Calcul des variations stochastiques sur l'espace de Poisson, applications à des E.D.P.S. paraboliques avec sauts et à certaines équations de Boltzmann

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Introduction

Cette thèse est consacrée au calcul des variations stochastiques pour des processus de saut, à ses applications aux équations aux dérivées partielles stochastiques paraboliques avec sauts d'une part, et aux équations de Boltzmann spatialement homogènes sans cutoff pour des molécules maxwelliennes d'autre part.
Ce travail est constitué de deux parties, comportant chacune trois chapitres. Chaque chapitre est indépendant des autres, en particulier pour ce qui est des notations, numérotations des théorèmes et des formules,... Ceci conduit bien sûr à des redondances, mais préserve l'homogénéité de chaque chapitre.

Dans le premier Chapitre, nous nous intéressons à une équation aux dérivées partielles stochastique (E.D.P.S.), conduite par un bruit blanc gaussien espace-temps, et par une mesure de Poisson compensée. Nous prouvons un résultat d'existence et d'unicité, puis nous étudions l'absolue continuité de la loi de la solution par rapport à la mesure de Lebesgue. Le deuxième Chapitre constitue en quelque sorte un "sous-produit" du premier : nous appliquons ses méthodes pour résoudre les mêmes questions concernant l'équation de Volterra stochastique avec sauts. Dans le troisième Chapitre, nous nous intéressons à une E.D.P.S. conduite par un bruit blanc gaussien espace-temps, et par une mesure de Poisson finie. Nous caractérisons le support de la loi de sa solution dans un espace de Skorokhod.

Dans le quatrième Chapitre, nous utilisons le calcul des variations stochastiques sur l'espace de Poisson, afin de prouver l'existence d'une solution régulière d'une équation de Boltzmann en dimension 2. Nous établissons un critère de stricte positivité de la densité pour des solutions d'E.D.S. avec sauts dans le cinquième Chapitre. Ceci nous permet de prouver la stricte positivité de la solution d'une équation de Kac ("caricature" unidimensionelle de l'équation de Boltzmann) dans le Chapitre 6.

Dans cette introduction, nous allons d'abord rappeler quelques définitions et résultats à propos des mesures de Poisson, et du calcul de Malliavin sur l'espace de Poisson. Nous décrirons ensuite les différents travaux constituant cette thèse, en essayant d'en dégager le contexte, les résultats, et les principales nouveautés.

1 Mesures ponctuelles de Poisson : rappels.

Dans tous les Chapitres de cette thèse, les équations étudiées comportent une intégrale par rapport à une mesure de Poisson, le plus souvent compensée. Rappelons donc les principales définitions que nous allons utiliser, qui figurent par exemple dans Jacod, Shiryaev, [23], ou Ikeda, Watanabe, [20].

Soit $(E, \mathcal{E})$ un espace de Blackwell (par exemple un espace Polonais, voir [23]), soit $T > 0$, et soit $(\Omega, \mathcal{F}, P)$ un espace de probabilités. Une mesure aléatoire $N(\omega, dt, dz)$ sur $[0,T] \times E$ est une mesure de Poisson (homogène) si les conditions suivantes sont vérifiées :

1. Pour tout $\omega \in \Omega$, $N(\omega, \{0\} \times E) = 0$. Pour tout $t \in]0,T]$, tout $\omega \in \Omega$, $N(\omega, \{t\} \times E) \in \{0,1\}$.

2. Si A et B sont deux éléments disjoints de $\mathcal{B}([0,T]) \otimes \mathcal{E}$, alors les variables aléatoires $N(A)$ et $N(B)$ sont indépendantes.

3. La mesure $\nu(A) = E(N(A))$ sur $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{E})$ est de la forme $dtq(dz)$, où q est une mesure positive σ-finie. La mesure ν est appelée "intensité" de N.

On peut vérifier que, si N est une telle mesure aléatoire, si A est un élément de $\mathcal{E}$ vérifiant $q(A) < \infty$, alors le processus $N_t(A) = N([0,t] \times A)$ est un processus de Poisson standard de paramètre $q(A)$.

Rappelons que la filtration canonique associée à la mesure N est donnée par

$$\mathcal{F}_t = \sigma \{ N(A) \ ; \ A \in \mathcal{B}([0,t]) \otimes \mathcal{E} \} \tag{1}$$

Ceci permet de définir les tribus prévisible et optionnelle (sur $\Omega \times [0,T]$) :

$$\mathcal{P} = \sigma \{ X \ ; \ X_t(\omega) \ \text{ processus càg } \ \{\mathcal{F}_t\} - \text{adapté sur } \ [0,T] \} \tag{2}$$

$$\mathcal{O} = \sigma \{ X \ ; \ X_t(\omega) \ \text{ processus càd } \ \{\mathcal{F}_t\} - \text{adapté sur } \ [0,T] \} \tag{3}$$

Toute mesure de Poisson peut s'écrire sous la forme

$$N(\omega, dt, dz) = \sum_{s \in [0,T]} \mathbb{I}_{D(\omega)}(s) \delta_{(s, \beta_s(\omega))}(dt, dz) \tag{4}$$

où β est un processus optionnel ($\mathcal{O}$-mesurable) à valeurs dans E, et D est un sous-ensemble aléatoire maigre de $[0,T]$, c'est à dire que $D(\omega) = \cup \{T_n(\omega)\}$ est une union au plus dénombrable de temps d'arrêt.

Mesures finies.

Dans le cas où $q(E) < \infty$, l'ensemble $D(\omega)$ est p.s. fini. La mesure de Poisson N est alors dite "finie", et peut s'écrire comme une somme finie de masses de Dirac :

$$N(\omega, dt, dz) = \sum_{i=1}^{\mu(\omega)} \delta_{(T_i(\omega), Z_i(\omega))}(dt, dz) \tag{5}$$

où les temps d'arrêt $0 < T_1 < ... < T_\mu$ sont les instants de saut du processus de Poisson $N_t(E)$. On sait que les variables aléatoires $T_i - T_{i-1}$ sont i.i.d. de loi exponentielle de paramètre $q(E)$, et que $\mu = N([0,T] \times E)$ suit une loi de Poisson de paramètre $Tq(E)$. Conditionnellement à $\mu, T_1, ..., T_\mu$, les Z_i sont i.i.d. de loi $q(dz)/q(E)$.

Mesures infinies.

Si $q(E) = \infty$, l'ensemble D est p.s. dense dans $[0,T]$. En revanche, on peut considérer une suite croissante de sous-ensembles E_p tels que $\cup E_p = E$ et pour chaque p, $q(E_p) < \infty$. Alors la restriction $N|_{[0,T]\times E_p}$ est une mesure de Poisson finie. De plus, cette approximation est souvent pratique, car $N = N|_{[0,T]\times E_p} + N|_{[0,T]\times E\setminus E_p}$ est la somme de deux mesures de Poisson indépendantes, et la deuxième tend vers 0 (dans un certain sens) quand p tend vers l'infini.
On peut aussi écrire, si $D = \cup_{n\in\mathbb{N}}\{T_n\}$, en posant $Z_i = \beta_{T_i}$,

$$N(\omega, dt, dz) = \sum_{i=1}^{\infty} \delta_{(T_i(\omega), Z_i(\omega))}(dt, dz) \tag{6}$$

mais cette fois, on ne peut plus ordonner les temps d'arrêt T_n selon l'ordre usuel sur $[0,T]$.

Compensateur.

La mesure de Poisson N admet pour compensateur son intensité $\nu(ds, dz) = dsq(dz)$, c'est à dire que pour toute fonction $W(\omega, s, z)$ prévisible (i.e. $\mathcal{P} \otimes \mathcal{E}$-mesurable) sur $\Omega \times [0,T] \times E$, vérifiant

$$E\left(\int_0^t \int_E |W(s,z)|q(dz)ds\right) < \infty \tag{7}$$

le processus

$$M_t = \int_0^t \int_E W(s,z)N(ds,dz) - \int_0^t \int_E W(s,z)q(dz)ds \tag{8}$$

est une $\{\mathcal{F}_t\}$-martingale càdlàg (l'intégrale par rapport à N ci-dessus est une intégrale de Stieljes, définie ω par ω). Si de plus,

$$E\left(\int_0^t \int_E W^2(s,z)q(dz)ds\right) < \infty \tag{9}$$

alors M est de carré intégrable, admet pour variation quadratique :

$$[M]_t = \int_0^t \int_E W^2(s,z)N(ds,dz) \tag{10}$$

et pour variation quadratique prévisible :

$$\langle M \rangle_t = \int_0^t \int_E W^2(s,z)q(dz)ds \tag{11}$$

Intégrale stochastique.

Posons $\tilde{N}(ds,dz) = N(ds,dz) - q(dz)ds$. Il est possible d'intégrer contre cette "mesure" des fonctions prévisibles qui ne sont pas intégrables pour $dsq(dz)$. Soit $W(\omega,s,z)$ une fonction prévisible sur $\Omega \times [0,T] \times E$ de carré intégrable pour $P(d\omega)dsq(dz)$, i.e. vérifiant (9). Si $q(E) = \infty$, l'intégrale $\int_0^t \int_E W(s,z)N(ds,dz)$ n'a en général pas de sens. Considérons donc une suite croissante de sous-ensembles E_p tels que $\cup E_p = E$ et pour chaque p, $q(E_p) < \infty$. Pour chaque p, on peut définir la $\{\mathcal{F}_t\}$-martingale càdlàg de carré intégrable

$$M_t^p = \int_0^t \int_{E_p} W(s,z)N(ds,dz) - \int_0^t \int_{E_p} W(s,z)q(dz)ds \tag{12}$$

Il est possible de vérifier (voir [20]) que M^p converge uniformément sur $[0,T]$ dans L^2 vers une martingale càdlàg de carré intégrable M_t, que nous noterons

$$M_t = \int_0^t \int_E W(s,z)\tilde{N}(ds,dz) \tag{13}$$

On peut aussi définir (voir [23]) M_t comme l'unique martingale purement discontinue dont les sauts sont donnés par

$$\Delta M_t = \mathbb{1}_D(t)W(t,\beta_t) \tag{14}$$

Cette intégrale s'appelle "intégrale stochastique" par rapport à la mesure de Poisson "compensée". Les crochets de M sont encore donnés par (10) et (11). On obtient en particulier l'isométrie L^2 : pour toute fonction prévisible W,

$$E\left[\left(\int_0^t \int_E W(s,z)\tilde{N}(ds,dz)\right)^2\right] = E\left[\int_0^t \int_E W^2(s,z)q(dz)ds\right] \tag{15}$$

Rappelons enfin l'inégalité de Burkholder pour cette intégrale : pour tout $p \geq 2$, il existe une constante C_p telle que pour tout fonction prévisible W,

$$E\left[\sup_{u\in[0,t]}\left|\int_0^u \int_E W(s,z)\tilde{N}(ds,dz)\right|^p\right] \leq C_p E\left[\left|\int_0^t \int_E W^2(s,z)N(ds,dz)\right|^{p/2}\right] \tag{16}$$

2 Calcul des variations stochastiques sur l'espace de Poisson.

A l'exception du Chapitre 3, toutes les parties de cette thèse font intervenir ce domaine. Le calcul de Malliavin, qui tente de définir une "dérivée par rapport à l'alea", a d'abord été introduit sur l'espace de Wiener, voir Malliavin, [25]. Initialement, l'idée était de résoudre le problème de Hörmander par des méthodes probabilistes, ce qui revient à résoudre le problème suivant. Considérons le processus de diffusion

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s \tag{17}$$

A quelles conditions sur les coefficients b et σ les lois des variables aléatoires X_t^x admettent-t-elles des densités $p_t(x,y)$ par rapport à la mesure de Lebesgue, éventuellement régulières en y, en (x,y), ou en (t,x,y) ?

Le calcul de Malliavin a permis de résoudre ce problème, mais aussi des questions similaires concernant bien d'autres fonctionnelles de Wiener ; il a débouché sur un vaste domaine de recherche, voir Malliavin, [26], ou Nualart, [30], pour des exposés complets sur le sujet.

Intéressons-nous maintenant à un processus de diffusion avec sauts Poissonniens :

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s + \int_0^t \int_E c(X_{s-}^x, z)\tilde{N}(ds, dz) \tag{18}$$

Bismut, [8], puis Bichteler, Jacod, [7] et Bichteler, Gravereaux, Jacod, [6], ont développé une méthode, consistant à "dériver" la mesure de Poisson par rapport à la taille de ses sauts, afin d'étudier l'existence de densités, éventuellement régulières, pour des variables aléatoires sur l'espace de Poisson. Cette méthode paraît la plus naturelle, mais elle présente un inconvénient majeur : elle ne permet de traiter que le cas des mesures de Poisson dont l'intensité admet une densité régulière par rapport à la mesure de Lebesgue.

Deux approches du calcul de Malliavin sur l'espace de Poisson ont en fait été développées par Bichteler, Gravereaux, et Jacod dans [6]. Les auteurs de cet article étudient l'existence et la régularité de densités pour des processus de diffusion avec sauts. L'une de ces méthodes consiste à définir un opérateur de "carré du champ" sur un domaine de variables aléatoires s'écrivant comme limites d'une suite de fonctionnelles "simples" de la mesure de Poisson, puis à prouver des formules d'intégration par parties sur ce domaine. La seconde approche consiste à "perturber" la mesure de Poisson à l'aide de fonctions prévisibles, à définir des "dérivées" dans L^2 de variables aléatoires perturbées, afin d'en déduire des formules d'intégration par parties. Tous les résultats obtenus découlent d'une accumulation de "petits sauts".

Dans toute cette thèse, à l'exception du Chapitre 3, nous adapterons, étendrons, ou raffinerons selon nos besoins ces méthodes, afin d'obtenir des résultats d'existence et de régularité d'autres fonctionnelles de Poisson, mais aussi pour étudier la stricte positivité des densités obtenues.

Mentionons enfin d'autres approches du calcul de Malliavin sur l'espace de Poisson. Pardoux, Carlen, [13], puis Denis, [14], ont construit un calcul variationnel par rapport aux instants de saut. Picard, [34], [35], construit une "dérivée discrète", en ajoutant ou en retirant des sauts. Ces approches permettent de traiter le cas où l'intensité de la mesure de Poisson n'est plus absolument continue par rapport à la mesure de Lebesgue. Citons enfin Privault, [37], qui étudie les fonctionnelles du processus de Poisson standard. Il construit un calcul variationnel à l'aide de "l'espace exponentiel", en exploitant le fait que les délais entre les sauts sont indépendants et identiquement distribués, de lois exponentielles.

3 E.D.P.S. paraboliques avec sauts.

Les E.D.P.S. paraboliques ont été introduites par Walsh, dans [46], afin de modéliser les potentiels électriques de membranes nerveuses. On s'intéresse aux potentiels des membranes d'un cylindre, et, pour des raisons de symétrie, on peut se ramener à une étude sur le segment $[0, 1]$. Notons $V(x, t)$ le potentiel en un point $x \in [0, 1]$ à l'instant t. En l'absence de stimulations, V

satisfait l'équation :

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V \tag{19}$$

Considérons maintenant le cas où il y a des stimulations. Soit donc $F(x,t)$ l'impulsion (l'arrivée ou le départ d'un ion, par exemple) au point x à l'instant t. Alors V satisfait :

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V + F \tag{20}$$

Walsh explique que F peut être modélisé par un processus ponctuel de Poisson : $F(x,t)$ est la taille du saut au point t,x d'une mesure de Poisson $N(dt,dx,dz)$ sur $[0,T]\times[0,1]\times I\!R$. Afin de résoudre (20), Walsh utilise la fonction de Green $G_t(x,y)$ associée, i.e. la solution fondamentale de (19) avec pour condition initiale une masse de Dirac en y (et avec des conditions au bord fixées) :

$$V(x,t) = \int_0^1 G_t(x,y)\mathcal{V}_0(y)dy + \int_0^t\int_0^1\int_{I\!R} G_{t-s}(x,y)N(ds,dy,dz) \tag{21}$$

Dans [46], Walsh prouve aussi un résultat de convergence, qui permet d'approximer la mesure de Poisson par un bruit blanc espace-temps $W(dx,ds)$, car les sauts (stimuli) sont très petits et très nombreux. Rappelons que W est défini comme un processus Gaussien $\{W(A)\ ;\ A \in \mathcal{B}([0,T]\times[0,1])\}$, indexé par les boreliens de $[0,T]\times[0,1]$, de covariance :

$$E(W(A)W(B)) = \iint_{A\cap B} ds\mu(dz) \tag{22}$$

où μ est une mesure positive finie sur $[0,1]$, le plus souvent choisie égale à la mesure de Lebesgue.

Dans [47], Walsh étudie une E.D.P.S. de la forme :

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + g(V) + f(V)W \tag{23}$$

Les solutions faibles sont définies comme des processus $V(x,t)$, adaptés pour la filtration du bruit blanc, et vérifiant l'équation d'évolution :

$$\begin{aligned} V(x,t) &= \int_0^1 G_t(x,y)\mathcal{V}_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)g(V(y,s))dyds \\ &\quad + \int_0^t\int_0^1 G_{t-s}(x,y)f(V(y,s))W(dy,ds) \end{aligned} \tag{24}$$

où G est le noyau de Green associé à l'E.D.P. $\partial_t V = \partial_{x^2} V$, avec des conditions au bord de type Neumann ou Dirichlet. Dans [47], Walsh prouve un théorème d'existence et d'unicité en adaptant les méthodes habituelles (Lemmes de Gronwall et Picard). Walsh prouve ensuite que sa solution est hölderienne en x,t, en appliquant le Lemme de Kolmogorov au membre de droite de (24). Depuis, cette équation a été largement explorée, voir par exemple [3], [4], [27], [33]...

Résumé du Chapitre 1

Dans le premier Chapitre, nous considérons l'E.D.P.S. sur $[0,T] \times [0,1]$:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + g(V) + f(V)W + h(V,.) * \tilde{N} \tag{25}$$

où W est un bruit blanc gaussien, et $\tilde{N}$ est la mesure compensée d'une mesure de Poisson sur $[0,T] \times [0,1] \times I\!R$, indépendante de W, d'intensité $dsdyq(dz)$, où q est σ-finie sur $I\!R$. Cette E.D.P.S., qui généralise (20) et (23), permet de modeliser les phénomènes décrits par Walsh, dans le cas où plusieurs types d'impulsions interviennent (des “tailles” et des “fréquences” différentes). L'équation d'évolution associée s'écrit :

$$\begin{aligned} V(x,t) &= \int_0^1 G_t(x,y)\mathcal{V}_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)g(V(y,s))dyds \\ &+ \int_0^t \int_0^1 G_{t-s}(x,y)f(V(y,s))W(dy,ds) \\ &+ \int_0^t \int_0^1 \int_{I\!R} G_{t-s}(x,y)h(V(y,s),z)\tilde{N}(ds,dy,dz) \end{aligned} \tag{26}$$

La condition initiale $\mathcal{V}_0$ est supposée déterministe, mesurable, et bornée sur $[0,1]$. Le noyau de Green G est celui de Walsh, associé à l'E.D.P. $\partial_t V = \partial_{x^2} V$, avec des conditions au bord de Neumann : on sait que $G_t(x,y)$ se comporte en gros comme

$$\frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{4t}\right) \tag{27}$$

Nous supposons que f et g sont Lipschitziennes de $I\!R$ dans $I\!R$, et que $h : I\!R \times I\!R \mapsto I\!R$ vérifie, pour une certaine fonction $\eta \in L^2(I\!R, q)$,

$$|h(0,z)| \leq \eta(z) \quad ; \quad |h(x,z) - h(y,z)| \leq |x-y|\eta(z) \tag{28}$$

Nous prouvons dans une première partie l'existence d'une unique solution faible $V(x,t)$ pour (25), admettant une version (dans un sens faible) prévisible, et satisfaisant :

$$\sup_{[0,1]\times[0,T]} E\left(V^2(x,t)\right) < \infty \tag{29}$$

Ce théorème d'existence et d'unicité est très simple à prouver, en utilisant les méthodes de Walsh, une fois que la validité des intégrales stochastiques a été établie. Par exemple, pour appliquer une itération de Picard, il faut être sûr que si Y est prévisible et borné dans L^2 sur $[0,1] \times [0,T]$, alors le processus

$$U(x,t) = \int_0^t \int_0^1 \int_{I\!R} G_{t-s}(x,y)h(Y(y,s),z)\tilde{N}(ds,dy,dz) \tag{30}$$

admet une version (faible) prévisible. Ceci n'est pas trivial, à cause de la présence du noyau. Nous résolvons ce problème de mesurabilité en approchant le semi-groupe G, de façon que U soit une limite $dxdt \otimes dP$-presque partout de processus prévisibles. Remarquons que nous sommes

obligés de définir une solution admettant une version (faible) prévisible : nous vérifions, à l'aide d'un exemple, que la solution n'a aucune chance d'avoir des trajectoires limitées à droite.
Le seul résultat de régularité "jointe" que nous avons obtenu est le suivant : si f est bornée et si $\eta \in L^1(I\!R, q)$, alors l'application

$$t \mapsto V(x,t)dx \tag{31}$$

est p.s. càglàd de $[0,T]$ dans l'espace $\mathcal{M}_b$ des mesures bornées (signées) sur $[0,1]$, muni de la topologie de la convergence étroite. On prouve aussi que ceci n'est en général plus vrai si on remplace $\mathcal{M}_b$ par $L^1([0,1])$.

Nous nous intéressons ensuite à l'absolue continuité de la loi de notre solution $V(x,t)$, pour $x \in [0,1]$ et $t > 0$. Nous supposons d'abord que $q(dz) = \mathbb{1}_O(z)\varphi(z)dz$, où O est un ouvert de $I\!R$ et φ est une fonction régulière sur O. Nous supposons aussi que les fonctions f, g, et h sont suffisamment régulières. Nous émettons enfin une hypothèse concernant le comportement de η et φ au bord de O, et une condition de non-dégénerescence du type

$$\forall\, x \in I\!R,\ f(x) \neq 0 \quad \text{ou} \quad \int_O \mathbb{1}_{\{z\in I\!R\ /h'_z(x,z)\neq 0\ \forall\ x\in I\!R\}}\varphi(z)dz = \infty \tag{32}$$

(notre deuxième hypothèse est en vérité plus forte, car elle fait intervenir, de manière implicite, le noyau de Green). Nous prouvons que sous ces hypothèses, la loi de $V(x,t)$ admet une densité dès que $t > 0$.

Afin de démontrer un tel résultat, nous construisons un calcul de Malliavin "partiel" sur l'espace de Poisson, et nous utilisons le calcul de Malliavin classique sur l'espace de Wiener associé au bruit blanc (voir Nualart, Zakai, [32], ou Nualart, [30]). Nous définissons donc deux opérateurs de dérivation, sur l'espace canonique produit. Le premier, $D^{1,0}_{\alpha,\tau}$, associé au bruit blanc, est standard. Le second, $D^{0,1}_{\alpha,\tau,\zeta}$, lié à la mesure de Poisson, tente de définir "proprement" la dérivée suivante : si X est une variable sur l'espace canonique Ω associé à N, si $\omega \in \Omega$, et si $(\alpha,\tau,\zeta) \in \text{supp } \omega$ (rappelons que tout $\omega \in \Omega$ est une mesure de comptage sur $[0,T] \times [0,1] \times O$ et que O est ouvert) :

$$D^{0,1}_{\alpha,\tau,\zeta}X(\omega) = \frac{\partial}{\partial\lambda}X(\omega - \delta_{(\alpha,\tau,\zeta)} + \delta_{(\alpha,\tau,\zeta+\lambda)})\Big|_{\lambda=0} \tag{33}$$

Pour cela, nous définissons un domaine $\mathcal{S}^{0,1}$ de variables simples, i.e. s'exprimant de manière simple en fonction de la mesure de Poisson, sur lequel $D^{0,1}_{\alpha,\tau,\zeta}$ est naturellement bien défini. En suivant le Chapitre 4 de Bichteler, Gravereaux, Jacod, [6], nous définissons un opérateur $L^{0,1}$ sur $\mathcal{S}^{0,1}$, dont les principales propriétés sont les suivantes :

1. $L^{0,1}$ est auto-adjoint dans L^2, i.e. si X et Y appartiennent à $\mathcal{S}^{0,1}$, alors
$$E(XL^{0,1}Y) = E(YL^{0,1}X) \tag{34}$$

2. Il existe une fonction bornée strictement positive $\rho \in L^1(I\!R, q)$ telle que pour tous X et Y dans $\mathcal{S}^{0,1}$,
$$\begin{aligned}\Gamma^{0,1}(X,Y) &= L^{0,1}(XY) - XL^{0,1}Y - YL^{0,1}X \\ &= \int_0^T \int_0^1 \int_O D^{0,1}_{\alpha,\tau,\zeta}XD^{0,1}_{\alpha,\tau,\zeta}Y\rho(\zeta)N(d\tau,d\alpha,d\zeta) \qquad (35)\end{aligned}$$

On considère ensuite sur $\mathcal{S}^{0,1}$ la norme

$$|||X|||_2 = E\left(X^2\right)^{\frac{1}{2}} + E\left(\int_0^T\int_0^1\int_O D^{0,1}_{\alpha,\tau,\zeta}X D^{0,1}_{\alpha,\tau,\zeta}Y\rho(\zeta)N(d\tau,d\alpha,d\zeta)\right)^{\frac{1}{2}} \tag{36}$$

Les propriétés 1 et 2 de $L^{0,1}$ permettent de fermer l'opérateur $D^{0,1}_{\alpha,\tau,\zeta}$, i.e. de l'étendre au domaine $\mathcal{D}^{0,1} = \overline{\mathcal{S}^{0,1}}^{|||\,|||_2}$, *bien que* $L^{0,1}$ *ne soit pas défini sur tout ce domaine.*

On construit de même les objets $\mathcal{S}^{1,0}$ et $\mathcal{D}^{1,0}$ associés au bruit blanc. Nous obtenons ainsi un domaine $\mathcal{D}$ de variables aléatoires sur l'espace canonique produit, dérivables au sens des opérateurs $D^{1,0}_{\alpha,\tau}$ et $D^{0,1}_{\alpha,\tau,\zeta}$.

Il est alors possible de prouver le critère suivant : si $X \in \mathcal{D}$, et si, p.s.,

$$\int_0^T\int_0^1\left(D^{1,0}_{\alpha,\tau}X\right)^2 d\alpha d\tau + \int_0^T\int_0^1\int_O\left(D^{0,1}_{\alpha,\tau,\zeta}X\right)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta) > 0 \tag{37}$$

alors la loi de X admet une densité par rapport à la mesure de Lebesgue. Pour cela, nous utilisons une méthode du type Bouleau, Hirsch, [9], dont la preuve a été simplifiée par Nualart, Zakai, [32].

Nous étudions ces opérateurs en détail, afin d'en exploiter les principales propriétés, pour pouvoir vérifier que $V(x,t)$ est "dérivable". Ceci est très technique, en particulier parce que $V(x,t)$ n'appartient pas à un espace L^p pour $p > 2$. Néanmoins, le théorème de convergence dominée de Lebesgue permet la plupart du temps de contourner ce défaut d'intégrabilité. Il faut de plus toujours prendre garde de n'intégrer que des processus prévisibles contre W et N. Enfin, nous sommes obligés de prouver une "isométrie" L^2 associée à notre opérateur $D^{0,1}_{\alpha,\tau,\zeta}$.

Une fois calculées les dérivées de la solution, il reste à vérifier que p.s., $\sigma(x,t) > 0$, où

$$\begin{aligned}\sigma(x,t) &= \int_0^T\int_0^1\left(D^{1,0}_{\alpha,\tau}V(x,t)\right)^2 d\alpha d\tau \\ &\quad + \int_0^T\int_0^1\int_O\left(D^{0,1}_{\alpha,\tau,\zeta}V(x,t)\right)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)\end{aligned} \tag{38}$$

Vu la différence de nature entre les deux intégrales ci-dessus, nous prouvons en fait que sous la première (resp. deuxième) hypothèse de non-dégénerescence, la première (resp. deuxième) intégrale est p.s. strictement positive.

Pour cela, nous localisons ces intégrales, c'est à dire que nous ne considérons que des intégrales au voisinage de t, afin d'exploiter l'explosion du noyau $G_{t-s}(x,y)$ en $s=t$.

Les nouveautés par rapport au travail de Bichteler, Gravereaux, Jacod, [6], sont les suivantes. Dans [6], la fonction φ est toujours supposée constante. De plus, Bichteler et al. ne définissent pas d'opérateur de dérivation. Ils travaillent directement à l'aide de $\Gamma^{0,1}$. Ceci occasionne une grosse perte d'information : on voit bien que la connaissance de $D^{0,1}$ est plus instructive que celle de $\Gamma^{0,1}$. Dans les cas des E.D.S., ceci n'a pas d'importance, car les dérivées (ou Γ) satisfont des E.D.S. linéaires, qu'on sait résoudre explicitement. Par contre, dans le cas des E.D.P.S., on

ne sait plus calculer explicitement les dérivées. Il est alors nécessaire de localiser les intégrales, ce qui est impossible en utilisant l'opérateur global Γ.
En utilisant des méthodes similaires à celles de Bichteler et al., [6], il paraît impossible d'établir la stricte positivité de $\sigma(x,t)$, sauf en supposant que V est un processus à variations finies, i.e. $f=0$, $\eta \in L^1(O, \varphi(z)dz)$, cf Saint Loubert Bié, [40].
Enfin, nous n'utilisons pas de formules d'intégration par parties pour prouver notre critère d'absolue continuité : nous n'exploitons que l'unicité des dérivées. En fait, nous avons étendu nos opérateurs de dérivation à un domaine "trop grand", au sens où L n'est en général pas défini sur $\mathcal{D}$. Or toutes les formules d'intégration par parties données dans [6] concernent l'opérateur L. Mais la solution V ne semble pas appartenir à un espace plus restreint.
L'avantage de cette méthode est claire : elle permet de traiter le cas d'un grand nombre de fonctionnelles de Poisson, sous des hypothèses assez faibles. L'inconvénient majeur est évident : on ne peut pas, avec cette méthode, étudier la régularité des densités, faute d'intégration par parties sur l'espace $\mathcal{D}$.

La principale nouveauté de ce travail est donc la construction et l'utilisation de ce calcul de Malliavin sur l'espace de Poisson, partiel et moins fort que celui de [6], mais plus précis.

Résumé du Chapitre 2

Dans le second chapitre, nous considérons l'équation de Volterra stochastique avec sauts :

$$X_t = x_0 + \int_0^t \sigma(s,t,X_s)dW_s + \int_0^t b(s,t,X_s)ds + \int_0^t \int_O h(s,t,X_s)\tilde{N}(ds,dz) \tag{39}$$

W est un mouvement brownien unidimensionel standard, et $\tilde{N}$ est une mesure de Poisson compensée, d'intensité $\varphi(z)dzds$ sur $[0,T]\times O$, où O est un ouvert de $\mathbb{R}$.

Les problèmes qui se posent (validité des intégrales stochastiques, existence, unicité d'une solution, calcul de Malliavin,...) sont exactement les mêmes que pour les E.D.P.S. : l'obstacle majeur est la présence du "t" dans les intégrales, ce qui implique en particulier que X n'est pas une semi-martingale. C'est pourquoi nous "vérifions" brièvement que les résultats du chapitre 1 s'adaptent.

Remarquons que dans cette étude, nous améliorons aussi le résultat d'absolue continuité de Bichteler, Jacod, [7] pour les diffusions avec saut (les fonctions σ, b, h ne dépendent pas de s,t) : notre méthode permet de supposer moins de régularité et d'intégrabilité, et permet d'étendre le résultat au cas où φ n'est plus constante.

Résumé du Chapitre 3

Dans le dernier chapitre de cette première Partie, nous considérons deux objets indépendants : un bruit blanc gaussien W et une mesure de Poisson finie N sur $[0,T]\times E$, où E est un espace Polonais, et l'intensité de N est de la forme $dtq(dz)$, où q est une mesure finie sur E. Considérons l'E.D.P.S. dont l'équation d'évolution s'écrit :

$$X(t,x) \quad = \quad \int_0^1 G_t(x,y)\mathcal{X}_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)b(X(s,y))dyds$$

$$+\int_0^t \int_0^1 G_{t-s}(x,y)\sigma(X(s,y))W(dy,ds)$$
$$+\int_0^t \int_E \int_0^1 G_{t-s}(x,y)g(X(s-,y),z)dy\ N(ds,dz) \tag{40}$$

La condition initiale $\mathcal{X}_0$ est supposée déterministe et continue sur $[0,1]$. Le noyau de Green G est celui de Walsh, (cf résumé du Chapitre 1), associé à l'E.D.P. $\partial_t V = \partial_{x^2} V$, avec des conditions au bord de Neumann. Nous supposons que b et σ sont Lipschitziennes de $I\!R$ dans $I\!R$, et que σ est de classe C^3. La fonction $g : I\!R \times E \mapsto I\!R$ est supposée aussi régulière qu'un produit de fonctions continues $\alpha(x)\beta(z)$.

Dans une première partie, nous esquissons la preuve d'un théorème d'existence et d'unicité : il existe un unique processus adapté $X(t,x)$ à valeurs dans $I\!D([0,T],\mathbf{C}([0,1]))$ et satisfaisant (40). Les problèmes de mesurabilité du Chapitre 1 n'apparaissent pas ici, puisque la solution recherchée est relativement régulière.

Notre but est ensuite de caractériser le support de la loi de X dans l'espace $I\!D([0,T],\mathbf{C}([0,1]))$ muni de la topologie de Skorokhod. Pour cela, nous considérons l'espace de Cameron-Martin associé au bruit blanc,

$$\mathcal{H} = \left\{ h(t,x) = \int_0^t \int_0^x \dot{h}(s,y)dyds \Big/ \dot{h} \in L^2([0,T]\times[0,1]) \right\} \tag{41}$$

et l'ensemble des mesures de comptage associé à N,

$$\mathcal{M} = \left\{ m(dt,dz) = \sum_{i=1}^n \delta_{(t_i,z_i)}(dt,dz) \Big/ \begin{array}{c} n \in I\!N,\ 0 < t_1 < \dots < t_n < T, \\ z_1,\dots,z_n \in \text{supp } q \end{array} \right\} \tag{42}$$

On définit ensuite le squelette associé à notre équation. Si $h \in \mathcal{H}$ et si $m \in \mathcal{M}$, on note $S(h,m)$ la solution de l'équation d'évolution (déterministe) où l'on a remplacé dans (40) $W(dy,ds)$ par $\dot{h}(s,y)dyds$ et $N(ds,dz)$ par $m(ds,dz)$. Notre résultat principal est le suivant :

$$\text{supp } P \circ X^{-1} = \overline{\{S(h,m)\ /\ h \in \mathcal{H}\ ,\ m \in \mathcal{M}\}} \tag{43}$$

où le support et l'adhérence sont relatifs à la topologie de Skorokhod sur $I\!D([0,T],\mathbf{C}([0,1]))$.

La preuve de ce résultat est constituée de quatre étapes. D'abord, un argument de localisation permet de supposer que σ est de classe C_b^3 et que g est bornée et uniformément continue en x.

Dans la deuxième étape, nous vérifions qu'il suffit de caractériser les supports de X_m et X_h pour tous $h \in \mathcal{H}$, $m \in \mathcal{M}$, où X_m (resp. X_h) est la solution de (40) où on a remplacé $N(ds,dz)$ par $m(ds,dz)$ (resp. $W(dy,ds)$ par $\dot{h}(s,y)dyds$). Pour décomposer le problème de la sorte, nous exploitons le fait que p.s., $N \in \mathcal{M}$, et nous utilisons l'espérance conditionnelle par rapport à N.

La troisième étape est dédiée à la caractérisation du support de X_h, pour un $h \in \mathcal{H}$ fixé. Pour cela, nous suivons la méthode de Simon, qui s'intéresse à des E.D.S. conduites uniquement par une mesure de Poisson (compensée ou non). L'inclusion directe est immédiate. Pour prouver l'inclusion difficile, nous fixons $m \in \mathcal{M}$. Nous construisons un ensemble $\Omega(m, \epsilon)$, de probabilité strictement positive, sur lequel "N est proche de m à ϵ près". Pour chaque $\omega \in \Omega(m, \epsilon)$, nous exhibons ensuite un changement de temps $\lambda : [0,T] \mapsto [0,T]$ tel que

$$\sup_{t,x} |X_h(\lambda(t), x) - S(h,m)(t,x)| \leq \epsilon \tag{44}$$

ce qui permet de conclure. Bien sur, une multitude de problèmes techniques apparaissent. Il faut gérer l'explosion du semi-groupe, le fait que $\dot{h} \in L^2(dyds)$ n'est pas borné ; la distance que nous utilisons n'est pas simple à manier, etc... Comme toujours, l'étude des E.D.P.S. est plus technique que celle des E.D.S.

Dans la quatrième étape, nous caractérisons le support de X_m. Nous remarquons d'abord que l'équation satisfaite par X_m ressemble beaucoup à celle de Walsh : on y a juste ajouté une "dérive discontinue". Notre méthode consiste donc à appliquer le théorème de support de Bally, Millet, Sanz, [3] (qui concerne l'équation de Walsh) entre les sauts de cette dérive. En fait, nous appliquons ce théorème de support à des lois conditionnelles, puisque les "conditions initiales" ne sont plus déterministes. Nous définissons donc des squelettes "conditionnels", et nous "recollons les morceaux", essentiellement à l'aide du raisonnement suivant : si le squelette est proche de la solution entre les sauts i et $i+1$, alors le squelette conditionnel est proche du squelette entre les sauts $i+1$ et $i+2$. Si de plus le squelette conditionnel est proche de la solution entre les sauts $i+1$ et $i+2$, la solution est donc proche du "vrai" squelette entre les sauts $i+1$ et $i+2$.

Dans une dernière partie, nous étendons notre résultat au cas où la mesure de Poisson est infinie, mais seulement quand le coefficient de diffusion σ est constant.

Les nouveautés de ce chapitre sont les suivantes. Le seul théorème de support pour des processus de sauts a été prouvé par Simon, [41], qui s'intéresse à des E.D.S. dirigées uniquement par une mesure de Poisson. Notre résultat semble donc constituer le seul théorème de support pour des E.D.P.S. avec sauts, et semble aussi être le seul théorème de support pour une équation conduite par deux objets indépendants mais différents.

Conclusion

Dégageons pour conclure quelques voies de recherche possibles.

Intéressons-nous d'abord à l'E.D.P.S. du Chapitre 1. La première question porte sur un problème resté assez flou : y a t-il un moyen plus simple de définir les solutions faibles. L'obstacle majeur est la régularité très faible d'éventuelles solutions.
La question suivante concerne la régularité de la densité obtenue, question techniquement difficile, car la solution de cette E.D.P.S. n'appartient à aucun espace L^p pour $p > 2$ (ou du moins pour $p \geq 3$).
Enfin, peut-on caractériser le support de la solution dans l'espace des fonctions càglàd sur $[0,T]$ à valeurs dans l'espace des mesures bornées (signées) sur $[0,1]$, muni de la topologie de la convergence étroite ?

Intéressons-nous maintenant à l'équation du Chapitre 3. Il serait intéressant, après avoir prouvé l'existence d'une densité continue pour la loi de $X(t,x)$, ce qui doit être très facile si σ ne s'annule jamais, d'essayer de caractériser l'ensemble des points de stricte positivité de cette densité de la manière suivante : si $X(t,x)$ a pour loi $p_{t,x}(y)dy$, si $z \in I\!\!R$,

$$p_{t,x}(z) > 0 \quad \Longleftrightarrow \quad \exists\, h \in \mathcal{H},\ m \in \mathcal{M}, \quad z = S(h,m)(t,x)$$

Peut-on, comme Millet, Sanz dans [28], utiliser le Théorème de Girsanov pour simplifier les preuves des Théorèmes de support pour des processus de sauts ?
Dans [41], Simon s'intéresse à des E.D.S. conduites par une mesure de Poisson compensée. Si nous parvenions à vérifier l'existence d'une solution de l'E.D.P.S. du chapitre 6 vivant dans $I\!\!D\left([0,T], \mathbf{C}([0,1])\right)$, en remplaçant N par une mesure de Poisson compensée, éventuellement sans bruit blanc, pourrait-on prouver un Théorème de support ? La méthode de Simon paraît dans ce cas difficile à utiliser, car il utilise fortement la structure Markovienne de sa solution.

4 Equations de Boltzmann spatialement homogènes.

L'équation de Boltzmann spatialement homogène décrit la densité $f(t,v)$ de particules ayant la vitesse v à l'instant t, dans un gaz suffisament dilué. Dans le cas de particules Maxwelliennes, une telle densité satisfait l'équation aux dérivées partielles :

$$\frac{\partial f}{\partial t} = \int_{v_* \in I\!\!R^3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \{f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)\}\, \beta(\theta) d\theta d\phi dv_* \tag{45}$$

où

$$v' = \frac{v+v_*}{2} - \frac{\| v - v_* \|}{2}\sigma \qquad ; \qquad v'_* = \frac{v+v_*}{2} + \frac{\| v - v_* \|}{2}\sigma \tag{46}$$

le vecteur unitaire σ ayant pour coordonnées θ et ϕ dans le système de coordonnées sphériques dont l'axe polaire a pour direction $v - v_*$. La "section efficace" β est une fonction paire et positive. Le membre de droite de cette équation est souvent appelé noyau de collision, et noté $K(f,f)$. Il "compte" les vitesses apparues après collision $(f(t,v')f(t,v'_*))$ et les vitesses disparues à cause d'une collision $(-f(t,v)f(t,v_*))$. En effet, v' et v'_* représentent les vitesses de deux particules après une collision d'angles θ et ϕ, si ces particules avaient pour vitesses v et v_* avant la collision.
Nous renvoyons à l'introduction de la thèse de Villani, [45], et au livre de Cercignani, Illner, Pulverenti, [11], pour plus de détails.

Afin d'étudier les équations de Boltzmann, nous utiliserons une approche probabiliste développée initialement par Tanaka, [44], puis par Desvillettes, Graham, Méléard, [17] et [19] : il est possible de construire un processus $V_t(\omega)$, solution (faiblement Markovienne) d'une équation différentielle non classique conduite par une mesure de Poisson, dont la loi $P_t = P \circ V_t^{-1}$ est solution "mesure" de (45). Cela entraîne que si pour tout $t > 0$, P_t admet une densité $f(t,v)$ par rapport à la mesure de Lebesgue, alors la fonction f satisfait (45).

En général, les analystes effectuent l'hypothèse de "cutoff angulaire de Grad" sur la section efficace (voir Villani, [45], p 15) :

$$\int_0^{\pi} \beta(\theta) d\theta < \infty \tag{47}$$

Cette hypothèse n'est pas réaliste d'un point de vue physique : dans le cas d'interactions en $1/r^s$, avec $s > 2$, la section efficace se comporte comme $\beta(\theta) \sim \theta^{-\frac{s+1}{s-1}}$ près de 0, d'où l'hypothèse naturelle

$$\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty \tag{48}$$

Notre approche probabiliste permet cette hypothèse, grâce au calcul stochastique "L^2".

Un autre avantage de l'approche probabiliste porte sur la condition initiale. Nous autorisons les conditions initiales "mesure", dès que cette mesure charge au moins deux points, ce qui est très faible. Les analystes considèrent en général des distributions initiales ayant une densité.

Résumé du Chapitre 4

Considérons l'équation de Boltzmann bidimensionelle suivante :

$$\frac{\partial f}{\partial t}(t,v) = \int_{v_* \in I\!R^2} \int_{\theta=-\pi}^{\pi} \{f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)\} \beta(\theta) d\theta dv_* \tag{49}$$

où, si R_θ est la rotation d'angle θ,

$$v' = \frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right) \quad ; \quad v'_* = \frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right) \tag{50}$$

La section efficace β satisfait (48).

En suivant la méthode de Desvillettes, Graham, Méléard, [17], nous associons à cette équation une E.D.S. non linéaire. Pour cela, nous considérons une mesure de Poisson $N(d\theta, d\alpha, ds)$ sur $[0,T]\times[0,1]\times[-\pi,\pi]$, d'intensité $\beta(\theta)d\theta d\alpha ds$. Le terme $[0,1]$ (qui correspond à $d\alpha$ dans la mesure de Poisson) joue le rôle d'un espace de probabilités auxiliaire, modélisant la non linéarité, sur lequel l'espérance sera notée E_α, et les lois $\mathcal{L}_\alpha$. Considérons la matrice

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix} \tag{51}$$

Si $V_t(\omega)$ et $W_t(\alpha)$ sont des processus (càdlàg, adaptés, et à valeurs dans $I\!R^2$) sur les espace Ω et $[0,1]$ respectivement, on dira que (V,W) est solution de l'E.D.S. non linéaire issue de V_0 si d'une part $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$, et d'autre part

$$\begin{aligned} V_t &= V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha))\tilde{N}(d\theta, d\alpha, ds) \\ &\quad - \frac{b}{2}\int_0^t \int_0^1 (V_{s-} - W_{s-}(\alpha)) d\alpha ds \end{aligned} \tag{52}$$

où $b = \int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)d\theta$.

Il est alors possible de vérifier que, si pour chaque $t > 0$, la loi de V_t (et donc celle de W_t) admet une densité $f(t,.)$ par rapport à la mesure de Lebesgue sur $I\!R^2$, alors f est solution de (49),

issue de $P_0 = \mathcal{L}(V_0)$.
En étendant les méthodes de Desvillettes, Graham, Méléard, [17], nous prouvons l'existence d'un couple (V, W) solution de l'E.D.S. non linéaire. Nous établissons aussi l'unicité en loi d'une telle solution.
Nous effectuons alors des hypothèses du type :

1. P_0 admet des moments de tous ordres, et n'est pas une masse de Dirac.
2. La section efficace vérifie (48), et, pour des constantes $k_0 > 0$, $r \in]1,3[$, et $\theta_0 \in]0,\pi[$,
$$\beta(\theta) \geq \frac{k_0}{|\theta|^r} \mathbb{1}_{\{|\theta| \leq \theta_0\}} \tag{53}$$
3. La section efficace est très petite au voisinage de π.

Le résultat principal du Chapitre 4 affirme que sous ces conditions, (49) admet une solution $f(t,v)$, continue sur $]0,T] \times I\!R^2$, et de classe C^∞ en $v \in I\!R^2$.

Remarquons que le meilleur résultat obtenu par les analystes semble être celui de Desvillettes, [16], qui prouve l'existence d'une solution $g(t,v)$, de classe $H^{1-\epsilon}$ en v (pour tout $\epsilon > 0$, tout $t > 0$), dans le cas où la distribution initiale admet une densité. L'approche probabiliste semble donc très efficace.

Afin d'établir un tel résultat, nous utilisons une approche de type Bismut du calcul de Malliavin sur l'espace de Poisson, pour vérifier que V_t admet une densité C^∞. Nous adaptons donc à notre contexte les méthodes développées par Bichteler, Gravereaux, Jacod, [6], qui consistent à perturber la mesure de Poisson, et donc le processus V_t, a dériver l'expression obtenue dans L^2, puis à prouver que le determinant de l'expression obtenue admet des moments inverses de tous ordres. La continuité jointe de f découle aussi du calcul de Malliavin.

Tout le début de ce travail (définition de l'E.D.S. non linéaire, des processus "perturbés", etc...) ressemble beaucoup aux travaux de Desvillettes, Graham, et Méléard, [17], [19], qui étudient un modèle unidimensionel. En revanche, le choix de la perturbation, et l'inversibilité dans tous les L^p de la matrice dérivée nécessitent beaucoup plus de technique. Nous devons vraiment utiliser les particularités de l'équation bidimensionelle de Boltzmann, en particulier pour contourner notre "ignorance" du processus W. Nous utilisons par exemple les conservations de l'énergie cinétique et de la quantité de mouvement.

Dans une dernière partie, nous simulons un système interactif de particules, dont la mesure empirique approche la "solution mesure" de l'équation de Boltzmann étudiée précédemment. Nous étendons ainsi l'approche Monte-Carlo de Graham, Méléard, [18], à cette situation sans cutoff.

Résumé des Chapitres 5 et 6

Dans le sixième Chapitre, nous nous intéressons à l'équation de Kac ("caricature" unidimensionelle de l'equation de Boltzmann homogène) :
$$\frac{\partial f}{\partial t}(t,v) = \int_{v_* \in I\!R} \int_{\theta=-\pi}^{\pi} \left\{ f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \right\} \beta(\theta) d\theta dv_* \tag{54}$$

où

$$v' = v\cos\theta - v_*\sin\theta \quad ; \quad v'_* = v\sin\theta + v_*\cos\theta \tag{55}$$

La section efficace β satisfait (48).

Cette fois, l'E.D.S. non linéaire associée est donnée par : $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$, et

$$V_t = V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} ((\cos\theta - 1)V_{s-} - \sin\theta W_{s-}(\alpha))\tilde{N}(d\theta, d\alpha, ds) - \frac{b}{2}\int_0^t V_{s-}ds \tag{56}$$

où $b = \int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)d\theta$, et où N, E_α, et $\mathcal{L}_\alpha$ sont définis comme au Chapitre 4. Graham et Méléard, [19], ont prouvé, sous conditions, l'existence d'une solution $f(t,v)$, de classe C^∞ en $v \in I\!R$, pour (54). Cette solution $f(t,.)$ est la densité de V_t.

Nous émettons les hypothèses suivantes :

1. La condition initiale P_0 admet des moments de tous ordres, et n'est pas une masse de Dirac en 0.
2. La section efficace vérifie (48), et, pour des constantes $k_0 > 0$, $r \in [2,3[$, et $\theta_0 \in]0,\pi[$,

$$\beta(\theta) \geq \frac{k_0}{|\theta|^r} 1\!\!I_{\{|\theta| \leq \theta_0\}} \tag{57}$$

Sous ces conditions, nous prouvons que pour tout $t > 0$, tout $v \in I\!R$, $f(t,v)$ est strictement positive. Ce résultat semble inconnu des analystes dans le cas sans cutoff, et pourrait peut-être servir à justifier la validité de certaines manipulations faisant intervenir l'entropie (dans laquelle apparaît le terme $\ln f(t,v)$).

Pour cela, nous utilisons encore le calcul de Malliavin sur l'espace de Poisson associé à N : nous prouvons et utilisons un critère de stricte positivité de la densité pour des fonctionnelles de Poisson, en suivant essentiellement les méthodes de Ben Arous, Léandre, [5], Bally, Pardoux, [4] (qui s'intéressent à l'espace de Wiener), et Bichteler, Jacod, [7] (qui établissent des critères d'existence et de régularité des densités pour des fonctionnelles de Poisson).

Il paraissait donc naturel de s'intéresser d'abord à des E.D.S. classiques conduites par des mesures de Poisson : au Chapitre 5, nous considérons l'équation

$$X_t = x_0 + \int_0^t g(X_{s-})ds + \int_0^t \int_O h(X_{s-}, z)\tilde{N}(ds, dz) \tag{58}$$

où N est une mesure de Poisson sur $[0,T] \times O$, d'intensité $\varphi(z)dzds$, où O est un ouvert de $I\!R$, et où φ est une fonction C^1 strictement positive sur O. Supposons que g et h soient suffisamment régulières, h bornée et suffisamment intégrable. L'idée principale du Chapitre 5 est de prouver que si $|h'_z(x,z)| \geq c(x)\delta(z)$, où $c > 0$ est continue sur $I\!R$, où δ est continue sur O, et où δ **n'appartient pas à** $L^1(O, \varphi(z)dz)$, alors pour chaque $t > 0$, (sous des conditions techniques supplémentaires) la loi de X_t est minorée par une mesure admettant une densité continue strictement positive partout.

Ce résultat semble nouveau : tous les travaux effectués dans cette direction traîtent du temps petit, voir Léandre, [24], Ishikawa, [21], et Picard, [36].

Décrivons brièvement notre méthode. L'idée de base est de construire, à l'aide du Théorème de Girsanov pour les mesures aléatoires, des processus "perturbés" $X_t(\lambda)$ et des changements de probabilité $P^\lambda = G(\lambda).P$, pour λ dans un voisinage de 0 (par exemple $[-1,1]$), vérifiant les propriétés suivantes :

1. $X_t(0) = X_t$ et $G(0) = 1$,
2. pour tout $\lambda \in [-1,1]$, la loi de $X(\lambda)$ sous P^λ est la même que celle de X sous P,
3. pour chaque ω, chaque t, les fonctions
$$\lambda \mapsto X_t(\omega,\lambda) \quad \text{et} \quad \lambda \mapsto G(\omega,\lambda) \tag{59}$$
sont régulières.

On peut alors écrire l'égalité, valable pour toute $f \in C_b^+(\mathbb{R})$,

$$E(f(X_t)) = \frac{1}{2} E\left[\int_{-1}^{1} f(X_t(\lambda))G(\lambda)d\lambda\right] \tag{60}$$

Fixons maintenant y_0 dans $\mathbb{R}$ et $t > 0$. Nous prouvons que sous certaines conditions (portant sur les dérivées première et seconde de l'application $\lambda \mapsto X_t(\omega,\lambda)$, donc sur le choix de la perturbation), il existe un ensemble Ω_0 de probabilité strictement positive, sur lequel X_t est proche de y_0, et sur lequel on peut utiliser un théorème d'uniforme inversion locale, afin d'effectuer, pour chaque $\omega \in \Omega_0$, le changement de variable $y = X_t(\lambda)$ dans (60). De la sorte, nous obtenons l'existence d'un processus $Z(y)$ sur un voisinage de y_0, vérifiant :

$$E(f(X_t)) \geq \frac{1}{2} \int f(y) E\left(Z(y) \mathbb{1}_{\Omega_0}\right) dy \tag{61}$$

Enfin, nous prouvons que la quantité $E\left(Z(y)\mathbb{1}_{\Omega_0}\right)$ est continue en y et strictement positive en $y = y_0$. En effectuant ce raisonnement pour chaque $y_0 \in \mathbb{R}$, nous concluons facilement.

Comparons maintenant notre méthode à celle de Bally, Pardoux, [4], qui s'intéressent à la stricte positivité de la densité pour une E.D.P.S. parabolique conduite par un bruit blanc gaussien. La preuve du critère s'adapte très facilement, la difficulté essentielle est son énoncé : il faut bien définir les perturbations, dérivées, changements de probabilité, etc... D'autant plus que nous ne pouvons plus nous contenter de perturbations déterministes.
En revanche, l'application du critère est beaucoup plus technique : le choix de la perturbation, en particulier, s'est avéré très délicat. Nous devons utiliser des temps d'arrêt au lieu de temps déterministes, etc... Tout est très "juste", et nous obtenons des hypothèses contraignantes et peu explicites (dans le cas de l'équation (58)).

Grâce à son expression explicite, l'application de cette méthode à l'équation de Kac s'est avérée beaucoup moins technique : malgré quelques difficultés dues à l'ignorance du processus W, il est plus agréable de travailler avec la fonction $\sin\theta$ qu'avec une fonction C_b^3 quelconque.

Remarquons enfin que, bien que le critère de stricte positivité s'adapte de manière triviale à des fonctionnelles n-dimensionelles de Poisson, son application à des processus n-dimensionels solutions d'E.D.S. semble très délicate.

Conclusion

La question naturelle portant sur le Chapitre 4 est l'extension à la dimension 3.

On peut aussi s'intéresser au cas de particules non maxwelliennes. Dans ce cas, l'équation s'écrit, en dimension 1 par exemple,

$$\frac{\partial f}{\partial t}(t,v) = \int_{v_*\in I\!R^3}\int_{\theta=-\pi}^{\pi} \{f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)\} B(|v-v_*|)\beta(\theta)d\theta dv_*$$

où

$$v' = v\cos\theta - v_*\sin\theta \quad ; \quad v'_* = v\cos\theta + v_*\sin\theta$$

Dans le cas où $\int_0^\pi \theta\beta(\theta)d\theta < \infty$, on peut associer à cette équation l'E.D.S. non linéaire :

$$\begin{aligned} V_t = V_0 + \int_0^t\int_0^1\int_{-\pi}^{\pi}\int_0^{B(|V_{s-}-W_{s-}(\alpha)|)} &[V_{s-}(1-\cos\theta) + W_{s-}(\alpha)\sin\theta]\, N(dz,d\theta,d\alpha,dt) \\ \mathcal{L}(V) = \mathcal{L}_\alpha(W) & \end{aligned} \tag{62}$$

où la mesure de Poisson N sur $[0,T]\times[0,1]\times[-\pi,\pi]\times I\!R^+$ a pour intensité $\beta(\theta)dzd\theta d\alpha dt$. Cette équation pose un problème fondammental : la fonction $1\!\!1_{\{z\leq B(|x|)\}}$ qui y apparaît n'est pas Lipschitzienne en x. De plus, en admettant l'existence d'une solution (V,W), cette fonction indicatrice rendrait difficile le calcul de Malliavin. Néanmoins, ces problèmes sont peut-être contournables. De plus, l'équation (62) pourrait en fait être remplacée par n'importe quelle E.D.S. non linéaire du type

$$\begin{aligned} V_t = V_0 + \int_0^t\int_0^1\int_{-\pi}^{\pi} &[V_{s-}(1-\cos\theta) + W_{s-}(\alpha)\sin\theta]\, M(d\theta,d\alpha,dt) \\ \mathcal{L}(V) = \mathcal{L}_\alpha(W) & \end{aligned} \tag{63}$$

où $M(d\theta,d\alpha,ds)$ est une mesure aléatoire de comptage (non Poissonnienne) sur $[0,T]\times[0,1]\times[-\pi,\pi]$ de compensateur $B(|V_{s-}-W_{s-}(\alpha)|)\beta(\theta)d\theta d\alpha ds$. Notre modélisation n'est peut-être pas la bonne...

Nous nous intéressons aussi à une équation de type Boltzmann, considérablement simplifiée mais non spatialement homogène :

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x,v) + v.\frac{\partial f}{\partial x}(t,x,v) &= \int_{\theta=-\pi}^{\pi}\{f(t,x,v+\theta) - f(t,x,v)\}\rho(t,x)\beta(\theta)d\theta \\ \rho(t,x) &= \int_{v\in I\!R} f(t,x,v)dv \end{aligned} \tag{64}$$

Cette fois, nous cherchons f sous la forme de la densité d'un couple de processus (X_t, V_t) satisfaisant les équations (si $\int_0^\pi \theta\beta(\theta)d\theta < \infty$) :

$$\begin{aligned} V_t &= V_0 + \int_0^t \int_0^{\rho(s,X_{s-})} \int_{-\pi}^{\pi} \theta N(d\theta, dz, ds) \\ X_t &= X_0 + \int_0^t V_s ds \qquad (65) \\ \mathcal{L}(X_t)(dx) &= \rho(t,x)dx \end{aligned}$$

L'existence d'une solution (V, X) pose un certain nombre de problèmes. En particulier, il faut manifestement, quelle que soit la suite de processus que nous voulons faire converger, utiliser le calcul de Malliavin à chaque étape. Nous sommes parvenus à construire des approximations de type "Euler", mais la convergence s'avère extrêmement compliquée.

Intéressons-nous maintenant aux Chapitres 5 et 6 : il serait intéressant de résoudre le même problème en dimension n. L'obstacle principal est le suivant. Nous utilisons un Théorème "d'uniforme inversion locale". C'est à dire qu'on applique uniformément le théorème d'inversion locale à des fonctions d'un ensemble du type (mais plus compliqué) :

$$\{g : I\!R \mapsto I\!R \ /c \leq |g'(0)| \leq K\} \qquad (66)$$

en dimension 1. Mais en dimension 2 (ou plus), cet ensemble doit être remplacé par

$$\left\{g : I\!R^2 \mapsto I\!R^2 \ /|\det g'(0)| \geq c \ , \ \| g'(0) \| \leq k\right\} \qquad (67)$$

C'est cette différence qui est la cause de notre échec : en dimension 1, le déterminant et la norme sont des quantités semblables, alors qu'il en va bien différemment en dimension 2. Ceci nous bloque car nous utilisons, en dimension 1, un temps d'arrêt du type

$$\tau = \inf\{t > 0 \ ; \ |X_t'(0)| \geq c\} \qquad (68)$$

On obtient ainsi $|X_\tau'(0)| \geq c$ p.s., et, pour une autre constante K, $|X_\tau'(0)| \leq K$ p.s. En dimension 2, on peut définir le temps d'arrêt soit pour assurer que $|\det X_\tau'(0)| \geq c$, soit pour assurer que $\| X_\tau'(0) \| \leq K$, mais l'un n'implique pas l'autre.

Dans le cas *particulier* de l'équation de Boltzmann bidimensionnelle du Chapitre 4, nous sommes néanmoins parvenus, dans un travail en préparation, à prouver un résultat de stricte positivité, sous des conditions similaires à celles du Chapitre 6, avec toutefois une restriction : le support de la distribution initiale P_0 doit être suffisamment grand.

Partie I

Etudes d'E.D.P.S. paraboliques avec sauts

Chapitre 1

Calcul de Malliavin pour une E.D.P.S. parabolique avec sauts Poissonniens

Résumé : Nous étudions une équation aux dérivées partielles stochastique de type parabolique, conduite par un bruit blanc gaussien et une mesure de Poisson. Après avoir défini les solutions faibles, nous vérifions un résultat d'existence et d'unicité. Nous construisons ensuite un calcul des variations stochastiques, afin de vérifier que sous certaines hypothèses de non-dégenerescence, la loi de la solution est absolument continue par rapport à la mesure de Lebesgue. Pour cela, nous introduisons deux opérateurs de dérivation, l'un lié au Bruit Blanc, l'autre à la mesure de Poisson. Le second est étudié en détail.

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1 Introduction.

Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ un espace probabilisé filtré, et L un réel positif. Munissons $\mathbb{R}^+ \times [0, L]$ et $\mathbb{R}^+ \times [0, L] \times \mathbb{R}$ de leurs tribus Boreliennes. Nous travaillerons dans presque tout ce travail dans le contexte suivant :

Hypothèse (I) :

1. W est un Bruit Blanc espace-temps sur $\mathbb{R}^+ \times [0, L]$ basé sur $dxdt$ (cf Walsh [47]) ;

2. N est une mesure de Poisson sur $I\!R^+ \times [0,L] \times I\!R$ d'intensité $\nu(dt,dx,dz) = dtdxq(dz)$, où q est une mesure σ-finie positive sur $I\!R$;
3. $\{\mathcal{F}_t\}_{t\geq 0}$ est la filtration canonique complète et càd associée aux objets indépendants W et N.

On notera $\tilde{N} = N - \nu$ la mesure de Poisson compensée de N.

On s'intéresse à l' E.D.P.S. suivante :

$$\left\{ \begin{array}{rcl} \dfrac{\partial V}{\partial t}(x,t) & = & \dfrac{\partial^2 V}{\partial x^2}(x,t) + g(V(x,t)) + f(V(x,t))\dot{W}_{x,t} + \displaystyle\int_{I\!R} h(V(x,t),z)\dot{\tilde{N}}_{x,t}(dz) \\ \dfrac{\partial V}{\partial x}(0,t) & = & \dfrac{\partial V}{\partial x}(L,t) = 0 \quad \forall t > 0 \\ V(x,0) & = & \mathcal{V}_0(x) \quad \forall x \in [0,L] \end{array} \right. \tag{1.1}$$

où f et g sont deux fonctions de $I\!R$ dans $I\!R$, h est une fonction de $I\!R \times I\!R$ dans $I\!R$, et $\mathcal{V}_0$ est une fonction aléatoire de $[0,L]$ dans $I\!R$. Les symboles $\dot{W}_{x,t}$ et $\dot{\tilde{N}}_{x,t}(dz)$ représentent les dérivées de Radon-Nikodym heuristiques de $W(dx,dt)$ et $\tilde{N}(dt,dx,dz)$ par rapport à la mesure de Lebesgue $dxdt$.

Les E.D.P.S. paraboliques conduites par un bruit blanc (i.e. (1.1) avec $h \equiv 0$) ont été introduites par Walsh dans [46] et [47]. Depuis, bien des travaux ont été réalisés dans ce domaine. Par contre, les propriétés des E.D.P.S. avec sauts sont très peu connues. Pourtant, dans [46], Walsh explique comment modéliser un phénomène neurophysiologique, ce qui l'a conduit a étudier les E.D.P.S. Il en ressort que l'équation qu'il propose, donnée par

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V + W \tag{1.2}$$

est obtenue en approximant une équation similaire conduite par un processus ponctuel de Poisson. Cette approximation est réaliste, car les sauts sont petits et nombreux, mais en aucun cas, le phénomène modélisé n'est continu. C'est pourquoi il paraît naturel d'étudier l'équation (1.1).

Dans la deuxième section, on s'intéressera à la définition de solutions faibles de (1.1). On prouvera dans la troisième section un théorème d'existence et d'unicité. Ces deux parties ne font que généraliser l'étude d'une E.D.P.S. déjà traitée par Walsh dans [47] : la difficulté vient de ce qu'on a ajouté le terme Poissonnien. Il est alors plus difficile de définir les solutions faibles de (1.1), car il faut faire attention de n'intégrer que des processus prévisibles.

La quatrième section traite le problème de l'existence d'une densité pour la loi de la solution de (1.1), quand l'intensité de N est "régulière". Cette partie est inspirée des articles de Bally, Pardoux, [4], de Bichteler, Gravereaux, Jacod, [6], et du livre de Nualart [30]. Bally et Pardoux prouvent l'existence d'une densité de classe C^∞ pour la solution de (1.1) dans le cas où $h = 0$. Bichteler, Gravereaux, et Jacod s'intéressent au calcul de Malliavin pour les processus de diffusion avec sauts Poissonniens. Enfin, le livre de Nualart, bien qu'il ne traite du calcul de Malliavin que dans un cadre purement Gaussien, a permis ici de prouver une multitude de

résultats techniques.
Enfin, la dernière section est constituée de quelques extensions de notre principal résultat.

Saint Loubert Bié s'intéresse dans sa thèse [40] à deux équations ressemblant à (1.1) : ces équations sont conduites par un bruit Poissonnien compensé ou non, et ne comportent pas d'intégrales stochastiques par rapport au Bruit Blanc. Dans le cas non compensé, il prouve plusieurs résultats remarquables à propos de la régularité de sa solution. Il parvient de plus à vérifier que la solution de son équation compensée est spatialement régulière (résultat qui s'étend à (1.1)).
Saint Loubert Bié construit aussi un calcul variationnel stochastique associé à son équation compensée, afin d'étudier l'existence d'une densité pour la loi de sa solution. Sa méthode conduit à des résultats relativement restreints, et ne peuvent pas s'étendre à (1.1), de par la présence du Bruit Blanc. Notre approche pour le calcul de Malliavin est assez différente, en particulier parce que nous n'utilisons pas de formule d'intégration par parties. Nous obtenons un théorème plus général et plus précis (nous comparerons ces résultats plus précisément au début de la Section 4).

2 Définition des solutions faibles.

On ne définira évidemment que des solutions au sens faible de (1.1), car une éventuelle solution n'a aucune chance d'être différentiable. Commençons par rappeler la définition suivante :

Définition 2.1 *Soit $Y = \{Y(y,s)\}$ un processus sur $[0,L] \times \mathbb{R}^+$. On dira que Y est*

- *mesurable s'il est $\mathcal{F} \otimes \mathcal{B}([0,L] \times \mathbb{R}^+)$-mesurable.*
- *prévisible s'il est $Pred \otimes \mathcal{B}([0,L])$-mesurable, où*

$$Pred = \sigma\left\{ X \ / \ X \text{ processus càg adapté sur } \mathbb{R}^+ \right\}$$

est une tribu sur $\Omega \times \mathbb{R}^+$.

- *une version faible de $X = \{X(y,s)\}_{[0,L]\times[0,T]}$ si $dP(\omega)dyds$-presque partout, $Y(y,s)(\omega) = X(y,s)(\omega)$.*

Ces définitions s'adaptent de manière évidente aux processus sur $[0,L] \times [0,T]$, où $T > 0$.

Précisons ensuite les définitions des intégrales stochastiques qu'on utilisera.

Définition 2.2 *Soit Y un processus admettant une version faible prévisible Y_-. Soit Φ une fonction mesurable telle que*

$$\int_0^T \int_0^L \int_{\mathbb{R}} E\left(\phi^2(Y(y,s),s,y,z)\right) q(dz)dyds < \infty \tag{2.1}$$

Alors on pose

$$\int_0^T \int_0^L \int_{\mathbb{R}} \phi(Y(y,s),s,y,z)\tilde{N}(ds,dy,dz) = \int_0^T \int_0^L \int_{\mathbb{R}} \phi(Y_-(y,s),s,y,z)\tilde{N}(ds,dy,dz) \tag{2.2}$$

La variable aléatoire obtenue ne dépend pas du choix de la version prévisible, à un ensemble $P(d\omega)$-négligeable près. On définit l'intégrale stochastique par rapport au bruit blanc de la même manière.

En utilisant la théorie classique de l'intégration stochastique, (voir Jacod, Shiryaev, [23] p 71-74, et Walsh, [47] p 292-298), on en déduit, puisque $Y_- = Y$ $dPdyds$-presque partout, que :

$$E\left[\left(\int_0^T\int_0^L\int_{I\!R}\Phi(Y(y,s),s,y,z)\tilde{N}(ds,dy,dz)\right)^2\right] = \int_0^T\int_0^L\int_{I\!R} E\left(\Phi^2(Y(y,s),s,y,z)\right)q(dz)dyds \tag{2.3}$$

$$E\left[\left(\int_0^T\int_0^L\Phi(Y(y,s),s,y)W(dy,ds)\right)^2\right] = \int_0^T\int_0^L E\left(\Phi^2(Y(y,s),s,y)\right)dyds \tag{2.4}$$

$$E\left[\left(\int_0^T\int_0^L\Phi(Y(y,s),s,y)dyds\right)^2\right] \le TL\int_0^T\int_0^L E\left(\Phi^2(Y(y,s),s,y)\right)dyds \tag{2.5}$$

Définissons maintenant les solutions faibles de (1.1). On suit ici le raisonnement de Walsh dans [47] (qui s'intéresse à la même équation avec $h=0$). Les hypothèses suivantes, suffisantes pour prouver l'existence et l'unicité d'une solution faible de (1.1), assurent en particulier la validité des intégrales ci-dessous.

Hypothèse (H) : f et g sont globalement lipschitziennes sur $I\!R$. h est mesurable sur $I\!R\times I\!R$, et il existe une fonction $\eta\in L^2(I\!R,q)$ positive sur $I\!R$ telle que :

$$\forall x,y,z\in I\!R \qquad |h(x,z)-h(y,z)|\le\eta(z)|x-y| \qquad \text{et} \qquad |h(0,z)|\le\eta(z)$$

Hypothèse (H_2) : $\mathcal{V}_0$ est $\mathcal{F}_0\otimes\mathcal{B}([0,L])$-mesurable, et est bornée dans L^2, c'est à dire que $\sup_{x\in[0,L]}E(\mathcal{V}_0^2(x))<\infty$.

Définition 2.3 *Soit $\{V(x,t)\}_{x\in[0,L],\,t\ge0}$ un processus admettant une version faible prévisible, et borné dans L^2 sur $[0,L]\times[0,T]$ pour tout $T>0$, c'est à dire tel que*

$$\forall T>0, \qquad \sup_{[0,L]\times[0,T]} E\left(V^2(y,s)\right)<\infty$$

Alors V est solution faible de (1.1) si pour tout $\Phi\in\mathcal{C}^\infty([0,L])$ vérifiant $\Phi'(0)=\Phi'(L)=0$, pour tout $t>0$, p.s., (on utilise les notations de la Définition 2.2)

$$\begin{aligned}\int_0^L V(x,t)\Phi(x)dx &= \int_0^L\mathcal{V}_0(x)\Phi(x)dx+\int_0^t\int_0^L V(x,s)\Phi''(x)dxds\\ &+\int_0^t\int_0^L f(V(x,s))\Phi(x)W(dx,ds)+\int_0^t\int_0^L g(V(x,s))\Phi(x)dxds\\ &+\int_0^t\int_0^L\int_{I\!R} h(V(x,s),z)\Phi(x)\tilde{N}(ds,dx,dz)\end{aligned} \tag{2.6}$$

La justification de cette définition est la suivante : si on multiplie l'équation (1.1) à droite et à gauche par une fonction test Φ de la forme de celles qui apparaissent dans la définition, si on intègre l'expression obtenue sur $[0,t]\times[0,L]$, et si on utilise la formule usuelle d'intégration par parties, on tombe sur l'égalité (2.6).

D'autre part, on cherche une solution admettant une version faible prévisible (et non càdlàg), pour des raisons qui figurent dans la remarque finale de cette section.

Cherchons maintenant une équation d'évolution associée à (1.1). Soit $G_t(x,y)$ la fonction de Green associée au système déterministe :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad ; \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \tag{2.7}$$

Ceci signifie que $G_t(x,y)$ est la solution du système avec pour condition initiale une masse de Dirac au point y. On peut calculer G explicitement :

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in Z} \left[\exp\left(\frac{-(y-x-2nL)^2}{4t} \right) + \exp\left(\frac{-(y+x-2nL)^2}{4t} \right) \right] \tag{2.8}$$

Les propriétés de G qu'on utilisera sont toutes dans l'Appendice.

Le lemme qui suit est fondamental pour la suite de ce travail (les fonctions f, g, et h qui y figurent satisfont l'hypothèse (H)) :

Lemme 2.4 *Soit $\{V(y,s)\}$ un processus admettant une version faible prévisible, borné dans L^2 sur $[0,L]\times[0,T]$. Alors les processus suivants admettent des versions faibles prévisibles :*

$$\begin{aligned} U_1(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) f(V(y,s)) W(dy,ds) \\ U_2(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g(V(y,s)) dyds \\ U_3(x,t) &= \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h(V(y,s),z) \tilde{N}(ds,dy,dz) \end{aligned}$$

De plus, ces processus sont bornés dans L^2 sur $[0,L]\times[0,T]$.

<u>Preuve du Lemme 2.4</u> : intéressons-nous par exemple à U_3. On peut approcher (cf Walsh [47], p 323) $G_t(x,y)$ de la manière suivante :

$$\sup_{x\in[0,L]} \int_0^T \int_0^L \left(G_t(x,y) - \sum_{k=0}^N \phi_k(x)\phi_k(y) e^{-\lambda_k t} \right)^2 dydt \longrightarrow^{N\to\infty} 0$$

où, $\phi_0 = \frac{1}{\sqrt{L}}$, $\phi_k(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{k\pi x}{L}\right)$, et $\lambda_k = \left(\frac{\pi}{L}\right)^2 k^2$. Considérons donc la suite de processus prévisibles définie par :

$$U_3^N(x,t) = \sum_{k=0}^N \phi_k(x) e^{-\lambda_k t} M_{t-}^k$$

où M^k est la martingale càdlàg de carré intégrable suivante :

$$M_t^k = \int_0^t \int_0^L \int_{I\!R} \phi_k(y) e^{\lambda_k s} h(V(y,s),z) \tilde{N}(ds,dy,dz)$$

Il est alors facile de voir, à l'aide de (2.3) et de (H), que

$$\sup_{x,t} E\left(\left(U_3(x,t) - U_3^N(x,t)\right)^2\right)$$

$$\leq K \sup_{x,t} \int_0^t \int_0^L \left(G_{t-s}(x,y) - \sum_{k=0}^N \phi_k(x)\phi_k(y)e^{-\lambda_k(t-s)}\right)^2 dyds$$

$$\leq K \sup_{x\in[0,L]} \int_0^T \int_0^L \left(G_t(x,y) - \sum_{k=0}^N \phi_k(x)\phi_k(y)e^{-\lambda_k t}\right)^2 dydt \longrightarrow^{N\to\infty} 0$$

D'autre part, U_3^N est prévisible, pour chaque N. La conclusion est alors immédiate. Pour vérifier que U_3 est borné dans L^2 sur $[0,L]\times[0,T]$, il suffit d'appliquer la formule (2.3), d'utiliser (H), puis l'Appendice (6.3).

Ecrivons enfin l'équation d'évolution associée à (1.1) :

Proposition 2.5 *Un processus V, admettant une version faible prévisible, borné dans L^2 sur $[0,L]\times[0,T]$ pour tout $T>0$, est solution faible de (1.1) si et seulement si pour tout x dans $[0,L]$, pour tout $t\geq 0$, p.s.,*

$$\begin{aligned} V(x,t) &= \int_0^L \mathcal{V}_0(y)G_t(x,y)dy + \int_0^t\int_0^L f(V(y,s))G_{t-s}(x,y)W(dy,ds) \\ &+ \int_0^t\int_0^L g(V(y,s))G_{t-s}(x,y)dyds \\ &+ \int_0^t\int_0^L\int_{I\!R} h(V(y,s),z)G_{t-s}(x,y)\tilde{N}(ds,dy,dz) \end{aligned} \tag{2.9}$$

La preuve de cette proposition est la même que dans le cas où $h=0$ (cf Walsh, [47]), le Théorème de Fubini utilisé s'appliquant en effet aussi bien à la mesure martingale associée à $\tilde{N}$ qu'à celle associée à W.

Il n'est pas habituel de travailler ainsi avec des processus admettant des versions faibles prévisibles. Dans un cadre "continu" ($h=0$), ces problèmes n'apparaissent pas. Pour étudier les diffusions avec sauts, (voir par exemple [6]), l'habitude est de chercher des solutions à trajectoires càdlàg p.s., et d'intégrer leurs limites à gauche (qui sont prévisibles). Ici, il est manifestement impossible de construire une solution de (2.9) dont les trajectoires sont càdlàg (ou plutôt càglàd) en temps sur $[0,T]$ pour chaque x dans $[0,L]$. En effet, ceci est déjà impossible dans le cas de l'équation considérablement simplifiée, sans terme de dérive, sans terme gaussien, avec une mesure de Poisson non compensée dont l'intensité $dtdxq(dz)$ vérifie $q(I\!R)<\infty$, avec une fonction h de la forme $h(x,z)=h(z)$, et avec condition initiale $\mathcal{V}_0\equiv 1$. En écrivant $N=\sum_{i=1}^{\mu}\delta_{(T_i,X_i,Z_i)}$, cette équation devient :

$$V(x,t) = 1 + \sum_{i=1}^{\mu} h(Z_i)G_{t-T_i}(x,X_i)1_{\{t>T_i\}}$$

Sous cette forme, on voit que pour chaque ω, l'application $t \mapsto V(\omega, X_1(\omega), t)$ n'est pas limité à droite, puisque $G_{t-T_1}(X_1, X_1)$ n'admet pas de limite finie lorsque t décroit vers T_1.

Dans [40], Saint Loubert Bié choisit une autre approche pour contourner ce problème : il vérifie que si $Y(x,t)$ est un processus progressif, alors le processus prévisible

$$Y_-(x,t) = \liminf_n \frac{n^2}{2} \int_{t-\frac{1}{n}}^{t} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} Y(y,s)dyds$$

est égal à $Y(x,t)$ $dxdtdP$-presque partout, et il choisit de définir des solutions progressivement mesurables.

3 Existence et unicité.

On s'intéresse maintenant à l'existence et à l'unicité des solutions faibles de (1.1). On ne traite pas ce problème dans un espace L^p avec $p > 2$, car l'inégalité de Burkholder appliquée au terme Poissonnien est difficile à manier, et parce qu'il est clair que la solution faible de (1.1) n'a aucune chance d'appartenir à un espace L^p, avec $p > 3$, puisque $\int_0^t \int_0^1 G^3_{t-s}(x,y)dyds = \infty$.

Théorème 3.1 *Sous les hypothèses (I), (H_2) et (H), il existe une unique solution faible de (1.1) au sens de la Définition 2.3. Pour être plus précis, si V et V' sont deux telles solutions de (1.1), alors pour tout x, t, p.s., $V(x,t) = V'(x,t)$.*

Preuve : nous suivons la preuve du Théorème 3.2 p 313 de Walsh [47]. L'unicité se vérifie facilement. Pour l'existence, la méthode de Walsh est légerement plus difficile à adapter, car nous cherchons une solution admettant une version faible prévisible.

1) Soit V et V' deux solutions faibles de (1.1), et $T > 0$. Soit

$$F(x,t) = E\left((V(x,t) - V'(x,t))^2\right), \qquad \text{et} \qquad \phi(t) = \sup_{x\in[0,L]} F(x,t)$$

On va montrer que ϕ est nulle sur $[0,T]$ à l'aide du lemme de Gronwall. Majorons $F(x,t)$, en utilisant les formules (2.3), (2.4), (2.5), puis (H) (K est une constante dont la valeur change d'une ligne à l'autre) :

$$\begin{aligned} F(x,t) &\leq K \int_0^t \int_0^L F(y,s) G^2_{t-s}(x,y) \left(1 + \int_{\mathbb{R}} \eta^2(z) q(dz)\right) dyds \\ &\leq K \int_0^t \phi(s) \frac{ds}{\sqrt{t-s}} \end{aligned}$$

où la dernière inégalité provient de l'Appendice (6.2). On obtient donc

$$\phi(t) \leq K \int_0^t \phi(s) \frac{ds}{\sqrt{t-s}}$$

En itérant une fois cette formule, en utilisant le Théorème de Fubini, puis l'inégalité, valable pour tout $0 \leq u < t$, $\int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} \leq 4$, on obtient :

$$\phi(t) \leq K \int_0^t \phi(s) ds$$

De plus, ϕ est bornée sur $[0,T]$ par hypothèse. Le lemme de Gronwall permet donc d'affirmer que $\phi \equiv 0$ sur $[0,T]$, et comme T est arbitraire, on en déduit $\phi \equiv 0$, puis $V(x,t) = V'(x,t)$ p.s. pour tout x, t.

2) On va construire une solution par la méthode de Picard. Posons

$$V^0(x,t) = \int_0^L \mathcal{V}_0(y) G_t(x,y) dy \tag{3.1}$$

V^0 est un processus prévisible (il est $\mathcal{F}_0$-mesurable). De plus, par l'inégalité de Cauchy-Schwarz, par (H_2), puis l'Appendice (6.2), V^0 est bornée dans L^2 sur $[0,L] \times [0,T]$:

$$\begin{aligned} E\left(\left(V^0(x,t)\right)^2\right) &= E\left[\left(\int_0^L \mathcal{V}_0(y) G_t(x,y) dy\right)^2\right] \\ &\leq E\left[\int_0^L \mathcal{V}_0^2(y) G_t(x,y) dy \times \int_0^L G_t(x,y) dy\right] \\ &\leq K \left(\int_0^L G_t(x,y) dy\right)^2 \leq K \end{aligned}$$

Ceci permet de définir par récurrence, à l'aide du Lemme 2.4, les approximations de Picard, admettant des versions faibles prévisibles, et bornées dans L^2 sur $[0,L] \times [0,T]$, suivantes :

$$\begin{aligned} V^{n+1}(x,t) &= V^0(x,t) + \int_0^t \int_0^L f(V^n(y,s)) G_{t-s}(x,y) W(dy,ds) \\ &\quad + \int_0^t \int_0^L g(V^n(y,s)) G_{t-s}(x,y) dy ds \\ &\quad + \int_0^t \int_0^L \int_{I\!R} h(V^n(y,s), z) G_{t-s}(x,y) \tilde{N}(ds,dy,dz) \end{aligned} \tag{3.2}$$

Nous allons montrer que V^n converge dans L^2. Posons

$$F^n(x,t) = E\left[\left(V^{n+1}(x,t) - V^n(x,t)\right)^2\right] \quad \text{et} \quad \phi^n(t) = \sup_{x \in [0,L]} F^n(x,t)$$

On vérifie exactement comme dans 1) que

$$\phi^{n+1}(t) \leq K \int_0^t \phi^n(s) \frac{ds}{\sqrt{t-s}} \tag{3.3}$$

puis

$$\phi^{n+2}(t) \leq C \int_0^t \phi^n(s)ds$$

Prouvons ensuite que ϕ^0 est bornée sur $[0,T]$. On montre toujours de la même manière que

$$\begin{aligned} F^0(x,t) &\leq K \int_0^t \int_0^L G^2_{t-s}(x,y) E\left(f^2(V^0(y,s)) + g^2(V^0(y,s))\right) dyds \\ &+K \int_0^t \int_0^L \int_{I\!R} G^2_{t-s}(x,y) E\left(h^2(V^0(y,s),z)\right) dyq(dz)ds \end{aligned}$$

Mais V^0 est bornée dans L^2 sur $[0,L] \times [0,T]$. Il est alors facile de vérifier que F^0, donc ϕ^0, puis ϕ^1 sont bornées, à l'aide de (H), de l'Appendice (6.3), puis (3.3). En utilisant le Lemme de Picard, on en déduit que les séries $\sum_n (\phi^{2n}(t))^{\frac{1}{2}}$ et $\sum_n (\phi^{2n+1}(t))^{\frac{1}{2}}$ convergent uniformément sur $[0,T]$. Il en est donc de même pour $\sum_n (\phi^n(t))^{\frac{1}{2}}$, et la suite V^n converge dans L^2, uniformément sur $[0,L] \times [0,T]$, vers une limite V bornée dans L^2 sur $[0,L] \times [0,T]$, qui admet une version faible prévisible. Enfin, un simple passage à la limite dans L^2 de (3.2) montre que V satisfait (2.9).

