Sleeping Beauty is a Double Halfer

(a short unfinished preprepreliminary preprepredraft)

Omer Adelman

LPSM, Sorbonne Université, Paris, France

May 2023

Abstract

The rational Sleeping Beauty is a double-halfer.

Contents

1 Introduction
2 Two misleading dogmas
3 According to Elga's analysis, $1 = 0$
4 Lewisian halfers and their partial agreement(s) with thirders
5 The principal mistake made by thirders (and by Lewisian halfers)
6 Rationality
7 The Sleeping Beauty problem revisited
8 Reflections on the reflection principle
9 And now?...

[Rédaction en cours]

1 Introduction

Adam Elga [8] wrote about the following Sleeping Beauty problem more than twenty years ago (in 2000).

Beauty, who is a perfectly rational person, is put to sleep on day 0 (Sunday). A fair coin is tossed. Beauty is awakened on day 1 (Monday) and, if Heads, this is her unique waking during the game. If Tails, she is given some drug that puts back to sleep and makes her forget Monday's waking, and she is awakened again on day 2 (Tuesday). Thus Heads implies a unique awakening, while Tails implies two. When awakened during the game, she knows she's in the game and she remembers the protocol, but she can't tell a Heads awakening from a Tails awakening or a first Tails awakening from a second one. When awakened during the game, what credence should she have in Heads?

Elga argues that the correct answer is 1/3: he is a *thirder*. Reading at least the first two sections of Elga's short and very readable paper would be a good idea.

The problem introduced above is the *traditional* Sleeping Beauty problem. We'll be also interested in the *general* Sleeping Beauty problem, where the random number of awakenings may be larger than two.

I show that the way thirders analyze the problem leads to a mathematical absurdity, and I say something about an implicite wrong assumption made by thirders (but not *only* by thirders). Then, after a brief discussion about rationality, I show that the first half of double halfism is right. (Adam Elga [8] almost established the second half (and my usage of the term "almost" is explained in section 5 below).)

A reasonably good background in the basics of probability theory might be useful to the reader.
Some portions of the text are not very mathematical, though. (Example: section 6.)

In section 8, I explain what made me confident that Beauty's answer must be one half when I first heard about the problem. Section 9 concludes with some remarks.

Let's start dryly with some conventions.
Convention: i, j, k and n are generic elements of the set $\mathbb{N} := \{1, 2, 3, \dots\}$ of *strictly* positive integers.
Convention: the random number of awakenings (during the game) is N, and it takes values in the set $\mathbb{N}$.
Convention: p_i is the probability that $N = i$ (so in the traditional story $p_1 = p_2 = 1/2$ and $p_i = 0$ for all $i > 2$).

Wait a second: when we say "probability", what is the probability measure we are talking about? On each stage of the game, Beauty finds herself in a particular... world (or state, or "epistemic state", or "centered world", as some authors put it). Let W^0 be Beauty's world just before she is put to sleep for the first time, and let W^1 be Beauty's world just after she is awakened (for the first time, say, though this doesn't matter). Later, we'll let $W^{2,k}$ be the world she finds herself in just after she is told "this is your awakening number k" (which, in the traditional story, applies to no k other than 1 or 2). The symbol "W" is unimportant, but the superscripts ^{0} and ^{1} (and ^{2,k}) will be useful: ^{0} will indicate we are talking about quantities perceived by Beauty while she is in W^0, etc. In particular, the probability that goes with W^0 is P^0, the expected value Beauty assigns to N while she is in the state W^0 is $E^0(N)$, etc. The subscript $_{\text{tr}}$ indicates we're dealing with the **tr**aditional problem (where $p_1 = p_2 = 1/2$ and $p_i = 0$ for all $i > 2$). And what we are after is the value of $P^1_{\text{tr}}(N = 1)$ which, according to Elga's analysis, satisfies

$$P^1_{\text{tr}}(N = 1) = \frac{1 \times P^0_{\text{tr}}(N = 1)}{\sum_i i \times P^0_{\text{tr}}(N = i)} = \frac{1 \times 1/2}{1 \times 1/2 + 2 \times 1/2} = 1/3.$$

⋆ ⋆ ⋆

Here are three **indisputable facts** (or *axioms*).

