SOME REMARKS ON THE REGULARITY OF CURRENTS SUPPORTED ON WERMER EXAMPLES

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1. Introduction

The purpose of this note is to complement the paper [Du2] by the author. It is self-contained and not intended for publication (so that it might not be free of minor errors).

To be specific, in [Du2] it was proved that there exists a positive closed current T in the unit bidisk, with $T \wedge T = 0$ and potential of class $C^{1,\alpha}$ for all $\alpha < 1$, which is not a laminar current. The core of [Du2] is the construction of a so-called *Wermer example* (see below for more details) supporting T. It differs from the original Wermer inductive construction [W] by an additional "subdivision" process. Here we explore the regularity that can be obtained without subdividing.

Nonetheless, we still consider a slightly more general construction than [W], since we allow the degrees to diverge arbitrarily fast to infinity. In this situation, we show (Theorem 4.1) that it is possible to have u –the potential of T– Hölder continuous of exponent α for every $\alpha < 1$. In the more classical case where the degree is only allowed to double at each step, u is never Hölder continuous, as we explained in [Du2, §4]. We show in Theorem 5.1 that essentially any sub-Hölder modulus of continuity can be reached in this case.

One possible source of motivation for not willing to subdivide is that it should lead to examples of extremal such currents, in the spirit of [Sł]. The (non trivial) details remain to be worked out.

2. Wermer examples

In this section we recall the construction of Wermer examples given in [Du2]. The only difference is that there is no "subdivision" step, and that the degree is multiplied by a varying number d_n at each step.

Recall that a subset X in $\mathbb{D} \times \mathbb{D}$ is *horizontal* if $\overline{X} \subset (\mathbb{D} \times D(0, 1-\varepsilon))$ for some $\varepsilon > 0$. A current is horizontal if its support is. Dividing the z coordinate by 2 we work the bidisk $D(0, 1/2) \times \mathbb{D}$ –this is convenient because for $z, z' \in D(0, 1/2)$, $|z - z'| < 1$.

Let first $(a_n)_{n\geq 1}$ be a sequence of points in $D(0, 1/4)$ such that (a_{2p}) and (a_{2p+1}) are dense in that disk. We put $A_n(z, w) = z - a_n$ if n is odd and $z + \frac{w}{10} - a_n$ if n is even. Note that $|A_n| \leq 1$ in $D(0, 1/2) \times \mathbb{D}$. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive real numbers and $(d_n)_{n\geq 1}$ a sequence of integers larger than 1, and consider the sequence of polynomials defined as follows:

$$P_0(z, w) = w, \text{ and } P_{n+1} = P_n^{d_{n+1}} - \varepsilon_{n+1} A_{n+1}, \tag{1}$$

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as well as the associated analytic subsets $\{P_n = 0\}$ in $D(0,1/2)\times\mathbb{D}$. If $(\delta_n)_{n\geq 0}$ is a sequence of positive real numbers, we set $X_n = \{|P_n| < \delta_n\}$. Let us also put $D_n = \prod_{i=0}^n d_i$, with the convention that $d_0 = 1$.

Lemma 2.1. *Fix a sequence of positive real numbers $(r_n)_{n\geq 1}$, decreasing to zero, with $r_n \leq \frac{1}{10}$ Let $(\delta_n)_{n\geq 0}$ be the sequence defined by $\delta_0 = 1/2$ and $\delta_{n+1} = \delta_n^{d_{n+1}} r_{n+1}/10$ and $(\varepsilon_n)_{n\geq 1}$ be defined by $\varepsilon_{n+1} = \delta_n^{d_{n+1}}/2$.*

Then the following properties hold for every $n \geq 1$:

(i.) $\overline{X_{n+1}} \subset X_n$ *in* $D(0,1/2)\times\mathbb{D}$*;*

(ii.) X_{n+1} *does not contain the graph of any holomorphic function over $D(a_{n+1}, r_{n+1})$, relative to the projection $\pi_0(z,w) = z$ if n if even, relative to $\pi_1(z,w) = z + \frac{w}{10}$ if n is odd;*

(iii.) for each $\alpha\in\mathbb{C}$ with $|\alpha|\leq\delta_n$, the analytic set $\{P_n = \alpha\}$ is horizontal in $D(0,1/2)\times\mathbb{D}$, of degree D_n and it is a graph over $\left\{\frac{3}{8}\leq |z|\leq \frac{1}{2}\right\}$.

Proof. Assuming that the constant δ_n has been chosen, then to ensure *(i.)* it is enough that

$$\delta_{n+1} + \varepsilon_{n+1} < \delta_n^{d_{n+1}}. \tag{2}$$

Let us also observe that since $X_0 = \{|P_0| < \delta_0 = 1/2\}$ is horizontal, the horizontality assertion in *(iv.)* follows from the fact that $X_n \subset X_0$.

