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Some remarks on the connectivity of Julia sets for 2-dimensional diffeomorphisms.

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Abstract. We explore the connected/disconnected dichotomy for the Julia set of polynomial automorphisms of $\mathbb{C}^2$. We develop several aspects of the question, which was first studied by E. Bedford and J. Smillie [**BS6, BS7**]. We introduce a new sufficient condition for the connectivity of the Julia set, that carries over for certain Hénon-like and birational maps. We study the structure of disconnected Julia sets and the associated invariant currents. This provides a simple approach to some results of Bedford-Smillie, as well as some new corollaries –the connectedness locus is closed, construction of external rays in the general case, etc.

We also prove the following theorem: a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ with connected Julia set must have attracting or repelling orbits. This is an analogue of a well known result in one dimensional dynamics.

Introduction

If $p : \mathbb{C} \to \mathbb{C}$ is a polynomial map, there exists a well known necessary and sufficient condition for the Julia set J_p (or equivalently the filled in Julia set K_p) to be connected. Namely J_p is connected iff all critical points have bounded orbits. In case p is hyperbolic, this is equivalent to saying that all critical points are attracted to periodic sinks (see [**CG**] for a general account).

If now $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial diffeomorphism, the Julia set has several analogues. Let K^+ (resp. K^-) be the set of points with bounded forward (resp. backward) orbit. Let $J^\pm = \partial K^\pm$. It is known that J^+ is the Julia set of f in the usual sense, for forward iteration. Let also $J = J^+ \cap J^-$ and J^* be the closure of the set of saddle orbits ($J^* \subset J$).

The sets $J^\pm$ are always connected. Indeed there exist invariant (1,1) currents $T^\pm$ with $\mathrm{Supp}(T^\pm) = J^\pm$, and with the additional property of being extremal as positive closed currents. Extremality easily implies that the sets $J^\pm = \mathrm{Supp}(T^\pm)$ are connected. On the other hand, the sets $J^\pm$ are not locally connected in general,

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even if f is hyperbolic on J. Indeed $J^{\pm}$ have laminar structure and, for instance if $f(z,w) = (aw + z^2 + c, az)$, with small a and large c, J^+ has totally disconnected transversals. We shall see later on that the connectivity properties of these transversals has to do with the connectivity of J itself.

Eric Bedford and John Smillie gave in [**BS6**] a necessary and sufficient condition for J to be connected. Replacing f by f^{-1} if necessary we may assume the (constant) Jacobian determinant $|\mathrm{Jac}\, f|$ is not larger than 1.

DEFINITION 0.1. f is *unstably connected* if for some saddle point p, $W^u(p) \cap K^+$ has no compact component (for the topology induced by the isomorphism $W^u(p) \simeq \mathbb{C}$).

An important issue in [**BS6**] is that this condition is independent of p.

THEOREM 0.2 ([**BS6**]). *If f is a polynomial diffeomorphism with $|\mathrm{Jac}\, f| \leq 1$, then*

$$J \textit{ is connected } \Leftrightarrow f \textit{ is unstably connected.}$$

A corollary, which was not realized there is that the connectedness locus is closed in parameter space (corollary 2.2).

There is also a notion of unstable critical point, which plays the role of an escaping critical point in one dimensional dynamics. Another result in [**BS6**] is

$$J \text{ is connected } \Leftrightarrow f \text{ has no unstable critical points.}$$

A salient feature in that paper is that if f is unstably connected, then $J^- \setminus K^+$ has the structure of a Riemann surface lamination. This allows to define external rays and study how landing of these influence the structure of J.

In this article, we give a sufficient condition for the connectedness of the Julia set of a polynomial diffeomorphism of $\mathbb{C}^2$. This condition, which is not directly related to unstable connectedness, was considered in the context of hyperbolic maps in [**BS7**].

We say K^+ is *transversely connected* if there exists a holomorphic disk V, with $\partial V \cap K^+ = \emptyset$ and $V \cap K^+ \neq \emptyset$ (a transversal to K^+) such that $V \cap K^+$ is connected. We prove (§1) that transversal connectedness implies J is connected. One interest of this result is that it also provides a simpler approach to the lamination structure of $J^- \setminus K^+$ and clarifies somehow the analogy between unstable and escaping critical points. Also the result is valid for certain Hénon-like and birational maps. Notice that images of $\mathbb{C}$ cannot be used in the context of Hénon-like maps, so the approach of [**BS6**] has to be modified.

In §2 we study the structure of the set J^- for an unstably disconnected polynomial diffeomorphism. We prove that the unstable current T^- is an integral of *closed* submanifolds in a large bidisk $\mathbb{B}$ containing K. This is strictly stronger than just being a laminar current in the sense of [**BLS**], but does not imply uniform laminarity. The obstruction to uniform laminarity is the phenomenon of *folding*, as considered in [**BS8**].

As a consequence, we can define external rays in this case. We prove that almost every ray lands and the landing measure is the maximal entropy measure.

Another corollary (Proposition 1.10 and corollary 2.5) is that if $\mathbb{B}$ is a sufficiently large bidisk,

$$f \text{ is unstably connected } \Leftrightarrow T^-|_{\mathbb{B}} \text{ is an extremal current.}$$

We study the expansion/contraction of the Poincaré metric along the leaves in this case, and derive some corollaries. This paragraph may also be seen as a gentle introduction to the notion of *quasi-expansion* [**BS8**].

In §3 we turn to hyperbolic polynomial diffeomorphisms. It is known that if f is hyperbolic on J then $\mathrm{Int}(K^+)$ is the union of finitely many sink basins. We prove that if f is hyperbolic and unstably connected (or equivalently J is connected and $|\mathrm{Jac} f| \leq 1$) then $\mathrm{Int}(K^+)$ is non empty, i.e. f does have attracting periodic orbits. This is analogous to the one dimensional case. This provides an alternate proof of the following fact [**BS7**, Corollary A.3]: the Julia set of a conservative and hyperbolic polynomial diffeomorphism cannot be connected.

1. Maps with connected Julia sets

1.1. Preliminaries on Hénon-like mappings. Our treatment of connectivity for polynomial diffeomorphisms was partly motivated by extending the results of [**BS6**] to the Hénon-like context. In particular we will use the Hénon-like formalism, even when considering Hénon maps.

We begin with some notation: let $\mathbb{B}$ be a bounded open set in $\mathbb{C}^2$ biholomorphic to a bidisk. We fix an isomorphism with the unit bidisk, and let $\partial_v\mathbb{B}$ (resp. $\partial_h\mathbb{B}$) be the "vertical" (resp. "horizontal") part of the boundary, $\partial_v\mathbb{B} = \{|z| = 1, |w| < 1\}$ (resp. $\partial_h\mathbb{B} = \{|z| < 1, |w| = 1\}$).

Definition 1.1. A Hénon-like mapping in $\mathbb{B}$ is an injective holomorphic map $f : N(\overline{\mathbb{B}}) \to \mathbb{C}^2$ (where $N(\overline{\mathbb{B}})$ is a neighborhood of $\overline{\mathbb{B}}$) such that $f(\mathbb{B}) \cap \mathbb{B} \neq \emptyset$, and satisfying:

i. $f(\partial_v\mathbb{B}) \cap \overline{\mathbb{B}} = \emptyset$;
ii. $f(\overline{\mathbb{B}}) \cap \partial\mathbb{B} \subset \partial_v\mathbb{B}$.

Notice that the source and target bidisks may differ; this was the case in the original definition of J. Hubbard and R. Oberste-Vorth [**HO2**]. A fundamental example of Hénon-like map is any polynomial automorphism of $\mathbb{C}^2$, while considered in a suitable large bidisk $\mathbb{B}$. Indeed it is known that there exists a system of coordinates (z, w) so that f is a composition of Hénon maps $f_j(z, w) = (aw + p_j(z), z)$, and for every large enough R, the restriction of f to the bidisk $\mathbb{B} = D(0, R)^2$ is Hénon-like. Moreover the nonwandering set is contained in $\mathbb{B}$ because as it is well known $\overline{V^+} = \left\{(z, w) \in \mathbb{C}^2,\ |z| \geq |w|,\ |z| \geq R\right\}$ lies inside the basin of attraction of a superattracting point at infinity for f. Respectively there is a corresponding $\overline{V^-}$ attracted to infinity by f^{-1}. Here f is naturally extended to a rational map of the projective plane.

Example 1.2. Perturbing such a map provides many examples of Hénon-like maps. For instance consider the polynomial birational map of $\mathbb{C}^2$ defined by

$$g(z, w) = ((a + b(z))w + p(z), az)$$

where $|a| \leq 1$, and $b(z)$ is a polynomial with $\deg(b(z)) \leq \deg(p(z)) - 2$. If the norm of b as a polynomial is small enough, g is a small perturbation of the Hénon map $(aw + p(z), az)$, so the restriction of g to $\mathbb{B}$ is Hénon-like . We will sketch a proof of the fact that nonwandering dynamics occurs only in $\mathbb{B}$. For ease of reading we restrict to the case where $a = 1$ (see Guedj [?] for similar computations with $\deg(p) = 2$ and $\deg(b) = 1$).

We consider g as a birational self-mapping of the projective plane $\mathbb{P}^2$ (where we use the homogeneous coordinates $[z : w : t]$). Simple explicit computations show that:

- The indeterminacy locus of g is $I^+ = [0 : 1 : 0]$ and the indeterminacy locus of g^{-1} is
$$I^- = \{[1 : 0 : 0]\} \cup \{[u_i : v_i : 1]\}$$
where the complex numbers v_i are the roots of $b(z) + 1$ and $u_i = p(v_i)$. Notive that $[u_i : v_i : 1] \in V^+$ if $b(z)$ has small norm.
- The point I^- is superattracting for g.
- If $\deg(b) \leq \deg(p) - 3$, the point $[1 : 0 : 0]$ is superattracting for g^{-1}. If $\deg(b) = \deg(p) - 2$ it is attracting if the leading coefficient of b is less than 1 in modulus.