Terminons cette section avec quelques remarques sur la régularité du processus solution :

Remarque 3.2 *Supposons (I), (H_2), (H), et qu'il existe $1 < p < 2$ tel que $\eta \in L^p(q)$ (η est la fonction qui apparaît dans (H)). Alors, si V est la solution faible de (1.1), et si $t > 0$, l'application $x \longrightarrow V(x,t)$ admet une version continue.*

Ceci découle du Lemme 6.6 de l'Appendice, du à Saint Loubert Bié [40]. Il suffit en effet de remarquer que le premier terme du membre de droite de (2.9) est continu (car $t > 0$) en espace, puis d'utiliser le lemme de Kolmogorov, et le Lemme 6.6 pour vérifier que les autres termes sont continus. On trouvera plus de précisions dans [40].

Signalons aussi d'autres résultats prouvés dans [40], dans le cas où la mesure de Poisson n'est plus compensée : supposons (I), (H_2), (H), $\eta \in L^1(q)$, $f = 0$, et que g est bornée. Alors pour chaque $t > 0$, la fonction $V(.,t)$ est p.s. continue sur $[0,L]$; et pour chaque $x \in [0,L]$, la fonction $V(x,.)$ est p.s. continue sur $[0,T]$. Ces résultats s'étendent facilement au cas où f n'est plus nulle, mais seulement bornée.

Une question subsiste : dans quel espace "vivent" les trajectoires (conjointement en x,t) de la solution de (1.1). Le résultat suivant est très faible, mais les trajectoires ne semblent pas appartenir à un "meilleur" espace. Nous ne sommes pas parvenus à nous débarrasser de l'hypothèse $\eta \in L^1(q)$. On note $\mathcal{M}_b$ l'ensemble des mesures bornées (signées) sur $[0,L]$.

Remarque 3.3 *Supposons (I), (H_2), (H), que $\eta \in L^1(q)$, et que f est bornée. Soit V l'unique solution faible de (1.1). Alors l'application $t \mapsto V(x,t)dx$ est presque sûrement càglàd de $[0,T]$ dans $\mathcal{M}_b$ muni de la topologie de la convergence étroite. De plus,*

$$E\left(\sup_{[0,T]} \parallel V(.,t) \parallel_{L^1([0,L])}\right) < \infty$$

Par contre, l'application $t \mapsto V(.,t)$ n'est en général pas presque sûrement limitée à droite de $[0,T]$ dans $L^1([0,L])$ muni de sa topologie forte.

<u>Preuve</u> : sous l'hypothèse $\eta \in L^1(q)$, on peut écrire, en posant $G(x) = g(x) - \int_{I\!R} h(x,z)q(dz)$,

$$V(x,t) = V^0(x,t) + V^1(x,t) + V^2(x,t) + V^3(x,t)$$

où

$$V^0(x,t) = \int_0^L G_t(x,y)\mathcal{V}_0(y)dy \quad ; \quad V^1(x,t) = \int_0^t \int_0^L G_{t-s}(x,y)f(V(y,s))W(dy,ds)$$

$$V^2(x,t) = \int_0^t \int_0^L G_{t-s}(x,y)G(V(y,s))dyds$$

$$V^3(x,t) = \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y)h(V(y,s),z)N(ds,dy,dz)$$

Comme p.s., $\mathcal{V}_0 \in L^2([0,L])$ et $G(V(y,s)) \in L^2([0,L]\times[0,T])$, il est connu que $V^0(.,t)$ et $V^2(.,t)$ sont presque sûrement continus de $[0,T]$ dans $L^1([0,L])$ muni de sa topologie forte, et que

$$E\left(\sup_{[0,T]} \| V^0(.,t) \|_{L^1([0,L])}\right) + E\left(\sup_{[0,T]} \| V^2(.,t) \|_{L^1([0,L])}\right) < \infty$$

Puisque f est bornée, il est d'autre part connu que V^1 est presque sûrement continu sur $[0,T]\times[0,L]$ et que

$$E\left(\sup_{[0,T]\times[0,L]} |V^1(x,t)|^2\right) < \infty$$

ce qui nous suffit amplement. Enfin, on s'intéresse à V^3. Pour cela, considérons une suite croissante E_p de sous ensembles de $I\!R$ vérifiant

$$\forall\, p, \quad q(E_p) < \infty \qquad \text{et} \qquad \cup_p E_p = I\!R$$

On pose

$$V_p^3(x,t) = \int_{[0,t[} \int_0^L \int_{E_p} G_{t-s}(x,y)h(V(y,s),z)N(ds,dy,dz)$$

Comme $q(E_p) < \infty$, la mesure de Poisson $N^p = N|_{[0,T]\times[0,L]\times E_p}$ peut s'écrire sous la forme $N^p(dt,dx,dz) = \sum_{i=1}^{\mu} \delta_{(T_i^p,X_i^p,Z_i^p)}(dt,dx,dz)$. On obtient ainsi

$$V_p^3(x,t) = \sum_{i=1}^{\mu} G_{t-T_i^p}(x,X_i^p)1_{\{t>T_i^p\}}h(V(X_i^p,T_i^p),Z_i^p)$$

Il est maintenant clair que pour chaque ω, la régularité de la trajectoire $V_p^3(\omega,x,t)$ est la même que celle de la fonction $G_{t-t_0}(x,x_0)1_{\{t>t_0\}}$, où $t_0 > 0$ et $x_0 \in [0,L]$ sont fixés. Cette fonction est càglàd de $[0,T]$ dans $\mathcal{M}_b$ muni de la topologie de la convergence étroite. Mais elle n'est pas limitée à droite de $[0,T]$ dans $L^1([0,L])$ muni de sa topologie forte, puisque $G_{t-t_0}(x,x_0)dx$

converge étroitement vers la masse de Dirac en x_0 quand t décroit vers t_0.
Il ne reste plus qu'à vérifier la convergence vers 0 quand p tend vers l'infini de

$$E\left(\sup_{[0,T]} \| V^3(.,t) - V_p^3(.,t) \|_{L^1([0,L])}\right)$$

Cette convergence s'obtient facilement, à l'aide de (H), et de l'hypothèse $\eta \in L^1(q)$.

4 Calcul de Malliavin.

On cherche maintenant à prouver l'existence d'une densité pour la loi de $V(x,t)$, où V est solution de (1.1) et (x,t) est fixé dans $[0,L]\times]0,\infty[$. On se placera sur $[0,L]\times[0,T]$, avec $T>0$, ce qui suffit puisque t est fixé.

Afin d'étudier ce problème, on émet des hypothèses supplémentaires. Tout d'abord, on suppose que l'intensité de N est "régulière" :

Hypothèse (M) :

1. W est un Bruit Blanc espace-temps sur $[0,T]\times[0,L]$ basé sur $dxdt$;
2. N est une mesure de Poisson sur $[0,T]\times[0,L]\times I\!R$ d'intensité

   $$\nu(ds,dy,dz) = \varphi(z)1_O(z)dsdydz$$

   où O est un ouvert de $I\!R$, et où φ est une fonction strictement positive de classe C^1 sur O.
3. $\{\mathcal{F}_t\}_{t\in[0,T]}$ est la filtration canonique complète et càd associée aux objets indépendants W et N.

On impose, afin de simplifier, des conditions sur $\mathcal{V}_0$ plus fortes que $(H)_2$:

Hypothèse (D) : la fonction initiale $\mathcal{V}_0$ est déterministe, mesurable, et bornée.

D'autre part, nous remplaçons l'hypothèse (H) par la suivante, plus forte :

Hypothèse (H') : f et g sont de classe C^1 sur $I\!R$, à dérivées bornées. La fonction $h(x,z)$ sur $I\!R\times O$ admet les dérivées partielles h'_z, $h''_{zx} = h''_{xz}$, et h'_x ; toutes sont continues sur $I\!R\times O$; il existe une constante K et une fonction $\eta \in L^2(O,\varphi(z)dz)$ telles que :

$$\forall x\in I\!R,\ \forall z\in O, \qquad |h'_z(0,z)| + |h''_{xz}(x,z)| \le K$$
$$\forall x\in I\!R,\ \forall z\in O, \qquad |h(0,z)| + |h'_x(x,z)| \le \eta(z)$$

On travaillera d'autre part avec une fonction ρ strictement positive et de classe C^1 sur O, bornée ainsi que sa dérivée, et satisfaisant

$$\rho \in L^1(O,\varphi(z)dz) \tag{4.1}$$

Ce "poids", qu'on peut choisir selon les paramètres de (1.1), permet en particulier aux fonctions bornées d'être intégrables pour $\rho(z)\nu(ds,dy,dz)$. La condition suivante est technique.

Hypothèse (S) : il existe une famille de fonctions K_ϵ positives et C^1 sur O, à support compact (dans O), bornées par 1, et satisfaisant

$$\forall\, z \in O, K_\epsilon(z) \longrightarrow_{\epsilon\to 0} 1 \quad ; \quad \int_O \left(K'_\epsilon(z)\right)^2 \eta^2(z)\rho(z)\varphi(z)dz \longrightarrow_{\epsilon\to 0} 0 \tag{4.2}$$

On suppose enfin une des hypothèses de non-dégénerescence suivantes :

Hypothèse (EW) : pour tout x dans $I\!R$, $f(x) \neq 0$.

Hypothèse $(EP1)$: la fonction f est identiquement nulle ; la fonction h'_x est positive ou nulle ; il existe une fonction $\hat{\eta}(z) \in L^1(O, \varphi(z)dz)$ telle que $h'_x(x,z) \leq \hat{\eta}(z)$ pour tout x dans $I\!R$ et tout z dans O ; et pour tout x dans $I\!R$,

$$\int_O 1_{\{h'_z(x,z)\neq 0\}}\varphi(z)dz = \infty$$

Hypothèse $(EP2)$: soit $\mathcal{H} = \left\{z \in O \;\Big/ \forall x \in I\!R,\ (h'_z(x,z))^2 > 0\right\}$. Il existe des constantes $C_0 > 0$, $r_0 \in]\frac{3}{4}, 1[$, et $\gamma_0 \geq 0$ telles que pour tout $\gamma \geq \gamma_0$,

$$\int_{\mathcal{H}} \left(1 - e^{-\gamma\rho(z)}\right)\varphi(z)dz \geq C_0 \times \gamma^{r_0} \tag{4.3}$$

L'hypothèse $(EP2)$ signifie simplement que l'ensemble $\mathcal{H}$ est grand. Dans le cas où h est de la forme $h(x,z) = c(x)\eta(z)$, $(EP2)$ est vérifiée si c ne s'annule jamais et si le support de η est grand. Le but de cette section est de vérifier le théorème suivant :

Théorème 4.1 *Supposons que (M), (D), (H'), et (S) soient satisfaites. Soit V l'unique solution faible de (1.1) au sens de la Définition 2.3, et soit $(x,t) \in [0,L]\times]0,T]$. Alors sous l'une des hypothèses (EW), $(EP1)$ ou $(EP2)$, la loi de $V(x,t)$ admet une densité par rapport à la mesure de Lebesgue sur $I\!R$.*

Nous supposerons donc dans toute la suite que les hypothèses (M), (D), (H'), et (S) sont satisfaites.

Nous allons utiliser deux opérateurs de dérivation. Le premier sera l'opérateur classique lié au Bruit Blanc, voir par exemple Nualart [30]. Le second opérateur, lié à la mesure de Poisson, est inspiré par le Chapitre IV de Bichteler et al., [6]. Ceux-ci s'intéressent au calcul de Malliavin pour les processus de diffusion avec sauts Poissonniens. Mais ils supposent que N admet une intensité du type $1_O(z)dsdydz$. De plus, ils ne définissent pas d'opérateur de dérivation. Ils travaillent à l'aide d'un "produit scalaire de dérivation", ce qui ne conduit ici qu'à des résultats très restreints : on ne pourrait vérifier le Théorème 4.1, en utilisant cette méthode, que sous une hypothèse du type $(EP1)$.

Nous utiliserons un critère d'absolue continuité du type "Bouleau, Hirsch" (voir [9]), qui permet de ne pas utiliser de formule d'intégration par parties (sauf bien entendu sur le domaine des variables simples, pour vérifier que nos opérateurs sont fermables). Ce critère facilite les preuves, et permet de réduire les hypothèses de régularité sur les paramètres de l'E.D.P.S. (1.1).

Le Théorème 4.1 établit deux résultats distincts : la loi de $V(x,t)$ admet une densité soit grâce au Bruit Blanc, soit grâce à la mesure de Poisson. Il paraît en effet difficile de formuler une condition de non-dégénerescence faisant intervenir à la fois f et h. Dans le cas des E.D.S., ceci est possible car la formule d'Itô, via les exponentielles de Doléans-Dade, permet de calculer explicitement la "dérivée" du processus solution.

L' hypothèse d'ellipticité (EW) paraît raisonnable. Dans [33], Pardoux et Zhang ont prouvé, dans le cas où $h=0$ et où la condition initiale est continue, l'existence d'une densité sous une hypothèse très faible : il suffit qu'il existe $y \in [0,L]$ tel que $f(\mathcal{V}_0(y)) \neq 0$. Mais ils utilisent fortement la continuité de leur solution.
L'hypothèse $(EP1)$ contient deux conditions : la première ($f=0$, $0 \leq h'_x \leq \hat{\eta} \in L^1(O, \varphi(z)dz)$) est assez restrictive, mais la seconde (pour tout $x \in I\!R$, $\int_O 1_{\{h'_z(x,z)\neq 0\}} \varphi(z)dz = \infty$) semble minimale, car elle apparaît déjà dans le cadre des diffusions avec sauts (voir Bichteler, Jacod, [7]).
Enfin, l'hypothèse $(EP2)$ est plus raisonnable au sens où elle permet la présence d'un bruit blanc, d'une mesure de Poisson "vraiment compensée", et n'impose pas de monotonie. Bichteler et al. utilisent dans [6] des hypothèses qui ressemblent un peu à $(EP2)$, pour prouver la régularité de la densité de leurs processus de diffusion (voir Assumption (SB) et Definition 2-20 p 13-14 dans [6]).

Afin de clarifier les hypothèses $(EP2)$ et (S), précisons quelques exemples d'application.

Remarque 4.2 *Si $O = I\!R$, alors (S) est satisfaite pour toutes φ, η, et tout choix de ρ.*

Preuve : il suffit de choisir une famille de fonctions positives et C^1 de la forme

$$K_\epsilon(z) = \begin{cases} 1 & \text{si} \quad |z| < 1/\epsilon \\ 0 & \text{si} \quad |z| > 1/\epsilon + 2 \end{cases}$$

telles que $|K'_\epsilon(z)| \leq 1_{\{|z| \in [1/\epsilon, 1/\epsilon+2\}}$. En utilisant le Théorème de Lebesgue et le fait que $\rho\eta^2 \in L^1(I\!R, \varphi(z)dz)$, (4.2) est immédiat.

Exemple 1 : supposons que $O = I\!R$, et que φ est une fonction C^1 sur $I\!R$ satisfaisant $K > \varphi > a$, pour certains $K > a > 0$. Considérons une fonction $h(x,z) = c(x)\eta(z)$, où c est une fonction strictement positive et C^1 sur $I\!R$ dont la dérivée est bornée. η doit être C^1 sur $I\!R$, appartenir à $L^2(I\!R, \varphi(z)dz)$, et η' doit être bornée. Si pour un certain $b \in I\!R$, $[b, \infty[\subset \{\eta' \neq 0\}$, alors (M), (H'), (S), et $(EP2)$ sont satisfaites :
grâce à la Remarque 4.2, il suffit de vérifier $(EP2)$. En choisissant $\rho(z) \geq z^{\frac{-7}{6}} 1_{\{z \geq b \vee 1\}}$, on voit que (4.3) est satisfaite, car $[b, \infty[\subset \mathcal{H}$, en utilisant

$$\forall\, x \in [0,1], \quad 1 - e^{-x} \geq x/2 \tag{4.4}$$

Exemple 2 : supposons que $O =]0,1[$, et que $\varphi(z) = 1/z^r$, pour un certain $r > 7/4$. Considérons une fonction $h(x,z) = c(x)\eta(z)$, où c est strictement positive et C^1 sur $I\!R$, avec c' bornée, et où $\eta(z) = z^\alpha$, pour un certain $\alpha > 1 \vee \frac{r-1}{2} \vee \frac{7-r}{6}$. Alors (M), (H'), (S), et $(EP2)$ sont satisfaites : (M) est satisfaite, ainsi que (H'), car $\alpha > 1 \vee \frac{r-1}{2}$. Il est clairement possible de choisir $\rho(z)$ de la forme

$$\rho(z) = \begin{cases} z^\beta & \text{si} \quad z \leq 1/4 \\ (1-z)^\delta & \text{si} \quad z \geq 3/4 \end{cases} \tag{4.5}$$

avec $\beta > 1 \vee (r-1)$ et $\delta \geq 1$. En utilisant (4.4), le fait que $\mathcal{H} =]0,1[$ et $\rho(z) \geq z^\beta 1_{]0,1/4[}(z)$, on voit que $(EP2)$ est satisfaite si $\beta < \frac{4}{3}(r-1)$.
Choisissons alors une famille K_ϵ de fonctions C^1 positives sur $]0,1[$, bornées par 1, et satisfaisant

$$K_\epsilon(z) = \begin{cases} 0 & \text{si} \quad z < \epsilon/2 \\ 1 & \text{si} \quad \epsilon < z < 1-\epsilon \\ 0 & \text{si} \quad 1-\epsilon/2 < z < 1 \end{cases} \quad ; \quad |K'_\epsilon(z)| \leq \frac{4}{\epsilon} 1_{]\epsilon/2,\epsilon[\cup]1-\epsilon,1-\epsilon/2[}(z)$$

Un calcul explicite montre que (S) est satisfaite si $\beta > r+1-2\alpha$ et si $\delta > 1$.
Puisque $\alpha > \frac{7-r}{6}$ et $r > 7/4$, il est possible de choisir β dans $](r-1), \frac{4}{3}(r-1)[\cap]1,\infty[\cap]r+1-2\alpha,\infty[$, et la conclusion s'ensuit.

Remarquons finalement que (S) est satisfaite pour tout O, η, ϕ, si ρ est de classe C^2_b, et si $\rho(z) + |\rho'(z)| \longrightarrow_{z\to\partial O} 0$. Bichteler, Gravereaux et Jacod émettent ce type de condition sur ρ dans [6].
Remarquons d'autre part que sous l'une des hypothèses $(EP1)$ ou $(EP2)$, on a forcément $\int_O \varphi(z)dz = \infty$: pour que $V(x,t)$ admette une densité grâce à la mesure de Poisson, il faut que celle-ci charge p.s. un nombre infini de points.

Dans sa thèse [40], Saint Loubert Bié s'intéresse à une équation du type (1.1) avec $f = 0$. Il parvient à prouver l'existence d'une densité sous des hypothèses moins restrictives que (M) et (D), et sous les hypothèses $(h1)$, et $(h2)$ ou $(h3)$, ci-dessous (les notations sont adaptées, ainsi que le cadre).
$\underline{(h1)}$: $f = 0$, g admet une dérivée bornée et lipschitzienne. h est de classe C^1 en x, et il existe $1 \leq p_0 < p_1 \leq 3$, et une fonction $\eta \in \cap_{]p_0,p_1[} L^p(O, \varphi(z)dz)$ tels que

$$|h(x,z)| \leq C(1+|x|)\eta(z) \qquad |h'_x(x,z)| \leq C\eta(z)$$

$$|h'_x(x,z) - h'_x(y,z)| \leq C|x-y|\eta(z)$$

h est de classe C^1 en z, et

$$|h'_z(x,z)| \leq C(1+|x|) \qquad |h'_z(x,z) - h'_z(y,z)| \leq C|y-x|$$

$\underline{(h2)}$: $h'_x = 0$ et $\int_{I\!R} 1\!\!1_{\{h'_z(z)\neq 0\}} \varphi(z)dz = \infty$
$\underline{(h3)}$: $\eta \in L^1(O, \varphi(z)dz)$, $h'_x \geq 0$, et une hypothèse du type $(EP1)$, mais dépendant directement du processus solution V.

L'hypothèse $(h1)$ est assez proche de (H'). L'hypothèse $(h2)$ est très restrictive, car elle signifie que la fonction h ne dépend pas de x. $(h3)$, bien qu'en fait comparable à $(EP1)$, est difficilement exploitable, car elle s'exprime à l'aide du processus solution. L'existence d'une densité pour la loi de $V(x,t)$ sous $(h2)$ ou $(h3)$ (comme sous $(EP1)$) utilise le fait que N est une mesure positive. Ceci ne peut pas s'étendre à (1.1) avec $f \neq 0$, car le Bruit Blanc W est signé.
C'est pourquoi dans notre travail, l'hypothèse la plus intéressante est clairement $(EP2)$.

4.1 Définition des opérateurs de dérivation.

Rappelons que l'on note $C_c^p(I\!R^d)$ l'ensemble des fonctions de classe C^p à support compact sur $I\!R^d$; et $C_b^p(I\!R^d)$ l'ensemble des fonctions de classe C^p sur $I\!R^d$ dont toutes les dérivées d'ordre 1 à p sont bornées (*ces fonctions ne sont pas nécessairement bornées*). Comme nous l'avons dit plus haut, nous allons définir deux opérateurs, l'un lié au Bruit Blanc et l'autre à la mesure Poissonnienne.

L'opérateur lié à la mesure de Poisson.

Commençons par définir le domaine des variables simples. Soit $\mathcal{CL}$ l'ensemble des fonctions l mesurables de $[0,T]\times[0,L]\times O$ dans $I\!R$, à support compact, C^2 sur O (en la troisième variable z), telles que l, l'_z, et l''_{zz} soient bornées sur $[0,T]\times[0,L]\times O$. On définit alors le domaine

$$\mathcal{S}^{0,1}=\left\{X=F(N(f_1),...,N(f_d))+a\Big/ d\geq 1,\ F\in C_c^2(I\!R^d),\ f_i\in\mathcal{CL},\ a\in I\!R\right\} \tag{4.6}$$

On a noté $N(f_i)=\int_0^T\int_0^L\int_O f_i(s,x,z)N(ds,dx,dz)$. Si $X\in\mathcal{S}^{0,1}$, si $\alpha\in[0,L]$, $\tau\in[0,T]$, et si $\zeta\in O$, on pose :

$$D^{0,1}_{\alpha,\tau,\zeta}X=\sum_{i=1}^d d_iF(N(f_1),...,N(f_d))(f_i)'_z(\alpha,\tau,\zeta) \tag{4.7}$$

Nous verrons dans le Lemme 4.3 ci-dessous que l'opérateur $D^{0,1}$ n'est défini qu'à un ensemble $N(d\tau,d\alpha,d\zeta)$-négligeable près. C'est à dire que nous pourrions remplacer $D^{0,1}_{\alpha,\tau,\zeta}X(\omega)$ par n'importe quoi à chaque fois que $N(\omega,\{\tau,\alpha,\zeta\})=0$.
Afin de comprendre le sens de la dérivée $D^{0,1}_{\alpha,\tau,\zeta}$, plaçons-nous sur l'espace canonique associé à N : Ω est l'ensemble des mesures de comptage sur $[0,T]\times[0,L]\times O$. Alors, pour chaque $X\in\mathcal{S}^{0,1}$ et chaque $\omega\in\Omega$,

$$1\!\!1_{\{\omega(\{\tau,\alpha,\zeta\})=1\}}D^{0,1}_{\alpha,\tau,\zeta}X(\omega)=1\!\!1_{\{\omega(\{\tau,\alpha,\zeta\})=1\}}\frac{\partial}{\partial\lambda}X\left(\omega-\delta_{(\tau,\alpha,\zeta)}+\delta_{(\tau,\alpha,\zeta+\lambda)}\right)\Big|_{\lambda=0}$$

Remarquons aussi qu'on peut définir cette dérivation d'une manière proche des méthodes de Bismut. On peut trouver une approche simple des méthodes "Bismutiennes" du calcul de Malliavin dans Bichteler, Jacod, [7]. Voir aussi Bismut, [8]. Soit Λ un voisinage de 0 dans $I\!R$, et soit $v(s,y,z)$ un élément de $L^2([0,T]\times[0,L]\times O,\varphi(z)dsdydz)$ tel que pour tout $\lambda\in\Lambda$, l'application

$$\gamma_v^\lambda(s,y,z)=(s,y,z+\lambda\rho(z)v(s,y,z))$$

soit bijective de $[0,T]\times[0,L]\times O$ dans lui-même. On définit alors la mesure aléatoire de comptage "perturbée"

$$\forall\ A\in\mathcal{B}([0,T]\times[0,L]\times O),\quad N_v^\lambda(A)=\int_0^T\int_0^L\int_O 1_A(\gamma_v^\lambda(s,y,z))N(ds,dy,dz)$$

Ceci fait penser au Théorème de Girsanov pour les mesures aléatoires (voir Jacod, Shiryaev, [23], p 157), utile à la méthode de Bismut.
Si $X=F(N(f_1),...,N(f_d))+a$ appartient à $\mathcal{S}^{0,1}$, on pose

$$X_v^\lambda=F(N_v^\lambda(f_1),...,N_v^\lambda(f_d))+a$$

Il est alors facile de vérifier que

$$\int_0^T \int_0^1 \int_O D^{0,1}_{\alpha,\tau,\zeta} X v(\tau,\alpha,\zeta)\rho(\zeta) N(d\tau,d\alpha,d\zeta) = \frac{d}{d\lambda} X^\lambda_v \Big|_{\lambda=0}$$

Afin d'étudier la fermabilité de l'opérateur $D^{0,1}_{\alpha,\tau,\zeta}$, nous introduisons un autre opérateur : pour $X \in \mathcal{S}^{0,1}$, on pose

$$\begin{aligned} L^{0,1}X &= \frac{1}{2}\sum_{i=1}^d d_i F(N(f_1),...,N(f_d)) N\left(\rho.(f_i)''_{zz} + \left(\rho\frac{\varphi'}{\varphi} + \rho'\right).(f_i)'_z\right) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d d_i d_j F(N(f_1),...,N(f_d)) N\left(\rho.(f_i)'_z.(f_j)'_z\right) \end{aligned} \tag{4.8}$$

Enfin, on définit le produit scalaire suivant : si X et Y sont dans $\mathcal{S}^{0,1}$,

$$\begin{aligned} \left\langle D^{0,1}X, D^{0,1}Y\right\rangle_{\rho N} &= \int_0^T \int_0^L \int_O D^{0,1}_{\alpha,\tau,\zeta} X D^{0,1}_{\alpha,\tau,\zeta} Y \rho(\zeta) N(d\tau,d\alpha,d\zeta) \\ &= L^{0,1}XY - XL^{0,1}Y - YL^{0,1}X \end{aligned} \tag{4.9}$$

Plus généralement, si $S_{\alpha,\tau,\zeta}(\omega)$ et $T_{\alpha,\tau,\zeta}(\omega)$ sont dans $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$, on notera

$$\langle S,T\rangle_{\rho N} = \int_0^T \int_0^L \int_O S_{\alpha,\tau,\zeta} T_{\alpha,\tau,\zeta} \rho(\zeta) N(d\tau,d\alpha,d\zeta) \quad \text{et} \quad \langle S\rangle_{\rho N} = \langle S,S\rangle_{\rho N}$$

Remarquons que si X et Y sont dans $\mathcal{S}^{0,1}$, alors X et Y sont bornés ; et $L^{0,1}X$, $L^{0,1}Y$, et $\langle D^{0,1}X, D^{0,1}Y\rangle_{\rho N}$ sont dans tous les L^p ($p<\infty$), tout ceci grâce à (4.1).

Vérifions ensuite que $L^{0,1}$ (et par conséquent $D^{0,1}$) est bien défini : le lemme suivant, évident à première vue, est en fait important, car un élément de $\mathcal{S}^{0,1}$ peut clairement s'exprimer de plusieurs façons.

Lemme 4.3 *Si $X = F(N(f_1),...,N(f_k)) \in \mathcal{S}^{0,1}$, et si $X \equiv 0$, alors $L^{0,1}X \equiv 0$, et donc $\langle D^{0,1}X\rangle_{\rho N} \equiv 0$.*

<u>Preuve</u> : on suit ici quasiment mot pour mot la preuve de Bichteler, Gravereaux, Jacod, [6] Proposition 9-3, b), p 113. On se place sur l'espace canonique associé à N : Ω est l'ensemble des mesures de comptage ω sur $[0,T]\times[0,L]\times O$. Soit ω un élément fixé de Ω, et soit (t,x,z) un point du support de ω. On pose

$$\omega' = \omega - \delta_{(t,x,z)} \quad \text{et} \quad \omega^\lambda = \omega' + \delta_{(t,x,z+\lambda)}$$

pour $\lambda \in \Lambda$, où Λ est un voisinage de 0 dans $I\!R$ satisfaisant $z+\Lambda \subset O$ (rappelons que O est ouvert). Alors $\omega' \in \Omega$ et $\omega^\lambda \in \Omega$ pour tout $\lambda \in \Lambda$. On définit ensuite

$$X_{t,x,z}(\lambda) = X(\omega^\lambda) = F(\omega'(f_1) + f_1(t,x,z+\lambda),...,\omega'(f_k) + f_k(t,x,z+\lambda))$$

Alors $X_{t,x,z}$ est identiquement nul, et de classe C^2 (en λ). On en déduit que

$$\frac{1}{\varphi(z)}\frac{\partial}{\partial\lambda}\left(\rho(z+\lambda)\varphi(z+\lambda)\times\frac{\partial}{\partial\lambda}X_{t,x,z}(\lambda)\right)\Big|_{\lambda=0}=0$$

Ceci peut aussi s'écrire

$$\begin{aligned}\sum_{i=1}^{k} d_iF(\omega(f_1),...,\omega(f_k))\left[\rho(z).(f_i)''_{zz}(t,x,z)+\left(\rho(z)\frac{\varphi'(z)}{\varphi(z)}+\rho'(z)\right).(f_i)'_z(t,x,z)\right]&\\ +\sum_{i,j=1}^{k} d_id_jF(\omega(f_1),...,\omega(f_k))\rho(z).(f_i)'_z(t,x,z).(f_j)'_z(t,x,z) &= 0\end{aligned}$$

Il ne reste plus qu'à sommer cette égalité sur tous les points (t,x,z) du support de ω pour obtenir $L^{0,1}X(\omega)=0$.

Prouvons ensuite que $L^{0,1}$ est auto-adjoint dans L^2 :

Lemme 4.4 *Si X et Y sont dans $\mathcal{S}^{0,1}$, alors*

$$E(XL^{0,1}Y)=E(YL^{0,1}X)=-\frac{1}{2}E\left(\left\langle D^{0,1}X,D^{0,1}Y\right\rangle_{\rho N}\right)\tag{4.10}$$

<u>Preuve</u> : ceci est encore adapté de [6], Proposition 9-3, d), p 113. On considère deux éléments de $\mathcal{S}^{0,1}$:

$$X=F(N(f_1),...,N(f_k))\quad\text{et}\quad Y=H(N(h_1),...,N(h_q))$$

Soit K un compact de O tel que les supports de toutes les fonctions f_i et h_j soient inclus dans $[0,T]\times[0,L]\times K$. On note $\{(T_i,X_i,Z_i)\}_{1\le i\le\mu}$ les points de $[0,T]\times[0,L]\times K$ chargés par N :

$$N|_{[0,T]\times[0,L]\times K}=\sum_{i=1}^{\mu}\delta_{(T_i,X_i,Z_i)}$$

On sait que μ est une variable aléatoire de Poisson de paramètre $TL\int_K\varphi(z)dz$, et que si $\mathcal{G}$ est la tribu engendrée par $(\mu,T_1,...,T_\mu)$, alors :

$$\mathcal{L}\left((X_1,Z_1),...,(X_\mu,Z_\mu)|\mathcal{G}\right)=\left(\frac{1_{[0,L]}(x)dx}{L}\otimes\frac{1_K(z)\varphi(z)dz}{\int_K\varphi(z)dz}\right)^{\otimes\mu}$$

On va vérifier que

$$E\left(XL^{0,1}Y+\frac{1}{2}\left\langle D^{0,1}X,D^{0,1}Y\right\rangle_{\rho N}\Big|\mathcal{G}\right)=0$$

Mais $XL^{0,1}Y+\frac{1}{2}\left\langle D^{0,1}X,D^{0,1}Y\right\rangle_{\rho N}$ est égal à

$$\frac{1}{2}F\left(\sum_{i=1}^{\mu}f_1(T_i,X_i,Z_i),...,\sum_{i=1}^{\mu}f_k(T_i,X_i,Z_i)\right)$$

$$\times \left\{ \begin{array}{l} \sum_{\alpha,\beta=1}^{q} d_\alpha d_\beta H\left(\sum_{i=1}^{\mu} h_1(T_i,X_i,Z_i),...,\sum_{i=1}^{\mu} h_q(T_i,X_i,Z_i)\right) \\ \qquad\qquad \times \sum_{n=1}^{\mu} \rho(Z_n).(h_\alpha)'_z(T_n,X_n,Z_n).(h_\beta)'_z(T_n,X_n,Z_n) \\ + \sum_{\alpha=1}^{q} d_\alpha H\left(\sum_{i=1}^{\mu} h_1(T_i,X_i,Z_i),...,\sum_{i=1}^{\mu} h_q(T_i,X_i,Z_i)\right) \\ \quad \times \sum_{n=1}^{\mu}\left[\rho(Z_n).(h_\alpha)''_{zz}(T_n,X_n,Z_n) + \left[\rho'(Z_n)+\rho(Z_n)\frac{\varphi'(Z_n)}{\varphi(Z_n)}\right].(h_\alpha)'_z(T_n,X_n,Z_n)\right] \end{array} \right\}$$

$$\begin{aligned} &+\frac{1}{2}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{q} d_\alpha F\left(\sum_{i=1}^{\mu} f_1(T_i,X_i,Z_i),...,\sum_{i=1}^{\mu} f_k(T_i,X_i,Z_i)\right) \\ &\qquad\qquad \times d_\beta H\left(\sum_{i=1}^{\mu} h_1(T_i,X_i,Z_i),...,\sum_{i=1}^{\mu} h_q(T_i,X_i,Z_i)\right) \\ &\qquad\qquad\qquad \times \sum_{n=1}^{\mu} \rho(Z_n).(f_\alpha)'_z(T_n,X_n,Z_n).(h_\beta)'_z(T_n,X_n,Z_n) \\ &= \psi(\mu,(T_1,X_1,Z_1),...,(T_\mu,X_\mu,Z_\mu)) \end{aligned}$$

où la dernière égalité est une définition. Il suffit alors de vérifier que quels que soient $\mu, T_1, ..., T_\mu, X_1, ...X_\mu$,

$$\int_{K^\mu} \psi(\mu,(T_1,X_1,z_1),...,(T_\mu,X_\mu,z_\mu)) \times \varphi(z_1)...\varphi(z_\mu)dz_1...dz_\mu = 0$$

Si $n \leq \mu$, on pose $\hat{z}_n = (z_1,...,z_{n-1},z_{n+1},...,z_\mu)$, et

$$\begin{aligned} Q^n_{\hat{z}_n}(z_n) &= F\left(\sum_{i=1}^{\mu} f_1(T_i,X_i,z_i),...,\sum_{i=1}^{\mu} f_k(T_i,X_i,z_i)\right) \\ &\quad \times \sum_{\alpha=1}^{q} d_\alpha H\left(\sum_{i=1}^{\mu} h_1(T_i,X_i,z_i),...,\sum_{i=1}^{\mu} h_q(T_i,X_i,z_i)\right)\rho(z_n)\varphi(z_n).(h_\alpha)'_z(T_n,X_n,z_n) \end{aligned}$$

On peut alors vérifier que

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{\mu}\frac{\partial}{\partial z}Q^n_{\hat{z}_n}(z_n) \times \varphi(z_1)...\varphi(z_{n-1})\varphi(z_{n+1})...\varphi(z_\mu) \\ &= \psi(\mu,(T_1,X_1,z_1),...,(T_\mu,X_\mu,z_\mu)) \times \varphi(z_1)...\varphi(z_\mu) \end{aligned}$$

Mais si z appartient à ∂K, alors $Q^n_{\hat{z}_n}(z) = 0$. Donc pour chaque n,

$$\int_K \frac{\partial}{\partial z} Q^n_{\hat{z}_n}(z)dz = 0$$

et le résultat s'ensuit facilement.

Le lemme suivant montre la fermabilité de l'opérateur $D^{0,1}_{\alpha,\tau,\zeta}$:

Lemme 4.5 *Soit Z_n une suite d'éléments de $\mathcal{S}^{0,1}$ qui converge vers 0 dans L^2. S'il existe $S_{\alpha,\tau,\zeta}(\omega) \in L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$ tel que*

$$E\left(\left\langle D^{0,1}Z_n - S\right\rangle_{\rho N}\right) \longrightarrow 0$$

alors S est nul au sens où $E(\langle S\rangle_{\rho N}) = 0$.

<u>Preuve</u> : quel que soit $X \in \mathcal{S}^{0,1}$, une simple application de l'inégalité de Cauchy-Schwarz (pour la mesure $P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta)$) montre que

$$E\left(\left\langle S, D^{0,1}X\right\rangle_{\rho N}\right) = \lim_n E\left(\left\langle D^{0,1}Z_n, D^{0,1}X\right\rangle_{\rho N}\right)$$

Mais d'après le Lemme 4.4, $E\left(\langle D^{0,1}Z_n, D^{0,1}X\rangle_{\rho N}\right) = -2E(Z_n L^{0,1}X)$. Comme Z_n tend vers 0 dans L^2, il est facile d'en déduire que $E\left(\langle S, D^{0,1}X\rangle_{\rho N}\right) = 0$.
En appliquant ceci pour $X = Z_k$, et en faisant tendre k vers l'infini, on obtient le résultat recherché.

L'opérateur lié au Bruit Blanc.

Le domaine des variables simples est cette fois défini par :

$$\mathcal{S}^{1,0} = \Big\{X = F(W(f_1), ..., W(f_k)) + a \;\Big/\; k \geq 1,\; F \in C_c^2(I\!R^k),$$
$$f_i \in L^2([0, L] \times [0, T]),\; a \in I\!R\Big\} \tag{4.11}$$

Ici, $W(f_i) = \int_0^T \int_0^L f_i(\alpha, \tau)W(d\alpha, d\tau)$. Si X est dans $\mathcal{S}^{0,1}$, on pose pour $\alpha \in [0, L]$ et $\tau \in [0, T]$:

$$D^{1,0}_{\alpha,\tau}X = \sum_{i=1}^k d_i F(W(f_1), ..., W(f_k)) f_i(\alpha, \tau) \tag{4.12}$$

On notera

$$\left\langle D^{1,0}X, D^{1,0}Y\right\rangle_{leb} = \int_0^T \int_0^L D^{1,0}_{\alpha,\tau}X D^{1,0}_{\alpha,\tau}Y d\alpha d\tau \tag{4.13}$$

Plus généralement, si $S_{\alpha,\tau}(\omega)$ et $T_{\alpha,\tau}(\omega)$ sont dans $L^2(P(d\omega)d\alpha d\tau)$, on notera

$$\langle S, T\rangle_{leb} = \int_0^T \int_0^L S_{\alpha,\tau}T_{\alpha,\tau} d\alpha d\tau \quad \text{et} \quad \langle S\rangle_{leb} = \langle S, S\rangle_{leb}$$

Remarquons que si X et Y sont dans $\mathcal{S}^{1,0}$, alors X, Y, et $\langle D^{1,0}X, D^{1,0}Y\rangle_{leb}$ sont bornés. Le lemme suivant figure dans le livre de Nualart [30] p 26.

Lemme 4.6 *Soit Z_n une suite d'éléments de $\mathcal{S}^{1,0}$ qui converge vers 0 dans L^2. S'il existe $S_{\alpha,\tau}(\omega) \in L^2(P(d\omega)d\alpha d\tau)$ tel que*

$$E\left(\left\langle D^{1,0}Z_n - S\right\rangle_{leb}\right) \longrightarrow 0$$

alors S est nul au sens où $E(\langle S\rangle_{leb}) = 0$.

Une interpretation possible de cette notion de dérivée est la suivante : considérons un élément h de $L^2([0,L]\times[0,T],dyds)$, et un voisinage Λ de 0 dans $I\!R^2$. On "perturbe" le Bruit Blanc en posant

$$W_h^\lambda(dy,ds) = W(dy,ds) + \lambda h(y,s)dyds$$

Ici encore, cette égalité fait penser au Théorème de Girsanov.
Si $X = F(W(f_1),...,W(f_k)) + a$ appartient à $\mathcal{S}^{1,0}$, on pose

$$X_h^\lambda = F(W_h^\lambda(f_1),...,W_h^\lambda(f_k)) + a$$

et on obtient :

$$\left\langle D^{1,0}X, h\right\rangle_{leb} = \frac{d}{d\lambda}X_h^\lambda\bigg|_{\lambda=0}$$

Les opérateurs sur l'espace produit.

On peut maintenant construire l'espace des variables simples associé à notre problème :

$$\mathcal{S} = \Big\{Y = H(X_1,...,X_d,Z_1,...,Z_k)) + a \;\Big/\; H \in C_c^2(I\!R^{d+k}),\ k+d\geq 1,$$

$$X_i \in \mathcal{S}^{1,0},\ Z_j \in \mathcal{S}^{0,1},\ a \in I\!R\Big\} \tag{4.14}$$

Si Y appartient à $\mathcal{S}$, on définit $D^{0,1}Y$ et $L^{0,1}Y$ (resp. $D^{1,0}Y$) comme dans les paragraphes précédents, en considérant les variables $X_1,...,X_d$ (resp. $Z_1,...,Z_k$) comme des constantes.
Pour être plus précis, plaçons-nous sur l'espace canonique produit associé à W et N. Chaque $\omega\in\Omega$ s'écrit $\omega = (\omega^W,\omega^N)$. Si Y appartient à $\mathcal{S}$, il est clair que pour chaque ω^N fixé, $Y_{\omega^N}(\omega^W) = Y(\omega^W,\omega^N)$ est dans $\mathcal{S}^{1,0}$. Nous définissons alors la dérivée de Y relative au Bruit Blanc par

$$D^{1,0}_{\alpha,\tau}Y(\omega^W,\omega^N) = \left(D^{1,0}_{\alpha,\tau}Y_{\omega^N}\right)(\omega^W)$$

On construit $D^{0,1}_{\alpha,\tau,\zeta}$ et $L^{0,1}$ de la même façon.
Les produits scalaires qui en découlent seront notés comme auparavant, et on voit que si X et Z sont dans $\mathcal{S}$, alors X, Z, et $\langle D^{1,0}X, D^{1,0}Z\rangle_{leb}$ sont bornés, et $L^{0,1}X$, $L^{0,1}Z$, et $\langle D^{0,1}X, D^{0,1}Z\rangle_{\rho N}$ sont dans tous les L^p $(p<\infty)$.

Si Z appartient à $\mathcal{S}$, on pose

$$|||Z|||_2 = \left[E\left(Z^2\right) + E\left(\left\langle D^{1,0}Z\right\rangle_{leb}\right) + E\left(\left\langle D^{0,1}Z\right\rangle_{\rho N}\right)\right]^{\frac{1}{2}} \tag{4.15}$$

Soit $\mathcal{D}_2$ la fermeture de $\mathcal{S}$ pour cette norme. Grâce aux deux propositions de fermabilité (4.5 et 4.6), les opérateurs $D^{1,0}_{\alpha,\tau}$ et $D^{0,1}_{\alpha,\tau,\zeta}$ s'étendent de manière unique à l'espace $\mathcal{D}_2$.

Remarque 4.7 *Nous avons étendu les opérateurs de dérivation à un domaine auquel appartiendra la solution faible de (1.1). Mais sur ce domaine, nous n'aurons pas de formule d'intégration par parties (du type (4.10)), car l'opérateur $L^{0,1}$ ne s'étend pas à $\mathcal{D}_2$. Néanmoins, la "dérivabilité" de la solution suffira pour vérifier le Théorème 4.1. Finalement, $L^{0,1}$ sert juste à la preuve de la fermabilité.*

Notation 4.8 *On notera* $\{(T_n, X_n, Z_n)\}_{n\geq 0}$ *les points de* $[0,T]\times[0,L]\times O$ *chargés par la mesure de Poisson, c'est à dire que*

$$N(dt, dx, dz) = \sum_{n=0}^{\infty} \delta_{(T_n, X_n, Z_n)}(dt, dx, dz) \tag{4.16}$$

On supposera que pour chaque n, T_n *est un temps d'arrêt, et* (X_n, Z_n) *est* $\mathcal{F}_{T_n}$*-mesurable.*

Remarque 4.9 *1. Soit* S *et* T *deux éléments de* $L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$*. Dans toute la suite, la notation "*$S_{\alpha,\tau,\zeta} = T_{\alpha,\tau,\zeta}$*" signifiera*

$$p.s., \quad \langle S - T\rangle_{\rho N} = 0$$

Ceci est encore équivalent à

$$\forall\, n \in I\!N, \quad S_{X_n,T_n,Z_n} = T_{X_n,T_n,Z_n} \quad p.s.$$

2. *La fermabilité de* $D^{0,1}$ *permet d'affirmer que si* $X \in \mathcal{D}_2$*, si* $Y = X$ *p.s., alors* $Y \in \mathcal{D}_2$*, et p.s.,*

$$\left\langle D^{0,1}X - D^{0,1}Y\right\rangle_{\rho N} = 0$$

3. *Soit* X *une variable aléatoire, éventuellement définie p.s. (*X *peut être une intégrale stochastique, une ésperance conditionnelle, ...) Pour prouver que* $X \in \mathcal{D}_2$ *et que pour certains* $S \in L^2(P(d\omega)d\alpha d\tau)$*,* $T \in L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$*,*

$$D^{1,0}_{\alpha,\tau}X = S_{\alpha,\tau} \quad et \quad D^{0,1}_{\alpha,\tau,\zeta}X = T_{\alpha,\tau,\zeta}$$

il suffit d'exhiber une suite d'éléments X_n *de* $\mathcal{S}$ *telle que*

$$E\left((X - X_n)^2\right) + E\left(\left\langle S - D^{1,0}X_n\right\rangle_{leb}\right) + E\left(\left\langle T - D^{0,1}X_n\right\rangle_{\rho N}\right)$$

tende vers 0.

4.2 Propriétés des opérateurs de dérivation.

Enonçons d'abord une proposition concernant les propriétés du domaine $\mathcal{D}_2$.

Proposition 4.10 $\mathcal{D}_2$ *muni du produit scalaire*

$$\langle Y, Z\rangle_{\mathcal{D}_2} = E(YZ) + E\left(\left\langle D^{1,0}Y, D^{1,0}Z\right\rangle_{leb}\right) + E\left(\left\langle D^{0,1}Y, D^{0,1}Z\right\rangle_{\rho N}\right) \tag{4.17}$$

est un espace de Hilbert.

Preuve : par construction, $\mathcal{D}_2$ est un $I\!R$-espace vectoriel fermé pour la norme $|||\ |||_2$, et il est facile de vérifier que $\langle\ ,\ \rangle_{\mathcal{D}_2}$ est un produit scalaire associé à cette norme. Enfin, $\mathcal{D}_2$ est complet car les espaces $L^2(\Omega, P)$, $L^2(\Omega \times [0,L] \times [0,T], P(d\omega)d\alpha d\tau)$ et $L^2(\Omega \times [0,L] \times [0,T] \times O, P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$ sont complets.

Poursuivons avec une propriété classique des opérateurs de dérivation.

Proposition 4.11 *Soit $Y=(Y_1,...,Y_n)\in\mathcal{D}_2^n$, et $F\in C_b^1(I\!R^n)$. Alors $Z=F(Y)\in\mathcal{D}_2$, et :*

$$D^{1,0}_{\alpha,\tau}Z=\sum_{i=1}^n d_iF(Y)D^{1,0}_{\alpha,\tau}Y_i \quad et \quad D^{0,1}_{\alpha,\tau,\zeta}Z=\sum_{i=1}^n d_iF(Y)D^{0,1}_{\alpha,\tau,\zeta}Y_i \tag{4.18}$$

<u>Preuve</u> : on supposera $n=1$, afin d'alléger les écritures. Cette preuve s'effectue en deux étapes : on suppose d'abord que F est de classe C_b^2, puis on étend le résultat par approximations.
<u>Etape 1</u> : supposons donc que F est de classe C_b^2. Soit Y_k une suite de $\mathcal{S}$ qui converge vers Y au sens de $|||\ |||_2$, et $Z_k=F(Y_k)$. Il est clair que $Z_k\in\mathcal{S}$ et que

$$D^{1,0}_{\alpha,\tau}Z_k=F'(Y_k)D^{1,0}_{\alpha,\tau}Y_k \quad \text{et} \quad D^{0,1}_{\alpha,\tau,\zeta}Z_k=F'(Y_k)D^{0,1}_{\alpha,\tau,\zeta}Y_k$$

par définition de $\mathcal{S}$, $D^{1,0}_{\alpha,\tau}$ et $D^{0,1}_{\alpha,\tau,\zeta}$. On pose donc

$$D^{1,0}_{\alpha,\tau}Z=F'(Y)D^{1,0}_{\alpha,\tau}Y \qquad \text{et} \qquad D^{0,1}_{\alpha,\tau,\zeta}Z=F'(Y)D^{0,1}_{\alpha,\tau,\zeta}Y$$

On peut supposer, quitte à extraire une sous-suite, que Y_k converge vers Y p.s. Il suffit alors de prouver que Z_k converge vers Z au sens de $|||\ |||_2$. Tout d'abord Z_k converge vers Z dans L^2, car

$$|Z_k-Z|\leq\parallel F'\parallel_\infty|Y_k-Y|$$

De plus, $\langle D^{0,1}Z_k-D^{0,1}Z\rangle_{\rho N}$ converge vers 0 dans L^1, car

$$\begin{aligned}E\left(\left\langle F'(Y_k)D^{0,1}Y_k-F'(Y)D^{0,1}Y\right\rangle_{\rho N}\right) \quad &\leq \quad \parallel F'\parallel_\infty^2 E\left(\left\langle D^{0,1}Y_k-D^{0,1}Y\right\rangle_{\rho N}\right)\\ &+E\left(\left\langle D^{0,1}Y\right\rangle_{\rho N}\times|F'(Y_k)-F'(Y)|\right)\end{aligned}$$

et il suffit d'appliquer le Théorème de convergence dominée (F' est bornée et continue), les hypothèses de convergence de Y_k vers Y (p.s., et pour $|||\ |||_2$), et d'utiliser le fait que $\langle D^{0,1}Y\rangle_{\rho N}$ est dans L^1. Enfin, $\langle D^{1,0}Z_k-D^{1,0}Z\rangle_{leb}$ converge vers 0 dans L^1 pour les mêmes raisons.
<u>Etape 2</u> : si F n'est que de classe C_b^1, il existe une suite de fonctions F_k de classe C_b^2 telle que pour tout k, $\parallel F_k'\parallel_\infty\leq 2\parallel F'\parallel_\infty$, telle que $F_k(0)=F(0)$, et telle que F_k et F_k' convergent simplement vers F et F'. Par l'étape 1, $F_k(Y)\in\mathcal{D}_2$ pour tout k, et il est facile de voir que la convergence dans $\mathcal{D}_2$ a lieu.

Notons maintenant

$$\begin{aligned}\mathcal{S}_b=\Big\{Z=F(W(f_1),...,W(f_m),N(g_1),...,N(g_d))\Big/ m+d\geq 1,\ F\in C_b^1(I\!R^{m+d})\\ f_i\in L^2([0,L]\times[0,T],dxdt),\ g_i\in\mathcal{CL}\Big\}\end{aligned} \tag{4.19}$$

et prouvons que cette ensemble est inclus dans $\mathcal{D}_2$.

Proposition 4.12 *1. Si Z appartient à $\mathcal{S}_b$, alors $Z\in\mathcal{D}_2$, et*

$$D^{1,0}_{\alpha,\tau}Z=\sum_{i=1}^m d_iF(W(f_1),...,W(f_m),N(g_1),...,N(g_d))f_i(\alpha,\tau) \tag{4.20}$$

$$D^{0,1}_{\alpha,\tau,\zeta}Z=\sum_{i=m+1}^{m+d} d_iF(W(f_1),...,W(f_m),N(g_1),...,N(g_d))(g_i)'_z(\alpha,\tau,\zeta) \tag{4.21}$$

2. En particulier, si $f_0 \in L^2([0,L]\times[0,T])$, alors $W(f_0)$ appartient à $\mathcal{D}_2$, et on a

$$D^{1,0}_{\alpha,\tau} W(f_0) = f_0(\alpha,\tau) \qquad et \qquad D^{0,1}_{\alpha,\tau,\zeta} W(f_0) = 0 \tag{4.22}$$

De même, si $g_0 \in \mathcal{CL}$, alors $N(g_0)$ et $\tilde{N}(g_0)$ sont dans $\mathcal{D}_2$, et

$$D^{1,0}_{\alpha,\tau} N(g_0) = D^{1,0}_{\alpha,\tau} \tilde{N}(g_0) = 0 \quad et \quad D^{0,1}_{\alpha,\tau,\zeta} N(g_0) = D^{0,1}_{\alpha,\tau,\zeta} \tilde{N}(g_0) = (g_0)'_z(\alpha,\tau,\zeta) \tag{4.23}$$

Preuve : comme dans la preuve précédente, on considère d'abord un élément

$$Z = F(W(f_1),...,W(f_m),N(g_1),...,N(g_d))$$

de $\mathcal{S}_b$ tel que F soit de classe C^2_b. Compactifions le support de F de la manière suivante : soit K_ϵ une suite de fonctions de classe C^2 sur $I\!R^{m+d}$, bornées par 1 à dérivées uniformément bornées, vérifiant :

$$K_\epsilon(x) = \begin{cases} 1 & \text{si } x \in B(0,1/\epsilon) \\ \\ 0 & \text{si } x \notin B(0,1+1/\epsilon) \end{cases}$$

Posons

$$Z_\epsilon = F(W(f_1),...,W(f_m),N(g_1),...,N(g_d))K_\epsilon(W(f_1),...,W(f_m),N(g_1),...,N(g_d))$$

Alors $Z_\epsilon \in \mathcal{S}$, et on a

$$D^{1,0}_{\alpha,\tau} Z_\epsilon = \sum_{i=1}^{m} (K_\epsilon d_i F + F d_i K_\epsilon)(W(f_1),...,W(f_m),N(g_1),...,N(g_d)) \times f_i(\alpha,\tau)$$

et

$$D^{0,1}_{\alpha,\tau,\zeta} Z_\epsilon = \sum_{i=m+1}^{m+d} (K_\epsilon d_i F + F d_i K_\epsilon)(W(f_1),...,W(f_m),N(g_1),...,N(g_d)) \times (g_i)'_z(\alpha,\tau,\zeta)$$

Il ne reste plus qu'à vérifier les convergences usuelles, ce qui est très facile à l'aide du Théorème de convergence dominée. (Rappelons que si $x \in I\!R^{m+d}$, alors $K_\epsilon(x)$ tend vers 1, et $d_i K_\epsilon(x)$ tend vers 0 pour chaque i, quand ϵ tend vers 0).
Si F n'est plus que de classe C^1_b, il suffit d'appliquer la Proposition 4.11 avec
$Y = (W(f_1),...,W(f_m),N(g_1),...,N(g_d))$ qui appartient à $\mathcal{D}_2^{m+d}$ par l'étape précédente (par exemple, $W(f_1) = Id(W(f_1))$, avec $Id \in C^\infty_b$).
Pour le 2., $W(f_0)$ et $N(g_0)$ ne posent pas de problème. Quant à $\tilde{N}(g_0)$, il suffit de remarquer que $\tilde{N}(g_0) = N(g_0) - \nu(g_0)$. Comme $\nu(g_0)$ est déterministe, on peut traiter $\tilde{N}(g_0)$ comme $N(g_0)$ pour ce qui est de la dérivation.

La proposition suivante permet "d'agrandir" l'ensemble $\mathcal{CL}$.

Proposition 4.13 *Soit ϕ une fonction mesurable sur $[0,T]\times[0,L]\times O$, de classe C^1 sur O, à support compact, et vérifiant :*

$$|\phi(s,y,z)| + |\phi'_z(s,y,z)| \le K$$

Alors $N(\phi)$ et $\tilde{N}(\phi)$ sont dans $\mathcal{D}_2$, et ont pour dérivées :

$$D^{1,0}_{\alpha,\tau} N(\phi) = D^{1,0}_{\alpha,\tau} \tilde{N}(\phi) = 0 \quad et \quad D^{0,1}_{\alpha,\tau,\zeta} N(\phi) = D^{0,1}_{\alpha,\tau,\zeta} \tilde{N}(\phi) = \phi'_z(\tau,\alpha,\zeta) \tag{4.24}$$

La preuve de cette proposition est très simple : il suffit de remarquer qu'il existe une suite de fonctions $\phi_k \in \mathcal{CL}$, uniformément bornées ainsi que leurs dérivées par rapport à z, dont le support est inclus dans celui de ϕ, et telles que ϕ_k et $(\phi_k)'_z$ convergent $dzdyds$-presque partout vers ϕ et ϕ'_z.

La proposition suivante montre que l'espérance conditionnelle d'une variable simple est encore dans $\mathcal{S}_b$, et permet de calculer ses dérivées.

Proposition 4.14 *1. Soit Z un élément de $\mathcal{S}$. On considère la martingale càdlàg $Z_s = E(Z|\mathcal{F}_s)$. Alors, pour chaque s, Z_s appartient à $\mathcal{S}_b$, est borné (par le même réel que Z), et admet pour dérivées :*

$$D^{1,0}_{\alpha,\tau} Z_s = E\left(D^{1,0}_{\alpha,\tau} Z \middle| \mathcal{F}_s \right) 1_{\{\tau \le s\}} \tag{4.25}$$

et

$$D^{0,1}_{\alpha,\tau,\zeta} Z_s = E\left(D^{0,1}_{\alpha,\tau,\zeta} Z \middle| \mathcal{F}_s \right) 1_{\{\tau \le s\}} \tag{4.26}$$

La deuxième égalité signifie que pour chaque $n \ge 0$,

$$D^{0,1}_{X_n,T_n,Z_n} Z_s = E\left(D^{0,1}_{X_n,T_n,Z_n} Z \middle| \mathcal{F}_s \right) 1_{\{T_n \le s\}} \tag{4.27}$$

On en déduit que pour tout α, τ, tout $n \in \mathbb{N}$, les processus $D^{1,0}_{\alpha,\tau} Z_s$ et $D^{0,1}_{X_n,T_n,Z_n} Z_s$ sont càdlàg et adaptés.

2. Avec les notations du 1., on a

$$E(Z_s^2) \le E(Z^2) \quad ; \quad E\left(\left\langle D^{1,0} Z_s \right\rangle_{leb} \right) \le E\left(\left\langle D^{1,0} Z \right\rangle_{leb} \right)$$

$$\text{et} \quad E\left(\left\langle D^{1,0} Z_s \right\rangle_{\rho N} \right) \le E\left(\left\langle D^{1,0} Z \right\rangle_{\rho N} \right) \tag{4.28}$$

3. Soit maintenant Y une variable de $\mathcal{D}_2$, $\mathcal{F}_s$-mesurable pour un $s \in [0,T]$. Alors, pour chaque α, τ, chaque n, on a $D^{1,0}_{\alpha,\tau} Y = D^{1,0}_{\alpha,\tau} Y 1_{\{\tau \le s\}}$ et $D^{0,1}_{X_n,T_n,Z_n} Y = D^{0,1}_{X_n,T_n,Z_n} Y 1_{\{T_n \le s\}}$, et ces variables sont $\mathcal{F}_s$-mesurables.

Preuve : 1) Soit $s \in [0,T]$ et

$$Z = F(W(f_1), ..., W(f_m), N(g_1), ..., N(g_d))$$

un élément de $\mathcal{S}$. Posons

$$\bar{f}_i(\alpha,\tau) = f_i(\alpha,\tau) 1_{\{\tau \le s\}} \quad ; \quad \hat{f}_i(\alpha,\tau) = f_i(\alpha,\tau) 1_{\{\tau > s\}}$$

et

$$\bar{g}_i(\alpha,\tau,\zeta) = g_i(\alpha,\tau,\zeta) 1_{\{\tau \le s\}} \quad ; \quad \hat{g}_i(\alpha,\tau,\zeta) = g_i(\alpha,\tau,\zeta) 1_{\{\tau > s\}}$$

Alors, par l'hypothèse (M)-3, pour tout i, j, $W(\bar{f}_i)$ et $N(\bar{g}_i)$ sont $\mathcal{F}_s$-mesurables, et $W(\hat{f}_i)$ et $N(\hat{g}_i)$ sont indépendants de $\mathcal{F}_s$. Donc

$$\begin{aligned} Z_s &= E\left(F(W(\bar{f}_1)+W(\hat{f}_1),...,N(\bar{g}_d)+N(\hat{g}_d))\middle| \mathcal{F}_s\right) \\ &= \int F(W(\bar{f}_1)+x_1,...,N(\bar{g}_d)+y_d))\mu(dx_1,...,dy_d) \\ &= H(W(\bar{f}_1),...,N(\bar{g}_d)) \end{aligned}$$

si μ est la loi de $(W(\hat{f}_1),...,N(\hat{g}_d))$ et si

$$H(X_1,...,Y_d) = \int_{\mathbb{R}^{m+d}} F(X_1+x_1,...,Y_d+y_d))\mu(dx_1,...,dy_d)$$

Or F est de classe C_c^1. Donc H est bornée, et, par le Théorème de Lebesgue, H est de classe C^1. De plus,

$$d_iH(X_1,...,Y_d) = \int d_iF(X_1+x_1,...,Y_d+y_d))\mu(dx_1,...,dy_d)$$

et H est de classe C_b^1. Donc Z_s appartient à $\mathcal{S}_b$, et la Proposition 4.12 implique que pour tout α, τ

$$D^{1,0}_{\alpha,\tau}Z_s = \sum_{i=1}^m d_iH(W(\bar{f}_1),...,N(\bar{g}_d))\bar{f}_i(\alpha,\tau)$$

et pour tout n,

$$D^{0,1}_{X_n,T_n,Z_n}Z_s = \sum_{i=m+1}^{m+d} d_iH(W(\bar{f}_1),...,N(\bar{g}_d))(\bar{g}_i)'_z(X_n,T_n,Z_n)$$

Mais $d_iH(W(\bar{f}_1),...,N(\bar{g}_d)) = E\left(d_iF(W(f_1),...,N(g_d)) \mid \mathcal{F}_s\right)$. On en déduit aisément (4.25) d'une part, et d'autre part que

$$\begin{aligned} D^{0,1}_{X_n,T_n,Z_n}Z_s &= \sum_{i=m+1}^{m+d} E\left[d_iF(W(f_1),...,N(g_d)) \mid \mathcal{F}_s\right](g_i)'_z(X_n,T_n,Z_n)1_{\{T_n\leq s\}} \\ &= E\left(\sum_{i=m+1}^{m+d} d_iF(W(f_1),...,N(g_d))(g_i)'_z(X_n,T_n,Z_n)1_{\{T_n\leq s\}} \;\middle|\; \mathcal{F}_s\right) \\ &= E\left(D^{0,1}_{X_n,T_n,Z_n}Z\middle|\mathcal{F}_s\right)1_{\{T_n\leq s\}} \end{aligned} \tag{4.29}$$

2) Prouvons par exemple la dernière inégalité, en utilisant la Notation 4.8, l'égalité (4.27), l'inégalité de Jensen, et le Théorème de Fubini pour intégrands positifs :

$$\begin{aligned} E\left(\left\langle D^{0,1}Z_s\right\rangle_{\rho N}\right) &= E\left[\sum_{n\geq 0} E\left\{D^{0,1}_{X_n,T_n,Z_n}Z \;\middle|\; \mathcal{F}_s\right\}^2 \rho(Z_n)1_{\{T_n\leq s\}}\right] \\ &\leq E\left[\sum_{n\geq 0} E\left\{\left(D^{0,1}_{X_n,T_n,Z_n}Z\right)^2 \;\middle|\; \mathcal{F}_s\right\}\rho(Z_n)1_{\{T_n\leq s\}}\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[E\left\{\sum_{n\geq 0}\left(D^{0,1}_{X_n,T_n,Z_n}Z\right)^2\rho(Z_n)1_{\{T_n\leq s\}}\,\middle|\,\mathcal{F}_s\right\}\right]\\
&= E\left[\sum_{n\geq 0}\left(D^{0,1}_{X_n,T_n,Z_n}Z\right)^2\rho(Z_n)1_{\{T_n\leq s\}}\right]\\
&= E\left(\left\langle D^{0,1}Z\right\rangle_{\rho N}\right) \qquad (4.30)
\end{aligned}
$$

3) Comme Y appartient à $\mathcal{D}_2$, il existe une suite $\{Z^k\}$ d'éléments de $\mathcal{S}$ qui converge vers Y dans $\mathcal{D}_2$. Il est alors facile de voir que la suite Z^k_s est de Cauchy dans $\mathcal{D}_2$. En effet, grâce au 2.,

$$|||Z^k_s - Z^{k+1}_s|||_{\mathcal{D}_2} \leq |||Z^k - Z^{k+1}|||_{\mathcal{D}_2}$$

D'autre part, Z^k_s converge vers Y dans L^2, toujours grâce à l'inégalité de Jensen :

$$E\left((Z^k_s - Y)^2\right) \leq E\left(\left(E(Z^k - Y|\mathcal{F}_s)\right)^2\right) \leq E\left((Z^k - Y)^2\right)$$

Ceci implique que Z^k_s converge vers Y dans $\mathcal{D}_2$. La conclusion s'ensuit facilement.

Dans un cadre Gaussien (ce qui revient à poser tout simplement $D^{0,1}\equiv 0$ ici), on sait que si X et Y sont dans $\mathcal{D}_2$, et si X et $\langle D^{0,1}X\rangle$ sont bornés, alors XY est dans $\mathcal{D}_2$, et la formule usuelle de dérivation d'un produit s'applique. Mais ici, $\langle D^{0,1}X\rangle_{\rho N}$ n'est que très rarement borné. Nous contournerons ce problème en utilisant la proposition suivante.

Proposition 4.15 *Soit X et Z deux éléments de $\mathcal{D}_2$. On suppose que X et Z sont indépendants. Alors XZ appartient à $\mathcal{D}_2$, et on peut appliquer la formule usuelle de dérivation d'un produit :*

$$D^{1,0}_{\alpha,\tau}XZ = XD^{1,0}_{\alpha,\tau}Z + ZD^{1,0}_{\alpha,\tau}X \quad \text{et} \quad D^{0,1}_{\alpha,\tau,\zeta}XZ = XD^{0,1}_{\alpha,\tau,\zeta}Z + ZD^{0,1}_{\alpha,\tau,\zeta}X \qquad (4.31)$$

Preuve : soit $\mathcal{G}$ (resp. $\mathcal{A}$) la tribu engendrée par X (resp. Z). Par hypothèse, ces tribus sont indépendantes. Soit X'^k une suite de $\mathcal{S}$ convergeant vers X dans $\mathcal{D}_2$. En adaptant la Proposition 4.14, on peut vérifier que $X^k = E(X'^k|\mathcal{G}) \in \mathcal{S}_b$, et que cette suite converge encore vers X dans $\mathcal{D}_2$. De plus, les variables X, X^k, $\langle D^{1,0}X\rangle_{leb}$, $\left\langle D^{1,0}X^k\right\rangle_{leb}$, $\left\langle D^{1,0}(X-X^k)\right\rangle_{leb}$, et $\langle D^{0,1}X\rangle_{\rho N}$, $\left\langle D^{0,1}X^k\right\rangle_{\rho N}$, $\left\langle D^{0,1}(X-X^k)\right\rangle_{\rho N}$ sont toutes $\mathcal{G}$-mesurables.
De même, il existe une suite Z^k de $\mathcal{S}_b$ convergeant vers Z dans $\mathcal{D}_2$, telle que toutes les variables associées à Z et Z^k (remplacer X par Z dans la liste ci-dessus) soient $\mathcal{A}$-mesurables.
Il est facile de voir que pour chaque k, X^kZ^k est encore dans $\mathcal{S}_b$ (car X^k et Z^k sont bornés), et que ses dérivées s'expriment comme dans l'énoncé (on n'utilise pas ici l'indépendance, mais seulement la forme de $\mathcal{S}_b$).
Or X^kZ^k converge vers XZ dans L^2, car

$$
\begin{aligned}
E\left((X^kZ^k - XZ)^2\right) &\leq E\left((X^k)^2(Z^k-Z)^2\right) + E\left((Z)^2(X^k-X)^2\right)\\
&= E\left((X^k)^2\right)\times E\left((Z^k-Z)^2\right) + E(Z^2)\times E\left((X^k-X)^2\right)
\end{aligned}
$$

par indépendance. De plus, on a par exemple

$$E\left(\left\langle X^k D^{0,1} Z^k - X D^{0,1} Z\right\rangle_{\rho N}\right)$$

$$\leq E\left((X^k)^2 \left\langle D^{0,1} Z^k - D^{0,1} Z\right\rangle_{\rho N}\right) + E\left(\left\langle D^{0,1} Z\right\rangle_{\rho N} (X^k - X)^2\right)$$

$$= E\left((X^k)^2\right) \times E\left(\left\langle D^{0,1} Z^k - D^{0,1} Z\right\rangle_{\rho N}\right) + E(\left\langle D^{0,1} Z\right\rangle_{\rho N}) \times E\left((X^k - X)^2\right) \longrightarrow 0$$

Les autres convergences se prouvent de la même façon, et la proposition est démontrée.

4.3 Un critère d'absolue continuité.

Enonçons maintenant le critère qu'on va utiliser pour prouver le Théorème 4.1. Ceci est une adaptation d'un résultat dû initialement à Bouleau, Hirsch, [9], dont la preuve a été simplifiée par Nualart, Zakai, [32]. La preuve figure aussi dans le livre de Nualart [30], p 87.

Théorème 4.16 *Soit $Z \in \mathcal{D}_2$. Soit*

$$\sigma = \left\langle D^{1,0} Z\right\rangle_{leb} + \left\langle D^{0,1} Z\right\rangle_{\rho N}$$

Alors, si $\sigma > 0$ p.s., la loi de Z admet une densité par rapport à la mesure de Lebesgue sur $I\!R$.

Preuve :
Etape 1 : supposons que $|Z| < 1$. Soit ϕ une fonction mesurable de $]-1,1[$ dans $[0,1]$, telle que $\displaystyle\int_{-1}^{1} \phi(y)dy = 0$. Le but est de montrer que $\phi(Z) = 0$ p.s.
On sait qu'il existe une suite $\{\phi_n\}$ de fonctions mesurables de $]-1,1[$ dans $[0,1]$, de classe C_b^1, telle que ϕ_n converge vers ϕ dx-p.s. et $P \circ Z^{-1}$-p.s. Posons

$$\Psi_n(y) = \int_{-1}^{y} \phi_n(x)dx \qquad \text{et} \qquad \Psi(y) = \int_{-1}^{y} \phi(x)dx$$

Il est alors clair que quel que soit n, $\Psi_n \in C_b^2$, donc à l'aide de la Proposition 4.11, $\Psi_n(Z) \in \mathcal{D}_2$ et

$$D^{1,0}_{\alpha,\tau} \Psi_n(Z) = \phi_n(Z) D^{1,0}_{\alpha,\tau} Z \quad \text{et} \quad D^{0,1}_{\alpha,\tau,\zeta} \Psi_n(Z) = \phi_n(Z) D^{0,1}_{\alpha,\tau,\zeta} Z$$

Comme ϕ_n converge vers ϕ dx-p.s., $\Psi_n(Z)$ converge vers $\Psi(Z)$ p.s., et par convergence dominée, la convergence a aussi lieu dans L^2. De plus, on sait que ϕ_n converge vers ϕ $P \circ Z^{-1}$-p.s., donc $\phi_n(Z)$ converge vers $\phi(Z)$ p.s. Donc

$$E\left(\left\langle \phi_n(Z) D^{0,1} Z - \phi(Z) D^{0,1} Z\right\rangle_{\rho N}\right) \leq E\left((\phi_n(Z) - \phi(Z))^2 \left\langle D^{0,1} Z\right\rangle_{\rho N}\right)$$

tend vers 0. La convergence concernant $D^{1,0}$ se prouve de la même façon, et on en déduit que $\Psi(Z) \in \mathcal{D}_2$, et

$$D^{1,0}_{\alpha,\tau} \Psi(Z) = \phi(Z) D^{1,0}_{\alpha,\tau} Y \qquad \text{et} \qquad D^{0,1}_{\alpha,\tau,\zeta} \Psi(Z) = \phi(Z) D^{0,1}_{\alpha,\tau,\zeta} Y$$

(ce qui n'était pas a priori évident, car Ψ n'est pas de classe C_b^1). Or l'hypothèse sur ϕ permet de dire que $\Psi(Z) = 0$. Ses dérivées sont donc nulles, et l'unicité des dérivées implique que presque sûrement,

$$\left\langle \phi(Z) D^{1,0} Z \right\rangle_{leb} + \left\langle \phi(Z) D^{0,1} Z \right\rangle_{\rho N} = 0$$

soit encore $\phi^2(Z)\sigma = 0$ p.s. Par conséquent, $\phi(Z) = 0$ p.s., et l'étape 1 est finie.

Etape 2 : plaçons nous maintenant dans le cas général (Z est une variable aléatoire à valeurs réelles). Soit F une bijection de $\mathbb{R}$ dans $]-1,1[$, de classe C_b^1, à dérivée strictement positive. Alors $F(Z)$ vérifie les hypothèses de l'étape 1, car par la Proposition 4.11, $F(Z) \in \mathcal{D}_2$, et

$$D^{1,0}_{\alpha,\tau} F(Z) = F'(Z) D^{1,0}_{\alpha,\tau} Z \qquad D^{0,1}_{\alpha,\tau,\zeta} F(Z) = F'(Z) D^{0,1}_{\alpha,\tau,\zeta} Z$$

Donc $F(Z)$ admet une densité, et il en va de même pour Z.

4.4 Dérivation et intégrales stochastiques.

Trois intégrales (aléatoires) apparaissent dans l'équation d'évolution (2.9). Afin de pouvoir utiliser le critère précédent, il faudra dériver la solution de (2.9). Les trois propositions principales de cette partie permettent de dériver ces intégrales.
Commençons par une remarque permettant d'éviter les confusions.

Remarque 4.17 *1. Soit Y et Y' deux versions faibles du même processus. Supposons que pour chaque y, chaque s, $Y(y,s) \in \mathcal{D}_2$. Alors pour presque tout Then for y, s,*

$$Y'(y,s) \in \mathcal{D}_2 \quad ; \quad \left\langle D^W Y(y,s) - D^W Y'(y,s) \right\rangle_{leb} + \left\langle D^N Y(y,s) - D^N Y'(y,s) \right\rangle_{\rho N} = 0 \ \text{ a.s.}$$

2. Soit Y un processus tel que pour chaque y, s, $Y(y,s) \in \mathcal{D}_2$. Supposons que pour chaque α, τ, $D^{1,0}_{\alpha,\tau} Y(y,s) = S_{\alpha,\tau}(y,s)$ $dPdyds$-presque partout (i.e. $S_{\alpha,\tau}$ est une version faible de $D^{1,0}_{\alpha,\tau} Y$), et que pour chaque n, $D^N_{X_n,T_n,Z_n} Y(y,s) = S'_{X_n,T_n,Z_n}(y,s)$ $dPdyds$-presque partout (i.e. S'_{X_n,T_n,Z_n} est une version faible de $D^N_{X_n,T_n,Z_n} Y$). Alors pour presque tout y, s, p.s.,

$$\left\langle D^W Y(y,s) - S(y,s) \right\rangle_{leb} + \left\langle D^N Y(y,s) - S'(y,s) \right\rangle_{\rho N} = 0$$

Preuve : 1. Ceci est immédiat, puisque pour presque tout y, s, p.s., $Y(y,s) = Y'(y,s)$.
2. Par exemple, grâce au Théorème de Fubini,

$$\int_0^T \int_0^L E\left(\left\langle D^W Y(y,s) - S(y,s) \right\rangle_{leb} \right) dyds$$

$$= \int_0^T \int_0^L d\alpha d\tau E \left(\int_0^T \int_0^L \left(D^{1,0}_{\alpha,\tau} Y(y,s) - S_{\alpha,\tau}(y,s) \right)^2 dyds \right) = 0$$

Définissons maintenant une classe de processus dont les intégrales seront dans $\mathcal{D}_2$.

Définition 4.18 *Soit Y un processus sur $[0,L] \times [0,T]$. On dira que Y est $\mathcal{D}_2$-prévisible-borné s'il vérifie les conditions suivantes :*

1. *Y est un processus prévisible.*

2. *Pour chaque y, s, $Y(y,s)$ appartient à $\mathcal{D}_2$, et $\sup_{y,s} |||Y(y,s)|||_2 < \infty$.*

3. *Pour chaque α, τ fixés, le processus $D^{1,0}_{\alpha,\tau} Y(y,s)$ admet une version faible prévisible sur $[0,L] \times [0,T]$ (et est nul pour $\tau > s$). L'application $\omega, \alpha, \tau, y, s \longrightarrow D^{1,0}_{\alpha,\tau} Y(y,s)(\omega)$ est globalement mesurable.*

4. *Pour chaque $n \geq 0$ fixé, le processus $D^{0,1}_{X_n,T_n,Z_n} Y(y,s)$ admet une version faible prévisible (et est nul pour $T_n > s$).*

Remarque 4.19 *Soit $Z \in \mathcal{S}$. Considérons la martingale càdlàg $Z_s = E(Z|\mathcal{F}_s)$. Ce processus est $\mathcal{D}_2$-prévisible-borné, et $|||Z_s|||_2 \leq |||Z|||_2$.*

Ceci est une conséquence immédiate de la Proposition 4.14-1.
La remarque suivante va permettre de "séparer" les variables y, s et ω dans les intégrales stochastiques.

Remarque 4.20 *$\mathcal{D}_2$ muni de la norme $||| \; |||_2$ est un espace de Hilbert sur $I\!R$, et $\mathcal{S}$ en est un sous espace vectoriel dense à base dénombrable. Soit $\{Z^k\}_{k\geq 0}$ une base orthonormée de $\mathcal{S}$. Alors tout élément Y de $\mathcal{D}_2$ se décompose de la manière suivante :*

$$Y = \sum_{k\geq 0} \lambda_k Z^k \quad \textit{avec} \quad \sum_{k\geq 0} \lambda_k^2 < \infty \quad \textit{où} \quad \lambda_k = \left\langle Y, Z^k \right\rangle_{\mathcal{D}_2} \tag{4.32}$$

<u>Preuve</u> : remarquons d'abord que $\mathcal{S}$ est à base dénombrable, car les espaces vectoriels $C^2_c(I\!R^m)$, $L^2([0,T] \times [0,L])$ et $\mathcal{CL}$ le sont.
Considérons $Y \in \mathcal{D}_2$. Y est limite d'une suite Y_n de $\mathcal{S}$. Décomposons chaque Y_n suivant la base $\{Z^k\}$:

$$Y_n = \sum_{k\geq 0} \lambda^n_k Z^k \quad \text{avec} \quad \sum_{k\geq 0} (\lambda^n_k)^2 < \infty \quad \text{où} \quad \lambda^n_k = \left\langle Y_n, Z^k \right\rangle_{\mathcal{D}_2}$$

Comme la suite Y_n est de Cauchy dans $\mathcal{D}_2$, il est facile de voir que la suite $\{\lambda^n\}_{n\geq 0} = \{\{\lambda^n_k\}_{k\geq 0}\}_{n\geq 0}$ est de Cauchy dans l^2 (l'ensemble des suites dont la série des carrés converge). En effet,

$$\sum_{k\geq 0} (\lambda^n_k - \lambda^m_k)^2 = |||Y_n - Y_m|||_2^2$$

Mais l^2 est complet, et donc λ^n converge vers un certain $\lambda = \{\lambda_k\}_{k\geq 0}$ dans l^2. Ceci implique que

$$|||Y_n - \sum_k \lambda_k Z^k|||_2^2 = \sum_{k\geq 0} (\lambda^n_k - \lambda_k)^2$$

tend vers 0. Y_n converge donc dans $\mathcal{D}_2$ vers $\sum_k \lambda_k Z^k$, et on obtient $Y = \sum_k \lambda_k Z^k$ par unicité de la limite. Enfin, $\lambda_k = \lim_n \left\langle Y_n, Z^k \right\rangle_{\mathcal{D}_2} = \left\langle Y, Z^k \right\rangle_{\mathcal{D}_2}$, quel que soit k, et la remarque est prouvée.

Appliquons alors la remarque 4.20 à un processus $\mathcal{D}_2$-prévisible-borné :

Lemme 4.21 *Soit $Y(y,s)$ un processus $\mathcal{D}_2$-prévisible-borné sur $[0,L] \times [0,T]$.*

1. *Par la Remarque 4.20, on peut décomposer Y dans $\mathcal{D}_2$:*
$$Y(y,s) = \sum_{k\geq 0} \phi_k(y,s) Z^k \quad \textit{avec} \quad \sup_{y,s} \sum_{k\geq 0} \phi_k^2(y,s) < \infty \tag{4.33}$$
où $\phi_k(y,s) = \left\langle Y(y,s), Z^k \right\rangle_{\mathcal{D}_2}$ est $\mathcal{B}([0,L]\times[0,T])$-mesurable.

2. *Si $Z_s^k = E(Z^k|\mathcal{F}_s)$, alors pour tout N,*
$$|||Y(y,s) - \sum_{k=0}^{N} \phi_k(y,s) Z_s^k|||_2 \leq |||Y(y,s) - \sum_{k=0}^{N} \phi_k(y,s) Z^k|||_2 \tag{4.34}$$
En particulier, $Y(y,s) = \sum_{k\geq 0} \phi_k(y,s) Z_s^k$ dans $\mathcal{D}_2$.

<u>Preuve</u> : 1. ϕ_k est mesurable par les propriétés des processus $\mathcal{D}_2$-prévisibles-bornés, et parce que, comme $Z^k \in \mathcal{S}$, les applications $D^{1,0}_{\alpha,\tau} Z^k(\omega)$ et $D^{0,1}_{X_n(\omega),T_n(\omega),Z_n(\omega)} Z^k(\omega)$ sont globalement mesurables.
2. se prouve exactement de la même manière que la deuxième partie de la Proposition 4.14, en utilisant le fait que pour $s > 0$ fixé, $Y(y,s)$ et Z_s^k ainsi que leurs dérivées sont $\mathcal{F}_s$-mesurables, et que ces dernières s'annulent pour $\tau > s$.

Nous avons aussi besoin de la Définition suivante :

Définition 4.22 *Soit $S_{\alpha,\tau,\zeta}(y,s)(\omega)$ une fonction sur $\Omega \times ([0,L]\times[0,T]\times O) \times ([0,L]\times[0,T])$. On dira que S appartient à la classe $\mathcal{DN}$ si :*

- *S est bornée dans $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$, i.e. $\sup_{y,s} E(\langle S(y,s)\rangle_{\rho N}) < \infty$.*
- *Pour chaque $n \geq 0$, le processus $S_{X_n,T_n,Z_n}(y,s)$ admet une version faible prévisible, et s'annule quand $T_n > s$.*

Ces conditions, clairement satisfaites par la dérivée relative à N d'un processus $\mathcal{D}_2$-prévisible-borné, sont nécessaires pour établir le lemme calculatoire (mais fondamental) 4.24.

Lemme 4.23 *Soit S une fonction de $\mathcal{DN}$. Alors le processus $\langle S(y,s)\rangle_{\rho N}$ admet une version faible prévisible.*

<u>Preuve</u> : sous la forme suivante,
$$\langle S(y,s)\rangle_{\rho N} = \sum_{n\geq 0} S^2_{X_n,T_n,Z_n}(y,s)\rho(Z_n) = \sum_{n\geq 0} S^2_{X_n,T_n,Z_n}(y,s)\rho(Z_n) 1_{\{T_n<s\}}$$
il est clair que $\langle S(y,s)\rangle_{\rho N}$ admet une version faible prévisible.

Le lemme technique suivant remplace l'isométrie L^2 constamment utilisée dans le cas des E.D.P.S. conduites par un Bruit Blanc. Les fonctions f, g, et h qui y figurent sont les paramètres de l'E.D.P.S. (1.1), et vérifient l'hypothèse (H').

