

AMSS Course :Bouttier-Di Francesco-Guitter bijection, tricks and admissibility.

The goal of text is to present a bijection, due to Bouttier, Di Francesco and Guitter, between bipartite planar maps and a class of labeled trees. Exercises 1 to 3 follow article [?]. In exercise 5, we use this bijection, and the intermediate combinatorial results of exercise 4, to deduce the admissibility criterion of weight sequences.

Exercise 1: From maps to planar mobiles.

Consider $m_\star = (m, \rho_\star)$ a bipartite planar map having a distinguished vertex ρ_\star . Perform the following operations:

- (i) Color in white every vertex of m_\star and label them by their distances to ρ_\star .
- (ii) Draw a black vertex inside each face.
- (iii) In each face consider the clockwise order around it. Note that each edge is travelled in its two directions when we follow the clockwise order of its two adjacent faces.
- (iv) For a face f and a white vertex adjacent to it, draw a new edge inside f between this vertex and the black vertex living inside f if the next white vertex in the clockwise order around f has a smaller label.
- (v) Erase the edges of m .

We obtain a graph embedded on the plane (it is not connected).

1. Apply operation (i) to (v) to this pointed bipartite planar map:

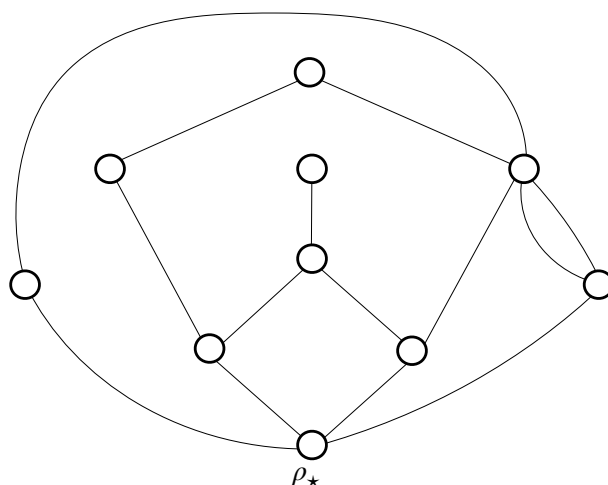


Figure 1: A pointed bipartite planar map

2. Show that ρ_\star is isolated on the resulting graph.

(vi) Erase ρ_\star .

We denote by Mob the concatenation of the operation (i) to (vi) (keeping the labels of the white vertices).

3. For every vertex u of m , let $A(u)$ denote the label of u (its distance to ρ_\star in m). Using the fact that m is bipartite show that if u, v are two neighboring vertices of m then:

$$|A(u) - A(v)| = 1.$$

4. Show that for each face f of m , the degree of the black vertex associated to f on $\text{Mob}(m_\star)$ is half of the perimeter of f .

5. Let v be a black vertex, and u_1, u_2 two neighbors of v consecutive in the clockwise order on $\text{Mob}(m_\star)$. Show that:

$$A(u_2) \geq A(u_1) - 1.$$

We call this property (P).

6. Show that $\text{Mob}(m_\star)$ does not have loops and using Euler's formula deduce that $\text{Mob}(m_\star)$ is a planar tree.

* A mobile is a tree with black and white vertices such that all the neighbors of a black vertex are white and viceversa. A well-labeled mobile is a mobile such that all the white vertices have a label and verify property (P). Finally we say that a well-labeled mobile is positif if its minimum label is 1 and that it is planar if it is equipped with a clockwise order (an embedding on the plane or the sphere). Note that, for every pointed bipartite planar map m_\star , $\text{Mob}(m_\star)$ is a positif well-labeled planar mobile.

Exercise 2: From mobiles to maps.

Consider \mathcal{T} a positif well-labeled planar mobile. All we are going to do here works for general well-labeled planar mobiles just by translating all the labels by the minimum label +1. A corner of \mathcal{T} is an angular sector delimited by a white vertex and two consecutive edges around this vertex. We define the following operations:

(i') Label each corner by the label of the associated white vertex.

(ii') For each corner C with label $A \geq 2$, denote by $s(C)$ the first encountered vertex in the clockwise order with label $A - 1$. Draw an edge between the vertex associated to C and the vertex associated to $s(C)$ without hitting the edges of \mathcal{T} and following the contour (clockwise order) of \mathcal{T} .

1. Show that the edges constructed in step (ii') are well defined and that we can draw them all without crossings.

(iii') Draw a new white vertex with label 0 in the infinite face.

(iv') For each corner C with label 1, draw an edge between the vertex associated to C and the new vertex with label 0 without hitting the other edges.

2. Show that this operation can be done without crossings.

(v') Erase all the edges of \mathcal{T} and all the black vertices. Denote by m the resulting embedded graph.

(vi') Point m at the unique vertex with label 0 and denote by m_\star the resulting pointed embedded graph.

