Total variation distance between two diffusions in small time with unbounded drift: application to the Euler-Maruyama scheme

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- We consider the following SDE in $\mathbb{R}^d$ along its one-step Euler-Maruyama scheme:

SDE and Euler scheme

$$X_0^x = x \in \mathbb{R}^d, \quad dX_t^x = b_1(X_t^x)dt + \sigma_1(X_t^x)dW_t,$$
$$\bar{X}_t^x = x + tb_1(x) + \sigma_1(x)W_t.$$

- We consider the total variation distance on $\mathcal{P}(\mathbb{R}^d)$ as

Total variation distance

$$\mathrm{d}_{\mathrm{TV}}(\pi_1, \pi_2) = \sup\left\{\int_{\mathbb{R}^d} f d\pi_1 - \int_{\mathbb{R}^d} f d\pi_1,\ f : \mathbb{R}^d \to [-1, 1] \text{ measurable}\right\}.$$

If π_1, π_2 have densities p_1, p_2 then

$$\mathrm{d}_{\mathrm{TV}}(\pi_1, \pi_2) = \int_{\mathbb{R}^d} \big|p_1(x) - p_2(x)\big| dx.$$

- **Objective:** give bounds for

$$\mathrm{d}_{\mathrm{TV}}(X_t^x, \bar{X}_t^x) \quad \text{as } t \to 0.$$

- Weak error asymptotics in small time for Monte Carlo simulation:

$$\mathbb{E}f(\bar{X}_t^x) - \mathbb{E}f(X_t^x) \quad \text{as } t \to 0, \; f \text{ measurable bounded.}$$

- Short time term in *Domino* strategies for weak error rates [Talay-Tubaro 1990]:

$$\begin{aligned}|\mathbb{E}f(\bar{X}_T^{x,N}) - \mathbb{E}f(X_T^x)| &= |\bar{P}_h \circ \cdots \circ \bar{P}_h f(x) - P_T f(x)| \\ &\le \sum_{k=1}^{n} \left|\bar{P}_h \circ \cdots \circ \bar{P}_h \circ (\bar{P}_h - P_h) \circ P_{T-kh} f(x)\right|,\end{aligned}$$

with $h = T/N$, P and $\bar{P}$ transition kernels of X and $\bar{X}$. We look for bounds for

$$(\bar{P}_h - P_h)g(x), \quad g : \mathbb{R}^d \to \mathbb{R}, \quad \text{as } h \to 0.$$

- [Bally-Talay 1996]: d_{TV} for the N-step Euler scheme $\bar{X}^{x,N}$ at fixed time horizon $T > 0$ and as $N \to \infty$:

$$\forall x \in \mathbb{R}^d, \ d_{TV}(X_T^x, \bar{X}_T^{x,N}) \le \frac{K(T)(1+|x|^Q)}{NT^q}.$$

  However we do not know whether $K(T)/T^q \to 0$ as $T \to 0$.

- [Gobet-Labart 2008] gives estimates for the transition densities p and $\bar{p}^N$:

$$\forall t \in (0,T], \ \forall x,y \in \mathbb{R}^d, \ |p(t,x,y) - \bar{p}^N(t,x,y)| \le \frac{K(T)T}{Nt^{(d+1)/2}} e^{-C|x-y|^2/t},$$

  but we cannot directly it for d_{TV}; taking $N = 1$ gives

$$d_{TV}(X_t^x, \bar{X}_t^x) = \int_{\mathbb{R}^d} |p(t,x,y) - \bar{p}^N(t,x,y)| dy \le K(T)Tt^{-1/2} \int_{\mathbb{R}^d} \frac{1}{t^{d/2}} e^{-C|x-y|^2/t}$$

  of order $t^{-1/2} \to \infty$.

- Difficulty of d_{TV}: If f is Lipschitz continuous then $|f(x) - f(y)| \le [f]_{\text{Lip}} |x-y|$, but if f is simply bounded measurable, we cannot bound $|f(x) - f(y)|$ in terms of $|x - y|$.