The following elementary statement is a consequence of Rouché's Theorem: *if $\delta < \varepsilon r$ and $d > 1$, there does not exist any holomorphic function f on $D(0,r)$ such that $\left|(f(\zeta))^d - \varepsilon\zeta\right| < \delta$ for $\zeta\in D(0,r)$.* Indeed assume such a function f exists. Reducing r slightly if necessary, we may assume that f is defined in the neighborhood of $\overline{D(0,r)}$. We see that on $\partial D(0,r)$ we have $\left|(f(\zeta))^d - \varepsilon\zeta\right| < |\varepsilon\zeta|$, so by Rouché's Theorem, f^d must have exactly one zero in $D(0,r)$, counting multiplicity, which is impossible.

From this we infer that to meet condition *(ii.)* it is enough that for every n,

$$\delta_{n+1} < \varepsilon_{n+1} r_{n+1} \tag{3}$$

Now it is clear that with our choice of (δ_n) and (ε_n), (2) and (3), whence *(i.)* and *(ii.)* hold.

It remains to check *(iii.)* We will prove by induction the following slightly stronger fact: let $U\subset\{3/8 < |z| < 1/2\}$ be a disk, and γ be a holomorphic function on $U\times\mathbb{D}$, such that $|\gamma|\leq\delta_n$; then the variety $\{P_n = \gamma\}$ is the union of D_n disjoint graphs over U.

This is true for $n = 1$. Assume the result holds for n, and consider the equation $P_{n+1} = \gamma$ where $|\gamma|\leq\delta_{n+1}$ in $U\times\mathbb{D}$, that is, $P_n^{d_{n+1}} = \gamma + \varepsilon_{n+1}A_{n+1}$. The right hand side does not vanish on $U\times\mathbb{D}$. Indeed $\gamma + \varepsilon_{n+1}(z - a_{n+1}) = 0$ is equivalent to $A_{n+1} = -\gamma/\varepsilon_{n+1}$, and with the choices that we have made,

$$\left|\frac{\gamma}{\varepsilon_{n+1}}\right| < \frac{r_n}{5} \leq \frac{1}{50} \text{ while } |A_{n+1}| > \frac{1}{40}.$$

In particular the function $\gamma + \varepsilon_{n+1}A_{n+1}$ admits d_{n+1}^{th} roots in $U\times\mathbb{D}$, which we may write as ξg, where ξ ranges over the d_{n+1}^{th} roots of unity. By (2) we have $|\xi g| < (\delta_{n+1}+\varepsilon_{n+1})^{1/d_{n+1}} < \delta_n$ our equation $P_{n+1} = \gamma$ is equivalent to $\left\{P_n = \xi g,\ \xi^{d_{n+1}} = 1\right\}$. We conclude by the induction hypothesis. □

Proposition 2.2. *Let P_n and X_n be as defined above and set $X = \bigcap_n X_n$. Then X is a polynomially convex horizontal subset in $D(0,1/2) \times \mathbb{D}$, and $X \cap (D(0,1/8) \times \mathbb{D})$ does not contain any holomorphic disk.*

Proof. The horizontality and polynomial convexity of X are obvious. By item *(ii.)* of the previous lemma applied to odd integers it is clear that $X \cap \{|z| < 1/4\}$ does not contain any piece of holomorphic graph over the z coordinate. So any holomorphic disk contained in $X \cap \{|z| < 1/8\}$ must be contained in a vertical line. Then this would be a graph over a certain open subset of $D(0,1/4)$, relative to the projection π_1, which again is impossible, still due to *(ii.)* in the Lemma. Thus $X \cap \{|z| < 1/8\}$ contains no holomorphic disk. □

From now on the free parameters are the sequences (d_n) and (r_n). In the next sections we adjust these parameters to obtain various moduli of continuity for the currents associated to the Wermer examples, that we construct just now.

3. Continuous potentials

Theorem 3.1. *Let P_n and δ_n be defined as in the previous section. Consider the sequence of psh functions $u_n = \frac{1}{D_n} \log \max(|P_n|, \delta_n)$. Then, if*

$$\sum_{n=1}^{\infty} \frac{|\log r_n|}{D_n} < \infty, \tag{4}$$

the sequence of currents $T_n = dd^c u_n$ converges to a horizontal positive closed current T such that

- *T has continuous potential and $T \wedge T = 0$;*
- *$\mathrm{Supp}(T) \cap (D(0,1/8) \times \mathbb{D})$ does not contain any holomorphic disk;*
- *T is uniformly laminar in $((D(0,1/2) \setminus D(0,3/8)) \times \mathbb{D}$.*

Proof. Recall the notation $X_n = \{|P_n| < \delta_n\}$ and $X = \bigcap X_n$. It is clear that (T_n) is a sequence of currents with locally uniformly bounded masses, and for the moment we let T be a cluster value of this sequence. Since $\mathrm{Supp}(T_n)$ is contained in ∂X_n, T has support in X, hence $\mathrm{Supp}(T) \cap (D(0,1/8) \times \mathbb{D})$ does not contain any holomorphic disk.

