CONTINUITY OF LYAPUNOV EXPONENTS FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

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Abstract. We prove two continuity theorems for the Lyapunov exponents of the maximal entropy measure of polynomial automorphisms of $\mathbb{C}^2$. The first continuity result holds for any family of polynomial automorphisms of constant dynamical degree. The second result is the continuity of the upper exponent for families degenerating to a 1-dimensional map.

1. Introduction

1.1. **The main results.** Let f be a polynomial diffeomorphism of $\mathbb{C}^2$. We define the dynamical degree d as $d = \lim_{n\to\infty}(\deg(f^n))^{1/n} > 1$. Then, according to the work of Bedford, Lyubich and Smillie [BS3, BLS1], f admits a natural invariant measure μ, with two non zero exponents of opposite signs $\chi^-(f) < 0 < \chi^+(f)$, which is the unique measure of maximal entropy $\log d$.

It follows from the work of Friedland and Milnor [FM] that the space Λ of all polynomial diffeomorphisms of dynamical degree d, modulo conjugacy in the group of polynomial diffeomorphisms, has the structure of a finite union of complex manifolds. Notice that conjugating by a polynomial diffeomorphism preserves Lyapunov exponents.

Our first main result is the following.

Theorem 1.1. *Consider the family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d > 1$ modulo conjugacy, as above. Then the Lyapunov exponents of the maximal entropy measure are continuous functions of λ.*

Bedford and Smillie [BS3] had previously shown that $\lambda \mapsto \chi^+(f_\lambda)$ is plurisubharmonic (psh for short), hence in particular upper semicontinuous. Since polynomial automorphisms of $\mathbb{C}^2$ have constant Jacobian, the sum

$$\chi^+(f_\lambda) + \chi^-(f_\lambda) = \log|\mathrm{Jac}(f_\lambda)| \tag{1}$$

is a pluriharmonic function of λ, so χ^- is plurisuperharmonic.

As a consequence of Young's formula [Y] for the dimension of μ,

$$\dim(\mu) = \log d\left(\frac{1}{\chi^+} - \frac{1}{\chi^-}\right), \tag{2}$$

we obtain the following corollary.

Corollary 1.2. *The Hausdorff dimension of μ_λ is a continuous function on parameter space.*

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An important issue in the theory of polynomial automorphisms of the (real or complex) plane is the study of their degenerations to one dimensional maps. The sample model is the family of *generalized Hénon mappings*

$$g_{p,b} : (z, w) \longmapsto (p(z) - bw, z),$$

degenerating to the 1-dimensional map $z \mapsto (z, p(z))$ when the Jacobian b tends to zero. Here p is a polynomial in z. Usually, a dynamical assumption (like hyperbolicity, Collet-Eckmann, or renormalizability) is made on p, and it is studied how this assumption influences the dynamics of the 2 dimensional mapping $g_{p,b}$ when b is small.

It is convenient for our purposes to conjugate $g_{p,b}$ (by a linear map) so that it rewrites as $(z, w) \mapsto (aw + p(z), az)$ and the degenerate form becomes $(p(z), 0)$. We work in the following general setting (see §3 for some examples): consider a complex manifold Λ, and a family of polynomial mappings of $\mathbb{C}^2$, depending holomorphically on λ, of the form

$$f_\lambda(z, w) = (p(z), 0) + R_\lambda(z, w), \tag{3}$$

with p monic and of degree d, and R_λ a polynomial mapping of $\mathbb{C}^2$, of degree $\leq d - 1$, vanishing identically for $\lambda = \lambda_0$. We assume that for λ outside some hypersurface $\Lambda_{\rm degen}$, f_λ is an automorphism of $\mathbb{C}^2$.

For $\lambda \in \Lambda_{\rm degen}$ close to zero, f_λ reduces to a 1 dimensional mapping of degree d, so it has a unique measure of maximal entropy $\log d$ (see Lemma 3.2), and we still denote by $\chi^+(f_\lambda)$ its unique (positive) Lyapunov exponent.

Our continuity result reads as follows. Notice that we make no hypothesis on p.

Theorem 1.3. *Assume that (f_λ) is a degenerating family of polynomial automorphims of the form* (3). *Then $\lambda \mapsto \chi^+(f_\lambda)$ is continuous at λ_0.*

As before, upper semicontinuity has been established in [BS3], so we only have to prove lower semicontinuity.

Observe further that by (1), $\chi^-(f_\lambda)$ tends to $-\infty$ as $\lambda \to \lambda_0$. Hence by combining Young's formula (2) and the formula of Mañe [Ma1] $\dim(\mu_p) = \log d/\chi^+(p)$, we conclude that under the assumptions of the previous theorem, $\lambda \mapsto \dim(\mu_\lambda)$ is continuous at λ_0.

1.2. **One dimensional maps.** To understand our motivations and methods, it is useful to make a détour through one dimensional dynamics. Let $(f_\lambda)_{\lambda\in\Lambda}$ be any holomorphic family of rational maps of $\mathbb{P}^1$, of degree d. For every λ, f_λ admits a unique measure of maximal entropy $\log d$ [L, FLM] with a unique positive exponent $\chi^+(f_\lambda) \geq \log d/2$. This defines a natural psh function $\lambda \mapsto \chi^+(f_\lambda)$ on parameter space. It is a theorem by Mañe [Ma1] that this function is continuous.

The importance of this function is underlined by the following remarkable connection between analysis and dynamics: the bifurcation locus of (f_λ) is precisely the support of the current $dd^c(\chi^+(f_\lambda))$ [DeM]. The geometry of this current and its successive powers is explored in [BB, DF], where the continuity of the potential plays an important technical role.

There are basically two ways of proving continuity in this setting. In a "direct" approach, Mañe [Ma1] considers the following integral formula for χ^+:

$$\chi^+(f_\lambda) = \int_{\mathbb{P}^1} \log \|Df_\lambda\| \, d\mu_\lambda, \tag{4}$$

where $\|\cdot\|$ is the operator norm associated to some Riemannian metric on $\mathbb{P}^1$. It is an easy fact that $\lambda \mapsto \mu_\lambda$ is continuous in the weak topology. This does not immediately imply the continuity of χ^+ because of the possible critical points of f_λ on $\mathrm{Supp}(f_\lambda)$. To overcome this difficulty, it is necessary to study the uniform integrability of functions with logarithmic singularities, which follows for instance by standard potential theoretic estimates. This method has been generalized for the sum of exponents of polynomial-like mappings in higher dimension by Pham [Ph].

The second approach uses more elaborate formulas for χ^+. The prototypical example is the following well known formula (due to Manning [Ma2], Przytycki [Pr] and Sibony [Si2]) for the Lyapunov exponent when f_λ is a polynomial

$$\chi^+(f_\lambda) = \log d + \sum_{c \text{ critical}} G_\lambda(c), \tag{5}$$

where G_λ is the dynamical Green function (see §3 for more details). The Hölder continuity of χ^+ follows immediately. This has been generalized to rational maps of $\mathbb{P}^1$ by DeMarco [DeM] and to certain sums of exponents in higher dimension in [BJ, BB].

1.3. **Methods.** Let us return to polynomial automorphisms of $\mathbb{C}^2$. There is an integral formula analogous to (4) for $\chi^+(f_\lambda)$:

$$\chi^+(f_\lambda) = \int \log \|Df_\lambda|_{e^u}\| \, d\mu_\lambda,$$

where e^u is the *unstable direction*, provided by the Osedelets Theorem, which is defined almost everywhere and varies measurably (see [BLS1] for an adapted presentation). It is unclear how to study the variation of this data as a function of λ.

On the other hand there is a formula, due to Bedford and Smillie [BS5], analogous to (5) in this context. We do not state this formula here (see §2.4 for details), but we indicate that it involves *unstable critical points*, that is, critical points of the Green function G^+ restricted to unstable manifolds. This seems to be an important notion regarding the geometry of the unstable lamination, but it is still poorly understood.

The main idea of the proof of Theorem 1.1 is to prove that the formula of [BS5] varies lower semicontinously. For this, we use a result of [Du3] on the geometry of the unstable lamination for unstably disconnected mappings (see §2.3 for the definition of unstable disconnectedness) that allows a precise counting of unstable critical points.

Likewise, in the proof of Theorem 1.3, we study the "convergence" of the formula of [BS5] to (5) when the Jacobian tends to zero. This involves a description of the geometry of the unstable lamination for perturbations of 1-dimensional maps, regardless of hyperbolicity.

We see that, besides the intrinsic interest of Theorems 1.1 and 1.3, another interesting point in the paper is the contribution to the understanding of the unstable lamination of polynomial diffeomorphisms.

1.4. **Outline.** The structure of the paper is as follows: sections 2 and 3 are respectively devoted to the proofs of Theorems 1.1 and 1.3.

As opposite to the 1-dimensional situation, one feature of unstable critical points is that they are invariant under the dynamics. In an Appendix, we briefly develop a notion of *fastest rate of escape* for critical points, generalizing the corresponding 1-dimensional notion. This is a natural level (of the Green function) where looking at unstable critical points, that provides a new dynamically defined function on parameter space. We also show that it gives an upper estimate for the Lyapunov exponent (Theorem A.2).

2. Proof of theorem 1.1

2.1. **Basics.** We will use some standard facts from the dynamics of polynomial automorphisms of $\mathbb{C}^2$. General references are [FM, BS1, BLS1, Si1]. As usual K^+ (resp. K^-) is the set of points with bounded forward (resp. backward) orbits. We denote the invariant currents by $T^\pm$ and the Green functions by $G^\pm$ so that $T^\pm = dd^cG^\pm$..

We also need the notion of *uniformly laminar current*. Recall that a positive (closed) current is uniformly laminar if locally, up to change of coordinates, it is a current of integration over a measured family of graphs over the unit disk. The reader is referred to [BLS1, Du2] for more details, and to [Dem] for basics on positive currents.

We now introduce a notion of *horizontality* of objects relative to a projection.