$(\mathrm{A}_{=1})$	for all $S \subseteq \mathbb{N}$,	$P^0(N \in S) = 1 \iff P^1(N \in S) = 1.$
$(\mathrm{A}_{=0})$	for all $S \subseteq \mathbb{N}$,	$P^0(N \in S) = 0 \iff P^1(N \in S) = 0.$
$(\mathrm{A}_{>0})$	for all $S \subseteq \mathbb{N}$,	$P^0(N \in S) > 0 \iff P^1(N \in S) > 0.$

(If you don't agree, you can discuss this with people around you, but not with me.)
(Easy exercise: please observe that $(\mathrm{A}_{=1})$ is equivalent to $(\mathrm{A}_{=0})$, which is equivalent to $(\mathrm{A}_{>0})$.)
The above three facts are implicit in the sequel.

Let's conclude the introduction by a definition (plus a notational convention).

If the event $\{N = k\}$ occurs, then the number of Beauty's awakenings (during the game) is k, and we'll say that each one of these awakenings is a k-awakening. Let N_k denote the number of Beauty's k-awakenings (so $N_k = k$ if $N = k$, and $N_k = 0$ otherwise). Observe that $N = \sum_k N_k$.

2 Two misleading dogmas

Thirders were led (or rather *mis*led) to their opinion by the rather intuitive **proportionality credo**, the dogma according to which, while in W^1, Beauty's credence in $\{N = k\}$ should be proportional to the "objective" expected value $E^0(N_k)$ of the number of her k-awakenings, so if $p_j > 0$, then

$$\frac{P^1(N = i)}{P^1(N = j)} \text{ should be equal to } \frac{E^0(N_i)}{E^0(N_j)} \ \Big(\text{which is of course equal to } \frac{ip_i}{jp_j}\Big).$$

Applied to the traditional Sleeping Beauty problem, the proportionality credo implies that $\frac{P^1_{\text{tr}}(N=2)}{P^1_{\text{tr}}(N=1)} = \frac{2\times(1/2)}{1\times(1/2)} = 2$, which implies that $P^1_{\text{tr}}(N = 2) = 2P^1_{\text{tr}}(N = 1)$; and since

$$P^1_{\text{tr}}(N = 1) = \frac{P^1_{\text{tr}}(N = 1)}{1} = \frac{P^1_{\text{tr}}(N = 1)}{P^0_{\text{tr}}(N \in \{1, 2\})} = \frac{P^1_{\text{tr}}(N = 1)}{P^1_{\text{tr}}(N \in \{1, 2\})} = \frac{P^1_{\text{tr}}(N = 1)}{P^1_{\text{tr}}(N = 1) + P^1(N = 2)}$$

this also implies that

$$P^1_{\text{tr}}(N = 1) = \frac{P^1_{\text{tr}}(N = 1)}{P^1_{\text{tr}}(N = 1) + 2P^1_{\text{tr}}(N = 1)} = 1/3.$$

Several authors seem to take the proportionality credo for granted. Some use this credo without even mentioning it. (Example: Aumann, Hart and Perry, whose thirdism is visible in footnote 1 on the first page of [3].) Some thirders *do* say something about what makes them adopt the proportionality credo. One argument goes more or less as follows: if one thousand copies of the original Sleeping Beauty experiment are carried out independently, then, with a probability close to one, there will be a unique awakening in about half of the copies, and exactly two awakenings in each one of the other copies, so the overall proportion of the 1-awakenings will be about $(1 \times 500)/(1 \times 500 + 2 \times 500) = 1/3$ – and this is supposed to become more convincing if "one thousand" is replaced by "one million", for example.

But thirders have more in their arsenal, and Elga describes the following attractive argument leading to the very same one third deduced above. That N be chosen before Beauty's first awakening is inessential: one awakening will be there anyway, so we can choose N some time *after* the first awakening (towards the end of day 1, say), and then, if $N > 1$, proceed to $N - 1$ cycles of putting Beauty to sleep while making her forget her previous awakening(s) and then awakening her again.