We have the integral representation

$$u_n = \frac{1}{D_n} \int_{\mathbb{R}/2\pi\mathbb{Z}} \log \left| P_n - \delta_n e^{i\theta} \right| d\theta, \text{ whence } T_n = \frac{1}{D_n} \int_{\mathbb{R}/2\pi\mathbb{Z}} [P_n = \delta_n e^{i\theta}] d\theta,$$

following from the well known formula $\log^+ |x| = \int \log \left| x - e^{i\theta} \right| d\theta$. From Lemma 2.1*(iii.)* we know that the varieties $\left\{P_n = \delta_n e^{i\theta}\right\}$ are graphs over $\left\{\frac{3}{8} < |z| < \frac{1}{2}\right\}$. It is classical that in this situation the laminar structure passes to the limit, thus T is uniformly laminar in $((D(0,1/2) \setminus D(0,3/8)) \times \mathbb{D}$.

By construction, we have that $u_n \geq \frac{1}{D_n} \log \delta_n$. Morevover by the definition of (δ_n) we have

$$\frac{1}{D_{n+1}} \log \delta_{n+1} - \frac{1}{D_n} \log \delta_n = \frac{1}{D_n} \log \frac{r_n}{10},$$

hence from (4) we infer that the sequence $\frac{1}{D_n} \log \delta_n$ decrases to a limit $c \in (-\infty, 0)$. On the other hand if $z \in D(0,1/2)$ is fixed, $P_n(z, \cdot)$ is a monic polynomial of degree D_n with all its roots in the unit disk (by the horizontality property), whence $\max_{\mathbb{D}} |P_n(z, \cdot)| \leq 2^{D_n}$. We therefore conclude that u_n is uniformly bounded. By the Hartogs Lemma we can extract from (u_n) a subsequence converging in $L^1_{\rm loc}$ to a bounded psh function u, with $T = dd^cu$. In a moment we will see that (u_n) converges uniformly, so that u is continuous. Let us first see why $T \wedge T = 0$. Indeed $u_n = \frac{1}{D_n}\log\delta_n$ is constant on $\overline{X_n}$, thus $u = c = \lim \frac{1}{D_n}\log\delta_n$ on X. In particular $T \wedge T = dd^c(uT) = dd^c(cT) = 0$.

It remains to show that (u_n) converges uniformly. For this we estimate $|u_{n+1} - u_n|$. We will give a uniform estimate of this quantity in a vertical slice $\{z = z_0\}$. In such a slice we have $X_{n+1} \Subset X_n \Subset \mathbb{D}$. Abusing notation we write w for (z_0, w).

If $w \in \overline{X_{n+1}}$, $u_n(w) = \frac{1}{D_n}\log\delta_n$ and $u_{n+1}(w) = \frac{1}{D_{n+1}}\log\delta_{n+1}$. Since $\delta_{n+1} = \delta_n^{d_{n+1}} r_{n+1}/10$ we infer that

$$|u_{n+1}(w) - u_n(w)| \leq \frac{1}{D_{n+1}} |\log r_{n+1}| .$$

If $w \notin X_n$, $u_{n+1}(w) = \frac{1}{D_{n+1}}\log|P_{n+1}|$ and $u_n(w) = \frac{1}{D_n}\log|P_n|$ with $|P_n| \geq \delta_n$. Using $P_{n+1} = P_n^{d_{n+1}} - \varepsilon_{n+1}A_{n+1}$ we infer

$$\left|\frac{1}{D_{n+1}}\log|P_{n+1}| - \frac{1}{D_n}\log|P_n|\right| = \left|\frac{1}{D_{n+1}}\log\left|1 - \frac{\varepsilon_{n+1}A_{n+1}}{P_n^{d_{n+1}}}\right|\right|.$$

In $D(0, 1/2) \times \mathbb{D}$, we have $|A_{n+1}| \leq 1$ so for $w \notin X_n$, we get $\left|\frac{\varepsilon_{n+1}A_{n+1}}{P_n^{d_{n+1}}}\right| \leq \frac{1}{2}$. We conclude that outside X_n we have $|u_{n+1} - u_n| \leq \dfrac{\log 2}{D_{n+1}} \leq \dfrac{1}{D_n}$.