In particular if b has small norm, $\overline{V^+}$ is contained in the basin of attraction of $[1 : 0 : 0]$. Indeed as a rational map on $\mathbb{P}^2$, g is a small perturbation of f, and I^+ is far from V^+. Similarly, $\overline{V^-}$ is in the basin of I^+ for g^{-1}. We conclude that every point outside $\mathbb{B}$ is wandering.

We call *horizontal* an object (form, current, subvariety) whose support stays (uniformly) away from the horizontal boundary $\partial^h\mathbb{B}$. Vertical currents, as well as horizontal and vertical submanifolds, are defined analogously. The *degree* of a horizontal subvariety V is by definition the number of intersection points of V with a generic vertical line. By definition, if L is a horizontal line, $d = \deg(f(L))$ is the degree of f. More generally if V is a horizontal submanifold of degree $\deg(V)$, $f(V) \cap \mathbb{B}$ is a horizontal submanifold of degree $d \deg(V)$.

There is a corresponding notion of degree for a horizontal positive closed currents T. The degree is defined as the mass of the intersection (wedge product) of T with a generic vertical line. A *normalized* current is a current of degree 1.

We now list some dynamical properties of Hénon-like mappings [**Du1**]:

Invariant currents: f acts by push forward on horizontal positive closed currents. Let $\mathcal{L} = 1_{\mathbb{B}} \frac{1}{d} f_*$ be the associated graph transform operator. If T is any horizontal normalized positive closed current, the sequence $\mathcal{L}^n T$ converges to the unique normalized $\mathcal{L}$ invariant current T^-. Moreover T^- has laminar structure and continuous potential. Similar results hold for pull backs of vertical currents.

Invariant measure: $\mu = T^+ \wedge T^-$ is the unique measure of maximal entropy $\log d$. It is mixing, hyperbolic, and describes the asymptotic distribution of periodic orbits.

If we let $K^\pm = \{x \in \mathbb{B},\ \forall n \geq 0\ f^{\pm n}(x) \in \mathbb{B}\} = \bigcap_{n \geq 0} f^{\mp n}(\mathbb{B})$ and $J^\pm = \partial K^\pm$, an interesting open question is whether $\mathrm{Supp}(T^\pm) = J^\pm$. Equality holds for polynomial diffeomorphisms of $\mathbb{C}^2$.

For the birational perturbations of Hénon maps considered above, the equality $J^+ = \mathrm{Supp}(T^+)$ is true. This is an easy consequence of the existence of the rate of escape function g^+, which is psh, nonnegative, continuous, and such that $dd^c g^+ =$

T^+ and $K^+ = \{g^+ = 0\}$. Here K^+ is the set of points with bounded orbits, and agrees inside $\mathbb{B}$ with the previously defined K^+ (see the discussion in example **??**).

1.2. A result on the intersection of positive closed currents. We begin with a result on the support of wedge product of positive closed currents, which has some independent interest. It is a very flexible generalization of [**BS3**, Proposition 2.3] and [**BS6**, Lemma 5.4].

THEOREM 1.3. *Let $\Omega \subset \mathbb{C}^2$ be a bounded open set, and $T_1 = dd^c u_1$, $T_2 = dd^c u_2$ be positive closed currents in Ω such that the wedge product $T_1 \wedge T_2$ is admissible. We moreover assume that*

$$\emptyset \neq \mathrm{Supp}(T_1) \cap \mathrm{Supp}(T_2) \subset\subset \Omega. \tag{1}$$

Then $\displaystyle\int_\Omega T_1 \wedge T_2 > 0$.

Remark that if Ω is exhausted by pseudoconvex open sets, hypothesis (1) implies that the wedge product $T_1 \wedge T_2$ is well defined (see Sibony [**Sib**]). The theorem has the following corollary.

COROLLARY 1.4. *If f is a Hénon-like map,* $\mathrm{Supp}(\mu)$ *intersects every connected component of* $\mathrm{Supp}(T^+) \cap \mathrm{Supp}(T^-)$.

PROOF. We may assume Ω is connected. Let χ be a test function with $\chi = 1$ in a neighborhood of $\mathrm{Supp}(T_1) \cap \mathrm{Supp}(T_2)$ and $\mathrm{Supp}(d\chi) \cap \mathrm{Supp}(T_1) \cap \mathrm{Supp}(T_2) = \emptyset$. Then

$$\int_\Omega T_1 \wedge T_2 = \int \chi dd^c u_1 \wedge dd^c u_2 = \int u_1 dd^c \chi \wedge dd^c u_2 < \infty \tag{2}$$

since u_1 is pluriharmonic near $\mathrm{Supp}(dd^c\chi) \cap \mathrm{Supp}(T_2)$. Let $T_1^\varepsilon = dd^c u_1^\varepsilon$ be the standard regularization of T_1. u_1^ε is obtained by the convolution of u_1 with a radial approximation of δ_0, with support in $B(0, \varepsilon)$. Reducing Ω slightly if necessary we may assume T_1^ε is well defined on Ω.

Now $u_1 - u_1^\varepsilon = 0$ in the set $\{p \in \Omega, d(p, \mathrm{Supp}(T_1)) > \varepsilon\}$. In particular if ε is small enough,$u_1 - u_1^\varepsilon = 0$ on $\mathrm{Supp}(dd^c\chi) \cap \mathrm{Supp}(T_2)$ and by (2) we get that $\int T_1 \wedge T_2 = \int T_1^\varepsilon \wedge T_2$. We do the same for T_2, hence $\int T_1 \wedge T_2 = \int T_1^\varepsilon \wedge T_2^\varepsilon$.

The next step is to prove that for small $\varepsilon > 0$, $\mathrm{Supp}(T_1^\varepsilon) \cap \mathrm{Supp}(T_2^\varepsilon) \neq \emptyset$ (of course $\mathrm{Supp}(T_1^\varepsilon) \cap \mathrm{Supp}(T_2^\varepsilon) \subset\subset \Omega$). Once again it is enough to show that $\mathrm{Supp}(T_1^\varepsilon) \cap \mathrm{Supp}(T_2) \neq \emptyset$. First, it is classical that there exists a nonnegative plurisubharmonic function v in $\Omega \setminus \mathrm{Supp}(T_2)$ tending to $+\infty$ on $\mathrm{Supp}(T_2)$. Indeed for every $\varepsilon > 0$, the function $u_2^\varepsilon - u_2$ is psh, nonnegative on $\Omega \setminus \mathrm{Supp}(T_2)$, and positive in the neighborhood of $\mathrm{Supp}(T_2)$. Pick a decreasing sequence $\varepsilon_k \to 0$ and just take

$$v = \sum_{k=0}^{\infty} C_k (u_2^{\varepsilon_k} - u_2),$$

where the constants C_k are adjusted so that $v \to +\infty$ on $\mathrm{Supp}(T_2)$ (the sum is locally finite in $\Omega \setminus \mathrm{Supp}(T_2)$).

Assume next that for every $\varepsilon > 0$, $\mathrm{Supp}(T_1^\varepsilon) \cap \mathrm{Supp}(T_2) = \emptyset$. We will use a Kontinuätzsatz-type argument. There exists a constant c such that in a neighborhood N of $\partial\Omega$, $\mathrm{Supp}(T_1) \cap N \subset \{v \leq c\}$. This also holds for T_1^ε for small ε, with a fixed c. Now if $\mathrm{Supp}(T_1^\varepsilon)\cap\mathrm{Supp}(T_2)=\emptyset$, v is well defined on $\mathrm{Supp}(T_1^\varepsilon)$ and by the maximum principle (see below), $\mathrm{Supp}(T_1^\varepsilon)\subset\{v\leq c\}$. Letting $\varepsilon\to 0$, we get a contradiction.

The maximum principle on a positive closed current is classical. The proof goes as follows: assume w is a smooth strictly psh function in U, such that $w<0$ on $\partial U\cap\mathrm{Supp}(T)$, and $w>0$ at some point $p\in\mathrm{Supp}(T)\cap U$. Then $w^+=\max(w,0)$ vanishes near $\mathrm{Supp}(T)\cap\partial U$ and $w^+=w$ near p. Then

$$0<\int_U dd^c w^+\wedge T=\int w^+\wedge dd^cT=0;$$

a contradiction.

So from now on we may assume that we are in the conditions of the theorem, with smooth T_1 and T_2. Smoothness is not enough to ensure $\int T_1\wedge T_2>0$, since T_1 and T_2 might be only semipositive on $\mathrm{Supp}(T_1)\cap\mathrm{Supp}(T_2)$. So assume T_1 and T_2 are smooth, and pick $p\in\mathrm{Supp}(T_1)\cap\mathrm{Supp}(T_2)$. Consider a small connected neighborhood N of id in the group SU_2 of rotations around p, and the current

$$T_1'=\int_N\theta_*T_1d\theta$$

where $d\theta$ is the normalized Haar measure in N. Elementary linear algebra shows that T_1' is strictly positive at p, so $\int T_1'\wedge T_2>0$.

It remains to prove that $\int T_1'\wedge T_2=\int T_1\wedge T_2$. For every rotation θ in N, θ is connected to the identity by a path in N, so θ_*T_1 is homotopic to T_1. The homotopy formula (see e.g. Simon [**Sim**]) asserts that

$$\theta_*T_1-T_1=dh_*(T_1\otimes[0,1])+h_*(dT_1\otimes[0,1]),$$

where $h(\cdot,t)$ is the homotopy connecting id and θ, and $T_1=T_1\mathbf{1}_\Omega$ is viewed as a current with boundary in the neighborhood of $\overline{\Omega}$. We infer that

$$\int\chi(\theta_*T_1-T_1)\wedge T_2=\int\chi dh_*(T\otimes[0,1])\wedge T_2+\int\chi h_*(dT_1\otimes[0,1])\wedge T_2.$$

The first integral on the right hand side vanishes because $d\chi=0$ near the support of $(h_*(T_1\otimes[0,1])\wedge T_2)$ and the second because $\chi=0$ near $\mathrm{Supp}\,(h_*(dT_1\otimes[0,1]))$. $\square$

1.3. Polynomial automorphisms. We begin the discussion on connectivity by considering transverse connectivity for polynomial automorphisms of $\mathbb{C}^2$. Fix a polynomial diffeomorphism f of degree d in $\mathbb{C}^2$ with non trivial dynamics. We will prove transversal connectedness of K^+ implies J is connected. We also directly recover the lamination structure of $J^-\backslash K^+$, and its ergodicity.