Now assume this alternative procedure is applied, and assume that a short while after each one of her awakenings Beauty is to learn that "this" is her awakening number so-and-so. If she learns that "this" is her awakening number k, she will not be in W^1 anymore, she will be in $W^{2,k}$. In particular, while in $W^{2,1}$, her credence in $\{N = i\}$ will be simply p_i (since N is still to be determined), so, according to Elga, A_1 standing for {"this" is my first awakening}, the awakened Beauty's computation continues by

$$P^1(A_1 \cap \{N = i\}) = P^1(A_1)P^1(N = i \mid A_1) = P^1(A_1)P^{2,1}(N = i) = P^1(A_1) \times p_i$$

In the traditional problem, where $\{N = 1\}$ = Face and $\{N = 2\}$ = Tails, this gives

$$P^1_{\rm tr}(A_1 \cap \{N = 1\}) = P^1_{\rm tr}(A_1) \times (1/2) = P^1_{\rm tr}(A_1 \cap \{N = 2\}),$$

or, in other words, $P^1_{\rm tr}$(Monday and Heads) = $P^1_{\rm tr}$(Monday) × (1/2) = $P^1_{\rm tr}$(Monday and Tails).

Now Elga (and most (if not all) all the other thriders, and many non thirders) also believe that, seen from W^1, given that $N = k$, "this" day's number has a conditional distribution which is uniform on the set $[1..k]$ of integers in the interval $[1, k]$. This is sometimes justified by an appeal to a **principle of indifference**, another credo dear to a large majority of those who have written about the Sleeping Beauty problem, a credo which is probably rooted in an intuition even more fundamental than the one supporting the proportionality credo. Living in the world W^1, if Beauty learns that $N = k$ and she has no other information, then she cannot distinguish "this" being day i from "this" being day j for whatever i and j not exceeding k, and the principle of indifference should render her (conditional) credence in {"this" day is day i} (= {I am "now" in my waking number i}) independent of i (and simply equal to $1/k$): for all $i \in [1..k]$, $P^1(A_i \mid N = k) = 1/k$; and, consequently, for all $i \in [1..k]$,

$$P^1(A_i \cap \{N = k\}) = P^1(N = k)P^1(A_i \mid N = k) = P^1(N = k)P^1(A_1 \mid N = k) = \frac{P^1(N = k)}{k}$$

If this is the way you think about these matters and you focus on the original Sleeping Beauty problem, you must deduce, like Elga did in 2000, that

$$P^1(\text{Monday and Heads}) = P^1(\text{Monday and Tails}) = P^1(\text{Tuesday and Tails})$$

and, realizing that $P^1_{\rm tr}$(Monday and Heads) + $P^1_{\rm tr}$(Monday and Tails) + $P^1_{\rm tr}$(Tuesday and Tails) = 1 and that, in W^1, "Heads" is synonymous to "Monday and Heads", you must conclude that $P^1_{\rm tr}$(Heads) = 1/3.

Some might want to replace A_i by $\{D = i\}$, where $D = 1$ if "this" is day 1, $D = 2$ if "this" is day 2, etc.

Too complicated? Too dense?

So let me say it again: please, read at least the first two sections of Elga's paper.

The title of the present section hasn't been justified yet: I still have to show that the proportionality credo and the principle of indifference shouldn't be trusted, and this is what I do in sections 3 and 5.

3 According to Elga's analysis, $1 = 0$

The proportionality credo tells us that $P^1(N = k)$ should be proportional to $E^0(N_k) = kp_k$, which easily entails that if F and G are finite subsets of $\mathbb{N}$ and $P^0(N \in G\}) > 0$, then

$$\frac{P^1(N \in F)}{P^1(N \in G)} = \frac{\sum_{i \in F} ip_i}{\sum_{j \in G} jp_j}$$

and that if $P^0(N \in G) = 1$, then

$$P^1(N \in F) = \frac{P^1(N \in F)}{1} = \frac{P^1(N \in F)}{P^1(N \in G)} = \frac{\sum_{i \in F} ip_i}{\sum_{j \in G} jp_j} = \frac{\sum_{i \in F} iP^0(N = i)}{E^0(N)}$$

If you are not comfortable with the above computation, please take your time (I can't give you mine).

