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October 2018

Feuille d'exercice N° 3

Intervalles de confiance & tests

Exercice 1. *Exponential distribution*
Let $X_1, \dots, X_n$ be i.i.d. random variables with exponential distribution $\mathcal{E}(\lambda)$ with parameter $\lambda > 0$ and density $f(x) = \lambda e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x)$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

1. Show that $1/\bar{X}_n$ tends to λ in probability as $n \to \infty$.
2. Determine the limit distribution of $\sqrt{n}(\bar{X}_n - 1/\lambda)$. Deduce the limit distribution of $\sqrt{n}(1/\bar{X}_n - \lambda)$.
3. Determine an *asymptotic confidence interval* $[\hat{a}_n, \hat{b}_n]$ for λ with significance level $1 - \alpha$, i.e. $\hat{a}_n$ and $\hat{b}_n$ are such that
$$\lim_{n\to\infty} \mathbb{P}\left(\lambda \in [\hat{a}_n, \hat{b}_n]\right) = 1 - \alpha \ .$$

Exercice 2. *Poisson distribution*
Let $X_n, n \geq 1$ be i.i.d. random variables with Poisson distribution $\mathcal{P}(\lambda)$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

1. Show that the estimator $\bar{X}_n$ of λ is unbiased, consistent and asymptotically normal.
2. By using
$$\sqrt{n}\left(\frac{\bar{X}_n - \lambda}{\sqrt{\lambda}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \ ,$$
and using Slutsky's lemma, show that the limit distribution of the following terms does not depend on λ. More precisely, show that
   (i) $\sqrt{n}\left(\dfrac{\bar{X}_n - \lambda}{\sqrt{\bar{X}_n}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$
   (ii) $\sqrt{n}\left(g(\bar{X}_n) - g(\lambda)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ with an appropriate choice of the function g.
3. Determine the corresponding asymptotic confidence intervals. Which one is the best?

Exercice 3. *A first example of a test*

1. We observe a realization x, from the uniform distribution $U[0,1]$ under H_0, and from the uniform distribution $U[2,3]$ under H_1. Propose a test of H_0 versus H_1 and compute its two error type rates.
2. Suppose now that we observe a realization x, from the uniform distribution $U[0,1]$ under H_0, and from the uniform distribution $U[0.5,2]$ under H_1. Consider the critical region of the test $R_c = \{x > c\}$ for some constant $c \in [0,2]$. Compute the two error type rates.

Exercice 4. *Test on the mean*
Assume that we observe $X_1, \ldots, X_n$, i.i.d. from the normal distribution $\mathcal{N}(\mu, 1)$. We want to test $H_0 : \mu = 0$ versus $H_1 : \mu = m$ where $m < 0$ is a fixed negative real number.

1. Give the critical region of a test of size α to test H_0 versus H_1.

2. Compute the power $m \in \mathbb{R}_- \mapsto \pi_n(m)$ of this test and draw its graph. Study the pointwise convergence of π_n as n tends to $+\infty$.

3. Now we consider the following alternative hypothesis that depends on n:

$$H_1 : \mu = -Cn^{-\gamma} ,$$

where $C > 0$ and $\gamma \in \mathbb{R}$. Study the behaviour of the power of the test as a function of γ when n tends to $+\infty$.

Exercice 5. *Quality control in a supermarket*
A customer of a supermarket measures the weight of 16 coffee packets of the same brandmark with nominal weight 500g. The measurement results are the following:

487.5,	500.1,	480.3,	519.8,	470.3,	500.2,	485.2,	499.4,
499.7,	503.1,	504.9,	480.7,	505.1,	494.7,	488.3,	473.3.

with $\bar{X} = 493.29$, $s_x = \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} = 12.82$. Assume that the measurements are i.i.d. realizations from a normal distribution with mean $\mu > 0$ and standard deviation σ.

1. Perform a statistical test to control the quality ($H_0 : \mu = 500; H_1 : \mu \neq 500$). Give the p-value of the test. What is the conclusion at a significance level of 5%? (Note that $\mathbb{P}(Z \leq 2.03) \approx 0.97$, for $Z \sim t_{15}$).

2. The customer is more interested in testing the hypotheses $H_0 : \mu \geq 500$ versus $H_1 : \mu < 500$. Give the p-value of the corresponding test. What is the conclusion at a significance level of 5%? Compare your result to the previous question. (Note that $\mathbb{P}(Z \leq 0.994) \approx 0.83$, for $Z \sim t_{15}$).