Definition 1.5. A *transversal* to K^+ is a holomorphic disk V such that $\partial V\cap K^+=\emptyset$ and $V\cap K^+\neq\emptyset$. K^+ is said to be transversely connected if there exists a transversal V such that $K^+\cap V$ is connected.

Notice that if $K^+\cap V$ has an isolated connected component (e.g. if $K^+\cap V$ has finitely many components) then K^+ is transversely connected. In [**BS7**] it was proved that if f is uniformly hyperbolic then K^+ is transversely connected iff J is connected. Here we extend the "only if" statement to the general case.

Theorem 1.6. *Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ and assume K^+ is transversely connected. Then the Julia set J is connected. Moreover $J^-\setminus K^+$ supports a unique Riemann surface lamination which is uniquely ergodic.*

The proof will proceed in several steps. Let $\mathbb{B} = D(0,R)^2$ be a large bidisk; $f|_{\mathbb{B}}$ is a Hénon-like map. Recall the open subset $V^+ = \left\{(z,w) \in \mathbb{C}^2,\ |z| > |w|\,,\ |z| > R\right\}$. It is classical that $f(\mathbb{B}) \subset \overline{\mathbb{B}} \cup V^+$ and $f(V^+) \subset V^+$.

Proof. We fix a transversal V such that $V \cap K^+$ is connected. Since $\partial V \cap K^+ = \emptyset$, ∂V escapes under iteration. So there exists $n_0 \geq 1$ such that $f^{n_0}(V) \cap \mathbb{B}$ is a horizontal submanifold in $\mathbb{B}$. Since $f^n(V \cap K^+) = f^n(V) \cap K^+$ is connected, by replacing V by $f^n(V) \cap \mathbb{B}$, we may assume V is a horizontal submanifold in $\mathbb{B}$ (i.e. $\partial V \subset \partial^v \mathbb{B}$), of degree $\deg(V)$. Here "submanifold" means a complex submanifold without boundary.

Step 1. If V is as above, then for every $n \geq 1$, $f^n(V) \cap \mathbb{B}$ is connected.

Let $\mathbb{B}_{-n} = \mathbb{B} \cap f^{-1}(\mathbb{B}) \cap \cdots \cap f^{-n}(\mathbb{B})$; from the fact that $f(V^+) \subset V^+$ we deduce that $f^n(V) \cap \mathbb{B} = f^n(V \cap \mathbb{B}_{-n})$. It then suffices to prove that for every n, $V \cap \mathbb{B}_{-n}$ is connected. For this, we just remark that K^+ intersects every connected component of $V \cap \mathbb{B}_{-n}$. Indeed, if U is such a component, then U is a (boundaryless) submanifold of $\mathbb{B}_{-n}$. So $f^n(U)$ is a non trivial horizontal submanifold in $\mathbb{B}$. It then follows from the Stokes theorem and degree considerations that $\int T^+ \wedge [f^n(U)] > 0$ (see e.g. [**Du1**, Prop. 2.7] or theorem 1.2), so $f^n(U)$ intersects K^+.

Step 2. The laminar structure of $J^- \setminus K^+$.

Recall first the *Riemann-Hurwitz formula*. Let Δ is a simply connected horizontal submanifold in $\mathbb{B}$ of degree δ (so Δ is a union of k holomorphic disks), and π denote the first projection $(z,w) \mapsto z$. Then the number of critical points of $\pi|_\Delta$ (vertical tangencies), counted with multiplicity, equals $\delta - k$.

Consider the transversal V as before; we have seen that that for $n \geq 1$, $f^n(V) \cap \mathbb{B}$ is a connected horizontal submanifold. Thus $f^n(V) \cap \mathbb{B} = f^n(V \cap \mathbb{B}_{-n})$ is a planar surface, and the maximum principle actually implies it is a disk. So by the Riemann-Hurwitz formula the number of vertical tangencies on $f^n(V) \cap \mathbb{B}$ equals $d^n \deg(V) - 1$.

Now increase R to get a larger bidisk $\mathbb{B}'$. For large enough n, $f^n(V) \cap \mathbb{B}'$ is also a horizontal disk in $\mathbb{B}'$, of the same degree $d^n \deg(V)$, and the same number of vertical tangencies. Thus all vertical tangencies of $f^n(V) \cap \mathbb{B}'$ are inside $\mathbb{B}$, or equivalently, the projection

$$\pi : f^n(V) \cap (\mathbb{B}' \setminus \mathbb{B}) \longrightarrow D(0,R') \setminus D(0,R)$$

is a covering. Hence for every simply connected open subset $Q \subset D(0,R') \setminus D(0,R)$, $f^n(V) \cap \pi^{-1}(Q)$ is the union of $d^n \deg(V)$ graphs. Recall that the sequence of currents

$$\frac{1}{d^n \deg(V)} [f^n(V) \cap \mathbb{B}']$$

converges to the unstable current T^-. It then follows (see e.g. [**BS5**]) that $T^-|_{\pi^{-1}(Q)}$ is a *uniformly laminar current*, made up of integration currents over the limiting graphs.

We have thus proved that in $\mathbb{B}' \setminus \mathbb{B}$, $J^- = \operatorname{Supp} T^-$ supports a lamination $\mathcal{L}^-$. Now if $\mathbb{B}'$ is so large that $(\mathbb{B}' \setminus \mathbb{B}) \cap J^-$ contains a fundamental domain for the action of f on $J^-\backslash K^+$, we obtain that $J^- \setminus K^+$ is laminated. The uniqueness of the lamination is obvious since J^- has empty interior.

Step 3. Unique ergodicity of the transverse measure.

The lamination $\mathcal{L}^-$ constructed above carries a foliation cycle T^-, so it has an invariant transverse measure. We prove it is uniquely ergodic, that is, the positive invariant transverse measure is unique. It implies of course ergodicity, i.e. any two transversals of positive measure are connected by holonomy. The lamination will actually be (uniquely) ergodic in each $\mathbb{B}' \setminus \mathbb{B}$.

We use here [**BS6**, Prop. 2.13], which is itself a slight extension of a result of J.E. Fornæss and N. Sibony [**FS**]. The claim is that any positive closed current S supported in $K^- \setminus \mathbb{B}$ is a multiple of T^-. In particular, $T^-|_{\mathbb{C}^2\setminus\mathbb{B}}$ is extremal in $\mathbb{C}^2 \setminus \mathbb{B}$. Since any invariant transverse measure induces a foliation cycle, which is a positive closed current supported in $K^- \setminus \mathbb{B}$, we get that the unique (up to a scalar multiple) invariant transverse measure is the one induced by T^-.

Now if ν is an invariant transverse measure for the lamination $\mathcal{L}^-|_{\mathbb{B}'\setminus\mathbb{B}}$, we prove it can be extended to a transverse measure on $\mathcal{L}^-|_{\mathbb{C}^2\setminus\mathbb{B}}$, hence it is again induced by T^-. Recall that for every simply connected open subset Q in $\mathbb{C}\setminus D(0,R)$, the leaves of the lamination in $\pi^{-1}(Q)$ are graphs over Q. Hence the vertical lines $\pi^{-1}(p)$, $p \in \mathbb{C}\setminus D(0,R)$ are global transversals. Given any two points $p \in D(0,R')\setminus D(0,R)$ and $q \in \mathbb{C}\setminus D(0,R)$, we may thus transport the transverse measure from $\pi^{-1}(p)$ to $\pi^{-1}(q)$ by using a simply connected $Q \subset \mathbb{C}\setminus D(0,R)$ containing p and q. So the transverse measure ν extends from $\mathcal{L}^-|_{\mathbb{B}'\setminus\mathbb{B}}$ to $\mathcal{L}^-|_{\mathbb{C}^2\setminus\mathbb{B}}$ and we get the desired conclusion.

Remark that since T^- has full support in J^-, the lamination is also minimal.

Step 4. Connectivity of J.

By corollary 1.3, every connected component of J contains a point of $J^* = \mathrm{Supp}(\mu)$, where $\mu = T^+ \wedge T^-$ is the maximal entropy measure. Similarly to [**BS6**], the connectivity of J will follow from the following fact "for every $\varepsilon > 0$, any two points p and q in J^* are joined by a path lying in the ε-neighborhood J_ε of J".

Indeed, fix $p, q \in J^*$, and $\varepsilon > 0$. For every $x \in J^- \setminus K^+$, $f^{-n}(x)$ converges to J, so by compactness there exists an integer n such that $f^{-n}((J^- \cap \mathbb{B})\setminus K^+) \subset J_\varepsilon$. By Pesin Theory, μ almost every point has a local unstable manifold W^u_{loc}, subordinate to a piece of unstable lamination with positive measure. Moving p and q slightly is necessary, we may assume this is true for p and q. Moreover, since the laminar structure of T^- is subordinate to the decomposition in unstable manifolds [**BLS**], for every local unstable manifold W^u_{loc}, $W^u_{loc} \cap (J^-\backslash K^+)$, which is non empty, is subordinate to the lamination $\mathcal{L}^-$ constructed above.