In $X_n \setminus X_{n+1}$, $u_{n+1} - u_n$ is harmonic and the two previous cases give us a bound for this function on $\partial X_n \cup \partial X_{n+1}$. So the maximum principle implies that $|u_{n+1}(w) - u_n(w)| \leq \frac{1}{D_n}|\log r_n|$ there. Summing up, we see that

$$|u_{n+1} - u_n| \leq \frac{1}{D_n}|\log r_n| \tag{5}$$

holds throughout $D(0, 1/2) \times \mathbb{D}$, which by (4) implies that (u_n) converges uniformly. □

4. Hölder continuity

Theorem 4.1. *Let $0 < \alpha < 1$ be arbitrary. Let P_n, δ_n, T and u be as above. Then it is possible to choose the sequences (d_n) and (r_n) so that u is Hölder continuous of exponent α in any given compact subset of $D(0, 1/2) \times \mathbb{D}$.*

In fact we prove that if (r_n) is a given sequence that does not decrease too rapidly (see (10) below), it is possible to choose (d_n) so that u is Hölder continuous of exponent α. It would also be possible to choose them in reverse order.

Actually it will be enough to estimate the modulus of continuity along lines close to be vertical. For this we estimate the derivatives of $u_n = \frac{1}{D_n}\log\max(|P_n|, \delta_n)$ along $z = cst$ and $z + \frac{1}{10}w = cst$. With a slight abuse, the notation u_n will stand for either $w \mapsto u_n(z_0, w)$ or $w \mapsto u_n(z_0 + \frac{1}{10}(w_0 - w), w)$, and likewise for P_n. We keep notation as in Section 2.

Lemma 4.2. *Set $M_n = \max_{D(0,3/8)\times\mathbb{D}} |\nabla u_n|$. Then for every n we have that $M_{n+1} \leq \dfrac{11M_n}{r_n}$.*

Proof. We work in a fixed line in the form $z = cst$ or $z + \frac{1}{10}w = cst$. In the latter case, we assume that it is not to close to the vertical boundary, so that it is vertically contained in our bidisk –working in $D(0,3/8)\times\mathbb{D}$ is enough. In this line we have $X_{n+1}\Subset X_n\subset\mathbb{D}$. In X_n, u_n is constant, hence

$$M_n=\max_{\mathbb{D}}|\nabla u_n|\leq\max_{|P_n|\geq\delta_n}\frac{1}{D_n}\frac{|P_n'|}{|P_n|},$$

where the notation u_n' stands for $\frac{\partial u_n}{\partial w}$. Let us give an inductive estimate on $\frac{|P_{n+1}'|}{|P_{n+1}|}$ in $|P_{n+1}|\geq\delta_{n+1}$.

Assume first that $w\in\{|P_n|\geq\delta_n\}$ and write

$$\frac{P_{n+1}'}{P_{n+1}}=d_{n+1}\frac{P_n'P_n^{d_{n+1}-1}}{P_{n+1}}+\frac{\star\varepsilon_{n+1}}{P_{n+1}}=d_{n+1}\frac{P_n'}{P_n}\left(1+\frac{\varepsilon_{n+1}A_{n+1}}{P_n^{d_{n+1}}}\right)^{-1}+\frac{\star\varepsilon_{n+1}}{P_{n+1}},$$

where $\star=0$, $-1/10$ or $1/10$ depending on the parity of n and the fact that our line is vertical or not. Simply applying the definitions and the inequality $|P_n|\geq\delta_n$ shows that

$$\left|\frac{P_{n+1}'}{P_{n+1}}(w)\right|\leq d_{n+1}\left|\frac{P_n'}{P_n}(w)\right|\left(1-\frac{1}{2}\right)^{-1}+\frac{\star\varepsilon_{n+1}}{\delta_{n+1}}\leq 2d_{n+1}M_n+\frac{1}{2r_{n+1}}. \tag{6}$$

Next, assume that $w\in X_n$. This time we write

$$\begin{aligned}\left|\frac{P_{n+1}'}{P_{n+1}}\right|\leq d_{n+1}\left|\frac{P_n'P_n^{d_{n+1}-1}}{P_{n+1}}\right|+\left|\frac{\star\varepsilon_{n+1}}{P_{n+1}}\right|&\leq d_{n+1}\frac{|P_n'P_n^{d_{n+1}-1}|}{\delta_{n+1}}+\frac{\star\varepsilon_{n+1}}{\delta_{n+1}}\\&\leq 10d_{n+1}\frac{|P_n'P_n^{d_{n+1}-1}|}{\delta_n^{d_{n+1}}r_{n+1}}+\frac{1}{2r_{n+1}}\\&\leq\frac{10d_{n+1}}{r_{n+1}}\frac{|P_n'|}{\delta_n}\frac{|P_n^{d_{n+1}-1}|}{\delta_n^{d_{n+1}-1}}+\frac{1}{2r_{n+1}}.\end{aligned}$$