Let's fix i for a while. Browsing through the papers in which so many authors expressed their thirdism, one sees that they believe that for all finite $F \subset \mathbb{N}$, if $P^0(N \in F) > 0$ (ie, if $\sum_{j \in F} p_j > 0$), then $P^1(N = i)/P^1(N \in F) = ip_i / \sum_{j \in F} jp_j$; so they should be convinced that if n is sufficiently large, then $P^1(N = i)/P^1(N \leq n) = ip_i / \sum_{j \leq n} jp_j$.
As n goes to infinity, $P^1(N \leq n)$ clearly converges to $P^1(N \in \mathbb{N})$, ie to 1. So thirders must accept that, as n goes to infinity, $P^1(N = i)/P^1(N \leq n)$ converges to $P^1(N = i)$ and that

$$P^1(N = i) = \lim_{n \to \infty} \frac{ip_i}{\sum_{j \leq n} jp_j} = \frac{ip_i}{\lim_{n \to \infty} \sum_{j \leq n} jp_j} = \frac{ip_i}{\sum_{j \in \mathbb{N}} jp_j} = \frac{ip_i}{E^0(N)}$$

Summary: I believe Elga himself will agree that, according his own analysis,

$$P^1(N = i) = \frac{i \times p_i}{E^0(N)}$$

for all i.
Please observe that the quantity $E^0(N)$ is not necessarily finite. (Exercise: exhibit a concrete choice of the probabilities $p_i = P^0(N = i)$ that renders $E^0(N)$ infinite.) Of course, if $E^0(N)$ is infinite, then thirders must admit that $P^1(N = i) = 0$ for all i and conclude that

$$1 = P^1(N \in \mathbb{N}) = \textstyle\sum_i P^1(N = i) = \sum_i 0 = 0.$$

(If you have doubts about $\sum_{i \in \mathbb{N}} 0$ being strictly smaller than 1, please note that $\sum_{i \in \mathbb{N}} 0 \leq \sum_{i \in \mathbb{N}} 5/10^i = 0.55555\ldots < 0.6$.)

If you understand the above and you're still a thirder, please raise your hand.

Now Elga derived the proportionality credo by a very simple argument relying on the principle of indifference (discussed in the next section). So those who abandon the proportionality credo should also get rid of their belief in the principle of indifference.

4 Lewisian halfers and their partial agreement(s) with thirders

As already mentioned, many non thirders believe in the principle of indifference. David Lewis was what I call a *Lewisian* halfer. Challenging Elga in a paper [5] entitled *Sleeping Beauty: Reply to Elga*, Lewis claimed that $P^1_{\text{tr}}(T_1) = P^1_{\text{tr}}(T_2) = 1/4$ (while thirders believe that $P^1_{\text{tr}}(T_1) = P^1_{\text{tr}}(T_2) = 1/3$). Anyway, both thirders and Lewisian halfers believe that $P^1_{\text{tr}}(T_1) = P^1_{\text{tr}}(T_2)$, and they share another belief: that $P^1_{\text{tr}}(\text{Heads}) < P^{2,1}_{\text{tr}}(\text{Heads})$. (Thirders: $P^1_{\text{tr}}(\text{Heads}) = 1/3 < 1/2 = P^{2,1}_{\text{tr}}(\text{Heads})$; Lewisian halfers: $P^1_{\text{tr}}(\text{Heads}) = 1/2 < 2/3 = P^{2,1}_{\text{tr}}(\text{Heads})$.) In fact, the inequality $P^1_{\text{tr}}(\text{Heads}) < P^{2,1}_{\text{tr}}(\text{Heads})$ seems extremely plausible. After all, while in W^1_{tr}, beauty cannot exclude that "this" is day 2, and it seems tempting to believe that $P^1_{\text{tr}}(A_2) = P^1_{\text{tr}}(\text{"this" is day 2}) > 0$ and that $P^1_{\text{tr}}(\text{Heads} \mid A_2) = 0$ (and of course that $0 < P^1_{\text{tr}}(A_1) < 1$) and to deduce that

$$P^1_{\text{tr}}(\text{Heads}) = P^1_{\text{tr}}(A_1) P^1_{\text{tr}}(\text{Heads} \mid A_1) + P^1_{\text{tr}}(A_2) \underbrace{P^1_{\text{tr}}(\text{Heads} \mid A_2)}_{= 0} < P^1_{\text{tr}}(\text{Heads} \mid A_1).$$

5 The principal mistake made by thirders (and by Lewisian halfers)

The **indifference principle** is a key tool in the analysis that led Elga (and many others) to thirdism.