Lemme 4.24 *Soit $Y(y,s)$ un processus admettant une version faible prévisible sur $[0,L]\times[0,T]$, et $S_{\alpha,\tau,\zeta}(y,s)$ une fonction de $\mathcal{DN}$. On pose, pour $\tau \leq t$:*

$$\begin{aligned}
T^a_{\alpha,\tau,\zeta}(x,t) &= \int_0^t\int_0^L G_{t-s}(x,y)f'(Y(y,s))S_{\alpha,\tau,\zeta}(y,s)W(dy,ds) \\
T^b_{\alpha,\tau,\zeta}(x,t) &= \int_0^t\int_0^L G_{t-s}(x,y)g'(Y(y,s))S_{\alpha,\tau,\zeta}(y,s)dyds \\
T^c_{\alpha,\tau,\zeta}(x,t) &= \int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(Y(y,s),z)S_{\alpha,\tau,\zeta}(y,s)\tilde{N}(ds,dy,dz) \qquad (4.35)
\end{aligned}$$

et $T^a_{\alpha,\tau,\zeta}(x,t) = T^b_{\alpha,\tau,\zeta}(x,t) = T^c_{\alpha,\tau,\zeta}(x,t) = 0$ si $\tau > t$. Ces trois fonctions sont encore dans $\mathcal{DN}$, et on a :

$$E\left[\langle T^a(x,t)\rangle_{\rho N}\right] = \int_0^t\int_0^L G^2_{t-s}(x,y)E\left[\{f'(Y(y,s))\}^2\langle S(y,s)\rangle_{\rho N}\right]dyds \qquad (4.36)$$

$$E\left[\left\langle T^b(x,t)\right\rangle_{\rho N}\right] \leq TL\int_0^t\int_0^L G^2_{t-s}(x,y)E\left[\{g'(Y(y,s))\}^2\langle S(y,s)\rangle_{\rho N}\right]dyds \qquad (4.37)$$

$$E\left[\langle T^c(x,t)\rangle_{\rho N}\right] = \int_0^t\int_0^L\int_O G^2_{t-s}(x,y)E\left[\{h'_x(Y(y,s),z)\}^2\langle S(y,s)\rangle_{\rho N}\right]\varphi(z)dzdyds \qquad (4.38)$$

Remarquons que les intégrales dans (4.35) ne sont pas réellement définies pour chaque α,τ,ζ. Une fois de plus, nous sous-entendons qu'il faut remplacer α,τ,ζ par X_n,T_n,Z_n.

<u>Preuve</u> : tout d'abord, il est clair que si les égalités (4.36), (4.37), (4.38) sont vérifiées, alors T^a, T^b, et T^c sont dans $\mathcal{DN}$. La mesurabilité et la prévisibilité sont immédiates, et (H'), l'Appendice (6.3), le fait que $\sup_{y,s}E(\langle S(y,s)\rangle_{\rho N}) < \infty$, et ces égalités conduisent à

$$\sup_{y,s}E\left(\langle T^a(y,s)\rangle_{\rho N}\right) + \sup_{y,s}E\left(\left\langle T^b(y,s)\right\rangle_{\rho N}\right) + \sup_{y,s}E\left(\langle T^c(y,s)\rangle_{\rho N}\right) < \infty$$

Prouvons par exemple l'égalité (4.38). En utilisant la Notation 4.8, puis en appliquant le Théorème de Fubini pour intégrands positifs, on obtient :

$$\begin{aligned}
E\left(\langle T^c(x,t)\rangle_{\rho N}\right) &= E\left(\sum_{n\geq 0}\rho(Z_n)\left[T^c_{X_n,T_n,Z_n}(x,t)\right]^2\right) \\
&= \sum_{n\geq 0}E\left(\rho(Z_n)\left[T^c_{X_n,T_n,Z_n}(x,t)\right]^2\right) \\
&= \sum_{n\geq 0}E\left[\left(\int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(Y(y,s),z)\sqrt{\rho(Z_n)}S_{X_n,T_n,Z_n}(y,s)\tilde{N}(ds,dy,dz)\right)^2\right]
\end{aligned}$$

Mais par hypothèse, $\sqrt{\rho(Z_n)}S_{X_n,T_n,Z_n}(y,s) = S_{X_n,T_n,Z_n}(y,s)\times\sqrt{\rho(Z_n)}1_{\{T_n<s\}}$ admet une version faible prévisible pour chaque n, et on peut appliquer la formule (2.3) :

$$E\left(\langle T^c(x,t)\rangle_{\rho N}\right)$$

$$= \sum_{n\geq 0} \int_0^t \int_0^L \int_O G_{t-s}^2(x,y) E\left(\{h'_x(Y(y,s),z)\}^2 \rho(Z_n) S^2_{X_n,T_n,Z_n}(y,s) \right) \varphi(z) dz dy ds$$

Enfin, une nouvelle application du Théorème de Fubini pour intégrants positifs permet de conclure :

$$\begin{aligned} & E\left(\langle T^c(x,t) \rangle_{\rho N} \right) \\ &= \int_0^t \int_0^L \int_O G_{t-s}^2(x,y) E\left(\{h'_x(Y(y,s),z)\}^2 \sum_{n\geq 0} \rho(Z_n) S^2_{X_n,T_n,Z_n}(y,s) \right) \varphi(z) dz dy ds \\ &= \int_0^t \int_0^L \int_O G_{t-s}^2(x,y) E\left(\{h'_x(Y(y,s),z)\}^2 \langle S(y,s) \rangle_{\rho N} \right) \varphi(z) dz dy ds \end{aligned}$$

Les équations (4.36) et (4.37) se vérifient de la même façon.

Nous sommes enfin dans la possibilité d'énoncer les trois propositions principales de cette sous-section.

Proposition 4.25 *Soit* $\{Y(x,t)\}_{x\in[0,L]\ t\in[0,T]}$ *un processus* $\mathcal{D}_2$*-prévisible-borné. On note*

$$U(x,t) = \int_0^t \int_0^L G_{t-s}(x,y) f(Y(y,s)) W(dy,ds)$$

Alors U *est aussi* $\mathcal{D}_2$*-prévisible-borné, et on a, si* Y_- *est une version faible prévisible de* Y *:*

$$\begin{aligned} D^{1,0}_{\alpha,\tau} U(x,t) &= G_{t-\tau}(x,\alpha) f(Y_-(\alpha,\tau)) 1_{\{\tau\leq t\}} \\ &\quad + \int_0^t \int_0^L G_{t-s}(x,y) f'(Y(y,s)) D^{1,0}_{\alpha,\tau} Y(y,s) W(dy,ds) \\ D^{0,1}_{\alpha,\tau,\zeta} U(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) f'(Y(y,s)) D^{0,1}_{\alpha,\tau,\zeta} Y(y,s) W(dy,ds) \end{aligned}$$

Proposition 4.26 *Soit* $\{Y(x,t)\}_{x\in[0,L]\ t\in[0,T]}$ *un processus* $\mathcal{D}_2$*-prévisible-borné. On note*

$$U(x,t) = \int_0^t \int_0^L G_{t-s}(x,y) g(Y(y,s)) dy ds$$

Alors U *est aussi* $\mathcal{D}_2$*-prévisible-borné, et on a :*

$$\begin{aligned} D^{1,0}_{\alpha,\tau} U(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g'(Y(y,s)) D^{1,0}_{\alpha,\tau} Y(y,s) dy ds \\ D^{0,1}_{\alpha,\tau,\zeta} U(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g'(Y(y,s)) D^{0,1}_{\alpha,\tau,\zeta} Y(y,s) dy ds \end{aligned}$$

Proposition 4.27 *Soit $\{Y(x,t)\}_{x\in[0,L]\ t\in[0,T]}$ un processus $\mathcal{D}_2$-prévisible-borné. On note*

$$U(x,t) = \int_0^t \int_0^L \int_O G_{t-s}(x,y)h(Y(y,s),z)\tilde{N}(ds,dy,dz)$$

Alors U est aussi $\mathcal{D}_2$-prévisible-borné, et on a, si Y_- est une version faible prévisible de Y :

$$\begin{aligned} D^{1,0}_{\alpha,\tau}U(x,t) &= \int_0^t \int_0^L \int_O G_{t-s}(x,y)h'_x(Y(y,s),z)D^{1,0}_{\alpha,\tau}Y(y,s)\tilde{N}(ds,dy,dz) \\ D^{0,1}_{\alpha,\tau,\zeta}U(x,t) &= G_{t-\tau}(x,\alpha)h'_z(Y_-(\alpha,\tau),\zeta)1_{\{\tau\le t\}} \\ &\quad + \int_0^t \int_0^L \int_O G_{t-s}(x,y)h'_x(Y(y,s),z)D^{0,1}_{\alpha,\tau,\zeta}Y(y,s)\tilde{N}(ds,dy,dz) \end{aligned}$$

Remarquons que les dérivées obtenues ne dépendent pas du choix de la version faible prévisible Y_- de Y. En effet, si Y^- est une autre version faible prévisible de Y, alors il est clair que pour tout x, t, p.s.,

$$\begin{aligned} &\int_0^t \int_0^L \left\{G_{t-\tau}(x,\alpha)f(Y_-(\alpha,\tau)) - G_{t-\tau}(x,\alpha)f(Y^-(\alpha,\tau))\right\}^2 d\alpha d\tau \\ + &\int_0^t \int_0^L \int_{I\!R} \left\{G_{t-\tau}(x,\alpha)h'_z(Y_-(\alpha,\tau),\zeta) - G_{t-\tau}(x,\alpha)h'_z(Y^-(\alpha,\tau),\zeta)\right\}^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta) = 0 \end{aligned}$$

Ces trois propositions se ressemblent beaucoup. Néanmoins, la troisième est sensiblement plus difficile, c'est pourquoi on ne prouvera que celle-ci. Il est d'abord nécessaire d'établir le lemme suivant, qui utilise l'hypothèse (S) :

Lemme 4.28 *Soit ϕ une fonction mesurable sur $[0,T]\times[0,L]\times O$, de classe C^1 sur O (en z), avec ϕ'_z bornée, et telle que*

$$|\phi(s,y,z)| \le K\eta(z)$$

(où $\eta \in L^2(O,\varphi(z)dz)$ est définie dans l'hypothèse (H')). Soit $(t,x) \in]0,T]\times[0,L]$ fixé, et

$$G(s,y,z) = G_{t-s}(x,y)\phi(s,y,z)1_{\{s<t\}}$$

alors $\tilde{N}(G) = \int_0^T \int_0^L \int_O G(s,y,z)\tilde{N}(ds,dy,dz)$ est élément de $\mathcal{D}_2$, et on peut calculer ses dérivées :

$$D^{1,0}_{\alpha,\tau}\tilde{N}(G) = 0 \qquad et \qquad D^{0,1}_{\alpha,\tau,\zeta}\tilde{N}(G) = G_{t-\tau}(x,\alpha)\phi'_z(\tau,\alpha,\zeta)1_{\{\tau<t\}}$$

Preuve : afin d'appliquer la Proposition 4.13, il faut approcher G par des fonctions bornées, à support compact, de classe C^1 en z, et à dérivée bornée : il faut tronquer G, et "compactifier" son support. Soit donc $\{\mathcal{T}_\epsilon\}$ une suite de fonctions de troncation de classe C^∞ sur $I\!R$, à dérivées premières bornées par 1, telles que

$$\mathcal{T}_\epsilon(u) = \begin{cases} u & \text{si } |u| \le \frac{1}{\epsilon} \\ 1+\frac{1}{\epsilon} & \text{si } u \ge 2+\frac{1}{\epsilon} \\ -1-\frac{1}{\epsilon} & \text{si } u \le -2-\frac{1}{\epsilon} \end{cases} \qquad \text{et} \qquad |\mathcal{T}_\epsilon(u)| \le |u|$$

Par (S), on sait d'autre part qu'il existe une famille de fonctions $\{K_\epsilon\}$ sur O, positives et C^1, bornées par 1, à support compact dans O, et vérifiant

$$\forall\, z \in O, K_\epsilon(z) \longrightarrow_{\epsilon\to 0} 1 \quad ; \quad \int_O \left(K'_\epsilon(z)\right)^2 \eta^2(z)\rho(z)\varphi(z)dz \longrightarrow_{\epsilon\to 0} 0 \tag{4.39}$$

On pose alors

$$G_\epsilon(s,y,z) = \mathcal{T}_\epsilon(G_{t-s}(x,y))\mathcal{T}_\epsilon(\phi(s,y,z))K_\epsilon(z)1_{\{s<t\}}$$

Il est clair que G_ϵ satisfait les hypothèses de la Proposition 4.13 : G_ϵ est bornée, à support compact, C^1 en z, et sa dérivée est bornée. On a

$$(G_\epsilon)'_z(s,y,z) = \mathcal{T}_\epsilon(G_{t-s}(x,y))1_{\{s<t\}}\left(\mathcal{T}'_\epsilon(\phi(s,y,z))\phi'_z(s,y,z)K_\epsilon(z) + \mathcal{T}_\epsilon(\phi(s,y,z))K'_\epsilon(z)\right)$$

Donc, par la Proposition 4.13, $\tilde{N}(G_\epsilon) \in \mathcal{D}_2$,

$$D^{1,0}_{\alpha,\tau}\tilde{N}(G_\epsilon) = 0 \quad \text{et} \quad D^{0,1}_{\alpha,\tau,\zeta}\tilde{N}(G_\epsilon) = (G_\epsilon)'_z(\alpha,\tau,\zeta)$$

Vérifions que $\tilde{N}(G_\epsilon)$ converge vers $\tilde{N}(G)$ dans $\mathcal{D}_2$ (avec les notations de l'énoncé, même si on n'est pas encore sûr que ces expressions sont bien les dérivées de $\tilde{N}(G)$). Tout d'abord, par la formule (2.3) :

$$\begin{aligned}
E\left(\left(\tilde{N}(G_\epsilon) - \tilde{N}(G)\right)^2\right) &= E\left(\left(\int_0^T\int_0^L\int_O (G(s,y,z) - G_\epsilon(s,y,z))\tilde{N}(dsdydz)\right)^2\right) \\
&= \int_0^T\int_0^L\int_O (G(s,y,z) - G_\epsilon(s,y,z))^2\varphi(z)dzdyds
\end{aligned}$$

Ce terme tend vers 0 par convergence dominée. En effet, pour chaque s,y,z, l'intégrand tend vers 0, et

$$(G(s,y,z) - G_\epsilon(s,y,z))^2 \leq 4G^2(s,y,z) \leq 4KG^2_{t-s}(x,y)\eta^2(z) \in L^1(\varphi(z)1_O(z)dzdyds)$$

De plus,

$$E\left(\left\langle D^{1,0}\tilde{N}(G) - D^{1,0}\tilde{N}(G_\epsilon)\right\rangle_{leb}\right) = 0$$

Enfin,

$$\begin{aligned}
&E\left(\left\langle D^{0,1}\tilde{N}(G) - D^{0,1}\tilde{N}(G_\epsilon)\right\rangle_{\rho N}\right) \\
&= E\left(\int_0^T\int_0^L\int_O \left(G'_z(\alpha,\tau,\zeta) - (G_\epsilon)'_z(\alpha,\tau,\zeta)\right)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)\right) \\
&= \int_0^T\int_0^L\int_O \left(G'_z(\alpha,\tau,\zeta) - (G_\epsilon)'_z(\alpha,\tau,\zeta)\right)^2\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau
\end{aligned}$$

Ceci tend aussi vers 0, par convergence dominée et grâce à (4.39) . Le lemme technique est prouvé.

Preuve de la Proposition 4.27

Etape 1 : si z est fixé, $h(\,.\,,z)$ est de classe C_b^1 sur $I\!R$. Donc, par la Proposition 4.11, pour tout (x,t,z) dans $[0,L]\times[0,T]\times I\!R$, $h(Y(x,t),z)\in\mathcal{D}_2$, et on a

$$D^{1,0}_{\alpha,\tau}h(Y(x,t),z)=h'_x(Y(x,t),z)D^{1,0}_{\alpha,\tau}Y(x,t)$$

$$\text{et}\quad D^{0,1}_{\alpha,\tau,\zeta}h(Y(x,t),z)=h'_x(Y(x,t),z)D^{0,1}_{\alpha,\tau,\zeta}Y(x,t)$$

De plus, vue (H'), on remarque que

$$\sup_{x,t}|||h(Y(x,t),z)|||_2\leq K\eta(z)$$

Décomposons alors $h(Y,z)$ dans $\mathcal{D}_2$, à l'aide d'une légère extension du Lemme 4.21 :

$$h(Y(x,t),z)=\sum_{k\geq 0}\phi_k(x,t,z)Z^k$$

où pour chaque k, ϕ_k est mesurable. On a de plus

$$\sup_{x,t}\sum_k\phi_k^2(x,t,z)=\sup_{x,t}|||h(Y(x,t),z)|||_2^2\leq K\eta^2(z)$$

Par le Théorème de Lebesgue, comme (par (H')) $|h'_z(x,z)|\leq K(1+|x|)$ et h''_{zx} est bornée, comme h'_z et h''_{zx} sont continues,

$$\begin{aligned}\phi_k(y,s,z)&=\left\langle h(Y(y,s),z),Z^k\right\rangle_{\mathcal{D}_2}\\&=E\left[h(Y(y,s),z)Z^k\right]+E\left[h'_x(Y(y,s),z)\left\langle D^{1,0}Y(y,s),D^{1,0}Z^k\right\rangle_{leb}\right]\\&\quad+E\left[h'_x(Y(y,s),z)\left\langle D^{0,1}Y(y,s),D^{0,1}Z^k\right\rangle_{\rho N}\right]\end{aligned}$$

est de classe C^1 en z, et sa dérivée est bornée. D'autre part, comme $h'_z(\,.\,,z)$ est de classe C_b^1, la Proposition 4.11 affirme que $h'_z(Y(y,s),z)$ appartient à $\mathcal{D}_2$, permet de calculer ses dérivées, puis de voir que (toujours à l'aide de (H')) :

$$\sup_{x,t,z}\sum_k\left((\phi_k)'_z(x,t,z)\right)^2=\sup_{x,t,z}|||h'_z(Y(x,t),z)|||_2^2<\infty$$

On a aussi

$$h'_z(Y(y,s),z)=\sum_{k\geq 0}(\phi_k)'_z(y,s,z)Z^k$$

dans $\mathcal{D}_2$. Toujours grâce au Lemme 4.21, si on pose $Z_s^k=E(Z^k|\mathcal{F}_s)$ et

$$\Psi^N(y,s,z)=\sum_{k=0}^N\phi_k(y,s,z)Z_s^k \tag{4.40}$$

on a les assertions suivantes :

$$\forall y,s,z \qquad |||\Psi^N(y,s,z) - h(Y(y,s),z)|||_2 \longrightarrow 0 \tag{4.41}$$

$$\sup_N \sup_{y,s} |||\Psi^N(y,s,z) - h(Y(y,s),z)|||_2^2 \leq K\eta^2(z) \tag{4.42}$$

$$\forall y,s,z \qquad |||(\Psi^N)'_z(y,s,z) - h'_z(Y(y,s),z)|||_2 \longrightarrow 0 \tag{4.43}$$

$$\sup_N \sup_{y,s,z} |||(\Psi^N)'_z(y,s,z) - h'_z(Y(y,s),z)|||_2^2 \leq \sup_{y,s,z} |||h'_z(Y(y,s),z)|||_2^2 < \infty \tag{4.44}$$

Afin de vérifier que U est bien dans $\mathcal{D}_2$, nous allons écrire U sous la forme $\sum_k \tilde{U}_k$, vérifier que pour chaque k, $\tilde{U}_k$ est bien dans $\mathcal{D}_2$, puis étudier la convergence de la série.

Etape 2 : montrons d'abord que pour chaque k,

$$\tilde{U}^k(x,t) = \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) Z_s^k \tilde{N}(dsdydz)$$

appartient à $\mathcal{D}_2$, et calculons ses dérivées. Il est vraiment nécessaire d'utiliser la suite Z_s^k, *car les processus $\phi_k(y,s,z)Z^k$ n'admettent pas a priori de versions faibles prévisibles.* On utilise une approximation du type Péano pour Z_s^k. Notons, si $0 \leq s \leq T$

$$s_n = \sup\left\{ \frac{i}{n}T \Big/ \frac{i}{n}T < s \right\} \vee 0$$

Approchons $\tilde{U}^k$ par

$$\begin{aligned}
\tilde{U}_n^k(x,t) &= \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) Z_{s_n}^k \tilde{N}(dsdydz) \\
&= \sum_{i=0}^{n-1} Z_{\frac{i}{n}T}^k \times \int_{[0,t]\cap]\frac{i}{n}T,\frac{i+1}{n}T]} \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z)\tilde{N}(dsdydz)
\end{aligned}$$

Comme $Z_{\frac{i}{n}T}^k$ appartient à $\mathcal{D}_2$ et est $\mathcal{F}_{\frac{i}{n}T}$-mesurable, comme l'application ϕ_k satisfait les hypothèses du Lemme 4.28, ce lemme et la Proposition 4.15 nous permettent d'affirmer que $\tilde{U}_n^k(x,t) \in \mathcal{D}_2$, et

$$\begin{aligned}
D_{\alpha,\tau,\zeta}^{0,1}\tilde{U}_n^k(x,t) &= \sum_{i=0}^{n-1} \int_{[0,t]\cap]\frac{i}{n}T,\frac{i+1}{n}T]} \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z)\tilde{N}(dsdydz) \times D_{\alpha,\tau,\zeta}^{0,1} Z_{\frac{i}{n}T}^k \\
&\quad + \sum_{i=0}^{n-1} Z_{\frac{i}{n}T}^k \times G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta) 1_{\{\tau\in]\frac{i}{n}T,\frac{i+1}{n}T]\cap[0,t[\}} \\
&= \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) D_{\alpha,\tau,\zeta}^{0,1} Z_{s_n}^k \tilde{N}(dsdydz) \\
&\quad + G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta) Z_{\tau_n}^k 1_{\{\tau\leq t\}}
\end{aligned}$$

et de même

$$D^{1,0}_{\alpha,\tau}\tilde{U}^k_n(x,t) = \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) D^{1,0}_{\alpha,\tau} Z^k_{s_n} \tilde{N}(dsdydz)$$

On pose donc

$$\begin{aligned} D^{0,1}_{\alpha,\tau,\zeta}\tilde{U}^k(x,t) &= \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) D^{0,1}_{\alpha,\tau,\zeta} Z^k_s \tilde{N}(dsdydz) \\ &\quad + G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta) Z^k_{\tau-} 1_{\{\tau\le t\}} \end{aligned}$$

et

$$D^{1,0}_{\alpha,\tau}\tilde{U}^k(x,t) = \int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z) D^{1,0}_{\alpha,\tau} Z^k_s \tilde{N}(dsdydz)$$

Il reste à vérifier les convergences au sens de $\mathcal{D}_2$. D'abord, par la formule (2.3), puis comme $|\phi_k| \le K\eta$,

$$\begin{aligned} &E\left[(\tilde{U}^k(x,t) - \tilde{U}^k_n(x,t))^2\right] \\ &= \int_0^t \int_0^L \int_O G^2_{t-s}(x,y)\phi^2_k(y,s,z) E\left((Z^k_s - Z^k_{s_n})^2\right)\varphi(z)dzdyds \\ &\le K \int_0^t \int_0^L G^2_{t-s}(x,y) E\left((Z^k_s - Z^k_{s_n})^2\right) dyds \end{aligned}$$

Ceci tend vers 0 par double convergence dominée : pour chaque s, $|Z^k_s - Z^k_{s_n}|$ tend vers 0 p.s., et est majoré par $2\sup_\omega |Z^k(\omega)| < \infty$. Donc le Théorème de convergence dominée montre que pour chaque s, $E\left((Z^k_s - Z^k_{s_n})^2\right)$ tend vers 0. Comme cette espérance est d'autre part majorée par une constante, et comme $G^2_{t-s}(x,y)$ est dans $L^1(dyds)$, le Théorème de convergence dominée appliqué à la mesure $dyds$ permet de conclure.
Ensuite,

$$\begin{aligned} &E\left[\left\langle D^{0,1}\tilde{U}^k(x,t) - D^{0,1}\tilde{U}^k_n(x,t)\right\rangle_{\rho N}\right] \\ &\le C \int_0^t \int_0^L \int_O G^2_{t-\tau}(x,\alpha)\left((\phi_k)'_z(\alpha,\tau,\zeta)\right)^2 E\left((Z^k_{\tau_n} - Z^k_{\tau-})^2\right)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau \\ &+ CE\Big[\int_0^t \int_0^L \int_O \Big(\int_0^t \int_0^L \int_O G_{t-s}(x,y)\phi_k(y,s,z)\left(D^{0,1}_{\alpha,\tau,\zeta} Z^k_{s_n} - D^{0,1}_{\alpha,\tau,\zeta} Z^k_s\right) \\ &\qquad \tilde{N}(ds,dy,dz)\Big)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\Big] \end{aligned}$$

Le premier terme tend vers 0 comme précédemment, et une adaptation du Lemme 4.24 permet de majorer le second par

$$\int_0^t \int_0^L \int_O G^2_{t-s}(x,y)\phi^2_k(y,s,z) E\left(\left\langle D^{0,1} Z^k_{s_n} - D^{0,1} Z^k_s\right\rangle_{\rho N}\right)\varphi(z)dzdyds$$

qui tend vers 0 pour les mêmes raisons (cette fois $\left\langle D^{0,1}Z^k_{s_n} - D^{0,1}Z^k_s\right\rangle_{\rho N}$ n'est plus majoré par une constante, mais par la variable aléatoire

$$X^k = 4\sup_{[0,T]} E\left[\left\langle D^{0,1}Z^k\right\rangle_{\rho N}\;\middle|\; \mathcal{F}_s\right]$$

qui appartient à $L^1(\Omega)$ par l'inégalité de Doob, puisque $\left\langle D^{0,1}Z^k\right\rangle_{\rho N} \in L^2(\Omega)$, car $Z^k \in \mathcal{S}$). Enfin, on prouve de même que $E\left[\left\langle D^{1,0}\tilde{U}^k(x,t) - D^{1,0}\tilde{U}^k_n(x,t)\right\rangle_{leb}\right]$ tend vers 0, et l'étape 2 est terminée.

Etape 3 : approchons maintenant $U(x,t)$ par

$$U^N(x,t) = \sum_{k=0}^{N} \tilde{U}^k(x,t)$$

Par l'étape 1, on sait que $U^N(x,t)$ appartient à $\mathcal{D}_2$, et on connait ses dérivées. Rappelons que Ψ^N est défini par (4.40). On a :

$$D^{1,0}_{\alpha,\tau}U^N(x,t) = \int_0^t\int_0^L\int_O G_{t-s}(x,y)D^{1,0}_{\alpha,\tau}\Psi^N(y,s,z)\tilde{N}(dsdydz)$$

et

$$\begin{aligned} D^{0,1}_{\alpha,\tau,\zeta}U^N(x,t) &= G_{t-\tau}(x,\alpha)(\Psi^N)'_z(\alpha,\tau,\zeta)1_{\{\tau\le t\}} \\ &\quad + \int_0^t\int_0^L\int_O G_{t-s}(x,y)D^{0,1}_{\alpha,\tau,\zeta}\Psi^N(y,s,z)\tilde{N}(dsdydz) \end{aligned}$$

On notera $D^{1,0}_{\alpha,\tau}U(x,t)$ et $D^{0,1}_{\alpha,\tau,\zeta}U(x,t)$ les expressions de l'énoncé, pour des raisons de notation, même si on n'est pas encore sûr que ce sont bien les dérivées de $U(x,t)$. D'abord, par (2.3),

$$\begin{aligned} & E\left[(U(x,t) - U^N(x,t))^2\right] \\ & \le \int_0^t\int_0^L\int_O G^2_{t-s}(x,y)E\left((h(Y(y,s),z) - \Psi^N(y,s))^2\right)\varphi(z)dydzds \\ & \le K\int_0^t\int_0^L\int_O G^2_{t-s}(x,y)|||h(Y(y,s),z) - \Psi^N(y,s)|||_2^2\varphi(z)dydzds \end{aligned}$$

Ceci tend vers 0 par convergence dominée, grâce aux équations (4.41) et (4.42). De plus,

$$\begin{aligned} & E\left[\left\langle D^{0,1}U(x,t) - D^{0,1}U^N(x,t)\right\rangle_{\rho N}\right] \\ & \le K\int_0^t\int_0^L\int_O G^2_{t-\tau}(x,\alpha)E\left((h'_z(Y_-(\alpha,\tau),\zeta) - (\Psi^N)'_z(\alpha,\tau,\zeta))^2\right)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau \\ & +KE\Big[\int_0^T\int_0^L\int_O\Big(\int_0^t\int_0^L\int_O G_{t-s}(x,y)(D^{0,1}_{\alpha,\tau,\zeta}h(Y(y,s),z) - D^{1,0}_{\alpha,\tau}\Psi^N(y,s,z)) \\ & \qquad \tilde{N}(ds,dy,dz)\Big)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big] \end{aligned}$$

Le premier terme tend vers 0 comme précédemment (grâce à (4.43) et (4.44)). Quant au second, on le majore à l'aide du Lemme 4.24 par

$$\int_0^t \int_0^L \int_O G_{t-s}^2(x,y) E\left(\left\langle D^{0,1}\Psi^N(y,s,z) - D^{0,1}h(Y(y,s),z)\right\rangle_{\rho N}\right)\varphi(z)dzdyds$$

qui tend encore vers 0 par convergence dominée.
On prouve de même la troisième convergence, et l'étape 3 est finie.

Etape 4 : il ne reste plus qu'à vérifier que $U(x,t)$ est $\mathcal{D}_2$ prévisible-borné. D'abord, U admet une version faible prévisible par le Lemme 2.4, ainsi que $D^{1,0}_{\alpha,\tau}U(x,t)$ si α, τ sont fixés. La mesurabilité globale de $D^{1,0}_{\alpha,\tau}U(x,t)$ et $D^{0,1}_{\alpha,\tau,\zeta}U(x,t)$ est vérifiée, et $D^{0,1}_{X_n,T_n,Z_n}U(x,t)$, pour $n \geq 0$ fixé, admet une version faible prévisible. U est borné dans L^2 par le Lemme 2.4, et $E(\langle D^{0,1}U(x,t)\rangle_{\rho N})$ est borné par le Lemme 4.24, (H'), et l'Appendice (6.3). De plus,

$$\begin{aligned}
&E\left(\left\langle D^{1,0}U(x,t)\right\rangle_{leb}\right)\\
&= E\left[\int_0^T \int_0^L \left(\int_0^t \int_0^L \int_O G_{t-s}(x,y)h'_x(Y(y,s),z)D^{1,0}_{\alpha,\tau}Y(y,s)\tilde{N}(ds,dy,dz)\right)^2 d\alpha d\tau\right]\\
&= \int_0^T \int_0^L E\left[\left(\int_0^t \int_0^L \int_O G_{t-s}(x,y)h'_x(Y(y,s),z)D^{1,0}_{\alpha,\tau}Y(y,s)\tilde{N}(ds,dy,dz)\right)^2\right] d\alpha d\tau\\
&\leq \int_0^T \int_0^L \int_0^t \int_0^L \int_O G_{t-s}^2(x,y)\eta^2(z)E\left(\left\{D^{1,0}_{\alpha,\tau}Y(y,s)\right\}^2\right)\varphi(z)dzdydsd\alpha d\tau\\
&= \int_0^t \int_0^L \int_O G_{t-s}^2(x,y)\eta^2(z)E\left(\left\langle D^{1,0}Y(y,s)\right\rangle_{leb}\right)\varphi(z)dzdyds
\end{aligned}$$

Ceci est clairement borné, car Y est $\mathcal{D}_2$-prévisible-borné, car $\eta \in L^2(O, \varphi(z)dz)$, et par l'Appendice (6.3). La preuve est achevée.

4.5 Dérivation de la solution.

Afin de pouvoir appliquer le Théorème 4.16, il faut prouver que V est dans $\mathcal{D}_2$, et calculer ses dérivées.

Théorème 4.29 *Sous les hypothèses (H'), (M), (D), et (S), si V est l'unique solution faible de (1.1) au sens de la Définition 2.3, alors V est $\mathcal{D}_2$-prévisible-borné, et, si V_- est une version faible pr'evisible de V,*

$$\begin{aligned}
D^{1,0}_{\alpha,\tau}V(x,t) &= G_{t-\tau}(x,\alpha)f(V_-(\alpha,\tau))1_{\{t\geq\tau\}} \qquad (4.45)\\
&+ \int_0^t \int_0^L G_{t-s}(x,y)f'(V(y,s))D^{1,0}_{\alpha,\tau}V(y,s)W(dy,ds)\\
&+ \int_0^t \int_0^L G_{t-s}(x,y)g'(V(y,s))D^{1,0}_{\alpha,\tau}V(y,s)dyds
\end{aligned}$$

$$+\int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)D^{1,0}_{\alpha,\tau}V(y,s)\tilde{N}(ds,dy,dz)$$

$$\begin{aligned} D^{0,1}_{\alpha,\tau,\zeta}V(x,t) &= G_{t-\tau}(x,\alpha)h'_z(V_-(\alpha,\tau),\zeta)1_{\{t\geq\tau\}} \\ &+\int_0^t\int_0^L G_{t-s}(x,y)f'(V(y,s))D^{0,1}_{\alpha,\tau,\zeta}V(y,s)W(dy,ds) \\ &+\int_0^t\int_0^L G_{t-s}(x,y)g'(V(y,s))D^{0,1}_{\alpha,\tau,\zeta}V(y,s)dyds \\ &+\int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)D^{0,1}_{\alpha,\tau,\zeta}V(y,s)\tilde{N}(ds,dy,dz) \end{aligned} \tag{4.46}$$

Toute la suite de cette sous-section est consacrée à la preuve de ce théorème.
Nous allons pour cela définir $D^{1,0}_{\alpha,\tau}V$ et $D^{0,1}_{\alpha,\tau,\zeta}V$ comme les solutions respectives de (4.45) et (4.46), puis nous vérifierons que ce sont bien les dérivées de V. Il faut donc montrer que ces équations ont bien une unique solution chacune.

Les équations (4.45) et (4.46) sont en fait des “systèmes d'équations”. En particulier, dans le cas de l'équation (4.46), nous ne cherchons pas à résoudre l'équation pour chaque α,τ,ζ fixé, mais pour chaque n fixé, en remplaçant α,τ,ζ par X_n,T_n,Z_n. La solution $D^{0,1}V(x,t)$ doit être considérée comme à valeurs dans $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$, pour chaque x,t.

Lemme 4.30 *1. L'équation (4.45) admet une unique solution*

$$\begin{aligned} x,t &\mapsto D^{1,0}V(x,t) \\ [0,L]\times[0,T] &\mapsto L^2(P(d\omega)d\alpha d\tau) \end{aligned}$$

telle que pour tout α,τ fixé, le processus $D^{1,0}_{\alpha,\tau}V(x,t)$ admette une version faible prévisible. La solution est unique au sens où si S et S' sont deux telles solutions de (4.45), alors

$$\sup_{x,t} E\left[\langle S(x,t)-S'(x,t)\rangle_{leb}\right]=0$$

2. L'équation (4.46) admet une unique solution

$$\begin{aligned} x,t &\mapsto D^{0,1}V(x,t) \\ [0,L]\times[0,T] &\mapsto L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta)) \end{aligned}$$

appartenant à $\mathcal{DN}$. La solution est unique au sens où si T et T' sont deux telles solutions de (4.46), alors

$$\sup_{x,t} E\left[\langle T(x,t)-T'(x,t)\rangle_{\rho N}\right]=0$$

Preuve : prouvons par exemple 2., et commençons par l'unicité. Soit S et T deux solutions dans $\mathcal{DN}$. Un calcul utilisant le Lemme 4.24 et l'hypothèse (H') montre que

$$\begin{aligned} E\left(\langle S(x,t)-T(x,t)\rangle_{\rho N}\right) &\leq K\int_0^t\int_0^L G_{t-s}^2(x,y)E\left(\langle S(x,t)-T(x,t)\rangle_{\rho N}\right)dyds \\ &\leq K\int_0^t \sup_y E\left(\langle S(y,s)-T(y,s)\rangle_{\rho N}\right)\frac{ds}{\sqrt{t-s}} \end{aligned}$$

où la deuxième inégalité provient de l'Appendice (6.2). En réitérant cette inégalité, puis en utilisant le lemme de Gronwall, on obtient

$$\sup_{y,s} E\left(\langle S(y,s)-T(y,s)\rangle_{\rho N}\right)=0$$

L'existence se montre à l'aide d'une itération de Picard. Soit

$$S^0_{\alpha,\tau,\zeta}(x,t)=G_{t-\tau}(x,\alpha)h'_z(V(\alpha,\tau),\zeta)1_{\{t>\tau\}}$$

Alors S^0 appartient à $\mathcal{DN}$. En effet, la mesurabilité globale est immédiate. De plus, si $n\geq 0$, le processus $S^0_{X_n,T_n,Z_n}(x,t)$ est càg et adapté, donc prévisible. Enfin, comme V_- est prévisible,

$$\sup_{x,t}E\left(\left\langle S^0(x,t)\right\rangle_{\rho N}\right)=\sup_{x,t}E\left(\int_0^t\int_0^L\int_O G_{t-\tau}^2(x,\alpha)(h'_z)^2(V_-(\alpha,\tau),\zeta)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau\right)$$

$$\leq K\sup_{x,t}\int_0^t\int_0^L\int_O G_{t-\tau}^2(x,\alpha)E\left(1+|V(\alpha,\tau)|^2\right)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau<\infty$$

par (H'), car $\rho\in L^1(O,\varphi(z)dz)$, car V est borné dans L^2, et par l'Appendice (6.3).
On pose ensuite, pour $n\geq 0$:

$$\begin{aligned} S^{n+1}_{\alpha,\tau,\zeta}(x,t) &= S^0_{\alpha,\tau,\zeta}(x,t)+\int_0^t\int_0^L G_{t-s}(x,y)f'(V(y,s))S^n_{\alpha,\tau,\zeta}(y,s)W(dy,ds) \\ &\quad+\int_0^t\int_0^L G_{t-s}(x,y)g'(V(y,s))S^n_{\alpha,\tau,\zeta}(y,s)dyds \\ &\quad+\int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)S^n_{\alpha,\tau,\zeta}(y,s)\tilde{N}(ds,dy,dz) \end{aligned} \tag{4.47}$$

Une récurrence utilisant le Lemme 4.24 montre que pour chaque n, S^n est dans $\mathcal{DN}$.
Le Lemme 4.24 et le Lemme de Picard permettent de vérifier que la série de terme général

$$\left[\sup_{x,t}E\left(\left\langle S^{n+1}(x,t)-S^n(x,t)\right\rangle_{\rho N}\right)\right]^{\frac{1}{2}}$$

converge. L'existence en découle.

Preuve du Théorème 4.29 : on doit une fois de plus considérer V comme la limite d'une suite V^n, montrer par récurrence qu'elle prend ses valeurs dans $\mathcal{D}_2$, puis étudier la convergence.

Considérons les approximations de Picard de la solution V définie dans la section 3 par les équations (3.1) et (3.2). Comme $\mathcal{V}_0$ est déterministe et borné, il est facile de voir que V^0 est $\mathcal{D}_2$-prévisible-borné. Un raisonnement par récurrence utilisant les Propositions 4.25, 4.26, et 4.27 montre que pour tout n, V^n est $\mathcal{D}_2$-prévisible-borné, et permet de calculer ses dérivées : si V^n_- est yune version faible prévisible de V^n,

$$\begin{aligned} D^{1,0}_{\alpha,\tau} V^{n+1}(x,t) &= G_{t-\tau}(x,\alpha) f(V^n_-(\alpha,\tau)) 1_{\{t\geq\tau\}} \\ &+ \int_0^t \int_0^L G_{t-s}(x,y) f'(V^n(y,s)) D^{1,0}_{\alpha,\tau} V^n(y,s) W(dy,ds) \\ &+ \int_0^t \int_0^L G_{t-s}(x,y) g'(V^n(y,s)) D^{1,0}_{\alpha,\tau} V^n(y,s) dyds \\ &+ \int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(V^n(y,s),z) D^{1,0}_{\alpha,\tau} V^n(y,s) \tilde{N}(ds,dy,dz) \end{aligned}$$

$$\begin{aligned} D^{0,1}_{\alpha,\tau,\zeta} V^{n+1}(x,t) &= G_{t-\tau}(x,\alpha) h'_z(V^n_-(\alpha,\tau),\zeta) 1_{\{t\geq\tau\}} \\ &+ \int_0^t \int_0^L G_{t-s}(x,y) f'(V^n(y,s)) D^{0,1}_{\alpha,\tau,\zeta} V^n(y,s) W(dy,ds) \\ &+ \int_0^t \int_0^L G_{t-s}(x,y) g'(V^n(y,s)) D^{0,1}_{\alpha,\tau,\zeta} V^n(y,s) dyds \\ &+ \int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(V^n(y,s),z) D^{0,1}_{\alpha,\tau,\zeta} V^n(y,s) \tilde{N}(ds,dy,dz) \end{aligned}$$

Il ne reste plus qu'à vérifier que, si $D^{1,0}_{\alpha,\tau}V(x,t)$ et $D^{0,1}_{\alpha,\tau,\zeta}V(x,t)$ sont définies comme solutions de (4.45) et (4.46), (au sens du Lemme 4.30), alors $V^n(x,t)$ converge vers $V(x,t)$ dans $\mathcal{D}_2$. On a déjà vu dans la preuve du Théorème 3.1 que $V^n(x,t)$ converge vers $V(x,t)$ dans L^2. Posons donc

$$G_n(x,t) = E\left[\left\langle D^{0,1}V(x,t) - D^{0,1}V^n(x,t)\right\rangle_{\rho N}\right] \quad \text{et} \quad \phi_n(t) = \sup_x G_n(x,t)$$

Il s'agit de prouver que pour chaque (x,t) dans $[0,L]\times[0,T]$, $G_n(x,t)$ tend vers 0. Majorons $G_{n+1}(x,t)$:

$$G_{n+1}(x,t) \leq K(I^n_1(x,t) + ... + I^n_7(x,t))$$

où :

$$I^n_1(x,t) = \int_0^t \int_0^L \int_O G^2_{t-\tau}(x,\alpha) E\Big(\left[h'_z(V^n(\alpha,\tau),\zeta) - h'_z(V(\alpha,\tau),\zeta)\right]^2\Big)\rho(\zeta)\varphi(\zeta) d\zeta d\alpha d\tau$$

$$\begin{aligned} I^n_2(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L & G_{t-s}(x,y)[f'(V(y,s)) - f'(V^n(y,s))] \\ & D^{0,1}_{\alpha,\tau,\zeta} V(y,s) W(dy,ds)\Big)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\Big] \end{aligned}$$

$$I_3^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L G_{t-s}(x,y)[g'(V(y,s)) - g'(V^n(y,s))]$$
$$D^{0,1}_{\alpha,\tau,\zeta}V(y,s)dyds\Big)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]$$

$$I_4^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L \int_O G_{t-s}(x,y)[h'_x(V(y,s),z) - h'_x(V^n(y,s),z)]$$
$$D^{0,1}_{\alpha,\tau,\zeta}V(y,s)\tilde{N}(ds,dy,dz)\Big)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]$$

$$I_5^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L G_{t-s}(x,y)f'(V^n(y,s))$$
$$\times[D^{0,1}_{\alpha,\tau,\zeta}V(y,s) - D^{0,1}_{\alpha,\tau,\zeta}V^n(y,s)]W(dy,ds)\Big)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]$$

$$I_6^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L G_{t-s}(x,y)g'(V^n(y,s))$$
$$\times[D^{0,1}_{\alpha,\tau,\zeta}V(y,s) - D^{0,1}_{\alpha,\tau,\zeta}V^n(y,s)]dyds\Big)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]$$

$$I_7^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L \int_O G_{t-s}(x,y)h'_x(V^n(y,s),z)$$
$$\times[D^{0,1}_{\alpha,\tau,\zeta}V(y,s) - D^{0,1}_{\alpha,\tau,\zeta}V^n(y,s)]\tilde{N}(ds,dy,dz)\Big)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]$$

Majorons ces termes un par un : h''_{zx} est bornée, donc

$$\big[h'_z(V^n(\alpha,\tau),\zeta) - h'_z(V(\alpha,\tau),\zeta)\big]^2 \leq K(V^n(\alpha,\tau) - V(\alpha,\tau))^2$$

Or

$$\sup_{\alpha,\tau} E\left((V^n(\alpha,\tau) - V(\alpha,\tau))^2\right) \longrightarrow 0$$

Donc, par l'Appendice (6.3), $I_1^n(x,t) \leq K_n \longrightarrow 0$.
Le Lemme 4.24 permet d'affirmer que $I_4^n(x,t)$ est égal à :

$$\int_0^t \int_0^L \int_O G^2_{t-s}(x,y)E\left[(h'_x(V(y,s),z) - h'_x(V^n(y,s),z))^2 \times \left\langle D^{0,1}V(y,s)\right\rangle_{\rho N}\right]\varphi(z)dzdyds$$

Appliquons alors l'inégalité de Hölder (pour la mesure $dyds$, avec $p = 5/4$ et $q = 5$), pour majorer $I_4^n(x,t)$ par :

$$\left[\int_0^t \int_0^L (G_{t-s}(x,y))^{5/2}dyds\right]^{4/5}$$
$$\times\left[\int_0^T \int_0^L \left[\int_O E\left((h'_x(V(y,s),z) - h'_x(V^n(y,s),z))^2 \left\langle D^{0,1}V(y,s)\right\rangle_{\rho N}\right)\varphi(z)dz\right]^5 dyds\right]^{1/5}$$

Le premier terme du produit est majoré par une constante (par l'Appendice (6.3), car $5/2 < 3$), et le second ne dépend plus de x, t (c'est pourquoi on a appliqué l'inégalité de Hölder). Notons donc ce terme K_n. Mais K_n tend vers 0 par (triple) convergence dominée. En effet comme h'_x est continue (en x) et majorée par η, comme $V^n(y,s)$ tend vers $V(y,s)$ dans L^2 (donc en probabilité), et comme $\langle D^{0,1}V(y,s)\rangle_{\rho N}$ est dans L^1 (pour y, s fixé), le Théorème de convergence dominée appliqué à P implique que l'espérance tend vers 0 pour chaque y, s, z. Mais cette espérance est d'autre part majorée par (à une constante près) $\eta^2(z) \in L^1(O, \varphi(z)dz)$. Le Théorème de convergence dominée pour la mesure $\varphi(z)dz$ permet donc d'affirmer que pour chaque y, s,

$$\left[\int_O E\left((h'_x(V(y,s),z) - h'_x(V^n(y,s),z))^2 \left\langle D^{0,1}V(y,s)\right\rangle_{\rho N}\right) \varphi(z)dz\right]^5$$

tend vers 0. Cette quantité est enfin bornée, et une dernière application du Théorème de convergence dominée (pour $dyds$) permet de conclure : K_n tend vers 0.
On majore I_2^n et I_3^n grâce au même procédé, et on obtient (C est une constante) :

$$I_2^n(x,t) + I_3^n(x,t) + I_4^n(x,t) \leq CK_n \longrightarrow 0$$

Après avoir utilisé le Lemme 4.24 et (H'), on voit que

$$I_7^n(x,t) \leq K \int_0^t \int_0^L G_{t-s}^2(x,y) G_n(y,s) dyds$$

Les mêmes majorations s'appliquent à $I_5^n(x,t)$ et $I_6^n(x,t)$, et on obtient finalement :

$$\begin{aligned} G_{n+1}(x,t) &\leq K_n + K \int_0^t \int_0^L G_n(y,s) G_{t-s}^2(x,y) dyds \\ &\leq K_n + K \int_0^t \phi_n(s) \frac{ds}{\sqrt{t-s}} \end{aligned}$$

avec $K_n \longrightarrow 0$. On en déduit que

$$\phi_{n+2}(t) \leq K'_n + K' \int_0^t \phi_n(s) ds$$

où K'_n tend encore vers 0. Comme ϕ_0 et ϕ_1 sont bornées grâce à l'étape 1, on peut appliquer le Lemme de Picard (ou plutôt une légère extension du Lemme de Picard) :

$$\sup_{[0,T]} \phi_n(t) \longrightarrow 0$$

Puis

$$\sup_{x,t} G_n(x,t) \longrightarrow 0$$

Un calcul similaire montre que

$$\sup_{x,t} E\left[\left\langle D^{1,0}V(x,t) - D^{1,0}V^n(x,t)\right\rangle_{leb}\right]$$

tend vers 0, et le Théorème 4.29 est prouvé.

4.6 Existence de la densité.

Nous sommes enfin armés pour prouver le Théorème 4.1. Soit donc $(x,t) \in [0,L]\times]0,T]$, et supposons que les hypothèses (M), (D), et (H') soient satisfaites. Par les Théorèmes 4.16 et 4.29, il suffit de vérifier que presque sûrement,

$$\begin{aligned}\sigma(x,t) &= \left\langle D^{1,0}V(x,t)\right\rangle_{leb} + \left\langle D^{0,1}V(x,t)\right\rangle_{\rho N} \\ &= \sigma^{1,0}(x,t) + \sigma^{0,1}(x,t)\end{aligned}$$

est strictement positif sous l'une des hypothèses (EW), $(EP1)$, ou $(EP2)$.

On ne sait pas calculer explicitement $\sigma(x,t)$. C'est pourquoi on doit séparer les cas : nous allons vérifier que sous (EW), $\sigma^{1,0}(x,t) > 0$ p.s., et que sous $(EP1)$ ou $(EP2)$, $\sigma^{0,1}(x,t) > 0$ p.s. La différence de nature entre les deux produits scalaires nous empêche d'établir un résultat sous une hypothèse de non-dégénerescence conjointe.

Existence de la densité sous $(EP1)$.

Comme prévu, nous nous intéressons à $\sigma^{0,1}$. Commençons par la remarque suivante.

Remarque 4.31 *Quel que soit $c \geq 0$, on peut supposer que $g' \geq c$.*

Preuve : il suffit d'utiliser la définition 2.3 des solutions faibles de (1.1), qui s'étend (cf Walsh [47]) aux fonctions test $\phi(x,t)$ de classe C^∞ sur $[0,L] \times [0,T]$ vérifiant $\phi'_x(0,t) = \phi'_x(L,t) = 0$ pour tout t :

$$\begin{aligned}&\int_0^L V(x,t)\phi(x,t)dx = \int_0^L \mathcal{V}_0(x)\phi(x,0)dx + \int_0^t\int_0^L V(x,s)\left(\phi''_{xx}(x,s) + \phi'_t(x,s)\right)dxds \\ &+ \int_0^t\int_0^L f(V(x,s))\phi(x,s)W(dx,ds) + \int_0^t\int_0^L g(V(x,s))\phi(x,s)dxds \\ &+ \int_0^t\int_0^L\int_O h(V(x,s),z)\phi(x,s)\tilde{N}(ds,dx,dz)\end{aligned}$$

Ceci peut se réécrire

$$\begin{aligned}&\int_0^L V(x,t)\phi(x,t)dx = \int_0^L \mathcal{V}_0(x)\phi(x,0)dx \\ &+ \int_0^t\int_0^L V(x,s)\left(\phi''_{xx}(x,s) - c\phi(x,s) + \phi'_t(x,s)\right)dxds \\ &+ \int_0^t\int_0^L f(V(x,s))\phi(x,s)W(dx,ds) + \int_0^t\int_0^L \left(g(V(x,s)) + cV(x,s)\right)\phi(x,s)dxds \\ &+ \int_0^t\int_0^L\int_O h(V(x,s),z)\phi(x,s)\tilde{N}(ds,dx,dz)\end{aligned}$$

Mais la fonction de Green associée au système

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - cu \ , \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

est $H_t(x,y) = e^{-ct}G_t(x,y)$. La solution faible V de (1.1) (et de manière équivalente de (2.9)), est donc aussi solution de l'équation (2.9) où l'on a remplacé $G_t(x,y)$ par $H_t(x,y)$ et $g(x)$ par $g(x)+cx$. Comme les deux noyaux G et H se comportent de la même manière, comme la fonction g' est bornée, la remarque est prouvée.

Remarquons ensuite que, comme V est $\mathcal{D}_2$-prévisible-borné, $D^{0,1}_{\alpha,\tau,\zeta}V(x,t) = 0$ dès que $\tau > t$ (voir la Proposition 4.14). De plus, par $(EP1)$, $f=0$, et $|h'_x| \leq \hat{\eta} \in L^1(O,\varphi(z)dz)$, donc en posant $G'(x) = g'(x) - \int_O h'_x(x,z)\varphi(z)dz$, on obtient :

$$\begin{aligned}
D^{0,1}_{\alpha,\tau,\zeta}V(x,t) &= G_{t-\tau}(x,\alpha)h'_z(V_-(\alpha,\tau),\zeta) \\
&\quad + \int_{]\tau,t[}\int_0^L G_{t-s}(x,y)G'(V(y,s))D^{0,1}_{\alpha,\tau,\zeta}V(y,s)dyds \\
&\quad + \int_{]\tau,t[}\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)D^{0,1}_{\alpha,\tau,\zeta}V(y,s)N(ds,dy,dz) \\
&\quad \text{si } \tau \leq t \\
D^{0,1}_{\alpha,\tau,\zeta}V(x,t) &= 0 \quad \text{si } \tau > t
\end{aligned}$$

Soit $S_{\alpha,\tau}(x,t)$ l'unique solution (au même sens que dans le Lemme 4.30-2.) de l'équation

$$\begin{aligned}
S_{\alpha,\tau}(x,t) &= G_{t-\tau}(x,\alpha) \\
&\quad + \int_{]\tau,t[}\int_0^L G_{t-s}(x,y)G'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\
&\quad + \int_{]\tau,t[}\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)N(ds,dy,dz) \\
&\quad \text{si } \tau \leq t \\
S_{\alpha,\tau}(x,t) &= 0 \quad \text{si } \tau > t
\end{aligned}$$

Par unicité, on remarque que

$$D^{0,1}_{\alpha,\tau,\zeta}V(x,t) = h'_z(V_-(\alpha,\tau),\zeta))S_{\alpha,\tau}(x,t)$$

au sens où

$$\sup_{x,t} E\left(\left\langle D^{0,1}V(x,t) - h'_z(V_-(\,.\,,\,.\,),\,.\,)S(x,t)\right\rangle_{\rho N}\right) = 0$$

ce qui implique bien entendu que pour chaque x, t, presque sûrement,

$$\sigma^{0,1}(x,t) = \langle h'_z(V_-(\,.\, , \,.\,), \,.\,)S(x,t)\rangle_{\rho N}$$

Par la Remarque 4.31, on peut supposer $G' \geq 0$ (choisir g' supérieure à $\int_O \hat{\eta}(z)\varphi(z)dz$). Comme par hypothèse h'_x est positive, un raisonnement par récurrence sur les approximations de Picard de $S_{\alpha,\tau}(x,t)$, puis un passage à la limite montre que pour chaque x, t, $P(d\omega)N(\omega, d\tau, d\alpha, d\zeta)$-presque partout,

$$S_{\alpha,\tau}(x,t) \geq G_{t-\tau}(x,\alpha)1_{\{\tau<t\}}$$

et il suffit de vérifier que pour tout $t > 0$, tout $x \in [0, L]$, p.s.,

$$\int_0^t \int_0^L \int_O G^2_{t-\tau}(x,\alpha)(h'_z(V_-(\alpha,\tau),\zeta))^2\rho(\zeta)N(d\tau,d\alpha,d\zeta) > 0$$

Mais pour tout $\tau \in [0, t[$, tout $\alpha \in [0, L]$ et tout $\zeta \in O$, $G^2_{t-\tau}(x,\alpha)\rho(\zeta) > 0$. Donc il suffit de montrer que pour tout $t > 0$, p.s.,

$$\int_0^t \int_0^L \int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\neq 0\}}N(d\tau,d\alpha,d\zeta) > 0$$

Considérons pour cela le temps d'arrêt

$$R = \inf\left\{ s > 0 \;\Big/\; \int_0^s \int_0^L \int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\neq 0\}}N(d\tau,d\alpha,d\zeta) > 0 \right\}$$

et prouvons que $R = 0$ p.s. :

$$\begin{aligned} &E\left(\int_0^R \int_0^L \int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\neq 0\}}\varphi(\zeta)d\zeta d\alpha d\tau\right) \\ &= E\left(\int_0^R \int_0^L \int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\neq 0\}}N(d\tau,d\alpha,d\zeta)\right) \leq 1 \end{aligned}$$

Ceci implique

$$\int_0^R \int_0^L \int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\neq 0\}}\varphi(\zeta)d\zeta d\alpha d\tau < \infty$$

p.s., ce qui contredit l'hypothèse

$$\forall x \in I\!R, \quad \int_O 1_{\{h'_z(x,\zeta)\neq 0\}}\varphi(\zeta)d\zeta = \infty$$

sauf si $R = 0$ p.s., et le Théorème 4.1 sous $(EP1)$ est démontré.

Existence de la densité sous $(EP2)$.

Comme sous $(EP1)$, on commence par écrire $D^{0,1}_{\alpha,\tau,\zeta}V(x,t) = h'_z(V_-(\alpha,\tau),\zeta)S_{\alpha,\tau}(x,t)$, où $S_{\alpha,\tau}(x,t)$ est l'unique solution (au sens du Lemme 4.30-2) de l'équation :

$$S_{\alpha,\tau}(x,t) \;\; = \;\; G_{t-\tau}(x,\alpha) + \int_{]\tau,t[} \int_0^L G_{t-s}(x,y)f'(V(y,s))S_{\alpha,\tau}(y,s)W(dy,ds)$$

$$
\begin{aligned}
&+\int_{]\tau,t[}\int_0^L G_{t-s}(x,y)g'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\
&+\int_{]\tau,t[}\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)\tilde{N}(ds,dy,dz) \\
&\text{si } \tau\leq t
\end{aligned} \tag{4.48}
$$

$$
S_{\alpha,\tau}(x,t) \quad = \quad 0 \quad \text{si } \tau > t
$$

Par unicité dans $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$, on obtient l'égalité presque sûre :

$$
\sigma^{0,1}(x,t)=\int_0^T\int_0^L\int_O S^2_{\alpha,\tau}(x,t)\left(h'_z(V_-(\alpha,\tau),\zeta)\right)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)
$$

Il est alors clair, par définition de l'ensemble $\mathcal{H}$ (voir $(EP2)$), que dès que

$$
\Sigma(x,t)=\int_0^T\int_0^1\int_{\mathcal{H}} S^2_{\alpha,\tau}(x,t)\rho(\zeta)N(d\tau,d\alpha,d\zeta)>0
$$

$\sigma^{0,1}(x,t)$ est aussi strictement positif. A cette fin, on décompose S en deux partie : la première va "exploser" autour de (x,t), alors que la seconde va rester suffisamment petite.

$$
S_{\alpha,\tau}(x,t)=G_{t-\tau}(x,\alpha)1_{\{\tau\leq t\}}+Q_{\alpha,\tau}(x,t)
$$

où $Q_{\alpha,\tau}(x,t)=0$ pour $\tau>t$, et, pour $\tau\leq t$,

$$
\begin{aligned}
Q_{\alpha,\tau}(x,t) \quad = \quad &\int_{]\tau,t[}\int_0^L G_{t-s}(x,y)f'(V(y,s))S_{\alpha,\tau}(y,s)W(dy,ds) \\
&+\int_{]\tau,t[}\int_0^L G_{t-s}(x,y)g'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\
&+\int_{]\tau,t[}\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)\tilde{N}(ds,dy,dz)
\end{aligned} \tag{4.49}
$$

On obtient ainsi, pour tout $\epsilon>0$ assez petit :

$$
\begin{aligned}
\Sigma(x,t) \quad &\geq \quad \int_{t-\epsilon}^t\int_0^L\int_{\mathcal{H}} S^2_{\alpha,\tau}(x,t)\rho(\zeta)N(d\tau,d\alpha,d\zeta) \\
&\geq \quad \frac{2}{3}\int_{t-\epsilon}^t\int_0^L\int_{\mathcal{H}} G^2_{t-\tau}(x,\alpha)\rho(\zeta)N(d\tau,d\alpha,d\zeta) \\
&\qquad -2\int_{t-\epsilon}^t\int_0^L\int_{\mathcal{H}} Q^2_{\alpha,\tau}(x,t)\rho(\zeta)N(d\tau,d\alpha,d\zeta) \\
&= \quad \frac{2}{3}A_\epsilon(x,t)-2B_\epsilon(x,t)
\end{aligned}
$$

Le lemme suivant montre que $B_\epsilon(x,t)$ est petit.

Lemme 4.32 *Il existe une constante $C_1 > 0$ telle que pour tout $\epsilon > 0$,*

$$E\left(B_\epsilon(x,t)\right) \leq C_1 \epsilon$$

Preuve : on peut directement majorer $E\left(B_\epsilon(x,t)\right)$ par

$$\begin{aligned}
KE&\left[\int_{t-\epsilon}^t \int_0^L \int_{\mathcal{H}} \left(\int_{]\tau,t[} \int_0^L G_{t-s}(x,y) f'(V(y,s)) S_{\alpha,\tau}(y,s) W(dy,ds)\right)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\right] \\
&+KE\left[\int_{t-\epsilon}^t \int_0^L \int_{\mathcal{H}} \left(\int_{]\tau,t[} \int_0^L G_{t-s}(x,y) g'(V(y,s)) S_{\alpha,\tau}(y,s) dyds\right)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\right] \\
&+KE\Big[\int_{t-\epsilon}^t \int_0^L \int_{\mathcal{H}} \left(\int_{]\tau,t[} \int_0^L \int_O G_{t-s}(x,y) h'_x(V(y,s),z) S_{\alpha,\tau}(y,s) \tilde{N}(ds,dy,dz)\right)^2 \\
&\qquad \rho(\zeta) N(d\tau,d\alpha,d\zeta)\Big]
\end{aligned}$$

En utilisant le Lemme 4.24 puis (H'), on obtient

$$E\left(B_\epsilon(x,t)\right) \leq K \int_{t-\epsilon}^t \int_0^L G^2_{t-s}(x,y) E\left(\left\langle S_{\alpha,\tau}(y,s) 1_{\mathcal{H}}(\zeta) 1_{[t-\epsilon,s[}(\tau)\right\rangle_{\rho N}\right) dyds$$

Mais pour $s \in]\tau,t[$,

$$\begin{aligned}
E\left(\left\langle S_{\alpha,\tau}(y,s) 1_{\mathcal{H}}(\zeta) 1_{[t-\epsilon,s[}(\tau)\right\rangle_{\rho N}\right) &\leq E\left(\left\langle S_{\alpha,\tau}(y,s) 1_{\mathcal{H}}(\zeta) 1_{[s-\epsilon,s[}(\tau)\right\rangle_{\rho N}\right) \\
&\leq KE\left(\int_{s-\epsilon}^s \int_0^L \int_{\mathcal{H}} G^2_{s-\tau}(y,\alpha) \rho(\zeta) N(d\tau,d\alpha,d\zeta)\right) \\
&\quad +KE\left(\int_{s-\epsilon}^s \int_0^L \int_{\mathcal{H}} Q^2_{\alpha,\tau}(y,s) \rho(\zeta) N(d\tau,d\alpha,d\zeta)\right) \\
&= K\left[I_1^\epsilon(y,s) + I_2^\epsilon(y,s)\right]
\end{aligned}$$

Un calcul similaire au précédent montre que

$$I_2^\epsilon(y,s) \leq K \int_{s-\epsilon}^s \int_0^L G^2_{s-s'}(y,y') E\left(\left\langle S_{\alpha,\tau}(y',s') 1_{\mathcal{H}}(\zeta) 1_{[s-\epsilon,s'[}(\tau)\right\rangle_{\rho N}\right) dy'ds' \leq K\sqrt{\epsilon}$$

par l'Appendice (6.4), et car S est définie comme vérifiant

$$\sup_{y,s} E\left(\langle S(y,s)\rangle_{\rho N}\right) < \infty$$

De plus,

$$I_1^\epsilon(y,s) = \int_{s-\epsilon}^s \int_0^L \int_{\mathcal{H}} G^2_{s-\tau}(y,\alpha) \rho(\zeta) \varphi(\zeta) d\zeta d\alpha d\tau \leq K\sqrt{\epsilon}$$

car $\rho \in L^1(O, \varphi(\zeta)d\zeta)$, et par l'Appendice (6.4).
On obtient ainsi

$$E\left(B_\epsilon(x,t)\right) \leq K\sqrt{\epsilon}\int_{t-\epsilon}^{t}\int_0^L G_{t-s}^2(x,y)dyds$$

et une dernière application de (6.4) conduit au résultat recherché.

Le prochain lemme montre que $E\left(e^{-\lambda A_\epsilon(x,t)}\right)$ est petit (quand λ est grand), donc que $A_\epsilon(x,t)$ est grand.

Lemme 4.33 *Il existe $\lambda_0 \geq 0$, $\epsilon_0 > 0$, et $K_0 > 0$, tels que pour tout $\lambda \geq \lambda_0$, tout $\epsilon \leq \epsilon_0$,*

$$\int_0^\epsilon\int_0^L\int_{\mathcal{H}}\left(1-e^{-\lambda G_s^2(x,y)\rho(z)}\right)\varphi(z)dzdyds \geq K_0\lambda^{r_0}\epsilon^{\frac{3-2r_0}{2}} \tag{4.50}$$

<u>Preuve</u> : remarquons d'abord que par l'Appendice (6.1), pour tout $s \in [\epsilon/2, \epsilon]$, tout $y \in [x-\sqrt{\epsilon}, x+\sqrt{\epsilon}]$, ($C > 0$ est une constante) :

$$G_s^2(x,y) \geq \frac{C}{\epsilon}$$

Le membre de gauche de (4.50) est donc minoré par

$$\int_{\mathcal{H}}\int_{\epsilon/2}^{\epsilon}\int_{x-\sqrt{\epsilon}}^{x+\sqrt{\epsilon}}\left(1-e^{-\lambda G_s^2(x,y)\rho(z)}\right)dyds\ \varphi(z)dz$$

$$\geq K\epsilon\sqrt{\epsilon}\int_{\mathcal{H}}\left(1-e^{-C\frac{\lambda}{\epsilon}\rho(z)}\right)\varphi(z)dz \geq K_0\epsilon^{\frac{3}{2}}\left(\frac{\lambda}{\epsilon}\right)^{r_0}$$

où la dernière inégalité, valide dès que $C\lambda/\epsilon \geq \gamma_0$, provient de l'hypothèse $(EP2)$.

Nous pouvons maintenant prouver que $\Sigma(x,t) > 0$ p.s. Remarquons que pour tout $\eta > 0$, tout $\epsilon > 0$, et tout $\lambda > 0$,

$$\begin{aligned} P\left(\Sigma(x,t) > 0\right) &\geq P\left(\frac{2}{3}A_\epsilon(x,t) > \eta\right) + P\left(2B_\epsilon(x,t) < \eta\right) - 1 \\ &\geq 1 - e^{\lambda\eta}E\left(e^{-\frac{2}{3}\lambda A_\epsilon(x,t)}\right) - \frac{2}{\eta}E\left(B_\epsilon(x,t)\right)\end{aligned}$$

or $E\left(B_\epsilon(x,t)\right) \leq C_1\epsilon$, et si $\epsilon < \epsilon_0$, si $\lambda \geq \frac{3}{2}\lambda_0$,

$$\begin{aligned} E\left(e^{-\frac{2}{3}\lambda A_\epsilon(x,t)}\right) &= \exp\left(-\int_{t-\epsilon}^{t}\int_0^L\int_{\mathcal{H}}\left(1-e^{-\frac{2}{3}\lambda G_{t-\tau}^2(x,\alpha)\rho(\zeta)}\right)\varphi(\zeta)d\zeta d\alpha d\tau\right) \\ &= \exp\left(-\int_0^{\epsilon}\int_0^L\int_{\mathcal{H}}\left(1-e^{-\frac{2}{3}\lambda G_{\tau}^2(x,\alpha)\rho(\zeta)}\right)\varphi(\zeta)d\zeta d\alpha d\tau\right) \\ &\leq \exp\left(-C_2\lambda^{r_0}\epsilon^{\frac{3}{2}-r_0}\right)\end{aligned}$$

par le Lemme 4.33. Ainsi, pour tout $\eta > 0$, tout $\epsilon < \epsilon_0$, tout $\lambda \geq \frac{3}{2}\lambda_0$,

$$P\left(\Sigma(x,t) > 0\right) \geq 1 - \exp\left(-C_2\lambda^{r_0}\epsilon^{\frac{3}{2}-r_0} + \lambda\eta\right) - 2C_1\frac{\epsilon}{\eta}$$

On choisit $\lambda = \eta^{-1} = \epsilon^{-\alpha}$ avec $\alpha > 0$. On obtient, pour tout $\epsilon > 0$ assez petit :

$$P\left(\Sigma(x,t) > 0\right) \geq 1 - \exp\left(1 - C_2\left(\frac{1}{\epsilon}\right)^{\alpha r_0 - \frac{3}{2} + r_0}\right) - 2C_1\epsilon^{1-\alpha}$$

Comme $r_0 > \frac{3}{4}$, on peut choisir $\alpha > 0$ tel que

$$\alpha r_0 - \frac{3}{2} + r_0 > 0 \quad ; \quad 1 - \alpha > 0$$

(prendre $\alpha = 3/4r_0$). En faisant tendre ϵ vers 0, on en déduit que $\Sigma(x,t) > 0$ p.s., puis que $\sigma^{0,1}(x,t) > 0$ p.s., et le Théorème 4.1 est prouvé sous $(EP2)$.

En comparant les preuves du Théorème 4.1 sous $(EP1)$ et sous $(EP2)$, on voit à quel point l'utilisation d'une dérivée "locale" est nécessaire : sous $(EP1)$, on considère l'objet $\langle D^{0,1}V(x,t)\rangle_{\rho N}$, et on n'a pas vraiment besoin de connaitre $D^{0,1}_{\alpha,\tau,\zeta}V(x,t)$ pour chaque α, τ, ζ. C'est d'une manière équivalente que travaille Saint Loubert Bié dans [40]. Sous $(EP2)$, on utilise par contre les expression locales des dérivées, ce qui permet d'exploiter "l'explosion" du noyau de Green, et donc de n'émettre aucune hypothèse de nullité, de monotonie, ou de positivité des fonctions f, g, h.

Existence de la densité sous (EW).

Nous allons prouver ici que $\sigma^{1,0}(x,t) > 0$ p.s. Le raisonnement qui suit est largement inspiré de celui de Bally et Pardoux dans [4], bien que ceux-ci utilisent la continuité de leur solution.
Considérons l'unique solution $S_{\alpha,\tau}(x,t)$ de l'équation (4.48), mais cette fois-ci au sens du Lemme 4.30-1 (ce n'est pas le même objet que dans le paragraphe précédent). Comme V est $\mathcal{D}_2$-prévisible-borné, $D^{1,0}_{\alpha,\tau}V(x,t) = 0$ dès que $\tau > t$. Un argument d'unicité (voir le Lemme 4.30-1.) conduit à l'égalité presque sûre :

$$\sigma^{1,0}(x,t) = \int_0^t\int_0^L \left(S_{\alpha,\tau}(x,t)\right)^2 f^2(V_-(\alpha,\tau))d\alpha d\tau$$

Donc grâce à l'hypothèse (EW), il suffit de montrer, pour conclure, que

$$\Sigma(x,t) = \int_0^t\int_0^L \left(S_{\alpha,\tau}(x,t)\right)^2 d\alpha d\tau > 0 \quad p.s.$$

Décomposons $S_{\alpha,\tau}(x,t)$:

$$S_{\alpha,\tau}(x,t) = G_{t-\tau}(x,\alpha)1_{\{\tau\leq t\}} + Q_{\alpha,\tau}(x,t)$$

où $Q_{\alpha,\tau}(x,y)$ est défini par l'équation (4.49). Alors, pour tout $\epsilon > 0$, suffisamment petit,

$$\begin{aligned}\Sigma(x,t) &\geq \frac{2}{3}\int_{t-\epsilon}^t\int_0^L G^2_{t-\tau}(x,\alpha)d\alpha d\tau - 2\int_{t-\epsilon}^t\int_0^L (Q_{\alpha,\tau}(x,t))^2 d\alpha d\tau \\ &= \frac{2}{3}J^\epsilon(x,t) - 2I^\epsilon(x,t)\end{aligned}$$

Dans l'Appendice (6.5), on peut voir que $J^\epsilon(x,t) \geq C\sqrt{\epsilon}$. De plus, un calcul similaire à la preuve du Lemme 4.32 montre que

$$E(I^\epsilon(x,t)) \leq K\epsilon$$

Donc

$$\begin{aligned} P(\Sigma(x,t) > 0) &\geq \sup_{\epsilon>0} P\left(2I^\epsilon(x,t) < \frac{2}{3}C\sqrt{\epsilon}\right) \\ &\geq \sup_{\epsilon>0}\left(1 - \frac{3E(I^\epsilon(x,t))}{C\sqrt{\epsilon}}\right) = 1 \end{aligned}$$

et le Théorème 4.1 est prouvé sous l'hypothèse (EW).

5 Extensions.

Nous présentons ici deux extensions du Théorème 4.1. Elles tentent de réduire les hypothèses portant sur la mesure de Poisson.

5.1 Extension par le Théorème de Girsanov.

Le but de la première extension est d'alléger l'hypothèse portant sur la mesure de Poisson, à l'aide du Théorème de Girsanov pour les mesures aléatoires. On cherche surtout à alléger l' hypothèse de régularité de la fonction φ. Nous allons voir qu'il suffit en fait que φ soit "très proche" d'une fonction C^1.

Emettons les hypothèses ci-dessous :

Hypothèse $(M)_1$:

1. et 3. sont les mêmes que dans (M) ;
2. N est une mesure de Poisson sur $[0,T] \times [0,L] \times \mathbb{R}$ d'intensité

$$\nu(ds,dy,dz) = \varphi(z)1_O(z)a(z)dsdydz$$

   où O est un ouvert de $\mathbb{R}$, et où φ est une fonction strictement positive de classe C^1 sur O. a est strictement positive, mesurable sur O, et satisfait

$$\int_O (a(z)-1)^2\,\varphi(z)dz < \infty$$

Hypothèse $(H')_1$: la même chose que (H'), sauf que $\eta \in L^2(O, a(z)\varphi(z)dz) \cap L^2(O, \varphi(z)dz)$.

La fonction ρ satisfait les conditions de la Section 4, et (EW), $(EP1)$, et $(EP2)$ ne changent pas (on ne remplace pas $\varphi(z)$ par $a(z)\varphi(z)$ dans ces hypothèses).

Remarque 5.1 *Sous $(M)_1$, (D), et $(H')_1$, l'équation (1.1) admet une unique solution faible V au sens de la Définition 2.3, grâce au Théorème 3.1. Si de plus (S) est vérifiée et si l'une des hypothèses (EW), $(EP1)$, ou $(EP2)$ est satisfaite, si $(x,t) \in [0,L]\times]0,T]$, la loi de $V(x,t)$ admet une densité par rapport à la mesure de Lebesgue sur $I\!R$.*

Preuve : Plaçons-nous sur l'espace canonique produit Ω associé au Bruit Blanc et à la mesure de Poisson. On considère sur Ω une probabilité P sous laquelle les processus canoniques W et N satisfont l'hypothèse (M). On considère ensuite l'E.D.P.S. (1.1) associée à ces objets et aux fonctions f, G, et h, où

$$G(v) = g(v) - \int_O h(v,z)(a(z)-1)\varphi(z)dz$$

Les fonctions f, G et h satisfont (H'). Cette équation admet donc une unique solution V prévisible et bornée dans L^2. De plus, on sait que (EW) ou $(EP1)$ ou $(EP2)$ est satisfaite, donc si $(x,t) \in [0,L]\times]0,T]$, la loi de $V(x,t)$ (sous P) admet une densité.
Intéressons-nous alors à la martingale exponentielle de Doléans-Dade suivante :

$$M_t = 1 + \int_0^t \int_0^L \int_O M_{s-}(a(z)-1)\tilde{N}(ds,dy,dz)$$

Cette martingale est strictement positive p.s., et de carré intégrable. Posons ensuite $P' = M_T.P$, qui est une probabilité équivalente à P. Par le Théorème de Girsanov pour les mesures aléatoires (cf Jacod, Shiryaev, [23], p 157), on sait que sous P', N est une mesure de Poisson d'intensité $a(z)\varphi(z)dz$. Il est d'autre part immédiat que W reste une Bruit Blanc basé sur $dxdt$ sous P' (par indépendance entre W et M_T). De plus, V satisfait sous P' l'équation suivante :

$$\begin{aligned} V(x,t) &= \int_0^L \mathcal{V}_0(y)G_t(x,y)dy + \int_0^t\int_0^L G_{t-s}(x,y)f(V(y,s))W(dy,ds) \\ &+ \int_0^t\int_0^L G_{t-s}(x,y)G(V(y,s),y,s)dyds \\ &+ \int_0^t\int_0^L\int_{I\!R} G_{t-s}(x,y)h(V(y,s),z)\tilde{N}(ds,dy,dz) \\ &+ \int_0^t\int_0^L\int_{I\!R} G_{t-s}(x,y)h(V(y,s),z)(a(z)-1)\varphi(z)dzdyds \end{aligned}$$

En utilisant l'expression de G, on en déduit que sous P', le processus V satisfait l'équation d'évolution (2.9) associé à W, N, et f,g,h. Soit enfin $(x,t) \in [0,L]\times]0,T]$. Comme $P \sim P'$, et comme la loi de $V(x,t)$ sous P admet une densité, on en déduit qu'il en va de même pour la loi de $V(x,t)$ sous P'. La remarque est vérifiée.

5.2 Une deuxième mesure de Poisson.

La deuxième extension consiste à ajouter une mesure de Poisson indépendante de la première. Elle s'appuie sur l'article [6] de Bichteler et al. Considérons l'équation d'évolution suivante.

$$V(x,t) = \int_0^L \mathcal{V}_0(y)G_t(x,y)dy + \int_0^t\int_0^L f(V(y,s))G_{t-s}(x,y)W(dy,ds)$$

$$+\int_0^t\int_0^L g(V(y,s))G_{t-s}(x,y)dyds$$

$$+\int_0^t\int_0^L\int_{I\!R} h(V(y,s),z)G_{t-s}(x,y)\tilde{N}(ds,dy,dz)$$

$$+\int_{[0,t[}\int_0^L\int_{I\!R} \sigma(V(y,s),u)G_{t-s}(x,y)\tilde{N}_1(ds,dy,du) \tag{5.1}$$

Enonçons nos hypothèses :

Hypothèse $(M)_2$:

1. La même chose que dans (M).
2. La même chose que dans (M).
3. N_1 est une mesure de Poisson sur $[0,T]\times[0,L]\times I\!R$ d'intensité $\nu_1(ds,dy,dz) = dsdyq(dz)$, où q est une mesure positive σ-finie sur $I\!R$.
4. $\{\mathcal{F}_t\}_{t\in[0,T]}$ est la filtration canonique complète et càd associée aux objets indépendants W, N et N_1.

Hypothèse $(H')_2$: la même chose que (H'). De plus, la fonction σ est de classe C^1 en x, et il existe $\alpha \in L^2(I\!R,q)$ telle que, pour tout $x\in I\!R$ et $u\in I\!R$,

$$|\sigma(0,u)| + |\sigma'_x(x,u)| \leq \alpha(u)$$

Hypothèse $(EP1)_2$: la même chose que $(EP1)$. De plus, σ'_x est positive, et il existe $\hat{\alpha}\in L^1(I\!R,q)$ telle que $|\sigma'_x| \leq \hat{\alpha}$.

La fonction ρ doit satisfaire les conditions de la Section 4.

Remarque 5.2 *Supposons $(M)_2$, (D), $(H')_2$, et (S). Soit V l'unique solution faible de (5.1). Sous l'une des hypothèses (EW), $(EP1)_2$, ou $(EP2)$, si $(x,t)\in[0,L]\times]0,T]$, la loi de $V(x,t)$ admet une densité par rapport à la mesure de Lebesgue sur $I\!R$.*

Ici, la loi de $V(x,t)$ admet une densité soit grâce à W, soit grâce à N. La mesure N_1, ne joue en fait qu'un rôle minime (comme par exemple N sous (EW)). Elle permet néanmoins d'ajouter un terme non "régulier" (l'hypothèse sur N_1 est assez faible), tant que l'hypothèse de non-dégénerescence ne concerne que N. Techniquement, on définit les opérateurs de dérivation liés à W et N comme dans la sous-section 4.1, et on choisit l'opérateur lié à N_1 identiquement nul (comme nous aurions pu choisir $D^{0,1}$ identiquement nul pour prouver le Théorème 4.1 sous (EW)). L'intégrale stochastique par rapport à N_1 se comporte alors en gros (pour ce qui est de la dérivation) comme l'intégrale par rapport à la mesure de Lebesgue.

6 Appendice.

Les résultats qui figurent ici sont des propriétés élémentaires du semi-groupe $G_t(x,y)$ défini dans la section 2 par l'équation (2.8). Dans la plupart des lemmes qui suivent, les inégalités sont vraies pour $(x,t) \in [0,L] \times]0,T]$, et la constante positive C_T ne dépend que du temps terminal T. Les trois premiers lemmes sont prouvés dans Walsh, [47].

Lemme 6.1 *On peut estimer le semi-groupe de la manière suivante :*

$$\frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\} \leq G_t(x,y) \leq \frac{C_T}{\sqrt{t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\}$$

Lemme 6.2 *Si $r > 0$, alors*

$$\int_0^L G_t^r(x,y)dy \leq C_T t^{\frac{1-r}{2}}$$

Lemme 6.3 *Si $0 < r < 3$, alors*

$$\int_0^t \int_0^L G_s^r(x,y)dyds \leq C_T$$

Les deux lemmes suivants sont prouvés par Bally et Pardoux dans l'Appendice de [4].

Lemme 6.4 *Si $r \in]0,3[$ alors pour tout $\epsilon > 0$,*

$$\int_{t-\epsilon}^t \int_0^L G_{t-s}^r(x,y)dyds \leq C_T \times \epsilon^{\frac{3-r}{2}}$$

Lemme 6.5 *Pour tout $\epsilon > 0$,*

$$\int_{t-\epsilon}^t \int_{x-\sqrt{\epsilon}}^{x+\sqrt{\epsilon}} G_{t-s}^2(x,y)dyds \geq K \times \sqrt{\epsilon}$$

Enfin, on trouvera une preuve du lemme suivant dans l'Appendice du premier chapitre de la thèse de St Loubert Bié [40] (il faut toutefois l'adapter, car le semi-groupe qui y apparaît n'est pas exactement le même qu'ici).

Lemme 6.6 *Si $1 < r < 2$, alors il existe $s(r) > 1$ tel que*

$$\int_0^t \int_0^L |G_{t-s}(x+h,y) - G_{t-s}(x,y)|^r dyds \leq C|h|^{s(r)}$$

Chapitre 2

Existence of the density for stochastic Volterra equations with jumps

Abstract : We study the solution of a stochastic Volterra equation with jumps. We use the Malliavin calculus in order to show that under some non-degeneracy assumptions, the law of this solution admits a density with respect to the Lebesgue measure. To this end, we introduce two derivative operators associated with the Brownian motion and the Poisson measure.

1 Introduction and statement of the main results.

Consider on the time interval $[0,T]$ the stochastic Volterra equation with jumps :

$$X_t = x_0 + \int_0^t \sigma(s,t,X_s^-)dW_s + \int_0^t b(s,t,X_s^-)ds + \int_0^t \int_{I\!R} h(s,t,X_s^-,z)\tilde{N}(ds,dz) \qquad (1.1)$$

where X^- is a predictable version of X, where W is a standard one-dimensional brownian motion, and $\tilde{N}$ is a compensated Poisson measure. The stochastic Volterra equation (without jumps) has been investigated for example by Nualart, Rovira, [31], who prove a large deviation principle.

We establish in this paper a theorem about the existence of a density for the law of X_t. This problem has been investigated by Bichteler and Jacod in [7], in the case of ordinary S.D.E.s, i.e. when σ, b, and h do not depend on s,t. Bichteler, Gravereaux and Jacod have extended in [6] their theory, in order to study the smoothness of the density in any dimension.
In [7], Bismut's approach of the Malliavin calculus is used. Bichteler et al. use in [6] Bismut's

and Malliavin's approaches, and they compare the two methods.
Our method is not so far from the second one in [6], but we define some "local" derivatives, which give more information. We also consider a more general Poisson measure.
At last, we use a criterion of absolute continuity looking like that of Bouleau and Hirsch. Thus the proofs are easier, and we can assume less regularity about σ, b, and h.

As a matter of fact, this paper has been inspired by Chapter 1. We have studied in Chapter 1 the Malliavin calculus for a parabolic S.P.D.E. driven by a white noise and a compensated Poisson measure, and the classical methods did fail. We thus had to build a more precise stochastic calculus of variations on the Poisson space. Since the method we used gives a slightly better result than the one of [7] in the case of equation (1.1) when the coefficients do not depend on s, t, and since it allows to solve the problem when X is not a semimartingale, we present it here.