Definition 2.1. *Let Ω be an open set, together with a holomorphic locally trivial fibration $\varphi : \Omega \to D$, where D is a disk in $\mathbb{C}$. We say that an analytic subset or current is* horizontal *relative to φ if* $\mathrm{Supp}(T)$ *is (locally uniformly) relatively compact in the fibers of φ. More specifically, we ask that for every $\zeta \in D$, there exists a neighborhood $D' \ni \zeta$ and a compact subset K of Ω such that for every $\zeta' \in D'$, $\varphi^{-1}(\zeta') \cap \mathrm{Supp}(T) \subset K$.*

The simplest case is when $\Omega = D \times \mathbb{C}$ or $D \times D$, and φ is the first projection. In this case we just say that T is horizontal in $D \times \mathbb{C}$ (resp. $D \times D$). Some basic properties of horizontal currents in this context can be found in [Sł, Du1] (see also [DS]).

It is clear that if V is a horizontal analytic subset, the number (couting multiplicities) of intersection points of V and the fibers of φ is locally (hence globally) constant. We call this number the *degree* of V relative to φ. When there is no danger of confusion, we drop the mention to φ and denote it by $\deg(V)$.

There is a related statement for currents. For simplicity we only consider the case where T admits a global potential in Ω.

Definition-Proposition 2.2. *If $T = dd^cu$ is a horizontal positive closed current relative to some $\varphi : \Omega \to D$, then for every $\zeta \in D$, the wedge product $T \wedge [\varphi^{-1}(\zeta)]$ is well defined, and its total mass does not depend on ζ. By definition, we call this mass the* slice mass *of T,* $\mathrm{sm}(T)$*.*

For instance, when V is a horizontal analytic set, $\mathrm{sm}([V]) = \deg(V)$.

Proof. Pick any fiber $F = \varphi^{-1}(\zeta)$. The potential u is pluriharmonic near ∂F, so u cannot be identically $-\infty$ on any connected component of F. Hence $u|_F \in L^1_{\mathrm{loc}}(F)$ and $dd^cu \wedge [F]$ is well defined.

Let $F_0 \Subset F$ be an open subset with smooth (oriented) boundary γ such that $\mathrm{Supp}(T) \cap F \subset F_0$. Then by Stokes' Theorem $\int_F dd^cu = \int_\gamma d^cu$. By assumption, the form d^cu is closed near γ, so by Stokes' Theorem again, the integral $\int_\gamma d^cu$ is insensitive to small perturbations of γ in other fibers. We conclude that $\int_{\varphi^{-1}(\zeta)} dd^cu$ is locally constant as a function of ζ. $\square$

Remark 2.3. To get the conclusion of the proposition, we do not really need φ to be a fibration. What is important is the existence of a smoothly varying γ enclosing $\mathrm{Supp}(T)$ in the fibers, so that we can apply Stokes' Theorem in the same way. This observation will be useful in the proof of Theorem A.2.

2.2. **Preparation.** Recall that the dynamical degree of a polynomial diffeomorphism f is defined by $d = \lim_{n\to\infty}(\deg(f^n))^{1/n} > 1$. By Friedland and Milnor [FM], a polynomial automorphism of $\mathbb{C}^2$ has non trivial dynamics if and only if its dynamical degree is larger than

1. Since the space of polynomial automorphisms of dynamical degree d is a finite union of complex manifolds, it will be enough to work in the following setting: we consider a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms of dynamical degree d, parameterized by some complex manifold Λ. Without loss of generality we assume that Λ is an open subset of $\mathbb{C}^N$ for some N, and $0\in\Lambda$. We denote the upper Lyapunov exponent by $\chi^+(f_\lambda)$ ($\chi^+(\lambda)$ for short). We will prove the continuity of χ^+ at $\lambda=0$.

In the next lemma we show that in the neighborhood of 0, the mappings f_λ may be written in a simpler form. We classically consider the extension of polynomial automorphisms as rational maps of the projective plane $\mathbb{P}^2$, and denote by $I^+(f)$ (resp $I^-(f)$) the indeterminacy point of f (resp f^{-1}); see [Si1] for more details. We also consider homogeneous coordinates $[z:w:t]$ on $\mathbb{P}^2$ so that our $\mathbb{C}^2$ imbeds as $(z,w)\mapsto[z:w:1]$.

Lemma 2.4. *There exists an open subset $\Lambda'\subset\Lambda$ containing 0, and a holomorphic family $(\varphi_\lambda)_{\lambda\in\Lambda'}$ of polynomial automorphisms of $\mathbb{C}^2$ such that for $\lambda\in\Lambda'$,*

$$\varphi_\lambda^{-1}f_\lambda\varphi_\lambda(z,w)=(z^d,0)+\text{ lower degree terms.}\tag{6}$$

Proof. It is well known [FM] that every polynomial automorphism of $\mathbb{C}^2$ with non trivial dynamics is conjugated in the group of polynomial automorphisms of $\mathbb{C}^2$ to a composition generalized complex Hénon mappings, which can be written under the form $(z,w)\mapsto(aw+p(z),az)$, with p monic and of degree d. Let φ_0 be such a conjugating map for f_0. We get that $I^+(\varphi_0^{-1}f_0\varphi_0)=[0:1:0]$ and $I^-(\varphi_0^{-1}f_0\varphi_0)=[1:0:0]$.

Now for small λ, $I^+(\varphi_0^{-1}f_\lambda\varphi_0)$ is close to $[0:1:0]$. Also, f_λ^{-1} depends holomorphically on λ. Indeed, we know that for each λ, f_λ^{-1} is a polynomial map of degree d, and it is of course completely determined by its restriction to a small open set. In such an open set, the holomorphic dependence of f_λ^{-1} on λ follows from the Implicit Function Theorem. In particular we get that $\lambda\mapsto I^-(\varphi_0^{-1}f_\lambda\varphi_0)$ is holomorphic.

Hence for small λ, $I^+(\varphi_0^{-1}f_\lambda\varphi_0)\neq I^-(\varphi_0^{-1}f_\lambda\varphi_0)$ and we get that $\varphi_0^{-1}f_\lambda\varphi_0$ has non trivial dynamics. Conjugating with a holomorphically varying linear map ℓ_λ, with $\ell_0=id$ we can ensure that $I^+((\varphi_0\ell_\lambda)^{-1}f_\lambda(\varphi_0\ell_\lambda))=[0:1:0]$ and $I^-((\varphi_0\ell_\lambda)^{-1}f_\lambda(\varphi_0\ell_\lambda))=[1:0:0]$.

Inspecting the higher order terms shows that

$$(\varphi_0\ell_\lambda)^{-1}f_\lambda(\varphi_0\ell_\lambda)(z,w)=(c(\lambda)z^d,0)+\text{ lower degree terms,}$$

with $c(0)=1$. For small λ, there exists a holomorphic determination of $c(\lambda)^{-1/(d-1)}$, and conjugating with $(z,w)\mapsto(c(\lambda)^{-1/(d-1)}z,w)$ gives the desired result. □

Lyapunov exponents are preserved under smooth conjugacy, so, slightly abusing notation, in the following we replace f_λ by $\varphi_\lambda^{-1}f_\lambda\varphi_\lambda$ and Λ by Λ', so that we can assume that the conclusion of the lemma holds.

Since polynomial automorphisms have constant Jacobian, $\chi^+(f)+\chi^-(f)=\log|\mathrm{Jac}(f)|$ varies continuously. Hence it is enough to consider the continuity of χ^+. Bedford and Smillie proved in [BS3] that $\lambda\mapsto\chi^+(\lambda)$ is plurisubharmonic, hence in particular upper semicontinuous. We recall their argument for completeness. Let $\|\cdot\|$ be any operator norm on the space of linear maps of $\mathbb{C}^2$. For $n\in\mathbb{N}$, let

$$\chi_n^+(\lambda)=\frac{1}{n}\int\log\|Df_\lambda^n\|\,d\mu_\lambda,$$

where Df is the tangent map. From the chain rule we get that χ_n^+ satisfies the following subadditivity property

$$(m+n)\chi_{m+n}^+ \leq m\chi_m^+ + n\chi_n^+.$$

In particular the sequence $\chi_{2^n}^+$ is decreasing.

On the other hand, for every n, $\lambda \mapsto \chi_n^+(\lambda)$ is continuous. This is a property of continuous variation of the maximal entropy measure, which itself easily follows from the joint continuity of the escape rate functions $G_\lambda^\pm(z,w)$ in the variable $(\lambda,(z,w))$ [BS1]. We conclude that χ^+ is a decreasing limit of continuous functions, hence upper semicontinuous.

Once upper semicontinuity is established, plurisubharmonicity follows from instance from the fact that χ^+ can be evaluated on saddle orbits [BLS2].

We infer that it is enough to prove that χ^+ is lower semicontinuous, that is,

$$\liminf_{\lambda\to 0} \chi^+(\lambda) \geq \chi^+(0). \tag{7}$$

Another result from [BS3] is that χ^+ is always bounded from below by $\log d$, so (7) is automatic when $\chi^+(0) = \log d$. Moreover, replacing f_0 by f_0^{-1} if necessary, we may assume that $|\mathrm{Jac}(f_0)| \leq 1$, in which case we have the following [BS5, BS6]

$$\chi^+(f_0) = \log(d) \Leftrightarrow J(f_0) \text{ is connected } \Leftrightarrow f_0 \text{ is unstably connected.}$$

We will explain what unstable connectedness means shortly. For the moment we conclude that Theorem 1.1 reduces to the following proposition.

Proposition 2.5. *Assume that $(f_\lambda)_{\lambda\in\Lambda}$ is a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$ of degree d, with $I^+(f_\lambda) = [0:1:0]$ and $I^-(f_\lambda) = [1:0:0]$. Assume further that $|\mathrm{Jac}(f_0)| \leq 1$ and f_0 is unstably disconnected. Then χ^+ is lower semicontinuous at f_0.*

2.3. **Laminar structure.** In this paragraph we explain some results of [Du3] on the laminar structure of T^- for unstably disconnected mappings.