In X_n we have $\frac{|P_n|}{\delta_n}\leq 1$ and by the maximum principle

$$\max_{X_n}\frac{|P_n'|}{\delta_n}=\max_{\partial X_n}\frac{|P_n'|}{\delta_n}=\max_{\partial X_n}\frac{|P_n'|}{|P_n|}\leq M_n,$$

so we conclude that in X_n we have

$$\left|\frac{P_{n+1}'}{P_{n+1}}\right|\leq\frac{10d_{n+1}}{r_{n+1}}M_n+\frac{1}{2r_{n+1}}. \tag{7}$$

From (6) and (7), we conclude that

$$M_{n+1}\leq\max(2M_n,\frac{10}{r_{n+1}}M_n)+\frac{1}{2r_{n+1}d_{n+1}}=\frac{1}{r_{n+1}}\left(10M_n+\frac{1}{2d_{n+1}}\right)\leq\frac{11}{r_{n+1}}M_n,$$

where the middle equality is true because $r_n\leq\frac{1}{10}$, and the second inequality holds because we can always assume that $M_n\geq 1$. □

In the course of the proof of Theorem 3.1 we have seen that

$$|u_n-u|\leq\sum_{k=n}^{\infty}\frac{1}{D_k}|\log r_k|$$

and we have just shown in the previous lemma that the Lipschitz constant of u_n satisfies

$$M_n = \max \left|u_n'\right| \leq \prod_{j=1}^{n} 11(r_j)^{-1}.$$

Without loss of generality in the following we replace M_n with $\prod_{j=1}^{n} 11(r_j)^{-1}$ Thus we have that

$$\left|u(w) - u(w')\right| \leq M_n \left|w - w'\right| + 2\sum_{k=n}^{\infty} \frac{1}{D_k} \left|\log r_k\right| \tag{8}$$

It is not a surprise that (8) leads to an estimate on the modulus of continuity of u. This is what we do afterwards. Let us first make a preliminary observation.

Lemma 4.3. *If $|\log r_n| = O(n^q)$ then there exists a constant C so that* $\sum_{k=n}^{\infty} \frac{|\log r_k|}{D_k} \leq \frac{Cn^q}{D_n}$.

Proof. Recall that for every n, $d_n \geq 2$. For large enough n we have that

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{|\log r_k|}{D_k} &\leq C \sum_{k=n}^{\infty} \frac{k^q}{D_k} = C \sum_{k=n}^{2n} \frac{k^q}{D_k} + C \sum_{k=2n}^{\infty} \frac{k^q}{D_k} \\
&\leq C \frac{(2n)^q}{D_n} \sum_{k=n}^{2n} \frac{1}{2^{k-n}} + \frac{C}{D_n} \sum_{k=2n}^{\infty} \frac{k^q}{2^{k-n}} \leq 2C \frac{(2n)^q}{D_n} + \frac{C}{D_n} \left[2^n \sum_{k=2n}^{\infty} \frac{1}{1.9^k} \right] \\
&\leq O\left(\frac{n^q}{D_n}\right) + \frac{C}{D_n} O\left(\frac{2^n}{1.9^{2n}}\right) = O\left(\frac{n^q}{D_n}\right)
\end{aligned}$$

whence the result. □

Lemma 4.4. *Assume that $|\log r_n| = O(n^q)$ and let ψ be any increasing function, defined in a neighborhood of $0 \in \mathbb{R}^+$, and such that*

$$\psi\left(\frac{1}{D_{n+1}M_{n+1}}\right) \geq \frac{1 + 2Cn^q}{D_n}, \text{ for large enough } n, \tag{9}$$

where C is as in the previous lemma. Then if u satisfies (8), it has a modulus of continuity $O(\psi)$.

Proof. Let w and w' be close to each other and let n be the integer such that

$$\frac{1}{D_{n+1}M_{n+1}} \leq \left|w - w'\right| < \frac{1}{D_n M_n}.$$

Then we have

$$\left|u(w) - u(w')\right| \leq M_n \left|w - w'\right| + 2\sum_{k=n}^{\infty} \frac{1}{D_k} \left|\log r_k\right| \leq \frac{1 + Cn^q}{D_n} \leq \psi(\left|w - w'\right|).$$

Since this holds only when w and w' are close enough, we conclude that in general the modulus of continuity is $O(\psi)$. □

Recall that $D_n = \prod_0^n d_j$ and $M_n = \prod_1^n 11(r_j)^{-1}$. We put $m_n = 11r_n^{-1}$ so that $M_n = \prod_1^n m_j$.

Lemma 4.5. *Let* $0 < \alpha < 1$ *and let* (r_n) *be a sequence decreasing to zero, satisfying*

$$|\log r_n| = O(n^q) \text{ and } |\log r_n| = o\left(\sum_{j=1}^{n} |\log r_j|\right). \tag{10}$$

Then there exists a sequence d_n *such that* (9) *holds with* $\psi(r) = r^\alpha$.