In Elga's paper, H_1 is assumed to be what Beauty perceives as

$$\{\text{Heads, and this actual waking of mine is my first waking}\},$$

T_1 is assumed to be what Beauty perceives as

$$\{\text{Tails, and this actual waking of mine is my first waking}\},$$

while T_2 stands for what Beauty perceives as

$$\{\text{Tails, and this actual waking of mine is my second waking}\}$$

(though Elga's original formulation is different).

Observing that being in T_1 is subjectively just like being in T_2, Elga deduces that even a highly restricted principle of indifference yields that Beauty ought then to have equal credence in each.

Well, I'd suggest a slightly less affirmative formulation, replacing

"ought then to have equal credence in each"

by something like

"if A is a meaningful formal assertion about a formal ordered pair of objects,
then whatever is true about Beauty's credence in $\mathrm{A}(T_1,\ T_2)$
is just as true about her credence in $\mathrm{A}(T_2,\ T_1)$,
and whatever is true about her credence in $\mathrm{A}(T_2,\ T_1)$
is just as true about her credence in $\mathrm{A}(T_1,\ T_2)$"

(but even such a pedantic formulation isn't completely rigorous).

This means that $(P^1_{\mathrm{tr}}(T_1), P^1_{\mathrm{tr}}(T_2)) = (1/5,\ 4/5)$ cannot hold: if there are real numbers a and b such that $(P^1_{\mathrm{tr}}(T_1), P^1_{\mathrm{tr}}(T_2)) = (a, b)$, then we must also have $(P^1_{\mathrm{tr}}(T_2), P^1_{\mathrm{tr}}(T_1)) = (a, b)$, so $(a, b) = (b, a)$ and $a = b$ and $P^1_{\mathrm{tr}}(T_1) = P^1_{\mathrm{tr}}(T_2)$. Of course, $P^1_{\mathrm{tr}}(T_1) = P^1_{\mathrm{tr}}(T_2) = a$ also implies that T_1 and T_2 both belong to $\mathcal{F}^1_{\mathrm{tr}}$, and here we can see a new door opened by the principle of indifference:

either T_1 and T_2 both belong to $\mathcal{F}^1_{\mathrm{tr}}$, **or non of them does**.

The measurability issue here is crying: had T_1 been an element of the σ-algebra $\mathcal{F}^1_{\mathrm{tr}}$, the real number one would have been equal to zero.

Sorry, dear thirders: T_1 *cannot* belong to $\mathcal{F}^1_{\mathrm{tr}}$.

(This means that in the degenerate case where N takes some $k > 1$ as its unique value, trying to attribute a concrete value to Beauty's credence in "this" being her first awakening (or her awakening number j for whatever specific $j \leq k$) is hopeless.)

Note 1. What would the superiorly intelligent Beauty think about "objects" like T_1 or about notions like "self" "subjectivity" or "here and now"? I don't know. Is Beauty a solipsist? I don't know. A possibility: having thought about these matters for a while, she might well consider that dwelling endlessly on such "metaphysical" issues is a hopeless endeavor. (I do *not* say or insinuate that she perceives these matters as nonsense or as uninteresting.)

Note 2. So the way thirders (or at least most thirders) analyze the problem entails an absurdity. This shows that their analysis is problematic, not that their conclusion is false. But their conclusion *is* false, and this is what I show below.

