

Isoperimetric inequalities in the Brownian plane*

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Abstract

We consider the model of the Brownian plane, which is a pointed non-compact random metric space with the topology of the complex plane. The Brownian plane can be obtained as the scaling limit in distribution of the uniform infinite planar triangulation or the uniform infinite planar quadrangulation and is conjectured to be the universal scaling limit of many others random planar lattices. We establish sharp bounds on the probability of having a short cycle separating the ball of radius r centered at the distinguished point from infinity. Then we prove a strong version of the spatial Markov property of the Brownian plane. Combining our study of short cycles with this strong spatial Markov property we obtain sharp isoperimetric bounds for the Brownian plane.

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1 Introduction

In recent years, much work and energy have been devoted to the study of discrete and continuous random geometry in dimension 2. In this paper we will study the Brownian plane \mathcal{M}_∞ , which appears as the scaling limit in distribution of the uniform infinite planar quadrangulation Q_∞ , in the local Gromov-Hausdorff sense and can also be interpreted as a Brownian map with infinite volume, see [8]. The Brownian plane is a random pointed and weighted boundedly compact length space homeomorphic to \mathbb{C} and is conjectured to be the universal scaling limit of other discrete models. The case of the uniform infinite planar triangulation of type I has been treated in [6]. We also mention that the Brownian plane is closely related to the Liouville quantum gravity surface called the quantum cone, see [20, Corollary 1.5].

The spaces Q_∞ and \mathcal{M}_∞ have a distinguished point, also called the root. Our first goal is to understand the probability of having a short injective cycle separating the ball of radius r centered at the root in \mathcal{M}_∞ from infinity. This will allow us to deduce isoperimetric inequalities for the Brownian plane. These results can then be extended to other models such as the Brownian sphere and the infinite Brownian disk. The study of short separating cycles starts in random planar geometry with the paper [19], where Krikun gave a construction of Q_∞ as the local limit of large finite planar quadrangulations. He also proved the existence, for every $r \in \mathbb{N}^*$, of cycles separating the ball of radius r of Q_∞ from infinity having length of order r . Krikun conjectured that it is not possible to find separating cycles with length of order smaller than r . In [15], Le Gall and Lehericy confirmed Krikun's conjecture by proving that for every $\delta > 0$, there exists a constant $c_\delta > 0$ such that for every $r \in \mathbb{N}^*$:

$$\mathbb{P}(L_r(Q_\infty) < \varepsilon r) < c_\delta \varepsilon^{2-\delta}$$

where $L_r(Q_\infty)$ stands for the infimum of the lengths of injective cycles disconnecting the ball of radius r of Q_∞ from infinity. They also proved that the probability $\mathbb{P}(L_r(Q_\infty) > ur)$ decreases exponentially fast when u goes to infinity.

This work can be seen as a continuous counterpart of this study. We are aiming at similar results for the Brownian plane. Thanks to the geometric properties of \mathcal{M}_∞ we get optimal results in the continuous setting. Since the Brownian plane is expected to be the universal scaling limit of random lattices such as the UIPQ and the UIPT, it is likely that these sharper results also have analogs for discrete models. Let us present our results more precisely. It should also be possible to adapt some of our techniques to the case of the UIPQ.

The Brownian plane \mathcal{M}_∞ is equipped with a root, which we denote by 0 , a distance Δ and a volume measure $|\cdot|$. The construction of \mathcal{M}_∞ on a probability space $(\Omega, \mathcal{F}, \Theta_0)$ is recalled in Section 2.4. For every $r > 0$, let $B_r(\mathcal{M}_\infty)$ denote the closed ball of radius r centered at 0 in \mathcal{M}_∞ . For every path $\gamma : [t, t'] \rightarrow \mathcal{M}_\infty$, we denote its length by $\Delta(\gamma)$ i.e.:

$$\Delta(\gamma) := \sup_{t=t_1 \leq t_2, \dots, \leq t_n=t'} \sum_{i=1}^{n-1} \Delta(\gamma(t_i), \gamma(t_{i+1})) \quad (1)$$

where the supremum is over all choices of the integer $n \geq 1$ and the finite sequence $t_1 \leq t_2 \leq \dots \leq t_n$ satisfying $(t_1, t_n) = (t, t')$. In this work a path has to be a continuous function. Moreover we say that a path

$\gamma : [t, t'] \rightarrow \mathcal{M}_\infty$ is a separating cycle if:

- for every $t \leq s < s' \leq t'$ we have $\gamma(s) = \gamma(s')$ if and only if $(s, s') = (t, t')$;
- the distinguished point 0 does not belong to the range of γ and there exists $r > 0$ such that for any path $\tilde{\gamma} : [s, s'] \rightarrow \mathcal{M}_\infty$ with $\tilde{\gamma}(s) = 0$ and $\tilde{\gamma}(s') \notin B_r(\mathcal{M}_\infty)$ we have:

$$\gamma([t, t']) \cap \tilde{\gamma}([s, s']) \neq \emptyset.$$

We will say that a separating cycle γ separates $B_r(\mathcal{M}_\infty)$ from infinity if it takes values in the complement of $B_r(\mathcal{M}_\infty)$. Recall that \mathcal{M}_∞ has a.s. the topology of \mathbb{C} and consequently it has only one end. So for every $r > 0$, we can consider the hull of radius r , i.e. the complement of the unique unbounded connected component of the complement of the closed ball of radius r centered at the distinguished point. We denote the hull of radius r by $B_r^\bullet(\mathcal{M}_\infty)$. For every $r > 0$ and any separating cycle γ that separates $B_r(\mathcal{M}_\infty)$ from infinity, an application of Jordan's theorem shows that the path γ has to take values in the complement of $B_r^\bullet(\mathcal{M}_\infty)$. We say that such a path γ separates $B_r^\bullet(\mathcal{M}_\infty)$ from infinity and we introduce the set \mathcal{C}_r of all cycles separating $B_r^\bullet(\mathcal{M}_\infty)$ from infinity, which is not empty since $B_r^\bullet(\mathcal{M}_\infty)$ is bounded. Remark that any separating cycle γ is in \mathcal{C}_r for r small enough and set:

$$L_r := \inf\{\Delta(\gamma) : \gamma \in \mathcal{C}_r\}.$$

One of the benefits of working in the continuous setting is the fact that the Brownian plane is scale invariant in distribution, i.e. for every $r > 0$, $(\mathcal{M}_\infty, 0, \Delta, |\cdot|) \stackrel{(d)}{=} (\mathcal{M}_\infty, 0, r\Delta, r^4|\cdot|)$ (see Section 2.4). In particular, the scaling invariance implies that:

$$L_r \stackrel{(d)}{=} rL_1.$$

Therefore we will focus on the variable L_1 . We will prove the following result:

Theorem 1.

(i) *We have*

$$\limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(L_1 > u))}{u} \leq - \sup_{s > 1} \frac{1}{2(s-1)} \log\left(\frac{s^2}{2s-1}\right).$$

Consequently, $\Theta_0(L_1 > u)$ decreases at least exponentially fast when u goes to ∞ .

(ii) *There exist two constants $0 < c_1 \leq c_2$ such that for every $\varepsilon > 0$:*

$$c_1(\varepsilon^2 \wedge 1) \leq \Theta_0(L_1 < \varepsilon) \leq c_2\varepsilon^2.$$

It may be possible to get a discrete version of Theorem 1 for the UIPQ by adapting our methods using the tree decomposition given in [7]. This decomposition is the discrete analog of the construction of the Brownian plane that we will present in the preliminaries. Let us mention that our methods also allow us to obtain upper and lower bounds on the probability of the event $\{L_1 < \varepsilon\}$ under various conditionings.

In Section 3.4 we prove a strong version of the spatial Markov property of the Brownian plane, which has been first derived in [17, Section 5.1]. The statement of this property requires some notation and we give the precise formulation of this property in Section 3.4. Combining Theorem 1 with this strong spatial Markov property we are able to study isoperimetric properties of the Brownian plane. Let us be more precise about this point.

We say that a closed subset A of \mathcal{M}_∞ is a (closed) Jordan domain if it is homeomorphic to the closed disk of the complex plane \mathbb{C} . Let \mathcal{K} be the set of all Jordan domains of \mathcal{M}_∞ whose interior contains the distinguished point of \mathcal{M}_∞ . For every $A \in \mathcal{K}$, we can define the length $\Delta(\partial A)$ of its boundary, as follows. We consider an injective cycle $g : [0, 1] \rightarrow \mathcal{M}_\infty$ such that $g([0, 1]) = \partial A$ and we set:

$$\Delta(\partial A) := \Delta(g).$$

This definition does not depend on the parameterization g . We can now state our result concerning isoperimetric inequalities in the Brownian plane. We will prove in Section 4 that:

Theorem 2. *For any nondecreasing function $f : \mathbb{R}_+ \rightarrow (0, \infty)$:*

(i) *We have*

$$\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0, \Theta_0\text{-a.s.}, \text{ if } \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} = \infty.$$

(ii) *We have*

$$\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0, \Theta_0\text{-a.s.}, \text{ if } \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

Theorem 2 can be extended to the infinite volume Brownian disk (see Corollary 4) and the Brownian map (since the Brownian map and the Brownian Plane are locally isometric [8, Theorem 1]). In [15], Le Gall and Lehericy use their study of short cycles to get an analog for the UIPQ of Theorem 2 for the special case $f(x) := x^{\frac{3}{4}+\delta}$ for any $\delta > 0$. We conclude this introduction by pointing out that the study of separating cycles appears naturally in other problems of random geometry; in the recent work [5] the authors use a class of separating cycles to obtain bijective enumerations of planar maps with three boundaries. They also discuss the statistics of the lengths of minimal separating loops in different discret models.

2 Preliminaries

The preliminaries are divided as follows. Section 2.1 gives a quick presentation of snake trajectories and the associated compact trees, we refer to [1, 10] for a more detailed description of these objects. Section 2.2 presents the Brownian snake excursion, which is the building block of the theory of Brownian geometry, and the special Markov property. Finally in Section 2.3 and 2.4 we introduce the notion of a coding triple and give the construction of the Brownian plane and the infinite volume Brownian disk; these last sections follow [17]. Before starting the preliminaries, let us introduce some standard notation.

Let (E, d) be a metric space.

- For $A \subset E$, we denote the closure (resp. the interior) of A in E by $\text{Cl}(A)$ (resp. $\text{Int}(A)$). Set $\partial A := \text{Cl}(A) \setminus \text{Int}(A)$.

- A path γ on E is a continuous function defined on an interval I of \mathbb{R} taking values in E . We say that the path γ separates two subsets A and B of E if the range of γ does not intersect $A \cup B$ and if any path starting at A and ending at B intersects the range of γ . The path γ is a geodesic on E if for every $s, t \in I$, $d(\gamma(s), \gamma(t)) = |s - t|$.

- We denote the length of a path γ by $d(\gamma)$. The definition of $d(\gamma)$ is the same as defined in (1) replacing Δ by d . We say that (E, d) is a length space if, for every $x, y \in E$, the distance $d(x, y)$ is the infimum of the quantities $d(\gamma)$ over all the paths γ on E starting at x and ending at y .

- If (E, d) is a length space and U is a path-connected subset of E , the intrinsic distance induced by d on U is the distance d_U on U defined as follows:

$$\forall x, y \in U, d_U(x, y) := \inf \{d(\gamma) : \gamma : [0, 1] \rightarrow U \text{ path with } (\gamma(0), \gamma(1)) = (x, y)\}.$$

Remark that d_U may take infinite values if U is not an open subset of E .

- We say that a compact (resp. boundedly compact) metric space is weighted if it is given with a finite (resp. finite on compact sets) measure, which is often called the volume measure. We denote by \mathbb{K} (resp. \mathbb{K}_∞) the set of all isometry classes of pointed and weighted compact (resp. boundedly compact) metric spaces equipped with the Gromov-Hausdorff-Prokhorov distance (resp. the local Gromov-Hausdorff-Prokhorov distance). Both \mathbb{K} and \mathbb{K}_∞ are Polish spaces.

Finally, we write $s \vee t := \max(s, t)$, $s \wedge t := \min(s, t)$ and by convention $\inf \emptyset := \infty$.

2.1 Snake trajectories and labeled trees

Let \mathcal{W} be the set of all continuous mappings $w : [0, \zeta_w] \rightarrow \mathbb{R}$, where $\zeta_w \geq 0$ is called the lifetime of w . We will write $\widehat{w} = w(\zeta_w)$ for the endpoint of w . For every $x \in \mathbb{R}$, we identify x with the map starting from x with 0 lifetime. Set $\mathcal{W}_x := \{w \in \mathcal{W} : w(0) = x\}$ and equip \mathcal{W} with the distance:

$$d_{\mathcal{W}}(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t \geq 0} |w(t \wedge \zeta_w) - w'(t \wedge \zeta_{w'})|.$$

Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $\omega : s \mapsto \omega_s$ from \mathbb{R}_+ into \mathcal{W}_x satisfying the following properties:

- $\omega_0 = x$ and the quantity $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$ is finite. The quantity $\sigma(\omega)$ is called the lifetime of ω . By convention $\sigma(\omega) := 0$ if $\omega_s = x$ for every $s \geq 0$;

- For every $s, s' \in \mathbb{R}_+$ with $s \leq s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \leq \min_{r \in [s, s']} \zeta_{\omega_r}$. This property is called the snake property.