Let us now state our assumptions :

Assumption (M) : W is a standard brownian motion on $[0,T]$. N is a Poisson measure on $[0,T] \times I\!R$, with intensity measure $\nu(ds,dz) = \varphi(z)1_O(z)dsdz$, where O is an open subset of $I\!R$, and φ is a strictly positive C^1 function on O. The probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is the canonical product space associated with the independent random elements W and N, and the filtration is càd and complete.

Assumption (H) : the functions σ and b are measurable on $\{0 \le s < t \le T\} \times I\!R$, and are C^1 in x, we will denote by σ' and b' the derivatives of σ and b with respect to x. The function h is measurable on $\{0 \le s < t \le T\} \times I\!R \times O$, and admits the continuous in (x,z) partial derivatives h'_z, h'_x, and $h''_{zx} = h''_{xz}$. There exists a function $a(s,t)$ on $\{0 \le s < t \le T\}$ meeting

$$\sup_{[0,T]} \int_0^t a^2(s,t)ds < \infty \tag{1.2}$$

and a function $\eta \in L^2(O, \varphi(z)dz)$ such that for all s,t,x,z,

$$|\sigma(s,t,0)| + |\sigma'(s,t,x)| + |b(s,t,0)| + |b'(s,t,x)| \le a(s,t) \tag{1.3}$$

$$|h'_z(s,t,0)| + |h''_{zx}(s,t,x)| \le a(s,t) \tag{1.4}$$

$$|h(s,t,0)| + |h'_x(s,t,x)| \le a(s,t)\eta(z) \tag{1.5}$$

Assumption (P) :

1. Any sequence $\{f_n(t)\}$ of positive functions on $[0,T]$ such that f_0 is bounded and such that
$$f_{n+1}(t) \le K \int_0^t f_n(s)a^2(s,t)ds \tag{1.6}$$
satisfies $\sup_{[0,T]} \sum_n \sqrt{f_n(t)} < \infty$.

2. If Y is a predictable bounded in L^2 process on $[0,T]$, then the following processes admit predictable versions (and are bounded in L^2 thanks to (H)) :

$$U_t^1 = \int_0^t \sigma(s,t,Y_s)dW_s \quad ; \quad U_t^2 = \int_0^t b(s,t,Y_s)ds$$

$$U_t^3 = \int_0^t \int_O h(s,t,Y_s,z)\tilde{N}(ds,dz) \tag{1.7}$$

Let us notice that (P)-1 is satisfied if $a^2(s,t) \le C(t-s)^\alpha$, with $\alpha > -1$.
Remark also that (P)-2 is satisfied if (H) holds, and if

$$\sigma(s,t,x) = \alpha(s,t)\tilde{\sigma}(x) \quad ; \quad b(s,t,x) = \beta(s,t)\tilde{b}(x) \quad ; \quad h(s,t,x,z) = \gamma(s,t)\tilde{h}(x,z) \tag{1.8}$$

For example, it suffices to approximate $\alpha(s,t)$ with $\sum_{k=0}^n \langle \alpha(.,t), e_k(.)\rangle_{L^2([0,T])} e_k(s)$, where $\{e_k\}$ is an orthonormal basis of $L^2([0,T])$.
At last, (P)-2 does also hold if σ, b, and h are sufficiently continuous in t : approximate $\sigma(s,t,x)$ with

$$\sigma_n(s,t,x) = \sum_{i=0}^{n-1} 1_{\{\frac{i}{n}T \ge s\}} \sigma\left(s, \frac{i}{n}T, x\right) 1_{[\frac{i}{n}T, \frac{i+1}{n}T[}(t) \tag{1.9}$$

The following proposition holds (see e.g. Ikeda and Watanabe, [20], p 230 for the case where σ, b and h do not depend on s,t.) :

Proposition 1.1 *Assume (M), (H), and (P). Then Equation (1.1) admits a unique solution X (admitting a predictable version X^-) such that $\sup_{[0,T]} E\left(X_t^2\right) < \infty$. If furthermore σ, b and h do not depend on s,t, then this solution is a.s. càdlàg, and we can choose $X_s^- = X_{s-}$.*

This proposition can be proved by using a Picard iteration, and a L^2-convergence. Assumption (P)-2 allows the Picard approximations to be well-defined, and these approximations converge thanks to (P)-1.

Let us now turn to our problem. We will need a C^1 function ρ on O, strictly positive, such that ρ and ρ' are bounded, and satisfying

$$\rho \in L^1(O, \varphi(z)dz) \tag{1.10}$$

We also need to state the following technical condition.

Assumption (B) : there exists a family of C^1 positive functions K_ϵ on O, with compact support (in O), bounded by 1, and such that

$$\forall\, z \in O, K_\epsilon(z) \longrightarrow_{\epsilon\to 0} 1 \quad ; \quad \int_O \left(K_\epsilon'(z)\right)^2 \eta^2(z)\rho(z)\varphi(z)dz \longrightarrow_{\epsilon\to 0} 0 \tag{1.11}$$

We will first prove a statement slightly more general than that of [7].

Assumption (S) : σ, b, and h do not depend on s,t.

In this case, we will assume the following non-degeneracy condition :

Assumption (NDS) : $A \cup B = I\!R$, where

$$A = \{x \in I\!R \ / \ \sigma(x) \neq 0\} \quad \text{and} \quad B = \left\{x \in I\!R \ / \ \int_O 1_{\{h'_z(x,z)\neq 0\}} \varphi(z)dz = \infty\right\} \tag{1.12}$$

Our first result is the following :

Theorem 1.2 *Assume (M), (H), (B), (S), and (NDS). Then for all $t > 0$, the law of X_t admits a density with respect to the Lebesgue measure on $I\!R$.*

Bichteler and Jacod establish in [7] the same statement as Theorem 1.2 under the assumptions (S), (M'), (H'), and (NDS), where (M') is the same as (M) with $\varphi \equiv 1$, and :
Assumption (H') : the functions $\sigma(x)$ and $b(x)$ are twice differentiable on $I\!R$, and their derivatives are bounded. The function $h(x,z)$ is twice differentiable on $I\!R \times O$, the partial derivatives h'_z, h''_{zz}, and h''_{zx} are bounded, and there exists $\eta \in L^2(O,dz) \cap L^4(O,dz)$ such that $|h(0,z)| \leq \eta(z)$, $|h'_x(x,z)| \leq \eta(z)$, and $|h''_{xx}(x,z)| \leq \eta(z)$.

In our next statements, we will have to consider two cases. The law of X_t will admit a density either thanks to the Brownian motion, if σ is non-degenerated, either thanks to the Poisson measure, if h is non-degenerated.

We first consider the case where the non degenerated term can be written as a product of functions. We fix $t > 0$, and we assume (M), (H), (P), (B), and $(NDM1(t))$ or $(NDM2(t))$ below :

Assumption $(NDM1(t))$: $\sigma(s,t,x) = \alpha(s,t)\tilde{\sigma}(x)$, $\tilde{\sigma}$ does never vanish. For every small $\epsilon > 0$, $\int_{t-\epsilon}^t \alpha^2(s,t)ds > 0$. At last,

$$\left(\int_{t-\epsilon}^t a^2(s,t) \left(\int_{t-\epsilon}^s a^2(u,s)du\right) ds\right) \times \left(\int_{t-\epsilon}^t \alpha^2(s,t)ds\right)^{-1} \longrightarrow_{\epsilon\to 0} 0 \tag{1.13}$$

Assumption $(NDM2(t))$: $h(s,t,x,z) = \beta(s,t)\tilde{h}(x,z)$. Let

$$\mathcal{H} = \left\{z \in O \ / \ \int_O 1_{\{\tilde{h}'_z(x,z)\neq 0\}} \varphi(z)dz = \infty \ \forall x \in I\!R\right\} \tag{1.14}$$

There exists a function $\gamma(\epsilon)$ growing to infinity when ϵ goes to 0, such that

$$\gamma(\epsilon) \int_{t-\epsilon}^t a^2(s,t) \left(\int_{t-\epsilon}^s a^2(u,s)du\right) ds \longrightarrow_{\epsilon\to 0} 0 \tag{1.15}$$

$$\int_{t-\epsilon}^t \int_{\mathcal{H}} \left[1 - e^{-\gamma(\epsilon)\beta^2(s,t)\rho(z)}\right] \varphi(z)dzds \longrightarrow_{\epsilon\to 0} \infty \tag{1.16}$$

Theorem 1.3 *Let $t > 0$ be fixed. Assume (M), (H), (P), (B), and $(NDM1(t))$ or $(NDM2(t))$. Then the law of X_t admits a density with respect to the Lebesgue measure on $I\!R$.*

We at last study the general case. We fix again $t > 0$. We will assume one of the following non-degeneracy conditions:

Assumption $(NDG1(t))$: there exists a function γ on $\{0 \le s > t \le T\}$ such that $|\sigma(s,t,x)| \ge \gamma(s,t)$. For every small $\epsilon > 0$, $\int_{t-\epsilon}^t \gamma^2(s,t)ds > 0$, and

$$\left(\int_{t-\epsilon}^t a^2(s,t)\left(\int_{t-\epsilon}^s a^2(u,s)du\right)ds\right)\times\left(\int_{t-\epsilon}^t \gamma^2(s,t)ds\right)^{-1} \longrightarrow_{\epsilon\to 0} 0 \tag{1.17}$$

Assumption $(NDG2(t))$: there exists a function ξ on $\{0 \le s > t \le T\}$ and a function δ on O such that $\rho(z){h'_z}^2(s,t,x,z) \ge \xi^2(s,t)\delta(z)$. There exists a function $\gamma(\epsilon)$ increasing to infinity when ϵ goes to 0, such that

$$\gamma(\epsilon)\int_{t-\epsilon}^t a^2(s,t)\left(\int_{t-\epsilon}^s a^2(u,s)du\right)ds \longrightarrow_{\epsilon\to 0} 0 \tag{1.18}$$

$$\int_{t-\epsilon}^t\int_O \left[1 - e^{-\gamma(\epsilon)\xi^2(s,t)\delta(z)}\right]\varphi(z)dzds \longrightarrow_{\epsilon\to 0} +\infty \tag{1.19}$$

We will prove the following :

Theorem 1.4 *Let $t > 0$ be fixed. Assume (M), (H), (P), (B), and $(NDG1(t))$ or $(NDG2(t))$. Then the law of X_t admits a density with respect to the Lebesgue measure on $I\!R$.*

In the next section, we define our derivative operators, we state some properties, we establish a criterion of absolute continuity, and we compute the derivatives of our solution process. This section is adapted from Chapter 1. We will thus explain only the main ideas.
We prove Theorem 1.2 in the third section, by checking (in a very simple way, thanks to the "local" derivatives), that the "scalar product of derivation" associated with X_t does never vanish a.s.
Theorem 1.3 is proved in Section 4. At last, we give a proof of Theorem 1.4 in Section 5.

Let us say a word about our assumptions. Of course, (NDS) is not surprising, since Theorem 1.2 does only generalises slightly the result of Bichteler, Jacod in [7]. Assumption $(NDM1(t))$ does not seem so bad, since it is nearly always satisfied if α and a behave in the same way, and do not vanish too much when s goes to t. Assumption $(NDM2(t))$ is not so stringent, since it deals with $\mathcal{H}$: it means that there exists a "large set" included in O on which $\tilde{h}'_z$ does not vanish. Assumption $(NDG1(t))$ corresponds to the usual strong ellipticity assumption ($(NDM1(t))$ is an ellipticity assumption). At last, $(NDG2(t))$ is quite stringent, since it does directly ask h'_z to be large. Nevertheless, Bichteler, Gravereaux and Jacod use in [6] similar conditions to study the regularity of the density (under (S)).

The conditions are more and more stringent when σ, b and h are more and more general, for the following reasons. In the first case, the derivatives of X_t satisfy a linear standard S.D.E., and thus can be computed explicitely. In the second case, they do not satisfy anymore a S.D.E., but can be written as a product. At last, we have to deal with their direct expression in the third context.

2 The derivative operators.

We build in this section two derivative operators. The first one is a "derivative with respect to the Poisson measure", and the second one is a "derivative with respect to the Brownian motion". We refer to Chapter 1 for the rigorous proofs of similar results.

Recall that $C_c^p(I\!R^d)$ (resp. $C_b^p(I\!R^d)$) is the set of C^p functions on $I\!R^d$ with compact support (resp. of which the derivatives of order 1 to p are bounded).

We begin with the Poisson measure. First, we denote by $\mathcal{CL}$ the set of measurable functions $l(s,z)$ on $[0,T]\times O$, with compact support, C^2 on O (in z), such that l, l'_z, and l''_{zz} are bounded. We define the domain of the "simple variables" by

$$\mathcal{S}^{0,1}=\left\{X=F(N(f_1),...,N(f_d))+a\Big/ d\geq 1,\ F\in C_c^2(I\!R^d),\ f_i\in\mathcal{CL},\ a\in I\!R\right\} \tag{2.1}$$

If $X\in\mathcal{S}^{0,1}$, we set :

$$\begin{aligned} L^{0,1}X &= \frac{1}{2}\sum_{i=1}^{d} d_iF(N(f_1),...,N(f_d))N\left(\rho.(f_i)''_{zz}+\left(\rho\frac{\varphi'}{\varphi}+\rho'\right).(f_i)'_z\right)\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{d} d_id_jF(N(f_1),...,N(f_d))N\left(\rho.(f_i)'_z.(f_j)'_z\right) \end{aligned} \tag{2.2}$$

and, for $\tau\in[0,T]$, and $\zeta\in O$,

$$D^{0,1}_{\tau,\zeta}X=\sum_{i=1}^{d} d_iF(N(f_1),...,N(f_d))(f_i)'_z(\tau,\zeta) \tag{2.3}$$

If $S_{\tau,\zeta}(\omega)$ and $T_{\tau,\zeta}(\omega)$ are in $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\zeta))$, we set

$$\langle S,T\rangle_{\rho N}=\int_0^T\int_O S_{\tau,\zeta}T_{\tau,\zeta}\rho(\zeta)N(d\tau,d\zeta)\quad\text{and}\quad \langle S\rangle_{\rho N}=\langle S,S\rangle_{\rho N} \tag{2.4}$$

Then we have to check the usual properties : first, it is immediate that

$$\left\langle D^{0,1}X,D^{0,1}Y\right\rangle_{\rho N}=L^{0,1}XY-XL^{0,1}Y-YL^{0,1}X \tag{2.5}$$

By adapting Bichteler et al., [6] Proposition 9-3, p 113, we check in the two lemmas below that $L^{0,1}$ is well-defined and self adjoint in L^2.

Lemma 2.1 *If $X=F(N(f_1),...,N(f_k))\in\mathcal{S}^{0,1}$, and if $X\equiv 0$, then $L^{0,1}X\equiv 0$.*

Lemma 2.2 *If X and Y are in $\mathcal{S}^{0,1}$, then*

$$E(XL^{0,1}Y)=E(YL^{0,1}X)=-\frac{1}{2}E\left(\left\langle D^{0,1}X,D^{0,1}Y\right\rangle_{\rho N}\right) \tag{2.6}$$

We deduce from (2.6) the following lemma, which shows that $D^{0,1}$ is closable.

Lemma 2.3 *Let Z_n be a sequence of $\mathcal{S}^{0,1}$ which goes to 0 in L^2. Assume that there exists $S_{\tau,\zeta}(\omega) \in L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\zeta))$ such that $E\left(\langle D^{0,1}Z_n - S\rangle_{\rho N}\right)$ goes to 0. Then $E(\langle S\rangle_{\rho N}) = 0$.*

We now define the derivative operator associated with the Brownian motion. First, we define the domain of the simple variables :

$$\mathcal{S}^{1,0} = \left\{X = F(W(f_1), ..., W(f_k)) + a \;\Big/\; k \geq 1,\ F \in C_c^2(I\!R^k),\ f_i \in L^2([0,T]),\ a \in I\!R\right\} \tag{2.7}$$

(where $W(f_i) = \int_0^T f_i(s)dW_s$). If X is in $\mathcal{S}^{0,1}$, if $\tau \in [0,T]$, we set :

$$D_\tau^{1,0}X = \sum_{i=1}^{k} d_i F(W(f_1), ..., W(f_k)) f_i(\tau) \tag{2.8}$$

If $S_\tau(\omega)$ and $T_\tau(\omega)$ are in $L^2(P(d\omega)d\tau)$, we set

$$\langle S, T\rangle_{leb} = \int_0^T S_\tau T_\tau d\tau \quad \text{and} \quad \langle S\rangle_{leb} = \langle S, S\rangle_{leb}$$

The following lemma can be found in Nualart [30] p 26.

Lemma 2.4 *Let Z_n be a sequence of $\mathcal{S}^{1,0}$ that goes to 0 in L^2. Assume that there exists $S_\tau(\omega) \in L^2(P(d\omega)d\tau)$ such that $E\left(\langle D^{1,0}Z_n - S\rangle_{leb}\right)$ goes to 0. Then $E(\langle S\rangle_{leb}) = 0$.*

Now we can build the operator on the product space. The smooth variables domain is

$$\mathcal{S} = \Big\{Y = H(X_1, ..., X_d, Z_1, ..., Z_k) + a \;\Big/\; k + d \geq 1,\ H \in C_c^2(I\!R^{d+k}),\ X_i \in \mathcal{S}^{1,0},$$
$$Z_j \in \mathcal{S}^{0,1},\ a \in I\!R\Big\} \tag{2.9}$$

If Y belongs to $\mathcal{S}$, we define $D^{0,1}Y$ and $L^{0,1}Y$ (resp. $D^{1,0}Y$) as previously, considering the variables $X_1, ..., X_d$ (resp. $Z_1, ..., Z_k$) as constants.

The scalar products are denoted as previously, and we see that if X and Z are in $\mathcal{S}$, then X, Z, and $\langle D^{1,0}X, D^{1,0}Z\rangle_{leb}$ are bounded ; and $L^{0,1}X$, $L^{0,1}Z$, and $\langle D^{0,1}X, D^{0,1}Z\rangle_{\rho N}$ are in L^p for all $p < \infty$.

If Z belongs to $\mathcal{S}$, we set

$$|||Z|||_2 = \left[E\left(Z^2\right) + E\left(\left\langle D^{1,0}Z\right\rangle_{leb}\right) + E\left(\left\langle D^{0,1}Z\right\rangle_{\rho N}\right)\right]^{\frac{1}{2}} \tag{2.10}$$

We denote by $\mathcal{D}_2$ the closure of $\mathcal{S}$ for this norm. Thanks to Lemmas 2.3 and 2.4, the operators $D_\tau^{1,0}$ et $D_{\tau,\zeta}^{0,1}$ can be extended to the space $\mathcal{D}_2$.

Remark 2.5 *We have extended $D^{1,0}$ and $D^{0,1}$ to $\mathcal{D}_2$, and our solution will belong to this space. But no integration by parts formula (like (2.6)) holds on $\mathcal{D}_2$, because $L^{0,1}$ can not be extended to this space. Nevertheless, the "differentiability" of our solution will allow us to prove Theorem 1.2.*

We now give the usual properties of our derivative operators.

Proposition 2.6 *$\mathcal{D}_2$, endowed with the following scalar product, is Hilbert :*

$$\langle Y,Z\rangle_{\mathcal{D}_2} = E(YZ) + E\left(\left\langle D^{1,0}Y, D^{1,0}Z\right\rangle_{leb}\right) + E\left(\left\langle D^{0,1}Y, D^{0,1}Z\right\rangle_{\rho N}\right) \tag{2.11}$$

Proposition 2.7 *Let Y be in $\mathcal{D}_2$ and let F be in $C_b^1(I\!R)$. Then $Z = F(Y)$ belongs to $\mathcal{D}_2$, $D_\tau^{1,0}Z = F'(Y)D_\tau^{1,0}Y$, and $D_{\tau,\zeta}^{0,1}Z = F'(Y)D_{\tau,\zeta}^{0,1}Y$.*

We at last state the absolute continuity criterion that we will use. This is adapted from Nualart [30], p 87.

Theorem 2.8 *Assume that Z belongs to $\mathcal{D}_2$, and set $\sigma = \langle D^{1,0}Z\rangle_{leb} + \langle D^{0,1}Z\rangle_{\rho N}$. Then, if $\sigma > 0$ a.s., the law of Z admits a density with respect to the Lebesgue measure on $I\!R$.*

In order to apply Theorem 2.8, we have to prove that X_t is in $\mathcal{D}_2$, and we need to compute its derivatives.

Proposition 2.9 *Assume (M), (H), (P), and (B). Then for all $t \in [0,T]$, X_t is in $\mathcal{D}_2$, and its derivatives satisfy the following equations : if $\tau > t$, $D_\tau^{1,0}X_t = D_{\tau,\zeta}^{0,1}X_t = 0$, and if $\tau \le t$,*

$$\begin{aligned} D_\tau^{1,0}X_t &= \sigma(\tau,t,X_\tau^-) + \int_\tau^t \sigma'(s,t,X_s^-)D_\tau^{1,0}X_s^- dW_s \\ &\quad + \int_\tau^t b'(s,t,X_s^-)D_\tau^{1,0}X_s^- ds + \int_\tau^t\int_O h_x'(s,t,X_s^-,z)D_\tau^{1,0}X_s^- \tilde{N}(ds,dz) \end{aligned} \tag{2.12}$$

(it is the unique solution of (2.12) satisfying $\sup_{[0,T]} E(\langle D^{1,0}X_t\rangle_{leb}) < \infty$)),

$$\begin{aligned} D_{\tau,\zeta}^{0,1}X_t &= h_z'(\tau,t,X_\tau^-,\zeta) + \int_\tau^t \sigma'(s,t,X_s^-)D_{\tau,\zeta}^{0,1}X_s^- dW_s \\ &\quad + \int_\tau^t b'(s,t,X_s^-)D_{\tau,\zeta}^{0,1}X_s^- ds + \int_\tau^t\int_O h_x'(s,t,X_s^-,z)D_{\tau,\zeta}^{0,1}X_s^- \tilde{N}(ds,dz) \end{aligned} \tag{2.13}$$

(it is the unique solution of (2.13) satisfying $\sup_{[0,T]} E(\langle D^{0,1}X_t\rangle_{\rho N}) < \infty$)).

The rigorous proof of a similar Proposition can be found in Chapter 1.

3 Absolute continuity under (NDS).

We assume (S) in the whole section. Thanks to Theorem 2.8 and Proposition 2.9, we just have to prove that under (M), (H), and (NDS), $m_t > 0$ a.s. for all $t > 0$, where

$$m_t = \left\langle D^{1,0}X_t\right\rangle_{leb} + \left\langle D^{0,1}X_t\right\rangle_{\rho N} \tag{3.1}$$

To this end, we first rewrite equations (2.12) and (2.13) in suitable forms. Let us consider the following L^2-martingale :

$$M_t = \int_0^t \sigma'(X_{s-})dW_s + \int_0^t b'(X_{s-})ds + \int_0^t\int_O h_x'(X_{s-},z)\tilde{N}(ds,dz) \tag{3.2}$$

Then we see that for all $0 \leq \tau \leq t$,

$$D_\tau^{1,0} X_t = \sigma(X_{\tau-}) + \int_\tau^t D_\tau^{1,0} X_{s-} dM_s \tag{3.3}$$

Thus, using the the Doléans-Dade exponential formula (see Jacod, Shiryaev, [23]), one easily checks that if

$$G_{t,\tau} = e^{M_t - M_\tau - \frac{1}{2}\int_\tau^t \sigma'^2(X_{s-})ds} \prod_{\tau \leq s \leq t} (1 + \Delta M_s) e^{-\Delta M_s} \tag{3.4}$$

then $D_\tau^{1,0} X_t = \sigma(X_{\tau-}) G_{t,\tau}$. We can compute $D_{\tau,\zeta}^{0,1} X_t$ in the same way, and we obtain :

$$m_t = \int_0^t \sigma^2(X_{\tau-}) G_{t,\tau}^2 d\tau + \int_0^t \int_O {h'_z}^2 (X_{\tau-}, \zeta) G_{t,\tau}^2 \rho(\zeta) N(d\tau, d\zeta) \tag{3.5}$$

<u>Proof of Theorem 1.2</u> : we fix $t > 0$. The process $G_{t,\tau}$ is not predictable (in τ), since it is not even adapted. We thus need to make it disappear. To this end, we consider the sequence of totally inaccessible stopping-times :

$$S_0 = 0\ , \quad S_n = \inf\{s > S_{n-1} \ / \ \Delta M_s = -1\} \tag{3.6}$$

It is clear that for $S_n < \tau < t < S_{n+1}$, $G_{t,\tau} \neq 0$. We will prove that if $t \in]S_n, S_{n+1}[$, then $\alpha_t^n + \beta_t^n > 0$, where

$$\alpha_t^n = \int_{]S_n, t]} \sigma^2(X_{\tau-}) d\tau \quad \text{and} \quad \beta_t^n = \int_{]S_n, t]} \int_O 1_{\{h'_z(X_{\tau-}, \zeta) \neq 0\}} N(d\tau, d\zeta) \tag{3.7}$$

Let $R_n = \inf\{s > S_n \ / \ \beta_s^n > 0\}$. Then $\beta_{t \wedge R_n}^n \leq 1$, thus $E(\beta_{t\wedge R_n}^n) \leq 1$, and hence

$$\int_{S_n}^{t \wedge R_n} \int_O 1_{\{h'_z(X_{\tau-}, \zeta) \neq 0\}} \varphi(\zeta) d\zeta d\tau < \infty \quad \text{a.s.} \tag{3.8}$$

Then we consider two cases : first, if $R_n = S_n$, then the definition of R_n yields that $\beta_t^n > 0$. Else, we deduce from (3.8) that for all $\tau \in]S_n, R_n \wedge t]$, $X_{\tau-} \notin B$ (recall that B is defined in (NDS)). From Assumption (NDS), it is clear that for all $\tau \in]S_n, R_n \wedge t]$, $X_{\tau-} \in A$ which of course implies that $\alpha_t^n > 0$.
In every case, $S_n < t < S_{n+1}$ implies that $\alpha_t^n + \beta_t^n > 0$. But it is easy to check that if $\alpha_t^n + \beta_t^n > 0$ and if $t > S_n$, then $m_t > 0$. Hence $m_t = \sum_n m_t 1_{]S_n, S_{n+1}[}(t) > 0$ a.s., which was our aim.

4 Absolute continuity under $(NDM1(t))$ or $(NDM2(t))$.

Here, the use of local derivatives will be really usefull, because we can not compute explicitely the derivatives of X : they do not satisfy a linear S.D.E. Let $t > 0$ be fixed. As in the previous section, we just have to prove that $m_t > 0$ a.s., where m_t is defined by equation (3.1).

<u>Proof of Theorem 1.3 under $(NDM2(t))$</u> : we will prove here that $\langle D^{0,1} X_t \rangle_{\rho N} > 0$ a.s. We first write $D_{\tau,\zeta}^{0,1} X_t = \tilde{h}'_z(X_\tau^-, \zeta) S_\tau(t)$, where S is the unique solution satisfying

$$\sup_{s \in [0,T]} E(\langle S_.(s) \rangle_{\rho N}) < \infty \tag{4.1}$$

of

$$\begin{aligned} S_\tau(t) &= \beta(\tau,t) + \int_\tau^t \sigma'(s,t,X_s^-)S_\tau^-(s)dW_s + \int_\tau^t b'(s,t,X_s^-)S_\tau^-(s)ds \\ &\quad + \int_\tau^t \int_O h'_x(s,t,X_s^-,z)S_\tau^-(s)\tilde{N}(ds,dz) \quad \text{if } \tau \le t \\ &= 0 \quad \text{if } \tau > t \end{aligned} \tag{4.2}$$

Consider

$$\Sigma_t = \int_0^t \int_{\mathcal{H}} S_\tau^2(t)\rho(\zeta)N(d\tau,d\zeta)$$

From $(NDM2(t))$, it is clear that if $\Sigma_t > 0$ a.s., then the theorem will be proved. First, if we set $S_\tau(t) = \beta(\tau,t) + Q_\tau(t)$,

$$\Sigma_t \ge \frac{2}{3}\int_{t-\epsilon}^t \int_{\mathcal{H}} \beta^2(\tau,t)\rho(\zeta)N(d\tau,d\zeta) - 2\int_{t-\epsilon}^t \int_{\mathcal{H}} Q_\tau^2(t)\rho(\zeta)N(d\tau,d\zeta) \ge \frac{2}{3}A_\epsilon(t) - 2B_\epsilon(t) \tag{4.3}$$

We thus see that for all $\epsilon > 0$, $\lambda > 0$, and $\eta > 0$,

$$\begin{aligned} P(\Sigma_t > 0) &\ge P\left(A_\epsilon(t) > \tfrac{3}{2}\eta\right) + P\left(B_\epsilon(t) < 2\eta\right) - 1 \\ &\ge 1 - e^{\frac{3}{2}\lambda\eta}E\left(e^{-\lambda A_\epsilon(t)}\right) - \tfrac{1}{2}\eta^{-1}E(B_\epsilon(t)) \end{aligned} \tag{4.4}$$

We choose $\lambda\eta = 1$, and $\lambda = \gamma(\epsilon)$, where γ is defined in $(NDM2(t))$. First, by assumption (see equation (1.16)),

$$E\left(e^{-\lambda A_\epsilon(t)}\right) = \exp\left(-\int_{t-\epsilon}^t \int_{\mathcal{H}} \left[1 - e^{-\gamma(\epsilon)\beta^2(s,t)\rho(z)}\right]\varphi(z)dz\right) \longrightarrow_{\epsilon\to 0} 0 \tag{4.5}$$

On the other hand, using **twice** inequalities like (we write here $N = \sum_n \delta_{(T_n,Z_n)}$) :

$$\begin{aligned} & E\left[\int_{t-\epsilon}^t \int_{\mathcal{H}} \left\{\int_\tau^t \int_O h'_x(s,t,X_s^-,z)S_\tau^-(s)\tilde{N}(ds,dz)\right\}^2 \rho(\zeta)N(d\tau,d\zeta)\right] \\ & = \sum_n E\left[\left\{\int_{t-\epsilon}^t \int_O h'_x(s,t,X_s^-,z)S_{T_n}^-(s)\sqrt{\rho(Z_n)}1_{\mathcal{H}}(Z_n)1_{\{t-\epsilon\le T_n<s\}}\tilde{N}(ds,dz)\right\}^2\right] \\ & \le \sum_n \int_{t-\epsilon}\int_O a^2(s,t)\eta^2(z)E\left(S_{T_n}^2(s)\rho(Z_n)1_{\mathcal{H}}(Z_n)1_{\{t-\epsilon\le T_n<s\}}\right)\varphi(z)dzds \\ & \le K\int_{t-\epsilon}^t a^2(s,t)E\left(\left\langle S_.(s)1_{\mathcal{H}}(.)1_{\{t-\epsilon\le .<s\}}\right\rangle_{\rho N}\right)ds \end{aligned} \tag{4.6}$$

and the fact that $\sup_{[0,T]} E(\langle S(s)\rangle_{\rho N}) < \infty$, one can check that

$$E(B_\epsilon(t)) \le C\int_{t-\epsilon}^t a^2(s,t)\int_{t-\epsilon}^s a^2(u,s)du\,ds \tag{4.7}$$

Thus, condition (1.15) yields that $\gamma(\epsilon)E(B_\epsilon(t))$ goes to 0 with ϵ. From this and equations (4.5) and (4.4), we deduce that $P(\Sigma_t > 0) = 1$, and thus that $m_t > 0$ a.s.

Proof of Theorem 1.3 under $(NDM1(t))$: in this case, our aim is to check that a.s.,

$$\left\langle D^{1,0} X_t \right\rangle_{leb} > 0$$

We first write $D^{1,0}_\tau X_t = T_\tau(t)\tilde{\sigma}(X^-_\tau)$, where T is the unique solution satisfying

$$\sup_{[0,T]} E(\langle T_\cdot(s)\rangle_{leb}) < \infty$$

of

$$\begin{aligned} T_\tau(t) &= \alpha(\tau,t) + \int_\tau^t \sigma'(s,t,X^-_s)T^-_\tau(s)dW_s + \int_\tau^t b'(s,t,X^-_s)T^-_\tau(s)ds \\ &\quad + \int_\tau^t \int_O h'_x(s,t,X^-_s,z)T^-_\tau(s)\tilde{N}(ds,dz) \quad \text{if } \tau \le t \\ &= 0 \quad \text{if } \tau > t \end{aligned} \tag{4.8}$$

Since $\tilde{\sigma}$ does never vanish, we just have to verify that $\Delta_t > 0$ a.s., where

$$\Delta_t = \int_0^t T^2_\tau(t)d\tau$$

First, writing $T_\tau(t) = \alpha(\tau,t) + R_\tau(t)$,

$$\Delta_t \ge \frac{2}{3}\int_{t-\epsilon}^t \alpha^2(\tau,t)d\tau - 2\int_{t-\epsilon}^t R^2_\tau(t)d\tau \ge \frac{2}{3}C_\epsilon(t) - 2D_\epsilon(t)$$

Since C_ϵ is deterministic, we see that for all $\epsilon > 0$,

$$P(\Delta_t > 0) \ge 1 - \frac{E(D_\epsilon(t))}{C_\epsilon(t)}$$

But one can check, by using a similar (but easier) argument as in the computation of $E(B_\epsilon(t))$ above (see the previous proof), that

$$E\left(D_\epsilon(t)\right) \le C\int_{t-\epsilon}^t a^2(s,t) \int_{t-\epsilon}^s a^2(u,s)du\, ds \tag{4.9}$$

Assumption (1.13) in $(NDM1(t))$ drives us to $P(\Delta_t > 0) = 1$, and the result follows.

5 Absolute continuity under $(NDG1(t))$ or $(NDG2(t))$.

This section is quite similar to the previous one. Let us first assume $(NDG2(t))$. We have to check that a.s.,

$$\Gamma^1_t = \int_0^t \int_O \left(D^{0,1}_{\tau,\zeta} X_t\right)^2 \rho(\zeta)N(d\tau,d\zeta) > 0 \tag{5.1}$$

The main difference with the previous section is that we can not write $D^{0,1}_{\tau,\zeta}X_t^-$ as a product, this is why our non-degeneracy condition is more stringent in this case. We write, for $\tau \le t$,

$$D^{0,1}_{\tau,\zeta}X_t = h'_z(\tau, t, X_\tau^-, \zeta) + U_{\tau,\zeta}(t) \tag{5.2}$$

Thus, thanks to Assumption $(NDG2(t))$,

$$\begin{aligned} \Gamma^1_t &\ge \frac{2}{3}\int_{t-\epsilon}^t \int_O {h'_z}^2(\tau, t, X_\tau^-, \zeta)\rho(\zeta)N(d\tau, d\zeta) - 2\int_{t-\epsilon}^t \int_O U^2_{\tau,\zeta}(t)\rho(\zeta)N(d\tau, d\zeta) \\ &\ge \frac{2}{3}\int_{t-\epsilon}^t \int_O \xi^2(\tau, t)\delta(\zeta)N(d\tau, d\zeta) - 2\int_{t-\epsilon}^t \int_O U^2_{\tau,\zeta}(t)\rho(\zeta)N(d\tau, d\zeta) \end{aligned} \tag{5.3}$$

and we can conclude exactly as in the proof of Theorem 1.3 under $(NDM2(t))$.

Under $(NDG1(t))$, we have to show that a.s.,

$$\Gamma^2_t = \int_0^t \left(D^{1,0}_\tau X_t\right)^2 d\tau > 0 \tag{5.4}$$

We thus split $D^{1,0}_\tau X_t$ into $\sigma(\tau, t, X_\tau^-) + V_\tau(t)$. Then, from $(NDG1(t))$,

$$\Gamma^2_t \ge \frac{2}{3}\int_{t-\epsilon}^t \gamma^2(\tau, t)d\tau - 2\int_{t-\epsilon}^t V^2_\tau(t)d\tau \tag{5.5}$$

and we can conclude again as in the proof of Theorem 1.3 under $(NDM1(t))$.

Chapitre 3

Support theorem for the solution of a white noise driven parabolic S.P.D.E. with temporal Poissonian jumps

Abstract : We study the weak solution X of a parabolic stochastic partial differential equation driven by two independant processes : a gaussian white noise, and a finite Poisson measure. We characterize the support of the law of X as the closure in $\mathbb{D}\left([0,T],\mathbf{C}([0,1])\right)$, endowed with its Skorokhod topology, of a set of weak solutions of ordinary partial differential equations.

1 Introduction.

Consider on $[0,T]\times[0,1]$ a space-time white noise $W(dx,dt)$ based on $dxdt$ (see Walsh, [47], p 269). Denote by (E,d) a Polish space, endowed with a positive finite measure q, and by $N(dt,dz)$ a Poisson measure on $[0,T]\times E$, with intensity measure $dtq(dz)$, independant of W. Our purpose is to study the following stochastic partial differential equation on $[0,T]\times[0,1]$

$$\frac{\partial X}{\partial t}(t,x)=\frac{\partial^2 X}{\partial x^2}(t,x)+b(X(t,x))+\sigma(X(t,x))\dot{W}_{x,t}+\int_E g(X(t-,x),z)\dot{N}_t(dz) \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial X}{\partial x}(t,0)=\frac{\partial X}{\partial x}(t,1)=0 \quad , \quad \forall\, t>0 \tag{1.2}$$

and deterministic initial condition $\mathcal{X}_0(x) \in \mathbf{C}([0,1])$. The symbol $\dot{N}_t(dz)$ (resp. $\dot{W}_{t,x}$) stands for the heuristical Radon-Nikodym density of $N(dt,dz)$ (resp. $W(dx,dt)$) with respect to the Lebesgue measure dt (resp. $dtdx$). We could also write, with abusive notations, $\dot{N}_t(dz)dtdx = dxN(dt,dz)$ and $\dot{W}_{x,t}dtdx = W(dx,dt)$.

We denote by $I\!D\left([0,T],\mathbf{C}([0,1])\right)$ the set of càdlàg functions from $[0,T]$ into $\mathbf{C}([0,1])$, endowed with the corresponding Skorokhod topology. In this paper, we characterize the support of the law of a weak solution X of equation (1.1) as the closure of a set of weak solutions of ordinary partial differential equations in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$.

Parabolic S.P.D.E.s driven by a white noise, i.e. equation (1.1) with $g \equiv 0$, have been introduced by Walsh, [46] and [47]. In [47], he defines his weak solutions, then he proves a theorem of existence, uniqueness and regularity. Since, various properties of Walsh's equation have been investigated : Malliavin calculus, large deviations, support theorem (see Bally, Millet, Sanz-Solé, [3]) ...
But Walsh builds his equation in order to model a discontinuous neurophysiological phenomenous. In [46], he explains that the white noise W approximates a Poisson point process. This approximation is realistic because there are many jumps, and the jumps are very small, but in any case, the observed phenomenous is discontinuous. However, S.P.D.E.s with jumps are much less known. In the case of temporal and spatial jumps, Saint Loubert Bié has studied in [40] the existence, uniqueness, regularity, and variational calculus. See also Chapter 1 for other results on the same subject. Nevertheless, no result about the "joint" regularity of the weak solutions has been proved in this case : we do not really know in which space the weak solution "lives", thus no support theorem may hold for the moment.

In the case of equation (1.1) with $\sigma \equiv 0$, but with $q(E) = \infty$, and with a compensated Poisson measure, Albeverio et al. have checked in [2] the existence and uniqueness of a "modified càdlàg" weak solution $u(t,x)$: a.s., u is continuous in x ; and u is right continuous and has left limits in $L^2(\Omega)$ in the variable t. One more time, we do not know in which space lies a.s. the weak solution.

Since Stroock and Varadhan established in [42] their famous support theorem for diffusion processes, there have been many investigations on the subject. In particular, Millet and Sanz have considerably simplified in [28] the proof of Stroock and Varadhan. But the only support theorem for jump processes seems to be that of Simon in [41], who studies a stochastic differential equation driven by a (compensated or not) infinite Poisson measure. In the case where the solution of his S.D.E. has finite variations, the statement and the proof of his main result is quite natural. But in the compensated case, the result is more astonishing, and the proof is quite hard.
Finally, let us mention that as far as we know, no support theorem seems to be known in the case of equations driven by two independant (but different) random elements.

This work is organized as follows. In the second section, we define the weak solutions of (1.1), by following the Walsh ideas, [47]. Using Ikeda and Watanabe's method, see [20], and applying Walsh's results, we sketch the proof of an existence and uniqueness result. We define the "squeleton" associated with equation (1.1), by using the Cameron-Martin space associated with W and the set of finite counting measures associated with N. Finally, we state our support theorem.

The third section is devoted to a simplification of the problem. First, we use a localization argument, in order to obtain weaker assumptions. Then we prove that it suffices to check two simpler support theorems. The first one is proved in the fourth section, and is related to an equation similar to (1.1), but without white noise : $\dot{W}_{t,x}$ is replaced by $\dot{h}(t,x)$, where h is an element of the Cameron-Martin space associated with W. The second one is proved in the fifth section, and deals with an equation without Poisson measure, but with an additional "jump drift". This concludes the proof of our main result.
The sixth section is devoted to an extension of our result to the case where the Poisson measure is a.s. infinite ($q(E)=\infty$), but where the diffusion coefficient is constant ($\sigma(x)=\sigma$).
Finally, one can find technical results in the Appendix lying at the end of the paper.

2 Framework.

Let us first define the weak solutions of (1.1). To this aim, we need some assumptions :

> Assumption (H) : the functions σ and b : $I\!R \mapsto I\!R$, satisfy a global Lipschitz condition. The function g : $I\!R \times E \mapsto I\!R$ is measurable on $I\!R \times E$, and for each $z \in E$, the map $g(.,z)$ is continuous on $I\!R$.

We define the weak solutions of (1.1) by following the Walsh ideas, [47], p 311-322. Consider the Green kernel $G_t(x,y)$ associated with the deterministic system :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ; \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0 \tag{2.1}$$

This kernel can be explicitely computed :

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in Z} \left[\exp\left(\frac{-(y-x-2nL)^2}{4t} \right) + \exp\left(\frac{-(y+x-2nL)^2}{4t} \right) \right] \tag{2.2}$$

If ϕ belongs to $\mathbf{C}([0,1])$, we set

$$G_t(\phi,x) = \begin{cases} \phi(x) & \text{if} \quad t=0 \\ \\ \displaystyle\int_0^1 G_t(x,y)\phi(y)dy & \text{if} \quad t>0 \end{cases} \tag{2.3}$$

The Appendix of this work contains technical results about this kernel. We endow our probability space $(\Omega, \mathcal{F}, P)$ with the canonical filtration associated with the independent random elements W and N :

$$\mathcal{F}_t = \sigma\left\{ W(A) \; ; \; A \in \mathcal{B}([0,1]\times[0,t]) \right\} \vee \sigma\left\{ N(B) \; ; \; B \in \mathcal{B}([0,t]\times E) \right\} \tag{2.4}$$

A process $X(t,x)$ on $[0,T]\times[0,1]$ is said to be adapted if for all $t \geq 0$, all $x \in [0,1]$, $X(t,x)$ is $\mathcal{F}_t$-measurable.

As Walsh, see also Saint Loubert Bié, [40], or Chapter 1, we define the weak solutions of (1.1) in the following sense.

Definition 2.1 *Consider an adapted process $X(t,x)$ on $[0,T]\times[0,1]$, lying a.s. in $I\!D([0,T],\mathbf{C}([0,1]))$. Then X is said to be a weak solution of (1.1) if and only if it satisfies the following evolution equation*

$$\begin{aligned} X(t,x) &= G_t(\mathcal{X}_0,x)+\int_0^t\int_0^1 G_{t-s}(x,y)\left[b(X(s,y))dyds+\sigma(X(s,y))W(dy,ds)\right] \\ &\quad +\int_0^t\int_E\int_0^1 G_{t-s}(x,y)g(X(s-,y),z)dy\ N(ds,dz) \end{aligned} \tag{2.5}$$

with the convention

$$\int_0^1 G_0(x,y)g(X(s-,y),z)dy = g(X(s-,x),z) \tag{2.6}$$

We now establish a result of existence and uniqueness of such a solution. Since $q(E)$ is finite, $N([0,T]\times E)$ is a.s. finite, and thus N can a.s. be written as

$$N(dt,dz)=\sum_{i=1}^{\mu}\delta_{(T_i,Z_i)}(dt,dz) \tag{2.7}$$

with $\mu\in I\!N$, $0<T_1<\ldots<T_\mu<T$, and $Z_1,\ldots,Z_\mu\in E$. Hence, equation (2.5) can be written as

$$\begin{aligned} X(t,x) &= G_t(\mathcal{X}_0,x)+\int_0^t\int_0^1 G_{t-s}(x,y)\left[b(X(s,y))dyds+\sigma(X(s,y))W(dy,ds)\right] \\ &\quad +\sum_{i=1}^{\mu}1\!\!1_{\{t\geq T_i\}}\int_0^1 G_{t-T_i}(x,y)g(X(T_i-,y),Z_i)dy \end{aligned} \tag{2.8}$$

Working recursively on the time intervals $[0,T_1[$, $[T_1,T_2[$,..., $[T_\mu,T]$, as Ikeda and Watanabe (proof of Theorem 9-1 p 231-232 in [20]), using Walsh's Theorems of existence, uniqueness, and regularity for equation (1.1) with $g\equiv 0$ (see [47], Theorem 3-2 p 313 and Corollary 3-4 p 317), and using the well-known estimates of the Green kernel stated in the Appendix, one can prove the following proposition :

Proposition 2.2 *Assume (H). Equation (1.1) admits a unique adapted solution $X(t,x)$ on $[0,T]\times[0,1]$, lying a.s. in $I\!D([0,T],\mathbf{C}([0,1]))$. The uniqueness holds in the sense where if Y is another adapted solution lying in $I\!D([0,T],\mathbf{C}([0,1]))$, then a.s., $\| X-Y\|_\infty=0$.*

We are now interested in the support of the law of X. Let us first recall the definition of the Skorokhod distance on $I\!D([0,T],\mathbf{C}([0,1]))$. On the set of the "changes of time",

$$\Lambda=\{\lambda\in\mathbf{C}([0,T])\ /\ \lambda(0)=0,\ \lambda(T)=T,\ \lambda \text{ is strictly increasing}\} \tag{2.9}$$

we consider the following norm

$$|||\lambda|||=\sup_{0\leq s<t\leq T}\left|\ln\left\{\frac{\lambda(t)-\lambda(s)}{t-s}\right\}\right| \tag{2.10}$$

The Skorokhod distance between two elements ϕ and ψ of $I\!D\left([0,T],\mathbf{C}([0,1])\right)$ is given by

$$\delta(\phi,\psi)=\inf_{\lambda\in\Lambda}\left\{\sup_{[0,T]\times[0,1]}|\phi(\lambda(t),x)-\psi(t,x)|+|||\lambda|||\right\} \tag{2.11}$$

$I\!D\left([0,T],\mathbf{C}([0,1])\right)$, endowed with δ, is a Polish space (see e.g. Jacod, Shiryaev, [23], p 289).

We now introduce some notations, describing the "supports" of W and N : we denote by

$$\mathcal{H}=\left\{h(t,x)=\int_0^t\int_0^x\dot{h}(s,y)dyds\ \Big/\ \dot{h}\in L^2([0,T]\times[0,1])\right\} \tag{2.12}$$

the Cameron-Martin space associated with W. We also consider the set of the finite counting measures on $[0,T]\times E$, of which the support is contained in $[0,T]\times\text{supp }q$:

$$\mathcal{M}=\left\{m(dt,dz)=\sum_{i=1}^n\delta_{(t_i,z_i)}(dt,dz)\ \Big/\ \begin{array}{c}n\in I\!N,\ 0<t_1<...<t_n<T,\\ z_1,...,z_n\in\text{supp }q\end{array}\right\} \tag{2.13}$$

with the convention $\sum_{i=1}^0=0$. Notice that for all $\omega\in\Omega$, $N(\omega)$ belongs to $\mathcal{M}$. But in general, (with abusive notation) $\dot{W}(\omega)\notin\mathcal{H}$, since $\dot{W}(\omega)$ is not even well-defined.

The following proposition, describes the "squeleton" associated with our evolution equation.

Proposition 2.3 *Assume (H). Let $h\in\mathcal{H}$ and $m\in\mathcal{M}$ be fixed. The following ordinary evolution equation admits a unique solution, which we denote by $S(h,m)$, lying in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$:*

$$\begin{aligned}S(h,m)(t,x)&=G_t(\mathcal{X}_0,x)+\int_0^t\int_0^1G_{t-s}(x,y)\Big[b(S(h,m)(s,y))dyds\\&\qquad+\sigma(S(h,m)(s,y))\dot{h}(s,y)dyds\Big]\\&\quad+\int_0^t\int_E\int_0^1G_{t-s}(x,y)g(S(h,m)(s-,y),z)dy\ m(ds,dz)\end{aligned} \tag{2.14}$$

This proposition can be proved as Proposition 2.2. Equation (2.14) is the same as (2.5), but we have replaced $W(dy,ds)$ and $N(ds,dz)$ by $\dot{h}(s,y)dyds$ and $m(ds,dz)$.

Finally, we recall the following standard remark :

Remark 2.4 *Let Z be a random variable with values in a Polish space A endowed with a distance α. Recall that the support* $\text{supp}_\alpha\ P\circ Z^{-1}$ *of the law of Z related to the distance α is the smaller closed subset F of (A,α) satisfying $P(Z\in F)=1$.*
Let B be a subset of A, and let $\overline{B}^\alpha$ be its closure in (A,α).

1. *If a.s., $Z\in\overline{B}^\alpha$, then*
$$\text{supp}_\alpha\ P\circ Z^{-1}\subset\overline{B}^\alpha \tag{2.15}$$

2. *If for all $b \in B$, all $\epsilon > 0$,*

$$P(\alpha(b, Z) < \epsilon) > 0 \tag{2.16}$$

then

$$\overline{B}^{\alpha} \subset \mathrm{supp}_{\alpha}\, P \circ Z^{-1} \tag{2.17}$$

In order to establish a support Theorem, we need the following assumptions.

Assumption $(S1)$: the function σ is C^3 on $I\!R$.

Assumption $(S2)$: for each $z_0 \in E$, each $n \in I\!N$,

$$\sup_{|x| \leq n} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{2.18}$$

For each $z_0 \in E$, each $n \in I\!N$, there exists a constant $\xi^n(z_0) > 0$, and a function $\psi^n_{z_0}(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that for all $|x| \leq n$, $|y| \leq n$,

$$\sup_{d(z,z_0) \leq \xi^n(z_0)} |g(x,z) - g(y,z)| \leq \psi^n_{z_0}(|x-y|) \tag{2.19}$$

Assumption $(S1)$ is nearly the same as that of Bally, Millet, Sanz, [3], who prove a support theorem in the case where $g \equiv 0$, and comes from a Taylor developpement of order 3. In fact they assume that σ is C^3_b, but a localisation procedure can be done (see the proof of Proposition 3.1 in the next section).
Notice that in the particular case where $g(x,z) = \alpha(x)\eta(z)$, $(S2)$ holds as soon as α is continuous on $I\!R$, and η is continuous on E.

Now we can state our main result :

Theorem 2.5 *Under (H), $(S1)$, and $(S2)$, if X denotes the unique weak solution of equation (1.1),*

$$\mathrm{supp}_{\delta}\, P \circ X^{-1} = \overline{\{S(h,m)\ /\ h \in \mathcal{H}\ ,\ m \in \mathcal{M}\}}^{\delta} \tag{2.20}$$

3 Simplification of the problem.

First, we "delocalize" $(S1)$ and $(S2)$, by using a standard argument. Consider the following assumptions, stronger than $(S1)$ and $(S2)$.

Assumption $(S'1)$: the function σ is C^3 on $I\!R$, bounded with its derivatives.

Assumption $(S'2)$: For all $z_0 \in E$,

$$\sup_{x \in I\!R} |g(x,z_0)| < \infty \quad ; \quad \sup_{x \in I\!R} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{3.1}$$

For all $z_0 \in E$, there exists $\xi(z_0) > 0$, and a function $\psi_{z_0}(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that for all $x, y \in I\!R$,

$$\sup_{d(z,z_0) \leq \xi(z_0)} |g(x,z) - g(y,z)| \leq \psi_{z_0}(|x-y|) \tag{3.2}$$

Remark that in the case where $g(x,z) = \alpha(x)\eta(z)$, then $(S'2)$ is satisfied when α is bounded and uniformly coninuous on $I\!R$, and when η is continuous on E.

Proposition 3.1 *If Theorem 2.5 holds under (H), $(S'1)$ and $(S'2)$, then it also holds under (H), $(S1)$ and $(S2)$.*

We will prove this proposition at the end of the section.

We now would like to check that Theorem 2.5 holds as soon as two easier support theorems are valid. The first one deals with equation 2.5 with a "deterministic" white noise, and the second one with a "deterministic" Poisson measure.

We first introduce some notations. If $h \in \mathcal{H}$ (resp. $m \in \mathcal{M}$), we denote by X_h (resp. X_m) the solution of equation (2.5) where we have replaced $W(dy,ds)$ by $\dot{h}(s,y)dyds$ (resp. $N(dt,dz)$ by $m(dt,dz)$). In other words,

$$\begin{aligned} X_h(t,x) &= G_t(\mathcal{X}_0,x) + \int_0^t\int_0^1 G_{t-s}(x,y)\left[b(X_h(s,y))dyds + \sigma(X_h(s,y))\dot{h}(s,y)dyds\right] \\ &\quad + \int_0^t\int_E\int_0^1 G_{t-s}(x,y)g(X_h(s-,y),z)dy\ N(ds,dz) \end{aligned} \tag{3.3}$$

$$\begin{aligned} X_m(t,x) &= G_t(\mathcal{X}_0,x) + \int_0^t\int_0^1 G_{t-s}(x,y)\left[b(X_m(s,y))dyds + \sigma(X_m(s,y))W(dy,ds)\right] \\ &\quad + \int_0^t\int_E\int_0^1 G_{t-s}(x,y)g(X_m(s-,y),z)dy\ m(ds,dz) \end{aligned} \tag{3.4}$$

We could also write, with abusive notations, $X_h = S(h,N)$, and $X_m = S(\dot{W},m)$. The next sections are devoted to the proof of the following propositions.

Proposition 3.2 *Assume (H) and $(S'2)$. Let $h \in \mathcal{H}$, $m \in \mathcal{M}$, and $\epsilon > 0$ be fixed. Then*

$$P\left(\delta(S(h,m),X_h) \le \epsilon\right) > 0 \tag{3.5}$$

We now denote by $\|\ \|_\infty$ the supremum norm on $[0,T]\times[0,1]$.

Proposition 3.3 *Assume (H), $(S'1)$ and $(S'2)$. Let $m \in \mathcal{M}$ be fixed. Then*

$$\mathrm{supp}_{\|\ \|_\infty}\, P\circ X_m^{-1} = \overline{\{S(h,m)\ /\ h\in\mathcal{H}\}}^{\|\ \|_\infty} \tag{3.6}$$

Let us remark that this second result implies the weaker one :

$$\mathrm{supp}_\delta\, P\circ X_m^{-1} = \overline{\{S(h,m)\ /\ h\in\mathcal{H}\}}^{\delta} \tag{3.7}$$

Assuming for a moment that these propositions hold, we prove our main result.

Proof of Theorem 2.5 : using Remark 2.4, we break the proof in two parts.

1) We first check that a.s., X belongs to $\overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}$. Consider the map from $\mathcal{M}$ to $[0,1]$, defined by

$$\phi(\mu) = P\left(X_\mu \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.8}$$

Let us first prove that a.s.,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\ \Big|\ \sigma(N)\right) = \phi(N) \tag{3.9}$$

where

$$\sigma(N) = \sigma\left\{N(A)\ ;\ A \in \mathcal{B}([0,T] \times E)\right\} \tag{3.10}$$

In order to understand (3.9), let us work with the canonical product space

$$(\Omega, \mathcal{F}, P) = (\Omega^W, \mathcal{F}^W, P^W) \otimes (\Omega^N, \mathcal{F}^N, P^N) \tag{3.11}$$

associated with W and N. Every element ω of Ω can be written as (ω^W, ω^N), where $\omega^W \in \mathbf{C}([0,T] \times [0,1])$ and $\omega^N \in \mathcal{M}$. Thus,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\ \Big|\ \sigma(N)\right)(\omega)$$

$$= \int \mathbb{I}_{\left\{X(\omega^W,\omega^N) \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right\}} dP^W(\omega^W) \tag{3.12}$$

But obviously, $X(\omega) = X(\omega^W, \omega^N) = X_{\omega^N}(\omega^W)$, where X_μ was defined by (3.4) for each $\mu \in \mathcal{M}$. Thus,

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\ \Big|\ \sigma(N)\right)(\omega)$$

$$= P^W\left(X_{\omega^N} \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.13}$$

Now, we notice, since for each $\mu \in \mathcal{M}$, X_μ is independent of N, that

$$\phi(\mu) = P^W\left(X_\mu \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) \tag{3.14}$$

Comparing (3.13) and (3.14), we deduce (3.9). Hence, we obtain

$$P\left(X \in \overline{\{S(h,m)\ /\ h \in \mathcal{H},\ m \in \mathcal{M}\}}^{\delta}\right) = E(\phi(N)) \tag{3.15}$$

Finally, it is clear from the definition of ϕ and from Proposition 3.3 that $\phi \equiv 1$. The conclusion follows easily.

2) We now fix $h \in \mathcal{H}$, $m \in \mathcal{M}$, and $\epsilon > 0$. We have to check that

$$P_0 = P\left(\delta(X, S(h,m)) \leq \epsilon\right) > 0 \tag{3.16}$$

First,

$$P_0 \geq P\left(\delta(X, X_h) \leq \epsilon/2 \ ; \ \delta(X_h, S(h,m)) \leq \epsilon/2\right) \tag{3.17}$$

Noticing that X_h is $\sigma(N)$-measurable, we see that

$$P_0 \geq E\left[\mathbb{1}_{\{\delta(X_h, S(h,m)) \leq \epsilon/2\}} P\left(\delta(X, X_h) \leq \epsilon/2 \mid \sigma(N)\right)\right] \tag{3.18}$$

But we know from Proposition 3.3 that for all $m \in \mathcal{M}$,

$$\psi(m) = P\left(\delta(X_m, S(h,m)) \leq \epsilon/2\right) > 0 \tag{3.19}$$

Working on the canonical product space as in 1), and noticing that for all $\omega = (\omega^W, \omega^N) \in \Omega$, $X(\omega) = X_{\omega^N}(\omega^W)$ and $X_h(\omega) = S(h, \omega^N)$ (all of this **without** abusive notation), we deduce that a.s.,

$$P\left(\delta(X, X_h) \leq \epsilon/2 \mid \sigma(N)\right) = \psi(N) > 0 \tag{3.20}$$

Thus, (3.16) holds as soon as

$$P\left(\delta(X_h, S(h,m)) \leq \epsilon/2\right) > 0 \tag{3.21}$$

which never fails, thanks to Proposition 3.2.

Provided we check Propositions 3.1, 3.2 and 3.3, Theorem 2.5 is be proved.

In order to prove Proposition 3.1, we begin with a uniqueness Lemma.

Lemma 3.4 *Consider some functions σ, b, g (resp. $\bar{\sigma}$, $\bar{b}$ and $\bar{g}$) satisfying (H), and denote by X (resp. $\bar{X}$) the corresponding unique weak solution of (1.1). Assume that for some $A \in \mathbb{R}^+$,*

$$\forall\, |x| \leq A, \quad \forall\, z \in E, \quad \sigma(x) = \bar{\sigma}(x) \quad , \quad b(x) = \bar{b}(x) \quad \text{and} \quad g(x,z) = \bar{g}(x,z) \tag{3.22}$$

Then there exists $\tilde{\Omega} \subset \Omega$ such that $P(\tilde{\Omega}) = 1$ and

$$\{\omega \in \Omega \ / \ \| X(\omega) \|_\infty \leq A\} \subset \left\{\omega \in \tilde{\Omega} \ / \ \| X(\omega) - \bar{X}(\omega) \|_\infty = 0\right\} \tag{3.23}$$

Proof of Lemma 3.4 : we consider the process $\tilde{X}$ defined by

$$\tilde{X}(\omega) = \begin{cases} X(\omega) & \text{if} \quad \| X(\omega) \|_\infty \leq A \\ \\ \bar{X}(\omega) & \text{else} \end{cases} \tag{3.24}$$

Then it is obvious that $\tilde{X}$ is solution of (1.1) with the parameters $\bar{\sigma}$, $\bar{b}$, $\bar{g}$. Proposition 2.2 yields the existence of a set $\tilde{\Omega} \subset \Omega$ such that $P(\tilde{\Omega}) = 1$ and such that each $\omega \in \tilde{\Omega}$ satisfies $\| \tilde{X}(\omega) - \bar{X}(\omega) \|_\infty = 0$. In particular, for each $\omega \in \tilde{\Omega}$ such that $\| X(\omega) \|_\infty \leq A$, we see that $X(\omega) = \tilde{X}(\omega) = \bar{X}(\omega)$, which was our aim.

Proof of Proposition 3.1 : we assume that theorem 2.5 holds under (H), $(S'1)$ and $(S'2)$, and we consider functions b, σ and g satisfying only (H), $(S1)$ and $(S2)$. We need a sequence of C_b^∞ functions $\phi_n : \mathbb{R}^+ \mapsto [0,1]$, satisfying :

$$\phi_n(x) = \begin{cases} 1 & \text{if} \quad |x| \leq n \\ \\ 0 & \text{if} \quad |x| \geq n+1 \end{cases} \tag{3.25}$$

Then the functions $\sigma_n(x) = \sigma(x)\phi_n(x)$ and $g_n(x,z) = g(x,z)\phi_n(x)$ clearly satisfy $(S'1)$ and $(S'2)$. Denote by X_n the solution of equation (2.5) with σ_n and g_n instead of σ and g. Lemma 3.4 yields that there exists $\tilde{\Omega} \subset \Omega$ such that $P(\tilde{\Omega}) = 1$ and for all $n \in I\!N$,

$$\{\omega \in \Omega \ / \parallel X(\omega) \parallel_\infty \leq n\} \subset \left\{\omega \in \tilde{\Omega} \ / \parallel X(\omega) - X_n(\omega) \parallel_\infty = 0\right\} \tag{3.26}$$

In the same way, we define $S_n(h,m)$, for $h \in \mathcal{H}$ and $m \in \mathcal{M}$, as the solution of equation (2.14) with σ_n and g_n instead of σ and g. We obtain, for all $n \in I\!N$,

$$\text{if} \quad \parallel S(h,m) \parallel_\infty \leq n \ \text{ or } \ \parallel S_n(h,m) \parallel_\infty \leq n, \ \text{ then } \ S(h,m) = S_n(h,m) \tag{3.27}$$

Since Theorem 2.5 holds under (H), $(S'1)$, and $(S'2)$, we know that for each $n \in I\!N$,

$$\text{supp}_\delta \ P \circ X_n^{-1} = \overline{\{S_n(h,m) \ / \ h \in \mathcal{H} \ , \ m \in \mathcal{M}\}}^\delta \tag{3.28}$$

Using Remark 2.4, Proposition 3.1 will hold if we check that on one hand,

$$P\left(X \in \overline{\{S(h,m) \ / \ h \in \mathcal{H} \ , \ m \in \mathcal{M}\}}^\delta\right) = 1 \tag{3.29}$$

and on the other hand that for all $h \in \mathcal{H}$, all $m \in \mathcal{M}$, all $\epsilon > 0$,

$$P\left(\delta(X, S(h,m)) \leq \epsilon\right) > 0 \tag{3.30}$$

Let us first prove (3.29). Let $\omega \in \tilde{\Omega}$ be fixed. Since $X(\omega)$ belongs to $I\!D\left([0,T], \mathbf{C}([0,1])\right)$, it is bounded, and there exists $n \in I\!N$ (depending on ω) such that

$$n \geq \parallel X(\omega) \parallel_\infty +1 \tag{3.31}$$

which yields $X(\omega) = X_n(\omega)$. But for all $\epsilon > 0$, we know from (3.28) that there exist $h \in \mathcal{H}$ and $m \in \mathcal{M}$ (depending on ω) such that

$$\delta(X_n(\omega), S_n(h,m)) \leq \epsilon \tag{3.32}$$

This and (3.31) yield (if $\epsilon \leq 1$), that $\parallel S_n(h,m) \parallel_\infty \leq n$, and thus that $S_n(h,m) = S(h,m)$. Hence,

$$\delta(X(\omega), S(h,m)) \leq \epsilon \tag{3.33}$$

which concludes the proof of (3.29), since $P(\tilde{\Omega}) = 1$.

In order to prove (3.30), we fix $h \in \mathcal{H}$, $m \in \mathcal{M}$, and $\epsilon > 0$. We consider $n \in I\!N$ such that

$$n \geq \parallel S(h,m) \parallel_\infty +1 \tag{3.34}$$

This way, if $\epsilon < 1$,

$$\begin{aligned} P\left(\delta(X, S(h,m)) \leq \epsilon\right) &= P\left(\delta(X, S_n(h,m)) \leq \epsilon\right) \\ &= P\left(\parallel X \parallel_\infty \leq n, \delta(X, S_n(h,m)) \leq \epsilon\right) \\ &= P\left(\delta(X_n, S_n(h,m)) \leq \epsilon\right) \end{aligned} \tag{3.35}$$

thanks to (3.26). From (3.28), this probability is strictly positive, which yields (3.30). Proposition 3.1 is proved.

4 The case where "W is deterministic".

This section is devoted to the proof of Proposition 3.2. We follow here partially the method of Simon [41], who studies the support of Poisson driven S.D.E.s (without Wiener term). The extension of his method to S.P.D.E.s drives to technical problems, essentially because we have to control the explosion of the Green kernel $G_t(x,y)$. Another new difficulty appears, because we have to add a second drift, in which the term $\dot{h}(s,y)$ belongs only to $L^2([0,T]\times[0,1])$.

In the whole section,

$$h(t,x)=\int_0^t\int_0^x \dot{h}(s,y)dyds \in \mathcal{H} \quad \text{and} \quad m(dt,dz)=\sum_{i=1}^n \delta_{(t_i,z_i)}(dt,dz)\in\mathcal{M} \tag{4.1}$$

are fixed. We set $t_0=0$, $t_{n+1}=T$, and

$$\zeta_0=\inf_{i=0,\ldots,n}|t_{i+1}-t_i|>0 \tag{4.2}$$

For simplicity, we set $S=S(h,m)$. We denote by $0<T_1(\omega)<\ldots<T_{\mu(\omega)}(\omega)$ the successive times of jump of $N(\omega)$, and by $Z_1(\omega)$, ..., $Z_{\mu(\omega)}(\omega)$ the size of its jumps. In other words,

$$N(\omega,dt,dz)=\sum_{i=1}^{\mu(\omega)}\delta_{(T_i(\omega),Z_i(\omega))}(dt,dz) \tag{4.3}$$

We recall that for all $\alpha\in]0,\zeta_0[$, and all $\xi>0$, the set

$$\Omega(\alpha,\xi)=\{\omega\in\Omega \;/\; \mu(\omega)=n,\ t_i-\alpha<T_i(\omega)<t_i,\ d(z_i,Z_i(\omega))\leq\xi\,\} \tag{4.4}$$

has a strictly positive probability. We will check that for all $\epsilon>0$, there exists $\alpha>0$, and $\xi>0$ such that for all $\omega\in\Omega(\alpha,\xi)$,

$$\delta(X_h(\omega),S)\leq\epsilon \tag{4.5}$$

which will imply Proposition 3.2.

In the whole section, the constant C depends only on h, m, and on the parameters (σ,b,g, $\mathcal{X}_0$, and T) of equation (1.1).

From now on, we consider $\omega\in\Omega(\alpha,\xi)$.

First, we choose $0<\alpha<\zeta_0/4$, and $0<\xi<\xi(z_1)\wedge\ldots\wedge\xi(z_n)$, where $\xi(z_i)$ was defined in assumption $(S'2)$. For some $\gamma\in]2\alpha,\zeta_0/8[$, which will be chosen later, we define the polygonal change of time $\lambda\in\Lambda$ by $\lambda(0)=0$, $\lambda(T)=T$, and for all $i\in\{1,\ldots,n\}$,

$$\lambda(T_i-\gamma)=T_i-\gamma \quad ; \quad \lambda(T_i)=t_i \quad ; \quad \lambda(T_i+\gamma)=t_i+\gamma \qquad \lambda(T_i+2\gamma)=T_i+2\gamma \tag{4.6}$$

Notice that all the properties below hold :

$$\text{for all } t\in[T_i,T_i+\gamma], \qquad \lambda(t)-t_i=t-T_i \tag{4.7}$$

$$\int_0^T \mathbb{1}_{\{\lambda(s)\neq s\}} ds \leq 3n\gamma \tag{4.8}$$

$$\text{for all } t\in[0,T], \quad \lambda(t)\geq t \quad \text{and} \quad \mathbb{1}_{\{\lambda(t)\geq t_i\}} = \mathbb{1}_{\{t\geq T_i\}} \tag{4.9}$$

$$\| \lambda - I \|_\infty \leq \alpha \tag{4.10}$$

Furthermore, it is easy to check that

$$|||\lambda||| \leq |\ln(1-\alpha/\gamma)| \vee |\ln(1+\alpha/\gamma)| \leq 2\alpha/\gamma \tag{4.11}$$

where the last inequality holds because $\alpha/\gamma \leq 1/2$. We have to prove that if $\alpha > 0$ and $\xi > 0$ are small enough then for some γ well chosen,

$$\| S(\lambda(t),x) - X_h(t,x) \|_\infty + |||\lambda||| \leq \epsilon \tag{4.12}$$

We now set $S_\lambda(t,x) = S(\lambda(t),x)$. Then, using (4.9), we see that for any $\omega \in \Omega(\alpha,\xi)$,

$$\begin{aligned}
&S_\lambda(t,x) - X_h(t,x) = G_{\lambda(t)}(\mathcal{X}_0,x) - G_t(\mathcal{X}_0,x)\\
&+\int_0^t\int_0^1 \Big(G_{\lambda(t)-s}(x,y) - G_{t-s}(x,y)\Big)\Big[b(S(s,y)) + \sigma(S(s,y))\dot h(s,y)\Big]\, dyds\\
&+\int_t^{\lambda(t)}\int_0^1 G_{\lambda(t)-s}(x,y)\Big[b(S(s,y)) + \sigma(S(s,y))\dot h(s,y)\Big]\, dyds\\
&+\int_0^t\int_0^1 G_{t-s}(x,y)\Big[\{b(S(s,y)) - b(S_\lambda(s,y))\}\\
&\qquad\qquad +\{\sigma(S(s,y)) - \sigma(S_\lambda(s,y))\}\dot h(s,y)\Big]dyds\\
&+\int_0^t\int_0^1 G_{t-s}(x,y)\Big[\{b(S_\lambda(s,y)) - b(X_h(s,y))\}\\
&\qquad\qquad +\{\sigma(S_\lambda(s,y)) - \sigma(X_h(s,y))\}\dot h(s,y)\Big]dyds\\
&+\sum_{i=1}^n \mathbb{1}_{\{t\geq T_i\}}\int_0^1 \Big(G_{\lambda(t)-t_i}(x,y) - G_{t-T_i}(x,y)\Big)\, g(S(t_i-,y),z_i)dy\\
&+\sum_{i=1}^n \mathbb{1}_{\{t\geq T_i\}}\int_0^1 G_{t-T_i}(x,y)\,[g(S(t_i-,y),z_i) - g(S(t_i-,y),Z_i)]\,dy\\
&+\sum_{i=1}^n \mathbb{1}_{\{t\geq T_i\}}\int_0^1 G_{t-T_i}(x,y)\,[g(S(t_i-,y),Z_i) - g(X_h(T_i-,y),Z_i)]\,dy\\
&= A(t,x) + \ldots + H(t,x)
\end{aligned} \tag{4.13}$$

We compute these terms one by one, still assuming that $\omega \in \Omega(\alpha,\xi)$.

Since $\lambda(t) = t$ for all $t \leq T_1 - \gamma$, and hence for all $t \leq 5\zeta_0/8$

$$|A(t,x)| \leq |A(t,x)| 1\!\!1_{\{t \geq 5\zeta_0/8\}} \leq \| \mathcal{X}_0 \|_\infty 1\!\!1_{\{t \geq 5\zeta_0/8\}} \int_0^1 |G_{\lambda(t)}(x,y) - G_t(x,y)| dy \tag{4.14}$$

Using the Appendix, (7.4), then (4.10), we see that

$$|A(t,x)| \leq C \frac{\lambda(t) - t}{(5\zeta_0/8)^{\frac{3}{2}}} \leq C \| \lambda - I \|_\infty \leq C\alpha \tag{4.15}$$

Using Cauchy-Schwarz's inequality, then the Appendix (7.5), and finally (4.10), we obtain

$$\begin{aligned} |B(t,x)| &\leq \left(\int_0^t \int_0^1 \left[b(S(s,y)) + \sigma(S(s,y)) \dot{h}(s,y) \right]^2 dyds \right)^{\frac{1}{2}} \\ &\qquad \times \left(\int_0^t \int_0^1 \left[G_{\lambda(t)-s)}(x,y) - G_{t-s}(x,y) \right]^2 dyds \right)^{\frac{1}{2}} \\ &\leq C \left(\sqrt{\lambda(t) - t} \right)^{\frac{1}{2}} \leq C\alpha^{\frac{1}{4}} \end{aligned} \tag{4.16}$$

Exactly in the same way, $|C(t,x)| \leq C\alpha^{\frac{1}{4}}$.

Using (H), we see that

$$|D(t,x)| \leq C \int_0^t \int_0^1 G_{t-s}(x,y) \, |S(s,y) - S_\lambda(s,y)| \left(1 + |\dot{h}(s,y)| \right) dyds \tag{4.17}$$

Thanks to Cauchy-Schwarz's inequality, and the Appendix (7.2),

$$\begin{aligned} |D(t,x)| &\leq C \left(\int_0^t \sup_{y \in [0,1]} |S(s,y) - S_\lambda(s,y)|^2 \, ds \int_0^1 G^2_{t-s}(x,y) dy \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^t 1\!\!1_{\{\lambda(s) \neq s\}} \frac{ds}{\sqrt{t-s}} \right)^{\frac{1}{2}} \end{aligned} \tag{4.18}$$

Using (4.8), we easily deduce that

$$|D(t,x)| \leq C \left(\sqrt{3n\gamma} \right)^{\frac{1}{2}} \leq C\gamma^{\frac{1}{4}} \tag{4.19}$$

The same computation drives us to

$$|E(t,x)| \leq C \left(\int_0^t \sup_{y \in [0,1]} |S_\lambda(s,y) - X_h(s,y)|^2 \frac{ds}{\sqrt{t-s}} \right)^{\frac{1}{2}} \tag{4.20}$$

Using (4.7), and (3.1) in $(S'2)$, we see that

$$|F(t,x)| \leq C \sum_{i=1}^n 1\!\!1_{\{t \geq T_i + \gamma\}} \sup_{x,y \in [0,1]} \left| G_{\lambda(t)-t_i}(x,y) - G_{t-T_i}(x,y) \right| \tag{4.21}$$

Thus, thanks to the Appendix (7.4),

$$|F(t,x)| \leq C \sum_{i=1}^{n} \mathbb{1}_{\{t \geq T_i + \gamma\}} \frac{|(\lambda(t) - t_i) - (t - T_i)|}{[(\lambda(t) - t_i) \wedge (t - T_i)]^{\frac{3}{2}}} \tag{4.22}$$

But $t \geq T_i + \gamma$ implies that $\lambda(t) - t_i \geq \lambda(T_i + \gamma) - t_i = \gamma$. Hence, thanks to (4.10) and since $\omega \in \Omega(\alpha, \xi)$,

$$|F(t,x)| \leq C \frac{\| \lambda - I \|_\infty + \sup_i |t_i - T_i|}{\gamma^{\frac{3}{2}}} \leq C\alpha/\gamma^{\frac{3}{2}} \tag{4.23}$$

Using (7.3) of the appendix, we deduce that

$$|G(t,x)| \leq \sum_{i=1}^{n} \sup_y |g(S(t_i-, y), z_i) - g(S(t_i-, y), Z_i)| \tag{4.24}$$

Thanks to (3.1) in $(S'2)$, recalling that for all i, $d(z_i, Z_i) \leq \xi$, we see that there exists a function $\varphi(\xi)$ from $\mathbb{R}^+$ into itself, decreasing to 0 when ξ decreases to 0, depending only on h, m, and on the parameters of equation (1.1), such that

$$|G(t,x)| \leq \varphi(\xi) \tag{4.25}$$

In the same way, but using (3.2) and the fact that $\xi \leq \xi(z_1) \wedge ... \wedge \xi(z_n)$, we easily prove the existence of a function $\beta(u) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, decreasing to 0 when u decreases to 0, such that

$$\begin{aligned} |H(t,x)| &\leq \sum_{i=1}^{n} \mathbb{1}_{\{t \geq T_i\}} \times \beta \left(\sup_{y \in [0,1]} |S(t_i-, y) - X_h(T_i-, y)| \right) \\ &\leq \sum_{i=1}^{n} \mathbb{1}_{\{t \geq T_i\}} \times \beta \left(\sup_{y \in [0,1]} |S_\lambda(T_i-, y) - X_h(T_i-, y)| \right) \end{aligned} \tag{4.26}$$

since $\lambda(T_i) = t_i$.

Finally, setting

$$I(t) = \sup_{y \in [0,1]} |S_\lambda(t,y) - X_h(t,y)| \tag{4.27}$$

and

$$K(\alpha, \gamma, \xi) = \alpha^{1/4}/\gamma^{3/2} + \gamma^{1/4} + \varphi(\xi) \tag{4.28}$$

we obtain :

$$I(t) \leq CK(\alpha, \gamma, \xi) + C \left(\int_0^t I^2(s) \frac{ds}{\sqrt{t-s}} \right)^{\frac{1}{2}} + C \sum_{i=1}^{n} \mathbb{1}_{\{t \geq T_i\}} \beta(I(T_i-)) \tag{4.29}$$

Hence

$$I^2(t) \leq CK^2(\alpha, \gamma, \xi) + C \int_0^t I^2(s) \frac{ds}{\sqrt{t-s}} + C \sum_{i=1}^{n} \mathbb{1}_{\{t \geq T_i\}} \beta^2(I(T_i-)) \tag{4.30}$$

Iterating once this formula, we get

$$\begin{aligned} I^2(t) &\le CK^2(\alpha,\gamma,\xi) + C\sum_{i=1}^n \mathbb{1}_{\{t\ge T_i\}}\beta^2(I(T_i-)) \\ &+ C\int_0^t \left[CK^2(\alpha,\gamma,\xi) + C\int_0^s I^2(u)\frac{du}{\sqrt{s-u}} + C\sum_{i=1}^n \mathbb{1}_{\{s\ge T_i\}}\beta^2(I(T_i-))\right]\frac{ds}{\sqrt{t-s}} \end{aligned} \tag{4.31}$$

Using Fubini's Theorem, and noticing that $\int_u^t \frac{ds}{\sqrt{t-s}\sqrt{s-u}} \le 4$, we deduce that

$$I^2(t) \le CK^2(\alpha,\gamma,\xi) + C\int_0^t I^2(u)du + C\sum_{i=1}^n \mathbb{1}_{\{t\ge T_i\}}\beta^2(I(T_i-)) \tag{4.32}$$

We now apply Gronwall's Lemma on $[0,T_1[$. This gives :

$$\sup_{[0,T_1[} I^2(t) \le CK^2(\alpha,\gamma,\xi)e^{CT} \le CK^2(\alpha,\gamma,\xi) \tag{4.33}$$

Thus, on $[0,T_2[$,

$$I^2(t) \le CK^2(\alpha,\gamma,\xi) + \beta^2(CK^2(\alpha,\gamma,\xi)) + C\int_0^t I^2(s)ds \tag{4.34}$$

Thanks to Gronwall's Lemma,

$$\sup_{[0,T_2[} I^2(t) \le \left(CK^2(\alpha,\gamma,\xi) + \beta^2(K^2(\alpha,\gamma,\xi))\right)e^{CT} \tag{4.35}$$

Iterating this argument, we deduce the existence of a function $\eta(u) : I\!R^+ \mapsto I\!R^+$, decreasing to 0 when u decreases to 0, such that

$$\sup_{[0,T]} I(t) \le \eta\left(K(\alpha,\gamma,\xi)\right) \tag{4.36}$$

Hence, there exists $\delta > 0$ such that if $K(\alpha,\gamma,\xi) \le \delta$, then $\sup_{[0,T]} I(t) \le \epsilon/2$. It now suffices to choose α, γ, ξ small enough, such that

$$K(\alpha,\gamma,\xi) \le \delta \quad ; \quad 2\alpha/\gamma \le \epsilon/2 \tag{4.37}$$

which will imply, for all $\omega \in \Omega(\alpha,\xi)$,

$$\delta(X_h(\omega), S) \le \| I(\omega) \|_\infty + |||\lambda(\omega)||| \le \epsilon \tag{4.38}$$

First, we choose $\xi \in]0, \xi(z_1) \wedge ... \wedge \xi(z_n)[$ small enough, in order to get $\varphi(\xi) \le \delta/3$. Then we choose γ in $]0, (\zeta_0/8) \wedge (\delta/3)^4[$. Finally, we choose

$$0 < \alpha < \gamma/2 \wedge \left(\delta\gamma^{\frac{3}{2}}/3\right)^4 \wedge \epsilon\gamma/4 \tag{4.39}$$

Proposition 3.2 is proved.

5 The case where N is "deterministic".

It remains to prove Proposition 3.3. In the whole section,

$$m(dt,dz)=\sum_{i=1}^{n}\delta_{(t_i,z_i)}(dt,dz)\in\mathcal{M} \tag{5.1}$$

is fixed. We set $t_0=0$, $t_{n+1}=T$.

We have to establish a support theorem for the solution of equation (3.4). Let us observe that this equation is not much different from that of Walsh [47]. Indeed, it does only contain one additional term, a "jump drift". Nevertheless, it is far from possible to use a method similar to that of Bally, Millet, Sanz-Solé in [3], who proved a support theorem for Walsh's equation, in particular because the solution of (3.4) does not lie in $\mathbf{C}([0,T]\times[0,1])$.

But the times of jump of the solution X_m of equation (3.4) are deterministic, and the associated squeleton $S(h,m)$ (m is fixed) has the same times of jump. Thus we do not need the Skorokhod topology : we will work with the stronger supremum norm on $[0,T]\times[0,1]$.

The method below consists in applying the result of Bally, Millet, and Sanz-Solé on each time interval $[t_i,t_{i+1}[$. To this end, we will define some processes X^i_m, which equal X_m only on $[t_i,t_{i+1}[\times[0,1]$, but also give information about the behaviour of X_m after t_{i+1}. We will also associate with X^i_m some deterministic squeletons $S^i_m(h)$. But we will apply the result of [3] to the conditional law of X^i_m with respect to $\mathcal{F}_{t_i}$ (for each i). Thus, we will have to define a non-deterministic "conditional squeleton" $T^i_m(h)$. Then we will develop a technical way to "paste the pieces".

