Metric on decorated spaces
X Escuela de Verano de Probabilidad y Procesos Estocásticos

Goal: To compare compact metric spaces with decorations.

1 Comparing embedded spaces.

Let (Y, δ) be a Polish space and E a subset of Y. We denote the ε–neighborhood of E by $E(\varepsilon)$, viz.

$$E(\varepsilon) := \{y \in Y : \ \delta(y, E) \leqslant \varepsilon\}.$$

We start recalling the definition of the Hausdorff distance, which allows to compare compact subsets of Y.

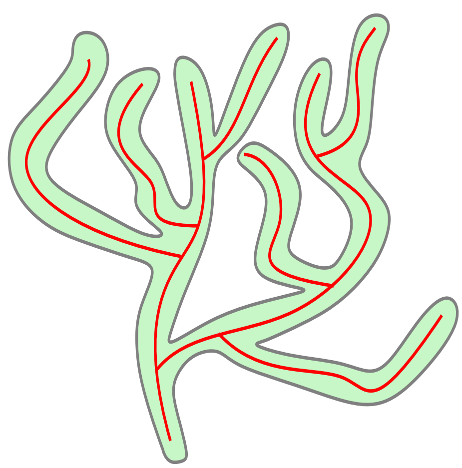


Figure 1: Illustration of a tree – in red – and its ε-neighbordhood in green.

For every E, E' compact subsets of Y, we set:

$$\delta_{\mathrm{H}}^{Y}(E, E') := \inf\left\{\varepsilon > 0 : \ E' \subset E(\varepsilon) \text{ and } E \subset E'(\varepsilon)\right\},$$

and remark that equivalently, we can express $\delta_{\mathrm{H}}^{Y}(E, E')$ as follows:

$$\delta_{\mathrm{H}}^{Y}(E, E') = \max\Big(\sup\left\{\delta(x, E') : \ x \in E\right\}, \sup\left\{\delta(y, E) : \ y \in E'\right\}\Big).$$

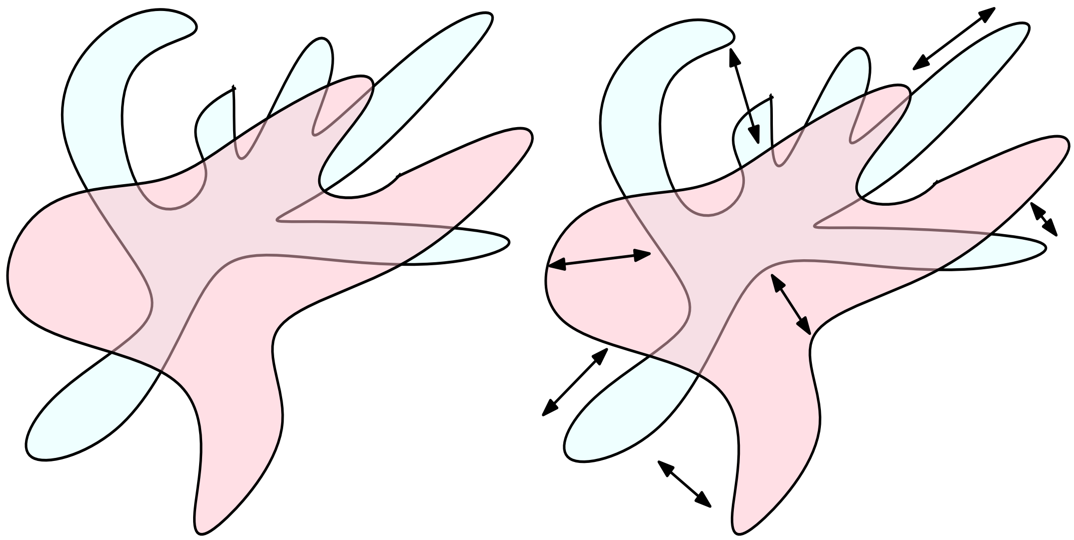


Figure 2: Illustration of δ_{H}.

Exercice 1. *Hausdorff distance.*
Let $\mathcal{K}(Y)$ be the set of all compact subsets of Y.
1. Show that $(\mathcal{K}(Y), \delta_{\mathrm{H}}^{Y})$ is a Polish space. The distance δ_{H}^{Y} is known as the Hausdorff distance.
2. In general, can we extend δ_{H}^{Y} into a distance on all the subsets of Y?
3. Let $(E_n : n \geqslant 0)$ be a sequence of compact subsets of Y converging to some compact subset E_∞ as $n \to \infty$, and assume that for every $n \geqslant 1$ the subset E_n is homeomorphic to E_0 . Do we have always that E_0 and E_∞ are homeomorphic ?