Let $M = \sup_{\mathbb{B}} \|df\|$. Since the lamination $\mathcal{L}^-$ is uniquely ergodic in a small neighborhood of $\partial^v\mathbb{B}$, its restriction to $\mathbb{B}\backslash K^+$ is ergodic, so we can find a path γ subordinate to a leaf of $\mathcal{L}^-|_{\mathbb{B}\setminus K^+}$, and joining two points p_1 and q_1, with $d(f^n(p), p_1) <$

$\frac{\varepsilon}{M^n}$ and. $d(f^n(q), q_1) < \frac{\varepsilon}{M^n}$. Hence $f^{-n}(\gamma)$ is a path contained in J_ε and joining $f^{-n}(p_1)$ and $f^{-n}(q_1)$, which are respectively ε-close to p and q. □

REMARK 1.7. There is a strong analogy between our condition and the condition of non escaping critical points in one variable dynamics. The role of "critical point" is played here by vertical tangencies. We indeed proved in step 2 that vertical tangencies do not escape a certain bidisk. See also the analysis in example 1.11 below.

A more precise notion of escaping critical point is developed in [**BS5, BS6**]. Escaping critical points are the critical points of G^+ restricted to unstable manifolds. Using the fact that the critical points of $G^+|_V$ are the vertical tangencies for the invariant projection φ^+ (the "Böttcher coordinate") it is not difficult to prove that if V is a transversal to K^+, then ∂V is compact and disjoint from K^+, so

$$V \cap K^+ \text{ is connected } \Leftrightarrow G^+|_V \text{ has no critical points.}$$

In the next proposition we relate transverse connectivity to unstable connectivity. It does not follow from Theorem 0.2 because of the assumption on the Jacobian.

PROPOSITION 1.8. *If K^+ is transversely connected, then f is unstably connected.*

From [**BS6**, Corollary 7.4] we deduce:

COROLLARY 1.9. *If K^+ is transversely connected, then $|Jac(f)| \leq 1$.*

PROOF. Assume f is unstably disconnected. So for some saddle periodic point p, $W^u(p) \cap K^+$ has a compact component C. The action of f on $W^u(p) \simeq \mathbb{C}$ (where 0 stands for p) is a nontrivial dilatation. So $0 \notin C$, and $\bigcup_{n\geq 1} f^{-n}(C)$ consists of infinitely many compact components of $W^u(p) \cap K^+$, close to 0.

On the other hand $[V] \wedge T^+ > 0$, so we claim that the stable manifold $W^s(p)$ has a transverse intersection point with V. A proof goes as follows (we use ideas from [**BLS**, §9]): $\frac{1}{d^n} f^n_*[V] \to cT^-$, where $c = \int [V] \wedge T^+$. Consider a Pesin box P of positive measure; the local stable manifolds associated to points form a piece of stable lamination of positive transverse measure, hence a uniformly laminar current [**BLS**, §8] $S^+ \leq T^+$, such that $T^- \wedge S^+ > 0$, because G^- cannot be harmonic on $W^s_{loc}(p)$. It is classical that S^+ then has continuous potential. Thus

$$\frac{1}{d^n} f^n_*[V] \wedge S^+ \to T^- \wedge S^+ > 0,$$

and we get that for large n, $f^n(V)$ has intersection points with a set of positive measure of disks in S^+ and most of them are transverse [**BLS**, Lemma 6.4].

So assume that $f^n(V)$ intersects $W^s_{loc}(x)$, for some $x \in P$, not necessarily periodic. By Poincaré recurrence, we may suppose that for infinitely many n_j, $f^n_j(x) \in P$, so by using the stable manifold theorem, we get that for large j, $f^{n+n_j}(V)$ contains a disk arbitrarily close to $W^u_{loc}(f^{n_j}x)$. Now if p is any saddle point, $W^s(p)$ has transverse intersection points with any set of positive measure of unstable disks [**BLS**, Lemma 9.1]. We conclude that $W^s(p)$ must intersect $f^{n+n_j}(V)$, hence V, transversely.

For $N \gg 1$, consider a set $C_1, \dots C_N$ of open and closed subsets of $W^u(p) \cap K^+$ close to p. For each C_i, consider a simple curve γ_i enclosing C_i, and such that

$\gamma_i \cap K^+ = \emptyset$. Choose local coordinates (x,y) so that $W^u_{loc}(p) \subset (y=0)$, and extend γ_i to a piece of 3-submanifold $\widetilde{\gamma}_i$, transverse to $W^u_{loc}(p)$, by adding vertical holomorphic disks. We assume the vertical disks are so small that $\widetilde{\gamma}_i \cap K^+ = \emptyset$

By the Lambda lemma, for large n, $f^n(V)$ contains a graph over $W^u_{loc}(p)$, and very close to it. So the curves $\widetilde{\gamma}_i \cap f^n(V)$ cut out N disks $\Delta_1 \dots \Delta_N$ in $f^n(V)$. Of course as $n \to \infty$ the disks Δ_i converge to $\mathrm{Int}(\gamma_i)$. This forces $\Delta_i \cap K^+$ to be non empty for large n: for instance use the fact that $[\mathrm{Int}(\gamma_i)] \wedge T^+ > 0$ (see step 1 above) and the continuity of the potential G^+.

We have proved that in any neighborhood of any transverse intersection point of $W^s(p) \cap V$, $K^+ \cap V$ has at least N connected components, which clearly contradicts transverse connectivity. □

REMARK 1.10. The proof of the proposition provides another approach to the fact that unstable connectivity is independent of the chosen unstable manifold.

In the next proposition we use the fact that if f is unstably connected, $J^- \setminus K^+$ has the structure described in Theorem 1.5 above (a uniquely ergodic lamination with foliation cycle T^-).

PROPOSITION 1.11. *If f is unstably connected then $T^-|_{\mathbb{B}}$ is an extremal current.*

PROOF. We have seen that the unstable current T^- is uniformly laminar in $\mathbb{C}^2 \setminus K^+$, and moreover if R and R' are sufficiently large the transverse measure in $\mathcal{L}^-|_{\pi^{-1}(A(R,R'))}$ is ergodic, where $A(R,R')$ denotes the annulus $D(0,R') \setminus D(0,R)$, and π is the first projection. Any positive closed current $S \leq T^-|_{\pi^{-1}(A(R,R'))}$ is uniformly laminar and subordinate to T^- (this is a result about analysis on laminations, see [**Du3**]). By ergodicity, we conclude that $T^-|_{\pi^{-1}(A(R,R'))}$ is extremal.

Let now S be a positive closed current in $\mathbb{B}$, with $S \leq T$. In particular $S = cT^-$ $(c \leq 1)$ in the neighborhood of $\partial^v\mathbb{B}$. Consider the positive closed current $\widetilde{S}$ on $\mathbb{C}^2$ defined by $\widetilde{S} = S$ in $\mathbb{B}$, and $\widetilde{S} = cT^-$ outside $\mathbb{B}$. Since T^- is extremal in $\mathbb{C}^2$, $\widetilde{S} = cT^-$ everywhere, and we conclude that $T^-|_{\mathbb{B}}$ is extremal. □

EXAMPLE 1.12. *If p is a hyperbolic polynomial with connected Julia set, then for small $|a|$, the Hénon map $f_a(z,w) = (aw + p(z), az)$ is transversely connected.*

We now proceed to give the proof, which is inspired by [**HO2**], from which the result may actually be extracted.

Let R be so large that $|z| = R$ implies $|p(z)| > 2R$, and let $\mathbb{B} = D(0,R)^2$. First, if $|a| \leq 1$, f is a Hénon-like map of degree d in $\mathbb{B}$ (see lemma 1.12 below). We will show that if L is any horizontal line in $\mathbb{B}$, and a is small enough, $L \cap K^+$ is connected. Since for such a L

$$L \cap K^+ = \bigcap L \cap \mathbb{B}_{-n} = \bigcap \overline{L \cap \mathbb{B}_{-n}},$$

and every component of $L \cap \mathbb{B}_{-n}$ intersects K^+, this is equivalent to saying that for every n, $L \cap \mathbb{B}_{-n}$ is connected. By the Riemann-Hurwitz formula (see step 2 above) this is in turn equivalent to the fact that $f^n(L) \cap \mathbb{B}$ has $d^n - 1$ vertical tangencies: "critical points do not escape $\mathbb{B}$".

We begin with a useful lemma.

LEMMA 1.13. *Let U_1 and U_2 be two topological disks such that $\overline{U_i} \subset D(0,R)$. Assume that*

- *$p : p^{-1}(U_2) \to U_2$ is a branched cover of some degree k.*
- $\mathrm{dist}(p(\partial U_1), U_2) > \delta$.

Then $f_a : U_1 \times D(0,R) \to U_2 \times D(0,R)$ is a Hénon-like (crossed) map of degree k as soon as $|a| \leq \min(\frac{\delta}{R}, 1)$.

Proof. Items *i.* and *ii.* of definition 1.1 easily hold. We check that $f_a : U_1 \times D(0,R) \to U_2 \times D(0,R)$ has degree k: if $u_2 \in U_2$, we need to prove that $f^{-1}(\{u_2\} \times D(0,R))$, which is a vertical submanifold in $U_1 \times D(0,R)$, has degree k. It is defined by the equation

$$\{(z,w),\ aw + p(z) = u_2\},$$

and clearly its intersection with $(w = 0)$ has k points counted with multiplicity. □

Assume now that p is a hyperbolic polynomial with connected Julia set. Fix a neighborhood N of J such that $p^{-1}(N) \subset\subset N$, and $p : p^{-1}(N) \to N$ is a covering of degree d. p is a strict expansion for the Poincaré metric of N. We denote by $U(z,\varepsilon)$ the ball of radius ε around z for the Poincaré metric of N. Fix ε small enough, so that for every $z \in J$, $U(z,\varepsilon)$ is a topological disk, and p is univalent on $U(z,\varepsilon)$. In particular,

$$p : p^{-1}(U(p(z),\varepsilon)) \cap U(z,\varepsilon) \to U(p(z),\varepsilon)$$

is a biholomorphism. Reducing ε once again, we may further assume that

$$\mathrm{dist}\big(p(\partial U(z,\varepsilon)), U(p(z),\varepsilon)\big) > \delta,$$

for some constant δ independent of z. By the preceding lemma, for $|a| \leq \frac{\delta}{R}$,

$$f_a : U(z,\varepsilon) \times D(0,R) \to U(p(z),\varepsilon) \times D(0,R)$$

is a Hénon-like map of degree 1. Let $U = \bigcup_{z \in J} U(z,\varepsilon)$.