Let f be a polynomial automorphism satisfying (6). Then if R is large enough, f stretches the horizontal direction in D_R^2 (here $D_R = D(0,R)$) in the sense that in D_R^2 the image of a horizontal manifold is horizontal, and the degree is multiplied by d.

The following definition was introduced in [BS6].

Definition 2.6. *f is* unstably disconnected *if for some saddle point p, $W^u(p)\cap K^+$ has a compact component (for the topology induced by the isomorphism $W^u(p)\simeq\mathbb{C}$).*

This condition is actually independent of the saddle point p [BS6] (see also [Du3, Prop. 1.8]). Now if V_0 is a disk in $W^u(p)$ such that $V_0 \cap K^+ \Subset V_0$, pushing V_0 sufficiently many times gives rise to a horizontal disk V of finite degree in D_R^2, lying inside $W^u(p)$.

In [Du3, Prop. 2.3] we proved that V is *subordinate to* T^- in the sense that there exist a non trivial uniformly laminar current $S \leq T^-$, made up of a family of disjoint horizontal disks of degree $\deg(V)$, and V is a member of this family, lying in the support of S. As an easy consequence, we get the following basic result on the structure of T^- [Du3, Th. 2.4].

Theorem 2.7. *If f is unstably disconnected, then there exists a sequence of currents T_k^- in D_R^2, with*

$$T^-|_{D_R^2} = \sum_{k=1}^{\infty} T_k^-,$$

and T_k^- is the integration current over a family of disjoint irreducible horizontal disks of degree k.

Of course the same holds for any sub-bidisk of the form $D \times D_R$, but we stress that *the decomposition depends on the bidisk.* Notice that, even if the degree is bounded, this need not imply that T^- is uniformly laminar. Indeed a sequence of disks of degree 2 may accumulate on a disk of degree 1, giving rise to some folding.

Definition 2.8. *We say that a horizontal current T in a bidisk has finite degree K if it is an integral of disjoint horizontal analytic subsets of degree $0 \leq k \leq K$. These analytic subsets are said to be subordinate to T.*

We denote by $\mathcal{F}_K(D \times \mathbb{C})$ the class of currents of finite degree K. This is relative to some projection which should be clear from the context.

We will need the following (expected) proposition.

Proposition 2.9. *Assume that (T_n) is a sequence of currents in $\mathcal{F}_k(D \times \mathbb{C})$, weakly converging to T, and whose supports are contained in a fixed vertically compact subset. Then $T \in \mathcal{F}_k$.*

Proof. For the proof, we say that two subvarieties are *compatible* if they do not have isolated intersection points. We work in the unit bidisk $\mathbb{B} = \mathbb{D}^2$. Assume that T_n is a sequence of (uniformly) horizontal currents of finite degree K, converging to T. Using Bishop's compactness theorem for curves of bounded volume [Bi] and the Hurwitz lemma, we will first construct a family of horizontal analytic sets to which T should be subordinate, and then indeed prove the subordination by using the Hahn-Banach Theorem.

The first observation is that if V_n is a sequence of curves in $\mathbb{B}$, with $V_n \subset \mathbb{D} \times D_{1-\varepsilon}$ for some ε, and of degree $\leq K$, then it has locally finite area (see e.g. [Sł, Lemma 3.6]), so there exists a converging subsequence by Bishop's Theorem. It is obvious that any cluster value must have degree $\leq K$. Notice that the limit may be singular. Another useful observation is that if (V_n) and (V'_n) are two converging sequences of varieties of degree $\leq K$, with $V_n \cap V'_n = \emptyset$, then by Hurwitz' Theorem, their limits are compatible.

Let $X_0 = \limsup(\mathrm{Supp}(T_n)) \supset \mathrm{Supp}(T)$, and $x_0 \in X_0$. There exists a sequence $x_n \in \mathrm{Supp}(T_n)$ converging to x_0; let $V_n(x_n)$ be the analytic set subordinate to T_n passing through x_n. There exists a subsequence $\varphi_0(n)$ so that $V_{\varphi_0(n)}(x_{\varphi_0(n)})$ converges to a $V(x_0) \ni x_0$ of degree $\leq K$.

Let now $X_1 = \limsup(\mathrm{Supp}(T_{\varphi_0(n)})) \supset \mathrm{Supp}(T)$. By construction, $x_0 \in X_1$. Let r_1 be the supremum of the radii of balls centered at points of X_1 and avoiding x_0, and let $x_1 \in X_1$ so that $x_0 \notin B(x_1, \frac{r_1}{2})$. Let $x_{\varphi_0(n)} \in \mathrm{Supp}(T_{\varphi_0(n)})$, $x_{\varphi_0(n)} \to x_1$, and $V_{\varphi_0(n)}(x_{\varphi_0(n)})$ be subordinate to $T_{\varphi_0(n)}$ through $x_{\varphi_0(n)}$. From $\varphi_0(n)$, extract a subsequence $\varphi_1(n)$ such that this sequence of analytic sets converges to $V_1 \ni x_1$. By construction, V_0 and V_1 are compatible, and have degree $\leq K$.

Inductively we construct a sequence of successive extractions $\varphi_k(n)$, a decreasing sequence of closed sets $X_k \supset \mathrm{Supp}(T)$, a sequence of points x_k with $x_0, \ldots, x_k \in X_k$, together with a family of compatible analytic subsets $V_0, \ldots, V_k$. Let then $X_{k+1} = \limsup(\mathrm{Supp}(T_{\varphi_k(n)}))$. The point x_{k+1} is chosen in the following way: let r_{k+1} be the supremum of radii of balls centered at points of X_{k+1} and avoiding $x_0, \ldots, x_k$. We choose x_{k+1} such that $x_0, \ldots, x_k \notin B(x_{k+1}, \frac{r_{k+1}}{2})$, and the extraction φ_{k+1}, and V_{k+1} as for the case $k = 0$.

Let $X = \bigcap_{k \geq 0} X_k \supset \mathrm{Supp}(T)$. It is an exercise to show that the sequence (x_k) is dense in X. Attached to each x_k there is an analytic set V_k of degree $\leq K$. Let $\mathcal{L}$ be the closure of the family (V_k); $\mathcal{L}$ is a family of compatible analytic sets of degree $\leq K$. In particular for every $x \in X$, there is a unique $V(x) \in \mathcal{L}$ containing x.

It remains to see that T is an integral of the varieties in $\mathcal{L}$, or equivalently, that $T \in \mathrm{Conv}(\mathcal{L})$. Assume not. Then by the Hahn-Banach Theorem, there exists a smooth test (1,1) form ϕ such that $\langle T, \phi\rangle < 0$ while $\langle [V], \phi\rangle > 0$ for every $V \in \mathcal{L}$. Let φ_∞ be a diagonal extraction of the φ_k, and for ease of notation denote by $\varphi_\infty(n)$ by n'. For large n', $\langle T_{n'}, \phi\rangle < 0$, so there exists an analytic set $V_{n'}$ subordinate to $T_{n'}$ such that $\langle [V_{n'}], \phi\rangle < 0$. Extract a further subsequence (still denoted by n') so that $V_{n'}$ converges to some V (V needn't be contained in $\mathrm{Supp}(T)$, this is the reason for the limsup above). We claim that $V \in \mathcal{L}$. Indeed, let $x_{n'} \in V_{n'}$ be convergent to x. Then $x \in X$. Now if $V \neq V(x)$, since near x, $V \cap V(x) = \{x\}$, V would have isolated intersection points with $V_{n'}$ for large n', a contradiction.

By construction, $V \in \mathcal{L}$ and $\langle [V], \phi\rangle \leq 0$ which is contradictory. This finishes the proof. □

2.4. **The formula of** [BS5]**.** In this paragraph we describe the formula of [BS5] for the Lyapounov exponents. The results in this section do not require unstable disconnectedness. We start with a temporary definition.

Definition 2.10. *An unstable critical point is a critical point of $G^+|_{W^u(p)}$, where p is some saddle periodic point.*

Every unstable manifold is equidistributed along the unstable current T^-, so, following [BS5], it is natural to define an unstable critical measure by extending the notion of critical point to Pesin unstable manifolds and integrate against the transverse measure. The formal definition is a bit delicate because of the weakness of the laminar structure of T^-. Here we provide a precise definition using the results of [Du2].

An embedded holomorphic disk Δ is said to be *subordinate* to T^- if there exists a uniformly laminar current $0 < S \leq T^-$ such that Δ lies inside a leaf of the lamination induced by S. We can now revise Definition 2.10.

Definition 2.11. *An unstable critical point is a critical point of $G^+|_\Delta$, where Δ is any disk subordinate to T^-.*

By *flow box* we mean a piece of lamination, biholomorphic to a union of graphs in the bidisk. In [Du2, Th. 1.1] we proved that if $\mathcal{L}$ is any flow box, T^- induces by restriction an invariant transverse measure on $\mathcal{L}$. Moreover it is clear from the construction of the laminar structure of T^- [BLS1, BS5] that there exists a countable family $(\mathcal{L}_i)$ of (overlapping) flow boxes so that every disk subordinate to T^- is contained in some union of $\mathcal{L}_i$. We say that $(\mathcal{L}_i)$ is a *complete system* of flow boxes associated to T^-.

Observe that by the Radon Nikodym Theorem, it is possible to define the supremum of two measures in the following way: if μ_1 and μ_2 are two σ-finite positive measures, there exist measurable bounded functions f_i, $i = 1, 2$ such that $\mu_i = f_i(\mu_1 + \mu_2)$. By definition $\sup(\mu_1, \mu_2) = \sup(f_1, f_2)(\mu_1 + \mu_2)$. We may hence define the supremum of a finite family of measures by induction, and if $(\mu_i)_{\{i\geq 1\}}$ is sequence of positive measures, we define $\sup(\mu_i)_{\{i\geq 1\}}$ as the increasing limit of $\sup(\mu_i)_{\{1\leq i\leq I\}}$ as $I \to \infty$. It is not clear of course that the limiting measure will have locally finite mass.

We are now ready to define the critical measure.

