AMSS Course: Metric on decorated spaces

Goal: To compare compact metric spaces with decorations.

1 Comparing embedded spaces.

Let (Y, δ) be a Polish space and E a subset of Y. We denote the ε–neighborhood of E by $E(\varepsilon)$, viz.

$$E(\varepsilon) := \{y \in Y : \delta(y, E) \leqslant \varepsilon\}.$$

We start recalling the definition of the Hausdorff distance, which allows to compare compact subsets of Y.

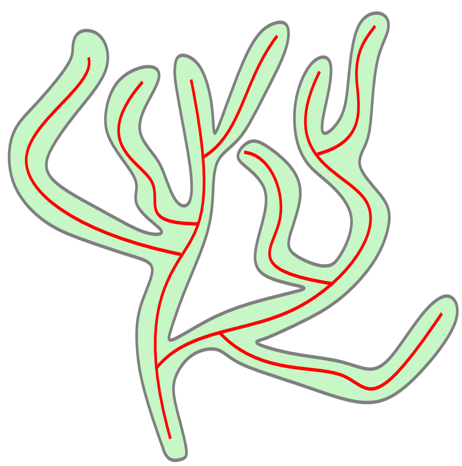

Figure 1: Illustration of a tree – in red – and its ε-neighbordhood in green.

For every E, E' compact subsets of Y, we set:

$$\delta_{\mathrm{H}}^{Y}(E, E') := \inf\left\{\varepsilon > 0 : E' \subset E(\varepsilon) \text{ and } E \subset E'(\varepsilon)\right\},$$

and remark that equivalently, we can express $\delta_{\mathrm{H}}^{Y}(E, E')$ as follows:

$$\delta_{\mathrm{H}}^{Y}(E, E') = \max\Big(\sup\left\{\delta(x, E') : x \in E\right\}, \sup\left\{\delta(y, E) : y \in E'\right\}\Big).$$

Exercice 1. *Hausdorff distance.*
Let $\mathcal{K}(Y)$ be the set of all compact subsets of Y.

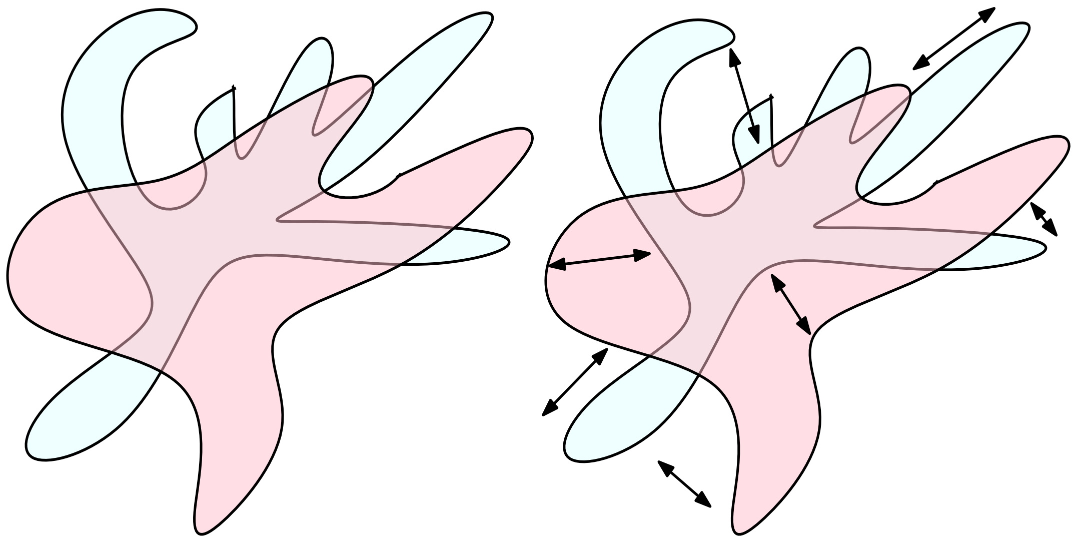

Figure 2: Illustration of δ_{H}.

