A NOTE ON THE RANK OF POSITIVE CLOSED CURRENTS

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The purpose of this note is to prove the following theorem, which was stated without proof in [Du].

Theorem 1. *Let T be a strongly positive closed current of bidimension (p,p) on $\mathbb{P}^k$, and assume that its trace measure has dimension $\dim(\sigma_T) < 4p$. Then σ_T-a.e. we have that*

$$p \leq \operatorname{rank}(T) \leq \frac{1}{2}\dim(\sigma_T). \tag{1}$$

We refer to [Le, De] or [Du, §1,2] for detailed basics on positive exterior algebra and positive currents. Here we just recall that a positive current T of bidimension (p,p) admits an *integral representation* in the sense that there exists a measurable field t_T of positive (p,p) vectors such that for any test (p,p)-form φ

$$\langle T, \varphi\rangle = \int \langle t_T, \varphi\rangle \sigma_T,$$

where σ_T is the trace measure. If $t_T(x)$ is well-defined (which happens σ_T-a.e.), the rank of T at x is the rank of t_T, that is the dimension of the smallest sub-vector space W of $T_x\mathbb{P}^k$ such that $t_T(x) \in \bigwedge^{p,p}(W)$. A positive (p,p) vector is decomposable if and only if its rank equals p.

Before starting the proof, we also recall the following result [Du, Corollary 2.5].

Theorem 2. *Let T be a strongly positive current of bidegree (q,q) in $\Omega \subset \mathbb{C}^k$. Assume that the family $(T^2_\varepsilon)_{\varepsilon>0}$ has locally uniformly bounded mass as $\varepsilon \to 0$.*
Then if $T^2_{\mathrm{ac}} = 0$, T has rank $< k$ a.e.

We refer to [Du] for the precise definition of T^2_{ac} for a positive current T.

Proof of Theorem 1. By definition $\operatorname{rank}(T) \geq p$ so only the inequality $\operatorname{rank}(T) \leq \frac{1}{2}\dim(\sigma_T)$ needs to be established. Let $\ell = \lfloor \frac{1}{2}\dim(\sigma_T)\rfloor + 1$. Let $q = k - p$.

If I (resp. L) is a linear subspace of dimension $k-\ell-1$ (resp. ℓ), such that $I \cap L = \emptyset$, we can consider the linear projection of center I, $\pi_I : \mathbb{P}^k \setminus I \to L$. If I is fixed, changing L amounts to post-composing π_I with a linear automorphism, so we may simply think of π_I as mapping $\mathbb{P}^k \setminus I$ onto $\mathbb{P}^\ell$.

For generic I the projection $(\pi_I)_*T$ is a well-defined positive current of bidimension (p,p) in $L \simeq \mathbb{P}^\ell$, in the sense that it satisfies the property $\langle (\pi_I)_*T, \varphi\rangle = \langle T, \pi_I^*\varphi\rangle$ for every test (p,p) form, and it has the same mass as T. Indeed for this it is enough to resove π_I by writing it as $\beta\circ\alpha^{-1}$, where α and β are holomorphic, and define $(\pi_I)_* = \beta_*\alpha^*$. The operator α^* is always well-defined on compact Kähler manifolds, even if it is not always continuous (see [DS] for

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details). Shortly we'll see that $(\pi_I)_*T$ is strongly positive. Notice that $\ell - p \leq \ell/2$, as follows from our assumption on $\dim(\sigma_T)$.

So from now on we consider I such that $(\pi_I)_*T$ is well-defined and $\sigma_T(I) = 0$, and we simply write π for π_I. We also denote by ω_L the restriction of ω to L. Fix a Borel set E such that $\sigma_T(E) = 1$ and $t_T(y)$ exists at every $y \in E$.