3. Show that m is connected. So m_\star is a pointed planar map.

4. Show that m is bipartite.

5. We denote by BFG the concatenation of operations (i') to (vi'). Apply BFG to this positif well-labeled planar mobile:

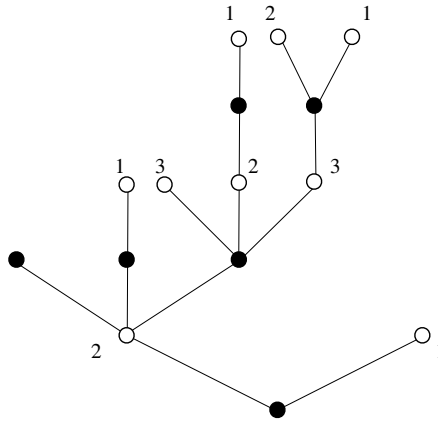


Figure 2: A positif well-labeled planar mobile.

Exercise 3: Bijection.

1. Consider m_\star a pointed bipartite map.

1.1. Show that $\#Edges(m_\star) = \#Edges(Mob(m_\star))$ and deduce that $\#Edges(m_\star) = \#Corners(Mob(m_\star))$.

1.2. Consider e an edge of m_\star . e connects two corners of $Mob(m_\star)$. Let $Mob^e(m_\star)$ be the map $Mob(m_\star)$ where we add the edge e . Note that $Mob^e(m_\star)$ has two faces and set \mathcal{C} the face not containing ρ_\star . Show that there is a unique vertex on \mathcal{C} realizing the minimum on \mathcal{C} and that this point is one of the extremities of e .

1.3. Deduce that $BFG(Mob(m_\star)) = m_\star$.

2. Consider \mathcal{T} a positif well-labeled planar mobile.

Fix v a black vertex of \mathcal{T} . For every C, C' two corners such that their associated vertices u_1, u_2 are successive neighbors of v and are in clockwise order, draw:

- The edge of $BGF(\mathcal{T})$ going from C to $s(C)$.
- For every integer $i \leq A(u_2) - A(u_1) - 1$, draw the edge going from $s^i(C')$ to $s^{i+1}(C')$.

2.1 What happens if we erase all the edges of \mathcal{T} containing v ?

2.2 Deduce that $Mob(BGF(\mathcal{T})) = \mathcal{T}$.

A plane tree is a rooted tree (at a vertex) with an ordering for the children of each vertex. Note that this definition is equivalent to the definition of a planar tree with a rooted edge. We say that a mobile is a plane mobile if its tree structure is a plane tree.

Exercise 4: Counting bridges and Janson & Stefansson's trick.

For every integer $l \geq 1$, let \mathcal{B}_l be the set of l -tuple (x_1, \dots, x_l) of integers such that:

$$\sum_{i=1}^l x_i = l.$$

1. Show that

$$\#\mathcal{B}_l = \binom{2l-1}{l}.$$

2. Let \mathcal{T} be a plane mobile rooted at a white vertex. We denote by ρ the root. Perform the following operations:

- Denote by r_1, \dots, r_j the children of ρ (in clockwise order) and by convention set $r_0 = r_{j+1} = \rho$. Following the contour of the mobile, draw without crossings, an edge going from r_i to r_{i+1} for every $i \in \llbracket 1, j \rrbracket$.

- Let $u \neq \rho$ be a white vertex. Denote by u_1, \dots, u_j its children and by u_0 its parent. For every $i \in \llbracket 0, j \rrbracket$, draw an edge going from u_i to u_{i+1} following the contour of the mobile and without crossings. Where by convention we take $u_{j+1} = u$.

- Erase the edges of \mathcal{T} , we obtain a tree as a consequence of Euler's formula. Root it at the black vertex r_1 .

We denote by JS this transformation, it was introduced by Janson and Stefansson in [?].

2.1. What happens with the white vertices after performing JS?

2.2. Let v be a black vertex of \mathcal{T} . Show that the number of children of v on $\text{JS}(\mathcal{T})$ is $\deg_{\mathcal{T}}(v)$.

2.3. Show that the map JS defines a bijection between plane mobiles rooted at a white vertex and plane trees where all the leaves are white and the rest of the vertices are black.

