Statistics with R

Chapter 4: Confidence intervals

Tabea Rebafka

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Outline

Definition I

In this chapter we consider the statistical model $\{\mathbb{P}_\theta, \theta \in \Theta\}$ with $\underline{\Theta \subset \mathbb{R}}$.

Confidence interval

Let $a(\cdot)$ and $b(\cdot)$ be real functions such that $a(\mathbf{x}) < b(\mathbf{x})$ for all $\mathbf{x}$. Let $0 < \alpha < 1$. The interval $[a(\mathbf{X}), b(\mathbf{X})]$ is called **confidence interval of level $1-\alpha$** for the parameter θ if

$$\mathbb{P}_\theta \left(a(\mathbf{X}) \leq \theta \leq b(\mathbf{X}) \right) \geq 1 - \alpha, \tag{1}$$

for all $\theta \in \Theta$. We denote $\mathrm{IC}_{1-\alpha}(\theta)$.

- $[a_n(\mathbf{X}_n), b_n(\mathbf{X}_n)]$ is an **asymptotic confidence interval of level** $1-\alpha$, if for all $\theta \in \Theta$,

$$\liminf_{n\to\infty} \mathbb{P}_\theta \left(a_n(\mathbf{X}_n) \leq \theta \leq b_n(\mathbf{X}_n) \right) \geq 1 - \alpha.$$

Definition II

- Note that here the interval $[a(\mathbf{X}), b(\mathbf{X})]$ is random and not the parameter θ.
- In general, we use $\alpha = 0.05$, and sometimes $\alpha = 0.01$ ou 0.1.
- The interval $\text{IC}_{1-\alpha}(\theta) = (-\infty, \infty)$ is always a confidence interval of level α. However, we are interested in *short* intervals.
- Confidence intervals that are valid for finite n are preferable to asymptotic confidence intervals.
- If θ is a vector of dimension $d > 1$, we search for **confidence regions** $\mathcal{C}(\mathbf{X}) \subset \mathbb{R}^d$ that contain the parameter vector θ with probability $1 - \alpha$ (at least).

Construction based pivots I

- A pivot is a statistic whose distribution does not depend on any unknown parameter.
- Pivots can be used to construct confidence intervals.

Construction based pivots II

Example

- Let $(X_1, \ldots, X_n)$ be i.i.d. random variables with normal distribution $\mathcal{N}(\mu, \sigma^2)$ with unknown mean $\mu \in \mathbb{R}$ and known variance $\sigma^2 > 0$.
- Compute a confidence interval of level $1 - \alpha$ for μ.

Construction based pivots III

Procedure

1. Find a point estimator $\hat{\theta}$ of θ.
2. Determine the distribution of the estimator $\hat{\theta}$.
3. Find a transformation $\mathcal{T}(\hat{\theta},\theta)$ of $\hat{\theta}$ such that $\mathcal{T}(\hat{\theta},\theta)$ is a pivot.
4. Determine the quantiles $q_{\alpha/2}$ and $q_{1-\alpha/2}$ of the distribution of $\mathcal{T}(\hat{\theta},\theta)$ such that

$$\mathbb{P}_\theta\left(q_{\alpha/2} \leq \mathcal{T}(\hat{\theta},\theta) \leq q_{1-\alpha/2}\right) = 1-\alpha/2-\alpha/2 = 1-\alpha.$$

5. Invert $\mathcal{T}$ (if possible), to deduce a confidence interval:

$$1-\alpha = \mathbb{P}_\theta\left(q_{\alpha/2} \leq \mathcal{T}(\hat{\theta},\theta) \leq q_{1-\alpha/2}\right) = \ldots = \mathbb{P}_\theta\left(A \leq \theta \leq B\right),$$

where $A = a(\hat{\theta}, q_{\alpha/2}, q_{1-\alpha/2}) = \tilde{a}(\mathbf{X})$ and
$B = b(\hat{\theta}, q_{\alpha/2}, q_{1-\alpha/2}) = \tilde{b}(\mathbf{X})$.