More generally we consider two general SDEs starting at the same point x with close coefficients:

$$X_0^x = x \in \mathbb{R}^d, \qquad dX_t^x = b_1(X_t^x)dt + \sigma_1(X_t^x)dW_t, \ t \in [0,T],$$
$$Y_0^x = x, \qquad dY_t^x = b_2(Y_t^x)dt + \sigma_2(Y_t^x)dW_t, \ t \in [0,T].$$

We define $\widetilde{\mathcal{C}}_b^k$ as the functions $\mathcal{C}^k$ with bounded derivatives but not bounded themselves. We say that σ is (uniformly) elliptic if

$$\exists \alpha > 0, \ \sigma\sigma^\top \geq \alpha I_d.$$

Theorem

Assume that $\sigma_i \in \mathcal{C}_b^{2r}$ for some $r \in \mathbb{N}$ and $b_i \in \widetilde{\mathcal{C}}_b^1$ and σ_i is elliptic. Then

$$\forall t \in [0,T], \ \forall x \in \mathbb{R}^d, \ \mathrm{d}_{\mathrm{TV}}(X_t^x, Y_t^x) \leq C(t^{1/2} + \Delta\sigma(x))^{2r/(2r+1)} + Ce^{c|x|^2} t^{1/2},$$

with $\Delta\sigma = |\sigma_1 - \sigma_2|$. In particular

$$\mathrm{d}_{\mathrm{TV}}(X_t^x, \bar{X}_t^x) \leq Ct^{r/(2r+1)} + Ce^{c|x|^2} t^{1/2}.$$

As $r \to \infty$, this gives a rate "almost" $t^{1/2}$.

A first regularization result

Let $\mathcal{W}_1$ be the L^1-Wasserstein distance with

$$\mathcal{W}_1(\pi_1, \pi_2) = \sup\left\{\int_{\mathbb{R}^d} f d\pi_1 - \int_{\mathbb{R}^d} f d\pi_2,\ f : \mathbb{R}^d \to \mathbb{R}\ \text{ 1-Lipschitz continuous}\right\}.$$

We show that we can bound the total variation with $\mathcal{W}_1$, provided that the laws are "regular enough".

Theorem

Let Z_1, Z_2 be random vectors in $L^1(\mathbb{R}^d)$ with densities p_1 and p_2. Then

$$\mathrm{d}_{\mathrm{TV}}(Z_1, Z_2) \leq C_d \mathcal{W}_1(Z_1, Z_2)^{2/3} \left(\int_{\mathbb{R}^d} (|\nabla^2 p_1(\xi)| + |\nabla^2 p_2(\xi)|) d\xi\right)^{1/3}.$$

Proof: For $\varepsilon > 0$ and $\zeta \sim \mathcal{N}(0, I_d)$ we have

$$\begin{aligned}\mathrm{d}_{\mathrm{TV}}(Z_1, Z_2) \leq\ & \mathrm{d}_{\mathrm{TV}}(Z_1, Z_1 + \sqrt{\varepsilon}\zeta) \\ & + \mathrm{d}_{\mathrm{TV}}(Z_1 + \sqrt{\varepsilon}\zeta, Z_2 + \sqrt{\varepsilon}\zeta) \\ & + \mathrm{d}_{\mathrm{TV}}(Z_2 + \sqrt{\varepsilon}\zeta, Z_2).\end{aligned}$$

For bounded f, we have

$$|\mathbb{E}f(Z_1+\sqrt{\varepsilon}\zeta)-\mathbb{E}f(Z_1)| = |\mathbb{E}_\zeta\varphi(\sqrt{\varepsilon}\zeta)-\varphi(0)|,$$
$$\varphi : y \mapsto \mathbb{E}_{Z_1}f(Z_1+y) = \int_{\mathbb{R}^d} f(\xi+y)p_1(\xi)d\xi = \int_{\mathbb{R}^d} f(\xi)p_1(\xi-y)d\xi.$$