Decoration. In our case of interest, the spaces are decorated. Specially, we consider quadruplets $(\mathbf{E}, g) := (E, \rho, \mu, g)$ where:

$$E \text{ is a compact subset of } Y, \rho \text{ is a point of } E,$$
$$\mu \text{ is a finite measure supported on } E \text{ and } g : E \to \mathbb{R}_+.$$

Remark. If we were only interested in comparing two elements (E, ρ, μ) and (E', ρ', μ') we could take:

$$\delta_{\mathrm{H}}^{Y}(E, E') \vee \delta(\rho, \rho') \vee \delta_{\mathrm{P}}^{Y}(\mu, \mu').$$

where δ_{P}^{Y} for the Prokhorov distance on the space of finite measures on Y. However, we need to take into account the decoration g.

Encoding the labels g with an hypograph. For technical reasons, we shall always impose that the function g is *upper semi-continuous* (abbreviated usc in the sequel), where we recall that a function $g : E \to \mathbb{R}_+$ is usc if and only if the set $\{x \in E : g(x) \geqslant r\}$ is closed for every $r \geqslant 0$. We say that $(\mathbf{E}, g)$ is an usc-decorated compact subset of Y and we denote the set of all $(\mathbf{E}, g)$ by $\mathcal{H}(Y)$. We need now to compare elements of $\mathcal{H}(Y)$. In this direction, we introduce the notion of hypograph. If $(\mathbf{E}, g)$ is an element of $\mathcal{H}(Y)$, the associated hypograph is the set:

$$\mathrm{Hyp}_g(E) := \{(x,r) : x \in E \text{ and } 0 \leqslant r \leqslant g(x)\} \subset E \times \mathbb{R}_+.$$

The hypograph $\mathrm{Hyp}_g(E)$ is naturally equipped with the distance:

$$\mathrm{d}_{\mathrm{Hyp}_g(E)}((x,r),(x',r')) := d_E(x,x') + |r - r'|, \qquad \text{for } (x,r),(x',r') \in \mathrm{Hyp}_g(E). \tag{1}$$

We stress that $\mathrm{Hyp}_g(E)$ is a compact subset of $Y \times \mathbb{R}_+$ – equipped with the product metric. The hypograph $\mathrm{Hyp}_g(E)$ entirely determines (E, g), since E is identified with $E \times \{0\}$, while $g(x) = \max\{r \geqslant 0 : (x,r) \in \mathrm{Hyp}_g(E)\}$ for every $x \in E$.

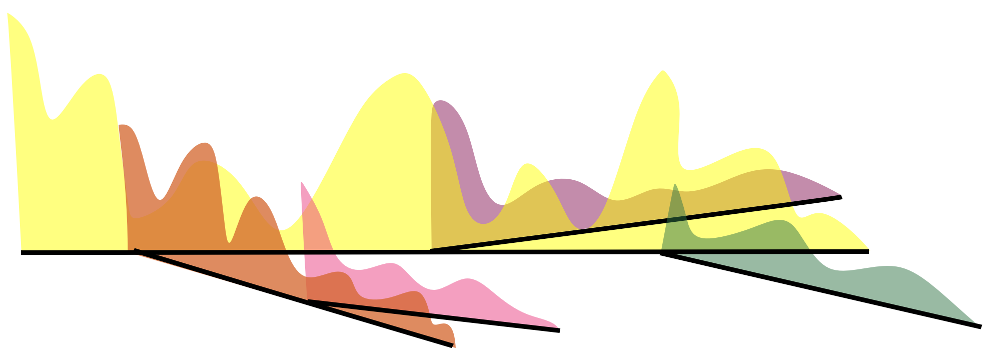


Figure 3: Representation of hypograph ontop of a tree.

Exercice 2. *Hypograph distance on $\mathcal{H}(Y)$.*
For every $(\mathbf{E}, g)$ and $(\mathbf{E}', g')$ two usc-decorated compact subset of Y, we set:

$$\mathrm{d}_{\mathcal{H}(Y)}((\mathbf{E},g),(\mathbf{E}',g')) := \delta_{\mathrm{H}}^{Y\times\mathbb{R}_+}\left(\mathrm{Hyp}_g(E), \mathrm{Hyp}_{g'}(E')\right) \vee \delta(\rho,\rho') \vee \delta_{\mathrm{P}}^{Y}(\mu,\mu'). \tag{2}$$

Show that $(\mathcal{H}(Y), \mathrm{d}_{\mathcal{H}(Y)})$ is a Polish space.