Recall that thanks to Remark 2.4, we have to prove on one hand that for all $h\in\mathcal{H}$, all $\epsilon>0$,

$$P\left(\parallel X_m-S(h,m)\parallel_\infty\leq\epsilon\right)>0 \tag{5.2}$$

and on the other hand that

$$P\left(X_m\in\overline{\{S(h,m)\ ;\ h\in\mathcal{H}\}}^{\parallel\,\parallel_\infty}\right)=1 \tag{5.3}$$

To this aim, we introduce some notations. First, if $S(t,x)$ belongs to $\mathbb{D}\left([0,T],\mathbf{C}([0,1])\right)$, and if $0\leq u<v\leq T$,

$$\parallel S\parallel_{[u,v]}=\sup_{t\in[u,v],\ x\in[0,1]}|S(t,x)| \tag{5.4}$$

We now define recursively, for i in $\{0,...,n\}$, the processes $X^i_m(t,x)$ on $[t_i,T]\times[0,1]$:

$$\begin{aligned}X^0_m(t,x) &= G_t(\mathcal{X}_0,x)+\int_0^{t_1\wedge t}\int_0^1 G_{t-s}(x,y)\Big[b(X^0_m(s,y))dyds+\\ &\qquad\qquad\sigma(X^0_m(s,y))W(dy,ds)\Big]\end{aligned} \tag{5.5}$$

and, for $i \in \{1,...,n\}$,

$$X^i_m(t,x) = X^{i-1}_m(t,x) + 1\!\!1_{\{t\geq t_i\}} \int_0^1 G_{t-t_i}(x,y) g(X^{i-1}_m(t_i-,y), z_i) dy \tag{5.6}$$
$$+ \int_{t_i}^{t_{i+1}\wedge t} \int_0^1 G_{t-s}(x,y) \Big[b(X^i_m(s,y)) dy ds + \sigma(X^i_m(s,y)) W(dy,ds) \Big]$$

Notice that for all i,

$$\text{for all } t \in [t_i, t_{i+1}[, \quad \text{all } x \in [0,1], \quad X^i_m(t,x) = X_m(t,x) \tag{5.7}$$

Indeed, it suffices to use a standard uniqueness argument. In the same way, we define, for $h \in \mathcal{H}$, the functions $S^i_m(h)$ on $[t_i, T] \times [0,1]$, by

$$S^0_m(h)(t,x) = G_t(\mathcal{X}_0, x) + \int_0^{t_1\wedge t} \int_0^1 G_{t-s}(x,y) \Big[b(S^0_m(h)(s,y)) dy ds \tag{5.8}$$
$$+ \sigma(S^0_m(h)(s,y)) \dot{h}(s,y) dy ds \Big]$$

and, for $i \in \{1,...,n\}$,

$$S^i_m(h)(t,x) = S^{i-1}_m(h)(t,x) + 1\!\!1_{\{t\geq t_i\}} \int_0^1 G_{t-t_i}(x,y) g(S^{i-1}_m(h)(t_i-,y), z_i) dy \tag{5.9}$$
$$+ \int_{t_i}^{t_{i+1}\wedge t} \int_0^1 G_{t-s}(x,y) \Big[b(S^i_m(h)(s,y)) dy ds + \sigma(S^i_m(h)(s,y)) \dot{h}(s,y) dy ds \Big]$$

Then, for all i,

$$\text{for all } t \in [t_i, t_{i+1}[, \quad \text{all } x \in [0,1], \quad S^i_m(h)(t,x) = S(h,m)(t,x) \tag{5.10}$$

Finally, we define the "conditional squeleton" associated with the conditional law of X^i_m with respect to $\mathcal{F}_{t_i}$:

$$T^i_m(h)(t,x) = X^{i-1}_m(t,x) + 1\!\!1_{\{t\geq t_i\}} \int_0^1 G_{t-t_i}(x,y) g(X^{i-1}_m(t_i-,y), z_i) dy \tag{5.11}$$
$$+ \int_{t_i}^{t_{i+1}\wedge t} \int_0^1 G_{t-s}(x,y) \Big[b(T^i_m(h)(s,y)) dy ds + \sigma(T^i_m(h)(s,y)) \dot{h}(s,y) dy ds \Big]$$

The function $T^i_m(h)$ is defined on $[t_i, T] \times [0,1]$. For all $t \in [t_i, T]$, all $x \in [0,1]$, $T^i_m(h)(t,x)$ is $\mathcal{F}_{t_i}$-measurable.

Then one can "nearly" use the Theorem of Bally, Millet, Sanz-Solé, [3] (see also Cardon-Weber, Millet, [10] for a more general setting), which yields the following result.

Proposition 5.1 *Assume (H) and $(S'1)$. Then, with the above notations, for all $i \in \{0,...,n\}$, the following conditional support theorem on $[t_i, T] \times [0,1]$ holds :*

$$\mathrm{supp}_{\|\ \|_{[t_i,T]}} \mathcal{L}\left(X^i_m \mid \mathcal{F}_{t_i} \right) = \overline{\{T^i_m(h) \ / \ h \in \mathcal{H}\}}^{\|\ \|_{[t_i,T]}} \tag{5.12}$$

In fact, the main theorem in [3] only yields the result for $i = 0$, with $\int_0^t$ instead of $\int_0^{t\wedge t_1}$. But conditioning is not a problem, and the initial values we obtain, for example

$$X_m^{i-1}(t,x) + \int_0^1 G_{t-t_i}(x,y)g(X_m^{i-1}(t_i-,y),z_i)dy$$

$$= X_m^{i-1}(t,x) + G_{t-t_i}\left(g(X_m^{i-1}(t_i-,.),z_i),x\right) \quad (5.13)$$

behave on $[t_i,T]$ exactly as $G_t(\mathcal{X}_0,x)$ on $[0,T]$, since they are $\mathcal{F}_{t_i}$-measurable, since $g(X_m^{i-1}(t_i-,.),z_i)$ is continuous on $[0,1]$, and since $X_m^{i-1}(t,x)$ is continuous on $[0,T]\times[0,1]$. Finally, it is clear that considering the integrals from t_i to $t\wedge t_{i+1}$ instead of 0 to t will not change much...

We now establish a Lemma, which will allow to paste the pieces. If $\| X_m^i(\omega) - S_m^i(h) \|_{[t_i,T]}$ is small, then the initial conditions associated with $S_m^{i+1}(h)$ and $T_m^{i+1}(h)(\omega)$ are near, and thus the distance between $S_m^{i+1}(h)$ and $T_m^{i+1}(h)(\omega)$ is small. We need this Lemma, because Proposition 5.1 gives an idea of the distance between $X_m^i(\omega)$ and $T_m^i(h)(\omega)$, but what we need to control is the distance between $S_m^i(h)$ and $X_m^i(\omega)$.

Lemma 5.2 *Assume (H), $(S'2)$. There exists a function $\gamma(x,u) : I\!R^+ \times I\!R^+ \mapsto I\!R^+$, such that for each x, $\gamma(x,u)$ decreases to 0 when u decreases to 0, and such that for all $\epsilon > 0$, all $i\in\{0,...,n-1\}$,*

$$\left\{\omega\in\Omega \ \Big/ \| X_m^i(\omega) - S_m^i(h) \|_{[t_i,T]}\le \epsilon\right\}$$

$$\subset \left\{\omega\in\Omega \ \Big/ \| S_m^{i+1}(h) - T_m^{i+1}(h)(\omega) \|_{[t_{i+1},T]}\le \gamma(\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2},\epsilon)\right\} \quad (5.14)$$

where $\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}^2 = \int_{t_{i+1}}^{t_{i+2}}\int_0^1 \dot{h}^2(s,y)dyds$.

<u>Proof</u> : Let ω belong to $\{\| X_m^i - S_m^i(h) \|_{[t_i,T]}\le\epsilon\}$. Then, for all t in $[t_{i+1},T]$, all x in $[0,1]$, using (H),

$$|S_m^{i+1}(h)(t,x) - T_m^{i+1}(h)(t,x)| \le |S_m^i(h)(t,x) - X_m^i(t,x)|$$

$$+\int_0^1 G_{t-t_{i+1}}(x,y)\left|g\left(X_m^i(t_{i+1}-,y),z_i\right) - g\left(S_m^i(h)(t_{i+1}-,y),z_i\right)\right|dy$$

$$+C\int_{t_{i+1}}^{t_{i+2}\wedge t}\int_0^1 G_{t-s}(x,y)\left|S_m^{i+1}(h)(s,y) - T_m^{i+1}(h)(s,y)\right|\left(1+|\dot{h}(s,y)|\right)dyds \quad (5.15)$$

We now set

$$F(t) = \sup_{x\in[0,1]}\left|S_m^{i+1}(h)(t,x) - T_m^{i+1}(h)(t,x)\right| \quad (5.16)$$

Using the assumption about ω, assumption $(S'2)$, the Appendix (7.3) and (7.2), and Cauchy-Schwarz's inequality, we get :

$$F(t)\le\epsilon+\psi_{z_i}(\epsilon)+C\left(1+\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}\right)\left(\int_{t_{i+1}}^t F^2(s)\frac{ds}{\sqrt{t-s}}\right)^{\frac{1}{2}} \quad (5.17)$$

where ψ_{z_i} was defined in assumption $(S'2)$. Hence,

$$F^2(t) \leq C\epsilon^2 + C\psi^2_{z_i}(\epsilon) + C\left(1+ \| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}\right)^2 \int_{t_{i+1}}^{t} F^2(s)\frac{ds}{\sqrt{t-s}} \tag{5.18}$$

Iterating once this formula (see the previous section, inequalities (4.30), (4.31), and (4.32) for more precisions), we obtain the existence of a function γ, satisfying the assumptions of the statement, such that

$$F^2(t) \leq \gamma\left(\| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}, \epsilon\right) + C\left(1+ \| \dot{h}|_{[t_{i+1},t_{i+2}]} \|_{L^2}\right)^2 \int_{t_{i+1}}^{t} F^2(s)ds \tag{5.19}$$

Gronwall's Lemma allows to conclude.

Proof of Proposition 3.3 : In order to simplify the notations, we assume in the sequel that $n=2$, i.e. that

$$m(dt,dz) = \delta_{(t_1,z_1)} + \delta_{(t_2,z_2)} \tag{5.20}$$

1) We fix $h \in \mathcal{H}$, and $\epsilon > 0$, and we check that

$$P_0 = P\left(\| X_m - S(h,m) \|_\infty \leq \epsilon\right) > 0 \tag{5.21}$$

First, using (5.7) and (5.10), we see that

$$\begin{aligned} P_0 \geq P\Big(&\| X^0_m - S^0_m(h) \|_{[0,T]} \leq \epsilon/3, \ \| X^1_m - S^1_m(h) \|_{[t_1,T]} \leq \epsilon/3, \\ & \| X^2_m - S^2_m(h) \|_{[t_2,T]} \leq \epsilon/3\Big) \end{aligned} \tag{5.22}$$

Noticing that for each i, X^i_m is $\mathcal{F}_{t_{i+1}}$-measurable and $S^i_m(h)$ is deterministic, we obtain, by conditionning our probability with respect to $\mathcal{F}_{t_2}$,

$$\begin{aligned} P_0 \geq E\Big[&\mathbb{1}_{\{\| X^0_m - S^0_m(h) \|_{[0,T]} \leq \epsilon/3\}} \mathbb{1}_{\{\| X^1_m - S^1_m(h) \|_{[t_1,T]} \leq \epsilon/3\}} \\ & \times P\left(\| X^2_m - S^2_m(h) \|_{[t_2,T]} \leq \epsilon/3 \,\Big|\, \mathcal{F}_{t_2}\right)\Big] \end{aligned} \tag{5.23}$$

On the other hand,

$$\begin{aligned} & P\left(\| X^2_m - S^2_m(h) \|_{[t_2,T]} \leq \epsilon/3 \,\Big|\, \mathcal{F}_{t_2}\right) \\ & \geq P\left(\| X^2_m - T^2_m(h) \|_{[t_2,T]} \leq \epsilon/6, \ \| T^2_m(h) - S^2_m(h) \|_{[t_2,T]} \leq \epsilon/6 \Big| \ \mathcal{F}_{t_2}\right) \\ & \geq \mathbb{1}_{\{\| T^2_m(h) - S^2_m(h) \|_{[t_2,T]} \leq \epsilon/6\}} P\left(\| X^2_m - T^2_m(h) \|_{[t_2,T]} \leq \epsilon/6 \Big| \ \mathcal{F}_{t_2}\right) \end{aligned} \tag{5.24}$$

since $S^2_m(h)$ is deterministic and $T^2_m(h)$ is $\mathcal{F}_{t_2}$-measurable. Using Proposition 5.1, we also know that a.s.,

$$P\left(\| X^2_m - T^2_m(h) \|_{[t_2,T]} \leq \epsilon/6 \Big| \ \mathcal{F}_{t_2}\right) > 0 \tag{5.25}$$

Hence, it suffices that $P_1 > 0$, where

$$P_1 = P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \leq \epsilon/3, \ \| X_m^1 - S_m^1(h) \|_{[t_1,T]} \leq \epsilon/3$$

$$\| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \leq \epsilon/6 \Big) \tag{5.26}$$

Thanks to Lemma 5.2, we know that for $\alpha > 0$ small enough,

$$\| X_m^1 - S_m^1(h) \|_{[t_1,T]} < \alpha \Longrightarrow \| T_m^2(h) - S_m^2(h) \|_{[t_2,T]} \leq \epsilon/6 \tag{5.27}$$

Thus,

$$P_1 \geq P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \leq \epsilon/3, \ \| X_m^1 - S_m^1(h) \|_{[t_1,T]} \leq \alpha \wedge \epsilon/3 \Big) \tag{5.28}$$

Iterating this argument, we see that P_0 is strictly positive as soon as $P_2 > 0$, where

$$P_2 = P\Big(\| X_m^0 - S_m^0(h) \|_{[0,T]} \leq \beta \Big) \tag{5.29}$$

for some $\beta > 0$ small enough. But it is clear that $S_m^0(h)$ identically equals $T_m^0(h)$. Thus, Proposition 5.1 implies that P_2 is strictly positive, and hence that (5.21) holds, which was our aim.

2) We still have to check that

$$P\left(X_m \in \overline{\{S(h,m)\ ,\ h \in \mathcal{H}\}}^{\|\ \|_\infty} \right) = 1 \tag{5.30}$$

We know from Proposition 5.1 that for almost all ω, say for all $\omega \in \bar{\Omega}$, with $P(\bar{\Omega}) = 1$,

$$X_m^0(\omega) \in \overline{\{T_m^0(h),\ h \in \mathcal{H}\}}^{\|\ \|_\infty} \quad ; \quad X_m^1(\omega) \in \overline{\{T_m^1(h)(\omega),\ h \in \mathcal{H}\}}^{\|\ \|_\infty}$$

$$X_m^2(\omega) \in \overline{\{T_m^2(h)(\omega),\ h \in \mathcal{H}\}}^{\|\ \|_\infty} \tag{5.31}$$

We now fix $\omega \in \bar{\Omega}$. There exists $h_n^0 \in \mathcal{H}$, $h_n^1 \in \mathcal{H}$, $h_n^2 \in \mathcal{H}$, (depending on ω) such that, for $i \in \{0,1,2\}$, when n goes to infinity,

$$\| X_m^i(\omega) - T_m^i(h_n^i)(\omega) \|_{[t_i,T]} \longrightarrow 0 \tag{5.32}$$

We now set

$$h_{n,k,q}(t,x) = h_n^0(t,x) 1\!\!\mathrm{I}_{[0,t_1]}(t) + h_k^1(t,x) 1\!\!\mathrm{I}_{[t_1,t_2]}(t) + h_q^2(t,x) 1\!\!\mathrm{I}_{[t_2,T]}(t) \tag{5.33}$$

We fix $\epsilon > 0$, and we prove that for n, k, q large enough,

$$\| X_m(\omega) - S(h_{n,k,q}, m) \|_{[0,T]} \leq \epsilon \tag{5.34}$$

which will suffice. One can easily check, using (5.7) and (5.10), that

$$\| X_m(\omega) - S(h_{n,k,q}, m) \|_{[0,T]} \leq A_n^0(\omega) + A_k^1(\omega) + A_q^2(\omega) + B_n^0(\omega) + B_k^1(\omega) + B_q^2(\omega) \tag{5.35}$$

where (if $i = 0,1,2$ and $l \in I\!\!N$)

$$A_l^i(\omega) = \| X_m^i(\omega) - T_m^i(h_l^i)(\omega) \|_{[t_i,T]} \tag{5.36}$$

and

$$B_l^i(\omega) = \| T_m^i(h_l^i)(\omega) - S_m^i(h_l^i) \|_{[t_i,T]} \tag{5.37}$$

First notice that B_n^0 vanishes identically. Thanks to Lemma 5.2, we know that

$$B_k^1(\omega) \leq \gamma\left(\| \dot{h}_k^1|_{[t_1,t_2]} \|_{L^2}, A_n^0(\omega)\right) \tag{5.38}$$

$$B_q^2(\omega) \leq \gamma\left(\| \dot{h}_q^2|_{[t_2,T]} \|_{L^2}, A_k^1(\omega) + B_k^1(\omega)\right) \tag{5.39}$$

i) First, we choose q large enough, in order that

$$A_q^2(\omega) \leq \epsilon/6 \tag{5.40}$$

Now that q is fixed, we consider $\alpha > 0$ such that

$$\gamma\left(\| \dot{h}_q^2|_{[t_2,T]} \|_{L^2}, \alpha\right) \leq \epsilon/6 \tag{5.41}$$

ii) Then we choose k in such a way that

$$A_k^1(\omega) \leq \epsilon/6 \wedge \alpha/2 \tag{5.42}$$

and we consider $\beta > 0$ such that

$$\gamma\left(\| \dot{h}_k^1|_{[t_1,t_2]} \|_{L^2}, \beta\right) \leq \epsilon/6 \wedge \alpha/2 \tag{5.43}$$

iii) Finally, we choose n such that

$$A_n^0(\omega) \leq \epsilon/6 \wedge \beta \tag{5.44}$$

We deduce from (5.44), (5.38), and (5.43) that

$$B_k^1(\omega) \leq \epsilon/6 \wedge \alpha/2 \tag{5.45}$$

Thanks to (5.45), (5.42), (5.41), and (5.39), we also see that

$$B_q^2(\omega) \leq \epsilon/6 \tag{5.46}$$

Finally, using (5.35), (5.44), (5.42), (5.40), (5.45), (5.46), we deduce (5.34). We thus have checked that for each $\omega \in \bar{\Omega}$, all $\epsilon > 0$, there exists $h \in \mathcal{H}$ such that

$$\| X_m(\omega) - S(h,m) \|_\infty \leq \epsilon \tag{5.47}$$

Since $P(\bar{\Omega}) = 1$, (5.30) holds, and Proposition 3.3 is proved.

6 Extension to the case of an a.s. infinite number of jumps when the diffusion coefficient is constant.

We now consider equation (1.1) in the following new setting : the diffusion coefficient is constant, $\sigma(x) \equiv \sigma$; but the positive measure q on E is only assumed to be σ-finite (*a priori*, $q(E) = \infty$). N is still a Poisson measure on $[0,T] \times E$, with intensity measure $dtq(dz)$. The evolution equation associated with equation (1.1) is still given by (2.5).

We also consider an increasing sequence of subsets E_p of E satisfying

$$q(E_p) < \infty \quad ; \quad \cup_{p \in I\!N} E_p = E \tag{6.1}$$

In order to obtain a result of existence and uniqueness, we state the following hypothesis :

Assumption (A) : the function σ is constant. The function b satisfies a global Lipschitz condition. There exists $\eta \in L^1(E,q)$ such that for all $x,y \in I\!R$, all $z \in E$,

$$|g(0,z)| \leq \eta(z) \quad ; \quad |g(x,z)-g(y,z)| \leq |x-y|\eta(z) \tag{6.2}$$

Proposition 2.2 yields that equation (1.1) with E_p instead of E admits a unique weak solution X^p lying in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$. Under (A), it is easy to check that there exists an adapted process X such that, when p goes to infinity,

$$E\left(\sup_{[0,T]\times[0,1]} |X(t,x)-X^p(t,x)|\right) \longrightarrow 0 \tag{6.3}$$

This way, we obtain the existence of an adapted weak solution X of equation (1.1) with our new setting. The uniqueness is straightforward under (A), and we can state the following proposition.

Proposition 6.1 *Assume (A). Equation (1.1) admits a unique weak solution $X(t,x)$, lying a.s. in $I\!D\left([0,T],\mathbf{C}([0,1])\right)$, and bounded in L^1.*

We now consider

$$\mathcal{M}_p = \left\{ m(dt,dz) = \sum_{i=1}^{n} \delta_{(t_i,z_i)}(dt,dz) \;\Big/\; \begin{array}{c} n \in I\!N,\ 0<t_1<...<t_n<T, \\ \\ z_1,...,z_n \in \text{supp } q \cap E_p \end{array} \right\} \tag{6.4}$$

and we set $\mathcal{M} = \cup_p \mathcal{M}_p$. The Cameron-Martin space $\mathcal{H}$ associated with W is still defined by (2.12). For each $m \in \mathcal{M}$ and $h \in \mathcal{H}$, we denote by $S(h,m)$ the unique solution of equation (2.14) (there is no difference with Proposition 2.3, since there exists p such that $m \in \mathcal{M}_p$). Since g is already Lipschitz, we assume (T) below instead of $(S2)$,

Assumption (T) : for each $z_0 \in E$, each $n \in I\!N$,

$$\sup_{|x|\leq n} |g(x,z)-g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{6.5}$$

For each z_0 in E, there exists $\xi(z_0)>0$ such that

$$\sup_{d(z,z_0)\leq \xi(z_0)} \eta(z) < \xi(z_0) \tag{6.6}$$

A function $g(x,z)=\alpha(z)\eta(z)$ clearly satisfies (A) and (T) if α is lipschitz, and $\eta \in L^1(E,q)$ is continuous. The aim of this section is to prove the following result.

Theorem 6.2 *Under (A) and (T), if X denotes the unique weak solution of equation (1.1),*

$$\text{supp}_\delta\, P \circ X^{-1} = \overline{\{S(h,m)\ /\ h \in \mathcal{H}\ ,\ m \in \mathcal{M}\}}^{\delta} \tag{6.7}$$

Since the method of Simon [41], combined with the previous sections, applies easily, we will only sketch the proof.

First, for the same reasons as in the previous sections, see Proposition 3.1, we can assume, additionally to (A) and (T), that for all $x \in I\!R$, all $z \in E$, $|g(x,z)| \leq \eta(z)$ and for each $z_0 \in E$,

$$\sup_{x \in I\!R} |g(x,z) - g(x,z_0)| \longrightarrow_{d(z,z_0)\to 0} 0 \tag{6.8}$$

Then, we notice that the direct inclusion $(\subset)$ of Theorem 6.2 is immediate, thanks to Theorem 2.5 (for X^p) and thanks to the convergence (6.3).

We now fix $p \in I\!N$, $h \in \mathcal{H}$, $m = \sum_{i=1}^{n} \delta_{(t_i,z_i)} \in \mathcal{M}_p$, and $\epsilon > 0$. We have to prove that

$$P\left(\delta(X, S(h,m)) \leq \epsilon\right) > 0 \tag{6.9}$$

To prove this, we will use three lemmas. The first one is a very particular case of the result of Bally, Millet, Sanz, [3].

Lemma 6.3 *Let $\alpha > 0$ be fixed, and let*

$$\Omega_0(\alpha) = \left\{\omega \in \Omega \;\Big/ \sup_{t,x} \left| \int_0^t \int_0^1 G_{t-s}(x,y) \left\{W(dy,ds) - \dot{h}(s,y)dyds\right\} \right| \leq \alpha \right\} \tag{6.10}$$

Then $P(\Omega_0(\alpha)) > 0$.

We now write the restriction $N^p = N|_{[0,T]\times E_p}$ (recall that p is fixed) as

$$N^p(ds,dz) = \sum_{i=1}^{\mu} \delta_{(T_i,Z_i)}(ds,dz) \tag{6.11}$$

The second lemma can be proved by using the same method as that of Proposition 3.2 (see Section 4).

Lemma 6.4 *Let $\beta > 0$ be fixed. There exists a set*

$$\Omega_1(\beta) \in \sigma\left\{N(A)\; ;\; A \in \mathcal{B}([0,T]\times E_p)\right\} \tag{6.12}$$

such that $P(\Omega_1(\beta)) > 0$, such that for each $\omega \in \Omega_1(\beta)$,

$$\mu(\omega) = n \quad ; \quad \forall i, \qquad d(z_i, Z_i(\omega)) \leq \xi(z_i) \tag{6.13}$$

and such that for some $\alpha > 0$ small enough, every $\omega \in \Omega_0(\alpha) \cap \Omega_1(\beta)$ satisfies

$$\delta(X^p(\omega), S(h,m)) \leq \beta \tag{6.14}$$

We will finally use the following result :

Lemma 6.5 *Let $\gamma > 0$ be fixed, and let*

$$\Omega_2(\gamma) = \left\{\omega \in \Omega \;\Big/ \int_0^T \int_{E/E_p} \eta(z) N(ds,dz) \leq \gamma \right\} \tag{6.15}$$

Then $P(\Omega_2(\gamma)) > 0$.

Proof of Lemma 6.5 : we set

$$\theta_p = \int_0^T \int_{E/E_p} \eta(z) N(ds, dz) \tag{6.16}$$

and, for $q > p$,

$$\theta_p^q = \int_0^T \int_{E_q/E_p} \eta(z) N(ds, dz) \tag{6.17}$$

We see that $\theta_p = \theta_p^q + \theta_q$, that for all q, $P(\theta_p^q = 0) > 0$, and that when q goes to infinity, θ_q goes to 0 in probability. Since for each q, θ_p^q is independent of θ_q, we can write

$$P(\theta_p \leq \gamma) \geq P(\theta_p^q = 0) P(\theta_q \leq \gamma) \tag{6.18}$$

and the lemma follows easily.

We finally sketch the proof of Theorem 6.2. An easy independance argument yields that for every $\alpha > 0$, $\beta > 0$, $\gamma > 0$, the set

$$\Omega_3(\alpha, \beta, \gamma) = \Omega_0(\alpha) \cap \Omega_1(\beta) \cap \Omega_2(\gamma) \tag{6.19}$$

has a strictly positive probability. We now have to choose α, β, γ in such a way that for all $\omega \in \Omega_3(\alpha, \beta, \gamma)$,

$$\delta(X(\omega), S(h, m)) \leq \epsilon \tag{6.20}$$

Let $\omega \in \Omega_3(\alpha, \beta, \gamma)$ be fixed. If α is small enough, we know from Lemma 6.4 that

$$\begin{aligned} \delta(X(\omega), S(h, m)) &\leq \| X(\omega) - X^p(\omega) \|_\infty + \delta(X^p(\omega), S(h, m)) \\ &\leq \| X(\omega) - X^p(\omega) \|_\infty + \beta \end{aligned} \tag{6.21}$$

We now set

$$V^p(t) = \sup_{x \in [0,1]} |X(t, x) - X^p(t, x)| \tag{6.22}$$

Using the Appendix, (A), (T), since $|g(x, z)| \leq \eta(z)$, and since ω belongs to $\Omega_3(\alpha, \beta, \gamma)$, we see that

$$\begin{aligned} V^p(t) &\leq C \int_0^t V^p(s) ds + C \sum_{i=1}^n 1_{\{t \geq T_i\}} V^p(T_i-) \eta(Z_i) \\ &\quad + \int_0^T \int_{E \setminus E_p} \eta(z) N(ds, dz) \\ &\leq C \int_0^t V^p(s) ds + C \sum_{i=1}^n 1_{\{t \geq T_i\}} V^p(T_i-) + \gamma \end{aligned} \tag{6.23}$$

Applying successively Gronwall's Lemma on the time intervals $[0, T_1[$, ..., $[T_n, T]$, we deduce that for all $\omega \in \Omega_3(\alpha, \beta, \gamma)$,

$$\sup_{[0,T]} V^p(\omega, t) \leq C\gamma \tag{6.24}$$

The conclusion follows easily.

Remark 6.6 *Of course, we are also interested in the case where $q(E) = \infty$ and σ is a function. In this case, it is possible to prove (under assumptions) that the sequence X_p of weak solutions of (1.1) where we have replaced E by E_p, converges to an adapted process $X(t,x)$ in the following sense :*

$$\sup_{t,x} E\left(|X(t,x) - X^p(t,x)|\right) \longrightarrow 0 \tag{6.25}$$

Once X is built, it might be possible to check that it admits a modification lying in $\mathbb{D}\left([0,T], \mathbf{C}([0,1])\right)$, by using the fact that X satisfies the evolution equation, but this is not immediate. If so, it seems natural to think that our support theorem extends to this case. However, everything will become much more technical. In particular, the direct inclusion is not obvious any more, since (6.3) does not seem to hold any more.

7 Appendix

We collect here well-known estimates about the Green kernel $G_t(x,y)$ associated with the deterministic system (2.1), and which has the expression (2.2). In all the inequalities below, the constant C depends only on the terminal time value T. The three first estimates can be found in [47], and the next ones are either easy consequences or classical estimates.

First, for all $x, y \in [0,1]$ and all $t \in [0,T]$,

$$\frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\} \le G_t(x,y) \le \frac{C}{\sqrt{t}} \exp\left\{\frac{-(y-x)^2}{4t}\right\} \tag{7.1}$$

For all $0 < t < T$, all $x \in [0,1]$,

$$\int_0^1 G_t^2(x,y)dy \le \frac{C}{\sqrt{t}} \tag{7.2}$$

and

$$\int_0^1 G_t(x,y)dy = 1 \tag{7.3}$$

For all $0 < s < t < T$, all $x, y \in [0,1]$, (see Lemma A3 in [12])

$$|G_t(x,y) - G_s(x,y)| \le C\frac{|t-s|}{s^{\frac{3}{2}}} \tag{7.4}$$

and (see Lemma B1 in [3])

$$\int_0^s \int_0^1 \left(G_{t-r}(x,y) - G_{s-r}(x,y)\right)^2 dydr + \int_s^t \int_0^1 G_{t-r}^2(x,y)dydr \le C\sqrt{t-s} \tag{7.5}$$

Finally, for all $\phi \in \mathbf{C}([0,1])$, the map

$$(t,x) \mapsto G_t(\phi, x) \tag{7.6}$$

is continuous on $[0,T] \times [0,1]$ (see Lemma A2 in [3] for a similar result).

Partie II

Etudes probabilistes de certaines équations de Boltzmann

Chapitre 4

Existence and regularity study for a 2-dimensional spatially homogeneous Boltzmann equation without cutoff by a probabilistic approach

Abstract : We consider a 2-dimensional homogeneous Boltzmann equation without cutoff, which we relate to a nonlinear stochastic differential equation. We prove the existence of a solution for this S.D.E., and we use the Malliavin calculus (or stochastic calculus of variations) to prove that the law of this solution admits a smooth density with respect to the Lebesgue measure on $I\!R$. This density satisfies the considered Boltzmann equation.

Une version courte de ce travail a été acceptée pour publication
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1 Introduction.

The Boltzmann equation describes the density of particles in a gas. This equation holds under the following assumptions : the gas is dilute enough (there are only two-particles collisions), the duration of a collision can be ignored, and the collision of two particles does not take into account any correlation between the two particles. We are using $f(t,r,v)$ to denote the density of particles which have the position $r \in I\!R^3$ and the velocity $v \in I\!R^3$ at the instant $t > 0$. The external force is denoted by F and the mass of the system by m. Then the Boltzmann equation can be written

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial r} + \frac{F}{m}.\frac{\partial f}{\partial v} = K^\beta(f,f)$$

where K^β is a collision kernel. We study here a simplified model : the 2-dimensional spatially homogeneous case. This means that we restrict our study to the case where f does not depend on the position of the particles. We furthermore assume that there is no external force, and that the particles take place in the plane. In this case, the Boltzmann equation can be written as follows :

$$\frac{\partial f}{\partial t}(t,v) = K_\beta(f,f)(t,v) \tag{1.1}$$

The collision kernel is given by :

$$K_\beta(f,f)(t,v) = \int_{v^* \in I\!R^2} \int_{-\pi}^{\pi} [f(t,c(v,v^*,\theta))f(t,c^*(v,v^*,\theta)) - f(t,v)f(t,v^*)]$$
$$\beta(\theta, |v-v^*|)d\theta dv^*$$

where, if R_θ is the θ-rotation centered at 0,

$$c(v,v^*,\theta) = \frac{v+v^*}{2} + R_\theta\left(\frac{v-v^*}{2}\right) \quad \text{and} \quad c^*(v,v^*,\theta) = \frac{v+v^*}{2} - R_\theta\left(\frac{v-v^*}{2}\right)$$

We will need the following computation of $c(v,v^*,\theta)$:

$$c(v,v^*,\theta) = \begin{pmatrix} c_x(v,v^*,\theta) \\ c_y(v,v^*,\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (v_x+v_x^*) + (v_x-v_x^*)\cos\theta - (v_y-v_y^*)\sin\theta \\ (v_y+v_y^*) + (v_y-v_y^*)\cos\theta + (v_x-v_x^*)\sin\theta \end{pmatrix}$$

In fact, $c(v,v^*,\theta)$ and $c^*(v,v^*,\theta)$ represent the velocities of two particles after their collision, if these particles had the velocities v and v^* before the collision, and if the angle due to the collision is θ. Consequently, the cross section may explode near $\theta = 0$, because of an accumulation of "grazing" collisions. We will assume that we are in a case of Maxwellian particles, i.e. that the cross section β depends only on θ, and is even : $\beta(\theta, |v-v^*|) = \beta(|\theta|)$. We will also suppose the physically reasonnable condition :

$$\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty \tag{1.2}$$

The Boltzmann equation "with cutoff", namely when $\int_0^\pi \beta(\theta)d\theta < \infty$, has been much investigated by the analysts. It is really more difficult to assume only (1.2), and the only analytical existence and regularity result under (1.2) is due to Desvillettes in [16].

A probabilistic approach using the underlying evolution Markov process allows to work under (1.2) thanks to the L^2-calculus. We obtain a slightly better existence result than Desvillettes, and our regularity result is much better. Desvilletes builds a solution $g(t,v)$ of (1.1), and he proves that for each $t > 0$, $f(t,.)$ is in $H^{1-\epsilon}(I\!R^2)$ for all $\epsilon > 0$. The solution $f(t,v)$ we build is continuous on $]0,T] \times I\!R^2$, and for each $t > 0$, $f(t,.)$ is in $C^\infty(I\!R^2)$.
Another advantage of a probabilistic approach is that we can assume that the initial data is a probability, and not necessarily a density of probability. At last, we give a (probabilistic) notion of uniqueness.

In order to define the weak solutions, we consider the following kernel, which depends on the test function $\phi \in C_b^2(I\!R^2)$ (the set of C^2 function on $I\!R^2$ of which the derivatives of order 0 to

2 are bounded) :

$$\begin{aligned} K_\beta^\phi(v,v^*) &= \int_{-\pi}^{\pi} \Big[\phi(c(v,v^*,\theta)) - \phi(v) - \phi_x'(v)\,(c_x(v,v^*,\theta)-v_x) \\ &\qquad\qquad -\phi_y'(v)\,(c_y(v,v^*,\theta)-v_y)\,\Big]\beta(\theta)d\theta \\ &\quad -\frac{b}{2}\Big[\phi_x'(v)(v_x-v_x^*)+\phi_y'(v)(v_y-v_y^*)\Big] \end{aligned} \tag{1.3}$$

where $b = \displaystyle\int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)d\theta$. This expression is well defined for every test function thanks to the assumption (1.2). Now we can define the weak solutions of (1.1).

Definition 1.1 *Let β be cross section, (even and positive on $[-\pi,\pi]\backslash\{0\}$) satisfying (1.2). Let P_0 be a probability on $I\!R^2$ that admits a moment of order 2. A positive function f on $I\!R^+\times I\!R^2$ is a weak solution of (1.1) with initial data P_0 if for every test function $\phi\in C_b^2(I\!R^2)$,*

$$\int_{v\in I\!R^2} f(t,v)\phi(v)dv = \int_{v\in I\!R^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in I\!R^2}\int_{v^*\in I\!R^2} K_\beta^\phi(v,v^*)f(s,v)f(s,v^*)dvdv^*ds \tag{1.4}$$

Let us explain this definition : a priori, we should look for weak solutions satisfying, for every test function,

$$\int_{v\in I\!R^2} f(t,v)\phi(v)dv = \int_{v\in I\!R^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in I\!R^2} K_\beta(f,f)(s,v)\phi(v)dvds$$

Let us subsitute $v' = c(v,v^*,\theta)$, $v'^* = c^*(v,v^*,\theta)$, and $\theta' = -\theta$ in the first part of $K_\beta(f,f)$. The Jacobian of this substitution is equal to 1, and an easy drawing shows that $v = c(v',v'^*,\theta')$, $v^* = c^*(v',v'^*,\theta')$ and $\theta = -\theta'$. We obtain :

$$\begin{aligned} &\int_{I\!R^2}\int_{I\!R^2}\int_{-\pi}^{\pi} f(t,c(v,v^*,\theta))f(t,c^*(v,v^*,\theta))\phi(v)\beta(\theta)d\theta dvdv^* \\ &\qquad = \int_{I\!R^2}\int_{I\!R^2}\int_{-\pi}^{\pi} f(t,v)f(t,v^*)\phi(c(v,v^*,\theta))\beta(\theta)d\theta dvdv^* \end{aligned}$$

and hence

$$\int_{v\in I\!R^2} f(t,v)\phi(v)dv = \int_{v\in I\!R^2}\phi(v)P_0(dv) + \int_0^t\int_{v\in I\!R^2}\int_{v^*\in I\!R^2} k_\beta^\phi(v,v^*)f(s,v)f(s,v^*)dvdv^*ds$$

where

$$k_\beta^\phi(v,v^*) = \int_{-\pi}^{\pi}\Big[\phi(c(v,v^*,\theta)) - \phi(v)\Big]\beta(\theta)d\theta$$

But this kernel does not make sense for every test function $\phi\in C_b^2(I\!R^2)$, except if we suppose that $\int_0^\pi \theta\beta(\theta)d\theta < \infty$. Consequently, we replace k_β^ϕ by K_β^ϕ, in which there is a compensated term. Notice that if $\int_0^\pi \theta\beta(\theta)d\theta<\infty$, then $\int_{-\pi}^{\pi}\sin\theta\beta(\theta)d\theta = 0$, and the two kernels are identical.

The method is partially adapted from the papers of L. Desvillettes, C. Graham and S. Méléard in [17] and [19], who solved this problem in dimension 1 : we first show that there exists a nonlinear stochastic differential equation associated with the equation (1.1). This means that if V_t is a solution of this S.D.E. and if V_t admits a density for each $t > 0$, then this density will be a weak solution of (1.1). The second section is devoted to the statement of the nonlinear S.D.E., to the existence and the uniqueness of a solution of this S.D.E., and to the study of some moment conservations for this solution, which can be related to physical conservations. The aim of the third section is to use the Malliavin Calculus in order to show the existence of a weak solution of (1.1), and to study the smoothness of this solution. We will use Bismut's approach of the Malliavin Calculus, by following the methods of Bichteler, Gravereaux, and Jacod in [6] and [7]. In the fourth section, we study the joint regularity of our solution. At last, the fifth section is devoted to approximate our weak solution of (1.1) with a simulable particle system, by applying a result of Graham and Méléard in [18].

The most difficult and original part of this paper is the proof of the regularity (see Lemmas 3.26, 3.27, and Theorem 3.28), for which we need to use the particular form of our two-dimensional nonlinear S.D.E.

In the sequel, β is a fixed cross section satisfying (1.2).

2 The probabilistic approach.

The whole section is an easy adaptation of the paper of Desvillettes, Graham and Méléard [17]. Nevertheless, we have to check all the results, because of a quite important difference between the nonlinear S.D.E. in dimension 1 and 2.
Since we are looking for a solution $f(t,v)$ which is a density of particles at each instant t, it is quite natural to relate $f(t,v)$ to the flow of marginals of a stochastic process. We restrict our study to the time interval $[0,T]$, where $T > 0$ is fixed.

Definition 2.1 *We will say that a flow $\{P_t\}_{t\in[0,T]}$ of probability measures on $I\!R^2$ such that P_0 admits a moment of order 2 is a weak solution of (1.1) with initial data P_0 if for every test function $\phi \in C_b^2(I\!R^2)$,*

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_0^t \left\langle K_\beta^\phi(v,v^*), P_s(dv)P_s(dv^*) \right\rangle ds \tag{2.1}$$

Remark 2.2 *If a flow $\{P_t\}_{t\in[0,T]}$ of probability measures on $I\!R^2$ is a weak solution of (1.1), and if for every $t \in]0,T]$, P_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $I\!R^2$, then f is a weak solution of (1.1) with initial data P_0 in the sense of Definition 1.1.*

Definition 2.3 *Let Q be a probability measure on $I\!D_T = I\!D([0,T], I\!R^2)$. We denote by $\{Q_t\}$ its marginal flow, and we assume that Q_0 admits a second order moment. Let X_t be the canonical process on $I\!D_T$. We will say that Q satisfies the nonlinear martingale problem with initial data Q_0 if for every $\phi \in C_b^2(I\!R^2)$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t \left\langle K_\beta^\phi(X_s,v^*), Q_s(dv^*) \right\rangle ds \tag{2.2}$$

is a square integrable Q-martingale.

Remark 2.4 *If Q satisfies the nonlinear martingale problem with initial data P_0, then its marginal flow is a weak solution of (1.1) in the sense of Definition 2.1.*

In order to state a nonlinear S.D.E. associated with the nonlinear martingale problem, we introduce some notations. Following Tanaka, [44], we will consider two probability spaces : the first one is an abstract space $(\Omega, \mathcal{F}, P)$, and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be some α-processes, the expectation under $d\alpha$ will be denoted E_α, and the laws $\mathcal{L}_\alpha$.

On $(\Omega, \mathcal{F}, P)$, we consider a Poisson measure $N(d\theta d\alpha dt)$ on $[-\pi,\pi]\times[0,1]\times[0,T]$ with intensity measure $\nu(d\theta d\alpha dt) = \beta(\theta) d\theta d\alpha dt$ and with compensated measure $\tilde{N}(d\theta d\alpha ds)$.
If Q is a probability on $I\!\!D_T$, and if $p \geq 1$, we will say that $Q \in \mathcal{P}_p(I\!\!D_T)$ if

$$\int_{x\in I\!\!D_T} \sup_{[0,T]} \| x(t) \|^p \, Q(dx) < \infty$$

A càdlàg adapted process Y_s on $[0,T]$ will be a $I\!\!L^p_T$-process if its law belongs to $\mathcal{P}_p(I\!\!D_T)$.

Definition 2.5 *Let $V_0(\omega) \in L^2(\Omega)$, let $Y_s(\omega)$ be a $I\!\!L^2_T$-process, and let $Z_s(\alpha)$ be a $I\!\!L^2_T$-α-process, every of these elements with values in $I\!\!R^2$. Then we denote by $V = \Phi(Y, Z, V_0, N)$ the process defined (and well defined) by*

$$\begin{aligned} V_t(\omega) &= V_0(\omega) + \int_0^t\int_0^1\int_{-\pi}^{\pi} \left[c(Y_{s-}(\omega), Z_{s-}(\alpha), \theta) - Y_{s-}(\omega)\right] \tilde{N}(d\theta d\alpha ds) \\ &\quad -\frac{b}{2}\int_0^t\int_0^1 (Y_s(\omega) - Z_s(\alpha))\, d\alpha ds \end{aligned} \tag{2.3}$$

This can also be written :

$$\begin{aligned} V_t^x &= V_0^x + \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi} \left[(Y_{s-}^x - Z_{s-}^x(\alpha))(\cos\theta - 1) - (Y_{s-}^y - Z_{s-}^y(\alpha))\sin\theta\right] \tilde{N}(d\theta d\alpha ds) \\ &\quad -\frac{b}{2}\int_0^t\int_0^1 (Y_s^x - Z_s^x(\alpha))\, d\alpha ds \end{aligned} \tag{2.4}$$

$$\begin{aligned} V_t^y &= V_0^y + \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi} \left[(Y_{s-}^y - Z_{s-}^y(\alpha))(\cos\theta - 1) + (Y_{s-}^x - Z_{s-}^x(\alpha))\sin\theta\right] \tilde{N}(d\theta d\alpha ds) \\ &\quad -\frac{b}{2}\int_0^t\int_0^1 (Y_s^y - Z_s^y(\alpha))\, d\alpha ds \end{aligned} \tag{2.5}$$

or, by using matrices :

$$V_t = V_0 + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)\,(Y_{s-} - Z_{s-}(\alpha))\,\tilde{N}(d\theta d\alpha ds) - \frac{b}{2}\int_0^t\int_0^1 (Y_s - Z_s(\alpha))\, d\alpha ds \tag{2.6}$$

where $A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}$.

Definition 2.6 *Let $\{V_t\}_{t\in[0,T]}$ be a $\mathbb{L}_T^2$-process and let $\{W_t\}_{t\in[0,T]}$ be a $\mathbb{L}_T^2$-α-process, with values in $\mathbb{R}^2$. We will say that (V,W) is a solution of the nonlinear S.D.E. with initial data V_0 if*

$$\mathcal{L}(V) = \mathcal{L}_\alpha(W) \qquad \text{and} \qquad V = \Phi(V,W,V_0,N)$$

We notice here that this S.D.E. is symmetric in V and W, which is not the case in dimension 1. This yields that the solution of this S.D.E. does not behaves in the same way when the dimension is 1 or 2. In particular the conservation of the momentum (i.e. $E(V_t) = E(V_0)$ for $t > 0$) will hold. Nevertheless, there will not be any fondamental change in this second section.

Remark 2.7 *If (V,W) is a solution of the nonlinear S.D.E. with initial data V_0, then its law $Q = \mathcal{L}(V) = \mathcal{L}_\alpha(W)$ is a solution of the nonlinear martingale problem with initial data $\mathcal{L}(V_0)$.*

Proof : it suffices to apply the Itô formula. If $\phi \in C_b^2(\mathbb{R}^2)$,

$$\begin{aligned}
\phi(V_t) &= \phi(V_0) + \int_0^t \phi'_x(V_{s-})dV_s^x + \int_0^t \phi'_y(V_{s-})dV_s^y \\
&\quad + \sum_{s\le t}\Big[\phi(V_s) - \phi(V_{s-}) - \phi'_x(V_{s-})\Delta V_s^x - \phi'_y(V_{s-})\Delta V_s^y\Big]
\end{aligned}$$

This can be written :

$$\begin{aligned}
\phi(V_t) = \phi(V_0) &- \frac{b}{2}\int_0^t\int_0^1 \Big[\phi'_x(V_s)(V_s^x - W_s^x(\alpha)) + \phi'_y(V_s)(V_s^y - W_s^y(\alpha))\Big]\, d\alpha ds \\
&+ \int_0^t\int_0^1\int_{-\pi}^{\pi} \Big[\phi'_x(V_{s-})\left(c_x(V_{s-},W_{s-}(\alpha),\theta) - V_{s-}^x\right) \\
&\qquad + \phi'_y(V_{s-})\left(c_y(V_{s-},W_{s-}(\alpha),\theta) - V_{s-}^y\right)\Big]\tilde{N}(d\theta d\alpha ds) \\
+ \int_0^t\int_0^1\int_{-\pi}^{\pi} &\Big[\phi(c(V_{s-},W_{s-}(\alpha),\theta)) - \phi(V_{s-}) - \phi'_x(V_{s-})\left(c_x(V_{s-},W_{s-}(\alpha),\theta) - V_{s-}^x\right) \\
&\qquad - \phi'_y(V_{s-})\left(c_y(V_{s-},W_{s-}(\alpha),\theta) - V_{s-}^y\right)\Big] N(d\theta d\alpha ds)
\end{aligned}$$

The expression in the last integral is in $L^1(\nu)\cap L^2(\nu)$. Hence :

$$\begin{aligned}
\phi(V_t) &= \phi(V_0) - \frac{b}{2}\int_0^t\int_0^1 \Big[\phi'_x(V_s)(V_s^x - W_s^x(\alpha)) + \phi'_y(V_s)(V_s^y - W_s^y(\alpha))\Big]\, d\alpha ds \\
&\quad + \int_0^t\int_0^1\int_{-\pi}^{\pi} \left[\phi(c(V_{s-},W_{s-}(\alpha),\theta) - \phi(V_{s-})\right]\tilde{N}(d\theta d\alpha ds) \\
&\quad + \int_0^t\int_0^1\int_{-\pi}^{\pi} \Big[\phi(c(V_s,W_s(\alpha),\theta) - \phi(V_s) - \phi'_x(V_s)\left(c_x(V_s,W_s(\alpha),\theta) - V_s^x\right) \\
&\qquad\qquad - \phi'_y(V_s)\left(c_y(V_s,W_s(\alpha),\theta) - V_s^y\right)\Big]\beta(\theta)d\theta d\alpha ds
\end{aligned}$$

Since $[\phi(c(V_{s-},W_{s-}(\alpha),\theta))-\phi(V_{s-})]$ is in $L^2(P\otimes\nu)$, the second integral is a square integrable martingale M_t. Then an easy computation shows that :

$$\begin{aligned}\phi(V_t) &= \phi(V_0)+M_t+\int_0^t\int_0^1 K_\beta^\phi(V_s,W_s(\alpha))d\alpha ds\\ &= \phi(V_0)+M_t+\int_0^t E_\alpha\left(K_\beta^\phi(V_s,W_s)\right)ds\end{aligned}$$

Finaly, since $\mathcal{L}_\alpha(W_s)=Q_s$, :

$$\phi(V_t)=\phi(V_0)+M_t+\int_0^t\left\langle K_\beta^\phi(V_s,v^*),Q_s(dv^*)\right\rangle ds$$

which was our aim.

2.1 Existence and uniqueness for the nonlinear S.D.E.

Let us first prove a usefull contraction formula for the map Φ (see Definition 2.5) :

Lemma 2.8 *Let Y_s and Y'_s be two $\mathbb{L}_T^2$-processes, let Z_s and Z'_s be two $\mathbb{L}_T^2$-α-processes, and let $V_0\in L^2(\Omega)$, every of these elements with values in $\mathbb{R}^2$. We consider*

$$V=\Phi(Y,Z,V_0,N)\qquad\text{and}\qquad V'=\Phi(Y',Z',V_0,N)$$

These are $\mathbb{L}_T^2$-processes, and the formula below holds (K is a constant depending only on T and β) :

$$E\left(\sup_{[0,t]}\parallel V_s-V'_s\parallel^2\right)\leq K\int_0^t\left[E\left(\parallel Y_s-Y'_s\parallel^2\right)+E_\alpha\left(\parallel Z_s-Z'_s\parallel^2\right)\right]ds \tag{2.7}$$

<u>Proof</u> : let us compute $V^x-V'^x$:

$$\begin{aligned}V_s^x-V'^x_s &= \frac{1}{2}\int_0^s\int_0^1\int_{-\pi}^{\pi}\Big([(Y_{u-}^x-Y'^x_{u-})-(Z_{u-}^x-Z'^x_{u-})]\times(\cos\theta-1)\\ &\qquad -[(Y_{u-}^y-Y'^y_{u-})-(Z_{u-}^y-Z'^y_{u-})]\times\sin\theta\Big)\tilde{N}(d\theta d\alpha ds)\\ &\quad -\frac{b}{2}\int_0^s[(Y_u^x-Y'^x_u)-E_\alpha(Z_u^x-Z'^x_u)]\,du\end{aligned}$$

Using Doob's and Cauchy-Schwarz's inequalities, we get

$$\begin{aligned}E\left(\sup_{[0,t]}(V_s^x-V'^x_s)^2\right) &\leq 8\int_0^t\int_0^1\int_{-\pi}^{\pi}E\Big(\Big\{[(Y_{u-}^x-Y'^x_{u-})-(Z_{u-}^x-Z'^x_{u-})]\times(\cos\theta-1)\\ &\qquad -[(Y_{u-}^y-Y'^y_{u-})-(Z_{u-}^y-Z'^y_{u-})]\times\sin\theta\Big\}^2\Big)\beta(\theta)d\theta d\alpha du\\ &\quad +\frac{b^2}{2}t\int_0^t E\left([(Y_u^x-Y'^x_u)-E_\alpha(Z_u^x-Z'^x_u)]^2\right)du\end{aligned}$$

Using Jensen's inequality for E_α, using $|\cos\theta - 1| + |\sin\theta| \in L^2(\beta(\theta)d\theta)$, and the same computation for $V_s^y - V'^y_s$, the contraction formula is immediate.

Now we solve the classical S.D.E. associated with the nonlinear S.D.E.

Proposition 2.9 *Let $V_0 \in L^2(\Omega)$, and let Z be a $\mathbb{L}^2_T$-α-process. Then the classical S.D.E. $V = \Phi(V, Z, V_0, N)$ admits a unique solution, that belongs to $\mathbb{L}^2_T$. Furthermore, the law of the solution depends only on $\mathcal{L}(V_0)$ and on the flow $\{\mathcal{L}_\alpha(Z_t)\}_{t\in[0,T]}$.*

Proof : the existence and the uniqueness for this kind of S.D.E. is classical, and follows from Lemma 2.8. In order to study the law of the solution, let us write the Poisson measure as $N = \sum_{s\in[0,T]} \mathbb{1}_D(s)\delta_{(\theta_s,\alpha_s,s)}$, and let us set $N^* = \sum_{s\in[0,T]} \mathbb{1}_D(s)\delta_{(\theta_s,Z_s(\alpha_s),s)}$. Then N^* is a Poisson measure on $[0,T]\times[-\pi,\pi]\times\mathbb{R}^2$ with intensity $\beta(\theta)d\theta\mathcal{L}_\alpha(Z_s)(dz)ds$. (Recall that Z_t is "ω-deterministic"). Then

$$V_t = V_0 + \int_0^t \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} (c(V_{s-}, z, \theta) - V_{s-})\, \tilde{N}^*(d\theta dz ds) - \frac{b}{2}\int_0^t V_s ds + \frac{b}{2}\int_0^t E_\alpha(Z_s)ds$$

and the law of V_t is entirely determined by $\mathcal{L}(V_0)$, by the intensity of N^*, and by $\{E_\alpha(Z_s)\}_{s\le T}$. The result follows.

We now define recursively the Picard iterations that will converge to a solution of the nonlinear S.D.E.

Definition 2.10 *Let $V_0 \in L^2$. Let V^0 be the process identically equal to V_0. Assuming that we have defined the $\mathbb{L}^2_T$-processes V^0,...,V^k, and the $\mathbb{L}^2_T$-α-processes Z^0,...,Z^{k-1}, we choose a $\mathbb{L}^2_T$-α-process Z^k satisfying*

$$\mathcal{L}_\alpha(Z^k|Z^{k-1},...,Z^0) = \mathcal{L}(V^k|V^{k-1},...,V^0)$$

then we set

$$V^{k+1} = \Phi(V^k, Z^k, V_0, N)$$

The following theorem shows the existence of a solution for the nonlinear S.D.E. (and hence for the nonlinear martingale problem), and begins the uniqueness.

Theorem 2.11 *Let $V_0 \in L^2(\Omega)$ be fixed. The sequences V^k and Z^k converge a.s. and in $\mathbb{L}^2_T$ to some processes V and W. The process V is in $\mathbb{L}^2_T$, and W is a $\mathbb{L}^2_T$-α-process. Furthermore,*

$$\mathcal{L}(V) = \mathcal{L}_\alpha(W) = P^\beta \qquad \text{and} \qquad V = \phi(V, W, V_0, N)$$

Hence (V, W) is a solution of the nonlinear S.D.E. with initial data V_0, and P^β is a solution of the nonlinear martingale problem with initial data $\mathcal{L}(V_0)$.
Furthermore, the law P^β does not depend on the possible choices for Ω, for N, for V_0, and for the Picard approximations, but only on $\mathcal{L}(V_0)$.

Proof : the convergence (we show that these sequences are Cauchy) is easy by using Lemma 2.8, and since for every k, $\mathcal{L}_\alpha(Z^k - Z^{k-1}) = \mathcal{L}(V^k - V^{k-1})$. Letting k go to infinity in the equality $V^{k+1} = \Phi(V^k, Z^k, V_0, N)$, we see that $V = \Phi(V, W, V_0, N)$. At last, $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ because the

sequences $\{V^k\}$ and $\{Z^k\}$ have the same law, and because the processes V^k and Z^k converge uniformly in L^2.
As in Proposition 2.9, we can check that the law of the sequence $\{V^k\}$ does not depend on the choices for Ω, N, V_0, and $\{Z^k\}$, but only on the laws of these elements.

We thus have built a solution for the nonlinear martingale problem with initial data P_0. Furthermore, we have seen that the law of this solution does not depend on the choices for the construction. We now prove the uniqueness in law for the nonlinear S.D.E. : it suffices to consider a fixed "space" (Ω, V_0, N), and to check that any solution of the nonlinear S.D.E. on this space have the law P^β.

Theorem 2.12 *Let Ω, $V_0 \in L^2(\Omega)$, and N be fixed. We consider the solution (V, W) (with $P^\beta = \mathcal{L}(V) = \mathcal{L}_\alpha(W)$) of the nonlinear S.D.E. that we have built in Theorem 2.11. We also assume that there exists another solution (U, Y), and we set $Q = \mathcal{L}(U) = \mathcal{L}_\alpha(Y)$. Then $Q = P^\beta$.*

Proof : we know that Q and P^β are in $\mathcal{P}_2(\mathbb{D}_T)$. Hence, for $\tau \in [0, T]$, we consider

$$\rho_\tau(P^\beta, Q) = \inf \left\{ E_\alpha \left(\sup_{[0,\tau]} \| W'_t - Y'_t \|^2 \right)^{\frac{1}{2}} \Big/ \mathcal{L}_\alpha(W') = P^\beta \quad et \quad \mathcal{L}_\alpha(Y') = Q \right\}$$

We will show that if $\tau_0 > 0$ is small enough, $\rho_{\tau_0}(P^\beta, Q) = 0$, which will yield that the restrictions of P^β and Q to $\mathbb{D}_{\tau_0}$ are identical. To this end, let us fix $\epsilon > 0$. There exist some α-processes W^ϵ and Y^ϵ of which the α-laws are respectively P^β and Q, and satisfying

$$\rho_\tau^2(P^\beta, Q) \le E_\alpha \left(\sup_{[0,\tau]} \| W_t^\epsilon - Y_t^\epsilon \|^2 \right) \le \rho_\tau^2(P^\beta, Q) + \epsilon$$

By Proposition 2.9, we can define $V^\epsilon = \Phi(V^\epsilon, W^\epsilon, V_0, N)$ and $U^\epsilon = \Phi(U^\epsilon, Y^\epsilon, V_0, N)$. Then, by using Lemma 2.8,

$$\begin{aligned} E\left(\sup_{[0,\tau]} \| V_s^\epsilon - U_s^\epsilon \|^2 \right) &\le K \int_0^\tau \left(E\left(\| V_s^\epsilon - U_s^\epsilon \|^2 \right) + E_\alpha \left(\| W_s^\epsilon - Y_s^\epsilon \|^2 \right) \right) ds \\ &\le K \int_0^\tau E\left(\| V_s^\epsilon - U_s^\epsilon \|^2 \right) ds + K \times \tau \left(\rho_\tau^2(P^\beta, Q) + \epsilon \right) \end{aligned}$$

Using Gronwall's Lemma, we obtain

$$E\left(\sup_{[0,\tau]} \| V_s^\epsilon - U_s^\epsilon \|^2 \right) \le \tau \left(\rho_\tau^2(P^\beta, Q) + \epsilon \right) \times K e^{KT} = C\tau \left(\rho_\tau^2(P^\beta, Q) + \epsilon \right)$$

But, since $\mathcal{L}_\alpha(W^\epsilon) = P^\beta = \mathcal{L}_\alpha(W)$, Proposition 2.9 yields that $\mathcal{L}(V^\epsilon) = P^\beta$. In the same way we get $\mathcal{L}(U^\epsilon) = Q$, and hence

$$\rho_\tau^2(P^\beta, Q) \le E\left(\sup_{[0,\tau]} \| V_s^\epsilon - U_s^\epsilon \|^2 \right) \le C\tau \left(\rho_\tau^2(P^\beta, Q) + \epsilon \right)$$

So we choose $\tau_0 > 0$ such that $C\tau_0 < 1$. Since ϵ is arbitrary, we deduce that $\rho_{\tau_0}(P^\beta, Q) = 0$, and thus that the restrictions of P^β and Q to $\mathbb{D}_{\tau_0}$ are identical.

We still have to iterate this result. If X is a process, we set $X_t^n = X_{t+n\tau_0}$. Then for every n, (V^n, W^n) (resp. (U^n, Y^n)) is a solution of the nonlinear S.D.E. with initial data $V_0^n = V_{n\tau_0}$ (resp. $U_0^n = U_{n\tau_0}$) and with Poisson measure $N^n - N_{n\tau_0}$. If we assume that V and U have the same law on $[0, n\tau_0]$, then $\mathcal{L}(V_0^n) = \mathcal{L}(U_0^n)$, and the first step of the proof shows that $\mathcal{L}(V^n) = \mathcal{L}(U^n)$ on $[0, \tau_0]$. Hence the laws of U and V are identical on $[0, (n+1)\tau_0]$, and we can conclude recursively.

2.2 Some conservations for the solution of the nonlinear S.D.E.

In this short subsection, Ω, N, and $V_0 \in L^2(\Omega)$ are fixed. We consider a solution (V, W) of the nonlinear S.D.E. with initial data V_0.

Proposition 2.13 *The conservations of momentum and of kinetic energy hold : for every $t \in [0,T]$,*

$$E(V_t) = E(V_0) \qquad \text{and} \qquad E\left(\| V_t \|^2\right) = E\left(\| V_0 \|^2\right)$$

Notice that the conservation of momentum does not hold in dimension 1.

Proof : in order to prove these equalities, it suffices to use the fact that the flow $P_t = \mathcal{L}(V_t)$ is a weak solution of (1.1) in the sense of Definition 2.1. Let us first consider the test function $\phi(v) = v_x$: it is easy to check that

$$K_\beta^\phi(v, v^*) = 0 - \frac{b}{2}(v_x - v_x^*)$$

Hence for every $s > 0$,

$$\left\langle K_\beta^\phi(v, v^*), P_s(dv)P_s(dv^*)\right\rangle = 0$$

and we obtain $\int_{I\!R^2} v_x P_t(dv) = \int_{I\!R^2} v_x P_0(dv)$. In the same way, $\int_{I\!R^2} v_y P_t(dv) = \int_{I\!R^2} v_y P_0(dv)$, and the conservation of momentum is proved.

Then we set $\phi(v) = v_x^2 + v_y^2$: since $K_\beta^\phi(v, v^*) = \frac{b}{2}\left(v_x^{*2} - v_x^2 + v_y^{*2} - v_y^2\right)$, it is clear that for every $s > 0$, $\left\langle K_\beta^\phi(v, v^*), P_s(dv)P_s(dv^*)\right\rangle = 0$, and we can conclude as above that the conservation of kinetic energy holds.

We now deduce a usefull corollary :

Corollary 2.14 *If $\mathcal{L}(V_0)$ is not a Dirac mass, then for every $t \in [0,T]$, $\mathcal{L}(V_t)$ is not a Dirac mass either.*

Proof : let us assume that there exists $t > 0$ and $X \in I\!R^2$ such that $\mathcal{L}(V_t) = \delta_X$. Then

$$\begin{aligned} 0 &= E\left(\| V_t - X \|^2\right) = E(\| V_t \|^2) + \| X \|^2 - 2\langle X, E(V_t)\rangle \\ &= E(\| V_0 \|^2) + \| X \|^2 - 2\langle X, E(V_0)\rangle = E\left(\| V_0 - X \|^2\right) \end{aligned}$$

which implies that $V_0 = X$ a.s.

We now state a standard proposition :

Proposition 2.15 *If we assume that V_0 admits moments of all orders, then for every $p \geq 1$, V is a $\mathbb{L}^p_T$-process, i.e.* $E\left(\sup_{[0,T]} \| V_t \|^p\right) < \infty$.

This is classical : it suffices to show that the Picard approximations of the definition 2.10 are in fact Cauchy in $\mathbb{L}^p_T$, by using the Appendix (6.4).

3 Existence and smoothness of a weak solution.

We now want to study the existence and the smoothness of a density with respect to the Lebesgue measure on $\mathbb{R}^2$ for the law of a solution of the nonlinear S.D.E. Indeed, if this density exists, it will satisfy (1.1) in the sense of Definition 1.1. We thus will use the stochastic calculus of variations (namely the Malliavin calculus). Bismut's methods are here easier than Malliavin's original approach. The papers of Bichteler, Jacod [7] and of Bichteler, Gravereaux, Jacod [6] explain the Malliavin calculus for diffusion processes with jumps in the case where the intensity measure of the Poisson measure is the Lebesgue measure ; and although we cannot apply directly their results, we will follow their methods. In [7], Bichteler and Jacod study the existence of a density for these processes in dimension 1, and Bichteler Gravereaux and Jacod extend in [6] the methods to the existence and the smoothness of this density in any finite dimension. This second paper is very complete, but the assumptions that yield the existence of a density are too much stringent, so that we have to use a mixed method to show the existence of a weak solution of (1.1).

First, let us state our assumptions.

Assumption (H) :

1. The initial data P_0 admits a moment of order 2, and is not a Dirac mass.
2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi, \pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0, \pi[$, and $r \in]1, 3[$ such that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r} \mathbb{1}_{[-\theta_0,\theta_0]}(\theta)$.
   We still assume $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$.

Assumption (S) :

1. All the moments of P_0 are finite.
2. The cross section β satisfies :

$$\left|\frac{\sin\theta}{1+\cos\theta}\right| \mathbb{1}_{|\theta|\in[\pi/2,\pi]} \in \cap_{p\geq 1} L^p(\beta(\theta)d\theta)$$

Then we state our main theorems.

Theorem 3.1 *Under the assumption (H), the equation (1.1) admits a weak solution with initial data P_0 in the sense of Definition 1.1.*

Theorem 3.2 *We assume (H) and (S), and we consider the weak solution $f(t, v)$ of the equation (1.1) with initial data P_0 built in Theorem 3.1. Then for each $t \in]0, T]$ fixed, $f(t, .)$ is of class C^∞ on $\mathbb{R}^2$.*

Let us notice that Assumption (H)-1 is natural. Indeed, if P_0 is a Dirac mass at $v_0 \in I\!R^2$, then all the particles have the initial velocity v_0, and there cannot be any collision. Hence $P_t = P_0$ for all t is a solution of (1.1) in the sense of Definition 2.1, and it is clear that in this case, P_t does not admit any density.
(H)-2 means that β sufficiently explodes at 0, and contains a regular part. We have chosen β_0 according to a physical behaviour.

If the angle of a collision between two particles is π, then these particles exchange their velocities, and this has no effect on the density $f(t,.)$. Thus if P_0 does not admit any density, and if $\beta(\theta)$ is concentrated near π, there cannot be any regularization property. Assumption (S)-2 means that β is very small near π.

In [16], the analyst Desvillettes states a comparable theorem under the following assumptions (here the initial data is a density of probability) :
Assumption (h1) : The cross section β is in $L^\infty_{loc}(]0,\pi[)$, and there exists $\beta_0 > 0$, $\beta_1 > 0$, and $\gamma \in]1,3[$ such that :

$$\beta_0|\theta|^{-\gamma} \leq \beta(\theta) \leq \beta_1|\theta|^{-\gamma}$$

Assumption (h2) : The initial data $f_0 \; : \; I\!R^2 \to I\!R^+$ satisfies :

$$\int_{I\!R^2} f_0(v)\left(1+|v|^2+|\ln f_0(v)|\right) dv < \infty$$

Theorem : *Under $(h1)$ and $(h2)$, the Boltzmann equation (1.1) admits a weak solution f satisfying, for every $t_0 > 0$, $\epsilon > 0$:*

$$f \in L^1_{loc}\left([t_0,\infty[, H^{1-\epsilon}(I\!R^2_v)\right) \cap L^\infty_{loc}\left([t_0,\infty[, H^{\frac{3-\gamma}{2}-\epsilon}(I\!R^2_v)\right)$$

Comparing this Theorem and Theorem 3.2 (see also Theorem 4.1 in nest section), we see how the probabilistic approach is efficient.

Let us come back to our method.

Notation 3.3 *In the whole section, Ω and N are fixed as in Section 2, and we assume at least (H). We also consider on Ω a random variable V_0 such that $\mathcal{L}(V_0) = P_0$, and a solution (V,W) of the nonlinear S.D.E. with initial data V_0 in the sense of Definition 2.6.*

3.1 The techniques.

The Malliavin Calculus is based on the *integration by parts settings* (IBPS). Of course, the IBPS needed for the existence of a density (which we will name *weak IBPS*) are less stringent than the ones used for the smoothness of the density.
In the next definition, we follow [6] p 27, and we introduce the weak IBPSs. Recall that $C^2_b(I\!R^d)$ (resp. $C^2_p(I\!R^d)$) is the set of C^2 functions on $I\!R^d$ of which all derivatives of order 0 to 2 are bounded (resp. have at most a polynomial growth).

Definition 3.4 *Let ϕ be a random variable with values in $I\!R^2$. We will say that $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS (resp. a weak IBPS) for ϕ if*

1. *σ is a random variable with values in $\mathcal{M}_2(I\!R)$ (the set of the 2×2-matrices on $I\!R$).*
2. *γ is a random variable with values in $I\!R^2$ such that $\gamma \in \cap_{p<\infty} L^p$ (resp. $\gamma \in L^2$).*
3. *$\mathcal{D}$ is a linear space of random variables contained in $\cap_{p<\infty} L^p$ (resp. L^2), and is stable under C_p^2 (resp. C_b^2).*
4. *$\delta = (\delta_1, \delta_2)$, where δ_i is a linear map on $\mathcal{D}$ such that if $n \geq 1$, if $F \in C_p^2(I\!R^n)$ (resp. $C_b^2(I\!R^n)$), and if $\psi = (\psi_1, ..., \psi_n) \in \mathcal{D}^n$, then*
$$\delta_i(F \circ \psi) = \sum_{j=1}^{n} d_j F(\psi) \delta_i(\psi_j)$$
5. *For every $g \in C_p^2(I\!R^2)$ (resp. $C_b^2(I\!R^2)$), for every $\psi \in \mathcal{D}$, for $j = 1, 2$ the following equality holds :*
$$E\left(\psi \sum_{i=1}^{2} d_i g(\phi) \sigma^{ij}\right) = E\left(g(\phi)[\psi \gamma^j + \delta_j(\psi)]\right) \tag{3.1}$$

We will use the following criteria :

Theorem 3.5 *Let ϕ be a random variable with values in $I\!R^2$. Assume that $(\sigma, \gamma, \mathcal{D}, \delta)$ is a weak IBPS for ϕ. If for each $i, j \in \{1, 2\}$, σ^{ij} is in $\mathcal{D}$, and if $\det \sigma \neq 0$ a.s., then the law of ϕ admits a density with respect to the Lebesgue measure on $I\!R^2$.*

Theorem 3.6 *Let ϕ be a random variable with values in $I\!R^2$. We assume that $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS for ϕ, and we consider the following sets :*

$$C_0 = \left\{\sigma^{ij}, \gamma^i \mid i, j \in \{1, 2\}\right\} \quad \text{and} \quad C_{n+1} = C_n \cup \{\delta_j(\psi) \mid j \in \{1, 2\},\ \psi \in C_n\}$$

Then ϕ admits a density of class C^∞ with respect to the Lebesgue measure on $I\!R^2$ provided for all $n \geq 0$, $C_n \subset \mathcal{D}$, and $(\det \sigma)^{-1} \in \cap_{p<\infty} L^p$.

Theorem 3.6 is proved in Bichteler, Gravereaux, Jacod [6] p 33, and Theorem 3.5 is also proved in [6] p 28 in the case where $(\sigma, \gamma, \mathcal{D}, \delta)$ is an IBPS for ϕ. But it is easy to see that they use only the fact $(\sigma, \gamma, \mathcal{D}, \delta)$ is a weak IBPS.

3.2 The perturbation.

The existence of the density for the law of a jump process is based on an accumulation of small jumps. Recalling that $\beta = \beta_0 + \beta_1$ and that β_0 explodes near 0, we will in fact be interested only in β_0. Hence, we suppose that the Poisson measure N splits into $N_0 + N_1$, where N_0 and N_1 are independent Poisson measures on $[0, T] \times [0, 1] \times [-\pi, \pi]$ with intensity measures

$$\nu_0(d\theta d\alpha ds) = \beta_0(\theta) d\theta d\alpha ds \quad ; \quad \nu_1(d\theta d\alpha ds) = \beta_1(\theta) d\theta d\alpha ds$$

We will denote by $\tilde{N}_0$ and $\tilde{N}_1$ the associated compensated measures. We also assume that our probability space is the canonical one associated with the independent random elements V_0, N_0, and N_1 :

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P) = (\Omega', \mathcal{F}', \{\mathcal{F}'\}, P') \otimes (\Omega^0, \mathcal{F}^0, \{\mathcal{F}^0_t\}, P^0) \otimes (\Omega^1, \mathcal{F}^1, \{\mathcal{F}^1_t\}, P^1) \tag{3.2}$$

An element $\omega \in \Omega$ can be written $\omega = (\omega', \omega^0, \omega^1)$, where ω' is a real number, and ω^0 and ω^1 are integer valued measures on $[0,T] \times [0,1] \times [-\pi, \pi]$.

Notation 3.7 *Although N_0 has its support in $[0,T] \times [0,1] \times [-\theta_0, \theta_0]$, we will still integrate against N_0 and $\tilde{N}_0$ on $[0,T] \times [0,1] \times [-\pi, \pi]$, even if the functions in the integrals are defined only on $[0,T] \times [0,1] \times [-\theta_0, \theta_0]$.*

Let us briefly present the method we will use to build an I.B.P.S. for V_t. We will first build a *perturbation*, in order to obtain a new family of integer valued random measures N_0^λ (for $\lambda \in \Lambda$, where Λ is a neighbourhood of 0 in $I\!R^2$). Of course, N_0^0 must equal N_0. Then we will build a family of probability measures $P^\lambda = G_t^\lambda.P$ on Ω, such that $\mathcal{L}(V_0, N_0^\lambda, N_1 | P^\lambda) = \mathcal{L}(V_0, N_0, N_1 | P)$. This way, we will obtain a perturbed process V_t^λ satisfying $\mathcal{L}(V_t^\lambda | P^\lambda) = \mathcal{L}(V_t | P)$, and thus $E(\phi(V_t^\lambda) G_t^\lambda) = E(\phi(V_t))$ for any borel bounded function ϕ on $I\!R^2$. Then we will differentiate this equality at $\lambda = 0$ (if ϕ is regular enough), by using a L^2-*derivative* of V_t^λ and G_t^λ. We will obtain something like

$$E\left((\phi'_x(V_t)\ \phi'_y(V_t)).DV_t\right) = -E(\phi(V_t) DG_t)$$

which looks like (3.1).

We now build the perturbation. Let ρ be a positive $C_b([-\theta_0, \theta_0])$ function satisfying :

$$\rho(\theta) \leq \left(ce^{-|\theta|^{-r'}}\right) \wedge \frac{|\theta|}{2} \wedge M \ ; \quad \rho(\theta) \overset{0}{\sim} ce^{-|\theta|^{-r'}} \quad ; \quad \{\rho = 0\} = \{-\theta_0, 0, \theta_0\} \tag{3.3}$$

where

$$r' = \frac{1}{8}(r-1) > 0 \quad ; \quad c = \frac{1}{4}\left(r 2^{r+1} \sup_{[0,\theta_0]} \left[\theta^{-1} e^{-|\theta|^{-r'}}\right]\right)^{-1} > 0 \quad \text{and} \quad M = \frac{1}{5} \tag{3.4}$$

In particular, we see that $\rho \in \cap_{p \geq 1} L^p(\beta_0(\theta) d\theta)$.

We also need a predictable function $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ from $\Omega \times [0,T] \times [-\theta_0, \theta_0] \times [0,1]$ to $I\!R^2$, such that for every ω, t, α, the map $\theta \longrightarrow v(\omega, t, \theta, \alpha)$ is of class C^1, and such that

$$\| v(\omega, t, \theta, \alpha) \| \vee \| v'(\omega, t, \theta, \alpha) \| \leq \rho(\theta) \tag{3.5}$$

where $v' = \begin{pmatrix} v'_x \\ v'_y \end{pmatrix}$ is the derivative of v with respect to θ. This function will be chosen at the end of the section.

We consider a neighbourhood $\Lambda \subset B(0,1)$ of 0 in $I\!R^2$. For $\lambda \in \Lambda$, we define the following *perturbation* :

$$\gamma^\lambda(\omega,t,\theta,\alpha) = \theta + \langle \lambda, v(\omega,t,\theta,\alpha)\rangle = \theta + \lambda_x v_x(\omega,t,\theta,\alpha) + \lambda_y v_y(\omega,t,\theta,\alpha) \qquad (3.6)$$

If Λ is small enough (which we assume), we can check that for every $\lambda \in \Lambda$, for every ω, t, α, the map $\theta \longrightarrow \gamma^\lambda(\omega,t,\theta,\alpha)$ is an increasing bijection from $[-\theta_0,\theta_0]$ into itself (by using $\| v' \| \leq \rho \leq M$ and $\rho(-\theta_0) = \rho(\theta_0) = 0$).

For $\lambda \in \Lambda$, we set $N_0^\lambda = \gamma^\lambda(N_0)$: if $A \subset [0,T] \times [0,1] \times [-\pi,\pi]$ is a Borel set,

$$N_0^\lambda(\omega, A) = \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1\!\!1_A(s, \gamma^\lambda(\omega,s,\theta,\alpha),\alpha) N_0(\omega, d\theta d\alpha ds)$$

We consider the shift S^λ defined (and entirely defined) by

$$V_0 \circ S^\lambda(\omega) = V_0(\omega)\ , \quad N_0 \circ S^\lambda(\omega) = N_0^\lambda(\omega)\ , \quad \text{and} \quad N_1 \circ S^\lambda(\omega) = N_1(\omega) \qquad (3.7)$$

We now look for a family of probability measures P^λ on Ω satisfying $P^\lambda \circ (S^\lambda)^{-1} = P$. To this end, we consider the following predictable real valued function on $\Omega \times [0,T] \times [-\theta_0,\theta_0] \times [0,1]$

$$Y^\lambda(\omega,t,\theta,\alpha) = \Big(1 + \lambda_x v'_x(\omega,t,\theta,\alpha) + \lambda_y v'_y(\omega,t,\theta,\alpha)\Big) \times \frac{\beta_0(\gamma^\lambda(\omega,t,\theta,\alpha))}{\beta_0(\theta)} \qquad (3.8)$$

If $\tilde\rho(\theta) = \rho(\theta) + r2^{r+1}\dfrac{\rho(\theta)}{|\theta|} + r2^{r+1}\rho(\theta)\dfrac{\rho(\theta)}{|\theta|}$, then

$$|Y^\lambda(t,\theta,\alpha) - 1| \leq \| \lambda \| \, \tilde\rho(\theta) \qquad (3.9)$$

Let us notice that $\tilde\rho \in \cap_{p\geq 1} L^p(\beta_0(\theta)d\theta)$, and that thanks to our choices for c and M, $\tilde\rho \leq \frac{1}{2}$. Then we consider the following square integrable Doléans-Dade martingale :

$$G_t^\lambda = 1 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} G_{s-}^\lambda (Y^\lambda(s,\theta,\alpha) - 1)\tilde N_0(d\theta d\alpha ds) \qquad (3.10)$$

Proposition 3.8 *G_t^λ is strictly positive for every $t \in [0,T]$. If P^λ is the probability measure defined by $P^\lambda = G_T^\lambda.P$, then $P^\lambda \circ (S^\lambda)^{-1} = P$.*

<u>Proof</u> : recall that if

$$M_t^\lambda = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha) - 1)\tilde N_0(d\theta d\alpha ds)$$

then, (see Jacod, Shiryaev, [23], p 59),

$$G_t^\lambda = e^{M_t^\lambda} \prod_{s \leq t} (1 + \Delta M_s^\lambda) e^{-\Delta M_s^\lambda}$$

Since $|Y^\lambda(s,\theta,\alpha) - 1| \leq \tilde\rho(\theta) \leq 1/2$ for $|\theta| \leq \theta_0$, the jumps of M^λ are greater than $-1/2$, and G_t^λ is strictly positive.

In order to prove the equality between the probability measures, recall the definition of the shift S^λ : we just have to show that $\mathcal{L}(N_0^\lambda|P^\lambda) = \mathcal{L}(N_0|P)$, $\mathcal{L}(N_1|P^\lambda) = \mathcal{L}(N_1|P)$, and $\mathcal{L}(V_0|P^\lambda) = \mathcal{L}(V_0|P)$.

Using the Girsanov Theorem (see the Appendix 6.5-1), with $P^\lambda|_{\mathcal{F}_t} = G_t^\lambda.P|_{\mathcal{F}_t}$, and noticing that

$$M_{P\otimes N_0}\left(\frac{G^\lambda}{G_-^\lambda}\mathbb{1}_{\{G_-^\lambda>0\}}\middle|\tilde{\mathcal{P}}\right) = Y^\lambda$$

we see that the compensator of N_0 under P^λ is $Y^\lambda.\nu_0$. Hence, the compensator of N_0^λ under P^λ is $\gamma^\lambda(Y^\lambda.\nu_0)$. But Y^λ has been chosen such that $\gamma^\lambda(Y^\lambda.\nu_0) = \nu_0$: let A be a Borel subset of $[0,T]\times[-\pi,\pi]\times[0,1]$. Then,

$$\gamma^\lambda(Y^\lambda.\nu_0)(A) = \iiint \mathbb{1}_A(t,\gamma^\lambda(t,\theta,\alpha),\alpha)Y^\lambda(t,\theta,\alpha)\beta_0(\theta)d\theta d\alpha dt$$

Let us subsitute $\theta' = \gamma^\lambda(t,\theta,\alpha)$, which implies $\beta_0(\theta')d\theta' = Y^\lambda(t,\theta,\alpha)\times\beta_0(\theta)d\theta$. We obtain

$$\gamma^\lambda(Y^\lambda.\nu_0)(A) = \iiint \mathbb{1}_A(t,\theta',\alpha)\beta_0(\theta')d\theta' d\alpha dt = \nu_0(A)$$

Hence, using the Appendix (6.5-2), $\mathcal{L}(N_0^\lambda|P^\lambda) = \mathcal{L}(N_0|P)$.

Since N_0 and N_1 are independent, they do never jump at the same time a.s. Thus, one can check that

$$M_{P\otimes N_1}\left(\frac{G^\lambda}{G_-^\lambda}\mathbb{1}_{\{G_-^\lambda>0\}}\middle|\tilde{\mathcal{P}}\right) = 1$$

and we can conclude that $\mathcal{L}(N_1|P^\lambda) = \mathcal{L}(N_1|P)$ as above.

At last, since V_0 is $\mathcal{F}_0$-measurable, it is easy to check that for every $\phi\in C_b(\mathbb{R}^2)$, $E^\lambda(\phi(V_0)) = E(\phi(V_0))$, and we finally obtain $\mathcal{L}(V_0|P^\lambda) = \mathcal{L}(V_0|P)$.

We now introduce the following derivatives :

Definition 3.9 *Recall that Λ is a neighbourhood of 0 in $\mathbb{R}^2$.*

1. *Let $\{X^\lambda\}_{\lambda\in\Lambda}$ be a family of real valued L^p random variables, for some $p\geq 2$. We will say that X^λ is L^p-differentiable at $\lambda=0$ if there exists a* **derivative** $DX = \begin{pmatrix} D^xX \\ D^yX\end{pmatrix}\in L^p$ *such that, when λ goes to 0,*

$$E\left(\left|X^\lambda - X^0 - \langle\lambda, DX\rangle\right|^p\right) = o\left(\|\lambda\|^p\right)$$

2. *Let $\{X^\lambda\}_{\lambda\in\Lambda}$ be a family of $\mathbb{R}^2$ valued L^p random variables, for some $p\geq 2$. We will say that X^λ is L^p-differentiable at $\lambda=0$ if there exists a derivative $DX = \begin{pmatrix} D^xX^x & D^yX^x \\ D^xX^y & D^yX^y\end{pmatrix}\in L^p$ such that, when λ goes to 0,*

$$E\left(\| X^\lambda - X^0 - DX.\lambda\|^p\right) = o\left(\|\lambda\|^p\right)$$

3. *We denote by* $\mathcal{D}$ *(resp.* $\mathcal{D}^\infty$*) the set of the real valued random variables* X *such that* $X^\lambda = X \circ S^\lambda$ *is* L^2*-differentiable (resp.* L^p*-differentiable for every* $p < \infty$*) at* 0*, and by* $\mathcal{D}_t$ *(resp.* $\mathcal{D}_t^\infty$*) its restriction to the set of the* $\mathcal{F}_t$*-measurable random variables.*

4. *Let now* $\{Y_t^\lambda\}_{\lambda\in\Lambda}$ *be a family of real valued* $\mathbb{L}_T^p$*-processes, for some* $p \geq 2$*. We will say that* Y^λ *is* L^p*-differentiable at* $\lambda = 0$ *if there exists a* $\mathbb{L}_T^p$*-process* $DY_t = \begin{pmatrix} D^x Y_t \\ D^y Y_t \end{pmatrix}$ *such that :*
$$E\left(\sup_{[0,T]} |Y_t^\lambda - Y_t^0 - \langle \lambda, DY_t\rangle |^p\right) = o\left(\| \lambda \|^p\right)$$

3.3 The perturbed equation.

We now describe the process $V_t^\lambda = V_t \circ S^\lambda$. The α-process W behaves here as a parameter.

Proposition 3.10 *The perturbed process* V^λ *satisfies the following equation under* P *(i.e.* $\tilde{N}_0 = N_0 - \nu_0$*) :*

$$\begin{aligned} V_t^\lambda &= V_0 - \frac{b}{2}\int_0^t \int_0^1 (V_s^\lambda - W_s(\alpha))d\alpha ds \qquad (3.11) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_1(d\theta d\alpha ds) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_0(d\theta d\alpha ds) \\ &- \int_0^t \int_0^1 \int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha) - 1)A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\beta_0(\theta)d\theta d\alpha ds \end{aligned}$$

<u>Proof</u> : we work here under P. The direct expression of V^λ is given by

$$\begin{aligned} V_t^\lambda &= V_0 - \frac{b}{2}\int_0^t \int_0^1 (V_s^\lambda - W_s(\alpha))d\alpha ds + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_1(d\theta d\alpha ds) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))(N_0^\lambda - \nu_0)(d\theta d\alpha ds) \end{aligned}$$

But the last term is equal to

$$\int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\tilde{N}_0(d\theta d\alpha ds)$$
$$- \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-}^\lambda - W_{s-}(\alpha))(\nu_0 - \gamma^\lambda(\nu_0))(d\theta d\alpha ds)$$

Since $\nu_0 - \gamma^\lambda(\nu_0) = \gamma^\lambda(Y.\nu_0) - \gamma^\lambda(\nu_0) = \gamma^\lambda((Y-1).\nu_0)$ (see Proposition 3.8), the proof is finished.