We denote the set of all snake trajectories starting at x by \mathcal{S}_x , and write $\mathcal{S} = \cup_{x \in \mathbb{R}} \mathcal{S}_x$ for the set of all snake trajectories. For every $\omega \in \mathcal{S}$ and $s \geq 0$, introduce the notation $W_s(\omega) := \omega_s$. The set \mathcal{S} is equipped with the distance:

$$d_{\mathcal{S}}(\omega, \omega') := |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

It is straightforward to verify that the space $(\mathcal{S}, d_{\mathcal{S}})$ is a Polish space. To simplify notation, for every $\omega \in \mathcal{S}$, we set

$$\omega_* := \inf\{\widehat{\omega}_s : s \geq 0\}.$$

It will be important for our study to associate a compact \mathbb{R} -tree \mathcal{T}_{ω} with every snake trajectory ω .

Let $\omega \in \mathcal{S}$ and define:

$$d_{\omega}(s, t) := \zeta_{\omega_s} + \zeta_{\omega_t} - 2 \inf_{r \in [s \wedge t, s \vee t]} \zeta_{\omega_r}$$

for every $s, t \in [0, \sigma(\omega)]$. Since $s \mapsto \zeta_{\omega_s}$ is continuous, d_{ω} is a continuous pseudo-distance on $[0, \sigma(\omega)]$. We define an equivalence relation $\approx_{d_{\omega}}$ by setting $s \approx_{d_{\omega}} t$ if $d_{\omega}(s, t) = 0$. The space $\mathcal{T}_{\omega} := [0, \sigma(\omega)] / \approx_{d_{\omega}}$ equipped with the distance induced by d_{ω} is a compact \mathbb{R} -tree. Let $p_{\omega} : [0, \sigma(\omega)] \rightarrow \mathcal{T}_{\omega}$ be the canonical projection and let V_{ω} be the pushforward of Lebesgue measure on $[0, \sigma(\omega)]$ under p_{ω} . We view the tree \mathcal{T}_{ω} as a pointed and weighted compact metric space, for which the volume measure is V_{ω} and the distinguished point is $\rho_{\omega} := p_{\omega}(0)$, which is called the root of \mathcal{T}_{ω} . For every $u \in \mathcal{T}_{\omega}$, set $\Lambda_u^{\omega} := \widehat{\omega}_t$ where t is any element of $p_{\omega}^{-1}(u)$. The quantity Λ_u^{ω} is well defined by the snake property and we interpret Λ_u^{ω} as a label assigned to u . The pair $(\mathcal{T}_{\omega}, (\Lambda_u^{\omega})_{u \in \mathcal{T}_{\omega}})$ is the labeled tree associated with the snake trajectory ω .

We will use the following standard nomenclature. Let \mathcal{T} be a compact tree. The multiplicity of a point $a \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \setminus \{a\}$. If the multiplicity of a is 1 (resp. > 2), a is called a leaf (resp. a branching point).

2.2 The Brownian snake excursion

To simplify notation, set $\widehat{W}_s(\omega) = \widehat{\omega}_s$ and $W_*(\omega) = \omega_*$ for every $\omega \in \mathcal{S}$. Fix $x \in \mathbb{R}$. The Brownian snake excursion measure \mathbb{N}_x is the unique σ -finite measure on \mathcal{S}_x that satisfies the following properties:

- The distribution of $s \mapsto \zeta_{\omega_s}$ is the Itô measure of positive excursions of linear Brownian motion, with the normalization:

$$\forall \varepsilon > 0, \mathbb{N}_x \left(\sup_{s \in [0, \sigma(\omega)]} \zeta_{\omega_s} > \varepsilon \right) = \frac{1}{2\varepsilon};$$

- Conditionally on $(\zeta_{\omega_s})_{s \geq 0}$, $(\widehat{W}_s(\omega))_{s \geq 0}$ is a Gaussian process with mean x and covariance function:

$$\forall s, s' \in [0, \sigma(\omega)], K(s, s') := \min_{r \in [s \wedge s', s \vee s']} \zeta_{\omega_r}.$$

Roughly speaking, conditionally on $(\zeta_s)_{s \geq 0}$, the process $(W_s)_{s \geq 0}$ evolves as follows. If ζ_s decreases, the path W_s is shortened from its tip, while if ζ_s increases, the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. We refer to [12] for a rigorous presentation. For every $x, y \in \mathbb{R}$ with $x < y$ we have:

$$\mathbb{N}_y(W_* < x) = \frac{3}{2(y-x)^2} \quad (2)$$

see [12, Chapter 6] for more details. To simplify notation, under $\mathbb{N}_x(d\omega)$ we will write σ for $\sigma(\omega)$ and $W_s(t)$ for $\omega_s(t)$.

Operations. We introduce a collection of elementary operations on \mathcal{S} .

- Translation:

For every snake trajectory ω and every $\lambda \in \mathbb{R}$, we will write $\omega + \lambda$ for the snake trajectory

$$(\omega + \lambda)_s(t) := \omega_s(t) + \lambda, \quad 0 \leq t \leq \zeta_{(\omega + \lambda)_s} := \zeta_{\omega_s}.$$

By construction for every $x \in \mathbb{R}$ the pushforward measure of \mathbb{N}_x under $\omega \mapsto \omega + \lambda$ is $\mathbb{N}_{x+\lambda}$.

- Scaling:

For every snake trajectory ω and every $\lambda \in \mathbb{R}_+^*$, we will write $\text{hom}_\lambda(\omega)$ for the snake trajectory defined by

$$\text{hom}_\lambda(\omega)_s(t) := \lambda \omega_{s\lambda^{-4}}(t\lambda^{-2}), \quad 0 \leq t \leq \zeta_{\text{hom}_\lambda(\omega)_s} := \lambda^2 \zeta_{\omega_{s\lambda^{-4}}}.$$

It is also easy to deduce from the scaling property of Brownian motion that for every $x \in \mathbb{R}$ the pushforward measure of \mathbb{N}_x under $\omega \mapsto \text{hom}_\lambda(\omega)$ is $\lambda^2 \mathbb{N}_{\lambda x}$. We will call this property the scaling property of the Brownian snake excursion.

- Truncation:

Let $(x, r) \in \mathbb{R}^2$ with $x > r$. For every $w \in \mathcal{W}_x$, let:

$$\text{hit}_r(w) := \inf\{t \in [0, \zeta_w] : w(t) = r\}$$

with the usual convention $\inf \emptyset = \infty$. Consider $\omega \in \mathcal{S}_x$, and for every $s \geq 0$ set:

$$\eta_s^{(r)}(\omega) := \inf \left\{ t \geq 0 : \int_0^t \mathbb{1}_{\zeta_{\omega_u} \leq \text{hit}_r(\omega_u)} du > s \right\}.$$

The snake trajectory $\text{tr}_r(\omega)$ defined by

$$\forall s \geq 0, (\text{tr}_r(\omega))_s := \omega_{\eta_s^{(r)}(\omega)}$$

is called the truncation of ω at level r . See [1, Proposition 10]. Roughly speaking, $\text{tr}_r(\omega)$ is obtained by removing those paths ω that hit r and then survive for a positive amount of time. Let $\mathcal{Y}_r(\omega) := \sigma(\text{tr}_r(\omega))$ which can be interpreted as the time spent by ω before hitting r and write \mathcal{H}_r^x for the σ -field on \mathcal{S}_x generated by $\text{tr}_r(W)$ and the class of all \mathbb{N}_x -negligible sets.

We now discuss the special Markov property of the Brownian snake excursion, which will be crucial in our study. The set:

$$\{s \geq 0 : \text{hit}_r(W_s) < \zeta_s\}$$

is open so it can be written as a union of disjoint open intervals $(a_i, b_i)_{i \in I}$ with I an indexing set that may be empty. For every $i \in I$, let $W^{(i)}$ be the snake trajectory defined by:

$$W_s^{(i)}(t) := W_{(a_i+s) \wedge b_i}(\zeta_{a_i} + t) \text{ for } 0 \leq t \leq \zeta_{(a_i+s) \vee b_i} - \zeta_{a_i}$$

for every $s \geq 0$. By definition the snake trajectories $(W^{(i)})_{i \in I}$ are the excursions of W below r . Note that the information about the paths W_s before hitting r is contained in the sigma-field \mathcal{H}_r^x . The exit measure at level r is the quantity:

$$\mathcal{Z}_r(\omega) := \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbb{1}_{\text{hit}_r(\omega_s) = \infty, \widehat{\omega}_s < r + \varepsilon}.$$

The previous lim inf is a well defined finite limit \mathbb{N}_x -a.e. (we refer to [14, Proposition 28] for a proof) and it is \mathcal{H}_r^x -measurable by [11, Proposition 2.3]. We now can give a formal statement of the special Markov property.

Special Markov property. Let $x, r \in \mathbb{R}$, such that $x > r$. Under \mathbb{N}_x , conditionally on \mathcal{H}_r^x , the point measure:

$$\sum_{i \in I} \delta_{W^{(i)}}(d\omega)$$

is Poisson with intensity $\mathcal{Z}_r \mathbb{N}_r(d\omega)$.

We refer to [13, Corollary 21] for a proof. It will be useful to note that for $r' < r < x$, if we replace $\mathbb{N}_x(d\omega)$ by $\mathbb{N}_x(d\omega \mid W_* > r')$, the last statement remains valid up to the replacement of $\mathcal{Z}_r \mathbb{N}_r(d\omega)$ by $\mathcal{Z}_r \mathbb{N}_r(d\omega \cap \{W_* > r'\})$. The Laplace transform of \mathcal{Z}_r is given by:

$$\mathbb{N}_x(1 - \exp(-\lambda \mathcal{Z}_r)) = \left(\lambda^{-\frac{1}{2}} + \sqrt{\frac{2}{3}}(x - r) \right)^{-2} \quad (3)$$

for every $\lambda \geq 0$. See e.g. formula (6) in [9]. Remark that the limit when λ goes to ∞ gives formula (2).

2.3 Coding triples and metric spaces

Infinite spine coding triples. An infinite spine coding triple is a triple $(w, \mathfrak{N}^+, \mathfrak{N}^-)$ such that:

- (i) $w : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function;
- (ii) $\mathfrak{N}^+ = \sum_{i \in I} \delta_{(t_i, \omega^i)}$ and $\mathfrak{N}^- = \sum_{i \in J} \delta_{(t_i, \omega^i)}$ are point measures on $(0, \infty) \times \mathcal{S}$ (I and J are two disjoint indexing sets) and for every $i \in I \cup J$, $\omega^i \in \mathcal{S}_{w(t_i)}$;
- (iii) the numbers $(t_i)_{i \in I \cup J}$ are distinct;

(iv) the functions

$$u \mapsto \beta_u^+ := \sum_{i \in I} \mathbb{1}_{t_i \leq u} \sigma(\omega^i), \quad u \mapsto \beta_u^- := \sum_{i \in J} \mathbb{1}_{t_i \leq u} \sigma(\omega^i)$$

take finite values, are monotone increasing on \mathbb{R}_+ , and tend to ∞ at ∞ ;

(v) for every $t > 0$ and $\varepsilon > 0$:

$$\#\{i \in I \cup J : t_i \leq t \text{ and } \sup_{s \in [0, \sigma(\omega^i)]} |\hat{\omega}_s^i - w_{t_i}| > \varepsilon\} < \infty.$$

We define a scaling operation for coding triples as follows; for every $\lambda > 0$

$$\text{hom}_\lambda \left(w, \sum_{i \in I} \delta_{(t_i, \omega^i)}, \sum_{i \in J} \delta_{(t_i, \omega^i)} \right) := \left(\lambda w(\cdot / \lambda^2), \sum_{i \in I} \delta_{(\lambda^2 t_i, \text{hom}_\lambda(\omega^i))}, \sum_{i \in J} \delta_{(\lambda^2 t_i, \text{hom}_\lambda(\omega^i))} \right).$$

An infinite spine coding triple belongs to the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$, where $M(\mathcal{S})$ stands for the space of all σ -finite measures μ on $(0, \infty) \times \mathcal{S}$ putting no mass on the set $\{(t, \omega) : \sigma(\omega) = 0\}$ and satisfying $\mu([0, t] \times \{\omega \in \mathcal{S} : \sigma(\omega) > \delta\}) < \infty$, for every $t \geq 0$ and $\delta > 0$. We equip the space $M(\mathcal{S})$ with the distance:

$$d_{M(\mathcal{S})}(\mu, \mu') := \sum_{n \geq 0} d_{\text{Pro}}(\mu(\cdot \cap \mathcal{S}_{(n)}), \mu'(\cdot \cap \mathcal{S}_{(n)})) \wedge 2^{-n},$$

where $\mathcal{S}_{(n)} = [0, 2^n] \times \{\omega \in \mathcal{S} : \sigma(\omega) > 2^{-n}\}$, and d_{Pro} stands for the Prokhorov metric inducing the weak topology on finite measures on $\mathbb{R}_+ \times \mathcal{S}$.

We also equip $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$ with the product metric and the associated Borel sigma-field.