The second requirement in (10) is likely to be superfluous since it is usually true when $|\log r_n|$ has subexponential growth.

Proof. If we put $\psi(r) = r^\alpha$, (9) is equivalent to

$$d_{n+1} \leq \frac{D_n^{\frac{1-\alpha}{\alpha}}}{M_{n+1}} \frac{1}{(1+2Cn^q)^{\frac{1}{\alpha}}}. \tag{11}$$

Fix $t > \frac{\alpha}{1-\alpha}$ and put $d_n = \lfloor (m_n)^t \rfloor + 1$, so that $D_n \geq M_n^t$, $D_n \approx M_n^t$ and $d_n \approx m_n^t$. Then for n large enough we have that

$$d_{n+1} \leq m_n^t + 1 \leq M_n^{\frac{t(1-\alpha)}{\alpha} - 1} \frac{1}{(1+2Cn^q)^{\frac{1}{\alpha}}} \leq \frac{D_n^{\frac{1-\alpha}{\alpha}}}{M_{n+1}} \frac{1}{(1+2Cn^q)^{\frac{1}{\alpha}}}$$

where the middle inequality follows from the fact that $\log m_n = o\left(\sum_{j=1}^n \log m_j\right)$ and $\frac{t(1-\alpha)}{\alpha} - 1 > 0$ (the $(1+2Cn^q)$ term has no influence because M_n has superexponential growth). This finishes the proof. □

We can now conclude the proof of Theorem 4.1. Of course what is mostly interesting is to understand the modulus of continuity in $D(0, \frac{1}{4}) \times \mathbb{D}$. Consider $p_i = (z_i, w_i)$, $i = 1, 2$ and assume that $|z_i| \leq \frac{3}{8}$, in which case the oblique lines $z + \frac{1}{10} w = z_i + \frac{1}{10} w_i$ are vertically contained in $D(0, \frac{1}{2}) \times \mathbb{D}$. If p_1 and p_2 are close enough to each other, the intersection point $p_3 = (z_2, w_1 + 10(z_1 - z_2))$ between $z + \frac{1}{10} w = z_1 + \frac{1}{10} w_1$ and $z = z_2$ belongs to $\overline{D}(0, \frac{3}{8}) \times \mathbb{D}$. For $i = 1, 2$ we observe that $\|p_i - p_3\| \leq 10 \|p_1 - p_2\|$. Choose sequences (r_n) and (d_n) such that so that Lemma 4.5 holds $\psi(r) = r^\alpha$. By Lemma 4.4, we then conclude that $|u(p_1) - u(p_2)| = O(\|p_1 - p_2\|^\alpha)$.

To estimate the modulus of continuity near the boundary of the bidisk, we have two possibilities: either we increase the constant 10 and work along lines of the form $z + \frac{1}{A} w = cst$ or we take advantage of the uniform laminarity of T near the boundary. Let us give some indications on the latter approach.

Let the p_i be such that $|z_i| > \frac{3}{8}$. Recall that T is uniformly laminar in $\left\{\frac{3}{8} < |z| < \frac{1}{2}\right\}$, with graphs as leaves. In [Du1, Lemma 6.4] we gave an estimate of the modulus of continuity of the potential of such a uniformly laminar current in terms of its modulus of continuity on vertical slices. From this estimate we conclude that if the modulus of continuity along vertical lines can be chosen to be Hölder with exponent arbitrary close to 1, then the same holds for the modulus of continuity of the current itself, which implies the result we seek. The details are left to the reader. □

In the next theorem we show that the modulus of continuity can be even closer to Lipschitz.

Theorem 4.6. *Let $0 < \alpha < 1$ be arbitrary. Let P_n, δ_n, T and u be as above. Fix real numbers $\alpha > 0$ and $1/2 < \beta < 1$. Define the sequences (d_n) and (r_n) as follows: let $1 < s < t < \frac{\beta}{1-\beta}$, $d_n = \min(\lfloor \exp(n^t) \rfloor, 2)$ and $r_n = \max\left(\exp(-n^{s-1}), \frac{1}{10}\right)$.*

Then u has modulus of continuity $O(\psi)$ in $D(0,1/4) \times \mathbb{D}$, with $\psi(r) = r\exp(\alpha\,|\log r|^\beta)$.

Of course the case $\beta = 1$ would correspond to Hölder continuity. A notable consequence of the estimate on the modulus of continuity is that $\dim_H(\operatorname{Supp} T) \geq 3$. Indeed it is well known that a current with Hölder continuous potential of exponent α cannot carry any mass on sets of dimension $< 2 + \alpha$.