1. Show that $(\mathcal{K}(Y), \delta_{\mathrm{H}}^{Y})$ is a Polish space. The distance δ_{H}^{Y} is known as the Hausdorff distance.
2. In general, can we extend δ_{H}^{Y} into a distance on all the subsets of Y?
3. Let $(E_n : n \geqslant 0)$ be a sequence of compact subsets of Y converging to some compact subset E_∞ as $n \to \infty$, and assume that for every $n \geqslant 1$ the subset E_n is homeomorphic to E_0 . Do we have always that E_0 and E_∞ are homeomorphic ?

Decoration. In our case of interest, the spaces are decorated. Specially, we consider quadruplets $\mathbf{E} := (E, \mu, \rho, \rho^\star)$ where:

$$E \text{ is a compact subset of } Y,\ \mu \text{ is a finite Borel measure supported on } E,$$
$$\rho, \rho^\star \text{ are two distinguished points of } E.$$

To compare two such elements $\mathbf{E}_1 := (E_1, \mu_1, \rho_1, \rho_1^\star)$ and $\mathbf{E}_2 := (E, \mu_2, \rho_2, \rho_2^\star)$ embedded in the same Polish space (Y, δ), we take:

$$\mathrm{d}_Y(\mathbf{E}_1, \mathbf{E}_2) := \delta_{\mathrm{H}}^{Y}(E_1, E_2) \vee \delta_{\mathrm{P}}^{Y}(\mu_1, \mu_2) \vee \delta(\rho_1, \rho_2) \vee \delta(\rho_1^\star, \rho_2^\star).$$

where δ_{P}^{Y} for the Prokhorov distance on the space of finite measures on Y.

2 Comparing non-embedded spaces.

We are more interested in the decorated space $\mathbf{E}$ by itself than in its particular embedding. For this reason, we introduce the equivalence relation:

$$\mathbf{E}_1 \sim \mathbf{E}_2 \iff \exists \text{ an isometry } \phi : E_1 \to E_2 \text{ such that } \left(\phi_\# \mu_1, \phi(\rho_1), \phi(\rho_1^\star)\right) = \left(\mu_2, \rho_2, \rho_2^\star\right).$$

We shall abuse notion and when no ambiguity is possible we still speak of decorated measured rooted compact metric spaces instead of their equivalence classes, and we denote the set of all isometry classes of such spaces by $\mathbb{H}$. To simplify notation, we write

$$\Phi(\mathbf{E}) = (\Phi(E), \Phi_\# \mu, \Phi(\rho), \Phi(\rho^\star)).$$

Our goal now is to introduce a convenient distance making $\mathbb{H}$ a Polish space. We set:

$$\mathrm{d}_{\mathbb{H}}\left(\mathbf{E}_1, \mathbf{E}_2\right) := \inf_{\substack{\Phi : E_1 \to Y \\ \Phi_2 : E_2 \to Y}} \mathrm{d}_Y\left(\Phi(\mathbf{E}), \Phi_2(\mathbf{E}_2)\right),$$

where the infimum is over all the Polish spaces (Y, δ) and all the isometry embeddings $\Phi : E_1 \to Y$ and $\Phi' : E_2 \to Y$. .

Theorem 1. *The map* $\mathrm{d}_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_+$ *is a distance on* $\mathbb{H}$ *and the space* $(\mathbb{H}, \mathrm{d}_{\mathbb{H}})$ *is Polish.*