Let $(w, \mathfrak{N}^+, \mathfrak{N}^-)$ be an infinite spine coding triple. We now introduce the infinite tree \mathcal{T}_∞ associated with $(w, \mathfrak{N}^+, \mathfrak{N}^-)$. For every $i \in I \cup J$, let (ζ_s^i) be the lifetime process associated with ω^i and $\sigma^i := \sigma(\omega^i)$. We write \mathcal{T}^i for the tree coded by ζ^i , i.e. $\mathcal{T}^i = \mathcal{T}_{\omega^i}$, and p_{ζ^i} for the canonical projection from $[0, \sigma^i]$ onto \mathcal{T}^i . The tree \mathcal{T}_∞ can be defined from the disjoint union:

$$[0, \infty) \cup \left(\bigcup_{i \in I \cup J} \mathcal{T}^i \right)$$

by identifying the point t_i of $[0, \infty)$ with $p_{\omega^i}(0)$ (that is, the root of \mathcal{T}^i) for every $i \in I \cup J$. The set $[0, \infty)$ is called the spine of \mathcal{T}_∞ . We equip \mathcal{T}_∞ with a natural distance $d_{\mathcal{T}_\infty}$ as follows. The restriction of $d_{\mathcal{T}_\infty}$ to the spine is the Euclidean distance in $[0, \infty)$ and the restriction on each tree \mathcal{T}^i is the tree distance d_{ω^i} . If $u \in \mathcal{T}^i$ and $t \in [0, \infty)$, we take $d_{\mathcal{T}_\infty}(u, t) = d_{\omega^i}(u, p_{\omega^i}(0)) + |t_i - t|$. If $u \in \mathcal{T}^i$ and $v \in \mathcal{T}^j$ with $i \neq j$, we take $d_{\mathcal{T}_\infty}(u, v) = d_{\omega^i}(u, p_{\omega^i}(0)) + |t_i - t_j| + d_{\omega^j}(v, p_{\omega^j}(0))$. Then $(\mathcal{T}_\infty, d_\infty)$ is a (non-compact) \mathbb{R} -tree. We can also assign a label Λ_u , to each u in \mathcal{T}_∞ as follows. If $t \in [0, \infty)$, we take $\Lambda_t := w(t)$. If $u \in \mathcal{T}^i$, we take $\Lambda_u := \Lambda_u^{\omega^i}$. In particular, we have $(\mathcal{T}^i, (\Lambda_u)_{u \in \mathcal{T}^i}) = (\mathcal{T}_{\omega^i}, (\Lambda_u^{\omega^i})_{u \in \mathcal{T}_{\omega^i}})$ for every $i \in I \cup J$. Moreover using property (v), one checks that the mapping $u \mapsto \Lambda_u$ is continuous on \mathcal{T}_∞ . Finally, we can define a natural volume measure $V_{\mathcal{T}_\infty}$ on \mathcal{T}_∞ as follows, $V_{\mathcal{T}_\infty}$ gives no mass to the spine and its restriction to \mathcal{T}^i is V_{ω^i} .

Roughly speaking, \mathcal{T}_∞ is obtained by gluing the trees \mathcal{T}^i along the spine and keeping their labels. It will be

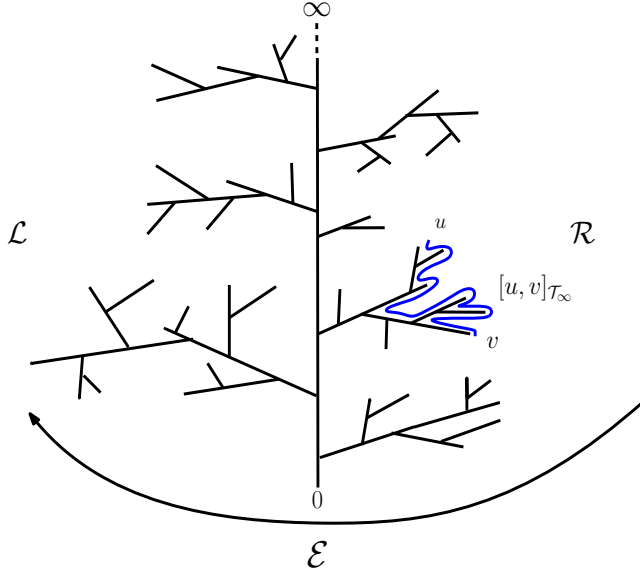


Figure 1: A representation of the tree \mathcal{T}_∞ .

important for our purposes to equip \mathcal{T}_∞ also with an order structure inherited from the coding triple. We define the left side of \mathcal{T}_∞ as the subset:

$$\mathcal{L} := [0, \infty) \cup \left(\bigcup_{i \in I} \mathcal{T}^i \right)$$

where again the point t_i is identified with $p_{\omega^i}(0)$ for $i \in I$, and we define the right side \mathcal{R} in the same way by replacing I by J . Remark that $\mathcal{L} \cap \mathcal{R} = [0, \infty)$. We write β_{u-}^+ and β_{u-}^- for the left limits of β^+ and β^- at u (and we take $\beta_{0-}^+ = \beta_{0-}^- = 0$ as convention). Note that if u is a discontinuity point of β^+ then there is a unique $i \in I$ such that $t_i = u$ and $\beta_u^+ - \beta_{u-}^+ = \sigma^i$ (and the same property is true for β^- replacing I by J).

We define the exploration process \mathcal{E}^+ of the left side of \mathcal{T}_∞ as follows.

For every $s \geq 0$, there is a unique u such that $\beta_{u-}^+ \leq s \leq \beta_u^+$. Then if there is an index $i \in I$ such that $t_i = u$, set

$$\mathcal{E}_s^+ := p_{\omega^i}(s - \beta_{t_i-}^+)$$

and if there is no such i , simply set $\mathcal{E}_s^+ := u$. We define similarly the exploration of the right side \mathcal{E}^- by replacing β^+ by β^- and I by J . Finally, let \mathcal{E} be the function from \mathbb{R} onto \mathcal{T}_∞ defined by:

$$\mathcal{E}_s := \begin{cases} \mathcal{E}_s^+ & \text{if } s \geq 0 \\ \mathcal{E}_{-s}^- & \text{if } s \leq 0 \end{cases}$$

Remark that \mathcal{E} is continuous and the volume measure on \mathcal{T}_∞ is the pushforward of Lebesgue measure on \mathbb{R} under the mapping $s \mapsto \mathcal{E}_s$. Moreover the left side of \mathcal{T}_∞ is $\{\mathcal{E}_s : s \geq 0\}$ and the right side is $\{\mathcal{E}_s : s \leq 0\}$. This exploration process allows us to define a notion of interval on \mathcal{T}_∞ . By convention, for every $s, t \in \mathbb{R}$

with $s < t$ we set $[t, s] := (-\infty, s] \cup [t, \infty)$. For every $u, v \in \mathcal{T}_\infty$ with $u \neq v$, let $[s, t]$ be the smallest interval such that $\mathcal{E}_s = u$ and $\mathcal{E}_t = v$. It is easy to check from the definition that there is always a smallest such interval. We put:

$$[u, v]_{\mathcal{T}_\infty} := \{\mathcal{E}_r : r \in [s, t]\}.$$

If $u = v$, take $[u, v]_{\mathcal{T}_\infty} = \{u\}$. Note that $[u, v]_{\mathcal{T}_\infty} \neq [v, u]_{\mathcal{T}_\infty}$ as long as $u \neq v$. See Figure 1 for an illustration.

By analogy with the case of compact trees, for every $u \in \mathcal{T}_\infty$, the multiplicity of u is the number of connected components of $\mathcal{T}_\infty \setminus \{u\}$. We will say that u is a leaf (resp. a branching point) if its multiplicity is 1 (resp. greater than 2). Remark that:

- 0 is the only leaf belonging to the spine.
- The branching points belonging to the spine are the points $(t_i)_{i \in I \cup J}$.
- For every $i \in I \cup J$, the multiplicity of $a \in \mathcal{T}^i \setminus \{t_i\}$ in \mathcal{T}_∞ is its multiplicity in \mathcal{T}^i .

Finally, for every $u, v \in \mathcal{T}_\infty$, we denote the unique geodesic segment of \mathcal{T}_∞ connecting u and v by $\llbracket u, v \rrbracket_{\mathcal{T}_\infty}$. We write $u \preceq v$ for $u, v \in \mathcal{T}_\infty$ if and only if $u \in \llbracket 0, v \rrbracket_{\mathcal{T}_\infty}$. In this case we say that u is an ancestor of v . We also write $\llbracket u, \infty \rrbracket_{\mathcal{T}_\infty}$ for the range of the unique geodesic from u to ∞ in \mathcal{T}_∞ .

From coding triples to metric spaces. Let $(w, \mathfrak{N}^+, \mathfrak{N}^-)$ be a coding triple and let $(\mathcal{T}_\infty, (\Lambda_v)_{v \in \mathcal{T}_\infty})$ be the associated labeled tree. We make the following assumption:

$$(H_1) : \begin{cases} \text{for every } v \in \mathcal{T}_\infty, \Lambda_v \geq 0; \\ \text{if } \Lambda_v = 0 \text{ then } v \text{ is a leaf;} \\ \Lambda_0 = 0; \\ \Lambda_{\mathcal{E}_t} \rightarrow \infty \text{ as } |t| \rightarrow \infty. \end{cases}$$

Set $\mathcal{T}_\infty^\circ := \{v \in \mathcal{T}_\infty : \Lambda_v > 0\}$ and $\partial\mathcal{T}_\infty := \mathcal{T}_\infty \setminus \mathcal{T}_\infty^\circ$. Remark that \mathcal{T}_∞° is path connected and dense in \mathcal{T}_∞ by (H_1) . The last assumption in (H_1) implies that $\inf_{[u, v]_{\mathcal{T}_\infty}} \Lambda$ is attained for every interval $[u, v]_{\mathcal{T}_\infty}$ of \mathcal{T}_∞ .

For every $u, v \in \mathcal{T}_\infty$ set:

$$\Delta^\circ(u, v) := \begin{cases} \Lambda_u + \Lambda_v - 2 \max\left(\inf_{[u, v]_{\mathcal{T}_\infty}} \Lambda, \inf_{[v, u]_{\mathcal{T}_\infty}} \Lambda\right) & \text{if } \max\left(\inf_{[u, v]_{\mathcal{T}_\infty}} \Lambda, \inf_{[v, u]_{\mathcal{T}_\infty}} \Lambda\right) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

We then let

$$\forall u, v \in \mathcal{T}_\infty^\circ, \Delta(u, v) := \inf_{u_1=u, u_2, \dots, u_n=v} \sum_{i=1}^{n-1} \Delta^\circ(u_i, u_{i+1}) \quad (4)$$

where the infimum is over all choices of the integer $n \geq 1$ and of the finite sequence u_1, \dots, u_n of elements of \mathcal{T}_∞ verifying $u_1 = u$ and $u_n = v$. Using the continuity of $u \mapsto \Lambda_u$ one verifies that the mapping $(u, v) \mapsto \Delta(u, v)$ takes finite values and is continuous on $\mathcal{T}_\infty^\circ \times \mathcal{T}_\infty^\circ$. Since $\Delta^\circ(u, v) \geq |\Lambda_u - \Lambda_v|$, we have for every $u, v \in \mathcal{T}_\infty^\circ$

$$\Delta(u, v) \geq |\Lambda_u - \Lambda_v|. \quad (5)$$

It is important to remark that Δ defines a pseudo-distance on \mathcal{T}_∞° . From now on we make the extra assumption that:

(H_2): The map $(u, v) \mapsto \Delta(u, v)$ has a continuous extension to $\mathcal{T}_\infty \times \mathcal{T}_\infty$

and we consider this continuous extension in what follows. For simplicity we keep the notation Δ for this continuous extension, which defines a pseudo-distance on \mathcal{T}_∞ . The associated equivalence relation is defined by $u \approx v$ iff $\Delta(u, v) = 0$. By abuse of notation, we write $\mathcal{T}_\infty/\Delta$ for $\mathcal{T}_\infty/\approx$. Note that the definition of $u \approx v$ makes sense for $u, v \in \mathcal{T}_\infty^\circ$ even if (H_2) does not hold and so we can still consider the space $\mathcal{T}_\infty^\circ/\approx$ in that case. We denote the canonical projection by $\Pi : \mathcal{T}_\infty \rightarrow \mathcal{T}_\infty/\Delta$ and, for every $x \in \mathcal{T}_\infty/\Delta$, we set $\Lambda_x := \Lambda_u$ where u is any preimage of x under Π (remark that the definition is unambiguous by (5)). We write $|\cdot|$ for the pushforward of V under Π , which defines a volume measure on $\mathcal{T}_\infty/\Delta$, and for simplicity we write 0 for the equivalence class of 0 in $\mathcal{T}_\infty/\Delta$. The metric space $(\mathcal{T}_\infty/\Delta, 0, \Delta, |\cdot|)$ is a weighted locally compact length space which is pointed at 0 , and we have:

$$\Delta(x, \Pi(\partial\mathcal{T}_\infty)) = \Lambda_x \tag{6}$$

for every $x \in \mathcal{T}_\infty/\Delta$. We refer [17, Subsection 4.1] for a proof of these two facts. For every $r \geq 0$, we write $B_r(\mathcal{T}_\infty/\Delta)$ for the set of all points $x \in \mathcal{T}_\infty/\Delta$ with $\Delta(x, \Pi(\partial\mathcal{T}_\infty)) \leq r$. By (6):

$$B_r(\mathcal{T}_\infty/\Delta) = \{x \in \mathcal{T}_\infty/\Delta : \Lambda_x \leq r\}.$$