Since we will study V^λ as a solution of (3.11), (we have no other information), we may need the following proposition :

Proposition 3.11 *For every $\lambda \in \Lambda$, the equation (3.11) admits one and only one solution $V^\lambda \in \mathbb{L}^2_T$. If furthermore $P_0 = \mathcal{L}(V_0)$ admits moments of all orders, then $V^\lambda \in \mathbb{L}^p_T$ for every $p < \infty$.*

Proof : let us prove the uniqueness. Let λ be fixed, and let U and U' be two $\mathbb{L}^2_T$ solutions of (3.11). Then

$$\begin{aligned}
U'^x_t - U^x_t &= -\frac{b}{2}\int_0^t (U'^x_s - U^x_s)ds \\
&+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[(U'^x_{s-} - U^x_{s-})(\cos\theta - 1) - (U'^y_{s-} - U^y_{s-})\sin\theta\Big]\tilde{N}_1(d\theta d\alpha ds) \\
&+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[(U'^x_{s-} - U^x_{s-})(\cos\gamma^\lambda(s,\theta,\alpha) - 1) \\
&\qquad\qquad -(U'^y_{s-} - U^y_{s-})\sin\gamma^\lambda(s,\theta,\alpha)\Big]\tilde{N}_0(d\theta d\alpha ds) \\
&-\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha) - 1)\Big[(U'^x_{s-} - U^x_{s-})(\cos\gamma^\lambda(s,\theta,\alpha) - 1) \\
&\qquad\qquad -(U'^y_{s-} - U^y_{s-})\sin\gamma^\lambda(s,\theta,\alpha)\Big]\beta_0(\theta)d\theta d\alpha ds
\end{aligned}$$

Let us notice that :

$$|\cos\theta - 1| \le \theta^2 \ ; \quad |\sin\theta| \le |\theta| \ ; \quad |\cos\gamma^\lambda(s,\theta,\alpha) - 1| \le |\gamma^\lambda(s,\theta,\alpha)|^2 \le (|\theta| + \rho(\theta))^2 \ ;$$
$$\text{and} \quad |\sin\gamma^\lambda(s,\theta,\alpha)| \le |\gamma^\lambda(s,\theta,\alpha)| \le (|\theta| + \rho(\theta))$$

and that all these upperbounds are in $\cap_{p\ge 2} L^p(\beta(\theta)d\theta)$. Furthermore,

$$|(Y^\lambda(s,\theta,\alpha) - 1)(\cos\gamma^\lambda(s,\theta,\alpha) - 1)| \le \| \lambda \| \, \tilde{\rho}(\theta)(|\theta| + \rho(\theta))^2$$

$$|(Y^\lambda(s,\theta,\alpha) - 1)\sin\gamma^\lambda(s,\theta,\alpha)| \le \| \lambda \| \, \tilde{\rho}(\theta)(|\theta| + \rho(\theta)) \tag{3.12}$$

and these upperbounds are in $\cap_{p\ge 1} L^p(\beta(\theta)d\theta)$. Hence, using the Appendix (6.4) for $p = 2$, we obtain

$$E\left(\sup_{[0,t]} |U'^x_s - U^x_s|^2\right) \le K\int_0^t \left[E\left(|U'^x_s - U^x_s|^2\right) + E\left(|U'^y_s - U^y_s|^2\right)\right] ds$$

The same computation holds for $(U'^y_s - U^y_s)$, and we finally get :

$$E\left(\sup_{[0,t]} \| U'_s - U_s \|^2\right) \le K\int_0^t E\left(\| U'_s - U_s \|^2\right) ds$$

This yields the uniqueness of a $\mathbb{L}^2_T$ solution by Gronwall's lemma. Furthermore, it is classical to prove the existence of a $\mathbb{L}^2_T$ (resp. $\mathbb{L}^p_T$) solution if P_0 admits a moment of order 2 (resp. of order p), by using a Picard iteration and the same computation as above.

3.4 An integration by parts setting for V^λ.

We begin with an obvious remark :

Remark 3.12 *For every $\omega, s, \theta, \alpha$, the function $\lambda \longrightarrow Y^\lambda(\omega, s, \theta, \alpha)$ is twice differentiable, and*

$$\frac{\partial}{\partial \lambda_x} Y^\lambda(s,\theta,\alpha) = v'_x(s,\theta,\alpha)\frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$
$$+\Big[1+\lambda_x v'_x(s,\theta,\alpha)+\lambda_y v'_y(s,\theta,\alpha)\Big]\times v_x(s,\theta,\alpha)\frac{\beta'_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$
$$\frac{\partial^2}{\partial \lambda_x^2} Y^\lambda(s,\theta,\alpha) = 2v'_x(s,\theta,\alpha)v_x(s,\theta,\alpha)\frac{\beta'_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$
$$+\Big[1+\lambda_x v'_x(s,\theta,\alpha)+\lambda_y v'_y(s,\theta,\alpha)\Big]\times v_x^2(s,\theta,\alpha)\frac{\beta''_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$
$$\frac{\partial^2}{\partial \lambda_x \partial \lambda_y} Y^\lambda(s,\theta,\alpha) = \Big[v'_x(s,\theta,\alpha)v_y(s,\theta,\alpha)+v'_y(s,\theta,\alpha)v_x(s,\theta,\alpha)\Big]\frac{\beta'_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$
$$+\Big[1+\lambda_x v'_x(s,\theta,\alpha)+\lambda_y v'_y(s,\theta,\alpha)\Big]\times v_x(s,\theta,\alpha)v_y(s,\theta,\alpha)\frac{\beta''_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}$$

Now we can differentiate G^λ.

Proposition 3.13 *The family $\{G^\lambda\}$ is L^p differentiable for every $p<\infty$, and has the following derivative*

$$DG_t = \begin{pmatrix} D^x G_t \\ \\ D^y G_t \end{pmatrix} = \begin{pmatrix} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \frac{\partial}{\partial \lambda_x} Y^\lambda(s,\theta,\alpha)\Big|_{\lambda=0} \tilde{N}_0(d\theta d\alpha ds) \\ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \frac{\partial}{\partial \lambda_y} Y^\lambda(s,\theta,\alpha)\Big|_{\lambda=0} \tilde{N}_0(d\theta d\alpha ds) \end{pmatrix} \tag{3.13}$$

<u>Proof</u> : let $p=2^q$, with $q \in I\!N^*$. We first notice, by using the Appendix (6.4) and since

$$\| \frac{\partial}{\partial \lambda} Y^\lambda(s,\theta,\alpha)\Big|_{\lambda=0} \| \le K\frac{\rho(\theta)}{|\theta|} \in \cap_{q\ge1} L^q(\beta_0(\theta)d\theta)$$

that $\sup_{[0,T]} \| DG_t \|$ is in L^p. On the other hand, it is easy to show that

$$\sup_\lambda E\left(\sup_{[0,T]} |G_t^\lambda|^p\right) < \infty$$

by using Gronwall's Lemma. Then, using the Appendix (6.4), since $|Y^\lambda - 1| \le \| \lambda \| \tilde{\rho}$, and since $\tilde{\rho} \in \cap_{p\ge1} L^p(\beta_0(\theta)d\theta)$, one can show that

$$E\left(\sup_{[0,T]} |G_t^\lambda - 1|^p\right) \le K \| \lambda \|^p$$

We now split $U_t^\lambda = G_t^\lambda - 1 - \langle \lambda, DG_t \rangle$ into :

$$\begin{aligned} U_t^\lambda &= \int_0^t \int_0^1 \int_{-\pi}^{\pi} G_{s-}^\lambda \left(Y^\lambda(s,\theta,\alpha) - 1 - \left\langle \lambda, \frac{\partial}{\partial \lambda} Y^\lambda(s,\theta,\alpha) \Big|_{\lambda=0} \right\rangle \right) \tilde{N}_0(d\theta d\alpha ds) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left(G_{s-}^\lambda - 1 \right) \left\langle \lambda, \frac{\partial}{\partial \lambda} Y^\lambda(s,\theta,\alpha) \Big|_{\lambda=0} \right\rangle \tilde{N}_0(d\theta d\alpha ds) \end{aligned}$$

But, using Remark 3.12 and the fact that $|\gamma^\lambda(s,\theta,\alpha)| \geq \frac{|\theta|}{2}$, we see that

$$\begin{aligned} \left| Y^\lambda(s,\theta,\alpha) - 1 - \left\langle \lambda, \frac{\partial}{\partial \lambda} Y^\lambda(s,\theta,\alpha) \Big|_{\lambda=0} \right\rangle \right| &\leq K \parallel \lambda \parallel^2 \times \sup_\lambda \parallel \frac{\partial^2}{\partial \lambda^2} Y(s,\theta,\alpha) \parallel \\ &\leq K \parallel \lambda \parallel^2 \times \frac{\rho^2(\theta)}{|\theta|^2} \end{aligned}$$

Furthermore,

$$\left| \left\langle \lambda, \frac{\partial}{\partial \lambda} Y^\lambda(s,\theta,\alpha) \Big|_{\lambda=0} \right\rangle \right| \leq K \parallel \lambda \parallel \times \frac{\rho(\theta)}{|\theta|}$$

Hence, since $\frac{\rho^2(\theta)}{|\theta|^2}$ and $\frac{\rho(\theta)}{|\theta|}$ are in $\cap_{p \geq 1} L^p(\beta_0(\theta)d\theta)$, and by using the Appendix (6.4), we obtain :

$$\begin{aligned} E\left(\sup_{[0,T]} |U_t^\lambda|^p \right) &\leq K \parallel \lambda \parallel^{2p} \int_0^t E\left(|G_s^\lambda|^p \right) ds + K \parallel \lambda \parallel^p \int_0^t E\left(|G_s^\lambda - 1|^p \right) ds \\ &\leq K \parallel \lambda \parallel^{2p} = o(\parallel \lambda \parallel^p) \end{aligned}$$

This means that G_t^λ is L^p differentiable at $\lambda = 0$. Of course, this holds in fact for every $p \geq 1$.

We also have to differentiate V^λ. Let us begin with a lemma.

Lemma 3.14 *Let V^λ be the solution of $E(\lambda)$. There exists $K \geq 0$ such that*

$$\sup_{\lambda \in \Lambda} E\left(\sup_{s \leq T} \parallel V_s^\lambda \parallel^2 \right) \leq K \qquad \textit{and} \qquad E\left(\sup_{s \leq T} \parallel V_s^\lambda - V_s \parallel^2 \right) \leq K \parallel \lambda \parallel^2$$

If furthermore $P_0 = \mathcal{L}(V_0)$ admits moments of all orders, this holds also in L^p for every $p < \infty$.

Proof : since $V_0 \in L^2$, and since W is a $\mathbb{L}_T^2$-α-process, one can easily check that

$$E\left(\sup_{s \leq T} \parallel V_s^\lambda \parallel^2 \right) \leq K + C \int_0^t E\left(\parallel V_s^\lambda \parallel^2 \right) ds$$

where K and C are some constants. Gronwall's Lemma yields the first inequality. The second inequality is less obvious. Let us split $V_t^{\lambda x} - V_t^x$:

$$V_t^{\lambda x} - V_t^x = -\frac{b}{2} \int_0^t (V_s^{\lambda x} - V_s^x) ds$$

$$+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[(\cos\theta-1)(V_{s-}^{\lambda\ x}-V_{s-}^x)-\sin\theta(V_{s-}^{\lambda\ y}-V_{s-}^y)\right]\tilde{N}_1(d\theta d\alpha ds)$$

$$+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[(\cos\theta-\cos\gamma^\lambda(s,\theta,\alpha))W_{s-}^x(\alpha)-(\sin\theta-\sin\gamma^\lambda(s,\theta,\alpha))W_{s-}^y(\alpha)\right]$$

$$\tilde{N}_0(d\theta d\alpha ds)$$

$$+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[(\cos\theta-1)(V_{s-}^{\lambda\ x}-V_{s-}^x)-\sin\theta(V_{s-}^{\lambda\ y}-V_{s-}^y)\right]\tilde{N}_0(d\theta d\alpha ds)$$

$$+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[(\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta)V_{s-}^{\lambda\ x}-(\sin\gamma^\lambda(s,\theta,\alpha)-\sin\theta)V_{s-}^{\lambda\ y}\right]\tilde{N}_0(d\theta d\alpha ds)$$

$$+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha)-1)\left[(\cos\gamma^\lambda(s,\theta,\alpha)-1)W_{s-}^x(\alpha)-\sin\gamma^\lambda(s,\theta,\alpha)W_{s-}^y(\alpha)\right]$$

$$\beta_0(\theta)d\theta d\alpha ds$$

$$-\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha)-1)\left[(\cos\gamma^\lambda(s,\theta,\alpha)-1)V_{s-}^{\lambda\ x}-\sin\gamma^\lambda(s,\theta,\alpha)V_{s-}^{\lambda\ y}\right]$$

$$\beta_0(\theta)d\theta d\alpha ds$$

Let us notice that $\cos\theta-1$ and $\sin\theta$ are in $\cap_{p\geq 2}L^p(\beta(\theta)d\theta)$, that

$$|\cos\theta-\cos\gamma^\lambda(s,\theta,\alpha)|\leq|\gamma^\lambda(s,\theta,\alpha)-\theta|\leq\parallel\lambda\parallel\rho(\theta)\ ;\quad|\sin\theta-\sin\gamma^\lambda(s,\theta,\alpha)|\leq\parallel\lambda\parallel\rho(\theta) \tag{3.14}$$

Using these inequalities, (3.12), and the fact that W is a $\mathbb{L}_T^2$-α-process, using the first part of the lemma, and Doob's and Cauchy-Schwarz's inequalities, we get

$$E\left(\sup_{[0,t]}(V_s^{\lambda x}-V_s^x)^2\right)\leq C\int_0^t E\left(\parallel V_s^\lambda-V_s\parallel^2\right)ds+K\parallel\lambda\parallel^2$$

The same computation holds for $V_t^{\lambda y}-V_t^y$, and we finaly obtain :

$$E\left(\sup_{[0,t]}\parallel V_s^\lambda-V_s\parallel^2\right)\leq C\int_0^t E\left(\parallel V_s^\lambda-V_s\parallel^2\right)ds+K\parallel\lambda\parallel^2$$

Gronwall's Lemma yields the conclusion.
If $p=2^q$, one can use the same computation, using the Appendix (6.4) instead of Doob's inequality. Of course, the result remains true for any $p<\infty$.

Notation 3.15 *We will denote in the sequel* $\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1 & y_2\end{pmatrix}=\begin{pmatrix}x_1y_1 & x_1y_2\\x_2y_1 & x_2y_2\end{pmatrix}$.

Theorem 3.16 *The family $\{V^\lambda\}$ is L^2-differentiable at $\lambda = 0$, and its derivative $DV = \begin{pmatrix} D^xV^x & D^yV^x \\ D^xV^y & D^yV^y \end{pmatrix}$ satisfies the equation :*

$$\begin{aligned} DV_t &= -\frac{b}{2}\int_0^t DV_s ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)DV_{s-}\tilde{N}(d\theta d\alpha ds) \\ &\quad + \int_0^t\int_0^1\int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)\tilde{N}_0(d\theta d\alpha ds) \\ &\quad - \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha))\left((v(s,.,\theta)\beta_0(.))'(\theta)\right)^T d\theta d\alpha ds \end{aligned} \tag{3.15}$$

This means that $E\left(\sup_{s\leq T} \| V_s^\lambda - V_s - DV_t.\lambda \|^2\right) = o(\| \lambda \|^2)$. If furthermore P_0 has moments of all orders, then V is L^p-differentiable for every $p < \infty$.

<u>Proof</u> : We set $Z_t^\lambda = V_t^\lambda - V_t - DV_t.\lambda$. Then

$$Z_t^{\lambda x} = V_t^{\lambda x} - V_t^x - \lambda_x D^xV_t^x - \lambda_y D^yV_t^x = a_t^\lambda + ... + g_t^\lambda$$

where :

$$a_t^\lambda = -\frac{b}{2}\int_0^t Z_s^{\lambda x} ds + \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[(\cos\theta - 1)Z_{s-}^{\lambda\ x} - \sin\theta Z_{s-}^{\lambda\ y}\right]\tilde{N}(d\theta d\alpha ds)$$

$$b_t^\lambda = \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[V_{s-}^{\lambda\ x}\left(\cos\gamma^\lambda(s,\theta,\alpha) - \cos\theta\right) + \sin\theta V_{s-}^x\left(\lambda_x v_x(s,\theta,\alpha) + \lambda_y v_y(s,\theta,\alpha)\right)\right] \tilde{N}_0(d\theta d\alpha ds)$$

$$c_t^\lambda = -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[V_{s-}^{\lambda\ y}\left[\sin\gamma^\lambda(s,\theta,\alpha) - \sin\theta\right] - \cos\theta V_{s-}^y\left(\lambda_x v_x(s,\theta,\alpha) + \lambda_y v_y(s,\theta,\alpha)\right)\right] \tilde{N}_0(d\theta d\alpha ds)$$

$$d_t^\lambda = -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi} W_{s-}^x\left[\cos\gamma^\lambda(s,\theta,\alpha) - \cos\theta + \sin\theta\left(\lambda_x v_x(s,\theta,\alpha) + \lambda_y v_y(s,\theta,\alpha)\right)\right] \tilde{N}_0(d\theta d\alpha ds)$$

$$e_t^\lambda = \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi} W_{s-}^x\left[\sin\gamma^\lambda(s,\theta,\alpha) - \sin\theta - \cos\theta\left(\lambda_x v_x(s,\theta,\alpha) + \lambda_y v_y(s,\theta,\alpha)\right)\right] \tilde{N}_0(d\theta d\alpha ds)$$

$$\begin{aligned} f_t^\lambda = -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}&\Big[V_{s-}^{\lambda\ x}(\cos\gamma^\lambda(s,\theta,\alpha) - 1)(Y^\lambda(s,\theta,\alpha) - 1)\beta_0(\theta) \\ &-V_{s-}^x(\cos\theta - 1)\left(\lambda_x\left[v_x(s,.,\alpha)\beta_0(.)\right]'(\theta) + \lambda_y\left[v_y(s,.,\alpha)\beta_0(.)\right]'(\theta)\right)\Big] d\theta d\alpha ds \end{aligned}$$

$$\begin{aligned} g_t^\lambda = -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}&\Big[V_{s-}^{\lambda\ y}\sin\gamma^\lambda(s,\theta,\alpha)(Y^\lambda(s,\theta,\alpha) - 1)\beta_0(\theta) \\ &- V_{s-}^y\sin\theta\left(\lambda_x\left[v_x(s,.,\alpha)\beta_0(.)\right]'(\theta) + \lambda_y\left[v_y(s,.,\alpha)\beta_0(.)\right]'(\theta)\right)\Big] d\theta d\alpha ds \end{aligned}$$

We upperbound these expressions one by one. First, it is easy to see that

$$E\left(\sup_{[0,t]}(a_s^\lambda)^2\right) \le C\int_0^t E\left(\| Z_s^\lambda \|^2\right) ds$$

Let us show that $E\left(\sup_{[0,t]}(b_s^\lambda)^2\right) \le C \| \lambda \|^4$. To this end, we split b^λ into

$$\begin{aligned} b_t^\lambda &= \frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta)(V_{s-}^{\lambda\ x}-V_{s-}^x)\tilde{N}_0(d\theta d\alpha da) \\ &+\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}V_{s-}^x\left(\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta+\sin\theta\left(\lambda_x v_x(s,\theta,\alpha)+\lambda_y v_y(s,\theta,\alpha)\right)\right) \\ &\qquad \tilde{N}_0(d\theta d\alpha ds) \end{aligned}$$

The first expression makes no problem, thanks to Lemma 3.14 and since

$$(\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta)^2 \le \| \lambda \|^2 \rho^2(\theta)$$

The second one is also easy, because

$$\begin{aligned} &|\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta+\sin\theta\left(\lambda_x v_x(s,\theta,\alpha)+\lambda_y v_y(s,\theta,\alpha)\right)| \\ &= |\cos\gamma^\lambda(s,\theta,\alpha)-\cos\theta-\cos'(\theta)\left(\gamma^\lambda(s,\theta,\alpha)-\theta\right)| \\ &\le \| \cos''\theta \|_\infty \times |\gamma^\lambda(s,\theta,\alpha)-\theta|^2 \le \| \lambda \|^2 \rho^2(\theta) \end{aligned}$$

Hence we have upperbounded b^λ, and the same computation works for c^λ, and are easier for d^λ and e^λ.

Now we study g^λ :

$$\begin{aligned} g_t^\lambda &= -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\sin\gamma^\lambda(s,\theta,\alpha)(Y^\lambda(s,\theta,\alpha)-1)(V_{s-}^{\lambda\ y}-V_{s-}^y)\beta_0(\theta)d\theta d\alpha ds \\ &\quad -\frac{1}{2}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[\sin\gamma^\lambda(s,\theta,\alpha)(Y^\lambda(s,\theta,\alpha)-1) \\ &\quad -\sin\theta\times\frac{\lambda_x\left[v_x(s,.,\alpha)\beta_0(.)\right]'(\theta)+\lambda_y\left[v_y(s,.,\alpha)\beta_0(.)\right]'(\theta)}{\beta_0(\theta)}\Big]V_{s-}^y\beta_0(\theta)d\theta d\alpha ds \\ &= -\frac{1}{2}A_t^\lambda-\frac{1}{2}B_t^\lambda \end{aligned}$$

Thanks to Cauchy-Schwarz's inequality,

$$\begin{aligned} E\left(\sup_{[0,t]}(A_s^\lambda)^2\right) &\le E\Big[\int_0^t\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha)-1)^2\beta_0(\theta)d\theta d\alpha ds \\ &\times\int_0^t\int_0^1\int_{-\pi}^{\pi}\sin^2\gamma^\lambda(s,\theta,\alpha)(V_{s-}^{\lambda\ y}-V_{s-}^y)^2\beta_0(\theta)d\theta d\alpha ds\Big] \le K \| \lambda \|^4 \end{aligned}$$

where the last inequality is due to Lemma 3.14, equation (3.9) and the following one :

$$\sin^2 \gamma^\lambda(s,\theta,\alpha) \leq (\gamma^\lambda(s,\theta,\alpha))^2 \leq \frac{9}{4}\theta^2 \in \mathrm{L}^1(\beta_0(\theta)d\theta)$$

In order to upperbound B^λ, we consider the map $f_{s,\theta,\alpha}(\lambda) = (Y^\lambda(s,\theta,\alpha) - 1)\sin\gamma^\lambda(s,\theta,\alpha)$. Since $Y^0 - 1 = 0$,

$$B_t^\lambda = \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[f_{s,\theta,\alpha}(\lambda) - f_{s,\theta,\alpha}(0) - \lambda_x \frac{\partial}{\partial \lambda_x} f_{s,\theta,\alpha}(0) - \lambda_y \frac{\partial}{\partial \lambda_y} f_{s,\theta,\alpha}(0) \right] V_{s-}^y \beta_0(\theta) d\theta d\alpha ds$$

Hence, if we show that there exists $z \in L^1(\beta_0(\theta)d\theta)$ satisfying

$$\sup_\Lambda \left| \frac{\partial^2}{\partial \lambda_x^2} f_{s,\theta,\alpha}(\lambda) \right| + \sup_\Lambda \left| \frac{\partial^2}{\partial \lambda_x \partial \lambda_y} f_{s,\theta,\alpha}(\lambda) \right| + \sup_\Lambda \left| \frac{\partial^2}{\partial \lambda_y^2} f_{s,\theta,\alpha}(\lambda) \right| \leq z(\theta)$$

we will have, thanks to Cauchy Schwarz's inequality and since V is a $\mathbb{L}_T^2$-process,

$$E\left(\sup_{[0,t]} (B_s^\lambda)^2 \right) \leq \| \lambda \|^4 \qquad \text{and} \quad \text{hence} \qquad E\left(\sup_{[0,t]} (g_s^\lambda)^2 \right) \leq \| \lambda \|^4$$

Let us prove the existence of z :

$$\begin{aligned}
\frac{\partial^2}{\partial \lambda_x^2} f_{s,\theta,\alpha}(\lambda) \quad &= \quad \frac{\partial^2}{\partial \lambda_x^2} Y^\lambda(s,\theta,\alpha) \times \sin\gamma^\lambda(s,\theta,\alpha) \\
&\quad + 2\frac{\partial}{\partial \lambda_x} Y^\lambda(s,\theta,\alpha) \times \frac{\partial}{\partial \lambda_x}\gamma^\lambda(s,\theta,\alpha) \times \cos\gamma^\lambda(s,\theta,\alpha) \\
&\quad - (Y^\lambda(s,\theta,\alpha) - 1) \times \left(\frac{\partial}{\partial \lambda_x}\gamma^\lambda(s,\theta,\alpha) \right)^2 \times \sin\gamma^\lambda(s,\theta,\alpha)
\end{aligned}$$

Since

$$|\gamma^\lambda(s,\theta,\alpha)| \leq |\theta| + \rho(\theta) \quad ; \quad |Y^\lambda(s,\theta,\alpha) - 1| \leq \tilde{\rho}(\theta)$$

$$\left| \frac{\partial}{\partial \lambda_x}\gamma^\lambda(s,\theta,\alpha) \right| \leq |v_x(s,\theta,\alpha)| \leq \rho(\theta) \quad ;$$

$$\left| \frac{\partial}{\partial \lambda_x} Y^\lambda(s,\theta,\alpha) \right| \leq K\rho(\theta) \left[\left| \frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)} \right| + \left| \frac{\beta_0'(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)} \right| \right] \quad ;$$

$$\text{and} \quad \left| \frac{\partial^2}{\partial \lambda_x^2} Y^\lambda(s,\theta,\alpha) \right| \leq K\rho^2(\theta) \left[\left| \frac{\beta_0'(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)} \right| + \left| \frac{\beta_0''(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)} \right| \right]$$

we see that

$$\left| \frac{\partial^2}{\partial \lambda_x^2} f_{s,\theta,\alpha}(\lambda) \right| \leq K\rho^2(\theta) \left[\frac{|\beta_0''(\gamma^\lambda(s,\theta,\alpha))| + |\beta_0'(\gamma^\lambda(s,\theta,\alpha))| + |\beta_0(\gamma^\lambda(s,\theta,\alpha))|}{\beta_0(\theta)} + \tilde{\rho}(\theta) \right]$$

Recalling that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r}\mathbb{1}_{|\theta|\leq\theta_0}$, that $|\theta|/2 \leq |\gamma^\lambda(s,\theta,\alpha)| \leq 3|\theta|/2$, and that $\rho(\theta) \leq ce^{-|\theta|^{-r'}}$, we obtain

$$\sup_\Lambda \left|\frac{\partial^2}{\partial\lambda_x^2} f_{s,\theta,\alpha}(\lambda)\right| \leq K\frac{\rho^2(\theta)}{\theta^2} \in L^1(\beta_0(\theta)d\theta)$$

The other derivatives can be studied in the same way, and B_t^λ, (and hence g_t^λ) is upperbounded. One can prove the same result for f^λ, and we finally get, (after the same computation for $Z_t^{\lambda y}$) :

$$E\left(\sup_{[0,t]} \| Z_s^\lambda \|^2\right) \leq K \| \lambda \|^4 + C\int_0^t E\left(\| Z_s^\lambda \|^2\right) ds$$

Gronwall's Lemma yields the result.
If all the moments of V_0 are finite, the same computation works in L^p for each $p < \infty$, by using the Appendix (6.4).

We can now state an IBPS for V_t.

Proposition 3.17 *Let $t \geq 0$. If $X \in \mathcal{D}_t$ (or if $X \in \mathcal{D}_t^\infty$, cf Definition 3.9-3.), we set $\delta_t(X) = -DX$. Under (H), $(DV_t, -DG_t, \mathcal{D}_t, \delta_t)$ is a weak IBPS for V_t. Under (H) and (S), $(DV_t, -DG_t, \mathcal{D}_t^\infty, \delta_t)$ is an IBPS for V_t.*

Proof : let us for example assume (H) and (S) and prove the second claim. DV_t is of course a $\mathcal{M}_2(\mathbb{R})$ valued random variable. Using Proposition 3.13, $-DG_t$ is a $\mathbb{R}^2$ valued random variable which is in $\cap_p L^p$. $\mathcal{D}_t^\infty$ is a linear space, and it is classical to show that if $X_1, ..., X_n$ are in $\mathcal{D}_t^\infty$, and if $F \in C_p^2(\mathbb{R}^n)$, then $F(X_1, ..., X_n) \in \mathcal{D}_t^\infty$, its derivative is given by :

$$DF(X_1, ..., X_n) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(X_1, ..., X_n)DX_i$$

It remains to prove that if $f \in C_p^2(\mathbb{R}^2)$, and if $X \in \mathcal{D}_t^\infty$, then $E(D_t) = 0$, where

$$D_t = DXf(V_t) + X\begin{pmatrix} f_x'(V_t) & f_y'(V_t)\end{pmatrix} DV_t + Xf(V_t)DG_t \tag{3.16}$$

By using the facts that $V_t \in \cap L^p$ and $f \in C_p^2(\mathbb{R}^2)$, it is standard and natural to show that

$$E\left(\left|X^\lambda f(V_t^\lambda)G_t^\lambda - Xf(V_t) - \langle\lambda, D_t\rangle\right|\right) = o(\| \lambda \|)$$

Hence,

$$\left|E\left(X^\lambda f(V_t^\lambda)G_t^\lambda\right) - E\left(Xf(V_t)\right) - \langle\lambda, E(D_t)\rangle\right| = o(\| \lambda \|)$$

But, since $X^\lambda f(V_t^\lambda) = Xf(V_t) \circ S^\lambda$ and since $P^\lambda \circ (S^\lambda)^{-1} = P$, we deduce that

$$E\left(X^\lambda f(V_t^\lambda)G_t^\lambda\right) = E\left(Xf(V_t)\right)$$

Hence $|\langle\lambda, E(D_t)\rangle| = o(\| \lambda \|)$, and $E(D_t) = 0$, which was our aim.

3.5 The choice of v and an explicit computation of DV.

In order to apply Theorems 3.5 and 3.6, we have to study the inversibility of DV_t. We will use the Doléans-Dade martingales, in order to obtain a suitable expression of DV_t. Then we will choose v, which is really more difficult in dimension 2 than in dimension 1. Only a good choice of v will allow DV_t to admit moments of all orders (see Theorem 3.28) : v must be "large" (this way, DV_t will be invertible) but also "small" (in particular, we need $\| v \| \leq \rho$).
We denote by I the unit matrix on $\mathbb{R}^2$.

Lemma 3.18 *One can rewrite the S.D.E. (3.15) in the following way :*

$$DV_t = \int_0^t dK_s.DV_{s-} + L_t$$

where $K_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\tilde{N}(d\theta d\alpha ds) - \frac{b}{2}tI$
and $L_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)N_0(d\theta d\alpha ds)$.

<u>Proof</u> : it suffices to prove that

$$\begin{aligned} &- \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha)) \left([v(s,.,\alpha)\beta_0]'(\theta) \right)^T d\theta d\alpha ds \\ &= \int_0^t \int_0^1 \int_{-\pi}^{\pi} A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)\beta_0(\theta) d\theta d\alpha ds \end{aligned}$$

This can be shown by using a (standard) integration by parts formula in the variable θ, and by noticing that

$$\forall \omega, s, \alpha \qquad v(\omega, s, -\theta_0, \alpha) = v(\omega, s, 0, \alpha) = v(\omega, s, \theta_0, \alpha) = 0$$

Now we can write DV in an explicit form.

Proposition 3.19 *Let M be the following Doléans-Dade martingale, taking its values in $\mathcal{M}_2(\mathbb{R})$:*

$$M_t = \int_0^t dK_s.M_{s-} + I \tag{3.17}$$

For all t, $(I + \Delta K_t)$ is a.s. invertible. We thus know (see Jacod, [22]) that for all s, M_s and M_{s-} are also a.s. invertible, and that $DV_t = M_t H_t$ where

$$\begin{aligned} H_t &= \int_0^t M_{s-}^{-1}(I + \Delta K_{s-})^{-1} dL_s \\ &= \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}^{-1}(I + A(\theta))^{-1}A'(\theta)(V_{s-} - W_{s-}(\alpha))v^T(s,\theta,\alpha)N_0(d\theta d\alpha ds) \end{aligned}$$

The only claim we need to show here is that for every t, $(I + \Delta K_t)$ is a.s. invertible. To this end, let us write our Poisson measure as

$$N = \sum_{s \in [0,T]} \mathbb{1}_D(s)\delta_{(s,\theta_s,\alpha_s)}$$

This way, we see that $I + \Delta K_s = I + A(\theta_s)\mathbb{1}_D(s)$ is invertible except if $\theta_s \in \{-\pi, \pi\}$, which never happens a.s.

We now choose v. First we need a positive C_b^∞ function δ on $[-\theta_0, \theta_0]$ such that ($C > 0$ is a constant) :

$$|\delta(\theta)| + |\delta'(\theta)| \leq \rho(\theta) \quad ; \qquad \{\delta = 0\} = \{-\theta_0, 0, \theta_0\} \quad ; \quad \delta(\theta) \overset{0}{\sim} Ce^{-|\theta|^{-2r'}} \tag{3.18}$$

We will also use a function on $\mathbb{R}^2 \times (\mathcal{M}_2(\mathbb{R})) \times [-\theta_0, \theta_0]$ with values in $\mathbb{R}^2$:

$$\bar{g}(x, y, \theta) = (A'(\theta)x)^T((I + A(\theta))^{-1})^T(y^{-1})^T$$

We consider the C^∞ function $h(x) = (1+ \| x \|^2)^{-1}$ from $\mathbb{R}^2$ to $]0, 1]$. At last, we will use a function k from $\mathcal{M}_2(\mathbb{R})$ to $[0, 1]$, such that $k(y) = 0$ if and only if $\det y = 0$, and such that the map

$$y \longrightarrow \begin{cases} (y^{-1})^T k(y) & \text{if} \quad \det y \neq 0 \\ \\ 0 & \text{if} \quad \det y = 0 \end{cases}$$

is C_b^∞ from $\mathcal{M}_2(\mathbb{R})$ to itself.

Then, the function on $\mathbb{R}^2 \times \mathcal{M}_2(\mathbb{R}) \times [-\theta_0, \theta_0]$ with values in $\mathbb{R}^2$ defined by

$$\begin{aligned} g(x, y, \theta) &= \bar{g}(x, y, \theta) h\left(A'(\theta)x\right) k\left(I + A(\theta)\right) k(y) \\ &= \frac{(A'(\theta)x)^T}{1+ \| A'(\theta)x \|^2} \times k(I + A(\theta)) \left((I + A(\theta))^{-1}\right)^T \times k(y)(y^{-1})^T \end{aligned}$$

is of class C_b^∞. We now set

$$\Delta(x, y, \theta) = g(x, y, \theta)\delta(\theta)$$

This function is of class C_b^∞, and satisfies, for every integers a, b, c, d, i, j, k, l :

$$\sup_{x,y} \left| \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial \theta^c} \Delta(x, y, \theta) \right| \in \bigcap_{p \geq 1} L^p \left(\left| \frac{\left(\beta_0^{(i)}(\theta)\right)^k}{\left(\beta_0^{(j)}(\theta)\right)^l} \right| d\theta \right)$$

Definition 3.20 *We set $v(s, \theta, \alpha) = \Delta\left(V_{s-} - W_{s-}(\alpha), M_{s-}, \theta\right)$. (This function satisfies the assumptions of the subsection 3.2).*

3.6 Higher derivatives.

In order to apply Theorems 3.5 and 3.6, we have either to differentiate DV (under (H)) or to differentiate infinitely DV and DG (under (H) and (S)). To this end, we first notice that M_t satisfies a quite similar (but easier) equation than V_t. Hence, since the initial condition $M_0 = I$ is deterministic, $M^\lambda = M \circ S^\lambda$ is L^p-differentiable at 0 for every $p < \infty$. Let us compute $v^\lambda(\omega, s, \theta, \alpha) = v(S^\lambda(\omega), s, \theta, \alpha)$: with the notations of the Definition 3.20,

$$v^\lambda(s, \theta, \alpha) = \Delta(V_{s-}^\lambda - W_{s-}(\alpha), M_{s-}^\lambda, \theta)$$

By using the expression of DV in Lemma 3.18, we can write $DV^\lambda = DV \circ S^\lambda$ as

$$\begin{aligned}
DV_t^\lambda &= -\frac{b}{2}\int_0^t DV_s^\lambda ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)DV_{s-}^\lambda \tilde{N}_1(d\theta d\alpha ds) \\
&+ \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\gamma^\lambda(s,\theta,\alpha))DV_{s-}^\lambda \tilde{N}_0(d\theta d\alpha ds) \\
&- \int_0^t\int_0^1\int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha)-1)A(\gamma^\lambda(s,\theta,\alpha))DV_{s-}^\lambda \beta_0(\theta)d\theta d\alpha ds \\
&+ \int_0^t\int_0^1\int_{-\pi}^{\pi} A'(\gamma^\lambda(s,\theta,\alpha))(V_{s-}^\lambda - W_{s-}(\alpha))\left(v^\lambda(s,\gamma^\lambda(s,\theta,\alpha),\alpha)\right)^T N_0(d\theta d\alpha ds)
\end{aligned}$$

One can show that under (H), the family DV^λ is L^2-differentiable at 0, by using the properties of v.

Assume now (H) and (S), and set $X_t = (DV_t, M_t, DG_t, V_t)$. Then X_t satisfies a S.D.E. with initial condition $X_0 = (0, I, 0, V_0)$. Using the properties of v, one can show that $X^\lambda = X \circ S^\lambda$ is L^p differentiable at 0 for every $p < \infty$, with $DX_t = (D^x X_t, D^y X_t)$. Hence, $DV_t \circ S^\lambda$, $M_t \circ S^\lambda$ and $DG_t \circ S^\lambda$ are L^p differentiable at 0 for every $p < \infty$.
Finally, we can iterate this method for $Y_t = (DX_t, X_t)$, and so on. We may state the following theorem :

Theorem 3.21 *Under (H), the derivative DV_t is in $\mathcal{D}_t$ for every $t \in [0,T]$. Under (H) and (S), V and G are infinitely L^p differentiable for every $p < \infty$.*

The first conditions of Theorems 3.5 and 3.6 are thus satisfied, and we still have to study the inversibility of DV_t.

3.7 Existence of a weak solution.

The following remark shows the way to prove that $DV_t = M_t.H_t$ is invertible. We already know from Proposition 3.19 that M_t is a.s. invertible for every $t \in [0,T]$.

Remark 3.22 *We consider the following symmetric nonnegative matrix :*

$$\Gamma(x,\theta) = (I + A(\theta))^{-1}(A'(\theta)x)(A'(\theta)x)^T\left((I+A(\theta))^{-1}\right)^T$$

Then we set

$$R_t = \int_0^t\int_0^1\int_{-\pi}^{\pi} \Gamma(V_{s-} - W_{s-}(\alpha),\theta) \times h\left(A'(\theta)(V_{s-} - W_{s-}(\alpha))\right) \times k(I + A(\theta)) \times k(M_{s-})$$
$$\delta(\theta)N_0(d\theta d\alpha ds)$$

This matrix is also symmetric, nonnegative, and is increasing for the strong order (on the set of symmetric nonnegative matrices : for every $s \le t$, $R_t - R_s$ is nonnegative). Furthermore, we now can write H as

$$H_t = \int_0^t M_{s-}^{-1} dR_s \left(M_{s-}^{-1}\right)^T$$

Hence, in order to show that H_t (and hence DV_t) is a.s. invertible, it suffices to prove that a.s., $R_t - R_s$ is invertible for every $0 \leq s < t \leq T$. At last, since the real valued expression in R_t is always in $]0,1]$, it suffices in fact to show that a.s., $\bar{R}_t - \bar{R}_s$ is invertible for all $0 \leq s < t \leq T$, where

$$\bar{R}_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Gamma(V_{s-} - W_{s-}(\alpha), \theta)\delta(\theta) N_0(d\theta d\alpha ds)$$

Theorem 3.23 *Let $t \in]0,T]$. Under (H), DV_t is a.s. invertible.*

Proof : recall that it suffices to show that a.s., for all $0 \leq s < t \leq T$, $\bar{R}_t - \bar{R}_s$ is invertible. We break the proof in several steps.

Step 1 : If Y is a (random) vector of $\mathbb{R}^2$ not equal to 0 an easy computation shows that for $\theta \in]-\pi, \pi[$,

$$\begin{aligned} Y^T \Gamma(V_{s-} - W_{s-}(\alpha), \theta) Y = \Big\{ & \frac{\sin\theta}{1+\cos\theta} \left[Y_x(V^x_{s-} - W^x_{s-}(\alpha)) + Y_y(V^y_{s-} - W^y_{s-}(\alpha)) \right] \\ & + \left[-Y_y(V^x_{s-} - W^x_{s-}(\alpha)) + Y_x(V^y_{s-} - W^y_{s-}(\alpha)) \right] \Big\}^2 \end{aligned} \tag{3.19}$$

Let us fix ω, s, and α. It is easy to see that if $V_{s-}(\omega) \neq W_{s-}(\alpha)$, then

$$\int \mathbb{I}_{\left\{\theta \in]-\pi,\pi[\;/\; Y^T(\omega)\Gamma(V_{s-}(\omega) - W_{s-}(\alpha), \theta) Y(\omega) = 0\right\}} d\theta = 0$$

Step 2 : Let $s > 0$ be fixed, and let Y be a (random) unit vector in $\mathbb{R}^2$ that is $\mathcal{F}_s$-measurable. The aim of this step is to show that a.s. $\forall t > s$, $Y^T(\bar{R}_t - \bar{R}_s)Y > 0$. To this end, we consider the following stopping time :

$$\begin{aligned} \tau(Y) &= \inf\left\{ t > s \Big/ Y^T(\bar{R}_t - \bar{R}_s) Y > 0 \right\} \\ &= \inf\left\{ t > s \Big/ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y)}(r,\theta,\alpha) N_0(d\theta d\alpha ds) > 0 \right\} \end{aligned}$$

where

$$B(Y) = \left\{ (r,\theta,\alpha) \Big/ r > s \quad \text{and} \quad Y^T \Gamma(V_{r-} - W_{r-}(\alpha), \theta) Y > 0 \right\}$$

(recall that $\bar{R}_u$ is "increasing"). It thus suffices to check that $\tau(Y) = s$ a.s. By assumption, $\mathcal{L}(V_0)$ is not a Dirac mass. By Lemma 2.14, for every $t > 0$, $\mathcal{L}(V_t) = \mathcal{L}_\alpha(W_t)$ is not a Dirac mass either. This implies that for every $r \geq 0$, for every ω,

$$\int_0^1 \mathbb{I}_{\{W_{r-}(\alpha) \neq V_{r-}(\omega)\}} d\alpha = P_\alpha(W_{r-} \neq V_{r-}(\omega)) > 0$$

Since $\int_{-\pi}^{\pi} \beta_0(\theta) d\theta = \infty$, and thanks to the first step, for all ω, for all $r > s$,

$$\begin{aligned} & \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{B(Y(\omega))}(r,\theta,\alpha) \beta_0(\theta) d\theta d\alpha \\ \geq & \int_0^1 \int_{-\pi}^{\pi} \mathbb{I}_{\{W_{r-}(\alpha) \neq V_{r-}(\omega)\}} \mathbb{I}_{B(Y(\omega))}(r,\theta,\alpha) \beta_0(\theta) d\theta d\alpha = \infty \end{aligned}$$

Consequently, except if $\tau(Y(\omega)) = s$,

$$\int_0^{\tau(Y(\omega))}\int_0^1\int_{-\pi}^{\pi} \mathbb{1}_{B(Y(\omega))}(r,\theta,\alpha)\beta_0(\theta)d\theta d\alpha dr = \infty$$

But a.s., by definition of $\tau(Y)$,

$$\int_0^{\tau(Y)}\int_0^1\int_{-\pi}^{\pi} \mathbb{1}_{B(Y)}(r,\theta,\alpha)N_0(d\theta d\alpha dr) \leq 1$$

Taking the expectations, we obtain

$$E\left(\int_0^{\tau(Y)}\int_0^1\int_{-\pi}^{\pi} \mathbb{1}_{B(Y)}(r,\theta,\alpha)\beta_0(\theta)d\theta d\alpha dr\right) \leq 1$$

Hence,

$$\int_0^{\tau(Y)}\int_0^1\int_{-\pi}^{\pi} \mathbb{1}_{B(Y)}(r,\theta,\alpha)\beta_0(\theta)d\theta d\alpha ds < \infty \quad \text{a.s.}$$

and thus $\tau(Y) = s$ a.s., which was our aim.

Step 3 : We now show that if $s > 0$ is fixed, then a.s., for all $t > s$, $\bar{R}_t - \bar{R}_s$ is invertible. We set $Ker_t = Ker(\bar{R}_t - \bar{R}_s)$. For each random unit vector Y in $I\!R^2$, that is $\mathcal{F}_s$-measurable, we know that a.s., for all $t > s$, $Y \notin Ker_t$. Hence, since Ker_t increases when t decreases, we obtain $Y \notin Ker_{s+} = \cup_{t>s} Ker_t$ a.s. Since Ker_{s+} is $\mathcal{F}_s$-measurable, and since this is true for every unit vector $\mathcal{F}_s$-measurable, we deduce that $Ker_{s+} = \{0\}$, and the step 3 is finished.

Step 4 : We just have to get a "better a.s.". First,

$$\text{a.s.} \quad \text{for all } s < t \quad \text{with} \quad s,t \in [0,T] \cap \mathbb{Q} \quad \bar{R}_t - \bar{R}_s \quad \text{is} \quad \text{invertible}$$

Since $\bar{R}_t$ is increasing, it is easy to drop the "$\cap\, \mathbb{Q}$", and the theorem follows.

Proof of Theorem 3.1 : it is immediate, thanks to Theorems 3.23 and 3.21, Proposition 3.17, Theorem 3.5, and Remarks 2.7, 2.4, and 2.2.

3.8 Smoothness of the weak solution.

We now have to study the inverse moments of DV_t's determinant. We use the notations of the previous subsection. Recall that $DV_t = M_t.H_t$, where M_t is the following Doléans-Dade martingale :

$$M_t = I - \frac{b}{2}\int_0^t M_s ds + \int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)M_{s-}\tilde{N}(d\theta d\alpha ds)$$

The expression of H is given by

$$H_t = \int_0^t\int_0^1\int_{-\pi}^{\pi} M_{s-}^{-1}\Gamma(V_{s-} - W_{s-}(\alpha),\theta)(M_{s-}^{-1})^T\zeta(V_{s-} - W_{s-}(\alpha), M_{s-},\theta)\delta(\theta)N_0(d\theta d\alpha ds)$$

where, for $x \in I\!R^2$ and $y \in \mathcal{M}_2(I\!R)$,

$$\Gamma(x,\theta) = (I + A(\theta))^{-1} \times (A'(\theta)x) \times (A'(\theta)x)^T \times ((I + A(\theta))^{-1})^T$$

and

$$\zeta(x,y,\theta) = h(A'(\theta)x) \times k(I + A(\theta)) \times k(y)$$

where h and k are defined in Subsection 3.5.

We first study the **inverse moments of** M_t.

Theorem 3.24 *Assume (H) and (S). For every $t \geq 0$, $(\det M_t)^{-1}$ admits moments of all orders.*

Proof : we will prove that under (S)-2,

$$\begin{aligned} M_t^{-1} &= I + \frac{b}{2}\int_0^t M_s^{-1} ds - \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}^{-1}(I + A(\theta))^{-1} A(\theta) \tilde{N}(d\theta d\alpha ds) \\ &\quad + \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}^{-1} A(\theta)(I + A(\theta))^{-1} A(\theta) \beta(\theta) d\theta d\alpha ds \end{aligned} \tag{3.20}$$

A simple computation shows that :

$$(I + A(\theta))^{-1} A(\theta) = \frac{\sin\theta}{\cos\theta + 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$A(\theta)(I + A(\theta))^{-1} A(\theta) = \frac{1}{2} \frac{\sin\theta}{\cos\theta + 1} \begin{pmatrix} -\sin\theta & 1 - \cos\theta \\ \cos\theta - 1 & -\sin\theta \end{pmatrix}$$

Thanks to Assumption (S)-2, and since $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$, one can check that

$$\frac{|\sin\theta|}{1 + \cos\theta} \in \cap_{p \geq 2} L^p(\beta(\theta)d\theta) \quad ; \quad \frac{\sin^2\theta + |\sin\theta(1 - \cos\theta)|}{1 + \cos\theta} \in \cap_{p \geq 1} L^p(\beta(\theta)d\theta)$$

Hence it is clear that M_t^{-1} (and thus its determinant) is well defined and admits moments of all orders (this S.D.E. is classical, and the initial data I is deterministic).
In order to check (3.20), we apply the Itô formula to the product $M_t.M_t^{-1}$, where M_t^{-1} is defined by (3.20)

$$\begin{aligned} M_t.M_t^{-1} &= M_0.M_0^{-1} + \int_0^t M_{s-}.dM_s^{-1} + \int_0^t dM_s.M_{s-}^{-1} + \sum_{s \leq t} \Delta M_s.\Delta M_s^{-1} \\ &= I + \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[A(\theta) M_{s-}.M_{s-}^{-1} - M_{s-}.M_{s-}^{-1}(I + A(\theta))^{-1} A(\theta)\right] \tilde{N}(d\theta d\alpha ds) \\ &\quad + \int_0^t \int_0^1 \int_{-\pi}^{\pi} M_{s-}.M_{s-}^{-1} A(\theta)(I + A(\theta))^{-1} A(\theta) \beta(\theta) d\theta d\alpha ds \\ &\quad - \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta) M_{s-}.M_{s-}^{-1}(I + A(\theta))^{-1} A(\theta) N(d\theta d\alpha ds) \end{aligned}$$

But I is also a solution of this S.D.E. Indeed,

$$\int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[A(\theta) - (I + A(\theta))^{-1} A(\theta)\right] \tilde{N}(d\theta d\alpha ds)$$

$$+\int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(I+A(\theta))^{-1}A(\theta)\beta(\theta)d\theta d\alpha ds$$
$$-\int_0^t\int_0^1\int_{-\pi}^{\pi} A(\theta)(I+A(\theta))^{-1}A(\theta)N(d\theta d\alpha ds)$$
$$=\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[A(\theta)-(I+A(\theta))^{-1}A(\theta)-A(\theta)(I+A(\theta))^{-1}A(\theta)\right]\tilde{N}(d\theta d\alpha ds)=0$$

and the proof is finished.

It is more difficult to prove that H_t **admits moments of all orders**. In fact, we will only study **the case where** $E(V_0)=0$ by using the Malliavin Calculus. The generalization (see the final proof of this section) will then follow from the uniqueness in law for the nonlinear S.D.E. We begin with a lemma.

Lemma 3.25 *The map* $(t,Y)\longrightarrow\mathcal{L}(\langle V_t,Y\rangle)$ *is weakly continuous on* $[0,T]\times\{Y\in I\!R^2 \mid \ \| Y \|=1\}$.

Proof : it classically suffices to consider C_b^2 functions. Let ψ be a C_b^2 function on $I\!R$, let $t\in[0,T]$, and let $\| Y \|=1$. We will show that if (t_n,Y_n) goes to (t,Y), then

$$\Delta_n=E\left(\psi(\langle V_t,Y\rangle)-\psi(\langle V_{t_n},Y_n\rangle)\right)$$

goes to 0. To this end, we set $\psi_Y(v)=\psi(\langle v,Y\rangle)$, which is a C_b^2 function on $I\!R^2$. Using the fact that the flow $\mathcal{L}(V_t)$ is a solution of (1.1) in the sense of Definition 2.1, we obtain :

$$\begin{aligned}\Delta_n &= E\left(\psi_Y(V_0)-\psi_{Y_n}(V_0)\right)+\int_0^{t_n}E\left[E_\alpha\left(K_\beta^{\psi_Y-\psi_{Y_n}}(V_u,W_u)\right)\right]du\\ &+ \int_{t_n}^t E\left[E_\alpha\left(K_\beta^{\psi_Y}(V_u,W_u)\right)\right]du\\ &= A_n+B_n+C_n\end{aligned}$$

Since ψ is globally Lipschitz and since V_0 is in L^2, it is clear that A_n goes to 0. On the other hand, a simple computation shows that for every $\phi\in C_b^2(I\!R^2)$, for every $v,v^*\in I\!R^2$,

$$|K_\beta^\phi(v,v^*)|\le C\ \|\phi''\|_\infty\times\|v-v^*\|^2+C\ \|\phi'\|_\infty\times\|v-v^*\|$$

Using the Lebesgue Theorem, the fact that V and W are $I\!L_T^2$-processes, we easily prove that B_n and C_n go to 0.

We now state a second lemma.

Lemma 3.26 *Assume* (H), (S), *and* $E(V_0)=0$. *Let* $t_0>0$ *be fixed. There exist* $\eta>0$, $q>0$, *and* $\xi>0$ *(depending on* t_0*) such that for every* $t\in[t_0,T]$, *for every* $X\in I\!R^2$, *for every unit vector* $Y\in I\!R^2$,

$$P_\alpha\left(\langle W_t-X,Y\rangle^2>\eta\ ,\ \|W_t\|^2<\xi\right)>q \tag{3.21}$$

Proof : since $\sup_{[0,T]} \| W_t \|$ is in $\cap_p L^p$, it suffices to show that there exists $\eta > 0$, $q > 0$ such that for every $t \in [t_0, T]$, for every $X \in I\!R^2$, for every $Y \in I\!R^2$ such that $\| Y \|= 1$,

$$P_\alpha \left(\langle W_t - X, Y \rangle^2 > \eta \right) > 2q$$

In order to check this claim, notice (by using Bienaymé Tchebichev's inequality) that there exists $\xi > 0$ such that for every t, $P_\alpha(\| W_t \|^2 \leq \xi) > 1 - q$. We now break the proof in several steps :

Step 1 : Let $t \geq t_0$ and $\| Y \|= 1$ be fixed. Thanks to the previous section, the law of W_t admits a density on $I\!R^2$, and hence the law of $\langle W_t, Y \rangle$ admits a density with respect to the Lebesgue measure on $I\!R$. By Proposition 2.13 and since $E(V_0) = 0$, we also know that $E_\alpha(W_t) = E_\alpha(W_0) = 0$, and hence $E_\alpha(\langle W_t, Y \rangle) = 0$. It is then easy to show that there exists $\eta(t, Y) > 0$ and $q(t, Y) > 0$ such that

$$P_\alpha \left(\langle W_t, Y \rangle > \sqrt{\eta(t, Y)} \right) > 2q(t, Y) \quad \text{and} \quad P_\alpha \left(\langle W_t, Y \rangle < -\sqrt{\eta(t, Y)} \right) > 2q(t, Y)$$

Step 2 : Using Lemma 3.25, Portemanteau's Theorem, and the step 1, it is classical to show that for every t in $[t_0, T]$, for every $\| Y \|= 1$, there exists a neighbourhood $\mathcal{V}(t, Y)$ of (t, Y) such that for every $(t', Y') \in \mathcal{V}(t, Y)$,

$$P_\alpha \left(\langle W_{t'}, Y' \rangle > \sqrt{\eta(t, Y)} \right) > 2q(t, Y)$$

Let us consider a finite covering $\cup_{i=1}^N \mathcal{V}(t_i, Y_i)$ of the compact set

$$[t_0, T] \times \{Y \in I\!R^2 \big/ \| Y \|= 1\}$$

If $\eta = \inf_{i \leq N} \eta(t_i, Y_i)$ and if $q = \inf_{i \leq N} q(t_i, Y_i)$, we obtain for all $t \geq t_0$ and $\| Y \|= 1$,

$$P_\alpha(\langle W_t, Y \rangle > \sqrt{\eta}) > 2q$$

In the same way, we get

$$P_\alpha(\langle W_t, Y \rangle < -\sqrt{\eta}) > 2q$$

for all $t \geq t_0$ and $\| Y \|= 1$.

Step 3 : At last, let X be in $I\!R^2$, $t \geq t_0$, and $\| Y \|= 1$ be fixed. If $\langle X, Y \rangle \leq 0$,

$$\begin{aligned} P_\alpha(\langle W_t - X, Y \rangle^2 > \eta) &\geq P_\alpha(\langle W_t - X, Y \rangle > \sqrt{\eta}) \geq P_\alpha(\langle W_t, Y \rangle > \sqrt{\eta} + \langle X, Y \rangle) \\ &\geq P_\alpha(\langle W_t, Y \rangle > \sqrt{\eta}) > 2q \end{aligned}$$

If $\langle X, Y \rangle \geq 0$, the same kind of argument does work, and the proof is finished.

We carry on with next lemma :

Lemma 3.27 *Assume* (H), (S), *and* $E(V_0) = 0$. *Let* $t_0 > 0$ *be fixed, and let* η, q, *and* ξ *be the strictly positive numbers associated with* t_0 *introduced in the previous lemma. If* $X \in \mathbb{R}^2$, $\| Y \| = 1$, *and* $s \geq t_0$, *we consider the set :*

$$\mathcal{H}_s(X,Y) = \Big\{(\theta,\alpha) \in [-\theta_0,\theta_0] \times [0,1] \Big/ \| W_s(\alpha) \|^2 \leq \xi \quad and \quad Y^T\Gamma(X - W_s(\alpha),\theta)Y \geq \eta\Big\} \tag{3.22}$$

Then for every even positive function z *on* $[-\theta_0,\theta_0]$,

$$\iint_{\mathcal{H}_s(X,Y)} z(\theta)\beta_0(\theta)d\theta d\alpha \geq q\int_0^{\theta_0} z(\theta)\beta_0(\theta)d\theta \tag{3.23}$$

Proof : let $X \in \mathbb{R}^2$, let $\| Y \| = 1$, and let $s \geq t_0$ be fixed. Recall (see equation (3.19) in the proof of Theorem 3.23) that :

$$Y^T\Gamma(X - W_s(\alpha),\theta)Y = \langle f(\theta)Y + PY, X - W_s(\alpha)\rangle^2$$

where

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad f(\theta) = \frac{\sin\theta}{\cos\theta + 1}$$

The function f is an increasing bijection from $]-\pi,\pi[$ to $\mathbb{R}$ satisfying $f(0) = 0$. We set

$$h_s(X,PY) = \Big\{\alpha \in [0,1] \Big/ \langle W_s(\alpha) - X, PY\rangle^2 > \eta \ , \ \| W_s(\alpha) \|^2 < \xi\Big\}$$

Thanks to Lemma 3.26, we know that $P_\alpha(h_s(X,PY)) > q$. We will show that if $\alpha \in h_s(X,PY)$, then $Y^T\Gamma(X - W_s(\alpha),\theta)Y \geq \eta$ either for all $\theta \in]0,\pi[$ or for all $\theta \in]-\pi,0[$. Then the lemma will be proved.

Let $\alpha \in h_s(X,PY)$. If $\langle Y, X - W_s(\alpha)\rangle = 0$, then

$$Y^T\Gamma(X - W_s(\alpha),\theta)Y = \langle PY, X - W_s(\alpha)\rangle^2 > \eta$$

for every θ. Else, $Y^T\Gamma(X - W_s(\alpha),\theta)Y \geq \eta$ for every θ such that $f(\theta) \in \mathbb{R}\backslash[x_1,x_2]$, where $x_1 \leq x_2$ are the solutions of

$$x^2 \times \langle Y, X - W_s(\alpha)\rangle^2 + 2x \times \langle Y, X - W_s(\alpha)\rangle \langle PY, X - W_s(\alpha)\rangle + \langle PY, X - W_s(\alpha)\rangle^2 - \eta = 0$$

Hence, it suffices to show that the signes of x_1 and x_2 are equal. But

$$x_1, x_2 = \frac{-\langle PY, X - W_s(\alpha)\rangle \pm \sqrt{\eta}}{\langle Y, X - W_s(\alpha)\rangle}$$

Since $\langle PY, X - W_s(\alpha)\rangle^2 \geq \eta$, the lemma follows.

Theorem 3.28 *Assume* (H), (S), *and* $E(V_0) = 0$. *For every* $t > 0$, $(\det H_t)^{-1}$ *admits moments of all orders (and thus so does* $(\det DV_t)^{-1}$*).*

Proof : we fix $t_0 > 0$, and we prove the theorem for every $t > t_0$, which of course suffices. Our aim is to apply Lemma 6.1 to the matrix FH_t, for a well-chosen $I\!R$-valued random variable F. We will use the notations of Lemmas 3.26 and 3.27.

Since $\theta_0 < \pi$, there exists $d_0 > 0$ such that, for every $|\theta| \leq \theta_0$,

$$|\det(I + A(\theta))| = \frac{1}{2}(1 + \cos\theta) \geq d_0$$

We choose k such that $k(y) = 1$ as soon as $|\det y| \geq d_0$. For every X in $I\!R^2$,

$$\| A'(\theta)X \|^2 = \frac{1}{4} \| X \|^2$$

Hence, if α is in any set $\mathcal{H}_s(X,Y)$, then

$$h(A'(\theta)(V_s - W_s(\alpha))) \geq \left(1 + \frac{1}{4}(\| V_s \|^2 + \xi)\right)^{-1}$$

Hence, for every $\| Y \| > 0$, a simple computation (using the Lemma 3.27) shows that for every $t \geq t_0$, $Y^T H_t Y$ is greater or equal than

$$\int_{t_0}^{t} \iint_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1^T} Y}{\|M_{s-}^{-1^T} Y\|}\right)} \| M_{s-}^{-1^T} Y \|^2 \times \eta \times \left(1 + \frac{1}{4}(\| V_{s-} \|^2 + \xi)\right)^{-1} \times k(M_{s-})$$
$$\delta(\theta) N_0(d\theta d\alpha ds)$$

Let us notice that the function on $\Omega \times [0,T] \times [-\pi,\pi] \times [0,1]$ defined by

$$\begin{aligned} \omega, s, \theta, \alpha \quad \longrightarrow \quad & 1\!\!1_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1^T} Y}{\|M_{s-}^{-1^T} Y\|}\right)}(\theta, \alpha) \\ = \; & 1\!\!1_{\left\{|\theta| \leq \theta_0,\ \|W_{s-}(\alpha)\|^2 \leq \xi,\ Y^T \frac{M_{s-}^{-1}}{\|M_{s-}^{-1^T} Y\|} \Gamma(V_{s-}(\omega) - W_{s-}(\alpha), \theta) \frac{M_{s-}^{-1^T}}{\|M_{s-}^{-1^T} Y\|} Y \geq \eta\right\}} \end{aligned}$$

is predictable, because V_{s-} and M_{s-}^{-1} are predictable, and because W is a measurable α-process. Let us define the following random variable :

$$F = \sup_{[0,T]} \left\{ \left(1 + \frac{1}{4}(\| V_s \|^2 + \xi)\right) \times \left(k(M_{s-}) \| M_{s-}^{-1^T} \|_{op}^2\right)^{-1} \right\}$$

where $\| M_{s-}^{-1^T} \|_{op}$ is the operator norm of $M_{s-}^{-1^T}$. Thus, for every $\| Y \| = 1$, $t \geq t_0$,

$$F \times Y^T H_t Y \geq \eta \int_{t_0}^{t} \iint_{\mathcal{H}_s\left(V_{s-}, \frac{M_{s-}^{-1^T} Y}{\|M_{s-}^{-1^T} Y\|}\right)} \delta(\theta) N_0(d\theta d\alpha ds)$$

In order to use the Appendix (6.1), we have to compute $E\left(e^{-\zeta F\times Y^T H_t Y}\right)$ for $\zeta>0$, $t\geq t_0$. To this end, we set

$$n_\zeta(s)=\frac{q\int_0^{\theta_0}\left(1-e^{-\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta}{\iint_{\mathcal{H}_s\left(V_{s-},\frac{M_{s-}^{-1\,T}Y}{\|M_{s-}^{-1\,T}Y\|}\right)}\left(1-e^{-\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta d\alpha}$$

Choosing δ even, and using Lemma 3.27, we see that $n_\zeta(s)\in]0,1[$ a.s. for every $s\geq t_0$, $\zeta>0$. Furthermore for every $\zeta>0$, the following function on $\Omega\times[t_0,T]\times[-\pi,\pi]\times[0,1]$ is predictable and takes its values in $[0,1]$:

$$g_\zeta(s,\theta,\alpha)=-\frac{1}{\zeta\delta(\theta)}\ln\left[1-n_\zeta(s)\left(1-e^{-\zeta\delta(\theta)}\right)\right]\mathbb{1}_{\mathcal{H}_s\left(V_{s-},\frac{M_{s-}^{-1\,T}Y}{\|M_{s-}^{-1\,T}Y\|}\right)}(\theta,\alpha)$$

Hence, for every $\|Y\|=1$, $t\geq t_0$, $\zeta>0$,

$$F\times Y^TH_tY\geq\eta\int_{t_0}^t\int_0^1\int_{-\pi}^{\pi}g_\zeta(s,\theta,\alpha)\delta(\theta)N_0(d\theta d\alpha ds)=\eta Z_t(\zeta)$$

Using Itô's formula,

$$\begin{aligned}
e^{-\zeta Z_t(\zeta)} &= 1-\zeta\int_0^t e^{-\zeta Z_{s-}(\zeta)}dZ_s(\zeta)+\sum_{s\leq t}\left[e^{-\zeta Z_s(\zeta)}-e^{-\zeta Z_{s-}(\zeta)}+\zeta e^{-\zeta Z_{s-}(\zeta)}\Delta Z_s(\zeta)\right]\\
&= 1-\int_{t_0}^t\int_0^1\int_{-\pi}^{\pi}e^{-\zeta Z_{s-}(\zeta)}\left(1-e^{-\zeta g_\zeta(s,\theta,\alpha)\delta(\theta)}\right)N_0(d\theta d\alpha ds)
\end{aligned}$$

Taking the expectations, and using the expression of g_ζ, we obtain for every $t\geq t_0$, $\zeta>0$,

$$\begin{aligned}
E(e^{-\zeta Z_t(\zeta)}) &= 1-E\left(\int_{t_0}^t\int_0^1\int_{-\pi}^{\pi}e^{-\zeta Z_{s-}(\zeta)}\left(1-e^{-\zeta g_\zeta(s,\theta,\alpha)\delta(\theta)}\right)\beta_0(\theta)d\theta d\alpha ds\right)\\
&= 1-q\int_0^{\theta_0}\left(1-e^{-\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta\times\int_{t_0}^t E(e^{-\zeta Z_s(\zeta)})ds
\end{aligned}$$

Thanks to the Appendix (6.2),

$$E(e^{-\zeta Z_t(\zeta)})=\exp\left(-q(t-t_0)\int_0^{\theta_0}\left(1-e^{-\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta\right)$$

and for every $\zeta>0$, $t\geq t_0$, $\|Y\|=1$,

$$E\left(\exp\left(-\zeta F\times Y^TH_tY\right)\right)\leq E\left(e^{-\eta\zeta Z_t(\eta\zeta)}\right)\leq\exp\left(-q(t-t_0)\int_0^{\theta_0}\left(1-e^{-\eta\zeta\delta(\theta)}\right)\beta_0(\theta)d\theta\right)$$

Recall that $\beta_0(\theta)=\dfrac{k_0}{|\theta|^r}\mathbb{1}_{|\theta|\leq\theta_0}$. We choose $\delta(\theta)\geq\frac{1}{\eta}e^{-|\theta|^{-2r'}}$ for small θ (with δ even and satisfying (3.18)). Thanks to the Appendix (6.3), there exists $C>0$ and $\zeta_0\geq0$ such that for every $\zeta\geq\zeta_0$,

$$\int_0^{\theta_0}(1-e^{-\eta\zeta\delta(\theta)})\beta_0(\theta)d\theta\geq C(\ln\zeta)^3$$

Thus for every $\zeta \geq \zeta_0$, $t \geq t_0$, and $\| Y \| = 1$,

$$E\left(\exp\left(-\zeta F Y^T H_t Y\right)\right) \leq \exp\left(-Cq(t-t_0)(\ln\zeta)^3\right)$$

Hence, for every $p \geq 0$, for all $t > t_0$,

$$E\left(\int_{X\in I\!R^2} \| X \|^p \exp\left(-X^T F H_t X\right) dX\right) = \int_{\rho=0}^{\infty}\int_{\|Y\|=1} \rho^p E\left(e^{-\rho^2 F Y^T H_t Y}\right) dY\, d\rho$$

$$\leq K\int_{\rho=0}^{\sqrt{\zeta_0}} \rho^p d\rho + K\int_{\rho=\sqrt{\zeta_0}}^{\infty} \rho^p \exp\left(-Cq(t-t_0)(\ln\rho^2)^3\right) d\rho < \infty$$

Thanks to the Appendix (6.1), this yields that for every $t > t_0$, $(\det F H_t)^{-1} = (F^2 \det H_t)^{-1}$ is in every L^p. But it is possible to choose k such that F has moments of all orders : $F \leq F_1 \times F_2$, where

$$F_1 = \sup_{[0,T]}\left(1 + \frac{1}{4} \| V_s \|^2 + \frac{\xi}{4}\right) \qquad \text{and} \qquad F_2 = \sup_{[0,T]}\left(k(M_s) \| M_s^{-1T} \|_{op}^2\right)^{-1}$$

We have already seen that F_1 has moments of all orders. In order to study F_2, let us first recall some norm inequalities for a symmetric positive matrix O :

$$|\det O|^2 \leq \| O \|^4 \leq 1 + \| O \|^8 \qquad |\det O| \times \| O^{-1} \|_{op} = \| O \|_{op} \geq \| O^{-1} \|^{-1}$$

We can choose k such that for every y,

$$k(y) \geq \frac{|\det y|^2}{1 + \| y \|^8}$$

(We still assume that $k(y) = 1$ if $\det y \geq d_0$). Hence,

$$F_2 \leq \sup_{[0,T]}\left(1 + \| M_s \|^8\right) \times \sup_{[0,T]} \| M_s^{-1} \|^2$$

Since M_s and M_s^{-1} are solutions of stochastic differential equations (with initial datum I), it is classical to show that they have moments of all orders, and we can say that F has moments of all orders. Thus :

$$E\left(|\det H_t|^{-p}\right) = E\left(|F|^{2p} \times |\det F H_t|^{-p}\right) \leq E\left(|F|^{4p}\right)^{\frac{1}{2}} E\left(|\det F H_t|^{-2p}\right)^{\frac{1}{2}} < \infty$$

We have proved that for $t > t_0$, $\det H_t$ admits some inverse moments of all orders, and the theorem follows.

Proof of Theorem 3.2 : using Theorem 3.28, Proposition 3.24, Theorem 3.21, Proposition 3.17, Theorem 3.6, the theorem is immediate when $E(V_0) = 0$.
We suppose now that V_0 is not centered. We denote by (V, W) (resp. (V', W')) a solution of the nonlinear S.D.E. with initial data V_0 (resp. $V_0' = V_0 - E(V_0)$). Since V_0 satisfies (H) and (S), so does V_0'. We thus know that for every $t > 0$, the law of V_t' admits a C^∞ density $f'(t,.)$ on $I\!R^2$, and that V_t admits a density $f(t,.)$ on $I\!R^2$. On the other hand, one can check that $(V - E(V_0), W - E(V_0))$ is a solution of the nonlinear S.D.E. with initial data V_0'. Hence, by Theorem 2.12, $\mathcal{L}(V_t - E(V_0)) = \mathcal{L}(V_t')$. This yields that $f(t,v) = f'(t, v - E(V_0))$, and the theorem follows.

4 Joint regularity.

We are now interested in the joint regularity of the weak solution f of (1.1) built in Theorem 3.1. For simplicity, we state a theorem under Assumptions (H) and (S) (see Section 3), although (S) could be relaxed.

Theorem 4.1 *Assume (H) and (S). Let $f(t,v)$ be the weak solution of (1.1) on $[0,T]$ with initial data P_0 built in Theorem 3.1. The map $(t,v) \longrightarrow f(t,v)$ is continuous on $]0,T] \times I\!R^2$.*

In the whole section, (V,W) is a solution of the nonlinear S.D.E. with initial data V_0, with $\mathcal{L}(V_0) = P_0$. By Theorems 3.1 and 3.2, and since (H) and (S) hold, we know that for every $t > 0$, the law of V_t admits a C^∞ density $f(t,.)$ with respect to the Lebesgue measure on $I\!R^2$.

In the case of a classical diffusion process with jumps X_t, Bichteler, Gravereaux and Jacod give in [6] a method to study the joint smoothness of $f(t,x)$, where $f(t,x)$ is the density of the law of X_t. Their method is based on the Malliavin Calculus, and on the smoothness of the maps $t \longrightarrow E(\psi(X_t))$ for any ψ sufficiently regular. In our case, these maps are only differentiable, because our S.D.E. is not time-homogeneous, and we thus cannot apply their method.
In the case of white noise driven parabolic S.P.D.E.s, Morien [27] studies also this problem. If $X(t,x)$ denotes his solution, he proves that for any φ sufficiently regular,

$$E\left(|\varphi'(X(t+h,x)) - \varphi'(X(t,x))|\right) \leq C|h|^\alpha \parallel \varphi \parallel_\infty$$

again by using the Malliavin theory, which yields regularity results for the densities. But the nonlinearity of our S.D.E. does not allow us to prove such an inequality.

The method we use here is based on the weak continuity of $t \to \mathcal{L}(V_t)$ and on Theorem 3.2. As in the proof of Theorem 3.2, **we assume that** $E(V_0) = 0$, the generalization beeing immediate by the uniqueness in law for the nonlinear S.D.E. (see Theorem 2.12). **We also fix** $t_0 > 0$, and we prove Theorem 4.1 on $[t_0,T] \times I\!R^2$, which of course suffices. We begin with a lemma.

Lemma 4.2 *Assume (H), (S), and $E(V_0) = 0$. For every multi-index α, there exists a constant C_{α,t_0} such that for every $g \in C_b^\infty(I\!R^2)$, for every $t \in [t_0,T]$,*

$$E\left(\partial_\alpha g(V_t)\right) \leq C_{\alpha,t_0} \parallel g \parallel_\infty \tag{4.1}$$

<u>Proof</u> : we just have to study the proof of Theorem 3.6 (which can be found in [6]). Let ϕ be a random variable with values in $I\!R^2$ satisfying the assumptions of Theorem 3.6, with the same notations. Then Bichteler et al. prove that for every multi-index α, there exists a constant K_α such that for every $g \in C_b^\infty(I\!R^2)$,

$$E\left(\partial_\alpha g(\phi)\right) \leq K_{\alpha,t_0} \parallel g \parallel_\infty$$

Following closely their proof, one can check that the constants K_α depends only on the moments of the elements of C_n $(n \in I\!N)$, and on the inverse moments of $\det \sigma$.
Let us come back to our problem : here we have a family $\phi_t = V_t$ of random variables satisfying the conditions of Theorem 3.6, with $\sigma_t = DV_t$. The sets C_n^t are composed with the derivatives

of all orders of V_t and G_t. Then one can check that for any n, for every $X_t \in C_n^t$, for all $p \geq 1$, (see Theorem 3.21),

$$\sup_{[0,T]} E\left(|X_t|^p\right) < \infty$$

Furthermore, following closely the proof of Theorems 3.28 and 3.24, one can see that for every p,

$$\sup_{[t_0,T]} E\left(|\det DV_t|^{-p}\right) < \infty$$

and the lemma follows.

We now prove that our weak solution f is equicontinuous :

Proposition 4.3 *For every v in $I\!R^2$,*

$$\sup_{s\in[t_0,T]} |f(s, v+k) - f(s,v)| \longrightarrow_{\|k\|\to 0} 0 \tag{4.2}$$

Proof : following Nualart [30] Lemma 2.1.5 p 88-89, and using Lemma 4.2, one can show that if $\mathcal{L}(V_t) = P_t$, and if $\hat{P}_t$ is the Fourier transform of P_t, then for every $t \in [t_0, T]$, $\left|\hat{P}_t(v)\right| \leq \frac{C_{(2,2),t_0}}{v_x^2 v_y^2} \wedge 1$: it suffices to apply Lemma 4.2 with $\alpha = (2,2)$ and with $g(y) = e^{i\langle v,y\rangle}$. In this case, f is the following inverse Fourier transform :

$$f(t,v) = \left(\frac{1}{2\pi}\right)^2 \int_{I\!R^2} e^{-i\langle y,v\rangle} \hat{P}_t(y)dy \tag{4.3}$$

Using Lebesgue's theorem and the uniform upperbound of $\hat{P}_t$, the Proposition is immediate.

Proof of Theorem 4.1 : recall that we assume that $E(V_0) = 0$, and that $t_0 \in]0,T]$ is fixed. Using the fact that $f(t,v)$ is a weak solution of (1.1), we see that the map $t \longrightarrow f(t,v)dv$ is weakly continuous on $]0,T]$.

Let $(t,v) \in [t_0,T] \times I\!R^2$. For all $\delta > 0$,

$$\begin{aligned}
|f(t+h, v+k) - f(t,v)| &\leq |f(t+h,v+k) - f(t+h,v)| \\
&\quad + \left|f(t+h,v) - (\pi\delta^2)^{-1} \int_{B(v,\delta)} f(t+h,w)dw\right| \\
&\quad + \left|(\pi\delta^2)^{-1} \int_{B(v,\delta)} \left(f(t+h,w) - f(t,w)\right) dw\right| \\
&\quad + \left|f(t,v) - (\pi\delta^2)^{-1} \int_{B(v,\delta)} f(t,w)dw\right| \\
&= a(t,h,v,k) + b(t,h,v,\delta) + c(t,h,v,\delta) + d(t,v,\delta)
\end{aligned}$$

Let $\epsilon > 0$ be fixed. Using Proposition 4.3, we see that there exists $\eta > 0$ such that for all h and for all $\| k \| \leq \eta$, $a(t,h,v,k) \leq \epsilon/3$. Using again this proposition, one can check that there exists $\delta > 0$ such that for all h, $b(t,h,v,\delta) + d(t,v,\delta) \le \epsilon/3$. At last, using the weak continuity of $t \to \mathcal{L}(V_t) = f(t,v)dv$, there exists γ (depending on δ) such that for all $|h| \le \gamma$,

$$c(t,h,v,\delta) = (\pi\delta^2)^{-1} \left| P(V_{t+h} \in B(v,\delta)) - P(V_t \in B(v,\delta)) \right| \le \epsilon/3$$

and the theorem follows.

5 Stochastic approximations.

We now forget the two previous sections, and come back to Sections 1 and 2. We will first show the uniqueness for the nonlinear martingale problem. The aim of this section is to approximate the solution of the nonlinear martingale problem (and thus the weak solution of (1.1)) associated with an uncutoffed cross section β with the empirical law of a simulable system of particles. To this end, we will apply directly a result of Graham and Méléard (in [18]), who discuss this problem in a very general context, but in the cutoffed case. Then we will show that the solution of the nonlinear martingale problem associated with a cutoffed cross section (converging to our cross section) converges to the solution of the nonlinear martingale problem associated with our cross section. Finally, we will give a simulation algorithm.

In the whole section, the initial data $P_0 \in \mathcal{P}_2(I\!R^2)$ and the cross section β satisfying $\int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta < \infty$ are fixed. As said previously, we first prove the uniqueness for the nonlinear martingale problem (2.2).

Proposition 5.1 *The nonlinear martingale problem with initial data P_0 (see Definition 2.3) admits a unique solution $Q \in \mathcal{P}_2(I\!D_T)$.*

Proof : we just have to prove the uniqueness. Since the uniqueness in law for the nonlinear S.D.E. holds (see Theorem 2.12), it suffices to prove that any solution of the nonlinear martingale problem is the law of a solution of the nonlinear S.D.E. Let Q be a solution of the nonlinear martingale problem with initial data P_0, and let X be the canonical process on $I\!D_T = I\!D([0,T], I\!R^2)$. We also need an α-process Y of which the α-law is Q.
Applying (2.2) with $\phi(v) = v_x$ and $\phi(v) = v_y$, we see that X admits the following decomposition under Q

$$X_t = X_0 - \frac{b}{2} \int_0^t \int_0^1 (X_s - Y_s(\alpha)) d\alpha ds + M_t \tag{5.1}$$

where M is a square integrable martingale. This decomposition is unique in the sense where if $X_t = X_0 + F_t + L_t$, if F is a predictable process with finite variations, and if L is a local martingale, then (see Jacod, Shiryaev, [23], p 43)

$$F_t = -\frac{b}{2} \int_0^t \int_0^1 (X_s - Y_s(\alpha)) d\alpha ds \; ; \quad L_t = M_t$$

Then we prove that the continuous martingale part X^c of X (defined as $X^c = M^c$) vanishes. To this end, we use the Itô formula : for all $\phi \in C_b^2(I\!R^2)$,

$$\phi(X_t) \;=\; \phi(X_0) + \int_0^t \phi'_x(X_{s-}) dX_s^x + \int_0^t \phi'_y(X_{s-}) dX_s^y$$

$$+\frac{1}{2}\sum_{i,j\in\{x,y\}}\int_0^t \phi''_{ij}(X_s)d\left\langle X^{ic},X^{jc}\right\rangle_s$$
$$+\sum_{s\leq t}\left[\phi(X_s)-\phi(X_{s_-})-\phi'_x(X_{s_-})\Delta X^x_s-\phi'_y(X_{s_-})\Delta X^y_s\right]$$

Comparing this formula with (2.2), it is clear that $X^{xc}=X^{yc}=0$. Hence M is purely discontinuous, and it is the compensated sum of jumps of X.
Let us now compute the Doob-Meyer bracket $\langle M^x,M^x\rangle$. To this end, we first show that the process A_t defined below is a martingale (under Q) :

$$A_t=\sum_{s\leq t}(\Delta X^x_s)^2-\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_x(X_s,Y_s(\alpha),\theta)-X^x_s\right)^2\beta(\theta)d\theta d\alpha ds \tag{5.2}$$

Since $X^{xc}=0$ and $dX^x_s=dM^x_s-\dfrac{b}{2}\displaystyle\int_0^1(X^x_s-Y^x_s(\alpha))d\alpha ds$, we obtain by using the Itô formula :

$$(X^x_t)^2-(X^x_0)^2=2\int_0^t X^x_{s_-}dM^x_s-b\int_0^t\int_0^1 X^x_s(X^x_s-Y^x_s(\alpha))d\alpha ds+\sum_{s\leq t}(\Delta X^x_s)^2 \tag{5.3}$$

On the other hand, (2.2) with $\phi(v)=v_x^2$ allows to say that

$$\begin{aligned}(X^x_t)^2-(X^x_0)^2-\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_x(X_s,Y_s(\alpha),\theta)-X^x_s\right)^2\beta(\theta)d\theta d\alpha ds&\\ +b\int_0^t\int_0^1 X^x_s(X^x_s-Y^x_s(\alpha))d\alpha ds&\end{aligned} \tag{5.4}$$

is a martingale. Thanks to (5.3) and (5.4), it is clear that A_t is a martingale.
We apply one more time the Itô formula :

$$(M^x_t)^2=2\int_0^t M^x_{s_-}dM^x_s+\sum_{s\leq t}(\Delta M^x_s)^2$$

Since $\Delta M^x_s=\Delta X^x_s$ for every s, and since A is a martingale, we see that

$$(M^x_t)^2-\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_x(X_s,Y_s(\alpha),\theta)-X^x_s\right)^2\beta(\theta)d\theta d\alpha ds$$

is also a martingale, which implies that

$$\langle M^x,M^x\rangle_t=\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_x(X_s,Y_s(\alpha),\theta)-X^x_s\right)^2\beta(\theta)d\theta d\alpha ds \tag{5.5}$$

In the same way, one can show that

$$\langle M^y,M^y\rangle_t=\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_y(X_s,Y_s(\alpha),\theta)-X^y_s\right)^2\beta(\theta)d\theta d\alpha ds \tag{5.6}$$

and

$$\langle M^x,M^y\rangle_t=\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(c_x(X_s,Y_s(\alpha),\theta)-X^x_s\right)\left(c_y(X_s,Y_s(\alpha),\theta)-X^y_s\right)\beta(\theta)d\theta d\alpha ds \tag{5.7}$$

We thus can build a Poisson measure N on $[0,T] \times [0,1] \times [-\pi,\pi]$ with intensity measure $\beta(\theta)d\theta d\alpha ds$, independent of X_0, such that

$$M_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (c(X_{s-}, Y_{s-}(\alpha), \theta) - X_{s-})\, N(d\theta d\alpha ds)$$

Hence $X = \Phi(X, Y, X_0, N)$ (recall Definition 2.5), and the proof is complete.

We now give some notations and definitions.

