon_k^{1-\gamma/2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\|x-y\|^{\gamma-2}f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s,$$

which tends to 0 as desired since $\sup_{\varepsilon\in(0,1]}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\|x-y\|^{\gamma-2}f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s<\infty$ by Proposition 5. □

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