It will also be useful to introduce for every $r > 0$, the set \mathcal{T}_∞^r of all points $u \in \mathcal{T}_\infty$ such that $\Lambda_u \geq r$ and $\Lambda_v > r$ for every $v \in \llbracket u, \infty \rrbracket_{\mathcal{T}_\infty} \setminus \{u\}$. Remark that \mathcal{T}_∞^r is an \mathbb{R} -tree. We define:

$$\mathcal{T}_\infty^{r,\circ} := \{u \in \mathcal{T}_\infty : \inf_{\llbracket u, \infty \rrbracket_{\mathcal{T}_\infty}} \Lambda > r\}$$

and we let the "boundary" $\partial\mathcal{T}_\infty^r$ be the set of all points $u \in \mathcal{T}_\infty$ such that $\Lambda_u = r$ and $\Lambda_v > r$ for every $v \in \llbracket u, \infty \rrbracket_{\mathcal{T}_\infty} \setminus \{u\}$. Set $\check{B}_r^\bullet(\mathcal{T}_\infty/\Delta) := \Pi(\mathcal{T}_\infty^r)$, $\check{B}_r^\circ(\mathcal{T}_\infty/\Delta) := \Pi(\mathcal{T}_\infty^{r,\circ})$ and:

$$B_r^\bullet(\mathcal{T}_\infty/\Delta) := \Pi\left(\{u \in \mathcal{T}_\infty : \inf_{\llbracket u, \infty \rrbracket_{\mathcal{T}_\infty}} \Lambda \leq r\}\right) = \Pi\left(\mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r,\circ}\right). \tag{7}$$

When there is no ambiguity, we will remove $\mathcal{T}_\infty/\Delta$ from the notation and write B_r^\bullet , \check{B}_r^\bullet and \check{B}_r° instead. In the next section we explain the geometric meaning of these sets and we will see that the notation B_r^\bullet is consistent with the one used in the introduction to designate the hull of the Brownian plane.

2.4 The Brownian plane and the infinite volume Brownian disk

In this section we give the construction of the Brownian plane and the infinite volume Brownian disk from random infinite spine coding triples. We also list some useful geometric properties of the Brownian plane.

2.4.1 The Brownian plane

We now consider a triple $(X, \mathfrak{L}, \mathfrak{R})$ such that:

- $X = (X_t)_{t \geq 0}$ is a nine-dimensional Bessel process started from 0;

- Conditionally on X , \mathfrak{L} and \mathfrak{R} are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity:

$$2dt\mathbb{N}_{X_t}(d\omega \cap \{\omega_* > 0\}).$$

It is easy to verify that $(X, \mathfrak{L}, \mathfrak{R})$ is a.s. a coding triple in the sense of Section 2.3, and the root of \mathcal{T}_∞ is the only point with zero label. We may assume that $(X, \mathfrak{L}, \mathfrak{R})$ is defined on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$ under the probability measure Θ_0 . As previously, we write $(\mathcal{T}_\infty, (\Lambda_v)_{v \in \mathcal{T}_\infty})$ for the associated infinite labeled tree. We note that $(X, \mathfrak{L}, \mathfrak{R})$ satisfies assumptions (H_1, H_2) , see [17, Section 4.2]. In fact, since the root of \mathcal{T}_∞ is the only point with zero label, it is possible to define directly the continuous extension of Δ to $\mathcal{T}_\infty \times \mathcal{T}_\infty$, just replacing Δ° in formula (4) by

$$\Delta^{\circ'}(u, v) := \Lambda_u + \Lambda_v - 2 \max \left(\inf_{[u, v]_{\mathcal{T}_\infty}} \Lambda, \inf_{[v, u]_{\mathcal{T}_\infty}} \Lambda \right).$$

See [17, Section 4.2] for more details. The Brownian plane is the space $(\mathcal{T}_\infty/\Delta, 0, \Delta, |\cdot|)$ under Θ_0 . To simplify notation, we denote this space, which is an element of \mathbb{K}_∞ , by \mathcal{M}_∞ . Remark that since $\partial\mathcal{T}_\infty = \{0\}$ we have $\Lambda_x = \Delta(0, x)$ for every $x \in \mathcal{M}_\infty$. Moreover, for every $\lambda > 0$, the pushforward of Θ_0 under hom_λ is Θ_0 . Consequently, the space $(\mathcal{T}_\infty/\Delta, 0, \lambda\Delta, \lambda^4|\cdot|)$ is also distributed as a Brownian plane. Another important property is that, Θ_0 -a.s., we have

(F): For every $u, v \in \mathcal{T}_\infty$, with $u \neq v$, we have $\Delta(u, v) = 0$ if and only if $\Delta^\circ(u, v) = \Delta^{\circ'}(u, v) = 0$. Moreover if $u \neq v$ and $\Delta(u, v) = 0$ then u and v must be leaves.

This fact is a classical result in Brownian geometry. The first part of (F) is derived in [9, Section 3.2]. The second part follows from the first part and the known results for the Brownian map (see [16, Lemma 3.2]). By formulas (16) and (17) in [9], the set $B_r^\bullet = B_r^\bullet(\mathcal{T}_\infty/\Delta)$ defined in (7) coincides with the hull of radius r of \mathcal{M}_∞ as defined in the introduction (Section 1), \check{B}_r° is the complement of the hull, and \check{B}_r^\bullet is the closure of \check{B}_r° . We also have

$$\partial B_r^\bullet = \partial \check{B}_r^\bullet = \Pi(\partial\mathcal{T}_\infty) \tag{8}$$

which is the range of an injective cycle (see the proof of [17, Theorem 31] for more details). We will equip the hull $B_r^\bullet = \Pi(\mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r, \circ})$ with the distance $\Delta^{(r)}$ defined as follows. First set

$$\forall u, v \in \mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r, \circ}, \Delta^{(r)}(u, v) := \inf_{\substack{u=u_1, u_2, \dots, u_n=v \\ u_2, \dots, u_{n-1} \in \mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r, \circ}}} \sum_{i=1}^{n-1} \Delta^{\circ'}(u_i, u_{i+1}). \tag{9}$$

By (F), we see that for every $u, v \in \mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r, \circ}$ we have $\Delta(u, v) = 0$ iff $\Delta^{(r)}(u, v) = 0$. In particular, we can define $\Delta^{(r)}$ on the hull B_r^\bullet taking for every $x, y \in B_r^\bullet$, $\Delta^{(r)}(x, y) := \Delta^{(r)}(u, v)$ where $u, v \in \mathcal{T}_\infty \setminus \mathcal{T}_\infty^{r, \circ}$ are any elements such that $(\Pi(u), \Pi(v)) = (x, y)$. By definition $\Delta^{(r)}$ is a distance on B_r^\bullet and it is not hard to verify that the restriction of $\Delta^{(r)}$ on $\text{Int}(B_r^\bullet)$ coincides with the intrinsic metric on $\text{Int}(B_r^\bullet)$ viewed as a subset of the metric space $(\mathcal{T}_\infty/\Delta, \Delta)$ (one can directly adapt the proof of [17, Lemma 30]). In other words, $\Delta^{(r)}$ is the continuous extension to B_r^\bullet of the intrinsic metric on $\text{Int}(B_r^\bullet)$. In what follows, we will always view B_r^\bullet as a (random) pointed and weighted compact metric space for the metric $\Delta^{(r)}$ (the volume measure is obviously the restriction of the volume measure on \mathcal{M}_∞ and the distinguished point is the same as in \mathcal{M}_∞).

Exit measures. We now introduce the exit measures of the infinite tree \mathcal{T}_∞ . For every $a \geq 0$, set

$$\tau_a := \sup\{t \geq 0 : X_t \leq a\} \quad (10)$$

which is Θ_0 -a.s. finite since $X_t \rightarrow \infty$, Θ_0 -a.s., when $t \rightarrow \infty$. We take $\tau_\infty := \infty$ by convention. For every $0 \leq s \leq t \leq \infty$, introduce the point measures $\mathfrak{L}^{s,t}$ and $\mathfrak{R}^{s,t}$ on $\mathbb{R}_+ \times \mathcal{S}$ defined as follows:

$$\int \Phi(\ell, \omega) \mathfrak{L}^{s,t}(d\ell d\omega) := \int_{\tau_s}^{\tau_t} \Phi(\ell - \tau_s, \omega) \mathfrak{L}(d\ell d\omega)$$

and

$$\int \Phi(\ell, \omega) \mathfrak{R}^{s,t}(d\ell d\omega) := \int_{\tau_s}^{\tau_t} \Phi(\ell - \tau_s, \omega) \mathfrak{R}(d\ell d\omega).$$

By the time reversal property of Bessel processes the process $(X_{(\tau_t - \ell) \vee 0})_{\ell \geq 0}$ is a Bessel process of dimension -5 started from t stopped when it hits 0 (see [21, Theorem 2.5]). Applying this property with t replaced by $t' > t$, we get that $(X_{(\tau_t - \ell) \vee 0})_{\ell \geq 0}$ and $(X_{(\tau_t + \ell)})_{\ell \geq 0}$ are independent. Consequently, for every $0 < t < \infty$:

$$((X_{(\tau_t + \ell)})_{\ell \geq 0}, \mathfrak{L}^{t,\infty}, \mathfrak{R}^{t,\infty}) \text{ and } ((X_{(\tau_t - \ell) \vee 0})_{\ell \geq 0}, \mathfrak{L}^{0,t}, \mathfrak{R}^{0,t}) \text{ are independent.}$$

We call this property the spine independence property of \mathcal{T}_∞ . For every $0 < r \leq s \leq t$ set:

$$Z_r^{s,t} := \int \mathcal{Z}_r(\omega) \mathfrak{R}^{s,t}(d\ell d\omega) + \int \mathcal{Z}_r(\omega) \mathfrak{L}^{s,t}(d\ell d\omega)$$

which is the total exit measure at level r accumulated by the snakes glued on $[\tau_s, \tau_t]$. To simplify notation, write $Z_r := Z_r^{r,\infty}$. The proof of [9, Lemma 4.2] gives the following formula, for every $\lambda \geq 0$:

$$\Theta_0(\exp(-\lambda Z_r^{s,t})) = \left(\frac{t}{s}\right)^3 \cdot \left(\frac{s-r + (r^{-2} + \frac{2}{3}\lambda)^{-\frac{1}{2}}}{t-r + (r^{-2} + \frac{2}{3}\lambda)^{-\frac{1}{2}}}\right)^3. \quad (11)$$

Consequently, computing the limit when λ goes to infinity, we obtain:

Proposition 1. *For every $0 \leq r \leq s \leq t < \infty$:*

$$\Theta_0(Z_r^{s,t} = 0) = \left(\frac{t}{s}\right)^3 \cdot \left(\frac{s-r}{t-r}\right)^3. \quad (12)$$

The special Markov property of the Brownian snake excursion implies that conditionally on $Z_r^{s,t}$ the excursions outside r of the snake trajectories ω^i with $t_i \in [\tau_s, \tau_t]$ are distributed as the atoms of a Poisson point measure with intensity:

$$Z_r^{s,t} \mathbb{N}_r(d\omega \cap \{\omega_* > 0\}).$$

We will use this property throughout the article. It will be also useful to note that the Laplace transform of Z_r can be deduced from formula (11) taking the limit when t goes to infinity with $s = r$. More precisely, for every $r > 0$ and $\lambda \geq 0$ we get:

$$\Theta_0(\exp(-\lambda Z_r)) = \left(1 + \frac{2\lambda r^2}{3}\right)^{-\frac{3}{2}}. \quad (13)$$

Equivalently Z_r follows a Gamma distribution with parameter $\frac{3}{2}$ and mean r^2 . The previous formula appears already in [9, Proposition 1.2], which also shows that $Z := (Z_r)_{r \geq 0}$ has a càdlàg modification, with only

negative jumps, and from now on we consider this modification. Furthermore, [9, Proposition 4.3] states that, for every $0 \leq r \leq s$ and $\lambda \geq 0$, we have

$$\Theta_0(\exp(-\lambda Z_r) | Z_s) = \left(\frac{s}{r + (s-r)(1 + \frac{2\lambda r^2}{3})^{\frac{1}{2}}} \right)^3 \cdot \exp\left(-\frac{3}{2} Z_s \left(\frac{1}{(s-r + (\frac{2\lambda}{3} + r^{-2})^{-\frac{1}{2}})^2} - \frac{1}{s^2} \right)\right). \quad (14)$$

We conclude this Subsection by giving a geometric interpretation of Z_r . One can derive from [18, Proposition 8] that

Lemma 1. Θ_0 -a.s. , for every $r > 0$ we have:

$$Z_r = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|. \quad (15)$$

For the sake of completeness we give a proof of Lemma 1, but we postpone it to the Appendix below to avoid to weigh down the preliminaries. It will be important for us to know that the convergence holds simultaneously for every $r > 0$. Roughly speaking, Z_r represents the length or perimeter (in a generalized sense) of ∂B_r^\bullet .