Note 3. When I said that Elga [8] *almost* established the second half of double halfism, I meant that he identifies what I present as $P^{2,1}_{\mathrm{tr}}(\text{Heads})$ (which is exactly $P^{2,1}_{\mathrm{tr}}(N = 1)$) with the P^1_{tr} conditional probability of Heads given $H_1 \cup T_1$, ie given "this" is Monday, which is illegitimate since Monday is not measurable in Beauty's world W^1_{tr}, a world in which H_1 is simply the event Heads ($= \{N = 1\}$), but in which T_1 and T_2 (and $H_1 \cup T_1$, and $H_1 \cup T_2$) cannot present any valid identity document during a police check. One might say that , in W^1_{tr}, the "object" $H_1 \cup T_1$ is the union of the event $H_1 \in \mathcal{F}^1_{\mathrm{tr}}$ with the disjoint nonevent T_1 (which does not belong to $\mathcal{F}^1_{\mathrm{tr}}$).
In fact, while in W^1_{tr}, it's better not to deal with T_1 at all.

6 Rationality

Beauty is a rational person.
What does this mean?
I cannot come up with a full answer, but I think we can agree Beauty is an expert in logic and in mathematics (if you believe that mathematics includes logic or that mathematics is a part of logic, have it your way, this is not our business here) and that she uses her skills in everyday life. When she is offered a bet, does she always decide according to the expected value of her gain? Not obvious, and not our problem here. Now since a rational person is a person, there are certainly things Beauty likes (GOOD, for example) and things she likes less (like BAD). The least we do know about Beauty's decision making is that *if she knows with certainty that* GOOD *or* BAD *will happen (but not both) and she prefers* GOOD *to* BAD *and she* ***must*** *make a choice between pushing button* A *and pushing button* B *and button* A *gives* GOOD *probability* α *and* BAD *probability* $1-\alpha$ *while button* B *gives* GOOD *probability* β *and* BAD *probability* $1-\beta$ *and* $\alpha > \beta$*, then, all other things being equal, Beauty pushes button* A.

Of course, Beauty is also *consistent* (and she knows she is): facing the same problem at each one of her waking sessions, her analysis of the situation is always the same; and if her reason and preferences make her act in a specific way on one of her wakings (and if her physical condition(s) are unchanged), then she acts in the very same way on each one of her wakings.

7 The Sleeping Beauty problem revisited [Section à réécrire complètement...]

7.1 If $p_1 > 1/2$, then $P^1(N=1) > 1/2$

Why would the rational Beauty accept to play a role in this crazy sleeping game (or experiment)? I didn't say why, I wanted to keep things nice, but it's time to reveal the cruel truth: Beauty is a prisoner, and she **must** play. On each awakening, she must push either button A, or button B: if she doesn't, too bad for her, the bad button is activated automatically.
Pushing button A stands for "I believe that $N = 1$", pushing button B is declaring "I believe that $N > 1$". If she is mistaken once, she is mistaken on each one of her N awakenings. If she is right once, then she is always right. And here is the deal: if your guess is good, then GOOD will happen; if you're wrong, too BAD for you. GOOD (or BAD) will happen only once, after the game is over. GOOD is better than BAD: freedom, wealth, health and happiness on one side, and I'd better not give details on how bad BAD is.

What does Beauty do? Well, if $P^1(N=1) > 1/2$, she pushes button A on each one of her awakenings and, of course, if $P^1(N=1) < 1/2$, then she pushes button B on each one of her awakenings. (For the moment, we don't deal with the case where $P^1(N=1) = 1/2$.)
Beauty knows the value of each p_k.
If $p_1 > 1/2$, then beauty knows perfectly well that pushing button A is her best choice, she knows this even when she is in W^1 and $E^0(N)$ is much larger than p_1. She is perfectly lucide, she can analyze the situation coldly, and she pushes button A, which is impossible had her W^1-credence in $\{N=1\}$ been strictly smaller than 1/2.

This argument will be explained in more detail in the unfinished section 7.2.
Also, if If $p_1 < 1/2$, then Beauty pushes button B, which is impossible had her W^1-credence in $\{N=1\}$ been strictly larger than 1/2.
A mathematical theorem (see section 7.2) then entails that, if $p_1 = 1/2$, then Beauty's W^1-credence in $\{N=1\}$ is exactly one half: $P^1(N=1) = 1/2$.