The main idea is to use some kind of **integration by parts** so that the derivatives w.r.t. ξ are taken with p_1 and not f. Then φ is $\mathcal{C}^2$ if p_1 is $\mathcal{C}^2$ and

$$\nabla^2\varphi(y) = \int_{\mathbb{R}^d} f(\xi)\nabla^2 p_1(\xi-y)d\xi,$$
$$\|\nabla^2\varphi\|_\infty \le \|f\|_\infty \int_{\mathbb{R}^d} |\nabla^2 p_1(\xi)|d\xi.$$

Then with a Taylor expansion, for some $\tilde{\zeta}\in(0,\zeta)$:

$$|\mathbb{E}f(Z_1+\sqrt{\varepsilon}\zeta)-\mathbb{E}f(Z_1)| = |\mathbb{E}\varphi(\sqrt{\varepsilon}\zeta)-\varphi(0)|$$
$$= |\cancel{\sqrt{\varepsilon}\mathbb{E}[\nabla\varphi(0)\zeta]} + (\varepsilon/2)\mathbb{E}[\nabla^2\varphi(\sqrt{\varepsilon}\tilde{\zeta})\zeta^{\otimes 2}]| \le (\varepsilon/2)\|\nabla^2\varphi\|_\infty \mathbb{E}|\mathcal{N}(0,I_d)|^2$$
$$\le C\varepsilon\|f\|_\infty \int_{\mathbb{R}^d} |\nabla^2 p_1(\xi)|d\xi.$$

The same way

$$|\mathbb{E}f(Z_2+\sqrt{\varepsilon}\zeta)-\mathbb{E}f(Z_2)| \le C\varepsilon\|f\|_\infty \int_{\mathbb{R}^d} |\nabla^2 p_2(\xi)|d\xi.$$

On the other hand:

$$|\mathbb{E}f(Z_1+\sqrt{\varepsilon}\zeta)-\mathbb{E}f(Z_2+\sqrt{\varepsilon}\zeta)| = |\mathbb{E}(f_\varepsilon(Z_1)-f_\varepsilon(Z_2))| \leq C[f_\varepsilon]_{\mathrm{Lip}}\mathcal{W}_1(Z_1,Z_2),$$

with f_ε is the convolution of f:

$$f_\varepsilon : y \mapsto \mathbb{E}f(y+\sqrt{\varepsilon}\zeta) = \frac{1}{(2\pi\varepsilon)^{d/2}}\int_{\mathbb{R}^d} f(\xi)e^{-|\xi-y|^2/(2\varepsilon)}d\xi$$

$$\nabla f_\varepsilon(y) = \frac{1}{(2\pi\varepsilon)^{d/2}}\int_{\mathbb{R}^d} f(\xi)\frac{\xi-y}{\varepsilon}e^{-|\xi-y|^2/(2\varepsilon)}d\xi = \frac{\varepsilon^{-1/2}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(y+\sqrt{\varepsilon}\xi)\xi e^{-|\xi|^2/2}d\xi$$
$$= \varepsilon^{-1/2}\mathbb{E}[f(y+\varepsilon\zeta)\zeta] \leq \|f\|_\infty\varepsilon^{-1/2}\mathbb{E}|\mathcal{N}(0,I_d)| \leq C\|f\|_\infty\varepsilon^{-1/2}.$$

so that

$$|\mathbb{E}f(Z_1+\sqrt{\varepsilon}\zeta)-\mathbb{E}f(Z_2+\sqrt{\varepsilon}\zeta)| \leq C\|f\|_\infty\varepsilon^{-1/2}\mathcal{W}_1(Z_1,Z_2)$$

$$\Longrightarrow \mathrm{d}_{\mathrm{TV}}(Z_1,Z_2) \leq C\varepsilon\int(|\nabla^2 p_1(\xi)|+|\nabla^2 p_2(\xi)|)d\xi + C\varepsilon^{-1/2}\mathcal{W}_1(Z_1,Z_2)$$

and we minimize in ε, giving the result.