Definition 2.12. *Let $\mathcal{L}$ be a flow box $\mathcal{L} = \bigcup_{t\in\tau} L_t$, where τ is a global transversal to $\mathcal{L}$, and write $T^-|_{\mathcal{L}} = \int_\tau [L_t] d\mu_{\mathcal{L}}(t)$, where $\mu_{\mathcal{L}}$ is the measure induced by T^- on τ.*

The critical measure restricted to $\mathcal{L}$ *is defined by*

$$\mu_c^-|_{\mathcal{L}} = \int_\tau \left[\mathrm{Crit}(G^+|_{\mathcal{L}_t})\right] d\mu_{\mathcal{L}}(t),$$

where $\mathrm{Crit}(G^+|_{\mathcal{L}_t})$ *is the sum of point masses at critical points of* G^+ *on* $\mathcal{L}_t$*, counting multiplicities.*

The global critical measure μ_c^- *is now defined by* $\mu_c^- = \sup(\mu_c|_{\mathcal{L}_i})$ *where* $(\mathcal{L}_i)$ *is a countable complete system of flow boxes associated to* T^-*, as above.*

This definition is independent of the choice of the system of flow boxes. Indeed if $(\mathcal{L}'_j)_j$ is another choice, the respective leaves of $(\mathcal{L}_i)_i$ and $(\mathcal{L}'_j)_j$ are compatible because $T^- \wedge T^- = 0$ (see also [Du2, Th. 1.1]), and the result follows by considering the system of flow boxes $(\mathcal{L}_i \cap \mathcal{L}'_j)_{i,j}$.

We can now state the main result of [BS5], upon which the proof of continuity will be based.

Theorem 2.13 ([BS5])**.** *The upper Lyapounov exponent of the maximal entropy measure satisfies*

$$\chi^+ = \log d + \int_{1 \le G^+ < d} G^+ d\mu_c^- = \log d + \int_P G^+ d\mu_c^-, \tag{8}$$

where P *is any measurable fundamental domain for* $f|_{\mathbb{C}^2 \setminus K^+}$*.*

2.5. **The total mass of the critical measure.** In this paragraph we consider an unstably disconnected polynomial automorphism f, satisfying (6). We will give a formula for the mass of the critical measure in certain domains.

We introduce the Böttcher function φ^+. If R is a positive real number, we classically denote by V_R^+ the forward invariant open set

$$V_R^+ = \{(z,w),\ |z| > R, |w| < |z|\}.$$

Recall that $f(z,w) = (z^d, 0) +$ *l.o.t.* near infinity. By analogy with the Böttcher coordinate in one variable dynamics, for large enough R and $(z,w) \in V_R^+$, we can define [HO]

$$\varphi^+(z,w) = \lim_{n\to\infty} (\pi_1 \circ f^n)^{\frac{1}{d^n}}$$

where the $(1/d^n)^{\text{th}}$ root is chosen so that $\varphi^+(z,w) = z + O(1)$ at infinity. It is a holomorphic function in V_R^+ satisfying the functional equation $\varphi^+ \circ f = (\varphi^+)^d$ and G^+ equals $\log|\varphi^+|$.

Geometrically, it should be understood as an invariant first projection near infinity. The following lemma is obvious.

Lemma 2.14. *Unstable critical points in* V_R^+ *are points of tangency between unstable manifolds and the fibers of* φ^+*. Moreover the multiplicity of a critical point as a critical point of* G^+ *and as a vertical tangency coincide.*

Since we are not going to consider φ^-, from now on we write φ for φ^+. For the same reason we drop the minus sign from μ_c^-. By Rouché's Theorem, (φ, w) is a coordinate system in V_R^+ for large enough R. Furthermore, if Q is a bounded simply connected open subset of $\{|z| > R + C\}$, where C is such that $|\varphi(z,w) - z| < C$ in V_R^+, then T^- is horizontal in $\varphi^{-1}(Q) \cap V_R^+$, and admits a decomposition like the one in Theorem 2.7, relative to φ. Indeed, consider such a decomposition in $\{|z|, |w| < R'\}$, with $R' \gg R$ and restrict it to $\varphi^{-1}(Q) \cap V_R^+$.

Another important remark is that since the fibers of φ are vertical graphs in V_R^+, T^- has slice mass 1 with respect to the projection φ. We now fix such a R, and for notational simplicity, we write $\varphi^{-1}(Q)$ for $\varphi^{-1}(Q)\cap V_R^+$.

The decomposition of Theorem 2.7 induces a formula for the mass of μ_c in $\varphi^{-1}(Q)$.

Proposition 2.15. *Let Q be a bounded simply connected open subset of $\{|z|>R\}$, and $T^-|_{\varphi^{-1}(Q)}=\sum_{k=1}^\infty T_k$ be the decomposition of T^- in $\varphi^{-1}(Q)$ relative to φ, as above. Then*

$$\mu_c(\varphi^{-1}(Q))=\sum_{k=1}^{\infty}\frac{k-1}{k}\mathrm{sm}(T_k).$$

Since $\sum \mathrm{sm}(T_k)=1$ this implies in particular (without using Theorem 2.13) that the critical measure has locally finite mass.

Proof. It is enough to compute the contribution of T_k for each k. Recall that T_k is made of horizontal disks of degree k over Q. Consider such a disk Δ and a uniformly laminar current $S\leq T_k$ supported in a small tubular neighborhood of Δ. Fix a vertical fiber $\varphi^{-1}(z)$ transverse to Δ, hence intersecting Δ in exactly k points. If $\mathrm{Supp}(S)$ is close enough to Δ, the same holds for every leaf of S. Hence, as a flow box, the transverse measure of S is $\frac{1}{k}\mathrm{sm}(S)$. Now, by the Riemann-Hurwitz formula, each leaf of S carries exactly $k-1$ critical points, counting multiplicities, so the contribution of S to the mass of the critical measure is $\frac{k-1}{k}\mathrm{sm}(S)$.

Exhausting T_k^- by such uniformly laminar currents finishes the proof. □

2.6. **Conclusion.** We return to the setting of §2.2, and consider a family of polynomial automorphisms satisfying the assumptions of Proposition 2.5. Under these assumptions, it is an easy fact that the escape radius R of the previous paragraph is locally uniformly bounded, and the resulting Böttcher function φ_λ is defined on a fixed V_R^+ and depends holomorphically on λ.

We claim that Proposition 2.5 follows from the following.

Proposition 2.16. *Under the above assumptions, let Q be a simply connected open subset of $\{|z|>R\}$, with piecewise smooth boundary. Then the critical mass of $\varphi_\lambda^{-1}(Q)$ is lower semicontinuous, that is (with obvious notation)*

$$\liminf_{\lambda\to 0}\mu_{c,\lambda}\left(\varphi_\lambda^{-1}(Q)\right)\geq \mu_{c,0}\left(\varphi_0^{-1}(Q)\right).$$

Proof of Proposition 2.5 assuming Proposition 2.16. By (8), it is enough to prove that

$$\lambda\longmapsto\int_{\left\{A\leq G_\lambda^+<dA\right\}}G_\lambda^+d\mu_{c,\lambda}$$

is lower semicontinuous at $\lambda=0$ for some A. We choose A so large that if $Q\subset\left\{e^A<|z|<e^{dA}\right\}$, then T^- is horizontal in $\varphi^{-1}(Q)\cap V_R^+$. Slightly moving A if necessary we can further assume that $\int_{\left\{G_0^+=A\right\}}G_0^+d\mu_{c,0}=0$.

Consider a sequence $(\mathcal{Q}_n)$ of subdivisions of the annulus $\left\{e^A<|z|<e^{dA}\right\}$ by simply connected and piecewise smoothly bounded pieces ("squares") of size smaller than $\frac{1}{n}$. We may further assume that $\mu_{c,0}\left(\varphi_0^{-1}(\partial\mathcal{Q}_n)\right)=0$. Then, since G_0^+ is continuous and constant along the fibers of φ_0,

$$\int_{\left\{A\leq G_0^+<dA\right\}}G_0^+d\mu_{c,0}=\lim_{n\to\infty}\sum_{Q\in\mathcal{Q}_n}\left(\inf_{\varphi_0^{-1}(Q)}G_0^+\right)\ \mu_{c,0}\left(\varphi_0^{-1}(Q)\right),$$

where the limit on the right hand side is increasing. For $\lambda \neq 0$, we have a similar result, except that the critical measure could charge the boundary of some subdivision, so that

$$\int_{\left\{A \leq G_\lambda^+ < dA\right\}} G_\lambda^+ d\mu_{c,\lambda} \geq \lim_{n\to\infty} \sum_{Q\in\mathcal{Q}_n} \left(\inf_{\varphi_\lambda^{-1}(Q)} G_\lambda^+\right) \mu_{c,\lambda}\left(\varphi_0^{-1}(Q)\right).$$

Let $h_n(\lambda)$ be the sum of the right hand side, and $h_\infty(\lambda)$ be its (increasing) limit. Notice further that since $G_\lambda^+ = \log|\varphi_\lambda|$, $\inf_{\varphi_\lambda^{-1}(Q)}(G_\lambda^+) = \inf_{z\in Q}(\log|z|)$ does not depend on λ. Hence by Proposition 2.16, $h_n(\lambda)$ is lower semicontinuous at 0 for every n. Lower semicontinuity is preserved under increasing limits so we get that

$$\liminf_{\lambda\to 0} \int_{\left\{A \leq G_\lambda^+ < dA\right\}} G_\lambda^+ d\mu_{c,\lambda} \geq \liminf_{\lambda\to 0} h_\infty(\lambda) \geq h_\infty(0) = \int_{\left\{A \leq G_0^+ < dA\right\}} G_0^+ d\mu_{c,0},$$

and the result follows. □

Proof of Proposition 2.16. We fix Q as in the statement of the proposition, and denote by $Q^\delta = \{z \in Q,\ \mathrm{dist}(z, \partial Q) > \delta\}$. For small δ, Q^δ is a topological disk.

The following lemma is easy and left to the reader.