2.4.2 The infinite volume Brownian disk

We keep the assumptions and notation of the preceding Subsection. Let $r > 0$ and set $\tilde{X}_t^{(r)} := X_{\tau_r+t} - r$. Let us also introduce the point measures $\tilde{\mathfrak{R}}_r$ and $\tilde{\mathfrak{L}}_r$ on $\mathbb{R}_+ \times \mathcal{S}$ defined by:

$$\int \Phi(t, \omega) \tilde{\mathfrak{L}}_r(dtd\omega) := \int \Phi(t, \text{tr}_0(\omega - r)) \mathfrak{L}^{r, \infty}(dt d\omega)$$

and

$$\int \Phi(t, \omega) \tilde{\mathfrak{R}}_r(dtd\omega) := \int \Phi(t, \text{tr}_0(\omega - r)) \mathfrak{R}^{r, \infty}(dt d\omega).$$

One easily checks that the triple $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ is a random infinite spine coding triple satisfying (H_1) . Moreover [17, Proposition 6] shows that there exists a unique collection of probability measures $(\Theta_z)_{z>0}$ on the space of coding triples such that for every $r > 0$:

$$\Theta_0(g(Z_r)F(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)) = \frac{3^{\frac{3}{2}}}{\sqrt{2\pi r}} \int_0^\infty dz z^{\frac{1}{2}} \exp(-\frac{3}{2r^2}z) g(z) \Theta_z(F). \quad (16)$$

and the pushforward of Θ_z by hom_λ is $\Theta_{\lambda^2 z}$ (for every $z, \lambda > 0$). In other words, conditionally on $Z_r = z$, the distribution of $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ is Θ_z . It is crucial that the preceding conditional distribution does not depend on r . Furthermore by [17, Lemma 16] an infinite coding triple distributed according to Θ_z satisfies a.s. (H_1, H_2) . Consequently we can consider the associated metric space and according to [17, Proposition 38] this space is the infinite volume Brownian disk with perimeter z . The infinite volume Brownian disk is a random element of \mathbb{K}_∞ and is a.s. homeomorphic to the complement of the open unit disk in the complex plane. It can also be obtained as scaling limit of random planar lattices with a boundary (see [2]). The boundary of the infinite volume Brownian disk is the set of points that have no neighborhood homeomorphic to the (open) disk. The infinite volume Brownian disk also satisfies a scale invariance property.

More precisely since the pushforward of Θ_z by hom_λ is $\Theta_{\lambda^2 z}$, if $(E, \rho_E, \Delta_E, |\cdot|_E)$ is an infinite volume Brownian disk with perimeter z , then $(E, \rho_E, \lambda \Delta_E, \lambda^4 |\cdot|_E)$ is an infinite volume Brownian disk with perimeter $\lambda^2 z$.

We now explain the geometric interpretation of $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ and the implications of (16) for the Brownian plane. First observe that the labeled tree associated with $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ can be identified with $(\mathcal{T}_\infty^r, (\Lambda_v - r)_{v \in \mathcal{T}_\infty^r})$, see the beginning of the proof of Theorem 29 in [17]. For every $r > 0$, let $\check{\Delta}^{(r)}$ be the intrinsic distance induced by Δ on \check{B}_r° and also write $|\cdot|_{\check{\Delta}^{(r)}}$ for the restriction of the volume measure $|\cdot|$ to \check{B}_r^\bullet . The following lemma is then a consequence of [17, Lemma 30] and the identification of $(\mathcal{T}_\infty^r, (\Lambda_v - r)_{v \in \mathcal{T}_\infty^r})$ with the labeled tree associated with $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$.

Lemma 2. Θ_0 -a.s., for every $r > 0$ such that $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ satisfies (H_2) the following properties hold:

- (i) The intrinsic distance $\check{\Delta}^{(r)}$ has a unique continuous extension to \check{B}_r^\bullet ;
- (ii) The space \check{B}_r^\bullet equipped with this continuous extension of $\check{\Delta}^{(r)}$, the measure $|\cdot|_{\check{\Delta}^{(r)}}$ and the distinguished point $\Pi(\tau_r)$ coincides as an element of \mathbb{K}_∞ with the metric space associated with $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$.

By (16), the coding triple $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ satisfies (H_2) , Θ_0 -a.s., for every fixed $r > 0$, and thus properties (i) and (ii) hold Θ_0 -a.s. when $r > 0$ is fixed. However, we point out that we do not claim that $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ satisfies (H_2) simultaneously for every $r > 0$. Consequently it is not clear whether $\check{\Delta}^{(r)}$ has a unique continuous extension to \check{B}_r^\bullet simultaneously for every $r > 0$, a.s.

On the other hand, we saw in Section 2.4.1 that the hull B_r^\bullet (equipped with the distance $\Delta^{(r)}$ defined in (9)) can be viewed as a random element of the space \mathbb{K} . In the next statement, we also view \check{B}_r^\bullet as an element of \mathbb{K}_∞ as explained in property (ii) of Lemma 2. The following theorem is essentially a reformulation of Theorems 29 and 31 in [17].

Theorem 3. Let $r > 0$. Then, conditionally on $Z_r = z$, the coding triple $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ is distributed according to Θ_z and is independent of B_r^\bullet . Consequently, conditionally on $Z_r = z$, the space \check{B}_r^\bullet is an infinite Brownian disk with perimeter z and is independent of B_r^\bullet .

The fact that conditionally on $Z_r = z$ the coding triple $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ is distributed according to Θ_z is just a reformulation of (16). Using property (ii) of Lemma 2, it follows that the conditional distribution of \check{B}_r^\bullet knowing that $Z_r = z$ is the law of the infinite volume Brownian disk with perimeter z . The conditional independence of \check{B}_r^\bullet and B_r^\bullet given Z_r is stated in [17, Theorem 31], and the slightly stronger conditional independence of $(\tilde{X}^{(r)}, \tilde{\mathfrak{L}}_r, \tilde{\mathfrak{R}}_r)$ and B_r^\bullet is also established at the end of the proof of this result.

We will refer to the last assertion of Theorem 3 as the spatial Markov property of \mathcal{M}_∞ . In Section 3.4 below, we will extend this property to the case of a random level r .

3 Separating cycles

In most of this section, we argue under Θ_0 and we use the following notation:

$$B_{r,s}^\circ := \text{Int}(B_s^\bullet \setminus B_r^\bullet) \text{ and } B_{r,s}^\bullet := \text{Cl}(B_{r,s}^\circ).$$

for every $r, s \in (0, \infty)$ with $r < s$. Our first goal is to study the quantity:

$$L_{r,s} = \inf\{\Delta(g) : g : [0, 1] \rightarrow B_{r,s}^\circ \text{ cycle separating } B_r^\bullet \text{ from } \infty\}. \quad (17)$$

Since \mathcal{M}_∞ has the topology of the complex plane \mathbb{C} , the quantity $L_{r,s}$ is well defined. Actually by construction it only depends on $B_{r,s}^\circ$ and the intrinsic distance induced by Δ on $B_{r,s}^\circ$. Let us briefly justify the measurability of the random variable $L_{r,s}$. We consider a dense sequence $(a_n : n \in \mathbb{N})$ in \mathcal{M}_∞ . Given $\alpha > 0$, we observe that $L_{r,s} < \alpha$ if and only if, for some $\delta > 0$, the following holds for every $\varepsilon > 0$: There exists a finite sequence $a_{n_1}, a_{n_2}, \dots, a_{n_p}, a_{n_{p+1}} = a_{n_1}$ such that $\Delta(a_{n_i}, (B_{r,s}^\circ)^c) \geq \delta$, $\Delta(a_{n_i}, a_{n_{i+1}}) < \varepsilon$ for every $1 \leq i \leq p$, and

$$\sum_{i=1}^p \Delta(a_{n_i}, a_{n_{i+1}}) < \alpha - \delta,$$

and such that for any other sequence a_{m_1}, \dots, a_{m_q} with $a_{m_1} \in B_r^\bullet$, $a_{m_q} \in \check{B}_s^\bullet$, and $\Delta(a_{m_j}, a_{m_{j+1}}) < \varepsilon$ for every $1 \leq j \leq q-1$, there exist $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ such that $\Delta(a_{n_i}, a_{m_j}) < \varepsilon$.

3.1 Geometric properties

We argue under Θ_0 and use the notation introduced in Section 2.3. In particular, $(\mathcal{E}_s)_{s \in \mathbb{R}}$ is the exploration function of the tree \mathcal{T}_∞ . Our goal here is to identify a subclass of separating cycles and then to show that we can restrict our study to this collection of paths which is easier to study.

For every $t \geq 0$, set $p_\infty^{(\ell)}(t) := \mathcal{E}_{\sup\{s \in \mathbb{R} : \Lambda_s = t\}}$ and $p_\infty^{(r)}(t) := \mathcal{E}_{\inf\{s \in \mathbb{R} : \Lambda_s = t\}}$. Remark that $p_\infty^{(\ell)}$ (resp. $p_\infty^{(r)}$) takes values in \mathcal{L} (resp. \mathcal{R}) since $\mathcal{E}_0 = 0$. By definition for every $s \leq t$, we have

$$\inf_{[p_\infty^{(\ell)}(s), p_\infty^{(\ell)}(t)]_{\mathcal{T}_\infty}} \Lambda = \inf_{[p_\infty^{(r)}(t), p_\infty^{(r)}(s)]_{\mathcal{T}_\infty}} \Lambda = s.$$

Consequently we have

$$\Delta^\circ(p_\infty^{(\ell)}(t), p_\infty^{(\ell)}(s)) = \Delta^\circ(p_\infty^{(r)}(t), p_\infty^{(r)}(s)) = |t - s| \quad (18)$$

for every $s, t > 0$. Moreover knowing that for every $t \geq 0$ and $u \in [p_\infty^{(\ell)}(t), p_\infty^{(r)}(t)]_{\mathcal{T}_\infty}$ we have $\Lambda_u \geq t$, we get that $\Delta^\circ(p_\infty^{(\ell)}(t), p_\infty^{(r)}(t)) = 0$ for every $t > 0$. We write

$$\gamma_\infty(t) := \Pi(p_\infty^{(\ell)}(t)) = \Pi(p_\infty^{(r)}(t)), \quad t \in [0, \infty).$$

By (5) and (18), γ_∞ is a geodesic path connecting 0 and ∞ . It can be shown that this geodesic path is the unique geodesic path connecting 0 and ∞ (see [8, Proposition 15]) but we will not use this result in this work. To simplify notation set $P^{(\ell)} := p_\infty^{(\ell)}(\mathbb{R}_+)$, $P^{(r)} := p_\infty^{(r)}(\mathbb{R}_+)$ and $P := P^{(\ell)} \cup P^{(r)}$. Remark that $\Pi(P)$ is the range of γ_∞ .

We define the left (resp. right) side of \mathcal{M}_∞ as the subset $\Pi(\mathcal{L})$ (resp. $\Pi(\mathcal{R})$).

Lemma 3. *The following properties hold Θ_0 -a.s.*

- (i) *The maps $\mathbb{R}_+ \ni t \mapsto \Pi(t)$ and $\mathbb{R}_+ \ni t \mapsto \gamma_\infty(t)$ are injective. Moreover $\Pi([0, \infty)) \cap \Pi(P) = \{0\}$.*
- (ii) *The sets $\text{Int}(\Pi(\mathcal{L}))$ and $\text{Int}(\Pi(\mathcal{R}))$ are the connected components of the complement of $\Pi([0, \infty)) \cup \Pi(P)$.*

Proof.

(i) Since $\mathbb{R}_+ \ni t \mapsto \gamma_\infty(t)$ is a geodesic path it has to be injective. Moreover, as the only leaf on the spine $[0, \infty)$ is 0, we can apply (F) to deduce that $t \in \mathbb{R}_+ \mapsto \Pi(t)$ is also injective and that $\Pi([0, \infty)) \cap \Pi(P) \subset \{0\}$.

(ii) As a simple consequence of (F), a point x belongs to the boundary of $\Pi(\mathcal{L})$ iff it belongs to $\Pi([0, \infty))$ or to $\Pi(P^{(\ell)}) = \Pi(P)$, and similarly if \mathcal{L} is replaced by \mathcal{R} . Consequently:

$$\text{Int}(\Pi(\mathcal{L})) = \Pi(\mathcal{L}) \setminus (\Pi([0, \infty)) \cup \Pi(P)) \quad , \quad \text{Int}(\Pi(\mathcal{R})) = \Pi(\mathcal{R}) \setminus (\Pi([0, \infty)) \cup \Pi(P)).$$

Thanks again to (F) we have $\text{Int}(\Pi(\mathcal{L})) \cap \text{Int}(\Pi(\mathcal{R})) = \emptyset$. Since \mathcal{M}_∞ has the topology of the complex plane and $\mathcal{M}_\infty \setminus (\Pi([0, \infty)) \cup \Pi(P))$ is the union of $\text{Int}(\Pi(\mathcal{L}))$ and $\text{Int}(\Pi(\mathcal{R}))$ the desired result follows. \square

Let us introduce the subclass of separating cycles that will play an important role. We define the set \mathcal{A} of all paths $\gamma : [t, t'] \mapsto \mathcal{M}_\infty$ such that:

- $\gamma(t) = \gamma(t')$ is in $\Pi(P)$ and γ does not hit 0;
- For every $t \leq s < s' \leq t'$, we have $\gamma(s) = \gamma(s')$ if and only if $(s, s') = (t, t')$;
- There exist two times $t_1 \leq t_2$ in $[t, t']$, such that $\gamma(t_1), \gamma(t_2) \in \Pi([0, \infty))$, $(\gamma(t))_{t \in [t_1, t_2]}$ does not intersect $\Pi(P)$, and $\gamma(s) \in \Pi(\mathcal{R})$ (resp. $\gamma(s) \in \Pi(\mathcal{L})$) for every $s \in [t, t_1]$ (resp. for every $s \in [t_2, t']$).