- For the Euler scheme:

$$\bar{X}^x_t = x + tb(x) + \sigma(x)W_t \sim \mathcal{N}\left(x + tb(x), t\sigma(x)\sigma(x)^\top\right),$$

$$p_{\bar{X}^x_t}(dy) = \frac{t^{-d/2}}{(2\pi \det(\sigma(x)\sigma(x)^\top))^{d/2}} \exp\left(-(\sigma(x)\sigma(x)^\top)^{-1} \cdot (y - x - tb(x))^{\otimes 2}/(2t)\right) dy,$$

$$\implies |\nabla^k_y p_{\bar{X}^x_t}(y)| \le \frac{Ce^{-c|y|^2/t}}{t^{(d+k)/2}},$$

$$\implies \int_{\mathbb{R}^d} |\nabla^k p_{\bar{X}^x_t}(y)| dy \le Ct^{-(d+k)/2} \int_{\mathbb{R}^d} e^{-c|y|^2/t} = Ct^{-(d+k)/2} t^{d/2} \int_{\mathbb{R}^d} e^{-c|y|^2} dy$$
$$= O(t^{-k/2}).$$

• The transition density from $(X_s = x)$ to $(X_t = y)$ denoted $p_X(s,t,x,y)$ satisfies the backward Kolmogorov PDE:

$$p_X(t,t,x,\cdot) = \delta_x, \ \ t \in [0,T],$$
$$\partial_s p_X(s,t,x,y) = \langle b_1(s,x), \nabla_x p_X(s,t,x,y)\rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma_1^\top(s,x)\nabla_x^2 p_X(s,t,x,y)\sigma_1(s,x)\right), \ \ s < t$$

If σ is elliptic and if $b_1, \sigma_1 \in \mathcal{C}_b^r$ then sub-Gaussian Aronson's bounds state that for every $m_0 = 0, 1$ and $0 \leq m_1 + m_2 \leq r$,

$$\|\nabla_x^{m_0+m_1}\nabla_y^{m_2} p_X(s,t,x,y)\| \leq \frac{Ce^{-c|y-x|^2/(t-s)}}{(t-s)^{(d+m_0+m_1+m_2)/2}}$$

How to deal with unbounded drift ?

- Recent advances in PDE theory [Menozzi-Pesce-Zhang 2021] give similar Aronson's bounds if we only have $b_1 \in \widetilde{C}_b^r$, however requires more regularity on σ_1 and not very clear for high order derivatives.
- Another method: we consider

$$d\widetilde{X}_t^x = \widetilde{b}_1^x(\widetilde{X}_t^x)dt + \sigma_1(\widetilde{X}_t^x)dW_t$$

where $\widetilde{b}_1^x$ is "cut" outside $\mathbf{B}(x,R)$, so bounded. Then since X_t^x leaves $\mathbf{B}(x,R)$ in small time with small probability, we have

$$\mathrm{d}_{\mathrm{TV}}(X_t^x, \widetilde{X}_t^x) \leq C(1+|b_1(x)|^2)t.$$

Using the Girsanov formula we obtain

$$\mathrm{d}_{\mathrm{TV}}(\widetilde{X}_t^x, \mathcal{M}(\sigma_1)_t^x) \leq Ce^{c|x|^2} t^{1/2}$$

where $\mathcal{M}(\sigma)$ is the martingale $d\mathcal{M}(\sigma)_t^x = \sigma(\mathcal{M}(\sigma)_t^x)dW_t$.