1. For $l \geq 1$, we set $\beta_l(\theta) = \beta(\theta)\mathbb{1}_{\{|\theta| \geq \frac{1}{l}\}}$ which is a cutoffed cross section : $\| \beta_l \|_1 = \int_{-\pi}^{\pi} \beta_l(\theta)d\theta < \infty$. We denote by Q (resp. Q_l) the solution of the nonlinear martingale problem associated with β (resp. β_l) with initial data P_0.

2. We endow $\mathbb{D}_T$ with the Skorokhod topology. We define three metrics on $\mathcal{P}(\mathbb{D}_T)$. First, the variation metric :

$$|p - q|_T = \sup \left\{ \int_{\mathbb{D}_T} \phi(x)p(dx) - \int_{\mathbb{D}_T} \phi(x)q(dx) \Big/ \phi \in B_b(\mathbb{D}_T, \mathbb{R}),\ \| \phi \|_\infty \leq 1 \right\}$$

   The next one corresponds to the weak convergence and convergence of the second moment :

$$\rho_T(p,q) = \inf \left\{ \left(\int_{\mathbb{D}_T \times \mathbb{D}_T} \| x - y \|_\infty^2 \, r(dx, dy) \right)^{\frac{1}{2}} \Big/ r \text{ has marginals } p \text{ and } q \right\}$$

   The last metric is weaker than ρ_T and $|\quad|_T$, and corresponds to the weak convergence :

$$\tilde{\rho}_T(p,q) = \inf \left\{ \left(\int_{\mathbb{D}_T \times \mathbb{D}_T} \| x - y \|_\infty^2 \wedge 1 \, r(dx, dy) \right)^{\frac{1}{2}} \Big/ r \text{ has marginals } p \text{ and } q \right\}$$

3. Let $n, l \geq 1$ be fixed. If $h \in \mathbb{R}^2$, if $i \in \{1, ..., n\}$, we set $h.e_i = (0, ..., h, ..., 0)$, where h stands at the i-th place. We denote by $V(\beta_l, n) = (V^1(\beta_l, n), ..., V^n(\beta_l, n)) \in \mathbb{D}([0,T], (\mathbb{R}^2)^n)$ a Markov process with infinitesimal generator

$$\frac{1}{n-1} \sum_{1 \leq i \neq j \leq n} \int_{-\pi}^{\pi} [\phi\,(v + (c(v_i, v_j, \theta) - v_i).e_i) - \phi(v)]\, \beta_l(\theta)d\theta$$

   (here $v \in (\mathbb{R}^2)^n$) and such that $\mathcal{L}(V_0(\beta_l, n)) = P_0^{\otimes n}$. This Markov process describes a particle system with a simple meanfield interaction. Finally, we denote by μ_n^l the empirical measure on $\mathbb{D}_T$:

$$\mu_n^l = \frac{1}{n} \sum_{i=1}^n \delta_{V^i(\beta_l, n)}$$

Applying directly Theorem 3.1 p 121 in Graham and Méléard, [18], we obtain the following proposition :

Proposition 5.2 *Let $l \geq 1$ be fixed. There exists a constant K independent of l and n such that*

$$\left|\mathcal{L}\left(V^1(\beta_l, n)\right) - Q_l\right|_T \leq K \frac{e^{\|\beta_l\|_1 T}}{n}$$

Moreover, μ_n^l converges in probability (when n tends to infinity) to Q_l for the weak convergence metric on $\mathcal{P}(\mathbb{D}_T)$, with an estimate in $K\dfrac{e^{\frac{1}{2}\|\beta_l\|_1 T}}{\sqrt{n}}$.

We want to approximate Q with μ_n^l, for l and n large enough. We thus have to check that Q_l goes to Q when l goes to infinity.

Proposition 5.3 *We set $b_l = \int_{-\pi}^{\pi} (1-\cos\theta)\beta_l(\theta)d\theta$ and $b = \int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)d\theta$. There exists a constant C depending only on P_0 and b such that*

$$\rho_T(Q, Q_l) \leq C\left((b-b_l) + (b-b_l)^2\right)$$

<u>Proof</u> : we consider on Ω two independent Poisson measures N_l and N_l^Δ on $[0,T]\times[0,1]\times[-\pi,\pi]$, with intensity measures $\beta_l(\theta)d\theta d\alpha ds$ and $(\beta(\theta)-\beta_l(\theta))d\theta d\alpha ds$. We set $N = N_l + N_l^\Delta$, which is a Poisson measure with intensity $\beta(\theta)d\theta d\alpha ds$. We also consider a random variable V_0 independent of N_l and N_l^Δ, such that $\mathcal{L}(V_0) = P_0$. In the same way than in Subsection 2.1 (see Definition 2.10 and Theorem 2.11), it is possible to build two $\mathbb{L}_T^2$-processes V and V^l, two $\mathbb{L}_T^2$-α-processes W and W^l satisfying :

$$V = \Phi(V, W, V_0, N, b)\ ; \quad V^l = \Phi(V^l, W^l, V_0, N_l, b_l)\ ; \quad \mathcal{L}_\alpha(W, W^l) = \mathcal{L}(V, V^l)$$

(We precise here the dependence in b of the map Φ). In particular, (V,W) is a solution of the nonlinear S.D.E. associated with β with initial data V_0. Thus $\mathcal{L}(V) = \mathcal{L}_\alpha(W) = Q$. In the same way, $\mathcal{L}(V^l) = \mathcal{L}_\alpha(W^l) = Q_l$, and hence

$$\rho_T(Q, Q_l) \leq E\left(\sup_{[0,T]} \| V_t - V_t^l \|^2\right)^{\frac{1}{2}}$$

The proposition follows easily, by using a computation quite similar to the proof of Proposition 2.8, and by using Proposition 2.13.

Using the previous propositions, the following theorem is immediate :

Theorem 5.4 *There exists some constants L_1 and L_2 such that for all l, n :*

$$\tilde{\rho}_T\left(\mathcal{L}\left(V^1(\beta_l, n)\right), Q_l\right) \leq L_1 \frac{e^{\|\beta_l\|_1 T}}{n} + L_2\left((b-b_l) + (b-b_l)^2\right)$$

Choosing a suitable sequence $l(n)$ going to infinity with n, such that $\dfrac{e^{\|\beta_{l(n)}\|_1 T}}{n} \longrightarrow 0$ when n tends to infinity, we deduce that $\tilde{\rho}_T\left(\mathcal{L}\left(V^1(\beta_{l(n)}, n)\right), Q_l\right)$ goes to 0.
With the same sequence $l(n)$, the empirical measure $\mu_n^{l(n)}$ converges in probability to Q for the weak convergence metric on $\mathcal{P}(\mathbb{D}_T)$.

It remains to simulate $V(n) = V(\beta_{l(n)}, n)$. Recall that this process describes an interacting system of n particles, and that for each i, $V^i(n)$ describes the evolution of the speed of the i-th particle. This process can be written as

$$V_t(n) = V_0(n) + \sum_{1\le i\neq j\le n} \int_0^t \int_{-\pi}^{\pi} \left[(c(V^i_{s-}, V^j_{s-}, \theta) - V^i_{s-}).e_i\right] \mu^{ij}(d\theta ds)$$

where $\mathcal{L}(V_0(n)) = P_0^{\otimes n}$, where μ^{ij} are independent Poisson measures on $[0,T] \times [-\pi, \pi]$ with intensity measures

$$\frac{\beta_{l(n)}(\theta)}{n-1} d\theta ds$$

independent of $V_0(n)$.

We of course assume that $V_0(n)$ is simulated, and we denote by

$$0 < T_1 < T_2 < ...$$

the successive times of jump of a standard Poisson process with parameter $n \parallel \beta_{l(n)} \parallel_1$. These times are modeling the times of collision.
Before the first collision, the velocities do not change, so that we set

$$\forall\ s < T_1, \quad V_s(n) = V_0(n)$$

Let us describe the first collision. First, we choose a couple (i,j) of particles, according to an uniform distribution on $\{(k,l) \in \{1,...,n\}^2 | k \neq l\}$. Then we choose the angle of collision θ by using a $\dfrac{\beta_{l(n)}(\theta)}{\parallel \beta_{l(n)} \parallel_1} d\theta$-distribution. We set

$$\begin{aligned} V^i_{T_1}(n) &= c(V^i_0(n), V^j_0(n), \theta) \\ V^k_{T_1}(n) &= V^k_0(n) \text{ if } k \neq i \end{aligned}$$

Since the velocities do not change between two collisions, we finally set $V_s(n) = V_{T_1}(n)$ for all $s \in [T_1, T_2[$.

Iterating this method (T_n goes to infinity a.s.), we can simulate $V(n)$, and thus approximate Q.

Notice here that we do not really simulate a gas : in the collisions between **two** particles, we do only change the speed of **one** particle. It would also work if we were changing both speeds (see the Bird algorithm in Graham and Méléard, [18]). However, the most natural system is this simple meanfield approach, (related to the Nanbu algorithm), since the expression of the martingale problem drives immediately to this algorithm.

6 Appendix.

We begin this annex by a lemma that can be found in [6], p 92 :

Lemma 6.1 *For every $p > 0$, there exists a constant C_p such that for every 2×2 symmetric positive matrix A,*

$$(det\ A)^{-p} \leq C_p \int_{X \in \mathbb{R}^2} \| X \|^{4p-2} e^{-X^T AX} dX$$

The following lemma is well-known, and can be shown as Gronwall's Lemma.

Lemma 6.2 *Let $0 \leq \epsilon < T < \infty$. Let g be a bounded function on $[\epsilon, T]$, and let a be a real number. Assume that for every $t \in [\epsilon, T]$,*

$$g(t) = 1 - a \int_\epsilon^t g(s) ds$$

Then $g(t) = e^{-a(t-\epsilon)}$ on $[\epsilon, T]$.

The next lemma is a simple computation :

Lemma 6.3 *Let $r \in]1, 3[$, let $r'' = \frac{1}{4}(r-1)$, and let $\epsilon > 0$. We set $\delta(\theta) = e^{-\theta^{-r''}}$. There exists a constant $C > 0$, a real number $\zeta_0 \geq 0$, such that for every $\zeta \geq \zeta_0$,*

$$\int_0^\epsilon \left(1 - e^{-\zeta\delta(\theta)}\right) \frac{d\theta}{\theta^r} \geq C(\ln \zeta)^3$$

Proof : we first notice that for every $x \in [0, 1]$, one has $1 - e^{-x} \geq \frac{x}{2}$. Furthermore, for every $\theta < 1$, $\delta^{-1}(\theta) = (\ln \theta^{-1})^{-\frac{1}{r''}}$. Hence, if ζ_0 is large enough (we need $\zeta_0^{-1} < 1$ and $\delta^{-1}(\zeta_0^{-1}) < \epsilon$), then for all $\zeta \geq \zeta_0$,

$$I(\zeta) = \int_0^\epsilon \left(1 - e^{-\zeta\delta(\theta)}\right) \frac{d\theta}{\theta^r} \geq \frac{\zeta}{2} \int_0^{\delta^{-1}(\zeta^{-1})} \frac{\delta(\theta)}{\theta^r} d\theta \geq \frac{\zeta}{2r''} \int_0^{\delta^{-1}(\zeta^{-1})} \frac{r''}{\theta^{r''+1}} \delta(\theta) \times \theta^{r''+1-r} d\theta$$

Since $r - r'' - 1 = \frac{3}{4}(r-1) > 0$, and since $\delta'(\theta) = \frac{r''}{\theta^{r''+1}} \delta(\theta)$, we obtain :

$$I(\zeta) \geq \frac{\zeta}{2r''} \times \left(\delta^{-1}(\zeta^{-1})\right)^{-\frac{3}{4}(r-1)} \times [\delta(\theta)]_0^{\delta^{-1}(\zeta^{-1})} = \frac{1}{2r''} (\ln \zeta)^3$$

which was our aim.

The following lemma is adapted from a lemma in the Appendix of [7]. We state it for N and β, but it can be obviously adapted to N_0 and β_0 or N_1 and β_1.

Lemma 6.4 *Let $Y(s, \alpha, \theta)$ be a predictable process such that $|Y(s, \alpha, \theta)| \leq |X(s, \alpha)| z(\theta)$. Then*

- *if z is in $\cap_{p \geq 2} L^p(\beta(\theta) d\theta)$, for every $p = 2^q$,*

$$E\left(\sup_{[0,t]} \left| \int_0^s \int_0^1 \int_{-\pi}^{\pi} Y(u, \alpha, \theta) \tilde{N}(d\theta d\alpha du) \right|^p\right) \leq C_p(z) \int_0^t \int_0^1 E\left(|X(s, \alpha)|^p\right) d\alpha ds$$

- *if z is in $L^1(\beta(\theta)d\theta)$, then for every $p < \infty$,*

$$E\left(\sup_{[0,t]}\left|\int_0^s\int_0^1\int_{-\pi}^{\pi} Y(u,\alpha,\theta)d\theta d\alpha du\right|^p\right) \leq C_p(z)\int_0^t\int_0^1 E\left(|X(s,\alpha)|^p\right)d\alpha ds$$

- *if z is in $\cap_{p\geq 1}L^p(\beta(\theta)d\theta)$, for every $p = 2^q$,*

$$E\left(\sup_{[0,t]}\left|\int_0^s\int_0^1\int_{-\pi}^{\pi} Y(u,\alpha,\theta)N(d\theta d\alpha du)\right|^p\right) \leq C_p(z)\int_0^t\int_0^1 E\left(|X(s,\alpha)|^p\right)d\alpha ds$$

We at last recall the Girsanov Theorem for random measures, reducing to our context the statement that can be found in the book of Jacod, Shiryaev, [23]. If P is a probability measure on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, we denote by $\mathcal{P}$ (resp. $\tilde{\mathcal{P}}$) the predictable σ-field on $\Omega\times[0,T]$ (resp. $\Omega\times[0,T]\times[-\pi,\pi]\times[0,1]$). Let μ be a $\tilde{\mathcal{P}}$-σ-finite random measure on $[0,T]\times[-\pi,\pi]\times[0,1]$. If W_t is an optional process on $[0,T]$, we denote by $M_{P\otimes\mu}\left(W|\,\tilde{\mathcal{P}}\right)$ the unique predictable function on $\Omega\times[0,T]\times[-\pi,\pi]\times[0,1]$ satisfying, for every predictable function U,

$$E\left(\int U(s,\theta,\alpha)M_{P\otimes\mu}\left(W|\,\tilde{\mathcal{P}}\right)(s,\theta,\alpha)\mu(d\theta d\alpha ds)\right) = E\left(\int U(s,\theta,\alpha)W_s\mu(d\theta d\alpha ds)\right)$$

We now state the Girsanov Theorem, and a well-known characterisation of the Poisson measures.

Theorem 6.5 *1. Let $P' << P$ be two probability measures on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, and assume that $P'|_{\mathcal{F}_t} = G_t.P|_{\mathcal{F}_t}$. Let μ be a $\tilde{\mathcal{P}}$-σ-finite random measure on $[0,T]\times[-\pi,\pi]\times[0,1]$, with compensator ν under P and ν' under P'. Then μ is $\tilde{\mathcal{P}}'$-σ-finite, and any positive version Y of $M_{P\otimes\mu}\left(\frac{G}{G_-}\mathbb{1}_{\{G_->0\}}\middle|\,\tilde{\mathcal{P}}\right)$ satisfies $\nu' = Y.\nu$.*

2. Any integer valued random measure μ on $[0,T]\times[-\pi,\pi]\times[0,1]$ satisfying $\mu(\{0\}\times[-\pi,\pi]\times[0,1]) = 0$ a.s., and of which the compensator ν is deterministic (under a probability measure Q on Ω) is a Poisson measure with intensity ν (under Q).

Chapitre 5

Strict Positivity of the Density for a Poisson Driven S.D.E.

Abstract : We consider a one-dimensional stochastic differential equation driven by a compensated Poisson measure. We assume that this equation admits a unique solution X_t. We prove that under a strong non-degeneracy condition, for each $t > 0$, the law of X_t is bounded below by a measure admitting a strictly positive continuous density with respect to the Lebesgue measure on $\mathbb{R}$. To this aim, we develop Bismut's approach of the Malliavin calculus for Poisson functionals.

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1 Introduction.

Consider the following stochastic differential equation :

$$X_t = x_0 + \int_0^t \int_O h(X_{s-}, z)\tilde{N}(ds, dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-}, u)\tilde{N}_1(ds, du) \tag{1.1}$$

where

<u>Assumption (M)</u> : N and N_1 are two independant Poisson measures on $\mathbb{R}^+ \times O$ and $\mathbb{R}^+ \times E$, where O is an open subset of $\mathbb{R}$ and E is a Blackwell space (see Jacod, Shiryaev, [23], p 65). The intensity measures of N and N_1 are respectively

$$\nu(ds, dz) = \varphi(z)dsdz \quad ; \quad \nu_1(ds, du) = dsq(du) \tag{1.2}$$

where φ is a strictly positive C^1 function on O, and q is a positive σ-finite measure on E. We denote by $\tilde{N}$ and $\tilde{N}_1$ the associated compensated measures.

We assume that equation (1.1) admits a unique solution, and we fix $T > 0$. Our problem is to prove, under some conditions, the existence of a strictly positive continuous function $\theta : I\!R \mapsto I\!R^+$ such that for all $f \in C_b^+(I\!R)$,

$$E(f(X_T)) \geq \int_{I\!R} f(y)\theta(y)dy \tag{1.3}$$

In particular, if the law of X_T admits a continuous density p_T with respect to the Lebesgue measure on $I\!R$, this will yield that $p_T(x) > 0$ for all $x \in I\!R$. In order to study this problem, we transpose to our context a method based on the Malliavin Calculus for Gaussian functionals, investigated by Ben Arous and Léandre in [5], and later by Bally and Pardoux in [4]. Bismut's approach of the Malliavin Calculus is used in [4]. Following this approach and the work of Bichteler and Jacod, [7], we will build a sequence of perturbations, then we will differentiate our perturbed process and study the obtained derivatives.

Comparing our work with that of Bally, Pardoux, we see that the big difficulty (and also the limit) of our work is that when we differentiate, we obtain integrals against the Poisson measure instead of the Lebesgue measure. We thus have to choose non deterministic perturbations and to deal with stopping times, which makes everything hard and drives to stringent conditions.

Of course, the result we obtain here in a quite general context is not completely satisfying : as for the existence of a smooth density, the assumptions we need are not very explicit. However, we have applied in Chapter 6 the present method for the case of a particular nonlinear S.D.E. in order to study the strict positivity of the solution of a Kac equation, and we have obtained quite a good result.

The present work is organized as follows. In the second section, we recall the assumptions given by Bichteler, Gravereaux, and Jacod in [6] under which the law of X_T admits a continuous density (in the case where $\varphi \equiv 1$), then we give our assumptions and we state our main result. The third section is devoted to the exposition of our notations and to the proof of the criterion of strict positivity we will use. Finally, we prove our main theorem in the last sections.

Let us now mention alternative methods that could be used to study our problem. First, one might use the Markov property of equation (1.1), in order to obtain minorations of the density. This method looks natural, but in fact, it seems difficult to apply, and probably necessitates more regularity of the density. Another idea could consist in applying the results of Simon, [41], who characterizes the support (in $I\!D([0,T], I\!R)$) of the law of X, and in using a method as that of Millet, Sanz, [29].

In the case of equation (1.1), all these methods might work and give different results. Anyway, our method, based on the stochastic calculus of variations, seems to be the only (probabilistic) way to prove that the solution of the Kac equation is strictly positive. Indeed, the S.D.E. associated with the Kac equation is nonlinear, thus it may be very difficult to use a Markov property or to prove a support theorem.

In [24], Léandre, studies the behaviour of the density of X_t when t goes to 0, at the points $x \neq x_0$ that can be reached in one jump by X_t. Ishikawa, [21], studies also other points, in the case where X_t has finite variations. Finally, let us mention Picard, [36], who studies the density in small time of X, at the points $x \neq x_0$ that X can reach in a finite number of jumps.

2 Statement of the main result.

This section is divided in three parts. In the first part, we recall a result of Bichteler et al. in [6]. We state our assumptions and results in the second one. At last, the third part deals with remarks and examples of applications.

2.1 Existence of a continuous density in the case where $\varphi \equiv 1$.

When $\varphi \equiv 1$, Bichteler, Gravereaux, and Jacod give in [6] a sufficient condition under which the law of X_T admits a continuous density. In fact, they do not prove a minimal assumption for this problem, since they are interested in the (at least) continuous differentiability of the density.
When φ is a function, the existence of a density for equation (1.1) has been studied in Chapter 2, see also Denis, [14] for a much more general intensity measure, but no continuity result seems to be known, even if the method of Bichteler et al. in [6] could probably be easily extended. Let us recall the assumptions in [6].

Assumption $(A-4)$: the function g is four times differentiable on $I\!R$, and its derivatives of order 1 to 4 are bounded. The function h is four times differentiable on $I\!R \times O$, the partial derivatives $h^{(n+q)}_{x^n z^q}$ are bounded as soon as $q \geq 1$ (with $n+q \leq 4$), and there exists a function $\eta \in \cap_{2 \leq p < \infty} L^p(O, dz)$ such that

$$|h(0,z)| + |h'_x(x,z)| + ... + |h^{(4)}_{x^4}(x,z)| \leq \eta(z) \tag{2.1}$$

For any $u \in E$, the function $f(.,u)$ is four times differentiable on $I\!R$, and there exists a function $\sigma \in \cap_{2 \leq p < \infty} L^p(E, q)$ such that

$$|f(0,u)| + |f'_x(x,u)| + ... + |f^{(4)}_{x^4}(x,u)| \leq \sigma(u) \tag{2.2}$$

Assumption (SC) : there exists $c_0 > 0$ such that identically,

$$1 + h'_x(x,z) \geq c_0 \quad ; \quad 1 + f'_x(x,u) \geq c_0 \tag{2.3}$$

In [6], a positive function δ on O is called (ζ, θ)-broad (for some fixed $\zeta \geq 0$, $\theta > 0$) if

$$\int_0^\infty \lambda^{\zeta - 1} \exp\left\{ -\theta \int_O \left(1 - e^{-\lambda \delta(z)}\right) dz \right\} d\lambda < \infty \tag{2.4}$$

and one also considers functions α on O satisfying

$$\alpha \geq 0 \quad ; \quad \alpha \text{ is } C^\infty_b \text{ on } O \quad ; \quad \alpha(z) \longrightarrow_{z \to \partial O} 0 \quad ; \quad \forall\, r \in I\!N,\ \frac{\partial^r \alpha}{\partial z^r} \in L^1(O, dz) \tag{2.5}$$

Their last hypothesis is :

Assumption $(SB)(\zeta,\theta)$: there exists $\epsilon > 0$, $q \geq 0$, a (ζ,θ)-broad function δ, and a function α satisfying (2.5) such that

$$\frac{h_z'^2(x,z)\alpha(z)}{(1+h_x'(x,z))^2} \geq \frac{\epsilon}{1+|x|^q}\delta(z) \tag{2.6}$$

We now can state the following partial result of [6] :

Theorem 2.1 *Let $T > 0$ be fixed. Assume (M) with $\varphi \equiv 1$, $(A-4)$, (SC), and $(SB)(\zeta,\theta)$ for some $\theta \leq T$, and some $\zeta > \frac{4}{\left[\frac{T}{\theta}\right]}$. Then the law of X_T admits a C^1 density $p_T(x)$ with respect to the Lebesgue measure on $I\!R$.*

2.2 Strict positivity of the density.

Let us now turn back to our problem. A classical way to write S.D.E.s consists in assuming that $h(x,z)$ splits into $\psi(x)\eta(z)$. Although we state a more general formulation, all the hypotheses below are especially adapted to this case.
The two first assumptions are quite similar to $(A-4)$ and (SC).

Assumption (H) : the function g is C^3 on $I\!R$, and its derivatives g', g'', and g''' are bounded. The function h admits the continuous partial derivatives $h^{(n+q)}_{x^n z^q}$ for $n,q \in \{0,1,2,3\}$ on $I\!R \times O$, the derivatives $h^{(n+q)}_{x^n z^q}$ are bounded as soon as $q \geq 1$, and there exists a function $\eta \in L^2(O,\varphi(z)dz) \cap L^\infty(O,\varphi(z)dz)$ such that

$$|h(0,z)| + |h_x'(x,z)| + |h_{xx}''(x,z)| + |h_{xxx}'''(x,z)| \leq \eta(z) \tag{2.7}$$

For any $u \in E$, the function $f(.,u)$ is C^3 on $I\!R$, and there exists a function $\sigma \in L^2(E,q) \cap L^\infty(E,q)$ such that

$$|f(0,u)| + |f_x'(x,u)| + |f_{xx}''(x,u)| + |f_{xxx}'''(x,u)| \leq \sigma(u) \tag{2.8}$$

Assumption (P) : there exists $c_0 > 0$ such that for all $x \in I\!R$, all $z \in O$,

$$1 + h_x'(x,z) \geq c_0 \tag{2.9}$$

For all $x \in I\!R$,

$$\int_E 1_{\{1+f_x'(x,u)=0\}} q(du) = 0 \tag{2.10}$$

In order to state our non-degeneracy condition, we introduce some notation.

Notation 2.2 *We denote by ∂O the boundary of O in $\bar{I\!R} = I\!R \cup \{-\infty,+\infty\}$.*

Notation 2.3 *We set*

$$\beta(z)=\sup\left\{\left|h^{(2+q)}_{x^q z^2}(x,z)\right| \ ; \ x\in I\!R,\ q\in\{0,1,2,3\}\right\} \tag{2.11}$$

Consider a C^1 positive function α on O such that $\|\alpha'\|_\infty<1$ and such that $\alpha(z)$ goes to 0 when z goes to the boundary of O (this implies that for all $z\in O$, $[z-\alpha(z),z+\alpha(z)]$ is contained in O). Then we set

$$\phi_\alpha(z)=\frac{\sup\{|\varphi'(w)|\ ;\ |w-z|\le\alpha(z)\}}{\varphi(z)} \quad ; \quad \xi_\alpha(z)=|\alpha'(z)|+3\alpha(z)\phi_\alpha(z) \tag{2.12}$$

$$\zeta_\alpha(z)=\sup\left\{\left|h^{(3+q)}_{x^q z^3}(x,w)\right| \ ; \ x\in I\!R\ ,\ q\in\{0,1,2,3\}\ ,\ |w-z|\le\alpha(z)\right\} \tag{2.13}$$

Our last hypothesis is :

Assumption (SP) :

1. There exist continuous functions $\psi>0$ on $I\!R$ and $\delta\ge 0$ on O such that :

$$\psi(x)\delta(z)\le|h'_z(x,z)|\le\delta(z) \tag{2.14}$$

$$\forall n\in\{1,2,3\},\quad |h^{(n+1)}_{x^n z}(x,z)|\le\delta(z) \tag{2.15}$$

Thanks to (H), ψ and δ are bounded.

2. There exists a sequence of C^1 positive functions α_n on O, a sequence of real numbers a_n decreasing to 0, a constant $d_0\in]0,1[$ such that, if $\xi_n=\xi_{\alpha_n}$, $\zeta_n=\zeta_{\alpha_n}$, (see Notation 2.3)

$$\alpha_n+\xi_n\in L^1(O,\varphi(z)dz)\ ;\ \|\alpha_n\|_\infty+\|\xi_n\|_\infty\le d_0\ ;\ \alpha_n(z)\longrightarrow_{z\to\partial O}0 \tag{2.16}$$

$$\forall\, n\ ,\ \{\delta=0\}\subset\{\alpha_n=0\} \tag{2.17}$$

and

$$a_n\int_O\alpha_n(z)\delta(z)\varphi(z)dz\longrightarrow+\infty \tag{2.18}$$

$$a_n\int_O\left[\alpha_n^2(z)\delta^2(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]\varphi(z)dz\longrightarrow 0 \tag{2.19}$$

Now we can state our main result :

Theorem 2.4 *Assume (M), (H), (P), (SP) and let $T>0$ be fixed. Then the law of X_T is bounded below by a measure that admits a strictly positive continuous density with respect to the Lebesgue measure on $I\!R$. This means that there exists a strictly positive continuous function $\theta: I\!R\mapsto I\!R^+$ such that for all $f\in C_b^+(I\!R)$,*

$$E(f(X_T))\ge\int_{I\!R}f(y)\theta(y)dy \tag{2.20}$$

In particular, if the law of X_T admits a continuous density p_T, then for every $x\in I\!R$, $p_T(x)>0$.

Let us say a word about N_1. We add an independent Poisson measure, in order to generalize our Theorem. Assume for example that the parameter h of equation (1.1) satisfies (SP) only on an open subset $A \subset O$. Then, replacing N by $N|_A$ and N_1 by $N_1 + N|_{O/A}$, the conditions might hold. The first part of Proposition 2.5 below will give an example for this kind of method.

The main supposition in (SP) is the following : $h'_z(x,.) \notin L^1(O, \varphi(z)dz)$ (obtained by (2.14), (2.16), and (2.18)). This assumption looks like (SB), but is much more stringent.
Somewhere in the proof of Theorem 2.4, we will need a function $v(s,z)$ of class C^1 on O such that $h'_z \times v \geq 0$. The first idea consists in choosing $v = h'_z \times w$, with w nonnegative, but this would drive us to the assumption $h'_z(x,.) \notin L^2(O, \varphi(z)dz)$, which is more stringent than (SP). This is why we state the following proposition.

Proposition 2.5 *1. Thanks to (2.17), we can assume that the sign of h'_z is constant on $I\!R \times O$.*

2. In (SP), we can assume that $d_0 \in]0,1[$ is as small as we want.

Proof :
1. We consider the solution X of equation (1.1). We assume only (SP), and we prove that X is the solution of another S.D.E. satisfying (SP), with $h'_z \geq 0$ (or $h'_z \leq 0$) identically. We first set

$$\mathcal{H} = \{z \in O \ / \ \delta(z) > 0\}$$

$$\mathcal{H}^+ = \{z \in \mathcal{H} \ / \ \forall x \in I\!R \ , \ h'_z(x,z) > 0\} \quad ; \quad \mathcal{H}^- = \{z \in \mathcal{H} \ / \ \forall x \in I\!R \ , \ h'_z(x,z) < 0\}$$

Since h'_z, and ψ are continuous, since ψ does never vanish, and since (2.14) is satisfied, we see that $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. We furthermore deduce that

$$\mathcal{H}^+ = \{z \in O \ / \ h'_z(0,z) > 0\} \tag{2.21}$$

is an open set. Of course, so is $\mathcal{H}^-$. From the definition of $\mathcal{H}$, and since $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, equation (2.18) yields the existence of a subsequence n_k such that either

$$a_{n_k} \int_{\mathcal{H}^+} \alpha_{n_k}(z)\delta(z)\varphi(z)dz \longrightarrow +\infty \tag{2.22}$$

or

$$a_{n_k} \int_{\mathcal{H}^-} \alpha_{n_k}(z)\delta(z)\varphi(z)dz \longrightarrow +\infty \tag{2.23}$$

Let us for example assume (2.22). We rewrite equation (1.1) as

$$\begin{aligned} X_t &= x_0 + \int_0^t \int_{\mathcal{H}^+} h(X_{s-},z)\tilde{N}^*(ds,dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-},u)\tilde{N}_1(ds,du) \\ &\quad + \int_0^t \int_{O\backslash\mathcal{H}^+} h^*(X_{s-},z)\tilde{N}_2(ds,dz) \end{aligned} \tag{2.24}$$

where $N^* = N|_{I\!R^+ \times \mathcal{H}^+}$, $N_2 = N|_{I\!R^+ \times (O \backslash \mathcal{H}^+)}$, and $h^*(x,z) = h(x,z)1_{z \in (O\backslash\mathcal{H}^+)}$. It is clear that our method would not fail with two independent Poisson measures (independent of N^*) instead of one. Furthermore, (M), (H), and (P) are clearly satisfied with the new coefficients. At last, the only problem to check that (SP) is still satisfied (with of course h'_z nonnegative on $\mathcal{H}^+$) is to see that for all n, $\alpha_n(z)$ goes to 0 when z goes to $\partial\,\mathcal{H}^+$.

Let $z_0 \in \partial\,\mathcal{H}^+$. Then Thanks to (2.21), we see that either $z_0 \in \partial O$ either $\lim_{z_0} h'_z(0,z) = 0$. In the first case, there is no problem. In the second case, we obtain, from (2.14), $\lim_{z_0} \delta(z) = 0$. Since $\{\delta = 0\} \subset \{\alpha_n = 0\}$, we immediately conclude.

2. It suffices to notice that we can replace α_n by $C \times \alpha_n$ in (SP), for any $C \in]0,1]$ fixed.

In the whole work, we will assume that $T > 0$ is fixed, that (M), (H), (P) and (SP) are satisfied. We will also suppose that h'_z is nonnegative on $\mathbb{R} \times O$.

2.3 Examples.

First, we give one "simple" way to obtain (SP) when $\varphi \equiv 1$. Similar methods may be used for any other particular φ.

Remark 2.6 *Assume that $\varphi \equiv 1$, that (SP)-1 and (SPI) below hold. Then (SP) is satisfied.*

Assumption (SPI) : there exists a positive C_b^1 function γ on O such that, for some $d_0 \in]0,1[$,

$$\|\gamma\|_\infty + \|\gamma'\|_\infty \le d_0 \quad ; \quad \gamma(z) \longrightarrow_{z \to \partial O} 0 \quad ; \quad \{\delta = 0\} \subset \{\gamma = 0\} \tag{2.25}$$

$$\delta\gamma \notin L^1(O,dz) \quad ; \quad \gamma^2\delta^2 + \gamma^2\beta + \gamma^3\zeta_\gamma \in L^1(O,dz) \tag{2.26}$$

where ζ_γ is defined in Notation 2.3.

Proof : in order to check this claim, choose a smooth version of

$$\alpha_n(z) = \begin{cases} \gamma(z) & \text{if} \quad z \in O\,,\ |z| \le n \\ \\ 0 & \text{if} \quad z \in O\,,\ |z| \ge n+1 \end{cases} \tag{2.27}$$

and set $a_n = \left(\int_O \alpha_n(z)\delta(z)dz \right)^{-\frac{1}{2}}$.

At last, we give two examples of function $h(x,z)$ satisfying (H), (P), and (SP).

Example 1 : we assume that $O =]1,\infty[$, that $\varphi \equiv 1$, and that $h(x,z) = c(x)\eta(z)$, where $\eta(z) = \sin z/z$, and where c is a strictly positive C^3 function on $\mathbb{R}$, bounded with all its derivatives. A simple computation shows that there exists $C < \infty$ such that $|\eta(z)| + ... + |\eta'''(z)| \le C/z$. Thus (H) is satisfied. If $\|c'\|_\infty < 1$, it is clear that (P) is satisfied. Else, we replace O by $]2\|c'\|_\infty, \infty[$, and we use the presence of N_1 as in the proof of Proposition 2.5-1. Condition (SP)-1 is satisfied with $\psi(x) = c(x)/A$, where $A = \|c\|_\infty + ... + \|c'''\|_\infty$, and with

$$\delta(z) = A|\eta'(z)| = A\left|\frac{z\cos z - \sin z}{z^2}\right| \tag{2.28}$$

Thus $\delta(z)$ behaves as $1/z$ on a "large" subset of O. More precisely, there exists a constant $K<\infty$ such that

$$\text{for all} \quad z \in \cup_{k\geq 1}]2k\pi - \pi/4, 2k\pi+\pi/4[\,, \quad \delta(z) \geq A\frac{\sqrt{2}}{2} \times \frac{z-1}{z^2} \geq \frac{K}{z} \tag{2.29}$$

Hence we can say that δ, β, and ζ_γ, defined in (SP), will behave as $1/z$. We now search a function γ such that h satisfies (SPI) : we need $\gamma\delta \notin L^1(O,dz)$, but $\gamma^2\beta+\gamma^3\zeta_\gamma \in L^1(O,dz)$. We thus will choose γ behaving as $1/\ln z$. We also need that $\{\delta=0\}\subset\{\gamma=0\}$. This will hold if the support of γ is contained in $\cup_{k\geq1}]2k\pi-\pi/4, 2k\pi+\pi/4[$. At last, γ has to be C^1, and bounded with its derivative. The function we search is given by $\gamma(z)=b_0\sum_{k\geq1}\gamma_k(z)$, where for each k, the function γ_k is C^1 on O, satisfies

$$\gamma_k(z) = \begin{cases} \dfrac{1}{\ln(k+1)} & \text{if} \quad z \in [2k\pi-\pi/8, 2k\pi+\pi/8] \\ \\ 0 & \text{if} \quad z \notin [2k\pi-\pi/4, 2k\pi+\pi/4] \end{cases} \tag{2.30}$$

and $0\leq\gamma_k\leq\dfrac{1}{\ln(k+1)}$, $|\gamma_k'|\leq\dfrac{16}{\pi}\dfrac{1}{\ln(k+1)}$. For b_0 small enough, $\parallel\gamma\parallel_\infty + \parallel\gamma'\parallel_\infty\leq d_0$, for some $d_0<1$. At last, it is clear that (2.26) is satisfied.

Example 2 : we now set $O=]0,\infty[$, and we consider the standard Lévy measure on O, $\varphi(z)dz = z^{-2}dz$. We assume that $h(x,z)=c(x)\eta(z)$, where c is as in example 1, and where $\eta(z)=z/(z+1)$. Assumptions (H) and (P) are obviously satisfied, at least if $\parallel c'\parallel_\infty<1$, and (SP)-1 is met with $\psi(x)=c(x)/A$, where $A=\parallel c\parallel_\infty + ... + \parallel c'''\parallel_\infty$, and with

$$\delta(z) = A|\eta'(z)| = A(1+z)^{-2} \tag{2.31}$$

We now consider a sequence of C^1 nonnegative functions α_n on O satisfying, for some $0<k<1/2$, $\alpha_n(z)\leq k(z\wedge 2)$, and

$$\alpha_n(z) = \begin{cases} 0 & \text{if} \quad z \in]0,1/n] \\ \\ kz & \text{if} \quad z\in]2/n,1] \\ \\ 0 & \text{if} \quad z\geq 2 \end{cases} \quad \text{and} \quad |\alpha_n'(z)| \leq \begin{cases} 0 & \text{if} \quad z\in]0,1/n] \\ \\ 4k & \text{if} \quad z\in]1/n,2] \\ \\ 0 & \text{if} \quad z\geq2 \end{cases} \tag{2.32}$$

One can easily check that for some constants B, C, $\phi_{\alpha_n}(z)\leq B/z$, and that $\xi_{\alpha_n}(z)\leq|\alpha_n'(z)| + C\alpha_n(z)/z$. Choosing k small enough, we see that $\parallel\alpha_n\parallel_\infty + \parallel\xi_{\alpha_n}\parallel_\infty\leq d_0$, for some $d_0<1$. Since α_n and ξ_{α_n} are bounded and vanish near 0, they belong to $L^1(]0,\infty[, z^{-2}dz)$. At last, the existence of a sequence a_n yielding (2.18) and (2.19) is immediate, since

$$\int_O \alpha_n(z)\delta(z)\varphi(z)dz \geq C\int_{2/n}^1 \frac{kz}{(1+z)^2}\times\frac{dz}{z^2} \geq C\int_{2/n}^1\frac{dz}{z} \longrightarrow_{n\to\infty} \infty \tag{2.33}$$

$$\int_O \left[\alpha_n^2(z)\delta^2(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]\varphi(z)dz \leq C\int_0^\infty \alpha_n^2(z)\frac{dz}{z^2} \tag{2.34}$$

$$\leq C\int_0^\infty \{z^2\wedge1\}\frac{dz}{z^2}\leq C \tag{2.35}$$

3 A criterion of strict positivity.

In the whole work, $\Omega = \Omega^N \times \Omega^{N_1}$ is the canonical product space associated with the independent random elements N and N_1. We will in fact be interested only in N.

This section contains two parts. We first introduce some general notations and definitions about Bismut's approach of the Malliavin Calculus on the Poisson space associated with N. We extend here the work of Bichteler, Jacod, [7], who work with $\varphi \equiv 1$. Then we adapt the criterion of strict positivity of Bally, Pardoux, [4] (which deals with the Wiener functionals) to the Poisson functionals.

Definition 3.1 *A predictable function $v(\omega, s, z)$ on $\Omega \times [0,T] \times O$ is said to be a* **perturbation** *if for all fixed ω, s, $v(\omega, s, .)$ is C^1 on O, if there exists two positive functions α and ρ on O such that,*

$$|v(\omega, s, z)| \leq \alpha(z) \quad ; \quad |v'(\omega, s, z)| \leq \rho(z) \tag{3.1}$$

and such that, if

$$\phi_\alpha(z) = \frac{\sup\{|\varphi'(w)| \ ; \ |w - z| \leq \alpha(z)\}}{\varphi(z)} \quad ; \quad \xi(z) = \rho(z) + 3\alpha(z)\phi_\alpha(z) \tag{3.2}$$

then, for some constant $c < 1$,

$$\alpha + \xi \in L^1 \cap L^\infty(O, \varphi(z)dz) \quad ; \quad \alpha(z) \longrightarrow_{z \to \partial O} 0 \quad ; \quad \xi(z) \leq c \tag{3.3}$$

We now consider a fixed perturbation v. For each $\omega \in \Omega$, $\lambda \in [-1,1]$ and $s \in [0,T]$, the map

$$z \mapsto \gamma^\lambda(\omega, s, z) = z + \lambda v(\omega, s, z) \tag{3.4}$$

is an increasing bijection from O to O, thanks to (3.1) and (3.3). We now set

$$Y^\lambda(\omega, s, z) = (1 + \lambda v'(\omega, s, z)) \times \frac{\varphi(\gamma^\lambda(\omega, s, z))}{\varphi(z)} \tag{3.5}$$

Then, a simple substitution shows that

$$\gamma^\lambda(Y^\lambda.\nu) = \nu \tag{3.6}$$

i.e. that for all Borel set $A \subset [0,T] \times O$,

$$\int_0^T \int_O 1_A(s, \gamma^\lambda(s,z)) Y^\lambda(s,z)\varphi(z)dzds = \int_0^T \int_O 1_A(s, z')\varphi(z')dz'ds \tag{3.7}$$

We also denote by $N^\lambda = \gamma^\lambda(N)$ the image measure of N by γ^λ

$$N^\lambda(A) = \int_0^T \int_O 1_A(s, \gamma^\lambda(s,z))N(ds, dz) \tag{3.8}$$

and by S^λ the shift on Ω defined by

$$N \circ S^\lambda(\omega) = N^\lambda(\omega) \quad ; \quad N_1 \circ S^\lambda(\omega) = N_1(\omega) \tag{3.9}$$

Then we consider the following martingale

$$M_t^\lambda = \int_0^t \int_O (Y^\lambda(s,z) - 1)\tilde{N}(ds,dz) \tag{3.10}$$

and its Doléans-Dade exponential (see Jacod, Shiryaev, [23], p 59)

$$G_t^\lambda = 1 + \int_0^t G_{s-}^\lambda dM_s^\lambda = e^{M_t^\lambda} \prod_{0\le s\le t} \left(1 + \Delta M_s^\lambda\right) e^{-\Delta M_s^\lambda} \tag{3.11}$$

which is clearly a square integrable martingale. Using the fact that Y^λ is always strictly positive, we see that G^λ is strictly positive a.s. We now set $P^\lambda = G_T^\lambda.P$. Thanks to (3.6), the Girsanov Theorem for random measures (see Jacod, Shiryaev, [23], p 157) shows that $P^\lambda \circ (S^\lambda)^{-1} = P$, i.e. that the law of (N^λ, N_1) under P^λ is the same as that of (N, N_1) under P.

It is easy to check that :

$$|Y^\lambda(s,z) - Y^\mu(s,z)| \le |\lambda - \mu|\xi(z) \tag{3.12}$$

We at last check the following lemma :

Lemma 3.2 *If v is a perturbation, and if G^λ is the associated exponential martingale, then a.s., the map $\lambda \mapsto G_T^\lambda$ is continuous.*

Proof : using (3.11), we obtain

$$G_T^\lambda = \exp\left[-\int_0^T \int_O \left(Y^\lambda(s,z) - 1\right)\varphi(z)dzds\right] \times \exp\left[\int_0^T \int_O \ln Y^\lambda(s,z)N(ds,dz)\right] \tag{3.13}$$

Using (3.12) and the fact that $\xi \in L^1(O, \varphi(z)dz)$, it is obvious that the first term in the product is a.s. continuous. Furthermore,

$$|Y^\lambda(s,z) - 1| = |Y^\lambda(s,z) - Y^0(s,z)| \le \xi(z) \le c < 1 \tag{3.14}$$

Thus for all $\lambda, \mu \in [-1,1]$,

$$\left|\ln\left(Y^\lambda(s,z)\right) - \ln\left(Y^\mu(s,z)\right)\right| \le \frac{1}{1-c}\left|Y^\lambda(s,z) - Y^\mu(s,z)\right| \le \frac{1}{1-c}|\lambda-\mu|\xi(z) \tag{3.15}$$

and the second term is also continuous.

We now give a criterion of strict positivity.

Theorem 3.3 *Let X be a real valued random variable on Ω, such that $P \circ X^{-1} = p(x)dx$, with p continuous on $I\!R$, and let $y_0 \in I\!R$. Assume that there exists a sequence v_n of perturbations such that, if $X^n(\lambda) = X \circ S_n^\lambda$, then for each n, the map*

$$\lambda \mapsto X^n(\lambda) \tag{3.16}$$

is a.s. twice differentiable on $[-1,1]$. Assume that there exists $c > 0$, $\delta > 0$, and $k < \infty$, such that for all $r \in]0,1]$,

$$\lim_{n\to\infty} P\left(\Lambda^n(r)\right) > 0 \tag{3.17}$$

where

$$\Lambda^n(r) = \left\{ |X - y_0| < r \ , \ \left|\frac{\partial}{\partial\lambda}X^n(0)\right| \geq c \ , \ \sup_{|\lambda|\leq\delta}\left[\left|\frac{\partial}{\partial\lambda}X^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X^n(\lambda)\right|\right] \leq k \right\} \tag{3.18}$$

Then there exists a continuous function $\theta_{y_0}(.) : I\!R \mapsto I\!R^+$ such that $\theta_{y_0}(y_0) > 0$ and such that for all $f \in C_b^+(I\!R)$,

$$E(f(X)) \geq \int_{I\!R} f(y)\theta_{y_0}(y)dy \tag{3.19}$$

In order to prove this criterion, we will use the following uniform local inverse Theorem, that can be found in Aida, Kusuoka, Stroock, [1] :

Lemma 3.4 *Let $c > 0$, $\delta > 0$, and $k < \infty$ be fixed. Consider the following set :*

$$\mathcal{G} = \left\{ g : I\!R \mapsto I\!R \ \Big/ \ |g'(0)| \geq c \ , \ \sup_{|x|\leq\delta}\left[|g(x)| + |g'(x)| + |g''(x)|\right] \leq k \right\} \tag{3.20}$$

Then there exists $\alpha > 0$ and $R > 0$ such that for every $g \in \mathcal{G}$, there exists a neigbourhood $\mathcal{V}_g$ of 0 contained in $]-R, R[$ such that g is a diffeomorphism from $\mathcal{V}_g$ to $]g(0) - \alpha, g(0) + \alpha[$.

Since this lemma deals with the behaviour of functions near 0, it can be obviously adapted to functions on $[-1, 1]$.

Proof of Theorem 3.3 :
Step 1 : first notice that for all $r \leq 1$, all n, and all $\omega \in \Lambda_n(r)$,

$$\sup_{|\lambda|\leq\delta}|X^n(\omega,\lambda)| \leq |X^n(\omega,0)| + \delta k = |X(\omega)| + \delta k \leq |y_0| + 1 + \delta k = k' \tag{3.21}$$

Thus, using Lemma 3.4, there exists $\alpha > 0$ and $R \in]0, 1]$, depending only on δ, c, k, and k', such that for all $r \leq 1$, all n, and all $\omega \in \Lambda_n(r)$, there exists $V_n(\omega)$ a neighbourhood of 0 contained in $]-R, R[$ such that the map $\lambda \mapsto X^n(\omega,\lambda)$ is a diffeomorphism from $V_n(\omega)$ to $]X^n(\omega,0) - \alpha, X^n(\omega,0) + \alpha[=]X(\omega) - \alpha, X(\omega) + \alpha[$.
Choosing α small enough, we can assume that $R \leq c/2k$. Thus, for all $\omega \in \Lambda_n(r)$ and $\lambda \in V_n(\omega)$, we have

$$\left|\frac{\partial}{\partial\lambda}X^n(\lambda)\right| \geq c/2 \tag{3.22}$$

We now fix $r < \alpha$, and we choose n large enough such that $P(\Lambda_n(r)) > 0$.

Step 2 : the perturbations have been built in order to obtain, for all λ and all $f \in C_b^+(I\!R)$,

$$E(f(X)) = E(f(X^n(\lambda))G_T^n(\lambda)) \tag{3.23}$$

Thus

$$\begin{aligned} E(f(X)) &= \frac{1}{2}\int_{-1}^{1} E(f(X^n(\lambda))G_T^n(\lambda))d\lambda \\ &\geq \frac{1}{2}E\left[\int_{V_n} f(X^n(\lambda))G_T^n(\lambda)d\lambda \times 1_{\Lambda_n(r)}\right] \end{aligned} \tag{3.24}$$

Using the first step, we substitute $y = X^n(\lambda)$, and we obtain :

$$\begin{aligned} E(f(X)) &\geq \frac{1}{2}E\left[\int_{]X-\alpha,X+\alpha[} f(y)\frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda}X^n(\{X^n\}^{-1}(y))\right|}dy \times 1_{\Lambda_n(r)}\right] \\ &\geq \int_{I\!R} f(y)E\left[\frac{1}{2}\psi(|X-y|)\left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda}X^n(\{X^n\}^{-1}(y))\right|}\right) \times 1_{\Lambda_n(r)}\right] dy \end{aligned} \tag{3.25}$$

where ψ is a continuous function on $I\!R^+$ such that $1_{[0,r]} \leq \psi \leq 1_{[0,\alpha]}$. Let

$$\theta_n(y) = E\left[\frac{1}{2}\psi(|X-y|) \times \left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{\left|\frac{\partial}{\partial\lambda}X^n(\{X^n\}^{-1}(y))\right|}\right) \times 1_{\Lambda_n(r)}\right] \tag{3.26}$$

Step 3 : on one hand, it is clear that $\theta_n(y_0) > 0$ (recall the definition of $\Lambda_n(r)$). On the other hand, one can show by using the Lebesgue Theorem and Lemma 3.2 that θ_n is continuous. This concludes the proof.

We at last state a usefull remark.

Remark 3.5 *Let X be a real-valued random variable on Ω. Suppose that for each $y_0 \in \text{supp}\, P\circ X^{-1}$, the assumptions of Theorem 3.3 hold. Then the law of X is bounded below by a measure with a continuous strictly positive density.*

Proof : for each y_0 in supp $P \circ X^{-1}$, we consider the continuous function θ_{y_0} built in Theorem 3.3. Since $\theta_{y_0}(y_0) > 0$, there exists a neighbourhood $\mathcal{W}_{y_0}$ of y_0 on which θ_{y_0} does not vanish. We easily deduce from (3.19) that for each y_0 in supp $P \circ X^{-1}$,

$$\mathcal{W}_{y_0} \subset \text{supp}\, P \circ X^{-1} \tag{3.27}$$

Thus supp $P \circ X^{-1}$ is an open set, and therefore is the whole real line.
For each $n \in \mathbf{Z}$, we thus can build a strictly positive continuous function θ^n on the compact set $[n, n+1]$ such that for all $f \in C_b^+(I\!R)$,

$$E(f(X)) \geq \int_{[n,n+1]} f(y)\theta^n(y)dy \tag{3.28}$$

One concludes easily, by choosing a strictly positive function θ on $I\!R$ such that for each $n \in \mathbf{Z}$, all $y \in [n, n+1]$, $\theta(y) \leq \theta^n(y)$.

In order to prove Theorem 2.4, we will of course apply the previous criterion. In the next section, we will consider a fixed perturbation v_n, and we will compute $X_t^n(\lambda) = X_t \circ S_n^\lambda$ and its derivatives for any $t \in [0,T]$. Section 5 is devoted to the explicit choice of the sequence

v_n of perturbations. In Section 6, we will prove that for each y_0 in $I\!R$, there exists a constant $C(y_0) > 0$ such that for any $r > 0$,

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| \geq C(y_0) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \tag{3.29}$$

At last, we will check in Section 7 that for some constant K,

$$P\left(\sup_{|\lambda|\leq 1}\left\{\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X_T^n(\lambda)\right|\right\} \leq K\right) \longrightarrow 1 \tag{3.30}$$

Since for all $y_0 \in \text{supp } P \circ X_T^{-1}$, for all $r > 0$, $P(X_T \in]y_0 - r, y_0 + r[) > 0$, we will easily conclude at the end of Section 7.

4 Computation of the derivatives of X.

Recall that

$$X_t = x_0 + \int_0^t \int_O h(X_{s-}, z)\tilde{N}(ds, dz) + \int_0^t g(X_{s-})ds + \int_0^t \int_E f(X_{s-}, u)\tilde{N}_1(ds, du) \tag{4.1}$$

We consider in this section a fixed perturbation v_n, and compute $X_t^n(\lambda) = X_t \circ S_n^\lambda$. Then we prove that for each t, $X_t^n(\lambda)$ is a.s. differentiable on $[-1, 1]$ and obtain $\dfrac{\partial}{\partial\lambda}X_t^n(\lambda)$. At last, we study the second derivative $\dfrac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)$.

4.1 The perturbed process.

The direct expression of $X_t^n(\lambda)$ is given by

$$\begin{aligned} X_t^n(\lambda) &= x_0 + \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \nu)(ds, dz) + \int_0^t g(X_{s-}^n(\lambda))ds \\ &\quad + \int_0^t \int_E f(X_{s-}^n(\lambda), u)\tilde{N}_1(ds, du) \end{aligned} \tag{4.2}$$

But

$$\begin{aligned} \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \nu)(ds, dz) &= \int_0^t \int_O h(X_{s-}^n(\lambda), z)(N_n^\lambda - \gamma_n^\lambda(\nu))(ds, dz) \\ &\quad + \int_0^t \int_O h(X_{s-}^n(\lambda), z)(\gamma_n^\lambda(\nu) - \nu)(ds, dz) \end{aligned}$$

$$\begin{aligned} &= \int_0^t \int_O h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z))\tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_O \left[h(X_{s-}^n(\lambda), \gamma_n^\lambda(s, z)) - h(X_{s-}^n(\lambda), z)\right]\varphi(z)dzds \end{aligned} \tag{4.3}$$

Finally,

Proposition 4.1 *For each $\lambda \in [-1,1]$, the perturbed process $X_t^n(\lambda) = X_t \circ S_n^\lambda$ is solution of the following equation :*

$$
\begin{aligned}
X_t^n(\lambda) &= x_0 + \int_0^t \int_O h(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) \tilde{N}(ds,dz) \\
&+ \int_0^t \int_O \left[h(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) - h(X_{s-}^n(\lambda), z) \right] \varphi(z) dz ds \\
&+ \int_0^t g(X_{s-}^n(\lambda)) ds + \int_0^t \int_E f(X_{s-}^n(\lambda), u) \tilde{N}_1(ds,du)
\end{aligned}
\tag{4.4}
$$

4.2 The first derivative.

We now would like to differentiate the paths of the map $\lambda \mapsto X^n(\lambda)$. Consider the following linear equation, that is obtained by differentiating formally (4.4).

$$
\begin{aligned}
\frac{\partial}{\partial \lambda} X_t^n(\lambda) &= \int_0^t \int_O h_x'(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda) \tilde{N}(ds,dz) \\
&+ \int_0^t \int_O h_z'(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) v_n(s,z) \tilde{N}(ds,dz) \\
&+ \int_0^t \int_O \left[h_x'(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) - h_x'(X_{s-}^n(\lambda), z) \right] \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda) \varphi(z) dz ds \\
&+ \int_0^t \int_O h_z'(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) v_n(s,z) \varphi(z) dz ds \qquad (4.5) \\
&+ \int_0^t g'(X_{s-}^n(\lambda)) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda) ds + \int_0^t \int_E f_x'(X_{s-}^n(\lambda), u) \frac{\partial}{\partial \lambda} X_{s-}^n(\lambda) \tilde{N}_1(ds,du)
\end{aligned}
$$

Proposition 4.2 *Fix an integer n. The map $\lambda \mapsto X_T^n(\lambda)$ is a.s. differentiable on $[-1,1]$, and its derivative $\frac{\partial}{\partial \lambda} X_T^n(\lambda)$ is the terminal value of the solution of equation (4.5).*

Proof : for simplicity, we drop the superscript n, since it is fixed. We break the proof in several steps.

Step 1 : we consider an increasing sequence O_k (resp. E_k) of subsets of O (resp. of E) such that O_k goes to O (resp. E_k goes to E), and for all k,

$$
\int_{O_k} \varphi(z) dz < \infty \qquad ; \qquad \int_{E_k} q(du) < \infty \tag{4.6}
$$

Then we denote by $\bar{X}_t^k(\lambda)$ and $\frac{\partial}{\partial \lambda} \bar{X}_t^k(\lambda)$ the solutions of (4.4) and (4.5) where we have replaced O and E by O_k and E_k. Using classical estimates, as Burkholder's inequality and Gronwall's Lemma, one can easily check that $\bar{X}_T^k(\lambda) - X_T(\lambda)$ and $\frac{\partial}{\partial \lambda} \bar{X}_T^k(\lambda) - \frac{\partial}{\partial \lambda} \bar{X}_T(\lambda)$ satisfy the assumptions of Lemma 8.1 of the Appendix. Thus there exists a subsequence such that a.s., when l

goes to infinity,

$$\sup_{|\lambda|\le 1}\left|\bar{X}^{k_l}_T(\lambda)-X_T(\lambda)\right|+\sup_{|\lambda|\le 1}\left|\frac{\partial}{\partial\lambda}\bar{X}^{k_l}_T(\lambda)-\frac{\partial}{\partial\lambda}X_T(\lambda)\right|\longrightarrow 0 \tag{4.7}$$

Step 2 : We now fix k, and we prove that there exists a random variable $Z^k<\infty$ a.s. such that for all λ,μ, and all $t\in[0,T]$,

$$\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right|\le|\mu|Z^k \tag{4.8}$$

Indeed, it is possible, using strongly the fact that $N([0,T]\times O_k)+N_1([O,T]\times E_k)<\infty$ a.s., that there exists a constant $C<\infty$ and a random variable $A^k<\infty$ a.s. such that for all $t\in[0,T]$ and all λ,μ,

$$\begin{aligned}\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right| \le\ & |\mu|A^k+C\int_0^t\left|\bar{X}^k_{s-}(\lambda+\mu)-\bar{X}^k_{s-}(\lambda)\right|ds\\ &+C\int_0^t\int_{O_k}\left|\bar{X}^k_{s-}(\lambda+\mu)-\bar{X}^k_{s-}(\lambda)\right|N(ds,dz)\\ &+C\int_0^t\int_{E_k}\left|\bar{X}^k_{s-}(\lambda+\mu)-\bar{X}^k_{s-}(\lambda)\right|N_1(ds,du)\end{aligned} \tag{4.9}$$

We denote by $0<S_1<...<S_\nu<T$ the times of jump of the process $N([0,t]\times O_k)+N_1([0,t]\times E_k)$. This way, we obtain

$$\begin{aligned}\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right| \le\ & |\mu|A^k+C\int_0^t\left|\bar{X}^k_{s-}(\lambda+\mu)-\bar{X}^k_{s-}(\lambda)\right|ds\\ &+C\sum_{i=1}^{\nu}\left|\bar{X}^k_{S_i-}(\lambda+\mu)-\bar{X}^k_{S_i-}(\lambda)\right|1_{\{t\ge S_i\}}\end{aligned} \tag{4.10}$$

Using Gronwall's Lemma on $[0,S_1[$, we obtain for all $t\in[0,S_1[$,

$$\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right|\le|\mu|A^k e^{CS_1}\le|\mu|A^k e^{CT} \tag{4.11}$$

This way, we can write for t in $[S_1,S_2[$

$$\begin{aligned}\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right| \le\ & |\mu|A^k+CS_1|\mu|A^k e^{CT}+C\int_{S_1}^t\left|\bar{X}^k_{s-}(\lambda+\mu)-\bar{X}^k_{s-}(\lambda)\right|ds\\ &+C|\mu|A^k e^{CT}\end{aligned} \tag{4.12}$$

Using again Gronwall's Lemma, we obtain for all $t\in[S_1,S_2[$,

$$\left|\bar{X}^k_t(\lambda+\mu)-\bar{X}^k_t(\lambda)\right|\le|\mu|\left[A^k+A^kCTe^{CT}+CA^ke^{CT}\right]e^{CT} \tag{4.13}$$

Iterating this argument, we obtain the existence of the announced random variable Z^k, and the second step is finished.

Step 3 : we again fix k, and we set

$$\Delta_t^k(\lambda,\mu) = \left|\bar{X}_t^k(\lambda+\mu) - \bar{X}_t^k(\lambda) - \mu\frac{\partial}{\partial\lambda}\bar{X}_t^k(\lambda)\right| \tag{4.14}$$

Then one can check, by using Step 2 and the same arguments as in this previous step, that there exists a random variable $U^k < \infty$ a.s. such that for all $t \in [0,T]$, and all λ, μ,

$$\Delta_t^k(\lambda,\mu) \le \mu^2 U^k \tag{4.15}$$

This of course implies that a.s., the map $\lambda \mapsto \bar{X}_T^k(\lambda)$ is differentiable on $[-1,1]$, and its derivative is $\dfrac{\partial}{\partial\lambda}\bar{X}_T^k(\lambda)$.

This and the convergence (4.7) yield that Proposition 4.2 is proved.

Rewriting equation (4.5), we obtain

Proposition 4.3 *for each $\lambda \in [-1,1]$, the process $\dfrac{\partial}{\partial\lambda}X_t^n(\lambda)$ is solution of the following S.D.E.*

$$\begin{aligned}
\frac{\partial}{\partial\lambda}X_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}(ds,dz) \\
&+ \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\varphi(z)dzds \\
&+ \int_0^t g'(X_{s-}^n(\lambda))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}_1(ds,du) \\
&+ \int_0^t\int_O h'_z(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))v_n(s,z)N(ds,dz)
\end{aligned} \tag{4.16}$$

This can also be written

$$\frac{\partial}{\partial\lambda}X_t^n(\lambda) = \int_0^t \frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)dK_s^n(\lambda) + H_t^n(\lambda) \tag{4.17}$$

where

$$\begin{aligned}
K_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))\tilde{N}(ds,dz) \\
&+ \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right]\varphi(z)dzds \\
&+ \int_0^t g'(X_{s-}^n(\lambda))ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\tilde{N}_1(ds,du)
\end{aligned} \tag{4.18}$$

and

$$H_t^n(\lambda) = \int_0^t\int_O h'_z(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))v_n(s,z)N(ds,dz) \tag{4.19}$$

Thanks to assumption (P), $1+\Delta K^n_{s-}(\lambda)$ does never vanish, thus the Doléans-Dade exponential martingale $\mathcal{E}(K^n(\lambda))$ is a.s. invertible, and we can write (see Jacod, [22]) :

$$\frac{\partial}{\partial\lambda}X^n_t(\lambda) = \mathcal{E}(K^n(\lambda))_t \times \int_0^t \mathcal{E}(K^n(\lambda))^{-1}_{s-}\left(1+\Delta K^n_s(\lambda)\right)^{-1} dH^n_s(\lambda) \tag{4.20}$$

Finally, since N and N_1 are independent, they never jump together a.s., and we obtain :

Remark 4.4 *The process* $\frac{\partial}{\partial\lambda}X^n_t(\lambda)$ *is given by*

$$\frac{\partial}{\partial\lambda}X^n_t(\lambda) = \mathcal{E}(K^n(\lambda))_t \times \int_0^t\int_O \mathcal{E}(K^n(\lambda))^{-1}_{s-}\frac{h'_z(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))}{1+h'_x(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))}v_n(s,z)N(ds,dz) \tag{4.21}$$

4.3 The second derivative.

Exactly as in the previous subsection, we obtain :

Proposition 4.5 *The second derivative* $\frac{\partial^2}{\partial\lambda^2}X^n_t(\lambda)$ *is solution of the following equation*

$$\frac{\partial^2}{\partial\lambda^2}X^n_t(\lambda) = \int_0^t \frac{\partial^2}{\partial\lambda^2}X^n_{s-}(\lambda)dK^n_s(\lambda) + L^n_t(\lambda) \tag{4.22}$$

where

$$\begin{aligned}
L^n_t(\lambda) \quad = \quad & 2\int_0^t\int_O h''_{xz}(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)v_n(s,z)N(ds,dz)\\
&+\int_0^t\int_O h''_{zz}(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))v_n^2(s,z)N(ds,dz)\\
&+\int_0^t\int_O h''_{xx}(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\tilde N(ds,dz)\\
&+\int_0^t\int_O \left[h''_{xx}(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z)) - h''_{xx}(X^n_{s-}(\lambda),z)\right]\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\varphi(z)dzds\\
&+\int_0^t g''(X^n_{s-}(\lambda))\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2 ds\\
&+\int_0^t\int_E f''_{xx}(X^n_{s-}(\lambda),u)\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\tilde N_1(ds,du)
\end{aligned} \tag{4.23}$$

We end this section with a classical remark.

Remark 4.6 *For each* $n\in I\!N$, *and each* $\lambda\in[-1,1]$, *the processes* $X^n_t(\lambda)$, $\frac{\partial}{\partial\lambda}X^n_t(\lambda)$, *and* $\frac{\partial^2}{\partial\lambda^2}X^n_t(\lambda)$ *are the solution of standard S.D.E.s. Thanks to* (H), *it is well-known that for each* λ, *each* n, *and each* $p<\infty$, *there exists a constant* $C(n,p,\lambda)$ *such that*

$$E\left(\sup_{[0,T]}|X^n_t(\lambda)|^p\right) + E\left(\sup_{[0,T]}\left|\frac{\partial}{\partial\lambda}X^n_t(\lambda)\right|^p\right) + E\left(\sup_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X^n_t(\lambda)\right|^p\right) \le C(n,p,\lambda) \tag{4.24}$$

5 Choice of the perturbation.

Let us first recall that, thanks to (4.18) and (4.21) :

$$\frac{\partial}{\partial\lambda}X_T^n(0)=\mathcal{E}(K)_T\times\int_0^T\int_O\mathcal{E}(K)_{s-}^{-1}\frac{h_z'(X_{s-},z)}{1+h_x'(X_{s-},z)}v_n(s,z)N(ds,dz) \tag{5.1}$$

where

$$K_t=\int_0^t\int_O h_x'(X_{s-},z)\tilde{N}(ds,dz)+\int_0^t g'(X_{s-})ds+\int_0^t\int_E f_x'(X_{s-},u)\tilde{N}_1(ds,du) \tag{5.2}$$

The main idea consists in choosing v_n in such a way that for some $0<\epsilon<K<\infty$, the probability

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right|\in[\epsilon,K]\;\Big/\;X_T\in[y_0-r,y_0+r]\right)$$

goes to 1 when n goes to infinity. In order to get rid of the random terms $\mathcal{E}(K)_T$ and $\mathcal{E}(K)_{s-}^{-1}$, we will choose $v_n(s,z)$ equal to 0 for $0\le s<T-a_n$, for some sequence a_n decreasing to 0, and we will use the a.s. continuity of $\mathcal{E}(K)$ at T. The term $(1+h_x'(X_{s-},z))^{-1}$ is not a problem, since it always belongs to $[1/(1+\parallel\eta\parallel_\infty),1/c_0]$, thanks to (P) and (H). Thanks to (SP) and proposition 2.5, $h_z'(X_{s-},z)\sim\psi(y_0)\delta(z)$ near T on the set $\{X_T\in[y_0-r,y_0+r]\}$. Then, choosing $v_n\sim\alpha_n$, we will use the fact that thanks to (2.18) in (SP),

$$\int_{T-a_n}^T\int_O\delta(z)\alpha_n(z)N(ds,dz)$$

goes a.s. to infinity when n goes to infinity. By this way, we will get a good lowerbound. The upperbound will be obtained by using a well-chosen stopping time.

Let us now describe the sequence v_n of perturbations. We will develop the arguments above in the next sections.

Definition 5.1 *Consider the sequences of increasing processes*

$$Z_t^n=\int_0^t\int_O\alpha_n(z)\delta(z)N(ds,dz)\quad;\quad U_t^n=\int_0^t\int_O\left[\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\right]N(ds,dz) \tag{5.3}$$

where a_n, α_n, δ, β, and ζ_n are defined in (SP). Define the stopping times

$$T_n=\inf\left\{t>T-a_n\,/Z_t^n-Z_{T-a_n}^n\ge l\right\}\quad;\quad R_n=\inf\left\{t>T-a_n\,/U_t^n-U_{T-a_n}^n\ge l\right\} \tag{5.4}$$

where $l>0$ is a strictly positive real number that we will choose (very small) in Section 7. We now set

$$v_n(s,z)=1_{[T-a_n,T\wedge T_n\wedge R_n]}(s)\alpha_n(z)sg(\mathcal{E}(K)_{s-}^{-1}) \tag{5.5}$$

where $sg(x)$ denotes the sign of x.

It is clear from (SP) that for each n, v_n is a perturbation in the sense of Definition 3.1. We will need the following key lemma :

Lemma 5.2 *When n goes to infinity,*

$$P(T_n \leq T < R_n) \longrightarrow 1 \tag{5.6}$$

Proof : first,

$$\begin{aligned}
P(T_n \leq T) &= P(Z_T^n - Z_{T-a_n}^n \geq l) \\
&\geq 1 - e^l E\left[\exp\left\{-(Z_T^n - Z_{T-a_n}^n)\right\}\right] \\
&\geq 1 - e^l \exp\left[-\int_{T-a_n}^T \int_O \left(1 - e^{-\delta(z)\alpha_n(z)}\right)\varphi(z)dzds\right] \\
&\geq 1 - e^l \exp\left\{-Ka_n \int_O \alpha_n(z)\delta(z)\varphi(z)dz\right\} \longrightarrow 1 \qquad (5.7)
\end{aligned}$$

We have used condition (2.18) of (SP), and the fact that, since $\| \delta\alpha_n \|_\infty \leq d_0 \| \delta \|_\infty$, there exists a constant K such that for all z,

$$1 - e^{-\delta(z)\alpha_n(z)} \geq K\delta(z)\alpha_n(z) \tag{5.8}$$

Furthermore, thanks to (2.19) in (SP),

$$\begin{aligned}
P(R_n \leq T) &= P(U_T^n - U_{T-a_n}^n \geq l) \\
&\leq \frac{1}{l}E\left(U_T^n - U_{T-a_n}^n\right) \\
&= \frac{1}{l}a_n \int_O \left[\alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\right]\varphi(z)dz \longrightarrow 0 \qquad (5.9)
\end{aligned}$$

The lemma is proved.

At last, we notice that since U^n and Z^n are càdlàg and increasing, since

$$\alpha_n(z)\delta(z) \leq d_0 \| \delta \|_\infty \quad ; \quad \alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z) \leq d_0 \left(\| \beta \|_\infty + \| \zeta_n \|_\infty\right)$$

for all $t \geq T - a_n$,

$$Z_{t\wedge T_n}^n - Z_{T-a_n}^n \leq l + d_0 \| \delta \|_\infty \quad ; \quad Z_{T_n}^n - Z_{T-a_n}^n \geq l \tag{5.10}$$

$$U_{t\wedge T_n}^n - U_{T-a_n}^n \leq l + d_0 \left(\| \beta \|_\infty + \sup_n \| \zeta_n \|_\infty\right) \quad ; \quad U_{T_n}^n - U_{T-a_n}^n \geq l \tag{5.11}$$

6 Lowerbound for the derivative at 0.

Thanks to our choice for v_n, since $1 + h'_x \geq 0$ thanks to (P) and since $h'_z \geq 0$ thanks to Proposition 2.5, we can write

$$\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| = |\mathcal{E}(K)_T| \times \int_{T-a_n}^{R_n\wedge T_n\wedge T} \int_O \left|\mathcal{E}(K)_{s-}^{-1}\right| \frac{h'_z(X_{s-},z)}{1 + h'_x(X_{s-},z)}\alpha_n(z)N(ds,dz) \tag{6.1}$$

This section is devoted to the proof of the following proposition :

Proposition 6.1 *For each $y_0 \in I\!R$ and each $l > 0$, there exists a constant $C(y_0, l) > 0$ such that for any $r \in]0,1]$,*

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| \geq C(y_0,l) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \tag{6.2}$$

<u>Proof</u> : thanks to (H), $1 + h'_x \leq 1+ \parallel \eta \parallel_\infty$, and thanks to (SP), $h'_z(x,z) \geq \psi(x)\delta(z)$. Hence,

$$\begin{aligned}
\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| &\geq \frac{1}{1+ \parallel \eta \parallel_\infty} |\mathcal{E}(K)_T| \times \int_{T-a_n}^{R_n \wedge T_n \wedge T} \int_O \left|\mathcal{E}(K)_{s-}^{-1}\right| \psi(X_{s-})\delta(z)\alpha_n(z)N(ds,dz) \\
&\geq \frac{1}{1+ \parallel \eta \parallel_\infty} \inf_{[T-a_n,T]} \left|\mathcal{E}(K)_T\mathcal{E}(K)_{s-}^{-1}\right| \times \inf_{[T-a_n,T]} \psi(X_{s-}) \\
&\qquad \times \int_{T-a_n}^{R_n \wedge T_n \wedge T} \int_O \delta(z)\alpha_n(z)N(ds,dz) \\
&= \frac{1}{1+ \parallel \eta \parallel_\infty} \times A_n \times B_n \times (Z^n_{T\wedge T_n \wedge R_n} - Z^n_{T-a_n})
\end{aligned} \tag{6.3}$$

where Z^n is defined by (5.3).

Since $\mathcal{E}(K)$ is a.s. continuous at T, since, thanks to (P), $\mathcal{E}(K)$ does never vanish a.s., it is clear that A_n goes a.s. to 1 when n goes to infinity.

Since, from (SP)-1, ψ is continuous, since X is a.s. continuous at T, it is clear that on the set $\{\omega \in \Omega \,;\, X_T(\omega) \in]y_0 - r, y_0 + r[\}$, a.s.,

$$\underline{\lim}\, B_n \geq \inf_{]y_0-2r, y_0+2r[} \psi(x) \geq \inf_{]y_0-2, y_0+2[} \psi(x) = \bar{\psi}(y_0) > 0 \tag{6.4}$$

At last, since $P(T_n < T < R_n)$ goes to 1 (see Lemma 5.2), and since $Z^n_{T_n} - Z^n_{T-a_n} \geq l$ a.s.,

$$P(Z^n_{T\wedge T_n} - Z^n_{T-a_n} \geq l) \longrightarrow 1 \tag{6.5}$$

Thus

$$P\left(\left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| \geq \frac{1}{2}\left(\frac{1}{1+ \parallel \eta \parallel_\infty} \times 1 \times \bar{\psi}(y_0) \times l\right) \quad \Big/ \quad X_T \in]y_0 - r, y_0 + r[\right) \longrightarrow 1 \tag{6.6}$$

and we have checked the first part of Criterion 3.3 for X_T.

7 Upperbound for the derivatives.

We have to prove now that there exists a constant $K < \infty$ such that, when n goes to infinity,

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.1}$$

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial^2}{\partial\lambda^2}X_T^n(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.2}$$

7.1 Upperbound for the first derivative.

Let us observe equation (4.16). Thanks to our choice for the perturbation v_n, it is clear that $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$ does vanish for $s \leq T - a_n$. Thus only the second and fifth terms in (4.16) seem not to go to 0. Hence, we consider the following S.D.E. : it looks like (4.16), but we have kept only the second and fifth terms.

$$\begin{aligned} I_t^n(\lambda) &= \int_0^t \int_O \left[h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda), z)\right] I_{s-}^n(\lambda) N(ds,dz) \\ &\quad + \int_0^t \int_O h'_z(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) v_n(s,z) N(ds,dz) \end{aligned} \tag{7.3}$$

Notice that the first term is an integral against $N(ds,dz)$ instead of $\varphi(z)dzds$. This comes from the fact that we will have to control the paths of $I_t^n(\lambda)$, and because our perturbation contains a stopping time which gives information about N. Furthermore, the difference between the integral against N and the one against $\varphi(z)dzds$ will go to 0.

Recall now that we can choose $d_0 \in]0,1[$ (see (SP), and Proposition 2.5) and $l > 0$ (see Definition 5.1) as small as we want. Remark that it is obvious from (H) that $\sup_n \parallel \zeta_n \parallel_\infty < \infty$. We will prove in this section the following estimate.

Proposition 7.1 *If we choose*

$$0 < l \leq \frac{1}{64} \qquad ; \qquad 0 < d_0 \leq \frac{1}{32\left(\parallel \delta \parallel_\infty + \parallel \beta \parallel_\infty + \sup_n \parallel \zeta_n \parallel_\infty\right)} \wedge 1 \tag{7.4}$$

then, when n goes to infinity,

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \leq 1\right) \longrightarrow 1 \tag{7.5}$$

In order to prove this, we will use the following lemmas.

Lemma 7.2 *Assume that l and d_0 are as in Proposition 7.1. Then for all n,*

$$\text{a.s.}\ , \qquad \sup_{|\lambda|\leq 1} \sup_{[0,T]} |I_t^n(\lambda)| \leq 1/2 \tag{7.6}$$

Lemma 7.3 *Assume that l and d_0 are as in Proposition 7.1. Let $p \in [1,\infty[$ be fixed. There exists a constant $C_p < \infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup_{[0,T]} |X_t^n(\lambda)|^p\right) \leq C_p \tag{7.7}$$

$$E\left(\sup_{[0,T]} |X_t^n(\lambda) - X_t^n(\mu)|^p\right) \leq C_p|\lambda-\mu|^p \tag{7.8}$$

$$E\left(\sup_{[0,T]} \left|\frac{\partial}{\partial\lambda}X_t^n(\lambda)\right|^p\right) \leq C_p \tag{7.9}$$

$$E\left(\sup_{[0,T]} \left|\frac{\partial}{\partial\lambda}X_t^n(\lambda) - \frac{\partial}{\partial\lambda}X_t^n(\mu)\right|^p\right) \leq C_p|\lambda-\mu|^p \tag{7.10}$$

Lemma 7.4 *Assume that l and d_0 are as in Proposition 7.1. Let $p \in [1,\infty[$ be fixed. There exists a constant $C_p < \infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup_{[0,T]} |I_t^n(\lambda) - I_t^n(\mu)|^p\right) \leq C_p|\lambda-\mu|^p \tag{7.11}$$

Furthermore, for all $\lambda \in [-1,1]$,

$$E\left(\sup_{[0,T]} \left|I_t^n(\lambda) - \frac{\partial}{\partial\lambda}X_t^n(\lambda)\right|^p\right) \longrightarrow_{n\to\infty} 0 \tag{7.12}$$

We now assume for a moment that these lemmas hold.

Proof of Proposition 7.1 : from estimates (7.10), (7.12), (7.11), and Lemma 8.1 of the Appendix, we deduce that, when n goes to infinity,

$$P\left(\sup_{|\lambda|\leq 1} \left|I_T^n(\lambda) - \frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \geq 1/2\right) \longrightarrow 0 \tag{7.13}$$

Thus, using (7.6),

$$\begin{aligned}
P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| \leq 1\right) &\geq P\left(\sup_{|\lambda|\leq 1}|I_T^n(\lambda)| \leq 1/2 \ ; \ \sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda) - I_T^n(\lambda)\right| \leq 1/2\right) \\
&\geq P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}X_T^n(\lambda) - I_T^n(\lambda)\right| \leq 1/2\right) \longrightarrow 1 \qquad (7.14)
\end{aligned}$$

and the proposition is proved.

Proof of Lemma 7.2 : we work here for a fixed $\omega \in \Omega$. Using the functions δ, β and ζ_n defined in (SP) and Notation 2.3, the fact that $|\gamma_n^\lambda(s,z) - z| \leq |v_n(s,z)|$ and $|v_n(s,z)| \leq \alpha_n(z)$, we obtain, for all $x \in I\!R$,

$$\begin{aligned}
\left|h'_z(x,\gamma_n^\lambda(s,z))\right| &\leq |h'_z(x,z)| + |\gamma_n^\lambda(s,z) - z| \times \sup_{|z-w|\leq|v_n(s,z)|} |h''_{zz}(x,w)| \\
&\leq \delta(z) + |v_n(s,z)| \times \left[|h''_{zz}(x,z)| + |v_n(s,z)| \times \sup_{|z-w|\leq|v_n(s,z)|}|h'''_{zzz}(x,w)|\right] \\
&\leq \delta(z) + |v_n(s,z)|\beta(z) + v_n^2(s,z)\zeta_n(z) \qquad (7.15)
\end{aligned}$$

and in the same way,

$$\left|h'_x(x,\gamma_n^\lambda(s,z)) - h'_x(x,z)\right| \leq |v_n(s,z)|\delta(z) + v_n^2(s,z)\beta(z) + |v_n(s,z)|^3\zeta_n(z) \tag{7.16}$$

Thus, since $|v_n(s,z)| \leq \alpha_n(z)1_{[T-a_n, T_n\wedge R_n\wedge T]}(s)$, we see that

$$\sup_{[0,t]}|I_s^n(\lambda)| \leq \int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O \left[\delta(z) + \alpha_n(z)\beta(z) + \alpha_n^2(z)\zeta_n(z)\right]$$

$$
\begin{aligned}
& \qquad\qquad \times \left[|I^n_{s-}(\lambda)| + 1\right] \alpha_n(z) N(ds,dz) \\
\leq\ & \left[1 + \sup_{[0,t]} |I^n_s(\lambda)|\right] \times \Big[\int_{T-a_n}^{T_n} \int_O \delta(z)\alpha_n(z) N(ds,dz) \\
& \qquad + \int_{T-a_n}^{R_n} \int_O \Big[\alpha_n^2(z)\beta(z) + \alpha_n^3(z)\zeta_n(z)\Big] N(ds,dz)\Big] \\
\leq\ & \left[1 + \sup_{[0,t]} |I^n_s(\lambda)|\right] \times [2l + d_0(\| \delta \|_\infty + \| \beta \|_\infty + \| \zeta_n \|_\infty)] \qquad (7.17)
\end{aligned}
$$

We have used here only the definitions of T_n and R_n. But we have chosen d_0 and l satisfying

$$
[2l + d_0(\| \delta \|_\infty + \| \beta \|_\infty + \| \zeta_n \|_\infty)] \leq 1/16 \qquad (7.18)
$$

we thus obtain

$$
\sup_{[0,t]} |I^n_s(\lambda)| \leq \frac{1}{16} + \frac{1}{16} \sup_{[0,t]} |I^n_s(\lambda)| \qquad (7.19)
$$

and the proof of (7.6) is finished.