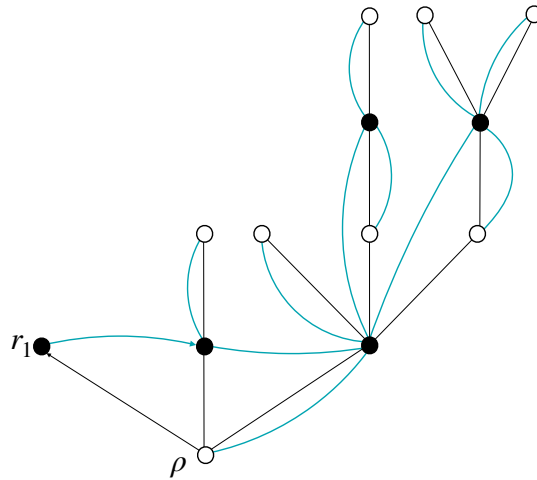


Figure 3: Janson & Stefansson's trick

* The goal of exercise 5 is to show that if a sequence $q := (q_k)_{k \geq 0}$ of non negative numbers is admissible if and only if the function:

$$f_q(x) := 1 + \sum_{k=1}^{\infty} \binom{2k-1}{k} q_k x^k$$

has a fix point.

Exercise 5: Enumeration results.

Fix l a positive integer. Set $N(k) := \binom{2k-1}{k}$, for every $k \geq 1$, and $N_0 := 1$.

1. Show that the map BFG defines a bijection between pointed bipartite planar maps with a distinguished face with perimeter $2l$ and positif well-labeled planar mobiles pointed at a black vertex with degree l .

Let m_\star^* be a pointed planar map with a root edge. We impose that the root face (the face incident on the right of the root edge) has perimeter $2l$.

Perform the following operations:

(vii) Unroot the map m_\star^* (forget where is the root edge) but keep track of the root face. Denote by \mathcal{T} the positif well-labeled planar mobile $\text{Mob}(m_\star^*)$ rooted at the black vertex living inside the root face.

(viii) Erase the root of \mathcal{T} , v_0 , and the edges containing it. We obtain a collection of l positif well-labeled mobiles. This collection is cyclically ordered. Root each new mobile at its unique white vertex which was a neighbor of v_0 in \mathcal{T} . Remark that the new mobiles are plane mobiles.

(ix) To order them, choose uniformly at random one of the new mobiles to be the first one. Translate each label by $-$ the label of the root of the first mobile.

We denote by $\text{Mob}^*(m_\star^*) := (\mathcal{T}_1, \dots, \mathcal{T}_p)$ the resulting forest of well-labeled pointed plane mobiles.

Fix $q = (q_k)_{k \in \mathbb{N}^*}$ a sequence of non negative numbers (such that $q \neq 0$). Let w_q denote the Boltzman measure associated to q . By convention we set $q_0 := 1$.

2. For $i \in \llbracket 1, p \rrbracket$, let u_i be the root of \mathcal{T}_i and $u_{p+1} = u_1$. Note that:

$$\forall i \in \llbracket 1, p \rrbracket, A(u_{i+1}) \geq A(u_i) - 1 \quad (P').$$

We say that an ordered collection of l well-labeled plane mobiles rooted at a white vertex is a positif well-labeled forest of rooted plane mobiles if the root of the first mobile is 0 and satisfies (P') . We extend Janson & Stefansson's trick to well-labeled forest of mobiles by erasing the labels and performing Janson & Stefansson's trick at each mobile of the forest (and keeping the order between the trees).

3. Show that the image measure of w_q on $\mathcal{M}_0^{(l)}$ by $\text{JS} \circ \text{Mob}^*$ is the measure \tilde{w}_q defined by:

$$\tilde{w}_q(\mathcal{F}) = 2N(l) \prod_{u \in \text{Vertex}(\mathcal{F})} N(\kappa_u) q_{\kappa_u}$$

for every forest of l plane trees \mathcal{F} (there is a direct bijection between trees and trees with leaves colored in white). Where $\text{Vertex}(\mathcal{F})$ stands for the set of vertices of \mathcal{F} and for every vertex u , κ_u stands for the number of children of u .

4. Show that q is admissible if and only if:

$$\sum_{T \in \text{Tree}} \prod_{u \in \text{Vertex}(T)} N(\kappa_u) q_{\kappa_u} < \infty$$

and deduce that $f_q(x) = x$ has a positive solution. Prove that if $f_q(x) = x$ has a positive solution, then q is admissible.

5. Suppose that q is admissible and write Z_q for smallest fix point of f_q . Take (T_1, \dots, T_l) a random variable distributed according to:

$$\frac{\tilde{w}_q}{w_q(\mathcal{M}_0^{(l)})}.$$

Show that (T_1, \dots, T_l) are iid Galton-Watson trees with offspring distribution:

$$\forall k \in \mathbb{N}, \mu_q(k) = Z_q^{k-1} N(k) q_k.$$

6. Deduce that:

$$W_0^{(l)} := w_q(\mathcal{M}_0^{(l)}) = 2N(l) Z_q^l.$$

References

- [1] J. BOUTTIER, P. DI FRANCESCO, AND E. GUITTER., Planar maps as labeled mobiles. *Electron. J. Combin.* (2004)
- [2] S. JANSON AND S.Ö.STEFÁNSSON. , Scaling limits of random planar maps with a unique large face. *The Annals of Probability*, (2015).