- Classical bounds give

$$\mathcal{W}_1(\mathcal{M}(\sigma_1)_t^x, \mathcal{M}(\sigma_2)_t^x) \leq C(t + \Delta\sigma(x)t^{1/2}).$$

- Applying the regularization theorem with $Z_i = \mathcal{M}(\sigma_i)_t^x$ gives

$$\mathrm{d}_{\mathrm{TV}}(\mathcal{M}(\sigma_1)_t^x, \mathcal{M}(\sigma_2)_t^x) \leq C(t^{1/2} + \Delta\sigma(x))^{2/3}$$

so that

$$\mathrm{d}_{\mathrm{TV}}(X_t^x, Y_t^x) \leq C(t^{1/2} + \Delta\sigma(x))^{2/3} + Ce^{c|x|^2} t^{1/2}.$$

(Assumptions:
– σ is elliptic, bounded, with bounded derivatives up to order 2
– ∇b is bounded)

We improve our regularization theorem. In the proof, we wrote $\varphi : y \mapsto \mathbb{E}f(Z_1 + y)$ and then for f bounded and $\zeta \sim \mathcal{N}(0, I_d)$:

$$|\mathbb{E}f(Z_1 + \sqrt{\varepsilon}\zeta) - \mathbb{E}f(Z_1)| = |\mathbb{E}\varphi(\sqrt{\varepsilon}\zeta) - \varphi(0)| \quad \text{of order } \varepsilon$$

using a Taylor expansion up to order 2.

Idea to improve: Taylor expansion of φ to some higher order $2r$ and we consider instead the linear combination

$$\left| \sum_{i=1}^{r} w_i \mathbb{E}f(Z_1 + \sqrt{\varepsilon/n_i}\zeta) - \mathbb{E}f(Z_1) \right|$$

where $w_i, n_i \in \mathbb{R}$ are well chosen so that the Taylor expansion terms anneals up to order $2r$.

Weighted multi-level Richardson-Romberg

- Assume that we want to estimate $\mathbb{E}[Z]$ for some Z (typically: $Z = F(X_T)$) by $\mathbb{E}[\bar{Z}^N]$ (typically: $\bar{Z}^N = F(\bar{X}_{T,N})$ the N-multi-step Euler-Maruyama scheme); assume that

$$\mathbb{E}[\bar{Z}^N] = \mathbb{E}[Z] + \frac{c_1}{N} + o\left(\frac{1}{N}\right), \quad N \to \infty$$

then

$$2\mathbb{E}[\bar{Z}^{2N}] - \mathbb{E}[\bar{Z}^N] = 2\mathbb{E}[Z] + \frac{2c_1}{2N} + o\left(\frac{1}{N}\right) - \mathbb{E}[Z] - \frac{c_1}{N} + o\left(\frac{1}{N}\right) = \mathbb{E}[Z] + o\left(\frac{1}{N}\right),$$

thus improving the convergence rate with the estimator $2\mathbb{E}[\bar{Z}^{2N}] - \mathbb{E}[\bar{Z}^N]$.

- More generally, assume

$$\mathbb{E}[\bar{Z}^N] = \mathbb{E}[Z] + \sum_{i=1}^{r} \frac{c_i}{N^i} + o\left(\frac{1}{N^r}\right), \quad N \to \infty$$

then with the estimator

$$\sum_{i=0}^{r} w_i \mathbb{E}[Z^{2^i N}]$$

for some well chosen $w_i \in \mathbb{R}$, we can obtain a convergence rate in $o(N^{-r})$.