Let us now consider a horizontal line L in $\mathbb{B}$. By the above argument, for every $|a| \leq \frac{\delta}{R}$, and every $n \geq 1$, all iterates $f_a^n(L)$ are graphs over U: indeed since $p : p^{-1}(N) \to N$ has degree d, the only contribution to $f_a^n(L)$ over U comes from $U \times D(0,R)$. Now for fixed n, if $a \to 0$, the vertical tangencies of $f_a^n(L)$ converge to the $d^n - 1$ (with multiplicities) critical values of p^n, located in K. Since the vertical tangencies cannot cross $U \times D(0,R)$, we conclude that in $\mathbb{B}$ $f_a^n(L)$ has $d^n - 1$ vertical tangencies in $K \times D(0,R)$, hence $f_a^n(L) \cap \mathbb{B}$ is connected. □

Remark 1.14. It can be proved that under these assumptions $J^+ \cap \mathbb{B}$ is laminated by vertical holomorphic disks, moving holomorphically with a. Moreover when $a \to 0$, the lamination converges to the trivial lamination of $J \times D(0,R)$. From this we conclude that any slice $J^+(f_a) \cap L$ is the image of $J(p)$ by a holomorphic motion.

1.4. Hénon-like mappings. We present some connectedness results in the Hénon-like setting[1]. The picture is much less precise than in the case of polynomial automorphisms, in particular we cannot prove that points in J are connected by paths subordinate to unstable manifolds.

[1]The main idea in the proof of the next theorem originates from a remark made to me by N. Sibony.

Let f be a Hénon-like map in $\mathbb{B}$. Notice first that if f is dissipative or conservative (i.e. $|\mathrm{Jac}(f)| \leq 1$), then K^- has measure zero, hence $J^- = K^-$. Indeed $f^{-1}K^- \subset\subset K^-$, so if f contracts volumes K^- cannot have positive measure.

If f is a birational perturbation of a dissipative Hénon map, as considered in §1.1, the equality $J^+ = \operatorname{Supp} T^+$ still holds.

THEOREM 1.15. *Let f be a Hénon-like map of degree $d > 1$ in $\mathbb{B}$, and assume that $J^- = K^-$.*

(1) If $J^+ = \operatorname{Supp}(T^+)$ and if K^+ is transversely connected, then $J = J^+ \cap J^-$ is connected.

(2) If $J^- = \operatorname{Supp}(T^-)$ and if there exists a transversal V to $\operatorname{Supp}(T^+)$ such that $\operatorname{Supp} T^+ \cap V$ is connected, then $\operatorname{Supp}(T^+) \cap \operatorname{Supp}(T^-)$ is connected.

PROOF. *(1)* If V is a transversal to K^+, then for n large enough, $\mathcal{L}^n[V]$ is a non trivial horizontal positive closed current in $\mathbb{B}$ (where $\mathcal{L}$ is the graph transform operator for currents, see §1.1). It follows that the sequence of currents $(\mathcal{L}^n[V])$ converges to cT^-, with $c > 0$. Let $\mu_n = \mathcal{L}^n[V] \wedge T^+ = (f^n)_*([V] \wedge T^+)$; since T^+ has continuous potential, $\mu_n \to c\mu$ as $n \to \infty$.

Since K^+ is holomorphically convex and V is a holomorphic disk, $V \setminus K^+$ has no compact components. It follows that if K^+ is transversely connected then $J^+ \cap V = \operatorname{Supp} T^+ \cap V$ is connected, and so does $f^n(J^+ \cap V)$.

On the other hand if J_ε is the ε-neighborhood of J, for large n we have

$$\operatorname{Supp} \mu_n \subset \operatorname{Supp}(T^+) \cap f^n(V) = f^n(J^+ \cap V) \subset J^+ \cap K^-_\varepsilon \subset J_\varepsilon.$$

We will conclude that $J = J^+ \cap J^-$ is connected by showing that every connected component of $J^+ \cap J^-$ contains a point of $\operatorname{Supp}(\mu)$.

So let J_1 be an open and closed subset of J, $J_1 = J \cap \Omega$, with $\partial\Omega \cap J = \emptyset$. We prove that $\mu(\Omega) > 0$. Since $J^- = K^-$, the points in $J^+ \cap \partial\Omega = \operatorname{Supp}(T^+|_{\overline{\Omega}}) \cap \partial\Omega$ escape under backwards iteration. So if ${}^t\mathcal{L}$ denotes the pull back graph transform operator, ${}^t\mathcal{L}^n(T^+|_\Omega)$ is closed and vertical for large n, and non trivial since $\operatorname{Supp}({}^t\mathcal{L}^n(T^+|_\Omega)) \cap J^- \neq \emptyset$. In particular ${}^t\mathcal{L}^n(T^+|_\Omega) \wedge T^- > 0$, hence $(T^+|_\Omega) \wedge T^- > 0$.

(2) The reasoning is similar. Let ψ be a nonnegative test function in V, with $\psi = 1$ near $\operatorname{Supp}(T^+)$, so that $\psi[V] \wedge T^+ = [V] \wedge T^+$. We claim that $\mathcal{L}^n\psi[V] \to cT^-$, with $c > 0$. Then, assuming the claim, $\mu_n = \mathcal{L}^n\psi[V] \wedge T^+$ converges to $c\mu$, and since $\operatorname{Supp}(T^-) = J^- = K^-$, for every $\varepsilon > 0$

$$\operatorname{Supp} \mu_n \subset \operatorname{Supp}(T^+) \cap f^n(V) = f^n(\operatorname{Supp}(T^+) \cap V) \subset (\operatorname{Supp}(T^+) \cap \operatorname{Supp}(T^-))_\varepsilon$$

for large n. We conclude by using corollary 1.3.

It remains to prove our claim. The difficulty is that we can not assume $\mathcal{L}^n[V]$ is closed after a few iterations. Nevertheless $c = \int T^+ \wedge [V] > 0$ by theorem 1.2, and it will be a consequence of [**Du1**] and [**DDS**] that $\mathcal{L}^n(\psi[V]) \to cT^-$.

An easy adaptation of proposition 4.8, and theorem 4.10 in [**Du1**] shows that if the mass of the sequence of currents $\mathcal{L}^n(\psi[V])$ is bounded, then $\mathcal{L}^n(\psi[V]) \to cT^-$. Let Θ be a smooth vertical positive closed current. From [**DDS**, Proposition 4.13], we may write ${}^t\mathcal{L}^n\Theta = dd^c u_n$, where (u_n) is a uniformly bounded sequence of psh functions. Then the sequence

$$\int \mathcal{L}^n(\psi[V]) \wedge \Theta = \int_V \psi dd^c u_n = \int_V u_n dd^c \psi$$

is uniformly bounded, and since in the neighborhood of K^+, masses may be evaluated by using vertical closed positive currents we get the result. □

Remark 1.16.

- If f is a polynomial diffeomorphism, by corollary 1.8, the assumption $J^- = K^-$ is a consequence of transversal connectedness.
- In case *(1)*, one may reproduce steps 1 and 2 in the proof of theorem 1.5, and infer that J^- is a lamination near $\partial^v\mathbb{B}$.

2. Maps with disconnected Julia sets

2.1. The connectedness locus is closed. Throughout this section we assume f is a polynomial automorphism of $\mathbb{C}^2$ of degree d, and we fix as before a bidisk $\mathbb{B}$ such that $f|_{\mathbb{B}}$ is Hénon-like of degree d.

Recall that f is said to be unstably disconnected if for some saddle point p, $W^u(p) \cap K^+$ has a compact component (for the "leafwise" topology induced by the isomorphism $W^u(p) \simeq \mathbb{C}$). We saw in proposition 1.7 that this is independent of the saddle point p. Recall also that if $|\mathrm{Jac}(f)| \leq 1$, J is disconnected iff f is unstably disconnected. Moreover, if $|\mathrm{Jac}(f)| < 1$, f is always stably disconnected, and if $|\mathrm{Jac}(f)| = 1$, f is stably disconnected iff f is unstably disconnected [**BS6**, Corollary 7.4].

Lemma 2.1. *If p is any saddle point, $W^u(p) \cap K^+$ has a compact component iff $W^u(p) \cap \mathbb{B}$ has a relatively compact component (for the leafwise topology).*

Proof. Without loss of generality assume p is fixed. Let first C be a compact component of $W^u(p) \cap K^+$, and γ be a Jordan curve in $W^u(p) \setminus K^+$ surrounding C, i.e. $C \subset \mathrm{Int}(\gamma)$. Then γ escapes under iteration, and for some n, $f^n(\mathrm{Int}(\gamma)) \cap \mathbb{B}$ is a horizontal disk in $\mathbb{B}$, which is relatively compact in $W^u(p)$.

Conversely assume C is a relatively compact component of $W^u(p) \cap \mathbb{B}$. Then C is a non trivial horizontal submanifold in $\mathbb{B}$. Thus C intersects K^+, and $C \cap K^+ \subset\subset C$ is a compact component of $W^u(p) \cap K^+$. □

We obtain as a corollary that the connectedness locus is closed in parameter space[2]. Because the parameter space of polynomial diffeomorphisms is not well understood, we state the result in terms of 1-parameter families.

Corollary 2.2. *Let $\{f_\lambda, \lambda \in \Lambda\}$ be a holomorphic 1-parameter family of polynomial diffeomorphisms. Then the connectedness locus*

$$\{\lambda \in \Lambda,\ J(f_\lambda) \text{ is connected}\}$$

is closed in Λ.