Proof of Lemma 7.3 : first of all, we omit the proofs of (7.7), (7.8), and (7.9), because they are similar but easier than the one of (7.10). In order to prove (7.10), we set :

$$
\Gamma^n_t(\lambda,\mu) = X^n_t(\lambda) - X^n_t(\mu) \quad ; \quad \theta^n_t(\lambda,\mu) = \frac{\partial}{\partial\lambda} X^n_t(\lambda) - \frac{\partial}{\partial\lambda} X^n_t(\mu) \qquad (7.20)
$$

Then, using the expression (4.16), we see that (the constant K depends only on p)

$$
\begin{aligned}
& E\left(\sup_{[0,t]} |\theta^n_s(\lambda,\mu)|^p\right) \\
& \leq KE\Big[\sup_{[0,t]} \Big| \int_0^u \int_O \Big(h'_x(X^n_{s-}(\lambda), \gamma^\lambda_n(s,z)) - h'_x(X^n_{s-}(\mu), \gamma^\lambda_n(s,z))\Big) \\
& \qquad\qquad \times \frac{\partial}{\partial\lambda} X^n_{s-}(\lambda) \tilde{N}(ds,dz)\Big|^p\Big] \\
& + KE\Big[\sup_{[0,t]} \Big| \int_0^u \int_O \Big(h'_x(X^n_{s-}(\mu), \gamma^\mu_n(s,z)) - h'_x(X^n_{s-}(\mu), \gamma^\lambda_n(s,z))\Big) \\
& \qquad\qquad \times \frac{\partial}{\partial\lambda} X^n_{s-}(\lambda) \tilde{N}(ds,dz)\Big|^p\Big] \\
& + KE\left[\sup_{[0,t]} \left| \int_0^u \int_O h'_x(X^n_{s-}(\mu), \gamma^\mu_n(s,z)) \times \theta^n_{s-}(\lambda,\mu) \tilde{N}(ds,dz)\right|^p\right] \\
& + 2^{p-1} E\left[\left| \int_0^t \int_O \Big|h'_x(X^n_{s-}(\lambda), \gamma^\lambda_n(s,z)) - h'_x(X^n_{s-}(\lambda), z)\Big| \, |\theta^n_{s-}(\lambda,\mu)| \varphi(z) dz ds\right|^p\right]
\end{aligned}
$$

$$+KE\left[\left|\int_0^t\int_O\left|h'_x(X^n_{s-}(\lambda),\gamma^\mu_n(s,z))-h'_x(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))\right|\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\mu)\right|\varphi(z)dzds\right|^p\right]$$

$$+KE\Big[\Big|\int_0^t\int_O\Big|h'_x(X^n_{s-}(\lambda),\gamma^\mu_n(s,z))-h'_x(X^n_{s-}(\lambda),z)-h'_x(X^n_{s-}(\mu),\gamma^\mu_n(s,z))$$

$$+h'_x(X^n_{s-}(\mu),z)\Big|\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\mu)\right|\varphi(z)dzds\Big|^p\Big]$$

$$+KE\left[\left|\int_0^t|g'(X^n_{s-}(\lambda))-g'(X^n_{s-}(\mu))|\times\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|ds\right|^p\right]$$

$$+KE\left[\left|\int_0^t|g'(X^n_{s-}(\mu))|\times|\theta^n_{s-}(\lambda,\mu)|\,ds\right|^p\right]$$

$$+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O(f'_x(X^n_{s-}(\lambda),u)-f'_x(X^n_{s-}(\mu),u))\times\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\tilde N_1(ds,du)\right|^p\right]$$

$$+KE\left[\sup_{[0,t]}\left|\int_0^u\int_O f'_x(X^n_{s-}(\mu),u)\times\theta^n_{s-}(\lambda,\mu)\tilde N_1(ds,du)\right|^p\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^t\int_O\left|h'_z(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))-h'_z(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\right|\times\alpha_n(z)N(ds,dz)\right|^p\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^t\int_O\left|h'_z(X^n_{s-}(\mu),\gamma^\mu_n(s,z))-h'_z(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\right|\times\alpha_n(z)N(ds,dz)\right|^p\right]$$

$$=J^{n,1}_t(\lambda,\mu)+...+J^{n,12}_t(\lambda,\mu)\tag{7.21}$$

First, we compute $J^{n,1}$. Using Burkholder's inequality, we obtain

$$J^{n,1}_t(\lambda,\mu)\ \leq\ KE\Big[\Big|\int_0^t\int_O\Big(h'_x(X^n_{s-}(\lambda),\gamma^\lambda_n(s,z))-h'_x(X^n_{s-}(\mu),\gamma^\lambda_n(s,z))\Big)^2\tag{7.22}$$

$$\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2N(ds,dz)\Big|^{p/2}\Big]$$

Using the functions η, δ, β, ζ_n defined in (SP) and (H), using the facts that $|v_n(s,z)|\leq\alpha_n(z)$ and $|\gamma^\lambda_n(s,z)-z|\leq|v_n(s,z)|$, we see that for all x,y in $I\!R$,

$$\left|h'_x(x,\gamma^\lambda_n(s,z))-h'_x(y,\gamma^\lambda_n(s,z))\right|\leq|x-y|\Big[\eta(z)+|v_n(s,z)|\delta(z)$$

$$+|v_n(s,z)|^2\beta(z)+|v_n(s,z)|^3\zeta_n(z)\Big]\tag{7.23}$$

Using Definition 5.1 of v_n, we see that $J^{n,1}_t(\lambda,\mu)$ is smaller than

$$KE\left[\left|\int_0^t\int_O(\Gamma^n_{s-}(\lambda,\mu))^2\times\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2\times\eta^2(z)N(ds,dz)\right|^{p/2}\right]$$

$$+KE\Big[\Big|\int_{(T-a_n)\wedge t}^{t}\int_{O}\left(\Gamma^n_{s-}(\lambda,\mu)\right)^2\times\left(\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right)^2$$

$$\times\left[\delta\alpha_n+\beta\alpha_n^2+\zeta_n\alpha_n^3\right]^2(z)N(ds,dz)\Big|^{p/2}\Big]$$

We now apply Lemma 8.2 for the first term. For the second term, we use Lemma 8.4-1, of which the conditions are satisfied thanks to (SP). This gives :

$$\begin{aligned} J^{n,1}_t(\lambda,\mu) &\le K\int_0^t E\left[|\Gamma^n_s(\lambda,\mu)|^p\times\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right]ds \\ &\quad+\frac{K}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left[|\Gamma^n_s(\lambda,\mu)|^p\times\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right]ds \end{aligned}\tag{7.24}$$

At last, we deduce from Cauchy-Schwarz's inequality, (7.8) and (7.9) that

$$J^{n,1}_t(\lambda,\mu)\le K|\lambda-\mu|^p\tag{7.25}$$

In order to estimate $J^{n,2}$, we will use Burkholder's inequality, then the fact that, since $|\gamma^\lambda_n(s,z)-\gamma^\mu_n(s,z)|\le|\lambda-\mu|\times|v_n(s,z)|$, for all $x\in I\!R$,

$$\begin{aligned}|h'_x(x,\gamma^\lambda_n(s,z))-h'_x(x,\gamma^\mu_n(s,z))|\le|\lambda-\mu|\times|v_n(s,z)|\times\Big[\delta(z)+|v_n(s,z)|\beta(z)\\ +|v_n(s,z)|^2\zeta_n(z)\Big]\end{aligned}\tag{7.26}$$

This way, $J^{n,2}_t(\lambda,\mu)$ is smaller than

$$\begin{aligned}&K|\lambda-\mu|^p\times E\Big[\Big|\int_{(T-a_n)\wedge t}^{t}\int_O\left[\delta(z)\alpha_n(z)+\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\right]^2\\ &\qquad\times\left|\frac{\partial}{\partial\lambda}X^n_{s-}(\lambda)\right|^2N(ds,dz)\Big|^{\frac{p}{2}}\Big]\\ &\le K|\lambda-\mu|^p\times\frac{1}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left(\left|\frac{\partial}{\partial\lambda}X^n_s(\lambda)\right|^p\right)ds\\ &\le K|\lambda-\mu|^p\end{aligned}\tag{7.27}$$

for the same reasons as in the computation of $J^{n,1}$.

We now notice that for any real number x,

$$\left|h'_x(x,\gamma^\lambda_n(s,z))\right|\le\eta(z)+|v_n(s,z)|\times\delta(z)+|v_n(s,z)|^2\times\beta(z)+|v_n(s,z)|^3\times\zeta_n(z)\tag{7.28}$$

Thus, exactly as for $J^{n,1}$, we obtain :

$$J^{n,3}_t(\lambda,\mu)\le K\int_0^t E\left[|\theta^n_s(\lambda,\mu)|^p\right]ds+\frac{K}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left[|\theta^n_s(\lambda,\mu)|^p\right]ds\tag{7.29}$$

We now are interested in $J^{n,4}$. First notice that

$$|h'_x(x,\gamma_n^\lambda(s,z)) - h'_x(x,z)| \leq |v_n(s,z)|\delta(z) + v_n^2(s,z)\beta(z) + |v_n(s,z)|^3\zeta_n(z) \tag{7.30}$$

Hence

$$\begin{aligned}
J_t^{n,4}(\lambda,\mu) &\leq 2^{p-1}E\Big[\Big|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)| \times \Big[\delta(z)\alpha_n(z) + \beta(z)\alpha_n^2(z) \\
&\qquad + \zeta_n(z)\alpha_n^3(z)\Big]\varphi(z)dzds\Big|^p\Big] \\
&\leq 4^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)| \times \delta(z)\alpha_n(z)\varphi(z)dzds\right|^p\right] \\
&\quad + 4^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{R_n\wedge t}\int_O |\theta_{s-}^n(\lambda,\mu)| \times \Big[\beta(z)\alpha_n^2(z) + \zeta_n(z)\alpha_n^3(z)\Big]\varphi(z)dzds\right|^p\right]
\end{aligned} \tag{7.31}$$

Using Assumption (SP), Lemma 8.5, and the definitions of T_n and R_n (see Definition 5.1), we deduce that

$$\begin{aligned}
J_t^{n,4}(\lambda,\mu) &\leq 4^{p-1}\left[2^{p-1}(l + d_0 \parallel \delta \parallel_\infty)^p \times E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)\right] \\
&\quad + 4^{p-1}\left[2^{p-1}(l + d_0(\parallel \beta \parallel_\infty + \parallel \zeta_n \parallel_\infty))^p \times E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right)\right] \\
&\quad + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\right) ds
\end{aligned} \tag{7.32}$$

But we have chosen l and d_0 satisfying (7.18). Thus

$$J_t^{n,4}(\lambda,\mu) \leq \frac{1}{2^p}E\left(\sup_{[0,t]}|\theta_s^n(\lambda,\mu)|^p\right) + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\right) ds \tag{7.33}$$

A similar computation allows us to write

$$J_t^{n,5}(\lambda,\mu) \leq K|\lambda-\mu|^p \times \left[KE\left(\sup_{[0,t]}\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right) + \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right) ds\right] \tag{7.34}$$

Thanks to (7.9), we obtain

$$J_t^{n,5}(\lambda,\mu) \leq K|\lambda-\mu|^p \tag{7.35}$$

Since

$$\begin{aligned}
&\left|h'_x(x,\gamma_n^\lambda(s,z)) - h'_x(x,z) - h'_x(y,\gamma_n^\lambda(s,z)) + h'_x(y,z)\right| \\
&\leq |x-y|\left[|v_n(s,z)|\delta(z) + v_n^2(s,z)\beta(z) + |v_n(s,z)|^3\zeta_n(z)\right]
\end{aligned} \tag{7.36}$$

one can check, as in the computation of $J^{n,4}$, that

$$\begin{aligned} J_t^{n,6}(\lambda,\mu) &\leq KE\left(\sup_{[0,t]}|\Gamma_s^n(\lambda,\mu)|^p\left|\frac{\partial}{\partial\lambda}X_s^n(\mu)\right|^p\right) \\ &\quad +\frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda)|^p\left|\frac{\partial}{\partial\lambda}X_s^n(\mu)\right|^p\right)ds \\ &\leq K|\lambda-\mu|^p \end{aligned} \tag{7.37}$$

thanks to the Cauchy-Schwarz inequality, and estimates (7.8) and (7.9).
Using (H), Lemma 8.2 for the Poissonian terms, one easily checks that

$$\begin{aligned} J_t^{n,7}(\lambda,\mu)+...+J_t^{n,10}(\lambda,\mu) &\leq K\int_0^t E\left(|\Gamma_s^n(\lambda,\mu)|^p\times\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right)ds \\ &\quad +K\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right)ds \end{aligned} \tag{7.38}$$

Using the Cauchy-Schwarz inequality, and estimations (7.8) and (7.9), we can conclude that

$$J_t^{n,7}(\lambda,\mu)+...+J_t^{n,10}(\lambda,\mu)\leq K|\lambda-\mu|^p+K\int_0^t E\left(|\theta_{s-}^n(\lambda)|^p\right)ds \tag{7.39}$$

It is easy to check that

$$|h'_z(x,\gamma_n^\lambda(s,z))-h'_z(y,\gamma_n^\lambda(s,z))|\leq|x-y|\times\left[\delta(z)+|v_n(s,z)|\times\beta(z)+|v_n(s,z)|^2\times\zeta_n(z)\right] \tag{7.40}$$

Thus $J_t^{n,11}(\lambda,\mu)$ is smaller than

$$\begin{aligned} & KE\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O|\Gamma_{s-}^n(\lambda,\mu)|\times\left[\delta(z)\alpha_n(z)+\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\right]N(ds,dz)\right|^p\right] \\ & \leq KE\left[\left|\int_{T-a_n}^{T_n}\int_O\delta(z)\alpha_n(z)N(ds,dz)\right|^p\times\sup_{[0,T]}|\Gamma_s^n(\lambda,\mu)|^p\right] \\ & +KE\left[\left|\int_{T-a_n}^{R_n}\int_O\left[\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\right]N(ds,dz)\right|^p\times\sup_{[0,T]}|\Gamma_s^n(\lambda,\mu)|^p\right] \\ & \leq K|\lambda-\mu|^p \end{aligned} \tag{7.41}$$

thanks to the definitions of T_n, R_n, and equation (7.8).

At last, since

$$|h'_z(x,\gamma_n^\lambda(s,z))-h'_z(x,\gamma_n^\mu(s,z))|\leq|\lambda-\mu|\times|v_n(s,z)|\times[\beta(z)+|v_n(s,z)|\zeta_n(z)] \tag{7.42}$$

we see that (thanks to the definition of R_n) :

$$J_t^{n,12}(\lambda,\mu)\leq KE\left[\left|\int_{T-a_n}^{R_n}\int_O|\lambda-\mu|\times\left[\beta(z)\alpha_n^2(z)+\zeta_n(z)\alpha_n^3(z)\right]N(ds,dz)\right|^p\right]\leq K|\lambda-\mu|^p \tag{7.43}$$

Finally, we obtain, for some constants K_1, K_2, K_3 depending only on p,

$$\begin{aligned} E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) &\leq K_1|\lambda-\mu|^p + K_2\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right) ds \\ &\quad + \frac{K_3}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda,\mu)|^p\right) ds \\ &\quad + \frac{1}{2^p} E\left[\sup_{[0,t]} |\theta_s^n(\lambda)|^p\right] \end{aligned} \tag{7.44}$$

Thus, since $p \geq 1$,

$$\begin{aligned} E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) &\leq 2K_1|\lambda-\mu|^p + 2K_2\int_0^t E\left(|\theta_s^n(\lambda,\mu)|^p\right) ds \\ &\quad + \frac{2K_3}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|\theta_s^n(\lambda,\mu)|^p\right) ds \end{aligned} \tag{7.45}$$

Furthermore, it is clear, from Remark 4.6, that for each λ, μ, n, the function $f(t) = E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right)$ is bounded on $[0,T]$. We thus can apply the extended Gronwall Lemma 8.6 proved in the Appendix, and conclude that there exists a constant C_p, independent of λ, μ, and n, such that

$$E\left(\sup_{[0,t]} |\theta_s^n(\lambda,\mu)|^p\right) \leq C_p|\lambda-\mu|^p \tag{7.46}$$

<u>Proof of Lemma 7.4</u> : we first check (7.12). We set $A_t^n(\lambda) = \frac{\partial}{\partial\lambda}X_t^n(\lambda) - I_t^n(\lambda)$. Writing $\varphi(z)dzds$ as $N(ds,dz) - \tilde{N}(ds,dz)$ in the expression (4.16) of $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$, we see that

$$\begin{aligned} A_t^n(\lambda) &= \int_0^t\int_O h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}(ds,dz) \\ &\quad + \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda), z)\right] A_{s-}^n(\lambda) N(ds,dz) \\ &\quad - \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda), \gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda), z)\right] \frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}(ds,dz) \\ &\quad + \int_0^t g'(X_{s-}^n(\lambda))\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)ds + \int_0^t\int_E f'_x(X_{s-}^n(\lambda),u)\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\tilde{N}_1(ds,du) \end{aligned} \tag{7.47}$$

But it is easy to check that $\frac{\partial}{\partial\lambda}X_t^n(\lambda)$ vanishes as soon as $t \leq T - a_n$. Hence, using Burkholder's inequality for the first, third, and fifth terms, using inequalities (7.28) and (7.30), the expression of v_n (see Definition 5.1), and (H), we see that

$$E\left(\sup_{[0,t]} |A_s^n(\lambda)|^p\right)$$

$$\leq KE\Big[\Big|\int_{(T-a_n)\wedge t}^{t}\int_O\Big[\eta(z)+\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\Big]^2 \times \Big|\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\Big|^2 N(ds,dz)\Big|^{\frac{p}{2}}\Big]$$

$$+2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge R_n\wedge t}\int_O\Big[\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\Big]\times|A_s^n(\lambda)|\,N(ds,dz)\right|^p\right]$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_O\Big[\alpha_n(z)\delta(z)+\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)\Big]^2\left|\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right|^2 N(ds,dz)\right|^{\frac{p}{2}}\right]$$

$$+K\int_{(T-a_n)\wedge t}^{t}E\left[\left|\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right|^p\right]ds$$

$$+KE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_E\sigma^2(u)\left|\frac{\partial}{\partial\lambda}X_{s-}^n(\lambda)\right|^2 N_1(ds,du)\right|^{\frac{p}{2}}\right]$$

$$\leq D_t^{n,1}(\lambda)+\ldots+D_t^{n,5}(\lambda) \tag{7.48}$$

First,

$$D_t^{n,1}(\lambda) \leq KE\left[\sup_{[0,T]}\left|\frac{\partial}{\partial\lambda}X_s^n(\lambda)\right|^p\right]^{\frac{1}{2}} \tag{7.49}$$

$$\times\ E\left[\left|\int_{T-a_n}^{T}\int_O\Big[\eta^2(z)+\alpha_n^2(z)\delta^2(z)+\alpha_n^4(z)\beta^2(z)+\alpha_n^6(z)\zeta_n^2(z)\Big]N(ds,dz)\right|^p\right]^{\frac{1}{2}}$$

Using Lemma 8.4, since from (SP),

$$a_n\int_O\Big[\eta^2(z)+\alpha_n^2(z)\delta^2(z)+\alpha_n^4(z)\beta^2(z)+\alpha_n^6(z)\zeta_n^2(z)\Big]\varphi(z)dz \longrightarrow_{n\to\infty} 0 \tag{7.50}$$

and using (7.9), it is clear that $D_t^{n,1}(\lambda)\leq K_n$, where the sequence K_n goes to 0. One can check in the same way that $D_t^{n,3}(\lambda)\leq L_n$, where L_n goes to 0. It is clear from (7.9) that $D_t^{n,4}(\lambda)\leq Ka_n$. Thanks to Lemma 8.2 and estimation (7.9), since $\sigma\in L^2(E,q)$, we see that $D_t^{n,5}(\lambda)\leq Ka_n$. At last, the definitions of T_n and R_n yield that

$$\begin{aligned}
D_t^{n,2}(\lambda) &\leq 4^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{T_n\wedge t}\alpha_n(z)\delta(z)|A_{s-}^n(\lambda)|N(ds,dz)\right|^p\right]\\
&\quad +4^{p-1}E\left(\left|\int_{(T-a_n)\wedge t}^{R_n\wedge t}[\alpha_n^2(z)\beta(z)+\alpha_n^3(z)\zeta_n(z)]|A_{s-}^n(\lambda)|N(ds,dz)\right|^p\right]\\
&\leq 4^pE\left(\sup_{[0,t]}|A_s^n(\lambda)|^p\right)\times\left[2l+d_0(\|\delta\|_\infty+\|\beta\|_\infty+\sup_n\|\zeta_n\|_\infty)\right]^p\\
&\leq \frac{1}{4^p}E\left(\sup_{[0,t]}|A_s^n(\lambda)|^p\right)
\end{aligned} \tag{7.51}$$

since d_0 and l satisfy (7.18). We finally can write

$$E\left(\sup_{[0,t]}|A_s^n(\lambda)|^p\right) \leq C_n + \frac{1}{4^p}E\left[\sup_{[0,t]}|A_s^n(\lambda)|^p\right] \tag{7.52}$$

where the sequence C_n goes to 0. Thus

$$E\left(\sup_{[0,t]}|A_s^n(\lambda)|^p\right) \leq 2C_n \longrightarrow 0 \tag{7.53}$$

when n goes to infinity.

The proof of (7.11) is quite similar to the one of (7.10).

7.2 Upperbound for the second derivative.

The method is exactly the same as in the previous subsection : we set

$$\begin{aligned} J_t^n(\lambda) &= \int_0^t\int_O \left[h'_x(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h'_x(X_{s-}^n(\lambda),z)\right] J_{s-}^n(\lambda)N(ds,dz) \\ &\quad +2\int_0^t\int_O h''_{xz}(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z))I_{s-}^n(\lambda)v_n(s,z)N(ds,dz) \\ &\quad +\int_0^t\int_O \left[h''_{xx}(X_{s-}^n(\lambda),\gamma_n^\lambda(s,z)) - h''_{xx}(X_{s-}^n(\lambda),z)\right]\left(I_{s-}^n(\lambda)\right)^2 N(ds,dz) \end{aligned} \tag{7.54}$$

and we state the following lemma, which can be proved as Lemmas 7.2, 7.3 and 7.4 :

Lemma 7.5 *Let d_0 and l be as in Proposition 7.1. Let $p \in [1,\infty[$ be fixed. There exists a constant $C_p < \infty$ such that for all λ, μ in $[-1,1]$ and all n,*

$$E\left(\sup_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)\right|^p\right) \leq C \ ; \ E\left(\sup_{[0,T]}\left|\frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda) - \frac{\partial^2}{\partial\lambda^2}X_t^n(\mu)\right|^p\right) \leq C|\lambda-\mu|^p \tag{7.55}$$

$$\text{a.s.,} \qquad \sup_{|\lambda|\leq 1}\sup_{[0,T]}|J_t^n(\lambda)| \leq \frac{1}{2} \tag{7.56}$$

$$\lim_n E\left(\sup_{[0,T]}\left|J_t^n(\lambda) - \frac{\partial^2}{\partial\lambda^2}X_t^n(\lambda)\right|^p\right) = 0 \ ; \ E\left(\sup_{[0,T]}|J_t^n(\lambda) - J_t^n(\mu)|^p\right) \leq C|\lambda-\mu|^p \tag{7.57}$$

Comparing equations (4.17) and (4.22), we see that in order to prove (7.55), we just have to check that

$$E\left(\sup_{[0,T]}|L_t^n(\lambda)|^p\right) \leq C_p \quad ; \quad E\left(\sup_{[0,T]}|L_t^n(\lambda) - L_t^n(\mu)|^p\right) \leq C_p|\lambda-\mu|^p \tag{7.58}$$

Similar simplifications may be used to prove (7.57). Let us just prove (7.56). Using Lemma 7.2, inequality (7.16), the same inequality for h''_{xx}, the expression of v_n, and inequality (7.18), we see that

$$\begin{aligned}
|J_t^n(\lambda)| &\leq \sup_{[0,t]} |J_s^n(\lambda)| \times \int_{T-a_n}^{T_n\wedge R_n} \left[\delta\alpha_n + \beta\alpha_n^2 + \zeta_n\alpha_n^3\right](z)N(ds,dz) \\
&\quad +2\sup_{[0,t]} |I_s^n(\lambda)| \times \int_{T-a_n}^{T_n\wedge R_n} \left[\delta\alpha_n + \beta\alpha_n^2 + \zeta_n\alpha_n^3\right](z)N(ds,dz) \\
&\quad +\sup_{[0,t]} |I_s^n(\lambda)|^2 \times \int_{T-a_n}^{T_n\wedge R_n} \left[\delta\alpha_n + \beta\alpha_n^2 + \zeta_n\alpha_n^3\right](z)N(ds,dz) \\
&\leq [2l + d_0(\| \delta \|_\infty + \| \beta \|_\infty + \| \zeta \|_\infty)] \times \left(\sup_{[0,t]} |J_s^n(\lambda)| + 2\times 1/2 + 1/4\right) \\
&\leq \frac{1}{16}\sup_{[0,t]} |J_s^n(\lambda)| + \frac{5}{64} \qquad (7.59)
\end{aligned}$$

This yields (7.56).

This Lemma allows to conclude, as in the previous subsection, that the proposition below holds.

Proposition 7.6 *Let d_0 and l be as in Proposition 7.1. Then, when n goes to infinity,*

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial^2}{\partial\lambda^2}X_T^n(\lambda)\right| \leq 1\right) \longrightarrow 1 \qquad (7.60)$$

7.3 Conclusion.

Fix y_0 in the support of $P\circ X_T^{-1}$. We have actually proved that there exists $C_1 > 0$ and $C_2 < \infty$ such that for all $r > 0$,

$$P\left(|X_T - y_0| < r \quad ; \quad \left|\frac{\partial}{\partial\lambda}X_T^n(0)\right| \geq C_1 \quad ; \quad \sup_\lambda \left|\frac{\partial}{\partial\lambda}X_T^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X_T^n(\lambda)\right| \leq C_2\right) \qquad (7.61)$$

goes to $P(|X_T - y_0| < r)$, that is strictly positive.

Thus, the assumptions of Theorem 3.3 are satisfied for any y_0 in the support of the law of X_T, and Remark 3.5 allows us to conclude that Theorem 2.4 is proved.

8 Appendix.

We begin this annex with a consequence of a standard limit theorem for continuous processes.

Lemma 8.1 *Let $\{Y_n(\lambda)\}_{\lambda\in[-1,1]}$ be a sequence of real valued processes. Assume that there exists a constant $C < \infty$ such that for all $\lambda, \mu \in [-1,1]$, all $n \in I\!N$,*

$$E\left(|Y_n(\lambda) - Y_n(\mu)|^2\right) \leq C|\lambda - \mu|^2 \qquad (8.1)$$

Suppose also that for all $\lambda \in [-1,1]$, when n goes to infinity,

$$E\left(|Y_n(\lambda)|^2\right) \longrightarrow 0 \tag{8.2}$$

Then the following convergence holds in probability :

$$\sup_{|\lambda|\le 1} |Y_n(\lambda)| \longrightarrow 0 \tag{8.3}$$

Proof : first notice that thanks to (8.1) and to the Kolmogorov criterion of continuity, see e.g. Revuz, Yor, [39], p 25, there exists a constant $K<\infty$ such that for all n,

$$E\left(\sup_{\lambda\neq\mu} \frac{|Y_n(\lambda)-Y_n(\mu)|}{|\lambda-\mu|^{\frac{1}{4}}}\right) \le K \tag{8.4}$$

Thus for all $\eta>0$,

$$\lim_{\delta\to 0}\sup_n P\left(\sup_{|\lambda-\mu|\le\delta} |Y_n(\lambda)-Y_n(\mu)| \ge \eta\right) \le \lim_{\delta\to 0} K\eta^{-1}\delta^{\frac{1}{4}} = 0 \tag{8.5}$$

Furthermore, the family of the laws of $Y_n(0)$ is clearly tight, thanks to (8.2). Thus, we deduce from Theorem 1.3.1 p 31 in [43] that the family of the laws of Y_n is tight.
On the other hand, it is obvious from (8.2) that for all λ_1, ..., λ_k in $[-1,1]$, $(Y_n(\lambda_1),...,Y_n(\lambda_k))$ goes to 0 in law when n goes to infinity. The standard limit Theorems for continuous processes yield that Y_n goes to 0 in law for the supremum norm, i.e. that

$$\sup_{[-1,1]} |Y_n(\lambda)| \longrightarrow 0 \tag{8.6}$$

in law. But the convergence in law to a constant implies the convergence in probability, and the Lemma is proved.

We now give five technical lemmas. We omit the proof of the first one, because it is similar to the one of Lemma 8.4 below.

Lemma 8.2 *Let $\eta\in L^2(O,\varphi(z)dz)\cap L^\infty(O,\varphi(z)dz)$, and let $p\ge 1$. There exist some constants C_p and $K_p(\eta)$ such that for every predictable process Y on $[0,T]$,*

$$\begin{aligned} E\left(\sup_{[0,t]}\left|\int_0^u\int_O \eta(z)Y_s\tilde{N}(ds,dz)\right|^p\right) &\le C_pE\left(\left|\int_0^t\int_O \eta^2(z)Y_s^2N(ds,dz)\right|^{\frac{p}{2}}\right) \\ &\le K_p(\eta)\int_0^t E\left(|Y_s|^p\right)ds \end{aligned} \tag{8.7}$$

Lemma 8.3 *Let γ_n be a sequence of positive functions on O and let a_n be a real valued sequence decreasing to 0, such that for some constant $C<\infty$,*

$$a_n\int_O \gamma_n(z)\varphi(z)dz \le C \tag{8.8}$$

Let $p\ge 1$. Then there exists a constant K_p such that for all predictable process Y on $[0,T]$,

$$E\left(\left|\int_{(T-a_n)\wedge t}^t\int_O \gamma_n(z)|Y_s|\varphi(z)dzds\right|^p\right) \le \frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right)ds \tag{8.9}$$

Proof : the left member of equation (8.9) is smaller than

$$E\left(\left|\int_{(T-a_n)\wedge t}^t |Y_s|ds \times \frac{C}{a_n}\right|^p\right) \leq \frac{K}{a_n^p} E\left[\left|\int_{(T-a_n)\wedge t}^t |Y_s|ds\right|^p\right] \tag{8.10}$$

Using Holder's inequality (for the measure ds), we obtain

$$\leq \frac{K}{a_n^p} \times \left|\int_{(T-a_n)\wedge t}^t ds\right|^{p-1} \times E\left[\int_{(T-a_n)\wedge t}^t |Y_s|^p ds\right] \leq \frac{K}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right) ds \tag{8.11}$$

which was our aim.

Lemma 8.4 *Consider a sequence of positive functions $\gamma_n \in L^1(O, \varphi(z)dz)$, such that $\parallel \gamma_n \parallel_\infty \leq k_0$, let a_n be a real valued sequence decreasing to 0, and let $p \geq 1$.*

1. *if $a_n \int_O \gamma_n(z)\varphi(z)dz \leq C$, then there exists K_p such that for all predictable process Y on $[0,T]$,*
$$E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|^p\right] \leq \frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^p\right) ds \tag{8.12}$$
2. *if $a_n \int_O \gamma_n(z)\varphi(z)dz \to 0$, then*
$$E\left[\left|\int_{T-a_n}^T \int_O \gamma_n(z)N(ds,dz)\right|^p\right] \to 0 \tag{8.13}$$

Proof : let us for example check 1. : we will prove (8.12) for every $p = 2^q$, recursively (on q). First, if $p = 1$,

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|\right] &= \int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)E\left(|Y_s|\right)\varphi(z)dzds \\ &\leq \int_{(T-a_n)\wedge t}^t E\left(|Y_s|\right) \times \frac{C}{a_n}ds \end{aligned} \tag{8.14}$$

Assume now that (8.12) holds for $p = 2^q$. Then

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|N(ds,dz)\right|^{2p}\right] &\leq KE\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|\tilde{N}(ds,dz)\right|^{2p}\right] \\ &+KE\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n(z)|Y_s|\varphi(z)dzds\right|^{2p}\right] \end{aligned} \tag{8.15}$$

Thanks to Burkholder's inequality and the inductive assumption, the first term is smaller than

$$K \times C_p E\left[\left|\int_{(T-a_n)\wedge t}^t \int_O \gamma_n^2(z)|Y_s|^2 N(ds,dz)\right|^p\right] \leq K \times k_0 \times C_p \times \frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^t E\left(|Y_s|^{2p}\right) ds \tag{8.16}$$

By using Lemma 8.3, the second term is smaller than

$$\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^{t} E\left(|Y_s|^{2p}\right)ds \tag{8.17}$$

which was our aim.

Lemma 8.5 *Let Y be a predictable process on $[0,T]$, let a_n be a sequence decreasing to 0, and let $l>0$ be fixed. Consider a sequence of positive functions γ_n on O such that*

$$\parallel \gamma_n \parallel_\infty \leq k_0 \qquad ; \qquad a_n\int_O \gamma_n^2(z)\varphi(z)dz \leq K \tag{8.18}$$

Consider also the following stopping time :

$$\tau_n = \inf\left\{t > T-a_n \;\Big/\; \int_{T-a_n}^{t}\int_O \gamma_n(z)N(ds,dz) \geq l\right\} \tag{8.19}$$

Then for all $p \in [1,\infty[$,

$$\begin{aligned} E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t}\int_O |Y_s|\gamma_n(z)\varphi(z)dzds\right|^p\right] &\leq 2^{p-1}\times|l+k_0|^p\times E\left(\sup_{[0,t]}|Y_s|^p\right)\\ &+\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left(|Y_s|^p\right)ds \end{aligned} \tag{8.20}$$

<u>Proof</u> : we first write $\varphi(z)dzds$ as $N(ds,dz)-\tilde{N}(ds,dz)$, to upperbound the left member of (8.20) with :

$$\begin{aligned} &2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t}\int_O |Y_s|\gamma_n(z)N(ds,dz)\right|^p\right]\\ &+2^{p-1}E\left[\left|\int_{(T-a_n)\wedge t}^{\tau_n\wedge t}\int_O |Y_s|\gamma_n(z)\tilde{N}(ds,dz)\right|^p\right] \end{aligned} \tag{8.21}$$

Thanks to Burkholder's inequality, this is smaller than

$$\begin{aligned} &2^{p-1}E\left[\sup_{[0,t]}|Y_s|^p\times\left|\int_{(T-a_n)}^{\tau_n}\int_O \gamma_n(z)N(ds,dz)\right|^p\right]\\ &+C_pE\left[\left|\int_{(T-a_n)\wedge t}^{t}\int_O Y_s^2\gamma_n^2(z)N(ds,dz)\right|^{\frac{p}{2}}\right] \end{aligned} \tag{8.22}$$

At last thanks to the definition of τ_n, Lemma 8.4, and (8.18) this is smaller than

$$2^{p-1}\times|l+k_0|^p\times E\left(\sup_{[0,t]}|Y_s|^p\right)+\frac{K_p}{a_n}\int_{(T-a_n)\wedge t}^{t}E\left(|Y_s|^p\right)ds \tag{8.23}$$

which was our aim.

At last, the next lemma is a consequence of Gronwall's Lemma.

Lemma 8.6 *If f is a positive and bounded function on $[0,T]$ satisfying, for some $a \in]0,T]$ and some positive constants C_1, C_2, and C_3 :*

$$f(t) \leq C_1 + C_2 \int_0^t f(s)ds + \frac{C_3}{a} \int_{(T-a)\wedge t}^t f(s)ds \tag{8.24}$$

then

$$\sup_{[0,T]} f(t) \leq C_1 \left[1 + C_2 T e^{C_2 T}\right] e^{C_2 T + C_3} \tag{8.25}$$

Proof : first, it is clear from Gronwall's lemma that

$$\sup_{[0,T-a]} f(t) \leq C_1 e^{C_2 T} \tag{8.26}$$

We now set $g(t) = f((T-a)+t)$, which is well defined on $[0,a]$. We obtain

$$g(t) \leq C_1 + C_2 C_1 e^{C_2 T}(T-a) + \left(C_2 + \frac{C_3}{a}\right) \int_0^t g(s)ds \tag{8.27}$$

Thus Gronwall's Lemma yields that

$$\sup_{[T-a,T]} f(t) = \sup_{[0,a]} g(t) \leq C_1 \left[1 + C_2 T e^{C_2 T}\right] e^{C_2 a + C_3} \tag{8.28}$$

Chapitre 6

Strict positivity of a solution to a one-dimensional Kac equation without cutoff

Abstract : We consider the solution of a one-dimensional Kac equation without cutoff built by Graham and Méléard in [19]. Recalling that this solution is the density of a Poisson driven nonlinear stochastic differential equation, we develop Bismut's approach of the Malliavin Calculus for Poisson functionals, in order to prove that this solution is strictly positive on $]0,\infty[\times I\!R$.

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1 Introduction.

We prove by a probabilistic approach the strict positivity of a solution of a one dimensional Kac equation without cutoff, in the case where the cross section does sufficiently explode. In the cutoff case, much more is known : Pulverenti and Wennberg, [38], have proved, by using analytic methods, the existence of a Maxwellian lowerbound. But their proof is based on the separation of the gain and loss terms, which typically cannot be done in the present case. No result seems to have been found by the analysts in the non cutoffed case.

The solution we study has been built by Graham and Méléard in [19]. This solution $f(t,v)$ can be related with the solution V_t of a Poisson driven nonlinear S.D.E. : for each $t>0$, $f(t,.)$ is the density of the law of V_t. We will thus study f as the density of a Poisson functional.

In Chapter 5, the strict positivity of the density for Poisson driven S.D.E.s is studied in the case where the intensity measure of the Poisson measure is the Lebesgue measure. The method is adapted from a paper of Bally and Pardoux, [4], which deals with a similar problem in the case of white noise driven S.P.D.E.s, i.e. with Wiener functionals. This method is based on Bismut's approach of the Malliavin Calculus, which consists in perturbing the processes, see e.g. Bichteler, Jacod, [7], for the case of classical diffusion processes with jumps. Nevertheless, we can not directly apply the results of Chapter 5. We can not either use exactly the same Malliavin Calculus as Bichteler and Jacod, because the intensity measure of our Poisson measure will not be the Lebesgue measure. We generalize a Malliavin Calculus adapted to our model, inspired by Graham and Méléard, [19].

Let us say a word about the difference between the techniques in the case of Wiener functionals and Poisson functionals. The main difference is that the Malliavin calculus does product integrals with respect to the Lebesgue measure in the first case, and with respect to the Poisson measure in the second case. We thus have to deal with random perturbations and with stopping times instead of deterministic perturbations and times. This is why the assumptions are very stringent in Chapter 5. Nevertheless, the method gives a quite good result in the case of the Kac equation without cutoff.

The present work is organized as follows. In Section 2, we recall the Kac equation, we give the results of Desvillettes, Graham, and Méléard in [17] and [19], who solved this equation, and we state our result. In Section 3, we define rigorously our "perturbations", and we state a criterion of strict positivity. At last, we apply this criterion in the next sections.

2 The Kac equation without cutoff, the main result.

The Kac equation deals with the density of particles in a gaz, and is a one-dimensional "caricature" of the famous spatially homogeneous Boltzmann equation. We denote by $f(t,v)$ the density of particles which have the velocity $v \in I\!R$ at the instant $t > 0$. Then

$$\frac{\partial f}{\partial t}(t,v) = \int_{v_* \in I\!R} \int_{\theta=-\pi}^{\pi} \left[f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \right] \beta(\theta) d\theta dv_* \tag{2.1}$$

where

$$v' = v\cos\theta - v_*\sin\theta \quad ; \quad v'_* = v\sin\theta + v_*\cos\theta \tag{2.2}$$

and β is a non cutoff cross section, i.e. an even and positive function on $[-\pi,\pi]\backslash\{0\}$ satisfying

$$\int_0^{\pi} \theta^2 \beta(\theta) d\theta < \infty \tag{2.3}$$

The case with cutoff, namely when $\int_0^{\pi} \beta(\theta) d\theta < \infty$, has been much investigated by the analysts, and they have obtained some existence, regularity and strict positivity results.
In [17] and [19], Desvillettes, Graham and Méléard give an existence and regularity result for such an equation, by using the probability theory. See also Desvillettes, [15] for another statement (using the Fourier Theory), and Desvillettes [16] or Chapter 4 for the 2-dimensional case. We are interested in this paper in the strict positivity of the solution of (2.1) built by Graham and

Méléard in [19]. Let us recall their main results.
First, we will consider solutions in the following (weak) sense.

Definition 2.1 *Let P_0 be a probability on $I\!R$ that admits a moment of order 2. A positive function f on $I\!R_*^+ \times I\!R$ is a weak solution of (2.1) with initial data P_0 if for every test function $\phi \in C_b^2(I\!R)$,*

$$\int_{v\in I\!R} f(t,v)\phi(v)dv = \int_{v\in I\!R} \phi(v)P_0(dv) + \int_0^t \int_{v\in I\!R}\int_{v^*\in I\!R} K^\phi(v,v_*)f(s,v)f(s,v^*)dvdv^*ds \quad (2.4)$$

where

$$K^\phi(v,v_*) = -bv\phi'(v) + \int_{-\pi}^{\pi} \Big\{\phi(v\cos\theta - v_*\sin\theta) - \phi(v) - [v(\cos\theta-1) - v_*\sin\theta]\,\phi'(v)\Big\}\beta(\theta)d\theta \quad (2.5)$$

and

$$b = \int_{-\pi}^{\pi} (1-\cos\theta)\beta(\theta)d\theta \quad (2.6)$$

Notice that b and the collision kernel K^ϕ are well defined thanks to (2.3).

In [17] and [19], one assumes that

Assumption (H) :

1. The initial data P_0 admits a moment of order 2, and is not a Dirac mass at 0.
2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi,\pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0,\pi/2[$, and $r \in]1,3[$ such that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r} 1_{[-\theta_0,\theta_0]}(\theta)$. We still assume $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$.

They also build the following random elements :

Notation 2.2 *We denote by N_0 and N_1 two independant Poisson measures on $[0,T]\times[0,1]\times[-\pi,\pi]$, with intensity measures :*

$$\nu_0(d\theta,d\alpha,ds) = \beta_0(\theta)d\theta d\alpha ds \quad ; \quad \nu_1(d\theta,d\alpha,ds) = \beta_1(\theta)d\theta d\alpha ds \quad (2.7)$$

and by $\tilde N_0$ and $\tilde N_1$ the associated compensated measures. We will write $N = N_0 + N_1$. We consider a real valued random variable V_0 independant of N_0 and N_1, of which the law is P_0. We also assume that our probability space is the canonical one associated with the independent random elements V_0, N_0, and N_1 :

$$(\Omega,\mathcal{F},\{\mathcal{F}_t\},P) = (\Omega',\mathcal{F}',\{\mathcal{F}'\},P')\otimes(\Omega^0,\mathcal{F}^0,\{\mathcal{F}_t^0\},P^0)\otimes(\Omega^1,\mathcal{F}^1,\{\mathcal{F}_t^1\},P^1) \quad (2.8)$$

We will consider $[0,1]$ as a probability space, denote by $d\alpha$ the Lebesgue measure on $[0,1]$, and denote by E_α and $\mathcal{L}_\alpha$ the expectation and law on $([0,1],\mathcal{B}([0,1]),d\alpha)$.

The following Theorem is proved in [17] (Theorem 3.6 p 11).

Theorem 2.3 *There exists a process $\{V_t(\omega)\}$ on Ω and a process $\{W_t(\alpha)\}$ on $[0,1]$ such that*

$$\begin{cases} V_t(\omega) = V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} [(\cos\theta - 1)V_{s-}(\omega) - (\sin\theta) W_{s-}(\alpha)]\, \tilde{N}(\omega, d\theta d\alpha ds) \\ \qquad -b \int_0^t V_{s-}(\omega) ds \\ \mathcal{L}_\alpha(W) = \mathcal{L}(V) \quad ; \quad E\left(\sup_{[0,T]} V_t^2\right) < \infty \end{cases} \tag{2.9}$$

At last, Graham and Méléard show in [19] the following theorem (see Theorem 1.6, Corollary 1.8, p 4)

Theorem 2.4 *Assume (H). Let (V,W) be a solution of (2.9). Then for all $t > 0$, the law of V_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $I\!R$. The obtained function f is a solution of the Kac equation (2.1) in the sense of Definition 2.1. Assume furthermore that P_0 admits some moments of all orders. Then for each $t > 0$, the function $f(t,.)$ is of class C^∞ on $I\!R$.*

Let us now give our assumption, which is more stringent than (H) : we need a stronger explosion of the cross section.

Assumption (SP) :

1. The initial data P_0 admits moments of all orders, and is not a Dirac mass at 0.
2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi,\pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0,\pi/2[$, and $r \in [2,3[$ such that $\beta_0(\theta) = \frac{k_0}{|\theta|^r} 1_{[-\theta_0,\theta_0]}(\theta)$. We still assume $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$.

Our result is the following :

Theorem 2.5 *Assume (SP), and consider the solution in the sense of Definition 2.1 of equation (2.1) built in Theorem 2.4. Then f is strictly positive on $]0,+\infty[\times I\!R$.*

In (SP), we do not really need the fact that P_0 has moments of all orders, but only the fact that the density $f(t,v)$ of the law of V_t built in Theorem 2.4 is continuous on $I\!R$ for each $t > 0$. Notice that our method does not work in the case where r belongs to $]1,2[$: we do really need a large explosion of the cross section at 0.

In the whole work, we will assume (SP), use notation 2.2, and consider a solution (V,W) of (2.9).

3 A criterion of strict positivity.

This section contains two parts. We first introduce some general notations and definitions about Bismut's approach of the Malliavin calculus on our Poisson space. We follow here Bichteler, Jacod, [7], and Graham, Méléard, [19]. Then we adapt the criterion of strict positivity of Bally, Pardoux, [4] (which deals with the Wiener functionals) to our probability space.

Definition 3.1 *A predictable function $v(\omega, s, \theta, \alpha)$ on $\Omega \times [0,T] \times [-\theta_0, \theta_0] \times [0,1]$ is said to be a "perturbation" if for all fixed ω, s, α, $v(\omega, s, ., \alpha)$ is C^1 on $[-\theta_0, \theta_0]$, and if there exists some even positive (deterministic) functions η and ρ on $[-\theta_0, \theta_0]$ such that*

$$|v(s,\theta,\alpha)| \leq \eta(\theta) \qquad ; \quad |v'(s,\theta,\alpha)| \leq \rho(\theta) \tag{3.1}$$

$$\eta(\theta) \leq \frac{|\theta|}{2} \quad ; \quad \eta(-\theta_0) = \eta(\theta_0) = 0 \tag{3.2}$$

$$\text{if } \xi(\theta) = \rho(\theta) + r2^{r+2}\frac{\eta(\theta)}{|\theta|} \quad \text{then} \quad \| \xi \|_\infty \leq \frac{1}{2} \quad \text{and} \quad \xi \in L^1(\beta_0(\theta)d\theta) \tag{3.3}$$

Notice that thanks to (3.3), η and ρ are in $L^1 \cap L^\infty(\beta_0(\theta)d\theta)$.

Consider now a fixed perturbation v. For $\lambda \in [-1,1]$ we set

$$\gamma^\lambda(s,\theta,\alpha) = \theta + \lambda v(s,\theta,\alpha) \tag{3.4}$$

Thanks to (3.1), (3.2), and (3.3), it is easy to check that for each $\lambda, s, \alpha, \omega$, $\gamma^\lambda(s,.,\alpha)$ is an increasing bijection from $[-\theta_0, \theta_0]\backslash\{0\}$ into itself. Then we denote by $N_0^\lambda = \gamma^\lambda(N_0)$ the image measure of N_0 by γ^λ : for any Borel subset A of $[0,T] \times [-\theta_0, \theta_0] \times [0,1]$,

$$N^\lambda(A) = \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A(s, \gamma^\lambda(s,\theta,\alpha), \alpha) N_0(d\theta d\alpha ds) \tag{3.5}$$

We also define the shift S^λ on Ω by

$$V_0 \circ S^\lambda = V_0 \quad ; \quad N_0 \circ S^\lambda = N_0^\lambda \quad ; \quad N_1 \circ S^\lambda = N_1 \tag{3.6}$$

We will need the following predictable function :

$$Y^\lambda(s,\theta,\alpha) = \frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}(1 + \lambda v'(s,\theta,\alpha)) \tag{3.7}$$

Then it is easy to check that for all $\lambda \in [-1,1]$,

$$\gamma^\lambda\left(Y^\lambda.\nu_0\right) = \nu_0 \tag{3.8}$$

and for all $\lambda, \mu \in [-1,1]$ (recall that ξ is defined in (3.3)),

$$\left|Y^\lambda(s,\theta,\alpha) - Y^\mu(s,\theta,\alpha)\right| \leq |\lambda - \mu| \times \xi(\theta) \tag{3.9}$$

We also consider the following martingale

$$M_t^\lambda = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (Y^\lambda(s,\theta,\alpha) - 1)\tilde{N}_0(d\theta d\alpha ds) \tag{3.10}$$

and its Doléans-Dade exponential (see Jacod, Shiryaev, [23])

$$G_t^\lambda = \mathcal{E}(M^\lambda)_t = e^{M_t^\lambda} \prod_{0\le s\le t} \left(1+\Delta M_s^\lambda\right) e^{-\Delta M_s^\lambda} \tag{3.11}$$

Since $|Y^\lambda - 1| \le \xi \le 1/2$, it is clear that G^λ is strictly positive on $[0,T]$ a.s. We now set $P^\lambda = G_T^\lambda.P$. Using equation (3.8), and the Girsanov Theorem for random measures (see Jacod, Shiryaev, [23], p 157) one can show that $P^\lambda \circ (S^\lambda)^{-1} = P$, i.e. that the law of (V_0, N_0^λ, N_1) under P^λ is the same as the one of (V_0, N_0, N_1) under P.

The following lemma can be proved as in Chapter 5 :

Lemma 3.2 *Let v be a perturbation, and G^λ the associated exponential martingale. Then a.s., the map $\lambda \mapsto G_T^\lambda$ is continuous on $[-1,1]$.*

We now give the criterion of strict positivity we will use.

Theorem 3.3 *Let X be a real valued random variable on Ω, such that $P\circ X^{-1} = p(x)dx$, with p continuous on $I\!R$, and let $y_0 \in I\!R$. Assume that there exists a sequence v_n of perturbations such that, if $X^n(\lambda) = X\circ S_n^\lambda$, then for all n, the map*

$$\lambda \mapsto X^n(\lambda) \tag{3.12}$$

is a.s. twice differentiable on $[-1,1]$. Assume that there exists $c>0$, $\delta>0$, and $k<\infty$, such that for all $r>0$,

$$\lim_{n\to\infty} P\left(\Lambda^n(r)\right) > 0 \tag{3.13}$$

where

$$\Lambda^n(r) = \left\{ |X-y_0|<r\ ,\ \left|\frac{\partial}{\partial\lambda}X^n(0)\right| \ge c\ ,\ \sup_{|\lambda|\le\delta}\left[\left|\frac{\partial}{\partial\lambda}X^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2}X^n(\lambda)\right|\right] \le k \right\} \tag{3.14}$$

Then $p(y_0)>0$.

The proof is exactly the same as that of Chapter 5.
We at last state a usefull remark.

Remark 3.4 *If X is a real valued random variable on Ω, admitting a continuous density p with respect to the Lebesgue measure on $I\!R$, and if for all $y\in \mathrm{supp}\ P\circ X^{-1}$, $p(y)>0$, then p is strictly positive on $I\!R$.*

Proof : since the support of the law of X is a closed set, we see that for $y \in \partial\ \{\mathrm{supp}\ P\circ X^{-1}\}$, $p(y)>0$. Assume that $(\mathrm{supp}\ P\circ X^{-1})^c \ne \emptyset$. Then there exists $\{y_k\}\subset (\mathrm{supp}\ P\circ X^{-1})^c$ such that $y_k \longrightarrow y \in \partial\ \{\mathrm{supp}\ P\circ X^{-1}\}$. Since p is continuous, we deduce that $p(y)=0$. Thus $(\mathrm{supp}\ P\circ X^{-1})^c = \emptyset$, and the proof is finished.

In order to prove Theorem 2.5, we will of course apply the previous criterion. In fact, we will only prove that $f(T,.)$ is strictly positive on $I\!R$, which suffices since T has been arbitrarily fixed. In the next section, we will consider a fixed perturbation v_n, and we will compute $V_t^n(\lambda)$ and its derivatives for any $t \in [0,T]$. Section 5 is devoted to the explicit choice of the sequence v_n of perturbations. In Section 6, we will prove (for some constant $\epsilon > 0$) that

$$\lim_{n\to\infty} P\left(\left|\frac{\partial}{\partial\lambda} V_T^n(0)\right| \geq \epsilon\right) = 1 \tag{3.15}$$

At last, we will check in Section 7 that for some constant K,

$$\lim_{n\to\infty} P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda} V_T^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2} V_T^n(\lambda)\right| \leq K\right) = 1 \tag{3.16}$$

Since for all $y_0 \in \text{supp } P\circ V_T^{-1}$, for all $r>0$, $P(V_T \in]y_0 - r, y_0 + r[) > 0$, we will easily conclude in Section 8.

4 Differentiability of the perturbed process.

In this section, we consider a fixed perturbation v_n. We compute $V_t^n(\lambda) = V_t \circ S_n^\lambda$, and we prove that for each t in $[0,T]$, this function is twice differentiable on $[-1,1]$.

4.1 The perturbed process.

Recalling that b is defined by (2.6), that $|\cos\theta - 1| \leq \theta^2$, and that (2.3) is satisfied, one can easily check that equation (2.9) can be written :

$$V_t = V_0 + \int_0^t\int_0^1\int_{-\pi}^{\pi} (\cos\theta - 1) V_{s-} N(d\theta d\alpha ds) - \int_0^t\int_0^1\int_{-\pi}^{\pi} (\sin\theta) W_{s-}(\alpha)\tilde{N}(d\theta d\alpha ds) \tag{4.1}$$

Hence, the perturbed process satisfies

$$\begin{aligned} V_t^n(\lambda) &= V_0 + \int_0^t\int_0^1\int_{-\pi}^{\pi} (\cos\theta - 1) V_{s-}^n(\lambda) N_0^{\lambda,n}(d\theta d\alpha ds) \\ &\quad + \int_0^t\int_0^1\int_{-\pi}^{\pi} (\cos\theta - 1) V_{s-}^n(\lambda) N_1(d\theta d\alpha ds) \\ &\quad - \int_0^t\int_0^1\int_{-\pi}^{\pi} (\sin\theta) W_{s-}(\alpha)\left[N_0^{\lambda,n}(d\theta d\alpha ds) - \beta_0(\theta) d\theta d\alpha ds\right] \\ &\quad - \int_0^t\int_0^1\int_{-\pi}^{\pi} (\sin\theta) W_{s-}(\alpha)\tilde{N}_1(d\theta d\alpha ds) \end{aligned} \tag{4.2}$$

But

$$\begin{aligned} &- \int_0^t\int_0^1\int_{-\pi}^{\pi} (\sin\theta) W_{s-}(\alpha)\left[N_0^{\lambda,n}(d\theta d\alpha ds) - \beta_0(\theta) d\theta d\alpha ds\right] \\ &= -\int_0^t\int_0^1\int_{-\pi}^{\pi} \sin\gamma_n^\lambda(s,\theta,\alpha) W_{s-}(\alpha)\tilde{N}_0(d\theta d\alpha ds) \end{aligned}$$

$$
\begin{aligned}
&\quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(\sin\gamma_n^\lambda(s,\theta,\alpha)-\sin\theta\right)W_{s-}(\alpha)\beta_0(\theta)d\theta d\alpha ds\\
&= \quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}(\sin\theta)W_{s-}(\alpha)\tilde N_0(d\theta d\alpha ds)\\
&\quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(\sin\gamma_n^\lambda(s,\theta,\alpha)-\sin\theta\right)W_{s-}(\alpha)N_0(d\theta d\alpha ds) \qquad (4.3)
\end{aligned}
$$

We finally obtain :

$$
\begin{aligned}
V_t^n(\lambda) &= V_0+\int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\gamma_n^\lambda(s,\theta,\alpha)-1)V_{s-}^n(\lambda)N_0(d\theta d\alpha ds)\\
&\quad +\int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\theta-1)V_{s-}^n(\lambda)N_1(d\theta d\alpha ds)\\
&\quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}(\sin\theta)W_{s-}(\alpha)\tilde N(d\theta d\alpha ds)\\
&\quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(\sin\gamma_n^\lambda(s,\theta,\alpha)-\sin\theta\right)W_{s-}(\alpha)N_0(d\theta d\alpha ds) \qquad (4.4)
\end{aligned}
$$

4.2 A Lipschitz property.

We study here the continuity of the map $\lambda\mapsto V_t^n(\lambda)$, which will be useful to study its differentiability. We set $U_t^n(\lambda,\mu)=V_t^n(\lambda)-V_t^n(\mu)$. This process satisfies :

$$
\begin{aligned}
U_t^n(\lambda,\mu) &= \int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\gamma_n^\lambda(s,\theta,\alpha)-1)U_{s-}^n(\lambda,\mu)N_0(d\theta d\alpha ds)\\
&\quad +\int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\theta-1)U_{s-}^n(\lambda,\mu)N_1(d\theta d\alpha ds)\\
&\quad +\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(\cos\gamma_n^\lambda(s,\theta,\alpha)-\cos\gamma_n^\mu(s,\theta,\alpha)\right)V_{s-}^n(\mu)N_0(d\theta d\alpha ds)\\
&\quad -\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(\sin\gamma_n^\lambda(s,\theta,\alpha)-\sin\gamma_n^\mu(s,\theta,\alpha)\right)W_{s-}(\alpha)N_0(d\theta d\alpha ds) \qquad (4.5)
\end{aligned}
$$

This equation is a linear S.D.E. If we set

$$
\begin{aligned}
K_t^n(\lambda) &= \int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\gamma_n^\lambda(s,\theta,\alpha)-1)N_0(d\theta d\alpha ds)\\
&\quad +\int_0^t\int_0^1\int_{-\pi}^{\pi}(\cos\theta-1)N_1(d\theta d\alpha ds) \qquad (4.6)
\end{aligned}
$$

then we can write (see Jacod, [22]) :

$$
U_t^n(\lambda,\mu)=\mathcal{E}(K^n(\lambda))_t\int_0^t\int_0^1\int_{-\pi}^{\pi}\mathcal{E}(K^n(\lambda))_{s-}^{-1}\times\frac{1}{\cos\gamma_n^\lambda(s,\theta,\alpha)}
$$

$$\times\Big\{V^n_{s-}(\mu)\left[\cos\gamma^\lambda_n(s,\theta,\alpha)-\cos\gamma^\mu_n(s,\theta,\alpha)\right]$$

$$-W_{s-}(\alpha)\left[\sin\gamma^\lambda_n(s,\theta,\alpha)-\sin\gamma^\mu_n(s,\theta,\alpha)\right]\Big\}N_0(d\theta d\alpha ds) \qquad (4.7)$$

where the Doléans-Dade exponential is given by (see Jacod, Shiryaev, [23]) :

$$\mathcal{E}(K^n(\lambda))_t = e^{K^n_t(\lambda)}\prod_{0\le u\le t}(1+\Delta K^n_u(\lambda))\,e^{-\Delta K^n_u(\lambda)} = \prod_{0\le u\le t}(1+\Delta K^n_u(\lambda)) \qquad (4.8)$$

But since any cosinus is in $[-1,1]$, it is clear that for all $s\le t$,

$$\left|\mathcal{E}(K^n(\lambda))_t\mathcal{E}(K^n(\lambda))^{-1}_{s-}\right| = \prod_{s\le u\le t}|1+\Delta K^n_u(\lambda)|\le 1 \qquad (4.9)$$

Furthermore, since $|\gamma^\lambda_n|\le\theta_0<\pi/2$, we see that

$$\left|\frac{1}{\cos\gamma^\lambda_n(s,\theta,\alpha)}\right|\le\frac{1}{\cos\theta_0}<\infty \qquad (4.10)$$

At last, since $|\gamma^\lambda_n(s,\theta,\alpha)|\le\frac{3}{2}|\theta|$,

$$\left|\cos\gamma^\lambda_n(s,\theta,\alpha)-\cos\gamma^\mu_n(s,\theta,\alpha)\right|\le\frac{3}{2}|\theta|\times|\lambda-\mu|\times|v_n(s,\theta,\alpha)|$$

$$\left|\sin\gamma^\lambda_n(s,\theta,\alpha)-\sin\gamma^\mu_n(s,\theta,\alpha)\right|\le|\lambda-\mu|\times|v_n(s,\theta,\alpha)| \qquad (4.11)$$

Hence, if

$$Y^n_t(\lambda)=\frac{1}{\cos\theta_0}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[\frac{3}{2}|\theta|\times|V^n_{s-}(\lambda)|+|W_{s-}(\alpha)|\right]\times|v_n(s,\theta,\alpha)|N_0(d\theta d\alpha ds) \qquad (4.12)$$

then for all λ,μ, $|U^n_t(\lambda,\mu)|\le|\lambda-\mu|\times Y^n_t(\lambda)$. In particular, this yields that for all λ,

$$|V^n_t(\lambda)|\le|V_t|+|U^n_t(\lambda,0)|\le|V_t|+Y^n_t(0) \qquad (4.13)$$

Finally, if

$$X^n_t=\frac{1}{\cos\theta_0}\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[\frac{3}{2}|\theta|\times|V_{s-}|+\frac{3}{2}|\theta|\times Y^n_{s-}(0)+|W_{s-}(\alpha)|\right]\times|v_n(s,\theta,\alpha)|N_0(d\theta d\alpha ds) \qquad (4.14)$$

then for all λ,μ,

$$|U^n_t(\lambda,\mu)|\le|\lambda-\mu|\times X^n_t \qquad (4.15)$$

Since we know from Theorem 2.4 that

$$E\left(\sup_{[0,T]}V_t^2\right)=E_\alpha\left(\sup_{[0,T]}W_t^2\right)<\infty \qquad (4.16)$$

we deduce, recalling that $|v_n(s,\theta,\alpha)|\le\eta_n\in L^1(\beta_0(\theta)d\theta)$, that :

$$\begin{aligned}
E\left(\sup_{[0,T]}|Y^n_t(0)|\right) &\leq \frac{1}{\cos\theta_0}\int_0^T\int_0^1\int_{-\theta_0}^{\theta_0}\left[\frac{3}{2}|\theta|\eta_n(\theta)E\left(|V_s|\right)+|W_s(\alpha)|\eta_n(\theta)\right]\beta_0(\theta)d\theta d\alpha ds \\
&\leq K\int_{-\theta_0}^{\theta_0}\eta_n(\theta)\beta_0(\theta)d\theta\times E\left(\sup_{[0,T]}|V_t|\right) \\
&\quad +K\int_{-\theta_0}^{\theta_0}\eta_n(\theta)\beta_0(\theta)d\theta\times E_\alpha\left(\sup_{[0,T]}|W_t|\right)<\infty
\end{aligned} \tag{4.17}$$

and, by using exactly the same computation,

$$E\left(\sup_{[0,T]}|X^n_t|\right)<\infty \tag{4.18}$$

Thus X^n_t is a.s. finished on $[0,T]$, and we can say that $V^n_t(\lambda)$ satisfies a Lipschitz property on $[-1,1]$ (for each t).

4.3 Differentiability.

We set (for the moment, this is only a notation) :

$$\frac{\partial}{\partial\lambda}V^n_t(\lambda)=-\mathcal{E}(K^n(\lambda))_t\int_0^t\int_0^1\int_{-\pi}^{\pi}\mathcal{E}(K^n(\lambda))^{-1}_{s-}\times\frac{1}{\cos\gamma^\lambda_n(s,\theta,\alpha)}$$
$$\times\Big\{V^n_{s-}(\lambda)\sin\gamma^\lambda_n(s,\theta,\alpha)+W_{s-}(\alpha)\cos\gamma^\lambda_n(s,\theta,\alpha)\Big\}v_n(s,\theta,\alpha)N_0(d\theta d\alpha ds) \tag{4.19}$$

We obtained this expression by differentiating formaly (4.4), and by using the same argument as in (4.7).

We set $D^n_t(\lambda,\mu)=V^n_t(\mu)-V^n_t(\lambda)-(\mu-\lambda)\dfrac{\partial}{\partial\lambda}V^n_t(\lambda)$. Let us compute $D^n_t(\lambda,\mu)$:

$$D^n_t(\lambda,\mu)=\mathcal{E}(K^n(\lambda))_t\int_0^t\int_0^1\int_{-\pi}^{\pi}\mathcal{E}(K^n(\lambda))^{-1}_{s-}\times\frac{1}{\cos\gamma^\lambda_n(s,\theta,\alpha)}$$
$$\times\Big\{V^n_{s-}(\mu)\times[\cos\gamma^\mu_n(s,\theta,\alpha)-\cos\gamma^\lambda_n(s,\theta,\alpha)+(\mu-\lambda)\sin\gamma^\lambda_n(s,\theta,\alpha)v_n(s,\theta,\alpha)]$$
$$+U^n_{s-}(\lambda,\mu)(\mu-\lambda)\sin\gamma^\lambda_n(s,\theta,\alpha)v_n(s,\theta,\alpha) \tag{4.20}$$
$$-W_{s-}(\alpha)\times[\sin\gamma^\mu_n(s,\theta,\alpha)-\sin\gamma^\lambda_n(s,\theta,\alpha)-(\mu-\lambda)\cos\gamma^\lambda_n(s,\theta,\alpha)v_n(s,\theta,\alpha)]\Big\}N_0(d\theta d\alpha ds)$$

Then a simple computation using equations (4.9), (4.10), (4.15), and something like (4.11) shows that if

$$S^n_t=\frac{1}{\cos\theta_0}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[\left(|V_{s-}|+X^n_{s-}\right)v^2_n(s,\theta,\alpha)+\frac{3}{2}|\theta|\times|v_n(s,\theta,\alpha)|\times X^n_{s-}$$
$$+\frac{3}{2}|\theta|\times|W_{s-}(\alpha)|\times v^2_n(s,\theta,\alpha)\Big]N_0(d\theta d\alpha ds) \tag{4.21}$$

then for all λ, μ,

$$|D_t^n(\lambda,\mu)| \leq (\lambda-\mu)^2 \times S_t^n \tag{4.22}$$

Using equations (4.16), (4.18), and the fact that

$$v_n^2(s,\theta,\alpha) + |\theta| \times |v_n(s,\theta,\alpha)| + |\theta| \times v_n^2(s,\theta,\alpha) \leq \left(\frac{1}{2}+\pi+\frac{1}{2}\pi\right)\eta_n(\theta) \in L^1(\beta_0(\theta)d\theta) \tag{4.23}$$

we see that

$$E\left(\sup_{[0,T]} |S_t^n|\right) < \infty \tag{4.24}$$

It is thus clear that $V_t^n(\lambda)$ is differentiable on $[-1,1]$, and that its derivative is $\frac{\partial}{\partial\lambda}V_t^n(\lambda)$.

4.4 Second differentiability.

One can check in the same way that $\frac{\partial}{\partial\lambda}V_t^n(\lambda)$ is differentiable, and that its derivative is given by

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}V_t^n(\lambda) = \mathcal{E}(K^n(\lambda))_t \int_0^t\int_0^1\int_{-\pi}^{\pi} \mathcal{E}(K^n(\lambda))_{s-}^{-1} \times \frac{1}{\cos\gamma_n^\lambda(s,\theta,\alpha)}\\ \times\Big\{-2\sin\gamma_n^\lambda(s,\theta,\alpha)\frac{\partial}{\partial\lambda}V_{s-}^n(\lambda)\times v_n(s,\theta,\alpha) - V_{s-}^n(\lambda)\cos\gamma_n^\lambda(s,\theta,\alpha)v_n^2(s,\theta,\alpha)\\ +W_{s-}(\alpha)\sin\gamma_n^\lambda(s,\theta,\alpha)v_n^2(s,\theta,\alpha)\Big\}N_0(d\theta d\alpha ds)\end{aligned} \tag{4.25}$$

4.5 Upperbounds.

We will soon use the following inequalities :

$$\left|\frac{\partial}{\partial\lambda}V_t^n(\lambda)\right| \leq R_t^n \tag{4.26}$$

where

$$\begin{aligned}R_t^n = \frac{1}{\cos\theta_0}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[(|V_{s-}|+Y_{s-}^n(0))\times\frac{3}{2}|\theta|\times|v_n(s,\theta,\alpha)|\\ +|W_{s-}(\alpha)|\times|v_n(s,\theta,\alpha)|\Big]N_0(d\theta d\alpha ds)\end{aligned} \tag{4.27}$$

and

$$\left|\frac{\partial^2}{\partial\lambda^2}V_t^n(\lambda)\right| \leq \Gamma_t^n \tag{4.28}$$

where

$$\begin{aligned}\Gamma_t^n = \frac{1}{\cos\theta_0}\int_0^t\int_0^1\int_{-\pi}^{\pi}\Big[3|\theta|\times R_{s-}^n\times|v_n(s,\theta,\alpha)| + (|V_{s-}|+Y_{s-}^n(0))\times v_n^2(s,\theta,\alpha)\\ +|W_{s-}(\alpha)|\times\frac{3}{2}|\theta|\times v_n^2(s,\theta,\alpha)\Big]N_0(d\theta d\alpha ds)\end{aligned} \tag{4.29}$$

5 Choice of the sequence of perturbations.

Recall that

$$\frac{\partial}{\partial\lambda}V_T^n(0) = -\mathcal{E}(K)_T \int_0^T \int_0^1 \int_{-\pi}^{\pi} \mathcal{E}(K)_{s-}^{-1} \times \frac{1}{\cos\theta} \times \left\{V_{s-}\sin\theta + W_{s-}(\alpha)\cos\theta\right\} \times v_n(s,\theta,\alpha) N_0(d\theta d\alpha ds) \tag{5.1}$$

where $K_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos\theta - 1) N(d\theta d\alpha ds)$.

The problem is now to choose v_n in such a way that for some $\epsilon > 0$, some $K < \infty$, the probability

$$P\left(\frac{\partial}{\partial\lambda}V_T^n(0) \in [\epsilon, K]\right)$$

goes to 1. First, we have to get rid of the random terms $\mathcal{E}(K)_T$ and $\mathcal{E}(K)_{s-}^{-1}$ in (5.1). To this end, we choose $v_n(s,\theta,\alpha)$ equal to 0 for $s \le T - a_n$, for some sequence a_n decreasing to 0, and we use the a.s. continuity of $\mathcal{E}(K)$ at T. Then we notice that the dominant term in $V_{s-}\sin\theta + W_{s-}(\alpha)\cos\theta$ is $W_{s-}(\alpha)\cos\theta$. We thus choose $v_n(s,\theta,\alpha)$ equal to 0 for $|\theta| \le 1/n$ (in order that $|v_n| \in L^1(\beta_0(\theta)d\theta)$) and equal to $k|\theta|$ for $|\theta| \in [2/n, \theta_1]$ for some $k > 0$ and $\theta_1 \le \theta_0$. This way,

$$\int_{T-a_n}^T \int_0^1 \int_{-\pi}^{\pi} |\sin\theta| \times |v_n(s,\theta,\alpha)| N_0(d\theta d\alpha ds)$$

will go to 0, but if a_n is well-chosen, since from (SP), $\theta \notin L^1(\beta_0(\theta)d\theta)$,

$$\int_{T-a_n}^T \int_0^1 \int_{-\pi}^{\pi} |v_n(s,\theta,\alpha)| N_0(d\theta d\alpha ds)$$

will go to infinity. Of course, this is not satisfying, but a stopping times will allow us to "cutoff" this second integral.

Let us now be precise. First, let us recall a Lemma that can be found in Graham, Méléard, [19] p 15.

Lemma 5.1 *There exists $0 < c < C < \infty$ and $q > 0$ such that for all $t \in [0,T]$,*

$$P_\alpha\left(c \le |W_t| \le C\right) \ge q \tag{5.2}$$

We will also need the following Lemma

Lemma 5.2 *One can build a sequence ϕ_n of positive, even, C^1 functions on $[-\theta_0, \theta_0]$ such that $\phi_n(-\theta_0) = \phi_n(\theta_0) = 0$, such that $\phi_n(\theta) \le k|\theta|$ for some $k \le 1/2$, such that if*

$$\xi_n(\theta) = |\phi_n'(\theta)| + r2^{r+2}\frac{\phi_n(\theta)}{|\theta|} \tag{5.3}$$

then $\xi_n \in L^1(\beta_0(\theta)d\theta)$ and $\xi_n \le 1/2$, and such that there exists a sequence a_n decreasing to 0 satisfying

$$a_n \int_{-\theta_0}^{\theta_0} \phi_n(\theta)\beta_0(\theta)d\theta \longrightarrow \infty \tag{5.4}$$

$$a_n \int_{-\theta_0}^{\theta_0} (|\theta|\phi_n(\theta) + \phi_n^2(\theta))\beta_0(\theta)d\theta \longrightarrow 0 \tag{5.5}$$

Proof : for any $k \in]0, 1/2]$, we clearly can build a sequence ϕ_n of even, positive and C^1 functions such that $\phi_n(\theta) \leq k|\theta|$, such that

$$\phi_n(\theta) = \begin{cases} 0 & \text{if} \quad |\theta| \leq 1/n \\ k|\theta| & \text{if} \quad |\theta| \in [2/n, \theta_0/2(1+k)] \\ 0 & \text{if} \quad |\theta| \in [\theta_0/(1+k), \theta_0] \end{cases} \tag{5.6}$$

and such that

$$|\phi_n'(\theta)| \leq \begin{cases} 0 & \text{if} \quad |\theta| \leq 1/n \\ 4k & \text{if} \quad |\theta| \in [1/n, 2/n] \\ k & \text{if} \quad |\theta| \in [2/n, \theta_0/2(1+k)] \\ 2k & \text{if} \quad |\theta| \in [\theta_0/2(1+k), \theta_0/(1+k)] \\ 0 & \text{if} \quad |\theta| \in [\theta_0/(1+k), \theta_0] \end{cases} \tag{5.7}$$

Then ξ_n is bounded, and vanishes near 0, it thus is in $L^1(\beta_0(\theta)d\theta)$. Furthermore, $\xi_n \leq 4k + r2^{r+2}k$, which is smaller than $1/2$ if we choose k small enough. We now choose

$$a_n = \left(\int_{-\theta_0}^{\theta_0} \phi_n(\theta)\beta_0(\theta)d\theta \right)^{-\frac{1}{2}} \tag{5.8}$$

We see that

$$\int_{-\theta_0}^{\theta_0} \phi_n(\theta)\beta_0(\theta)d\theta \geq 2\int_{2/n}^{\theta_0/2(1+k)} k\theta \times \frac{k_0}{\theta^r}d\theta = 2kk_0 \int_{2/n}^{\theta_0/2(1+k)} \theta^{1-r}d\theta \tag{5.9}$$

goes to infinity when n goes to infinity, since r is greater than 2. Hence a_n goes to 0, and condition (5.4) is satisfied. On the other hand,

$$a_n \int_{-\theta_0}^{\theta_0} (|\theta|\phi_n(\theta) + \phi_n^2(\theta))\beta_0(\theta)d\theta \leq Ka_n \int_0^{\theta_0} \theta^{2-r}d\theta \leq Ka_n \tag{5.10}$$

which goes to 0 since $r < 3$. The Lemma is proved.

We now define a stopping time that will allow the derivative at 0 not to be too large. Consider the following process :

$$Z_t^n = \int_0^t \int_{c \leq |W_s(\alpha)| \leq C} \int_{-\pi}^{\pi} \phi_n(\theta) N_0(d\theta d\alpha ds) \tag{5.11}$$

We fix $l > 0$, and we set

$$T_n = \inf\{t > T - a_n \ / \ Z_t^n - Z_{T-a_n}^n \geq l\} \tag{5.12}$$

We now can define our sequence of perturbations ($sg(x)$ denotes the signe of x).

$$v_n(s,\theta,\alpha) = 1_{[T-a_n, T_n \wedge T]}(s) 1_{\{c \le |W_{s-}(\alpha)| \le C\}} sg(\mathcal{E}(K)_{s-}) sg(W_{s-}(\alpha)) \phi_n(\theta) \tag{5.13}$$

For each n, v_n is a perturbation (see Definition 3.1), since it is predictable, and since it satisfies (3.1), (3.2), and (3.3) thanks to Lemma 5.2.

We at last prove the essential following convergence :

$$\lim_{n\to\infty} P(T_n < T) = 1 \tag{5.14}$$

Indeed,

$$\begin{aligned} P(T_n < T) &\ge P(Z_T^n - Z_{T-a_n}^n \ge l) \ge 1 - e^l E\left(e^{-(Z_T^n - Z_{T-a_n}^n)}\right) \\ &\ge 1 - e^l \exp\left\{ - \int_{T-a_n}^T \int_{c \le |W_{s-}(\alpha)| \le C} \int_{-\pi}^{\pi} \left(1 - e^{-\phi_n(\theta)}\right) \beta_0(\theta) d\theta d\alpha ds \right\} \\ &\ge 1 - e^l \exp\left\{ -a_n \times q \times \frac{1}{2} \int_{-\pi}^{\pi} \phi_n(\theta) \beta_0(\theta) d\theta \right\} \end{aligned} \tag{5.15}$$

which goes to 1 thanks to equation (5.4). We have used Lemma 5.1 and the fact that since ϕ_n is smaller than 1, $1 - e^{-\phi_n} \ge \phi_n/2$.

6 The derivative at 0 is large enough.

Thanks to our choice for the perturbation v_n, we can write

$$\begin{aligned} \left| \frac{\partial}{\partial \lambda} V_T^n(0) \right| &= |\mathcal{E}(K)_T| \times \left| \int_{T-a_n}^{T_n \wedge T} \int_{c \le |W_{s-}(\alpha)| \le C} \int_{-\theta_0}^{\theta_0} \left| \mathcal{E}(K)_{s-}^{-1} \right| \right. \\ &\quad \left. \times \left\{ (\tan\theta) V_{s-} sg(W_{s-}(\alpha)) + |W_{s-}(\alpha)| \right\} \phi_n(\theta) N_0(d\theta d\alpha ds) \right| \\ &\ge |\mathcal{E}(K)_T| \int_{T-a_n}^{T_n \wedge T} \int_{c \le |W_{s-}(\alpha)| \le C} \int_{-\theta_0}^{\theta_0} \left| \mathcal{E}(K)_{s-}^{-1} \right| \times c \times \phi_n(\theta) N_0(d\theta d\alpha ds) \\ &\quad - |\mathcal{E}(K)_T| \int_{T-a_n}^{T} \int_0^1 \int_{-\theta_0}^{\theta_0} \left| \mathcal{E}(K)_{s-}^{-1} \right| \times |\tan\theta| \times |V_{s-}| \times \phi_n(\theta) N_0(d\theta d\alpha ds) \\ &\ge A_n - B_n \end{aligned} \tag{6.1}$$

First A_n is larger than

$$\inf_{[T-a_n, T]} \left| \mathcal{E}(K)_T \mathcal{E}(K)_{s-}^{-1} \right| \times c \times (Z_{T\wedge T_n}^n - Z_{T-a_n}^n) \tag{6.2}$$

But $\mathcal{E}(K)$ is a.s. continuous (and does not vanish) at T, thus the first term in the product goes a.s. to 1. Furthermore, using equations (5.12) and (5.14), we see that

$$\lim_{n\to\infty} P\left(Z_{T\wedge T_n}^n - Z_{T-a_n}^n \ge l\right) = 1 \tag{6.3}$$

It is thus clear that

$$\lim_{n\to\infty} P\left(A_n \geq cl/2\right) = 1 \tag{6.4}$$

On the other hand,

$$B_n \leq \sup_{[T-a_n,T]} \left|\mathcal{E}(K)_T \mathcal{E}(K)_{s-}^{-1}\right| \times \frac{1}{\cos\theta_0} \times \int_{T-a_n}^{T}\int_0^1\int_{-\theta_0}^{\theta_0} |V_{s-}| \times |\theta| \phi_n(\theta) N_0(d\theta d\alpha ds) \tag{6.5}$$

First, we have already seen (see equation (4.9)) that the first term in the product is always smaller than 1. The last term goes to 0 in L^1, thanks to (5.5) and (4.16), since

$$E\left[\int_{T-a_n}^{T}\int_0^1\int_{-\theta_0}^{\theta_0} |V_{s-}| \times |\theta|\phi_n(\theta) N_0(d\theta d\alpha ds)\right]$$

$$\leq E\left(\sup_{[0,T]} |V_{s-}|\right) \times a_n \int_{-\theta_0}^{\theta_0} |\theta|\phi_n(\theta)\beta_0(\theta) d\theta \tag{6.6}$$

Hence B_n goes to 0 in probability, and we finally deduce that

$$\lim_{n\to\infty} P\left(\left|\frac{\partial}{\partial\lambda} V_T^n(0)\right| \geq cl/4\right) = 1 \tag{6.7}$$

The first part of our criterion is satisfied.

7 The derivatives are not too large.

We still have to check that there exists $K < \infty$ such that

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda} V_T^n(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.1}$$

and

$$P\left(\sup_{|\lambda|\leq 1}\left|\frac{\partial^2}{\partial\lambda^2} V_T^n(\lambda)\right| \leq K\right) \longrightarrow 1 \tag{7.2}$$

We refer to Section 4 for the notations. In order to prove (7.1), we just have to check that $P\left(R_T^n \leq K\right)$ goes to 1 (see (4.26) and (4.27)). First, we will need the following preliminary estimate (L is a constant independant of n) :

$$E\left[\sup_{[0,T]} Y_t^n(0)\right] \leq L \tag{7.3}$$

But this expectation is smaller than (M is a constant)

$$\begin{aligned} & ME\left[\int_{T-a_n}^{T_n\wedge T}\int_{c\leq|W_{s-}(\alpha)|\leq C}\int_{-\pi}^{\pi} \left[|\theta| \times |V_{s-}| + |W_{s-}(\alpha)|\right] \phi_n(\theta) N_0(d\theta d\alpha ds)\right] \\ \leq\ & ME\left[\int_{T-a_n}^{T}\int_0^1\int_{-\theta_0}^{\theta_0} |\theta| \times |V_{s-}|\phi_n(\theta) N_0(d\theta d\alpha ds)\right] \\ & +ME\left[\int_{T-a_n}^{T_n}\int_{c\leq|W_{s-}(\alpha)|\leq C}\int_{-\pi}^{\pi} \phi_n(\theta) N_0(d\theta d\alpha ds)\right] \end{aligned} \tag{7.4}$$

Thanks to the definition (5.12) of T_n, the second term is smaller than $M(l+\parallel \phi_n \parallel_\infty)$. But ϕ_n is always smaller than $1/2$, and thus the second term is smaller than $M(l+1/2)$. On the other hand, the first term is smaller than

$$M\int_{T-a_n}^{T}\int_0^1\int_{-\theta_0}^{\theta_0} E\left(|V_{s-}|\right)\times|\theta|\phi_n(\theta)\beta_0(\theta)d\theta d\alpha ds$$

$$\leq ME\left(\sup_{[0,T]}|V_t|\right)\times a_n\int_{-\theta_0}^{\theta_0}|\theta|\phi_n(\theta)\beta_0(\theta)d\theta \tag{7.5}$$

which goes to 0, thanks to (5.5) and (4.16). Inequality (7.3) is satisfied.

We now write R_t^n as $\dfrac{1}{\cos\theta_0}\left(3R_t^{n,1}+R_t^{n,2}\right)$, where

$$R_t^{n,1}=\int_0^t\int_0^1\int_{-\pi}^{\pi}\left(|V_{s-}|+Y_{s-}^n(0)\right)\times|\theta|\times|v_n(s,\theta,\alpha)|N_0(d\theta d\alpha ds) \tag{7.6}$$

$$R_t^{n,2}=\int_0^t\int_0^1\int_{-\pi}^{\pi}|W_{s-}(\alpha)|\times|v_n(s,\theta,\alpha)|N_0(d\theta d\alpha ds) \tag{7.7}$$

It is clear, thanks to the definitions of v_n and T_n, and since $\phi_n\leq 1/2$, that $R_T^{n,2}\leq C(l+1/2)$. On the other hand, (4.16), (7.3) and (5.5) yield that $R_T^{n,1}$ goes to 0 in L^1. Hence, $P\left(R_T^n\leq 2C(l+1/2)\right)$ goes to 1, and (7.1) is satisfied.

Notice that we have proved in particular that there exists a constant L independant of n such that

$$E\left(\sup_{[0,T]}R_t^n\right)\leq L \tag{7.8}$$

In order to prove (7.2), we have to check that $P\left(\Gamma_T^n\leq K\right)$ goes to 1 (recall (4.28) and (4.29)). Thanks to (5.5), (4.16), (7.3), and (7.8), we see that

$$E\left(\Gamma_T^n\right)\longrightarrow 0 \tag{7.9}$$

which gives immediately the result.

Notice that we do not need to choose l (see the definition of T_n, (5.12)) : this might look strange, but in fact, it is natural. First, if l is large, then the derivative at 0 will be more easily large, but the first and second derivatives will be less easily bounded on $[-1,1]$. As a second reason, notice that we use a sequence of perturbations that would make explode $\dfrac{\partial}{\partial\lambda}V_T^n(0)$ if we did not use T_n.

8 Conclusion.

We have found some constants $\epsilon>0$ and $K<\infty$ such that

$$\lim_{n\to\infty}P\left(\left|\frac{\partial}{\partial\lambda}V_T^n(0)\right|\geq\epsilon\ ;\ \sup_{|\lambda|\leq 1}\left|\frac{\partial}{\partial\lambda}V_T^n(\lambda)\right|\leq K\ ;\ \sup_{|\lambda|\leq 1}\left|\frac{\partial^2}{\partial\lambda^2}V_T^n(\lambda)\right|\leq K\right)=1 \tag{8.1}$$

Let now y_0 be a point of the support of the law of V_T. Then for any $r > 0$,

$$P\left(|V_T - y_0| \le r \; ; \; \left|\frac{\partial}{\partial\lambda}V_T^n(0)\right| \ge \epsilon \; ; \; \sup_{|\lambda|\le 1}\left|\frac{\partial}{\partial\lambda}V_T^n(\lambda)\right| \le K \; ; \; \sup_{|\lambda|\le 1}\left|\frac{\partial^2}{\partial\lambda^2}V_T^n(\lambda)\right| \le K\right)$$

$$\longrightarrow_{n\to\infty} P\left(|V_T - y_0| \le r\right) > 0 \tag{8.2}$$

Theorem 3.3 allows us to say that $f(T, y_0) > 0$. Since we know from Theorem 2.4 that $f(T, .)$ is continuous on $I\!R$, Remark 3.4 allows us to deduce that for all $y \in I\!R$, $f(T, y) > 0$. At last, since $T > 0$ has been arbitrarily fixed, this holds for any $T > 0$, and the proof of Theorem 2.5 is finished.

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