$$
\begin{aligned}
d_{TV}(Z_1, Z_2) &= \sup_{\|f\|_\infty \le 1} |\mathbb{E}f(Z_1) - \mathbb{E}f(Z_2)| \\
&\le \sup_{\|f\|_\infty \le 1} |\mathbb{E}f(Z_1) - \sum_{i=1}^{r} w_i \mathbb{E}f(Z_1 + \sqrt{2^{-(i-1)}\varepsilon}\zeta)| \\
&\quad + |\sum_{i=1}^{r} w_i \mathbb{E}f(Z_1 + \sqrt{2^{-(i-1)}\varepsilon}\zeta) - \sum_{i=1}^{r} w_i \mathbb{E}f(Z_2 + \sqrt{2^{-(i-1)}\varepsilon}\zeta)| \\
&\quad + |\sum_{i=1}^{r} w_i \mathbb{E}f(Z_2 + \sqrt{2^{-(i-1)}\varepsilon}\zeta) - \mathbb{E}f(Z_2)|
\end{aligned}
$$

For bounded f and $\zeta \sim \mathcal{N}(0, I_d)$:

$$|\sum_{i=1}^{r} w_i \mathbb{E} f(Z_1 + \sqrt{2^{i-1}\varepsilon}\zeta) - \mathbb{E} f(Z_1)| = |\sum_{i=1}^{r} w_i \mathbb{E}\varphi(\sqrt{2^{i-1}\varepsilon}\zeta) - \varphi(0)|$$

with

$$\nabla^k \varphi(y) = \nabla^k \int f(\xi + y) p_1(\xi) d\xi = \nabla^k \int f(\xi) p_1(\xi - y) d\xi$$
$$= (-1)^k \int f(\xi) \nabla^k p_1(\xi - y) d\xi \leq C \|f\|_\infty \int |\nabla^k p_1(\xi)| d\xi.$$

By Taylor expansion up to order $2r$, and setting $\sum_{i=1}^r w_i = 1$:

$$\left(\sum_{i=1}^{r} w_i \mathbb{E} f(Z_1 + \sqrt{2^{-(i-1)}\varepsilon}\zeta)\right) - \mathbb{E} f(Z_1) = \sum_{i=1}^{r} w_i \left(\mathbb{E}\varphi(\sqrt{2^{-(i-1)}\varepsilon}\zeta) - \varphi(0)\right)$$
$$= \sum_{i=1}^{r} w_i \left(\sum_{k=1}^{2r-1} \frac{\nabla^k \varphi(0)}{k!} (2^{-(i-1)}\varepsilon)^{k/2} \mathbb{E}[\zeta^{\otimes k}] + \frac{(2^{-(i-1)}\varepsilon)^r}{(2r)!} \mathbb{E}\left[\nabla^{2r}\varphi(\sqrt{\varepsilon}\tilde{\zeta}_i) \cdot \zeta^{\otimes 2r}\right]\right)$$
$$= \left(\sum_{k=1}^{r-1} \frac{\nabla^{2k}\varphi(0)}{(2k)!} \mathbb{E}[|\mathcal{N}(0, I_d)|^{2k}] \varepsilon^k \sum_{i=1}^r 2^{-k(i-1)} w_i\right) + \left(\sum_{i=1}^r w_i \frac{(2^{-(i-1)}\varepsilon)^r}{(2r)!} \mathbb{E}[\nabla^{2r}\varphi(\sqrt{\varepsilon}\tilde{\zeta}_i) \cdot \zeta^{\otimes 2r}]\right)$$

where $\tilde{\zeta}_i \in (0, \xi)$ (from Taylor-Lagrange).

We now choose (w_i) as the unique solution to the $r \times r$ Vandermonde system

$$\sum_{i=1}^{r} w_i 2^{-(i-1)k} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}, \quad k = 0, 1, \dots, r-1,$$

giving

$$\begin{aligned} \sum_{i=1}^{r} w_i \mathbb{E} f(Z_1 + \sqrt{2^{i-1}\varepsilon}\zeta) - \mathbb{E} f(Z_1) &= \frac{\varepsilon^r}{(2r)!} \sum_{i=1}^{r} w_i 2^{-(i-1)r} \mathbb{E}[\nabla^{2r}\varphi(\sqrt{\varepsilon}\tilde{\zeta}_i) \cdot \zeta^{\otimes 2r}] \\ &\leq C\|f\|_\infty \left(\int |\nabla^{2r} p_1(\xi)| d\xi \right) \mathbb{E}[|\mathcal{N}(0, I_d)|^{2r}] \varepsilon^r \sum_{i=1}^{r} w_i 2^{-(i-1)r} \\ &\leq C\|f\|_\infty \left(\int |\nabla^{2r} p_1(\xi)| d\xi \right) \varepsilon^r. \end{aligned}$$