Proof. We prove the disconnectedness locus is open. So suppose that at $0 \in \Lambda$, $J(f_0)$ is disconnected. Reducing Λ slightly if necessary, we may assume all f_λ are Hénon-like in $\mathbb{B}$.

f_0 is both stably and unstably disconnected. We prove stable disconnectedness persists under perturbation. By the preceding lemma, there is a saddle point p_0, such that $W^u(p_0) \cap \mathbb{B}$ has a compact component. Saddle points and their unstable manifolds vary continuously (holomorphically) under perturbation, so for λ close to 0, there is a holomorphic family p_λ of saddle points corresponding to p_0, and

[2] This result stemmed out during a conversation with Eric Bedford and Misha Lyubich at the Snowbird conference in June 2004.

every relatively compact part of $W^u(p)$ can be followed holomorphically. So for λ close to 0, $W^u(p_\lambda)\cap\mathbb{B}$ has a compact component and we are done. $\square$

2.2. Structure of T^-. The currents $T^\pm$ have laminar structure in general. We will show that if f is unstably disconnected, $T^-|_{\mathbb{B}}$ is an integral of horizontal submanifolds of $\mathbb{B}$. In other words, almost every leaf of $T^-|_{\mathbb{B}}$ is a finite branched cover over the basis $\mathbb{D}$ for the natural vertical projection.

Recall that f is always stably or unstably disconnected, so the result always apply to at least one of T^+ or T^-.

If T is a laminar current, we say that a disk Δ is subordinate to T if there exists a non trivial uniformly laminar current $S\leq T$ such that Δ lies inside a leaf of S. See [**Du2**] for definitions and related results. We begin with a useful proposition.

PROPOSITION 2.3. *If p is a saddle point, any relatively compact disk $\Delta\subset W^u(p)$ is subordinate to T^-.*

PROOF. If $W^u_{loc}(p)$ denotes any neighborhood of p in $W^u(p)$, pulling back Δ by f if necessary, it is enough to prove $W^u_{loc}(p)$ is subordinate to T^-.

We saw in the course of the proof of proposition 1.7 that if Q is a Pesin box, the local unstable lamination $\mathcal{L}^u(Q)$ supports a non trivial uniformly laminar current subordinate to T^-. Moreover if p is any saddle point, $W^s(p)$ has transverse intersection points with $\mathcal{L}^u(Q)$. Thus there exists a uniformly laminar current

$$S_0=\int[\Delta_\alpha]d\nu_0(\alpha)\leq T^-$$

where the Δ_α are small disks intersecting $W^s(p)$ transversely at one point.

By the Lambda lemma for every α, $f^n(\Delta_\alpha)$ contains graphs over $W^u_{loc}(p)$, arbitrary close to p when n is large. So there exists a uniformly laminar current S_j subordinate to $d^{-j}(f^j)_*S_0\leq T^-$, and made up of graphs over $W^u_{loc}(p)$, close to p. Extracting a subsequence if necessary we may assume the S_j have disjoint supports. Define a uniformly laminar current S in the neighborhood of p as $S=\sum c_jS_j$, where the constants $c_j\leq 1$ are adjusted so that the series converges. $W^u_{loc}(p)$ is subordinate to S, and $S\leq T^-$ so this solves the problem. $\square$

The structure theorem for the current $T^-|_{\mathbb{B}}$ follows easily. It is stated in $\mathbb{B}$, however the bidisk may be arbitrary large. A consequence is that J^- is locally the limit of the union of an increasing sequence of laminations.

THEOREM 2.4. *If f is unstably disconnected, then there exists a family of uniformly laminar currents T^-_k in $\mathbb{B}$, respectively made up of horizontal disks of degree k, such that*

$$T^-|_{\mathbb{B}}=\sum_{k=1}^{\infty}T^-_k.$$

PROOF. Lemma 2.1 asserts that for any unstable manifold $W^u(p)$, $W^u(p)\cap\mathbb{B}$ has a relatively compact component. Such a component is a horizontal submanifold of some degree k in $\mathbb{B}$. By proposition 2.3 this horizontal submanifold is subordinate to T^-, so there exists a uniformly laminar T_0 made up of submanifolds of degree k in $\mathbb{B}$, with $0\leq T_0\leq T^-|_{\mathbb{B}}$. Notice that by the maximum principle, all submanifolds involved here are holomorphic disks.

The sequence of cut-off iterates $\frac{1}{d^n}(f^n)_*(T_0)|_{\mathbb{B}} \leq T^-$ converges to $T^-|_{\mathbb{B}}$, and each of these currents is uniformly laminar and made up of global submanifolds of bounded degree. Any uniformly laminar current subordinate to T is of the form hT, where $0 \leq h \leq 1$ is a measurable function, locally constant along the leaves. So we can write $\frac{1}{d^n}(f^n)_*(T_0)|_{\mathbb{B}}$ as $h_n T^-|_{\mathbb{B}}$, where $0 \leq h_n \leq 1$, and the convergence result says that $h_n \to 1$, T^--a.e. Hence the sequence of currents

$$T_n = \max\left(T_0, \ldots, d^{-n}(f^n)_*(T_0)|_{\mathbb{B}}\right) = \max(h_0, \ldots, h_n)T^-|_{\mathbb{B}}$$

increases to $T^-|_{\mathbb{B}}$. It is clear that each T_n is uniformly laminar, and made of submanifolds of bounded degree, so T^- has the desired structure. □

Since none of the T_k^- can be extremal we get the following corollary.

COROLLARY 2.5. *If f is unstably disconnected, then $T^-|_{\mathbb{B}}$ is not an extremal current.*

REMARK 2.6. The decomposition of T^- into an integral of extremal components can easily be deduced from the structure theorem. It appears that the extremal components of $T^-|_{\mathbb{B}}$ are irreducible horizontal submanifolds.

In case $f(\mathbb{B}) \cap \mathbb{B}$ is disconnected, a similar corollary already appears in [**DDS**, §5.2]. More specifically, if $f(\mathbb{B}) \cap \mathbb{B} = U_1 \cup \cdots \cup U_m$ $(m > 1)$ each point in K^+ (resp. K^-) can be assigned an itinerary $\alpha \in \{1, \ldots, m\}^{\mathbb{N}}$. This induces decompositions of the currents $T^\pm|_{\mathbb{B}}$ in terms of itinerary sequences.

Each open set U_i comes equipped with a multiplicity $d_i = \deg f(L) \cap U_i$, where L is a horizontal line in $\mathbb{B}$. If at least one partial degree d_i is larger than one, then, with respect to a natural measure on $\{1, \ldots, m\}^{\mathbb{N}}$ related to the d_i, for almost every symbolic sequence α, $d_{\alpha(0)} \cdots d_{\alpha(n)} \to \infty$. From this it follows that in general the decomposition of $T^-|_{\mathbb{B}}$ given in [**DDS**] is not the extremal decomposition of T^-.

Another corollary is the construction of external rays in the unstably disconnected case. External rays for unstably connected maps were constructed in [**BS6**] using the lamination structure of $J^- \setminus K^+$.

COROLLARY 2.7. *There exists a measured family $(\mathcal{E}, \nu)$ of external rays, defined as gradient lines of G^+ restricted to unstable manifolds. Almost every ray lands and if e denotes the endpoint mapping, then $e_*\nu$ equals the maximal entropy measure μ.*

PROOF. Pick a generic leaf M of $T^-|_{\mathbb{B}}$, so M is a horizontal disk in $\mathbb{B}$ of finite degree. Reducing M a little if necessary, it has finite area, and we may assume ∂M if of the form $\{G^+ = r\}$ for some $r > 0$, and does not intersect the critical set of $G^+|_M$.

On $M \setminus K^+$, $G^+|_M$ is a positive harmonic function, so outside the critical points of $G^+|_M$, one may flow along the gradient lines (in the sense of decreasing G^+). If $\{G^+ = s\}$ is conveniently oriented, the 1-form $d^c(G^+|_M)|_{\{G^+=s\}}$ defines a positive measure on $M \cap \{G^+ = s\}$ which is invariant by the flow. If $s < r$, only finitely many gradient lines issued from $\{G^+ = r\}$ hit a critical point of $G^+|_M$ before attaining $\{G^+ = s\}$. This defines a measurable map $e_{r,s} : \{G^+ = r\} \to \{G^+ = s\}$ such that

$$(e_{r,s})_* d^c(G^+)|_{\{G^+=r\}\cap M} = d^c(G^+)|_{\{G^+=s\}\cap M}.$$

That almost every ray lands is a consequence of the fact that M has finite area, see Bedford-Jonsson [**BJ**, §7] for a proof.

At the level of the whole unstable lamination, we get a map $e_{r,s} : \{G^+ = r\} \to \{G^+ = s\}$, defined almost everywhere with respect to the measure $d^cG^+|_{\{G^+=r\}} \wedge dd^cG^-$. Indeed, notice that $d^cG^+|_{\{G^+=r\}} \wedge dd^cG^-$ is the integral of the family of measures $d^cG^+|_{\{G^+=r\}\cap M}$ with respect to the transverse measure induced by T^-. Moreover, the maps $e_{r,s}$ preserve the transverse measure

$$(e_{r,s})_* d^cG^+|_{\{G^+=r\}} \wedge dd^cG^- = d^cG^+|_{\{G^+=s\}} \wedge dd^cG^-.$$

On the other hand [**BS6**]

$$d^cG^+|_{\{G^+=r\}} \wedge dd^cG^- = dd^c \max(G^+, r) \wedge dd^cG^- \underset{r\to 0}{\longrightarrow} \mu = dd^cG^+ \wedge dd^cG^-,$$

so the landing measure is the maximal entropy measure. □

2.3. The Poincaré metric along the leaves. We are interested in the expansion properties of f with respect to the Poincaré metric along the leaves of $T^-|_{\mathbb{B}}$. We will use the structure of $T^-|_{\mathbb{B}}$ to derive (rough) estimates on the Poincaré metric.