The proof is identical to that of Theorem 4.1, except for Lemma 4.5, which has to be replaced by the following:

Lemma 4.7. *If (d_n) and (r_n) are defined as in the statement of Theorem 4.6, then (9) holds with $\psi(r) = r\exp(\alpha\,|\log r|^\beta)$.*

Proof. Observe first that $\log d_n = n^t + o(1)$ and $|\log r_n| = O(n^{s-1})$ so that the estimate of Lemma 4.3 holds. With ψ as above, (9) is equivalent to

$$d_{n+1}M_{n+1} \leq \frac{\exp\left(\alpha\,|\log D_{n+1} + \log M_{n+1}|^\beta\right)}{1 + 2Cn^{s-1}},$$

that is,

$$\log d_{n+1} + \log M_{n+1} \leq \alpha \left|\sum_{k=1}^{n+1} \log d_k + \log M_{n+1}\right|^\beta - \log(1 + 2Cn^{s-1}). \tag{12}$$

Since $\sum_1^n k^{s-1} \sim \frac{n^s}{s} \leq n^s$, with our definition of r_n we have that $\log M_n = \sum_1^n \log\frac{11}{r_k} = o(n^t)$, so the left hand side of (12) is $n^t + o(n^t)$. On the other hand the right hand side is $\frac{\alpha}{(t+1)^\beta} n^{(t+1)\beta} + o(n^{t+1})$, and since $t < (t+1)\beta$ we conclude that (12) indeed holds. □

5. When $d_n = 2$ for all n

Theorem 5.1. *Let P_n, δ_n, T and u be as defined in Sections 2 and 3. Assume that $d_n = 2$ for all n. Let ψ be a real increasing function, defined in a neighborhood of $0 \in \mathbb{R}^+$, with $\psi(0) = 0$ and such that*

$$\lim_{r \to 0} \frac{\log \psi(r)}{\log r} = 0. \tag{13}$$

Then there exists a sequence (r_n) so that u has a modulus of continuity $O(\psi)$ in $D(0, \frac{1}{4}) \times \mathbb{D}$.

Observe that if $\log\psi(r)/\log r$ converges to $\alpha > 0$ as $r \to 0$, then $\psi = O(r^\beta)$ for every $\beta < \alpha$, so that (13) corresponds to a sub-Hölder modulus of continuity. Observe also that the slower $\frac{\log\psi(r)}{\log r}$ converges to zero, the better the modulus of continuity is, so the result is mostly interesting when $\frac{\log\psi(r)}{\log r}$ converges to zero slowly. In the course of the proof, we will actually be led to to replace it with a slower function.

It will be enough to restrict to slowly decreasing sequences (r_n), so from now on we assume that $r_n \geq \frac{1}{10n}$ for all $n \geq 1$.

Let us recollect a few facts. Of course now $D_n = 2^{n+1}$, so (8) rewrites as

$$\big|u(w) - u(w')\big| \le M_n \left|w - w'\right| + 2\sum_{j=n}^{\infty} \frac{1}{2^j} \left|\log r_j\right|. \tag{14}$$

for all n, with $M_n = \prod_{j=1}^n m_j = \prod_{j=1}^n 11(r_j)^{-1}$. Since $r_n \ge 1/10n$, we have the following easy estimate on the second term on the right hand side

$$\sum_{j=n}^{\infty} \frac{1}{2^j} \left|\log r_j\right| \le \sum_{j=n}^{\infty} \frac{|\log j|}{2^j} = \sum_{j=n}^{2n} \frac{|\log j|}{2^j} + \sum_{j=2n}^{\infty} \frac{|\log j|}{2^j} \le (\log n + \log 2) \sum_{j=n}^{2n} \frac{1}{2^j} + \sum_{j=2n}^{\infty} \frac{1}{1.9^j}$$
$$\le \frac{\log n}{2^{n-1}} + \frac{2}{2^{n-1}} \le \frac{3\log n}{2^n},$$

where the inequalities hold only for large enough n (e.g. $n \ge 100$ will do). We conclude that for large n,

$$\big|u(w) - u(w')\big| \le M_n \left|w - w'\right| + \frac{6\log n}{2^n}. \tag{15}$$

Lemma 4.4 may now be restated as follows.

Lemma 5.2. *Let ψ be any increasing function, defined in a neighborhood of $0 \in \mathbb{R}^+$, and such that for large enough n*

$$\psi\left(\frac{1}{2^n M_n}\right) \ge \left(2\frac{M_n}{M_{n-1}} + 6\log n\right)\frac{1}{2^n}. \tag{16}$$

Then if u satisfies (15), it has a modulus of continuity $O(\psi)$.

When $M_n = A^n$ the lemma is classical and its conclusion is that u is Hölder continuous. In our case M_n has superexponential growth and the game will be to choose (m_n) slow enough so that ψ can be chosen arbitrary close to Hölder.