The first claim is that $\sigma_{\pi_*T} \ll \pi_*\sigma_T$. Indeed observe first that $\pi^*(\omega_L^p) \ll \omega^p$, thus $T \wedge \pi^*(\omega_L^p) \ll T \wedge \omega^p$. Next, we have the formulas $\sigma_{\pi_*T} = \pi_*\left(T \llcorner \pi^*\omega_L^p\right)$ and $\pi_*\sigma_T = \pi_*\left(T \llcorner \omega^p\right)$ and the result easily follows.

From this we deduce that $\dim((\pi_I)_*T) \leq \dim(\sigma_T) < 2\ell$. Indeed, since π is locally Lipschitz outside I, $\mathrm{HD}(\pi_I(E)) \leq \mathrm{HD}(E)$, and $\pi_I(E)$ is a set of full mass for $\sigma_{(\pi_I)_*T}$.

Conversely for generic I, $\pi_*\sigma_T \ll \sigma_{\pi_*T}$. Indeed, if not, there is a set A of positive σ_T mass such that if $x \in A$

$$t_T(x) \llcorner \pi^*(\omega_L^p) = \langle t_T(x), \pi^*(\omega_L^p) \rangle = \langle \pi_*(t_T(x)), \omega_L^p \rangle = 0,$$

thus $\pi_*(t_T(x)) = 0$. This means that the decomposable vectors making up $t_T(x)$ are not in general position with respect to the fibers of π_L. More precisely, if t is such a vector, $\mathrm{Span}(t)$ will not be transverse to the fiber, which has dimension $k - \ell < k - p$. This can only happen for a set of projections of zero measure (see Lemma 3 below). We conclude that the existence of such a A is not possible for generic I. From now on we assume that I is chosen so that $\pi_*\sigma_T \ll \sigma_{\pi_*T}$, and we let $h \in L^1_{\mathrm{loc}}(\sigma_{\pi_*T})$ such that $\pi_*\sigma_T = h\sigma_{\pi_*T}$.

We can now describe the tangent vectors to π_*T. Recall that the measure σ_T can be disintegrated along the fibers of the projection π as follows. If f is a measurable function we have that

$$\int f(x)\sigma_T(x) = \int_L \left(\int_{\pi^{-1}(z)} f(x)\sigma_T(x|\pi^{-1}(z)) \right) (\pi_*\sigma_T)(z),$$

with the usual notation $\sigma_T(\cdot|\pi^{-1}(z))$ for the conditional measure of σ_T on the fiber.

If now φ is a test (p,p) form on L, we have

$$\begin{aligned}
\langle \pi_*T, \varphi \rangle = \langle T, \pi^*\varphi \rangle &= \int \langle t_T(x), (\pi^*\varphi)(x) \rangle \, \sigma_T(x) \\
&= \int_L \left(\int_{\pi^{-1}(z)} \langle t_T(x), (\pi^*\varphi)(x) \rangle \, \sigma_T(x|\pi^{-1}(z)) \right) (\pi_*\sigma_T)(z) \\
&= \int_L \left(\int_{\pi^{-1}(z)} \langle \pi_*(t_T(x)), \varphi(\pi(x)) \rangle \, \sigma_T(x|\pi^{-1}(z)) \right) (\pi_*\sigma_T)(z) \\
&= \int_L \left\langle \widetilde{t}(z), \varphi(z) \right\rangle (\pi_*\sigma_T)(z) \text{ where } \widetilde{t}(z) = \int_{\pi^{-1}(z)} \pi_*(t_T(x))\sigma_T(x|\pi^{-1}(z)) \\
&= \int_L \left\langle h(z)\widetilde{t}(z), \varphi(z) \right\rangle \sigma_{\pi_*T}(z).
\end{aligned}$$

We see that the last integral is actually the integral representation of π_*T, so for σ_{π_*T} a.e. z,

$$t_{\pi_*T}(z) = h(z)\widetilde{t}(z) = h(z) \int_{\pi^{-1}(z)} \pi_*(t_T(x))\sigma_T(x|\pi^{-1}(z)). \tag{2}$$

This implies in particular that π_*T is strongly positive, since t_{π_*T} is a.s. an average of strongly positive (p,p) vectors.