Notice that we cannot rule out the possibility of singular leaves on a set of zero transverse measure (the disks constructed in theorem 2.4 are smooth). This does not affect the treatment of the Poincaré metric, since if Γ is a singular irreducible holomorphic disk of degree k, there exists a parametrization $\phi : \mathbb{D} \xrightarrow{\sim} \Gamma$ such that $\pi \circ \phi$ is a branched cover of degree k. Indeed just consider the uniformization $\mathbb{D} \to \widehat{\Gamma}$ of the desingularization of Γ. Pushing forward by ϕ defines unambiguously the Poincaré metric on Γ. Here we shall treat horizontal holomorphic disks regardless of their regular or singular nature; this actually has to be considered only in proposition 2.14.

Assume M is a horizontal disk of degree k in $\mathbb{B}$. Let $\mathbb{B}_0$ and $\mathbb{B}_1$ be "vertical sub-bidisks" of the form $\mathbb{B}_i = \mathbb{B} \cap \{|z| < R_i\}$, with $R_0 < R_1$. It is classical that there exists a constant C such that

$$k\pi R_1^2 \leq \mathrm{Vol}(M \cap \mathbb{B}_1) \leq Ck\pi R_1^2.$$

The following nice result is [**BS8**, Theorem 3.1].

THEOREM 2.8. *Let M be a horizontal disk of degree k as above. Assume $M_0 \subset M_1$ are respective connected components of $M \cap \mathbb{B}_0$ and $M \cap \mathbb{B}_1$. Then $M_1 \setminus M_0$ is an annulus and there exists a constant C depending only on R_0 and R_1 such that*

$$\mathrm{Modulus}(M_1 \setminus M_0) \geq \frac{C}{k}$$

We denote by ρ_M the Poincaré (Kobayashi) metric of the submanifold M in $\mathbb{B}$.

COROLLARY 2.9. *There exist constants C and $\lambda > 1$, depending on (k, R_0, R_1) such that the hyperbolic diameter of M_0 in M_1 is bounded by C and $\rho_{M_0} \geq \lambda\rho_{M_1} \geq \rho_M$.*

PROOF. We may think of M_1 as being the unit disk $\mathbb{D}$. If $E \subset\subset \mathbb{D}$ define (see McMullen [**McM**, §2.3])

$$\mathrm{Mod}(E, \mathbb{D}) = \sup \{\mathrm{Modulus}(A),\ A \subset \mathbb{D} \text{ is an annulus enclosing } E\}.$$

Then the hyperbolic diameter of E and $\mathrm{Mod}(E, \mathbb{D})$ are related by

$$(\mathrm{diam}_{\mathbb{D}}(E) \to \infty) \Leftrightarrow (\mathrm{Mod}(E, \mathbb{D}) \to 0).$$

So by the previous theorem there exists a constant $C(k, R_0, R_1)$ such that

$$\mathrm{diam}_{\mathbb{D}}(M_0) \leq C.$$

Moving M_0 by an isometry of $\mathbb{D}$ if necessary, we may assume that $0 \in M_0$. The last estimate then implies the existence of a radius $r(k, R_0, R_1) < 1$ such that $M_0 \subset D(0, r)$, and the result follows.

We may actually give an explicit estimate: when the diameter is large the following holds

$$C\frac{1}{\mathrm{diam}_{\mathbb{D}}(E)} \geq \mathrm{Mod}(E, \mathbb{D}) \geq Ce^{-\mathrm{diam}_{\mathbb{D}}(E)},$$

so if k is large $\mathrm{diam}_{\mathbb{D}}(E) \leq Ck$. Since

$$\rho(0, z) = \frac{1}{2}\log\frac{1+|z|}{1-|z|}$$

we get $r \leq 1 - e^{-Ck}$, hence the estimate

$$\rho_{M_0} \geq \frac{1}{1-e^{-Ck}}\ \rho_{M_1} \geq \rho_M$$

on the Poincaré metric. □

Using non uniform expansion along the leaves for the Poincaré metric allows to recover the following result from [**BS6**]. Notice that the current T^+ induces by restriction a measure on every unstable leaf. These measures are equivalent to the "unstable conditionals" of μ [**BLS**, Prop 3.1].

THEOREM 2.10. *If f is unstably disconnected, then for μ-a.e. p, almost every component of $K^+ \cap W^u(p)$ is a point, that is, for $T^+|_{W^u(p)}$-a.e. x, the connected component of x in $K^+ \cap W^u(p)$ is $\{x\}$.*

PROOF. For a μ-generic point p, let $K^{+,u}(p)$ be the connected component of $K^+ \cap W^u(p)$ containing p. Since the unstable conditionals of μ are induced by T^+ it is enough to prove that for μ-a.e. p, $K^{+,u}(p) = \{p\}$.

Theorem 2.4 provides us with a decomposition $T^-|_{\mathbb{B}} = \sum T_k^-$. Fix an integer K so that $T_{\leq K}^- = \sum_{k \leq K} T_k^-$ is nontrivial. The measure $\mu_K = T_{\leq K}^- \wedge T^+ \leq \mu$ has positive mass, and there exists a set E of total μ_K mass, such that if $p \in E$, the connected component $\Gamma^u(p)$ of $W^u(p) \cap \mathbb{B}$ containing p is a horizontal disk of degree $k \leq K$. since K is arbitrary it suffices to show that for $p \in E$, $K^{+,u}(p) = \{p\}$.

There exists a radius R_1 such that $f^{-1}(\mathbb{B}) \cap \mathbb{B} \subset \mathbb{B} \cap \{|z| < R_1\}$. So by the previous corollary, if $p \in E$, the hyperbolic diameter of $K^+ \cap \Gamma^u(p)$ is uniformly bounded by a constant D, and if p, $f^k(p) \in E$, $f^{-k} : \Gamma^u(f^k(p)) \to \Gamma^u(p)$ is a contraction of factor a least λ. Moreover for almost every $p \in E$ there are infinitely many n such that $f^n(p) \in E$. Label these values as n_j, $j \geq 1$. For such a n_j, $f^{-n_j} : \Gamma^u(f^{n_j}(p)) \to \Gamma^u(f(p))$ is a contraction of factor λ^j for the Poincaré metric. Of course $f(K^{+,u}(p)) = K^{+,u}(f(p))$. Thus the hyperbolic diameter of $K^{+,u}(p)$ is bounded by D/λ^j and the result follows. □

REMARK 2.11. The proof incidentally shows that in the unstably disconnected case, f always has positive Lyapounov exponent with respect to the Poincaré metric.

If J is disconnected, f is both stably and unstably disconnected. The previous result suggests the following:

Question 2.12. *If f is a polynomial diffeomorphism with disconnected Julia set, is it true that almost every connected component of J is a point?*

We have no general answer to this question, except in the case of hyperbolic maps.

Corollary 2.13. *If J is disconnected and f is hyperbolic on J, then almost every component of J is a point.*

Proof. if f is hyperbolic on J, it has local product structure. So locally near p, J is homeomorphic to $(J^+ \cap W^u_{loc}(p)) \times (J^- \cap W^s_{loc}(p))$; also the measure μ is the product of its stable and unstable conditionals. The previous theorem applied in the stable and unstable directions implies the corollary. □

In the next two propositions, we examine the very special situation where the submanifolds involved in the decomposition of $T^-|_{\mathbb{B}}$ have bounded degree. In this case we say $T^-|_{\mathbb{B}}$ has *no degree growth*. We do not know whether there are examples exhibiting such a phenomenon, besides perturbations of polynomials with totally disconnected Julia sets (horseshoes). It seems very likely to us that real mappings on the boundary of the horseshoe locus indeed satisfy this assumption (see [**BS10**, Prop. 3.3]). We also believe that the condition should hold when f is hyperbolic and has no attracting orbits.

Proposition 2.14. *If $T^-|_{\mathbb{B}}$ has no degree growth, then*

(1) f is uniformly expanding with respect to the Poincaré metric along the leaves of $T^-|_{\mathbb{B}}$.
(2) For every unstable manifold $W^u(p)$, $K^+ \cap W^u(p)$ is totally disconnected. As a consequence no component of $\mathrm{Int}(K^+)$ intersects J^-. In particular f has no attracting orbits, and no attracting rotation sets.
(3) $J = J^*$ $(:= \mathrm{Supp}\,\mu)$.

It is proved in [**BS8**, §4] that the first item of the proposition implies "quasi-expansion". Quasi-expansion means f is uniformly expanding with respect to a metric related to subsets of the form $W^u(p) \cap \{G^+ \leq 1\}$ in unstable manifolds; this is 2-dimensional analogue of semi-hyperbolicity in the sense of Carleson, Jones and Yoccoz [**CJY**].

Proof. Assume $T^-|_{\mathbb{B}}$ has no degree growth, i.e. all the disks occurring in the decomposition of $T^-|_{\mathbb{B}}$ have degree $\leq K$. The set of horizontal subvarieties of degree $\leq K$ is compact for the Hausdorff topology. Moreover if a sequence of horizontal holomorphic disks Δ_n converges to Δ in the Hausdorff topology, consider a sequence of parametrizations $\phi_n : \mathbb{D} \to \Delta_n$, normalized for instance by $\pi \circ \phi_n(0) = 0$. The cluster values of the sequence (ϕ_n) are finite branched coverings $\phi : \mathbb{D} \to \Delta$ (use for instance the fact that $\pi \circ \phi_n$ is a Blaschke product; see also [**CJY**, Lemma 2.2]). In particular, consideration of the Euler characteristic shows Δ is a disk. Since $\mathrm{Supp}\, T^- = J^-$, $J^- \cap \mathbb{B}$ is then the union of a family of (possibly singular) horizontal disks of degree $\leq K$, namely all cluster values of the disks of the decomposition of $T^-|_{\mathbb{B}}$. Moreover, unstable manifolds being disjoint, by the Hurwitz Theorem any two limiting disks Δ_1 and Δ_2 are disjoint or equal. So through $p \in J^-$ there is a unique disk $\Gamma^u(p)$, of degree $\leq K$.