Likewise

$$\sum_{i=1}^{r} w_i \mathbb{E} f(Z_2 + \sqrt{2^{i-1}\varepsilon}\zeta) - \mathbb{E} f(Z_2) \leq C\|f\|_\infty \left(\int |\nabla^{2r} p_2(\xi)| d\xi \right) \varepsilon^r.$$

On the other hand

$$\left|\sum_{i=1}^{r} w_i \mathbb{E} f(Z_1+\sqrt{2^{-(i-1)}\varepsilon}\zeta) - \sum_{i=1}^{r} w_i \mathbb{E} f(Z_2+\sqrt{2^{-(i-1)}\varepsilon}\zeta)\right|$$

$$\leq \sum_{i=1}^{r} w_i [f_{2^{-(i-1)}\varepsilon}]_{\mathrm{Lip}} \mathcal{W}_1(Z_1,Z_2) \leq C\|f\|_\infty \varepsilon^{-1/2}\mathcal{W}_1(Z_1,Z_2)\sum_{i=1}^{r}|w_i|2^{(i-1)/2}$$

$$\leq C\|f\|_\infty \varepsilon^{-1/2}\mathcal{W}_1(Z_1,Z_2).$$

At the end:

$$\mathrm{d}_{\mathrm{TV}}(Z_1,Z_2) \leq C\left(\int (|\nabla^{2r}p_1(\xi)| + |\nabla^{2r}p_2(\xi)|)d\xi\right)\varepsilon^r + C\varepsilon^{-1/2}\mathcal{W}_1(Z_1,Z_2)$$

and we optimize in ε, giving

$$\mathrm{d}_{\mathrm{TV}}(Z_1,Z_2) \leq C_d \mathcal{W}_1(Z_1,Z_2)^{2r/(2r+1)}\left(\int_{\mathbb{R}^d} (|\nabla^{2r}p_1(\xi)| + |\nabla^{2r}p_2(\xi)|)d\xi\right)^{1/(2r+1)}.$$

- The sub-Gaussian bounds from PDE theory :

$$|\nabla_y^k p_{\mathcal{M}(\sigma_1)_t^x}(y)| \leq \frac{Ce^{-c|y|^2/t}}{t^{(d+k)/2}} \implies \int (|\nabla^{2r} p_1(\xi)| + |\nabla^{2r} p_2(\xi)|) d\xi = O(t^{-r})$$

- Recall:

$$\mathcal{W}_1(\mathcal{M}(\sigma_1)_t^x, \mathcal{M}(\sigma_2)_t^x) \leq C(t + \Delta\sigma(x) t^{1/2}).$$

$$\implies \mathrm{d}_{\mathrm{TV}}(X_t^x, Y_t^x) \leq C(t^{1/2} + \Delta\sigma(x))^{2r/(2r+1)} + Ce^{c|x|^2} t^{1/2}$$

(Assumptions:
– σ is elliptic, bounded, with bounded derivatives up to order $2r$
– ∇b is bounded)

Conclusion:

- With a good linear combination, we can improve the convergence rate from $t^{1/3}$ to $t^{r/(2r+1)}$, $r \in \mathbb{N}$.
- We believe that such weighted multi-level methods could be applied to other problems, to improve the convergence rate.
- The general bound we obtained on $\mathrm{d}_{\mathrm{TV}}(Z_1, Z_2)$ could be applied to other problems to give bounds on the weak error knowing bounds on the strong error.

Thank you for your attention!