We are now in position to conclude the proof of the theorem. We argue by contradiction, so let us assume that there exists a set A of positive trace mass such that $t_T(x)$ has rank $\geq \ell$ for $x \in A$. Let $S = T|_A$, and consider the current $\pi_* S$ on L. Then $\pi_* S$ satisfies the assumptions of Theorem 2, since it is dominated by the positive closed current $\pi_* T$. Since $\dim(\sigma_{\pi_* S}) < 2\ell$, we infer that $(\pi_* S)_{\rm ac} = 0$, therefore $\mathrm{rank}(\pi_* S) < \ell$ a.e.

Now by (2), for a.e. z, $t_{\pi_* S}(z)$ is an average of $\pi_*(t_S(x))$ with $x \in \pi^{-1}(z)$. Thus by Lemma 3 *i.* below, $\mathrm{rank}(\pi_*(t_S(x))) < \ell$ for $\sigma_S(\cdot|\pi^{-1}(z))$-a.e. x. On the other hand, by Lemma 3 *ii.*, if I is chosen generically, $\mathrm{rank}(\pi_*(t_S(x))) \geq \ell$, σ_S-a.e. This contradiction finishes the proof. □

Lemma 3. *Let V be a Hermitian complex vector space with associated (1,1) form β.*

i. Let $(t_\alpha)_{\alpha\in\mathcal{A}}$ be a measurable family of strongly positive (p,p) vectors of trace 1, and μ be a probability measure on $\mathcal{A}$. Let $t = \int_{\mathcal{A}} t_\alpha d\mu(\alpha)$. If $\mathrm{rank}(t) < \dim V$, then for a.e. α, $\mathrm{rank}(t_\alpha) < \dim V$.

ii. Let $p < \ell$ and fix a complex subspace L of dimension ℓ. If K is a supplementary subspace to L, we denote by $\pi_{K,L}$ be the projection onto L with kernel K. Let t be a strongly positive (p,p) vector of rank $r \geq \ell$. Then there exists a set $\mathcal{E}(t)$ of zero Lebesgue measure in the corresponding Grassmannian such that, if $K \notin \mathcal{E}(t)$, $\mathrm{rank}((\pi_{K,L})_*(t)) = \ell$.

Proof of Lemma 3. i. Recall that $\mathrm{rank}(t) = \mathrm{rank}(t \llcorner \beta^{p-1})$ so it is enough to prove the result for positive (1,1) vectors, that is, nonnegative Hermitian matrices. But in this context the result is obvious, as follows for instance from the concavity of $M \mapsto (\det(M))^{1/k}$.

ii. We use the following fact: if $p \leq \ell$ and W is a p-dimensional subspace, then the set of K's such that $\pi_{K,L}|_W : W \to L$ is injective is open and of full measure.

Fix a decomposition $t = \sum_{k=1}^s t_k$ of t as a sum of decomposable vectors. Since $p < \ell$, by the previous observation we can assume that for each k, $\pi_{K,L}|_{\mathrm{Span}(t_k)}$ is injective Furthermore, let us choose ℓ linearly independent vectors $e_1, \ldots, e_\ell$ belonging to $\bigcup_{k=1}^s \mathrm{Span}(t_k)$. We may assume that $\pi_{K,L}|_{\mathrm{Vect}(e_1,\ldots,e_\ell)}$ is injective as well. Thus, $(\pi_{K,L})_*(t_k)$ is a non-trivial decomposable element of $\bigwedge^{p,p}(L)$, and by our second requirement $\mathrm{rank}(\sum(\pi_{K,L})_*(t_k)) \geq \ell$, whence the result. □

Remark 4. It is clear from the proof that a sharper condition for $\mathrm{rank}(T) < \ell$ a.e. is that for a generic linear projection π onto $\mathbb{P}^\ell$, $\pi_* \sigma_T$ is singular w.r.t. Lebesgue measure.

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