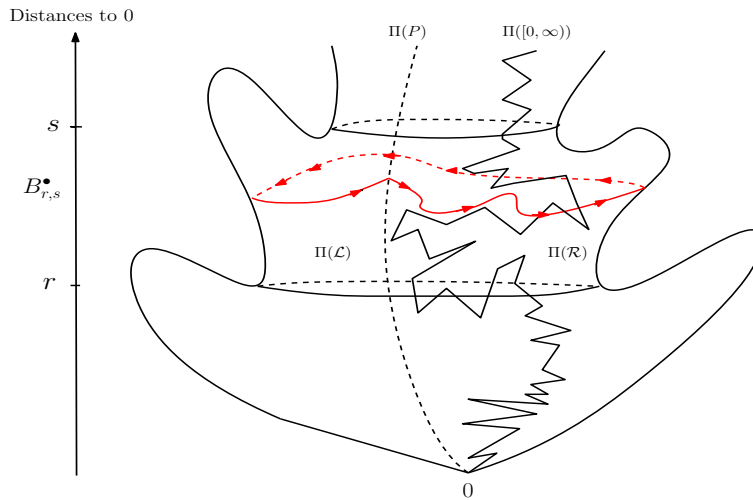


Figure 2: Cactus representation of the Brownian plane. The vertical distances represent the distances to the root 0. In red a path of \mathcal{A} taking values in $B_{r,s}^{\circ}$.

Using Lemma 3 and the fact that \mathcal{M}_∞ is homeomorphic to \mathbb{C} , one easily verifies that any path γ in \mathcal{A} is a separating cycle. So to give an upper bound for $L_{r,s}$ for $r < s$, it is sufficient to construct a path $\gamma \in \mathcal{A}$ taking values in $B_{r,s}^{\circ}$. See Figure 2 for an illustration. In the next lemma we explain why we can restrict our attention to the subclass \mathcal{A} .

Lemma 4. Θ_0 -a.s. for every separating cycle γ there exists a path γ' in \mathcal{A} such that $\Delta(\gamma') \leq \Delta(\gamma)$. Moreover if γ takes values in $B_{r,s}^\circ$ then the path γ' also takes values in $B_{r,s}^\circ$.

Proof. Let $(\gamma(t))_{t \in [0,1]}$ be a separating cycle. Since γ does not hit 0, there exist $r < s$ such that γ takes values in $B_{r,s}^\circ$. In what follows we fix $r < s$ such that γ stays in $B_{r,s}^\circ$. Our goal is to show that there exists $\gamma' \in \mathcal{A}$ taking values in $B_{r,s}^\circ$ such that $\Delta(\gamma') \leq \Delta(\gamma)$. First notice that since the path $(\gamma_\infty(t))_{t \geq 0}$ connects 0 and ∞ , the range of γ has to intersect $\Pi(P) = \gamma_\infty(\mathbb{R}_+)$. Without loss of generality we may and will assume that $\gamma(0) = \gamma(1) \in \Pi(P)$. Let $(t_i, t'_i)_{i \in \mathcal{I}}$ be the connected components of $\{t \in [0, 1] : \gamma(t) \notin \Pi(P)\}$ and to simplify notation set $\gamma^i := \gamma|_{[t_i, t'_i]}$. Remark that γ^i hits $\Pi(P)$ only at times t_i and t'_i . In particular, since γ does not hit 0 we can use Lemma 3 to obtain that for every $i \in \mathcal{I}$ there exists $\varepsilon > 0$ such that:

$$\forall t \in [0, \varepsilon], \gamma^i(t_i + t) \in \Pi(\mathcal{R}) \text{ or } \forall t \in [0, \varepsilon], \gamma^i(t_i + t) \in \Pi(\mathcal{L}).$$

In the first case we say that γ^i starts in $\Pi(\mathcal{R})$ and in the second case that γ^i starts in $\Pi(\mathcal{L})$. Similarly by Lemma 3 there exists $\varepsilon > 0$ such that:

$$\forall t \in [0, \varepsilon], \gamma^i(t'_i - t) \in \Pi(\mathcal{R}) \text{ or } \forall t \in [0, \varepsilon], \gamma^i(t'_i - t) \in \Pi(\mathcal{L}).$$

In the first case we say that γ^i ends in $\Pi(\mathcal{R})$ and in the second case that γ^i ends in $\Pi(\mathcal{L})$. Then since γ is a separating cycle we claim that

(C): there exists $i \in \mathcal{I}$ such that γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$, or starts in $\Pi(\mathcal{L})$ and ends in $\Pi(\mathcal{R})$.

Let us explain why (C) holds. Thanks to Lemma 3, we can find a homeomorphism $h : \mathcal{M}_\infty \rightarrow \mathbb{C}$ such that $h(\Pi(\mathbb{R}_+)) = \mathbb{R}_-$, $h(\gamma_\infty(\mathbb{R}_+)) = \mathbb{R}_+$, and $h(\Pi(\mathcal{L}))$ and $h(\Pi(\mathcal{R}))$ are the upper and lower half-planes. In particular, since $h(0) = 0$, the injective cycle $h \circ \gamma$ separates 0 from infinity in the complex plane \mathbb{C} . Let $(\rho(t), \theta(t))_{t \in [0,1]}$ be two continuous functions from $[0, 1]$ into \mathbb{R}_+ such that $h(\gamma(t)) = \rho(t) \exp(i\theta(t))$. Since $\gamma(0) \in \gamma_\infty(\mathbb{R}_+)$, we can take $\theta(0) = 0$ which determines in a unique way the pair $(\rho(t), \theta(t))_{t \in [0,1]}$. Since $h \circ \gamma$ separates 0 from infinity in \mathbb{C} , an application of Jordan's theorem shows that we must have $\theta(1) \in \{2\pi, -2\pi\}$. Suppose that $\theta(1) = 2\pi$ for definiteness and set $s_0 := \sup\{t \in [0, 1] : \theta(t) = 0\}$ and $s_1 = \inf\{t \in [s_0, 1] : \theta(t) = 2\pi\}$. Necessarily there exists i such that $\gamma^i = (\gamma(r))_{r \in [s_0, s_1]}$, and the "excursion" γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$ or conversely.

Let us derive the lemma from the claim (C). Up to replacing γ by $(\gamma(1-t))_{t \in [0,1]}$, we can assume that there exists $i \in \mathcal{I}$ such that γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$. Since $\gamma(0), \gamma^i(t_i), \gamma^i(t'_i) \in \Pi(P)$, we can consider the geodesic path g (resp. g') taking values in $\Pi(P)$ starting at $\gamma(0)$ and ending at $\gamma(t_i)$ (resp. starting at $\gamma(t'_i)$ and ending at $\gamma(1) = \gamma(0)$). The concatenation of g' , γ^i and g gives a path γ' in \mathcal{A} , which is shorter than γ . Moreover as γ takes values in $B_{r,s}^\circ$ we have $\gamma(0), \gamma^i(t_i), \gamma^i(t'_i) \in \gamma_\infty((r, s))$ and consequently g and g' takes values in $\gamma_\infty((r, s))$. We conclude that the concatenation of g' , γ^i and g takes values in $B_{r,s}^\circ$. \square

3.2 Lower bound for the tail of L_1 near 0

In this section, for every $0 < r < s$, we construct an explicit path in \mathcal{A} taking values in $B_{r,s}^\circ$. This gives an upper bound for $L_{r,s}$. We will use this bound to obtain Theorem 1 (i) and the lower bound for point (ii) of

Theorem 1.

We start with some notation. Let $u, v \in \mathcal{T}_\infty$ and let $t, t' \in \mathbb{R}$ be chosen in a unique way so that $\mathcal{E}_t = u$, $\mathcal{E}_{t'} = v$ and $[t, t']$ is as small as possible (recall our special convention for $[t, t']$ when $t > t'$). Suppose that $t \leq t'$. Recall that $[u, v]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : r \in [t, t']\}$. Let $M_{u,v} := \inf_{[u,v]_{\mathcal{T}_\infty}} \Lambda$ and, for every $0 \leq r \leq \Lambda_u - M_{u,v}$, set

$$\gamma_{u,v}(r) := \Pi\left(\mathcal{E}_{\inf\{r' \in [t, t'] : \Lambda_{r'} = \Lambda_u - r\}}\right)$$

and for every $\Lambda_u - M_{u,v} < r \leq \Lambda_u + \Lambda_v - 2M_{u,v}$

$$\gamma_{u,v}(r) := \Pi\left(\mathcal{E}_{\sup\{r' \in [t, t'] : \Lambda_{r'} = r + 2M_{u,v} - \Lambda_u\}}\right).$$

By construction $\Delta^\circ(\gamma_{u,v}(r_1), \gamma_{u,v}(r_2)) = |r_1 - r_2|$ as soon as $r_1, r_2 \in (0, \Lambda_u - M_{u,v})$ or $r_1, r_2 \in (\Lambda_u - M_{u,v}, \Lambda_u + \Lambda_v - 2M_{u,v})$. In particular, applying (5), we deduce that the restriction of $\gamma_{u,v}$ to $[0, \Lambda_u - M_{u,v}]$ or $[\Lambda_u - M_{u,v}, \Lambda_u + \Lambda_v - 2M_{u,v}]$ is a geodesic. Hence $\gamma_{u,v}$ is a path with length $\Lambda_u + \Lambda_v - 2 \inf_{[u,v]_{\mathcal{T}_\infty}} \Lambda$. Remark that the range of $\gamma_{u,v}$ is contained in $\Pi([u, v]_{\mathcal{T}_\infty})$. In particular, if $u, v \in \mathcal{R}$ (resp. $u, v \in \mathcal{L}$) the range of $\gamma_{u,v}$ is contained in $\Pi(\mathcal{R})$ (resp. $\Pi(\mathcal{L})$) since $[u, v]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : r \in [t, t']\}$.

Finally, for $r < s$ set:

$$K_r^s := \{u \in \partial\mathcal{T}_\infty^s : \text{there exists } v \in \mathcal{T}_\infty \text{ such that } u \in [v, \infty[_{\mathcal{T}_\infty} \text{ and } \Lambda_v \leq r\}.$$

The continuity of the label function $u \mapsto \Lambda_u$ from \mathcal{T}_∞ into \mathbb{R} and the last assumption in (H_1) imply that the set K_r^s is finite Θ_0 -a.s. Recall the definition (10) of $(\tau_a)_{a \geq 0}$ and remark that τ_s is the unique element of K_r^s belonging to the spine $[0, \infty)$. Write $N_r^s := \#K_r^s - 1$ for the number of elements of K_r^s not belonging to the spine.

Proposition 2. *For every $r < s$, Θ_0 -a.s., we have:*

$$L_{r,s} \leq 2(N_r^s + 1)(s - r). \quad (19)$$

Proof. Denote the elements of K_r^s by $u_1, \dots, u_{N_r^s+1}$ in such a way that

$$\inf\{t \in \mathbb{R} : \mathcal{E}_s = u_i\} < \inf\{t \in \mathbb{R} : \mathcal{E}_s = u_j\}$$

if $i < j$. Recall that τ_s is the only point of the spine in K_r^s . Let $1 \leq k \leq N_r^s + 1$ be the unique index such that $u_k = \tau_s$. Fix $i \in \{1, \dots, N_r^s + 1\}$ with $i \neq k$. Next set, for every $i \in \{1, \dots, N_r^s + 1\}$,

$$f_i := \inf\{t \in \mathbb{R} : \mathcal{E}_t = u_i\} \text{ and } \ell_i := \sup\{t \in \mathbb{R} : \mathcal{E}_t = u_i\}.$$