Lemma 2.17. *If $\delta > 0$ and Q are fixed, then there exists a neighborhood N of $0 \in \Lambda$ depending only on δ and Q, such that if $\lambda \in N$ and C is any horizontal curve of degree k in $\varphi_\lambda^{-1}(Q)$ (relative to the projection φ_λ), then $C \cap \varphi_0^{-1}(Q^\delta)$ is a horizontal curve of degree k relative to φ_0.*

Observe that if C is a disk, and if $C \cap \varphi_0^{-1}(Q^\delta)$ is disconnected, then by the maximum principle it is a union of disks.

Recall the notation $\mathcal{F}_K(\varphi_\lambda^{-1}(Q))$ for the set of currents of finite degree K over Q relative to the projection φ_λ, which was introduced after Definition 2.8. By the previous lemma, if δ is fixed and λ is small enough, then

$$S \in \mathcal{F}_K(\varphi_\lambda^{-1}(Q)) \Rightarrow S|_{\varphi_0^{-1}(Q^\delta)} \in \mathcal{F}_K(\varphi_0^{-1}(Q^\delta)).$$

An important further remark is that the slice mass of S is invariant under small perturbations of the transversal, so in particular it does not depend on λ.

We will prove that for every $\delta > 0$,

$$\liminf_{\lambda\to 0} \mu_{c,\lambda}\left(\varphi_\lambda^{-1}(Q)\right) \geq \mu_{c,0}\left(\varphi_0^{-1}(Q^\delta)\right), \tag{9}$$

whence the desired result by letting $\delta \to 0$. For this, we use decompositions $T^-(\lambda) = \sum T_k(\lambda)$ with the following conventions:

- for $\lambda \neq 0$, the decomposition is relative to $\varphi_\lambda^{-1}(Q)$ (and the projection φ_λ);
- for $\lambda = 0$, the decomposition is relative to $\varphi_0^{-1}(Q^\delta)$ (and the projection φ_0).

We also use the following notation: $m_k(\star) = \mathrm{sm}(T_k(\star))$, $T^-_{\leq K}(\star) = \sum_{1\leq k\leq K} T_k(\star)$, and $M_{\leq K}(\star) = \mathrm{sm}(T_{\leq K}(\star))$, where $\star$ stands for 0 or λ.

The first observation is that the locus of unstable disconnectivity is open (see [Du3, §2.1], this actually follows from the discussion below). In particular for small λ, Proposition 2.15 applies.

As $\lambda \to 0$, by Proposition 2.9 every cluster value S of $T^-_{\leq K}(\lambda)$ in $\varphi_0^{-1}(Q^\delta)$ belongs to $\mathcal{F}_K(\varphi_0^{-1}(Q^\delta))$ and satisfies $S \leq T^-(0)$, so $S \leq T^-_{\leq K}(0)$. From this we get that

$$\limsup_{\lambda\to 0} M_{\leq K}(\lambda) \leq M_{\leq K}(0). \tag{10}$$

We want to prove that

$$\liminf_{\lambda\to 0} \sum_{k=1}^{\infty} \frac{k-1}{k} m_k(\lambda) \geq \sum_{k=1}^{\infty} \frac{k-1}{k} m_k(0). \tag{11}$$

Since $T^-(\star)$ has total slice mass 1, we infer that $\sum_{k=1}^\infty m_k(\star) = 1$. In particular,

$$\sum_{k=1}^{\infty} \frac{k-1}{k} m_k(\star) = 1 - \sum_{k=1}^{\infty} \frac{m_k(\star)}{k}. \tag{12}$$

We now make the following classical "integration by parts" on series: $m_k(\star) = M_{\leq k}(\star) - M_{\leq k-1}(\star)$, with the convention that $M_0 = 0$, so that

$$\sum_{k=1}^{\infty} \frac{m_k(\star)}{k} = \sum_{k=1}^{\infty} \frac{M_{\leq k}(\star)}{k(k+1)}.$$

So by (10) we get that

$$\limsup_{\lambda\to 0} \sum_{k=1}^{\infty} \frac{m_k(\lambda)}{k} \leq \sum_{k=1}^{\infty} \frac{m_k(0)}{k},$$

which from (12) implies (11). This concludes the proof. $\square$

3. Extended parameter space

In this section we prove that the upper Lyapunov exponent is continuous when the (f_λ) degenerate to a 1-dimensional map. In a preliminary subsection, we start with some general considerations on degenerating families of polynomial automorphisms.

3.1. **Degenerating families of polynomial automorphisms.** There is no classification of degenerating families of polynomial automorphisms in the literature, so we will consider a general, but possibly non exhaustive, situation.

Consider as parameter space a neighborhood Λ of the origin in $\mathbb{C}^n$, and a family of polynomial mappings of $\mathbb{C}^2$, depending holomorphically on λ, of the form

$$f_\lambda(z, w) = (p(z), 0) + R_\lambda(z, w), \tag{13}$$

with p monic and of degree d, R_λ a polynomial mapping of degree $\leq d-1$ with $R_0 \equiv 0$, and such that for λ outside some hypersurface Λ_{degen}, f_λ is an automorphism of $\mathbb{C}^2$.

A typical such situation is as follows: fix integers $d_1, \ldots, d_m$, with $d_i \geq 2$, and let $d = d_1 \cdots d_m$. Consider a family of polynomial automorphisms

$$f_\lambda = f_{m,\lambda} \circ \cdots \circ f_{1,\lambda} \text{ , where } f_{i,\lambda} = (a_{i,\lambda} w + p_{i,\lambda}(z), a_{i,\lambda} z), \tag{14}$$

where $a_{i,\lambda}$ are holomorphic functions of λ, $p_{i,\lambda}$ are monic polynomials of degree d_i, depending holomorphically on λ, and $a_{i_0,0} = 0$ for some i_0. By conjugating with an appropriate composition of the f_i, without loss of generality we may assume that $i_0 = m$. After this conjugacy, this family of automorphisms is of the form (13).

Recall from [FM] that any polynomial automorphism of $\mathbb{C}^2$ with non trivial dynamics is of the form (14), up to conjugacy.

In the next lemmas, we collect some well known results on the extension of the dynamics to $\Lambda_{\rm degen}$. Recall that $V_R^+ = \{(z,w),\ |z| > |w|\,,\ |z| > R\}$, and π_1 denotes the first projection in $\mathbb{C}^2$.

Lemma 3.1. *For $\lambda \in \Lambda$, let $G_\lambda^+(z,w) = \lim_{n\to\infty} \log^+ \|f_\lambda^n(z,w)\|$. Then G^+ is continous on Λ as a function of (λ, z, w).*

Also, reducing Λ if necessary, there exists a fixed $R > 0$ so that the Böttcher function $\varphi_\lambda(z,w) = \lim_{n\to\infty}(\pi_1 \circ f^n(z,w))^{\frac{1}{d^n}}$ is well defined in V_R^+, and is jointly holomorphic in (λ, z, w).

Proof. We sketch the proof for completeness. Extending the polynomial mappings f_λ to the projective plane $\mathbb{P}^2$ (with coordinates $[z : w : t]$), we find that the indeterminacy set I_λ^+ is constant and equal to $[0 : 1 : 0]$. Also, f_λ maps the line at infinity with I^+ deleted on $[1 : 0 : 0]$ so when $\lambda \notin \Lambda_{\rm degen}$, the indeterminacy set of f_λ^{-1} is $[1 : 0 : 0]$. In particular f_λ is regular in the sense of Sibony [Si1], and has entropy $\log d$.

There are several ways of proving the continuity of $(\lambda, z, w) \mapsto G_\lambda^+(z,w)$. Following [Si1], we use the homogeneous lift F_λ of f_λ to $\mathbb{C}^3$ (see [Si1] for more details). Let $\pi : \mathbb{C}^3 \to \mathbb{P}^2$ be the natural projection. For λ close to 0, there exists uniform constants c and C such that if $\pi(p) \in \mathbb{P}^2$ is far away from I^+, $c\,\|p\|^d \le F_\lambda(p) \le C\,\|p\|^d$. Moreover, every point escapes any small neighborhood of I^+ after finitely many iterations of f_λ, whenever λ is in $\Lambda_{\rm degen}$ or not. Indeed when $\lambda \notin \Lambda_{\rm degen}$ this is classical, and when $\lambda \in \Lambda_{\rm degen}$, this follows from the fact that the image of f_λ is a variety not going through I^+. We conclude that the limit defining $G_\lambda^+(z,w)$ is locally uniform.

The second assertion of the lemma is a simple consequence of the construction of the Böttcher function [HO] and is left to the reader. □

A consequence which will be important for us is that when $\lambda = 0$, both G_0^+ and φ_0 depend only on z, so that at the limit the unstable critical points become points where unstable manifolds have vertical tangencies.

Lemma 3.2. *If $\lambda \in \Lambda_{\rm degen}$ is close enough to zero, then $f_\lambda(\mathbb{C}^2) = M_\lambda$ is a (possibly singular) subvariety of degree $\le d$, and $f_\lambda|_{M_\lambda}$ is conjugated to a polynomial map of degree d.*

Moreover $T_\lambda^- := [M_\lambda]/\deg(M_\lambda)$ is the unique closed positive current invariant under $\frac{1}{d}(f_\lambda)_*$, and the maximal entropy measure of f_λ is $T_\lambda^+ \wedge T_\lambda^- = dd^c G_\lambda^+ \wedge T_\lambda^-$.

It is clear that M_λ is irreducible. We do not know any example where M_λ is singular. As a simple illustration, let

$$g_{a,b}(z,w) = (aw + q(z), az) \circ (bw + r(z), bz).$$

This is a degenerating family of polynomial automorphisms of the form (13). Here the parameter space is $\Lambda = \mathbb{C}^2_{a,b}$, $\Lambda_{\rm degen} = \{ab = 0\}$, $p = q \circ r$ and $d = \deg(q)\deg(r)$. If $b = 0$ and $a \neq 0$, $M_\lambda = M_{a,0} = \{(q(t), at), t \in \mathbb{C}\}$ has degree $\deg(q)$.