As in the proof of the previous theorem, by corollary 2.9, $f^{-1} : \Gamma^u(f(p)) \to f^{-1}\Gamma^u(f(p)) \subset\subset \Gamma^u(p)$ is a strict contraction for the Poincaré metric, with factor λ independent of p. Moreover $K^+ \cap \Gamma^u(p)$ has uniformly bounded diameter. Since $f(K^{+,u}(p)) = K^{+,u}(f(p))$ it follows that every connected component of $K^+ \cap \Gamma^u(p)$ is point. In particular, no component of $\mathrm{Int}(K^+)$ can intersect J^-. The corollaries in *(2)* are straightforward.

For *(3)*, let Δ be a disk in $\mathbb{B}$ subordinate to some $\Gamma^u(p)$, such that $\Delta \cap J^+ \neq \emptyset$. Assume all forward iterates $f^n(\Delta)$ remain in some vertical sub-bidisk $\mathbb{B}' \subset \mathbb{B}$. Then $f^n(\Delta) \subset \Gamma^u(f^n(p)) \cap \mathbb{B}'$, which contradicts the strict expansion.

We deduce that if $p \in J = J^+ \cap J^-$, the function G^+ is not harmonic in any neighborhood of p in $\Gamma^u(p)$. The function G^+ being continuous, if Γ_n converges to $\Gamma^u(p)$ with multiplicity k in the sense of currents ($k \leq K$), $[\Gamma_n] \wedge dd^c G^+$ is positive in the neighborhood of p, thus $T^- \wedge dd^c G^+ > 0$ in the neighborhood of p and $p \in J^*$. □

If moreover $T^+|_{\mathbb{B}}$ has no degree growth, we obtain stronger results.

PROPOSITION 2.15. *Assume further $T^-|_{\mathbb{B}}$ and $T^+|_{\mathbb{B}}$ have no degree growth. Then:*

(1) The connected components of $J^+ \cap \mathbb{B}$ (resp. $J^- \cap \mathbb{B}$) are horizontal disks of bounded degree.
(2) If M is any piece of complex curve, not included in J^+, then $J^+ \cap M$ is totally disconnected. The same holds for J^-.
(3) $\mathrm{Int}(K^+) = \mathrm{Int}(K^-) = \emptyset$.
(4) $J = K$ is totally disconnected.

PROOF. Let $p \in J$. We showed in the preceding proposition that $K^{+,u}(p) = \{p\}$, so let γ be a small loop in $\Gamma^u(p)$ surrounding p and avoiding K^+, so that $\mathrm{Int}(\gamma) \cap K^+$ is open and closed in $K^+ \cap \Gamma^u(p)$ (Here $\mathrm{Int}(\gamma)$ is the bounded connected component of γ^c).

Since $T^+|_{\mathbb{B}}$ has no degree growth, $J^+ \cap \mathbb{B}$ is the disjoint union of a family of vertical disks Γ^s with bounded degree. The disks Γ^s move continuously in the Hausdorff topology. From this we deduce that the union

$$\bigcup_{q \in \mathrm{Int}\,\gamma} \Gamma^s(q)$$

is both open and closed in $J^+ \cap \mathbb{B}$. As γ shrinks to p, these graphs converge to $\Gamma^s(p)$. Hence $\Gamma^s(p)$ is the connected component of p in $J^+ \cap \mathbb{B}$.

Item *(2)* is then obvious. Similarly, if $\mathrm{Int}(K^+) \neq \emptyset$, let L be a horizontal line intersecting $\mathrm{Int}(K^+)$. Then $\mathrm{Int}(K^+) \cap L$ is a bounded open set, and $\partial(\mathrm{Int}(K^+) \cap L) \subset J^+ \cap L$ which is totally disconnected, a contradiction.

For *(4)*, let $J(p)$ be the connected component of J containing p. Then $J(p) \subset \Gamma^s(p) \cap \Gamma^u(p)$ which is a finite set. We conclude that $J(p) = \{p\}$. □

3. Hyperbolic maps with connected J

In this last section we give an analogue to the familiar result in one dimensional dynamics, that for hyperbolic maps connectedness is related to the existence of attracting orbits. Here the Jacobian determinant tells whether attracting or repelling orbits must arise.

THEOREM 3.1. *Let f be a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$, with $|\mathrm{Jac}(f)| \leq 1$. Assume J is connected. Then* $\mathrm{Int}(K^+)$ *is not empty, that is, f has attracting periodic orbits.*

COROLLARY 3.2 ([**BS7**] Corollary A.3). *If f is hyperbolic with connected Julia set, then f is not conservative, i.e.* $|\mathrm{Jac}(f)| \neq 1$.

We actually do believe the analogy with one dimensional dynamics is more complete.

QUESTION 3.3. *If f is a hyperbolic polynomial diffeomorphism with* $|\mathrm{Jac}(f)| \leq 1$ *and no attracting orbits, does it follow that J is totally disconnected?*

Our opinion is that the answer is yes. More precisely we think that for such a f, the currents $T^{\pm}|_{\mathbb{B}}$ should have no degree growth, in the sense of propositions 2.14 and 2.15 above.

Before embarking to the proof we recall some results of [**BS7**]. As before, let

$$V^+ = \{(z,w),\ |z| > |w|\,,\ |z| > R\}\,.$$

It is proved in [**HO1**] that for large enough R, and for an appropriate choice of $(d^n)^{\mathrm{th}}$-root, the sequence $(f^n)^{\frac{1}{d^n}}$ converges in V^+ to a holomorphic function φ^+ (the "Böttcher coordinate") such that $\varphi^+ \circ f = \varphi^d$ and

$$\varphi^+(z,w) = z + O(1) \text{ as } V^+ \ni z \to \infty. \tag{3}$$

Hence φ^+ may serve as an invariant first projection at infinity. We saw in section 1 that when f is unstably connected, the unstable leaves are graphs over the first coordinate near $\partial^v\mathbb{B}$ for large R. By (3), Rouché's Theorem implies these are also graphs for the invariant projection φ^+ –this is the exact meaning of having no unstable critical points.

The level lines of φ^+ define a holomorphic foliation in V^+, and pulling back by f, we get an invariant holomorphic foliation $\mathcal{F}^+$ in $U^+ = \mathbb{C}^2 \setminus K^+$. If f is uniformly hyperbolic and unstably connected, the leaves of the unstable foliation are transverse to those of $\mathcal{F}^+$ in V^+. By using the Lambda lemma, we get the following result.

PROPOSITION 3.4 ([**BS7**] Prop. 2.7). *If f is unstably connected and hyperbolic, the stable lamination $\mathcal{L}^s$ of J^+ and the holomorphic foliation $\mathcal{F}^+$ fit together into a lamination of $J^+ \cup U^+$.*

We study the extension of this foliation to $\mathbb{P}^2$.

LEMMA 3.5. *$\mathcal{F}^+$ extends as a holomorphic foliation of $\overline{V^+} \subset \mathbb{P}^2$ by adding the line at infinity as a leaf.*

PROOF. The proof is a simple change of coordinates. Let $[Z : W : T]$ be the homogeneous coordinates in $\mathbb{P}^2$ such that $z = \frac{Z}{T}$ and $w = \frac{W}{T}$. In V^+ we put $u = \frac{w}{z} = \frac{W}{Z}$ and $v = \frac{1}{z} = \frac{T}{Z}$. In coordinates (u, v), V^+ becomes a bidisk $B_{u,v} = \{|u| < 1,\ |v| < 1/R\}$, with the line at infinity as $(v = 0)$. By (3) we get (with an obvious abuse of notation)

$$\varphi^+(u, v) = \frac{1}{v} + \delta(u, v),$$

where δ is a bounded holomorphic function. By Riemann extension, δ extends as a holomorphic function through the line $(v = 0)$.

Consider the level line $\left\{\varphi^+ = \frac{1}{c}\right\}$ for small c. This equation rewrites as $v(1 - cv\delta(u, v)) = c$. For fixed u, the equation has only one solution in v for small c, depending holomorphically on c. This means that in $B_{u,v}$ the level line $\left\{\varphi^+ = \frac{1}{c}\right\}$ is a graph, tending to $(v = 0)$ as $c \to 0$, which is the desired result. □

We can now proceed to the proof[3] of theorem 3.1. Under the hypotheses of the theorem, f is unstably connected. Assume $\mathrm{Int}(K^+) = \emptyset$. Then $\mathcal{L}^s \cup \mathcal{F}^+$ becomes a lamination of $\mathbb{C}^2$, holomorphic outside J^+. We prove it is globally holomorphic.

For this, we may adapt a result of E. Ghys [**G**], or use the following direct argument. A theorem of R. Bowen and D. Ruelle [**BR**] asserts that $J^+ = W^s(J)$ has Lebesgue measure zero. Fix a flow box for the lamination $\mathcal{F}^+ \cup \mathcal{L}^s$, intersecting J^+. Then for almost every transverse holomorphic disk T, $J^+ \cap T$ has zero measure. We take two such transversals T_1 and T_2. The holonomy map $h : T_1 \to T_2$ is holomorphic in the full measure subset $T_1 \setminus J^+$, and globally quasiconformal, so it has L^2_{loc} derivatives. By Weyl's lemma, h is holomorphic.

We conclude that $\mathcal{F}^+ \cup \mathcal{L}^s$ defines a holomorphic foliation in $\mathbb{C}^2$. This foliation further extends to $\overline{V^+} \subset \mathbb{P}^2$ by adding the line at infinity, by the preceding lemma. Pulling back by f (viewed as a birational map on $\mathbb{P}^2$), allows to extend it to $\mathbb{P}^2 \setminus I(f) = \mathbb{P}^2 \setminus [0 : 1 : 0]$. We have thus obtained a singular holomorphic foliation of $\mathbb{P}^2$, preserved by the dynamics. This is a contradiction due to a theorem of M. Brunella [**Bru**]. □

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