2 Comparing non-embedded spaces.

We are more interested in the decorated space $(\mathbf{E}, g)$ by itself than in its particular embedding. For this reason, we introduce the equivalence relation:

$$(\mathbf{E},g) \sim (\mathbf{E}',g') \iff \exists \text{ an isometry } \phi : E \to E' \text{ such that } \left(\phi(\rho), \phi_{\#}\mu, g \circ \phi^{-1}\right) = (\rho', \mu', g').$$

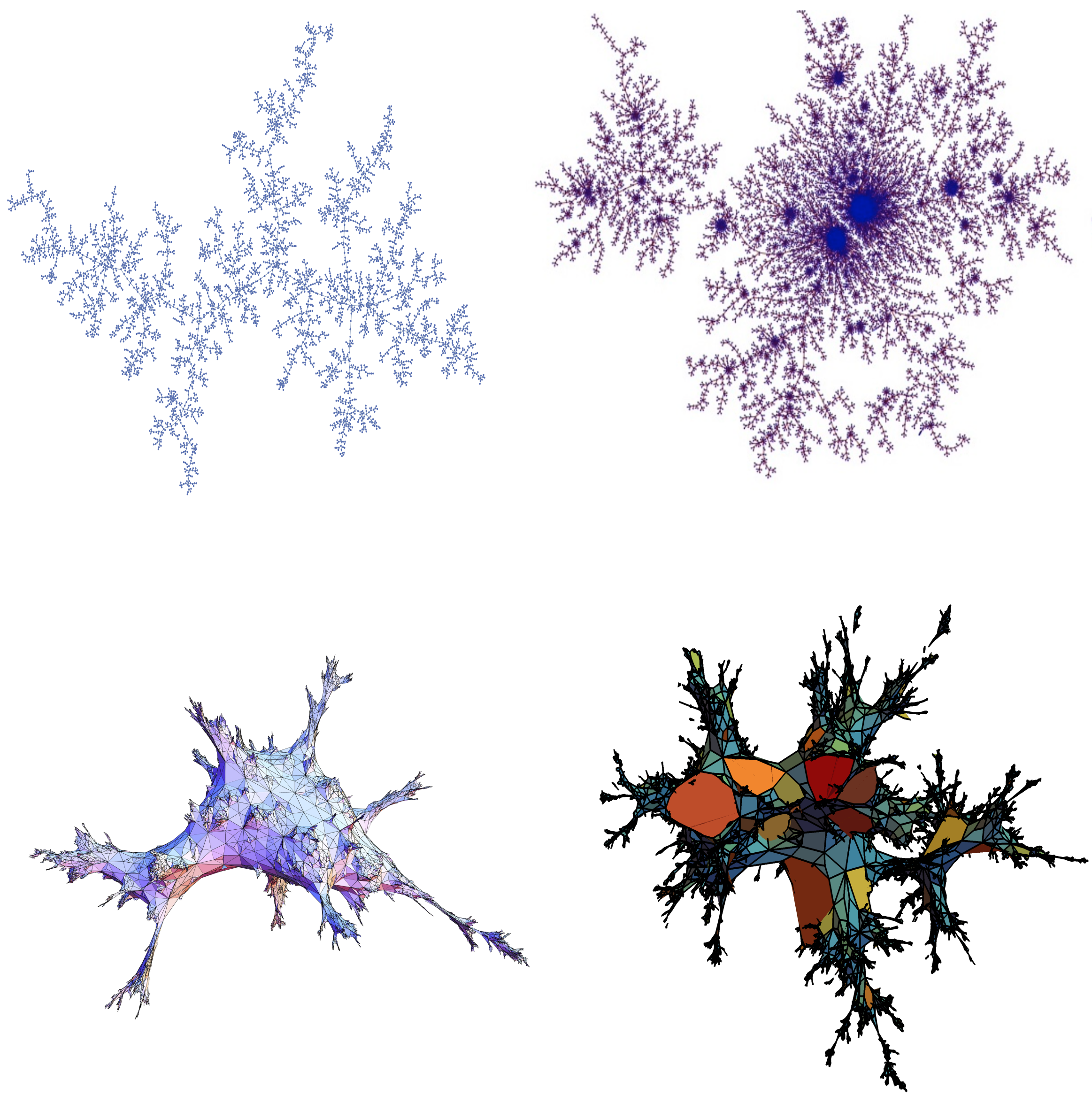
We shall abuse notion and when no ambiguity is possible we still speak of usc-decorated measured rooted compact metric spaces instead of their equivalence classes, and we denote the set of all isometry classes of such spaces by $\mathbb{H}$. To simplify notation, we write $\Phi(\mathbf{E}) = (\Phi(E), \Phi(\rho), \Phi_{\#}\mu)$