Since u_i cannot be a leaf we have $f_i < \ell_i$. Moreover, the sequence $(f_i)_{1 \leq i \leq n}$ is increasing. Remark that, for every $1 \leq i \leq N_r^s$, we have $\ell_i < f_{i+1}$ and $[\ell_i, f_{i+1}]$ is the smallest interval such that $\mathcal{E}_{\ell_i} = u_i$ and $\mathcal{E}_{f_{i+1}} = u_{i+1}$. Therefore we can consider the path $\gamma_{u_i, u_{i+1}}$ as defined before the proposition. By construction, labels of points of the form \mathcal{E}_t with $t \in [\ell_i, f_{i+1}]$ are greater than r , and it follows that the length of the path $\gamma_{u_i, u_{i+1}}$ is smaller than $2(s - r)$. Finally set

$$R = \min\left(\inf_{(-\infty, f_1]} \Lambda; \inf_{[\ell_{N_r^s+1}, \infty)} \Lambda\right)$$

which is greater than r by construction. Set $u_0 := p_\infty^{(r)}(R)$ and $u_{N_r^s+2} := p_\infty^{(\ell)}(R)$. Note in particular that $\Pi(u_0) = \Pi(u_{N_r^s+2})$. Again we can consider the paths γ_{u_0, u_1} and $\gamma_{u_{N_r^s+1}, u_{N_r^s+2}}$. Let γ be the cycle obtained by concatenating the paths $(\gamma_{u_i, u_{i+1}})_{0 \leq i \leq N_r^s+1}$. It is straightforward to verify using property (F) that γ is an injective cycle. The paths γ_{u_0, u_1} and $\gamma_{u_{N_r^s+1}, u_{N_r^s+2}}$ have length $s - R \leq s - r$ and the paths $(\gamma_{u_i, u_{i+1}})_{1 \leq i \leq N_r^s}$ have length smaller than $2(s - r)$. Hence the length of γ is smaller than $2(N_r^s + 1)(s - r)$. Moreover, by a preceding remark, the range of $\gamma_{u_i, u_{i+1}}$ is contained in $\Pi(\mathcal{R})$ when $i < k$ and contained in $\Pi(\mathcal{L})$ when $i \geq k$. Consequently, if t_0 is the time at which γ visits the point $\Pi(u_k)$, we have $\gamma(t) \in \Pi(\mathcal{R})$ when $t \leq t_0$ and $\gamma(t) \in \Pi(\mathcal{L})$ when $t \geq t_0$. We conclude that the path γ is in \mathcal{A} and in particular γ is a separating path. Finally, by construction the path γ visits only points v such that $\Lambda_v \in (r, s]$. Since $\gamma(0) = \gamma_\infty(R)$ does not belong to B_r^\bullet , it follows that γ takes values in $B_s \setminus B_r^\bullet$. But now remark that γ hits the boundary ∂B_s only at the times at which it visits $\Pi(u_1), \dots, \Pi(u_{N_r^s+1})$. Since N_r^s is, Θ_0 -a.s., finite we deduce that γ hits the boundary ∂B_s a finite number of times. Consequently, by an approximation procedure, for every $\varepsilon > 0$ we can find $\gamma' \in \mathcal{A}$ taking values in $\text{Int}(B_s^\bullet) \setminus B_r^\bullet \subset B_{r,s}^\circ$ such that $\Delta(\gamma') < \Delta(\gamma) + \varepsilon$. Consequently by the definition of $L_{r,s}$ as an infimum (17) we deduce that

$$L_{r,s} \leq \Delta(\gamma) \leq 2(N_r^s + 1)(s - r).$$

□

The proposition shows that it is enough to control N_r^s in order to get an upper bound for $L_{r,s}$. Moreover since the Brownian plane is scale invariant we have:

$$N_r^s \stackrel{(d)}{=} N_1^{\frac{s}{r}}.$$

So we consider only N_1^s with $s > 1$. Thanks to (13), the law of N_1^s can be easily determined:

Proposition 3. *For $s > 1$ and $\lambda \geq 0$ we have*

$$\Theta_0(\exp(-\lambda N_1^s)) = \left(1 + (1 - \exp(-\lambda)) \frac{2s - 1}{(s - 1)^2}\right)^{-\frac{3}{2}}$$

Proof. Let $s > 1$, and write $\mathfrak{L}^{s,\infty} := \sum_{i \in I_s} \delta_{s_i, \omega^i}$; $\mathfrak{R}^{s,\infty} := \sum_{i \in J_s} \delta_{s_i, \omega^i}$. Let \tilde{I}_s (resp. \tilde{J}_s) be the set of all indices $i \in I_s$ (resp. $j \in J_s$) such that $\omega_*^i < s$. For every $i \in \tilde{I}_s \cup \tilde{J}_s$, write $(\omega^{i,k})_{k \in \mathbb{N}}$ for the excursions of ω^i below s . By the special Markov property, conditionally on Z_s ,

$$\sum_{i \in \tilde{I}_s \cup \tilde{J}_s} \sum_{k \in \mathbb{N}} \delta_{\omega^{i,k}}$$

is a Poisson point measure with intensity $Z_s \mathbb{N}_s(\cdot \cap \{W_* > 0\})$. Moreover by definition:

$$N_1^s = \#\{(i, k) \in (\tilde{I}_s \cup \tilde{J}_s) \times \mathbb{N} : \omega_*^{i,k} \leq 1\}.$$

So conditionally on Z_s , N_1^s is distributed as a Poisson variable with intensity $Z_s \mathbb{N}_s(0 < W_* \leq 1)$. We can then apply (2) to obtain:

$$\mathbb{N}_s(0 < W_* \leq 1) = \frac{3}{2} \left(\frac{1}{(s - 1)^2} - \frac{1}{s^2} \right) = \frac{3}{2} \frac{2s - 1}{s^2 (s - 1)^2}. \quad (20)$$

Using (13) we get that for $\lambda \geq 0$:

$$\begin{aligned}\Theta_0(\exp(-\lambda N_1^s)) &= \mathbb{E}[\exp(-(1 - \exp(-\lambda))Z_s \mathbb{N}_s(0 < W_* \leq 1))] \\ &= \left(1 + (1 - \exp(-\lambda)) \frac{2s-1}{(s-1)^2}\right)^{-\frac{3}{2}}.\end{aligned}$$

□

Let us list some immediate properties of N_1^s .

Lemma 5.

- (i) For every $s > 1$ we have $\Theta_0(N_1^s = 0) = (\frac{s-1}{s})^3$.
- (ii) The law of $(s-1)^2 N_1^s$ under Θ_0 converges weakly to a Gamma distribution with parameter $\frac{3}{2}$ and mean $3/2$ when $s \downarrow 1$. Furthermore, N_1^s tends to 0 as $s \rightarrow \infty$, Θ_0 -a.s.
- (iii) For every $s > 1$ and $q < \log(\frac{s^2}{2s-1})$ there exists a constant C_q such that for all $r > 0$:

$$\Theta_0(N_1^s > r) < C_q \exp(-qr).$$

Furthermore:

$$\limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(N_1^s > u))}{u} = -\log\left(\frac{s^2}{2s-1}\right).$$

- (iv) For every $s > 1$ we have $\Theta_0(N_1^s) = \frac{3}{2} \frac{2s-1}{(s-1)^2}$.

Proof.

- (i) By Proposition 3, we have

$$\Theta_0(N_1^s = 0) = \lim_{\lambda \rightarrow \infty} \Theta_0(\exp(-\lambda N_1^s)) = \left(\frac{s-1}{s}\right)^3$$

(we can also use the fact that $\Theta_0(N_1^s = 0) = \Theta_0(Z_1^{s,\infty} = 0)$ and Proposition 1).

- (ii) Again using Proposition 3, we obtain:

$$\Theta_0(\exp(-\lambda(s-1)^2 N_1^s)) = \left(1 + (1 - \exp(-\lambda(s-1)^2)) \frac{2s-1}{(s-1)^2}\right)^{-\frac{3}{2}}.$$

When s goes to 1 the Laplace transform converges to $\lambda \mapsto (1 + \lambda)^{-\frac{3}{2}}$ which is the Laplace transform of a Gamma distribution with parameter $3/2$ and mean $3/2$. The fact that N_1^s tends to 0 as $s \rightarrow \infty$, Θ_0 a.s., immediately follows from the property $\Lambda_{\mathcal{E}_t} \rightarrow \infty$ as $|t| \rightarrow \infty$.

- (iii) For every $\lambda > 0$ we have:

$$\Theta_0(\exp(\lambda N_1^s)) = \Theta_0(\exp(Z_s(\exp(\lambda) - 1)\mathbb{N}_s(0 < W_* \leq 1)))$$

because, conditionally on Z_s , the variable N_1^s is distributed as a Poisson random variable with intensity $Z_s \mathbb{N}_s(0 < W_* \leq 1)$. But Z_s is a Gamma random variable with parameter $3/2$ and mean s^2 so the previous expectation is finite if and only if

$$(\exp(-\lambda) - 1)\mathbb{N}_s(0 < W_* < 1) < 3/(2s^2)$$

or equivalently $\lambda < \log(\frac{s^2}{2s-1})$. The first part of (iii) then follows from the Markov inequality. On the other hand if we had:

$$\limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(N_1^s > u))}{u} \leq \alpha < -\log(\frac{s^2}{2s-1})$$

this would contradict $\mathbb{E}[\exp(\log(\frac{s^2}{2s-1})N_1^s)] = \infty$. This gives the second assertion of (iii).

(iv) We use again the fact that, under Θ_0 , conditionally on Z_s , N_1^s is distributed as a Poisson random variable with intensity $Z_s \mathbb{N}_s(0 < W_* \leq 1)$. We have:

$$\Theta_0(N_1^s) = \mathbb{N}_s(0 < W_* \leq 1)\Theta_0(Z_s) = \frac{3}{2} \frac{2s-1}{s^2(s-1)^2} \Theta_0(Z_s) = \frac{3}{2} \frac{2s-1}{(s-1)^2}$$

where in the second equality we use (20) and in the last equality we use the fact that Z_s has mean s^2 . \square

As a direct consequence we derive Theorem 1 (i) from Lemma 5.

Proof of Theorem 1 (i). By (19), for every $s > 1$ we have $L_1 \leq 2(s-1)(N_1^s + 1)$ Θ_0 -a.s. Let $s > 1$. Lemma 5 gives that :

$$\begin{aligned} \limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(L_1 > u))}{u} &\leq \limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(N_1^s > \frac{u}{2(s-1)} - 1))}{u} \\ &\leq -\frac{1}{2(s-1)} \log(\frac{s^2}{2s-1}). \end{aligned}$$

Since this holds for every $s > 1$, we obtain:

$$\limsup_{u \rightarrow \infty} \frac{\log(\Theta_0(L_1 > u))}{u} \leq -\sup_{s>1} \frac{1}{2(s-1)} \log(\frac{s^2}{2s-1}).$$

\square

The rest of this section is devoted to the proof of the lower bound appearing in Theorem 1 (ii). The proof relies again in (19) but in a more technical way. We state the following slightly stronger result:

Proposition 4. *There exists a positive constant, c_1 , such that for every $\varepsilon \in [0, 1]$ and $r > 0$:*

$$\Theta_0(L_{r,3r} < \varepsilon r) \geq c_1 \varepsilon^2.$$

The factor 3 is arbitrary and we will see in the proof that it can be replaced by any constant greater than 1. It will be useful in what follows to note that for every $0 < r < t$, we have $\{N_r^t = 0\} = \{Z_r^{t,\infty} = 0\}$ Θ_0 -a.s. We are going to deduce Proposition 4 from (19) and the following result:

Lemma 6. *There exists a positive constant c_1 such that for every $r > 0$ and $m \in \mathbb{N}^*$:*

$$\Theta_0\left(\bigcup_{i=0}^{m-1} \{N_{(m+i)r}^{(m+i+1)r} = 0\}\right) \geq \frac{c_1}{m^2}$$

Proof. By the scaling invariance of \mathcal{M}_∞ we can take $r = 1$. For $m \geq 2$, we have:

$$\begin{aligned} \Theta_0\left(\bigcup_{i=0}^{m-1}\{N_{m+i}^{m+i+1} = 0\}\right) &= \Theta_0(N_m^{m+1} = 0) \\ &+ \sum_{k=0}^{m-2} \Theta_0\left(\{N_{m+k+1}^{m+k+2} = 0\} \cap \bigcap_{i=0}^k \{N_{m+i}^{m+i+1} > 0\}\right). \end{aligned} \quad (21)$$

Moreover, for every $k \in \{0, \dots, m-2\}$:

$$\begin{aligned} \Theta_0\left(\{N_{m+k+1}^{m+k+2} = 0\} \cap \bigcap_{i=0}^k \{N_{m+i}^{m+i+1} > 0\}\right) & \\ &= \Theta_0\left(\{Z_{m+k+1}^{m+k+2, \infty} = 0\} \cap \bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \\ &= \Theta_0(Z_{m+k+1}^{m+k+2, \infty} = 0) \Theta_0\left(\bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \end{aligned} \quad (22)$$

where the first equality comes from the fact that $N_{m+k+1}^{m+k+2} = 0$, which is equivalent to $Z_{m+k+1}^{m+k+2, \infty} = 0$, implies $Z_t^{m+k+2, \infty} = 0$ for every $t \leq m+k+1$, so that, on the event $\{N_{m+k+1}^{m+k+2} = 0\}$, we have $Z_{m+i}^{m+i+1, \infty} = Z_{m+i}^{m+i+1, m+k+2}$ for every $0 \leq i \leq k$. The second equality in (22) is a consequence of the spine independence property of \mathcal{T}_∞ (see Section 2.4). The idea now is to prove that for every integer $k \geq 0$:

$$\Theta_0\left(\bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \geq \prod_{i=0}^k \Theta_0(Z_{m+i}^{m+i+1, m+k+2} > 0). \quad (23)$$