I am not the first to come up with such an idea. When I told Laurent Delable about it, he pointed out that he himself had described a scenario of a similar nature towards the end of his thesis [7] (this starts on page 246). But Laurent Delabre himself told me that his own position is... complex and that he considers Beauty herself as a complex person whose answer to the question "What is the value of your

$P^1_{\mathrm{tr}}(\text{Heads})$?" is not a clear one half neither a clear one third. (More details can be found in the last chapter of [7].) At any rate, I am not aware of authors who describe the same argument *and* say clearly what is said in our section 6 about rationality.

7.2 Some more details...

So we have our $p_1 = P^0(N = 1)$, p_2, p_3 & co. But how do we choose N?
No coins here: let's do it as follows.
Let λ be the Lebesgue measure on the unit interval I (equipped with its Borel σ-algebra $\mathcal{B}(I)$), let I_k be pairwise disjoint Borel subsets of I such that, for all k, $\lambda(I_k) = p_k$, and assume that the union of the sets I_k is exactly I. (If I is the *half-open* unit interval $[0, 1)$ and, for all k, $I_k = [\sum_{j<k} p_j,\ (\sum_{j<k} p_j) + p_k)$, then the Borel sets I_k are in fact *intervals*, and they satisfy our requirements.) We can assume that the value of U is determined while Beauty is asleep, say before her first awakening. Now let U be a random variable taking values in the unit interval and uniform over this interval. We can now decide that $N = k$ exactly when $U \in I_k$ so, for example $\{N = 5\} = \{U \in I_5\}$.
Beauty knows all that. She has no other information about U (and she ignores the "actual" value of U).
Of course, if G is some Borel subset of the unit interval, then $P^0(U \in G) = \lambda(G)$. I claim that more is true, and that

$$\text{if } G \text{ is a Borel subset of the unit interval, then } P^1(U \in G) = P^0(U \in G) = \lambda(G).$$

In the sequel, S will be some "special" fixed Borel subset of the I, and Beauty knows S and, in particular, she knows $\lambda(S)$.

This time, pushing button A stands for "I believe that $U \in S$", while pushing button B is declaring "I believe that $U \notin S$". And again, if she is mistaken once, she is mistaken on each one of her N awakenings, etc.
What does Beauty do? Well, if $P^1(U \in S) > 1/2$, she pushes button A on each one of her awakenings (and, of course, if $P^1(U \in S) < 1/2$, then she pushes button B on each one of her awakenings).
It might be useful to have a notation for the laws of U in the world W^0 and W^1. We follow the traditional usage, so P^0_U will be the law of U in the world W^0, and P^1_U will be the law of U in the world W^1: for all Borel $B \subseteq I$, $P^0_U(B) = P^0(U \in B) = \lambda(B)$, and $P^1_U(B) = P^1(U \in B)$.
The measures P^0_U and P^1_U are defined on the set $\mathcal{B}(I)$ of Borel subsets of the unit interval I, and P^0_U is of course diffuse, in the sense that, for all $A \in \mathcal{B}(I)$, if $P^0_U(A) > 0$, then there exists some $B \in \mathcal{B}(I)$ such that $B \subset A$ and $0 < P^0_U(B) < P^0_U(A)$; and it is not difficult to see that, in this case, for all $A \in \mathcal{B}(I)$, $\{P^0_U(B) : B \in \mathcal{B}(I),\ B \subseteq A\} = [0,\ P^0_U(A)]$.

⋮

Theorem
If μ and ν are probability measures on the same measurable space equipped with the σ-algebra $\mathcal{G}$ and the measure μ is diffuse
and, for all $G \in \mathcal{G}$, $\mu(G) > 1/2 \implies \nu(G) \geq 1/2$,
then $\mu = \nu$.

Exercise Prove it!

The proof, and other details (and sections 8 and 9) are postponed to June (or July?), ie, to a version of the present paper which will not be such a prepreprelimininary preprepredraft.