Proof. Assume $\lambda \in \Lambda_{\rm degen}$ and is close to zero. We first prove that $f_\lambda(\mathbb{C}^2) = M_\lambda$ is a subvariety of $\mathbb{C}^2$ close to $\{w = 0\}$. It is convenient to consider the meromorphic extension of f_λ to $\mathbb{P}^2$.

For $\lambda \in \Lambda_{\rm degen}$, f_λ is not an automorphism. Since f_λ is approximated by automorphisms, it has constant Jacobian. If the Jacobian was a non zero constant, f_λ would be a local diffeomorphism of finite degree. Now if for some $q \in \mathbb{C}^2$, $f_\lambda^{-1}(q)$ was a finite set with at least two elements, this would persistently hold in the neighborhood of λ in Λ, which is not possible. We conclude that for $\lambda \in \Lambda_{\rm degen}$, f_λ has zero Jacobian.

Consider a generic line $\overline{L}$ in $\mathbb{P}^2$ not going through the indeterminacy set of f_λ. Then $f_\lambda(\overline{L})$ is an irreducible analytic subset of degree $\leq d$ in $\mathbb{P}^2$. Let $L = \overline{L} \cap \mathbb{C}^2$. Since a generic L must intersect every fiber of f_λ, except possibly finitely many of them, we conclude that $f_\lambda(\mathbb{C}^2) = f_\lambda(L) = M_\lambda$ is irreducible and of degree $\leq d$.

Furthermore, $f_\lambda|_L : L \to M_\lambda$ gives rise to a finite-to-one map $\mathbb{C} \to M_\lambda$. If M_λ is smooth, this directly imply that M_λ is isomorphic to $\mathbb{C}$. If not, we only get that the desingularization of M_λ is isomorphic to $\mathbb{C}$.

We compute the topological degree of $f_\lambda|_{M_\lambda}$. Fix a large bidisk V_R such that every point of $M_\lambda \setminus D_R^2$ escapes to infinity, and M_λ is horizontal in D_R^2. A generic vertical line L^v in D_R^2 intersects M_λ in $\deg(M_\lambda)$ points. $f_\lambda^{-1}(L^v) \cap D_R^2$ is a vertical curve of degree d, thus intersecting M_λ at $d\deg(M_\lambda)$ points, that are the preimages of the previous ones under $f_\lambda|_{M_\lambda}$. It follows that the degree of $f_\lambda|_{M_\lambda}$ is d.

We conclude that f_λ induces a holomorphic self map of (the desingularization of) M_λ of degree d, that is, a polynomial of degree d on $\mathbb{C}$.

The fact that $[M_\lambda]/\deg(M_\lambda)$ is the only current invariant under $\frac{1}{d} f_*$ is obvious. And by the functional equation $G_\lambda^+ \circ f_\lambda = dG_\lambda^+$, we infer that $dd^c(G_\lambda^+|_{M_\lambda})$ is a non atomic measure of constant Jacobian d relative to f_λ, hence the unique measure of maximal entropy. □

3.2. **Continuity of the upper exponent.** Whenever degenerate or not, for small λ, f_λ has a unique measure of maximal entropy $\log d$, with only one positive exponent, still denoted by $\chi^+(f_\lambda)$.

For convenience we recall the statement of the continuity result.

Theorem 3.3. *Assume that (f_λ) is a degenerating family of polynomial automorphims of the form* (13). *Then $\chi^+(f_\lambda)$ converges to $\chi^+(p)$ as $\lambda \to 0$.*

Proof. As before, upper semicontinuity has been established in [BS3], so we only prove lower semicontinuity. Again, the tools will be the formula (8) of [BS5], and the Manning-Przytycki formula for $\chi^+(p)$ [Ma2, Pr]

$$\chi^+(p) = \log d + \sum_{c \text{ critical}} G_p(c). \tag{15}$$

Of course, only escaping critical points contribute to the sum.

The starting point is to rewrite the Manning-Przytycki formula in the light of the formula of Bedford and Smillie. Let p be monic and of degree d, and define the critical measure $\mu_{c,0}$ for p as follows (the subscript 0 is used because $p = f_0$)

$$\mu_{c,0} = \sum_{c \text{ critical escaping}} \mathrm{mult}_p(c) \sum_{k\geq 0} \frac{1}{d^k} \delta_{f^k(c)},$$

where $\mathrm{mult}_p(c)$ is the multiplicity of c as a critical point (e.g. $\mathrm{mult}_{z^2}(0) = 1$).

Let $G_{\max}$ be the maximum of $G_p(c)$ over all critical points. By the invariance relation for G_p, we immediately get the following rewriting of (15):

$$\forall A \geq G_{\max},\ \chi^+(p) = \log d + \int_{A \leq G_p < dA} G_p d\mu_{c,0}. \tag{16}$$

We will have to understand how unstable critical points of f_λ degenerate to the escaping postcritical points of f_0.

First, a brief outline of the proof, which will comprise several sublemmas. We start with the easier situation where λ approaches 0 along the hypersurface Λ_{degen} of degenerate parameters (Proposition 3.4). In a second step, we give an interpretation of the critical measure for p in terms of counting ramified and unramified inverse branches of some domain Q outside K_p (Lemma 3.5 and Corollary 3.6). The third step is to make the connection with 2-dimensional dynamics, by inspecting the geometry of iterated submanifolds, as projected along the Böttcher fibration (Lemmas 3.7 and 3.8). We then conclude the proof by using the decomposition of T^- given by Theorem 2.7.

Step 1. We first settle the continuity problem along Λ_{degen}.

Proposition 3.4. *If $\lambda \in \Lambda_{\mathrm{degen}}$ is close enough to zero, then f_λ reduces to a 1-dimensional map with entropy $\log d$, and $\chi^+(f_\lambda) \to \chi^+(f_0) = \chi^+(p)$ as $\lambda \to 0$ along Λ_{degen}*

Proof. We use formulas (8) and (16) –it is possible to give a direct proof using (4) in the spirit of [Ma1], leading however to delicate potential theoretic estimates.

Let $\lambda \in \Lambda_{\mathrm{degen}}$ be close to 0, and L be a line close to $\{w = 0\}$. Then by Lemma 3.2, $f_\lambda(L) = f_\lambda(\mathbb{C}^2)$ is a subvariety of some degree $\deg(M_\lambda)$, which is close to $\{w = 0\}$ on compact sets. Let $k = \deg(M_\lambda)$. Since $f_\lambda(z, w)$ is close to $(p(z), 0)$, M_λ is a union of graphs over $\{w = 0\}$ for large $|z|$. More precisely if $R > \max\{p(c),\ c \text{ critical}\}$, then for any simply connected open set $Q \subset \{|z| > R\}$ and λ small enough (depending on Q) $M_\lambda \cap \pi_1^{-1}(Q)$ is the union of k graphs over Q –see below Lemma 3.7 for more details on a similar argument.

For $\lambda \in \Lambda_{\mathrm{degen}}$, we can define a critical measure associated to the current T^-_λ and the function G^+_λ, exactly in the same way as in Definition 2.12. Of course for $\lambda = 0$ we obtain $\mu_{c,0}$. For general λ, however, there is an extra $1/k$ factor against each Dirac mass, coming from the normalization of T^-_λ.

In virtue of the description of M_λ given above and Lemma 3.1, we infer that if A is large enough and chosen so that $\{G_p = A\}$ avoids the postcritical set of p,

$$\int_{A \le G^+_\lambda < dA} G^+_\lambda d\mu_{c,\lambda} \longrightarrow \int_{A \le G_p < dA} G_p d\mu_{c,0}, \text{ as } \lambda \to 0 \text{ along } \Lambda_{\mathrm{degen}}.$$

The only remaining issue is to show that $\log d + \int_{A \le G_\lambda < dA} G^-_\lambda d\mu_{c,\lambda}$ is indeed the Lyapunov exponent of f_λ.

For this, let $\phi = (\phi_1, \phi_2) : \mathbb{C} \to M_\lambda$ be a parameterization. Outside the singular set of M_λ, ϕ is 1-1, so $\phi_1 : \mathbb{C} \to \mathbb{C}$ is a holomorphic map of degree k, that is, a polynomial of degree k. Normalize so that $\phi_1(t) = t^k + l.o.t.$; the global map f_λ induces a polynomial map p_λ on the t variable, in the sense that $f_\lambda \circ \phi = \phi \circ p_\lambda$. With the normalization done, p_λ is monic. Let G_ϕ be its Green function in the t coordinate. Since $G^+_\lambda(\phi(t)) = k \log |t| + O(1)$ at infinity, and $G^+_\lambda \circ \phi$ satisfies the same functional equation as G_ϕ, we get that $G^+_\lambda \circ \phi = kG_\phi$.

In the t coordinate, we can apply formula (16) in the fundamental domain $\{A \le kG_\phi < dA\}$, which yields

$$\chi^+(p_\lambda) = \chi^+(f_\lambda|_{M_\lambda}) = \log d + \int_{A \le kG_\phi < dA} G_\phi d\mu_{c,\phi} = \log d + \int_{A \le G^+_\lambda \circ \phi < dA} G^+_\lambda \circ \phi \frac{1}{k} d\mu_{c,\phi}$$

where $\mu_{c,\phi}$ is the critical measure associated to p_λ. This finishes the proof, because $\phi_* \left(\frac{1}{k}\mu_{c,\phi}\right)$ equals $\mu_{c,\lambda}$. □

Step 2. From now on, for ease of reading, it is understood that for $\lambda \neq 0$, f_λ is an automorphism, while f_0 is the degenerate map, that is, we restrict ourselves to $\Lambda \setminus \Lambda_{\mathrm{degen}}$.

In the next lemma, which is purely 1-dimensional dynamics, we give a precise counting of ramified inverse branches outside the filled Julia set K_p.