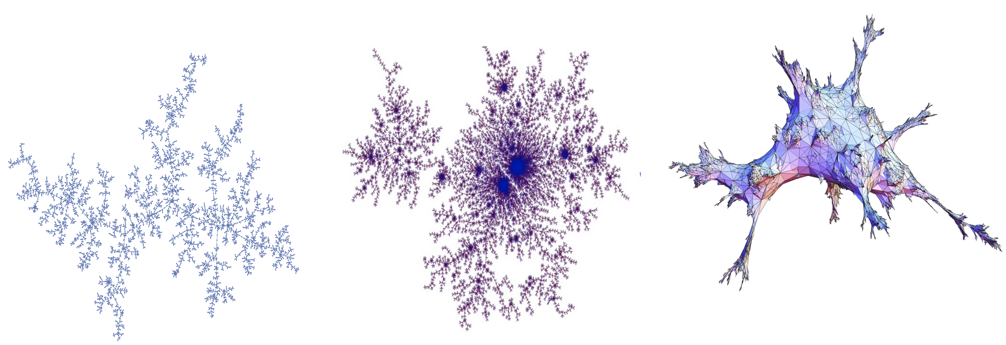

Figure 3: Examples of large combinatorial random structures

The distance $\mathrm{d}_{\mathbb{H}}$ is known as the bi-pointed Gomov-Prokhorov distance. The proof of the theorem relies on the following intermediate result:

Lemma 1. *Fix $(\varepsilon_n)_{n\geqslant 1}$ a sequence in $\mathbb{R}_+$ and $\mathbf{E}_n$ a sequence in $\mathbb{H}$ such that*

$$\mathrm{d}_{\mathbb{H}}(\mathbf{E}_n, \mathbf{E}_{n+1}) < \varepsilon_n.$$

Then there exists a Polish space (Z, δ) and isometric embeddings $\phi_1, \phi_2, ...$ respectively from $E_1, E_2...$ into Z such that:

$$\mathrm{d}_{\mathcal{H}(Y)}\big(\phi_n(\mathbf{E}_n), \phi_{n+1}(\mathbf{E}_{n+1})\big) < \varepsilon_n.$$

This result is an adaptation of [2, Lemma 5.7].

Exercice 3. *Proof of the theorem.*
Deduce the theorem from Lemma 1.

Exercice 4. *Proof of the technical lemma.*
We use the notation of the statement of the lemma. First, remark that by definition, for every $n \geqslant 1$, we can find a Polish space (Y_n, δ_n), and two isometric embeddings $\Phi_n : E_n \to Y_n$ and $\Phi'_n : E_{n+1} \to Y_n$ such that:

$$\bar{\varepsilon}_n := \mathrm{d}_{\mathcal{H}(Y)}\big(\Phi_n(\mathbf{E}_n), \Phi'_n(\mathbf{E}_{n+1})\big) < \varepsilon_n.$$

Next, we introduce the disjoint union $Z := \bigsqcup_{n\geqslant 1} E_n$, and we endow Z with the metric δ defined as the biggest distance verifying $\delta(x,y) = d_{E_n}(x,y)$ if $(x,y) \in E_n^2$ and $\delta(x,y) := \delta_n\big(\Phi_n(x), \Phi'_n(y)\big) + (\varepsilon_n - \bar{\varepsilon}_n)/2$ for $x, y \in E_n \times E_{n+1}$, the term $(\varepsilon_n - \bar{\varepsilon}_n)/2$ ensures that δ does not identify two points $(x,y) \in E_n \times E_{n+1}$.

1. Show that (Z, δ) is separable.

For simplicity with slightly abuse of notation we still write (Z, δ) for its completion. Now we let $\phi_n : E_n \to Z$ be the canonical embedding.

2. Prove that for every $n \geqslant 1$, we have:

$$\mathrm{d}_{\mathcal{H}(Y)}\big(\phi_n(\mathbf{E}_n), \phi_{n+1}(\mathbf{E}_{n+1})\big) < \varepsilon_n.$$

References

[1] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.*American Mathematical Society, Providence, RI, 2001.

[2] A. Greven, P. Pfaffelhuber, A. Winter. Convergence in distribution of random metric measure spaces (-coalescent measure trees). *Probab. Theory Related Fields*, 145(1-2):285–322, 2009