Merci pour votre attention,
Omer Adelman

REFERENCES

[1a] Frank Arntzenius (2002)
Reflections on Sleeping Beauty. *Analysis*, **62** (1), 53-62
Link: https://www.jstor.org/stable/3329069

[1b] Frank Arntzenius (2003)
Some Problems for Conditionalization and Reflection. *The Journal of Philosophy*, **100** (7), 356-370
Link: https://www.jstor.org/stable/3655783

[2] Robert J. Aumann, Sergiu Hart, Motty Perry (1997)
The forgetful passenger. *Games and Economic Behavior*, **20** (1), 117-120
Link: http://www.ma.huji.ac.il/raumann/pdf/Forgetful Passenger.pdf

[3] Nick Bostrom (2007)
Sleeping Beauty and Self-Location: A Hybrid Model. *Synthese*, **157** (1), 59-78
Link: https://www.jstor.org/stable/27653543

[4] Darren Bradley (2003)
Sleeping Beauty: a note on Dorr's argument for 1/3. *Analysis*, **63** (3), 266-268
Link: https://philpapers.org/archive/BRASBA-2.pdf

[5] Laurent Delabre, Léo Gerville-Réache (2015)
Insaisissable Belle au bois dormant. *Philosophia Scientiae*, **19** (1), 251-269
Link: https://www.cairn.info/revue-philosophia-scientiae-2015-1-page-251.htm

[6] Laurent Delabre (2015)
Un jeune paradoxe : la Belle au bois dormant. *Implications Philosophiques*, **19** (1), 251-269
Link: https://www.implications-philosophiques.org/belleauboisdormant/

[7] Laurent Delabre (2019)
L'interprétation double de la probabilité et les problèmes d'auto-localisation (PhD thesis)
Link: https://theses.hal.science/tel-03234602/document

[8] Adam Elga (2000)
Self-locating belief and the Sleeping Beauty problem, *Analysis*, **60** (2), 143-147
Link: https://www.princeton.edu/ adame/papers/sleeping/sleeping.pdf

[9] Paul Franceschi (2010)
A Two-Sided Ontological Solution to the Sleeping Beauty Problem, *PhilSci-Archive* (Preprint)
Link: http://philsci-archive.pitt.edu/8357/4/sb-en.pdf

[10] Berry Groisman (2008)
The End of Sleeping Beauty's Nightmare. *British Journal for the Philosophy of Science*, **59** (3), 409-416
Link: https://arxiv.org/pdf/0806.1316.pdf

[11] David Lewis (2001)
Sleeping Beauty: Reply to Elga. *Analysis*, **61** (3), 171-176
Link: http://www.fitelson.org/probability/lewis_sb.pdf

[12] Ioannis Mariolis (2014)
Revealing the Beauty behind the Sleeping Beauty Problem (Preprint)
Link: https://arxiv.org/pdf/1409.3803.pdf

[13] Bradley Monton (2002)
Sleeping Beauty and the Forgetful Bayesian. *Analysis*, **62** (1), 47-53
Link: https://philpapers.org/archive/MONSBA.pdf

[14] Michele Piccione, Ariel Rubenstein (1997)
On the interpretation of decision problems with imperfect recall. *Games and Economic Behavior* **20** (1) 3–24
Link: https://arielrubinstein.tau.ac.il/papers/53.pdf

[15] Jeffrey S. Rosenthal (2009)
A mathematical analysis of the Sleeping Beauty problem, *The Mathematical Intelligencer*, **31** (3), 32-37
Link: http://probability.ca/jeff/ftpdir/beauty.pdf

[16] Roger White (2006)
The generalized Sleeping Beauty problem : A challenge for thirders. *Analysis*, **66** (2), 114-119
Link: http://web.mit.edu/rog/www/papers/sleeping_beauty.pdf

[17] Peter Winkler (2017)
The Sleeping Beauty Controversy, *The American Mathematical Monthly*, **124** (7), 579-587
Link: https://faculty.cord.edu/andersod/sleeping-beauty.pdf

[18] And, of course, Wikipedia on the Sleeping Beauty problem
English: https://en.wikipedia.org/wiki/Sleeping_Beauty_problem
French: https://fr.wikipedia.org/wiki/Problème_de_la_Belle_au_bois_dormant

Omer Adelman
LPSM (= Laboratoire de Probabilités, Statistique et Modélisation),
Sorbonne Université, 4 Place Jussieu, 75005 Paris, France
Email address: Omer.Adelman@sorbonne-universite.fr
URL: https://perso.lpsm.paris/~oadelman