Lemma 3.5. *Let Q be an open topological disk lying outside K_p, intersecting every orbit at most once. Let $n \geq 1$ and $c_1, \ldots, c_q$ be the critical points falling in Q after at least one and at most n iterations, and j_k be the unique integer such that $p^{j_k}(c_k) \in Q$.*

Then the sum of the multiplicities of the critical points of $p^n : p^{-n}(Q) \to Q$ is

$$\sum_{k=1}^{q} \mathrm{mult}_z(c_k) d^{n-j_k}. \tag{17}$$

Proof. The preimage $p^{-n}(Q)$ is a union of topological disks. The Riemann Hurwitz formula asserts that the number of these disks is precisely d^n minus the sum of the multiplicities of the critical points of $p^n : p^{-n}(Q) \to Q$.

Recall the notation $Q^\delta = \{z \in Q,\ \mathrm{dist}(z, \partial Q) > \delta\}$. Since Q lies outside K_p, there exists $\delta > 0$ such that $Q \setminus Q^\delta$ does not intersect the n^{th} image of the critical set. In particular, replacing Q by Q^δ, we can assume that there are no postcritical points on ∂Q.

The topology of $p^{-n}(Q)$ is thus stable under small perturbations of p in the space of polynomials of degree d. In particular, without affecting the sum in (17), we may always assume that the c_k are critical points of multiplicity 1 and that there are no critical orbit relations between the c_k (i.e. the grand orbits of the c_k are disjoint).

Now subdivide Q into finitely many topological disks Q_k, each of which intersecting the postcritical set in a single point, and so that no postcritical point is in the boundary of any Q_k. Counting the sum of multiplicities of the critical points of $p^n : p^{-n}(Q) \to Q$ boils down to doing the computation for each Q_k and summing over k.

For a single Q_k, (17) is obvious. Indeed, for $j \leq j_k$, $p^{-j}(Q_k)$ consists of d^j univalent preimages of Q_k. Exactly one of the components of $p^{-j_k}(Q_k)$ contains the critical point c_k (which is simple by assumption). So among the components of $p^{-n}(Q_k)$, exactly d^{n-j_k} are double branched covers of Q_k under p^n, while p^n is univalent on the remaining ones. This concludes the proof. □

We have the following corollary.

Corollary 3.6. *Under the assumptions of the previous lemma, assume further that $Q \subset \{A < G_p < dA\}$, where $A > G_{\max} = \max\{G_p(c),\ c \text{ critical}\}$, and let N be so large that $d^{-N}A < \min\{G_p(c),\ c \text{ escaping}\}$. Then the sum of the multiplicities of the critical points of $p^N : p^{-N}(Q) \to Q$ is $d^N \mu_{c,0}(Q)$.*

Step 3. We turn to 2-dimensional dynamics. We fix a large R as in the statement of Lemma 3.1, and such that moreover $\varphi_\lambda|_{\{w=0\}}$ is univalent on $\{|\zeta| > R - C\}$ for every $\lambda \in \Lambda$. This is possible because of the uniformity of the $O(1)$ in $\varphi_\lambda(\zeta, 0) = \zeta + O(1)$. Here C is a constant such that $|\varphi_\lambda(z, w) - z| < C$ on V_R^+ for all λ.

Let A be such that $\sup_{D_R^2} G_\lambda < A$ for every λ, and such that there is no postcritical point of p on $\{G_p = A\}$. Then fix an integer N such that

- $A/d^{N-1} < \min\{G_p(c), c \text{ escaping}\}$,
- $d^{N-1} G_\lambda > A$ on $\overline{V_R^+}$ for all λ.

These quantities will be kept fixed throughout the proof.

As before, we project along the fibers of the Böttcher function. As opposite to the previous section, we do not use φ_λ itself but rather a projection which is π_1 when $\lambda = 0$. Recall that $\varphi_\lambda|_{\{w=0\}}$ is univalent on $\{|\zeta| > R\}$. The projection along the fibers of φ_λ onto $\{w = 0\}$ is $\pi_{1,\lambda} := (\varphi_\lambda|_{\{w=0\}})^{-1} \circ \varphi_\lambda$, which is well defined in V_R^+ and converges to π_1 when $\lambda \to 0$.

The next lemma establishes the basic connection between 1- and 2-dimensional critical points.

Lemma 3.7. *Let Q be an open topological disk in $\{A < G_p < dA\}$, intersecting every orbit of p at most once. Let $\delta > 0$ be such that $Q \setminus Q^{3\delta}$ does not intersect the postcritical set of p. Let N be the above fixed integer.*

Then there exists a neighborhood Λ' of $0 \in \Lambda$ such that if $\lambda \in \Lambda'$ and L is any horizontal line of the form $L = \{w = w_0\}$, with $|w_0| \leq R$, then:

i. The total number of tangencies between $f_\lambda^N(L)$ and the fibers of the form $\pi_{1,\lambda}^{-1}(\zeta)$, with $\zeta \in Q^\delta$ (resp. of vertical tangencies of $f_\lambda^N(L)$ over Q^δ), counted with multiplicity is exactly $d^N \mu_{c,0}(Q)$.

ii. There are no tangencies between $f_\lambda^N(L)$ and the fibers $\pi_{1,\lambda}^{-1}(\zeta)$ for $\zeta \in Q^\delta \setminus \overline{Q^{2\delta}}$.

Proof. For simplicity let us consider genuine vertical tangencies first. From the expression (13) of f_λ, we get that

$$f_\lambda^N(z, w) = (p^N(z) + P_{\lambda,N}(z, w), Q_{\lambda,N}(z, w)),$$

where $P_{\lambda,N}$ and $Q_{\lambda,N}$ vanish for $\lambda = 0$. If we fix a horizontal line $L = \{w = w_0\}$, with $|w_0| \leq R$, then $\pi_1 \circ f_\lambda^n|_L$ is the map $z \mapsto p^N(z) + P_\lambda^N(z, w_0)$, which is a close perturbation of p^N if λ is small.

There are no critical points of p^N in $p^{-N}(Q \setminus Q^{3\delta})$ so, as in Lemma 3.5 above, the total number of critical points over Q^δ, counted with multiplicity, is stable under small perturbations. From Corollary 3.6 we conclude that the number of vertical tangencies of $f_\lambda^N(L)$ over Q^δ equals $d^N \mu_{c,0}(Q)$ if λ is small enough (assertion *i.*). The same argument says that there will be no vertical tangency over $Q^\delta \setminus \overline{Q^{2\delta}}$ (assertion *ii.*).

The projection $\pi_{1,\lambda}$ is a small perturbation of π_1 when λ is small. Hence by the same reasoning as above we conclude that the number of tangencies between $f_\lambda^N(L)$ and the fibers $\pi_{1,\lambda}^{-1}(\zeta)$, $\zeta \in Q^\delta$ equals $d^N \mu_{c,0}(Q)$ if λ is small. Also there will be no tangencies over $Q^\delta \setminus \overline{Q^{2\delta}}$ □

After horizontal lines, we now investigate how $f_\lambda^N(D_R^2 \cup \overline{V_R^+})$ wraps around over Q, where Q is as in the previous lemma. Observe that by the second condition in the choice of N, $f_\lambda^N(\overline{V_R^+})$ does not intersect $\pi_{1,\lambda}^{-1}(Q)$, so considering $f_\lambda^N(D_R^2)$ is enough.

Let M be a horizontal submanifold relative to some fibration $\pi : \pi^{-1}(D) \to D$ (cf. §2.1), and $U(M)$ be some neighborhood of M. We say that $U(M)$ is a *fiberwise trivial neighborhood of M over $D' \subset D$* if the projection π makes $U(M)$ a trivial fibration over M in $D' \times \mathbb{C}$. In other words, $U(M)$ is fiberwise trivial over D' if for $\zeta \in D'$, the intersection of $U(M)$ with the fiber $\pi^{-1}(\zeta)$ consists of $\deg(M)$ topological disks, each of which containing a single point of $\pi^{-1}(\zeta) \cap M$.

Lemma 3.8. *Let Q, R, A, δ , N as in Lemma 3.7, and $\lambda \in \Lambda$ be so small that the conclusions of Lemma 3.7 hold. Then if $L \subset D_R^2$ is any horizontal line, $f_\lambda^N(D_R^2)$ is a neighborhood of $f_\lambda^N(L)$, which is fiberwise trivial over $Q^\delta \setminus \overline{Q^{2\delta}}$, relative to the projection $\pi_{1,\lambda}$.*

Proof. Recall from the previous lemma that for every horizontal line $L_{w_0} = \{w = w_0\}$ in D_R^2, $f_\lambda^N(L)$ has no tangencies with the fibers $\pi_{1,\lambda}^{-1}(\zeta)$ for $\zeta \in Q^\delta \setminus \overline{Q^{2\delta}}$. Fix such a fiber $F_{\zeta_0} = \pi_{1,\lambda}^{-1}(\zeta_0)$ and let us analyze $f_\lambda^N(L_{w_0}) \cap F_{\zeta_0}$ when w_0 ranges across D_R. The intersection consists of d^N points, moving holomorphically and without collision, because f_λ is a diffeomorphism and all intersections are always transverse. We conclude that $f_\lambda^N(D_R^2) \cap F_{\zeta_0}$ is the union of d^N topological disks, each of which containing a single point of $f_\lambda^N(L_{w_0})$, for every w_0, which was the assertion to be proved. □

The following easy lemma will be useful for bounding the topology of the leaves of T^- from below.

Lemma 3.9. *Let M be a horizontal submanifold, relative to some fibration $\pi : \pi^{-1}(D) \to D$, with no tangencies with the fibers of π over some neighborhood of ∂D. Assume that both M and the base D are isomorphic to disks. Let $U(M)$ be some neighborhood of M, fiberwise trivial over a neighborhood of ∂D.*

Let $M' \subset U(M)$ be any horizontal submanifold. Then $\deg(M)$ *divides* $\deg(M')$.

Proof. By assumption, $\pi : M \to D$ is a covering over an annulus of the form $D \setminus D'$. Fix a point $\zeta_0 \in D \setminus D'$ and a loop γ at ζ_0, generating the fundamental group of $D \setminus D'$.