and $\Phi(\mathbf{E}, g) = (\Phi(E), g \circ \Phi^{-1})$. Our goal now is to introduce a convenient distance making $\mathbb{H}$ a Polish space. We set:

$$\mathrm{d}_{\mathbb{H}}\big((\mathbf{E}, g), (\mathbf{E}', g')\big) := \inf_{\substack{\Phi: E \to Y \\ \Phi': E' \to Y}} \mathrm{d}_{\mathcal{H}(Y)}\big(\Phi(\mathbf{E}, g), \Phi'(\mathbf{E}', g')\big),$$

where the infimum is over all the Polish spaces (Y, δ) and all the isometry embeddings $\Phi : E \to Y$ and $\Phi' : E' \to Y$. We stress that when g and g' are identically zero, the quantity $\mathrm{d}_{\mathbb{H}}\big((\mathbf{E}, g), (\mathbf{E}', g')\big)$ is the classical Gromov-Hausdorff-Prokhorov distance between $\mathbf{E}$ and $\mathbf{E}'$.

Theorem 1. *The map* $\mathrm{d}_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_+$ *is a distance on* $\mathbb{H}$ *and the space* $(\mathbb{H}, \mathrm{d}_{\mathbb{H}})$ *is Polish.*



Examples of large combinatorial random structures

The proof of the theorem relies on the following intermediate result:

Lemma 1. *Fix* $(\varepsilon_n)_{n\geqslant 1}$ *a sequence in* $\mathbb{R}_+$ *and* $(\mathbf{E}_n, g_n)$ *a sequence in* $\mathbb{H}$ *such that*

$$\mathrm{d}_{\mathbb{H}}((\mathbf{E}_n, g_n), (\mathbf{E}_{n+1}, g_{n+1})) < \varepsilon_n.$$

Then there exists a Polish space (Z, δ) *and isometric embeddings* $\phi_1, \phi_2, ...$ *respectively from* $E_1, E_2...$ *into* Z *such that:*

$$\mathrm{d}_{\mathcal{H}(Y)}\big(\phi_n(\mathbf{E}_n, g_n), \phi_{n+1}(\mathbf{E}_{n+1}, g_{n+1})\big) < \varepsilon_n.$$

This result is an adaptation of [2, Lemma 5.7].

Exercice 3. *Proof of the theorem.*
Deduce the theorem from Lemma 1.

Exercice 4. *Proof of the technical lemma.*
We use the notation of the statement of the lemma. First, remark that by definition, for every $n \geqslant 1$, we can find a Polish space (Y_n, δ_n), and two isometric embeddings $\Phi_n : E_n \to Y_n$ and $\Phi'_n : E_{n+1} \to Y_n$ such that:

$$\bar{\varepsilon}_n := \mathrm{d}_{\mathcal{H}(Y)}\big(\Phi_n(\mathbf{E}_n, g_n), \Phi'_n(\mathbf{E}_{n+1}, g'_{n+1})\big) < \varepsilon_n.$$

Next, we introduce the disjoint union $Z := \bigsqcup_{n\geqslant 1} E_n$, and we endow Z with the metric δ defined as the biggest distance verifying $\delta(x, y) = d_{E_n}(x, y)$ if $(x, y) \in E_n^2$ and $\delta(x, y) := \delta_n\big(\Phi_n(x), \Phi'_n(y)\big) + (\varepsilon_n - \bar{\varepsilon}_n)/2$ for $x, y \in E_n \times E_{n+1}$, the term $(\varepsilon_n - \bar{\varepsilon}_n)/2$ ensures that δ does not identify two points $(x, y) \in E_n \times E_{n+1}$.

1. Show that (Z, δ) is separable.

For simplicity with slightly abuse of notation we still write (Z, δ) for its completion. Now we let $\phi_n : E_n \to Z$ be the canonical embedding.

2. Prove that for every $n \geqslant 1$, we have:

$$\mathrm{d}_{\mathcal{H}(Y)}\big(\phi_n(\mathbf{E}_n, g_n), \phi_{n+1}(\mathbf{E}_{n+1}, g_{n+1})\big) < \varepsilon_n.$$

References

[1] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.*American Mathematical Society, Providence, RI, 2001.

[2] A. Greven, P. Pfaffelhuber, A. Winter. Convergence in distribution of random metric measure spaces (-coalescent measure trees). *Probab. Theory Related Fields*, 145(1-2):285–322, 2009