Proof. Let w and w' be close to each other and let n be the integer such that

$$\frac{1}{2^n M_n} \le \left|w - w'\right| < \frac{1}{2^{n-1} M_{n-1}}.$$

Then we have

$$\big|u(w) - u(w')\big| \le \frac{2M_n}{M_{n-1}}\frac{1}{2^n} + \frac{6\log n}{2^n} \le \psi(\left|w - w'\right|).$$

Since this holds only when w and w' are close enough, we conclude that in general the modulus of continuity is $O(\psi)$. □

Lemma 5.3. *Let ψ be an increasing function, defined in a neighborhood of $0 \in \mathbb{R}^+$, and satisfying the assumption (13) of Theorem 5.1. Then there exists $\widetilde{\psi} \le \psi$ satisfying (13) and a sequence of radii r_j tending to zero such that (16) holds for $\widetilde{\psi}$.*

Proof. Let $\alpha(r) = \log\psi(r)/\log r$ so that $\psi(r) = r^{\alpha(r)}$. Consider the reciprocal ψ^{-1} of ψ and write it as $\psi^{-1}(r) = r^{\beta(r)}$. It is easy to check that β tends to infinity as $r \to 0$, and that

$$\psi_1 \le \psi_2 \text{ iff } \alpha_1 \ge \alpha_2 \text{ iff } \beta_1 \le \beta_2.$$

Fix a real number $\theta \in (0,1)$, and recall that $M_n = \prod_1^n m_j = \prod_1^n 11(r_j)^{-1}$. We want to find ψ and a sequence (m_j) tending to infinity, with

$$\psi\left(\frac{1}{2^n M_n}\right) = \frac{1}{(2-\theta)^n} \text{ and } m_n = \frac{M_n}{M_{n-1}} = o\left(\frac{2^n}{(2-\theta)^n}\right), \tag{17}$$

so that in particular (16) holds. For this we have to modify ψ a little bit. Write $\psi^{-1}(r) = r^{\beta(r)}$. We easily obtain that for all n,

$$\psi\left(\frac{1}{2^n M_n}\right) = \frac{1}{(2-\theta)^n} \Leftrightarrow M_n = \left(2^n \psi^{-1}((2-\theta)^{-n})\right)^{-1}$$
$$\Leftrightarrow m_n = 2(2-\theta)^{(n-1)(\beta((2-\theta)^n)-\beta((2-\theta)^{n-1}))+\beta((2-\theta)^{n-1})}.$$

From this equation we see that $m_n \to \infty$ (indeed $\beta((2-\theta)^{n-1})$ does). To meet the second requirement in (17), we use the following elementary lemma, whose proof is left to the reader.

Lemma 5.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $\lim_{+\infty} f = +\infty$. Then there exists a function $\widetilde{f}$ with the same properties, and such that moreover: f is derivable and increasing, $\widetilde{f} \leq f$, $\widetilde{f}(x) = o(x)$ and $\lim_{+\infty} \widetilde{f}' = 0$.*

Apply this lemma to $f(s) = \beta((2-\theta)^s)$, thus obtaining a certain $\widetilde{f}$, and consider $\widetilde{\beta}$ such that $\widetilde{f}(s) = \widetilde{\beta}((2-\theta)^s)$. In particular $\widetilde{\beta}((2-\theta)^n) - \widetilde{\beta}((2-\theta)^{n-1}) \to 0$ and $\widetilde{\beta}((2-\theta)^n) = o(n)$.

Notice that $r^{\widetilde{\beta}(r)}$ is increasing, so this modification gives rise to a function $\widetilde{\psi} \leq \psi$ satisfying (13) defined by $\widetilde{\psi}^{-1}(r) = r^{\widetilde{\beta}(r)}$. Replacing ψ by $\widetilde{\psi}$ in the above computation then gives us a sequence m_n such that (17) holds with $\widetilde{\psi}$ instead of ψ. This finishes the proof. □

The proof of Theorem 5.1 is now exactly the same as that of Theorem 4.1. Let $p_i = (z_i, w_i)$, $i = 1, 2$ with $|z_i| \leq \frac{3}{8}$, and $p_3 \in \overline{D}(0, \frac{3}{8}) \times \mathbb{D}$ be the intersection point $p_3 = (z_2, w_1 + 10(z_1 - z_2))$ between $z + \frac{1}{10}w = z_1 + \frac{1}{10}w_1$ and $z = z_2$. We have that $\|p_i - p_3\| \leq 10\,\|p_1 - p_2\|$.

Let ψ be as in the statement of Theorem 5.1 and $\psi_0 = \frac{1}{2}\psi(\cdot/10)$. Choose now a sequence of radii r_j so that Lemma 5.3 holds for ψ_0. By Lemma 5.2, we then infer that $|u(p_1) - u(p_2)| = O(\widetilde{\psi}_0(10\,\|p_1 - p_2\|)) = O(\psi(\|p_1 - p_2\|))$. □

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