Lifting γ to M yields a holonomy map of the fiber $M \cap \pi^{-1}(\zeta_0)$, which is a cycle on the $\deg(M)$ points of the fiber because M has a single boundary component.

Now, over some neighborhood of ∂D, π makes $U(M)$ a locally trivial fibration over M, and M' is horizontal with respect to this fibration. In particular, the number of points of M' in each component of $U(M) \cap \pi^{-1}(\zeta)$, counting multiplicities, is a constant k. We conclude that M' has degree $k\deg(V)$ over D. □

Step 4: conclusion. Let A and N be as defined before Lemma 3.7. There are finitely many postcritical points of p in $\{A < G_p < dA\}$. For each of these postcritical points, we fix a topological disk Q containing it, and satisfying the requirements of Lemma 3.7. We will prove that for every small enough λ,

$$\liminf_{\lambda \to 0} \mu_{c,\lambda}\left(\pi_{1,\lambda}^{-1}(Q^\delta)\right) \geq \mu_{c,0}(Q).$$

As in the proof of Theorem 1.1, this implies the lower semicontinuity of the exponent.

Consider one of the disks Q. By Lemma 3.7, for small λ, $f_\lambda^N(L) \cap \pi_{1,\lambda}^{-1}(Q)$ is the union of finitely many horizontal submanifolds $V_1, \ldots, V_q$, with the property that:

- the V_i have no tangencies with the fibers of $\pi_{1,\lambda}$ over $Q^\delta \setminus \overline{Q^{2\delta}}$;
- the total number of tangencies over Q^δ equals $d^N \mu_{c,0}(Q)$.

By the Maximum Principle, the V_i are topological disks, so by the Riemann-Hurwitz formula, we get that

$$\#\text{tangencies over } Q^\delta = d^N \mu_{c,0}(Q) = \sum_{i=1}^{q} (\deg(V_i) - 1). \tag{18}$$

Consider now the current T_λ^- in $\pi_{1,\lambda}^{-1}(Q)$, which is horizontal and of slice mass 1 relative to the projection $\pi_{1,\lambda}$. The support of T_λ^- is contained in $D_R^2 \cup \overline{V_R^+}$, hence by invariance it is contained in $f_\lambda^N(D_R^2 \cup \overline{V_R^+})$. Also, $f_\lambda^N(D_R^2 \cup \overline{V_R^+}) \cap \pi_{1,\lambda}^{-1}(Q) = f_\lambda^N(D_R^2) \cap \pi_{1,\lambda}^{-1}(Q)$ is a neighborhood of $f_\lambda^N(L)$, which is fiberwise trivial over $Q^\delta \setminus \overline{Q^{2\delta}}$ by Lemma 3.8. In particular, if $\zeta_0 \in Q^\delta \setminus \overline{Q^{2\delta}}$, the intersection of this open set with the fiber $\pi_{1,\lambda}^{-1}(\zeta_0)$ is the union of d^N topological disks. We claim that the slice mass of T_λ^- in each of these disks is d^{-N}.

Indeed, since the sum is 1, it is enough to bound each slice mass from below by d^{-N}. Let Δ be any of the components of $f_\lambda^N(D_R^2) \cap \pi_{1,\lambda}^{-1}(\zeta_0)$. Then $f_\lambda^{-N}(\Delta)$ is a vertical disk in D_R^2, so the mass of its intersection with T_λ^- is a non zero integer. By the invariance property of T_λ^-, we thus get that the mass of $T_\lambda^- \wedge [\Delta]$ is kd^{-N}, with $k \geq 1$, which yields the desired result.

As in the proof of Theorem 1.1, we now use the decomposition of T_λ^- given by Theorem 2.7. We need to prove that f_λ is unstably disconnected for small λ. Indeed Q contains a postcritical point, so $\mu_{c,0}(Q) > 0$. Hence by (18), at least one of the V_i has degree > 1. Let $U(V_i)$ be the connected component of $f_\lambda^N(D_R^2) \cap \pi_{1,\lambda}^{-1}(Q^\delta)$ containing it. The restriction of T_λ^- to $U(V_i)$ has slice mass $\deg(V_i)d^{-N}$ by the previous claim, so it is non zero. In particular $K_\lambda^- \cap U(V_i) \neq \emptyset$. If f_λ was unstably connected, then by [BS6], K_λ^- would be the support of a lamination by graphs over $\pi_{1,\lambda}$. This clearly contradicts Lemma 3.9.

We now give a quantitative version of this argument. Consider any of the horizontal disks V_i, and let $U(V_i)$ as above. Recall that the restriction of T_λ^- to $U(V_i)$ has slice mass $\deg(V_i)d^{-N}$. By Theorem 2.7, we have a decomposition relative to the projection $\pi_{1,\lambda}$ over Q^δ,

$$T_\lambda^-|_{U(V_i)} = \sum_{k=1}^\infty T_k, \text{ with } \sum_{k=1}^\infty \mathrm{sm}(T_k) = \deg(V_i)d^{-N}.$$

Now by Lemma 3.9 above, the smallest possible degree in the decomposition is $\deg(V_i)$. Also the function $k \mapsto \frac{k-1}{k}$ is increasing. Thus by Proposition 2.15, we can bound the critical mass of $T_\lambda^-|_{U(V_i)}$ from below:

$$\mu_{c,\lambda}(U(V_i)) = \sum_{k=\deg(V_i)}^\infty \frac{k-1}{k}\mathrm{sm}(T_k) \geq \frac{\deg(V_i)-1}{\deg(V_i)} \sum_{k=\deg(V_i)}^\infty \mathrm{sm}(T_k) = (\deg(V_i)-1)d^{-N}.$$

By summing over i, we conclude that the critical mass of $\pi_{1,\lambda}^{-1}(Q^\delta)$ is bounded from below by $\sum_i(\deg(V_i)-1)d^{-N} = \mu_{c,0}(Q)$. This concludes the proof. $\square$

Appendix A. The fastest escaping critical point

For the purpose of studying parameter families of polynomial automorphisms of $\mathbb{C}^2$, it is useful to have natural dynamically defined functions on parameter space. Lyapunov exponents of the maximal entropy measures are such functions. Here we define a notion of "fastest rate of escape for critical points", which is a natural generalization of $G_{\max} = \max\{G_p(c),\ c \text{ escaping}\}$ when p is a polynomial in $\mathbb{C}$. In the space of polynomials, this defines a psh function which plays an important role in [DF]. Observe also that $\chi^+(p) \leq \log d + (d-1)G_{\max}$ as it easily follows from the Manning-Przytycki formula.

So let f be a polynomial automorphism of $\mathbb{C}^2$, normalized so that $f(z,w)) = (z^d, 0) + l.o.t.$ Recall that if U is any open set avoiding K^+, due to the functional equation for φ^+, there exists an integer N so that $(\varphi^+)^N$ is well defined on U. Our definition of $G^+_{\max}(f)$ is inspired by 1-dimensional dynamics.

Definition A.1. *We define the fastest escape rate $G^+_{\max}(f)$ as the infimum of $R > 0$ such that there exists an extension of φ^+ to a neighborhood of $K^- \cap \{G^- > R\}$.*

For instance, Bedford and Smillie prove in [BS6] that if $G^+_{\max} = 0$, then f is unstably connected, hence has no unstable critical points. It is likely that $f \mapsto G^+_{\max}(f)$ is psh, but we could not prove it.

As in dimension 1, there is an upper estimate for χ^+ using $G^+_{\max}$. This should be compared with the estimate given for horseshoes in [BS5, Theorem A.4][1].

Theorem A.2. $\chi^+(f) \leq \log d + d \cdot G^+_{\max}(f)$

Proof. The estimate is based on Proposition 2.15. Let $R > G^+_{\max}$ be such that the critical measure puts no mass on $\{G^+ = R\}$. Then φ^+ is well defined on some neighborhood of $K^- \cap \{R < G^+ < dR\}$, and maps onto the annulus $A = \left\{e^R < |\zeta| < e^{dR}\right\}$. As before we write φ for φ^+. Then

$$\chi^+(f) = \log d + \int_{R \leq G^+ < dR} G^+ d\mu_c = \log d + \int_{\varphi^{-1}(A)} G^+ d\mu_c.$$

We claim that $\mu_c^-\left(\varphi^{-1}(A)\right) \leq 1$, which implies that $\chi^+(f) \leq \log d + dR$. Since R can be arbitrary close to $G^+_{\max}$, this will finish the proof.

The first claim is that in $\varphi^{-1}(A)$, T^- admits a decomposition $T^- = \sum T_k$ where T_k is made of submanifolds of degree k relative to φ. Indeed, by the discussion preceding Proposition 2.15, we know that such a decomposition exists in $M \leq G^+ < dM$ for large M. Since the existence of such a decomposition is invariant under the diffeomorphism f, we get the result in the original annulus by iterating sufficiently many times.

Now consider a neighborhood of $K^- \cap \{R < G^+ < dR\}$ of the form $\{R < G^+ < dR\} \cap \{G^- < \varepsilon\}$ where φ is well defined. The projection φ needn't be a locally trivial fibration onto A, however we leave the reader check that the assumption of Remark 2.3 is satisfied. In particular, we can say that T^- and the T_k are horizontal currents relative to the projection φ and the slice mass is invariant.

Cut the annulus A by a radial line L such that $\varphi^{-1}(L)$ has zero mass for the critical measure. Then $Q := A \setminus L$ is a simply connected open set, and we have a decomposition (different from the previous one) $T^- = \sum T_k$ of T^- over Q relative to φ. By Proposition 2.15, we infer that

$$\mu_c(\varphi^{-1}(Q)) = \sum_{k=1}^{\infty} \frac{k-1}{k} \mathrm{sm}(T_k).$$

Near infinity, the slice mass of T^- relative to φ is 1, so by invariance of the slice mass, this is also true in A. We conclude that $\sum_{k=1}^{\infty} \mathrm{sm}(T_k) = 1$, hence $\mu_c(\varphi^{-1}(Q)) \leq 1$. □

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