Let us explain how to obtain this inequality. Let $k > 0$, then:

$$\begin{aligned} \Theta_0\left(\bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) &= \Theta_0\left(\bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \\ &- \Theta_0\left(\{Z_{m+k}^{m+k+1, m+k+2} = 0\} \cap \bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \\ &= \Theta_0\left(\bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \\ &- \Theta_0\left(\{Z_{m+k}^{m+k+1, m+k+2} = 0\} \cap \bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+1} > 0\}\right) \end{aligned}$$

where the second equality is a consequence of the fact that $Z_{m+k}^{m+k+1, m+k+2} = 0$ implies that for every $i < k$ $Z_{m+i}^{m+i+1, m+k+2} = Z_{m+i}^{m+i+1, m+k+1}$. We now can apply the spine independence property to obtain that

$\Theta_0\left(\bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right)$ is equal to

$$\Theta_0\left(\bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) - \Theta_0(Z_{m+k}^{m+k+1, m+k+2} = 0) \Theta_0\left(\bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+1} > 0\}\right).$$

Now using the property $\{Z_{m+i}^{m+i+1, m+k+1} > 0\} \subset \{Z_{m+i}^{m+i+1, m+k+2} > 0\}$ for $i = 0, \dots, k-1$, we derive

$$\Theta_0\left(\bigcap_{i=0}^k \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \geq \Theta_0\left(\bigcap_{i=0}^{k-1} \{Z_{m+i}^{m+i+1, m+k+2} > 0\}\right) \Theta_0(Z_{m+k}^{m+k+1, m+k+2} > 0).$$

We can then iterate this argument to obtain (23). By combining (22) and (23) we deduce that:

$$\Theta_0\left(\{N_{m+k+1}^{m+k+2} = 0\} \cap \bigcap_{i=0}^k \{N_{m+i}^{m+i+1} > 0\}\right) \geq \Theta_0(Z_{m+k+1}^{m+k+2, \infty} = 0) \prod_{i=0}^k \Theta_0(Z_{m+i}^{m+i+1, m+k+2} > 0).$$

On the other hand, Proposition 1 states that for $0 < r < t < s$,

$$\Theta_0(Z_r^{t, s} = 0) = \left(\frac{s}{t}\right)^3 \left(\frac{t-r}{s-r}\right)^3$$

and taking the limit when s goes to ∞ , we obtain $\Theta_0(Z_r^{t, \infty} = 0) = \left(\frac{t-r}{t}\right)^3$. It follows that:

$$\Theta_0\left(\{N_{m+k+1}^{m+k+2} = 0\} \cap \bigcap_{i=0}^k \{N_{m+i}^{m+i+1} > 0\}\right) \geq \frac{1}{(m+k+2)^3} \prod_{i=0}^k \left(1 - \left(\frac{m+k+2}{m+i+1}\right)^3 \frac{1}{(k+2-i)^3}\right).$$

Then, for $m \geq 3$ and $k \in \{0, \dots, m-2\}$:

$$\begin{aligned} \prod_{i=0}^k \left(1 - \left(\frac{m+k+2}{m+i+1}\right)^3 \frac{1}{(k+2-i)^3}\right) &\geq \left(1 - \frac{1}{8} \left(\frac{m+k+2}{m+k+1}\right)^3\right) \prod_{i=0}^{k-1} \left(1 - \left(\frac{2m}{m}\right)^3 \frac{1}{(k+2-i)^3}\right) \\ &\geq \left(1 - \frac{1}{8} \left(\frac{5}{4}\right)^3\right) \prod_{i=3}^{\infty} \left(1 - \frac{8}{i^3}\right) \end{aligned}$$

which is a positive constant not depending on m . Let \tilde{c}_1 denote this constant. By applying the previous inequality to (21), we obtain that for every $m \geq 3$:

$$\begin{aligned} \Theta_0\left(\bigcup_{i=0}^{m-1} \{N_{m+i}^{m+i+1} = 0\}\right) &\geq \sum_{k=0}^{m-2} \Theta_0\left(\{N_{m+k+1}^{m+k+2} = 0\} \cap \bigcap_{i=0}^k \{N_{m+i}^{m+i+1} > 0\}\right) \\ &\geq \sum_{k=0}^{m-2} \frac{\tilde{c}_1}{(m+k+2)^3} \end{aligned}$$

which gives us the lower bound in the lemma. □

Proposition 4 follows now easily.

Proof of Proposition 4. By scaling we only need to prove the proposition for $r = 1$. Let $\varepsilon \in [0, 1)$ and set $s = \frac{\varepsilon}{2}$ and $m = \lceil \frac{2}{\varepsilon} \rceil$. Then the bound (19) and the fact that $L_{u', v'} \leq L_{u, v}$ if $[u', v'] \subset [u, v]$ give:

$$L_{1,3} \leq 2 \left(1 + \min_{i \in \{0, \dots, m-1\}} N_{(m+i)s}^{(m+i+1)s}\right) s$$

Θ_0 -a.s.. Consequently $\Theta_0(L_{1,3} \leq \varepsilon) \geq \Theta_0\left(\bigcup_{i=0}^{m-1} \{N_{(m+i)s}^{(m+i+1)s} = 0\}\right)$. The desired result follows directly from Lemma 4. □

3.3 Upper bound for the tail of L_1 near 0

We are going to deduce the upper bound for Theorem 1 (ii) from the following result

Proposition 5. *For every $\delta > 0$, there exists a constant α_δ such that for every $r \geq 1$:*

$$\Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) \leq \exp(1 - \alpha_\delta Z_{r+2\delta}).$$

We need to introduce some notation in order to prove Proposition 5. Let $u \in \mathcal{T}_\infty \setminus [0, \infty)$ such that u is not a leaf. Set $f_u := \inf\{t \in \mathbb{R}_+ : \mathcal{E}_t = u\}$ and $\ell_u := \sup\{t \in \mathbb{R}_+ : \mathcal{E}_t = u\}$. We consider the subtree $\mathcal{T}_\infty^{(u)} := \{\mathcal{E}_t : t \in [f_u, \ell_u]\}$ which is also equal to the set of all points $v \in \mathcal{T}_\infty$ such that $u \preceq v$. Remark that for every $v_1, v_2 \in \mathcal{T}_\infty^{(u)}$ there are two possibilities, either $[v_1, v_2]_{\mathcal{T}_\infty} \subset \mathcal{T}_\infty^{(u)}$ and in this case $0 \notin [v_1, v_2]_{\mathcal{T}_\infty}$ or $0 \in [v_1, v_2]_{\mathcal{T}_\infty}$. Consequently $\Delta^\circ(v_1, v_2)$ only depends on the subtree $(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v \in \mathcal{T}_\infty^{(u)}})$. For every $v, w \in \mathcal{T}_\infty^{(u)}$, set:

$$\tilde{\Delta}_u(v, w) := \inf_{\substack{v=v_1, \dots, v_n=w \\ v_1, \dots, v_n \in \mathcal{T}_\infty^{(u)}}} \sum_{i=1}^{n-1} \Delta^\circ(v_i, v_{i+1}) \quad (24)$$

where the infimum is over all choices of the integer $n \geq 1$ and all the finite sequences u_0, \dots, u_n of elements of $\mathcal{T}_\infty^{(u)}$ verifying $v_1 = v$ and $v_n = w$. Remark that $\tilde{\Delta}_u$ defines a continuous pseudo-distance on $\mathcal{T}_\infty^{(u)}$ (since $v \mapsto \Lambda_v$ is continuous) and that $\Delta(v, w) \leq \tilde{\Delta}_u(v, w)$ for every $v, w \in \mathcal{T}_\infty^{(u)}$. By the previous remark and since by (F), for every $v, w \in \mathcal{T}_\infty^{(u)}$, we have $\Delta(v, w) = 0$ if and only if $\Delta^\circ(v, w) = 0$, we see that $\tilde{\Delta}_u$ defines a distance on $\Pi(\mathcal{T}_\infty^{(u)})$ (and we keep the notation $\tilde{\Delta}_u$ for this distance). To simplify notation, we introduce the set $A_u := \Pi(\mathcal{T}_\infty^{(u)})$ and the paths $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ defined as follows. For every $0 \leq t \leq \Lambda_u - \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v$, take:

$$\gamma_u^{(1)}(t) := \Pi(\inf\{r \in [f_u, \ell_u] : \Lambda_r = \Lambda_u - t\})$$

and

$$\gamma_u^{(2)}(t) := \Pi(\sup\{r \in [f_u, \ell_u] : \Lambda_r = \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v + t\}).$$

By construction and (5), $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ are two geodesic paths. We also observe that $\min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v < \Lambda_u$ as a consequence of [16, Lemma 3.2] (it is important to notice that this holds simultaneously for all $u \in \mathcal{T}_\infty \setminus [0, \infty)$ such that u is not a leaf). Moreover, we have $\gamma_u^{(1)}(\Lambda_u - \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v) = \gamma_u^{(2)}(0)$ and $\gamma_u^{(1)}(0) = \gamma_u^{(2)}(\Lambda_u - \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v)$. From property (F), we get that the concatenation of $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ is an injective cycle with length $2\Lambda_u - 2 \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v$. We denote this path by γ_u . It will be useful to remark that γ_u^1 and γ_u^2 are also geodesic paths for $\tilde{\Delta}_u$. Recall the notation Δ_{A_u} for the intrinsic distance induced by Δ on A_u .

Lemma 7. *Θ_0 -a.s. for every $u \in \mathcal{T}_\infty \setminus [0, \infty)$ such that u is not a leaf, the set A_u is homeomorphic to the closed unit disk of \mathbb{C} and its boundary is the range of γ_u . Moreover we have:*

$$\Delta_{A_u} = \tilde{\Delta}_u. \quad (25)$$

The main interest of (25) is the fact that the function $\tilde{\Delta}_u$ only depends on $(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v \in \mathcal{T}_\infty^{(u)}})$. Recall the definition of Δ° above (4), and remark that Δ° on $\mathcal{T}_\infty^{(u)}$ does not change if we shift all the labels by any $a > -\min_{\mathcal{T}_\infty^{(u)}} \Lambda$. This gives that $\tilde{\Delta}_u$ can also be defined from the labeled tree $(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v \in \mathcal{T}_\infty^{(u)}} + a)$ for any $a > -\min_{\mathcal{T}_\infty^{(u)}} \Lambda$.

Proof. Let $u \in \mathcal{T}_\infty \setminus [0, \infty)$ such that u is not a leaf. Since γ_u is an injective cycle, Jordan's theorem implies that the complement of the range of γ_u has two connected components, namely a bounded connected component U_1 and an unbounded connected component U_2 . Moreover, the closure of U_1 is homeomorphic to the closed unit disk. The first assertion of the lemma then follows from the fact that $\text{Cl}(U_1) = A_u$, which is easy and left to the reader. Let us turn to the second part of the lemma. We start by showing that $\Delta_{A_u}(v, w) \leq \tilde{\Delta}_u(v, w)$ for every $v, w \in \mathcal{T}_\infty^{(u)}$. Let $v, w \in \mathcal{T}_\infty^{(u)}$. Up to interchanging v and w , we can suppose that

$$\Delta^\circ(v, w) = \Lambda_v + \Lambda_w - 2 \min_{[v, w]_{\mathcal{T}_\infty}} \Lambda,$$

and $v = \mathcal{E}_s$, $w = \mathcal{E}_t$ with $s \leq t$ and $[u, w]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : s \leq r \leq t\}$. We can then consider the path $\gamma_{v, w}$ introduced at the beginning of Section 3.2. The length of $\gamma_{v, w}$ is $\Delta^\circ(v, w)$ and its range is a subset of $\Pi([v, w]_{\mathcal{T}_\infty})$. Since $[v, w]_{\mathcal{T}_\infty} \subset \mathcal{T}_\infty^{(u)}$, we deduce that $\gamma_{v, w}$ takes values in A_u . From the definition (24) we obtain that $\Delta_{A_u}(v, w) \leq \tilde{\Delta}_u(v, w)$ for every $v, w \in \mathcal{T}_\infty^{(u)}$.

Let us prove the reverse inequality. Let $\gamma : [0, 1] \rightarrow A_u$ be a path. It is enough to show that $\Delta(\gamma) \geq \tilde{\Delta}_u(\gamma(0), \gamma(1))$. We deal with two separate cases.

- Case 1: We assume that, for every $t \in (0, 1)$, $\gamma(t) \notin \partial A_u$. Let us start by showing that γ is also continuous for $\tilde{\Delta}_u$. In order to prove this, remark that the identity function $(A_u, \tilde{\Delta}_u) \mapsto (A_u, \Delta)$ is a bijection and that it is also continuous, since $\Delta \leq \tilde{\Delta}_u$. Moreover, as $\mathcal{T}_\infty^{(u)}$ is compact, the continuity of Π implies that A_u is compact for the quotient topology. Since Δ and $\tilde{\Delta}_u$ are continuous on $\mathcal{T}_\infty^{(u)} \times \mathcal{T}_\infty^{(u)}$ we derive that $(A_u, \tilde{\Delta}_u)$ and (A_u, Δ) are both compact. So the identity function $(A_u, \tilde{\Delta}_u) \mapsto (A_u, \Delta)$ is a continuous bijection between compact spaces which implies that it is also an homeomorphism. We deduce that γ is continuous for $\tilde{\Delta}_u$. In particular, we have

$$\Delta(\gamma) = \lim_{\varepsilon \rightarrow 0} \Delta(\gamma|_{[\varepsilon, 1-\varepsilon]}) \quad \text{and} \quad \tilde{\Delta}_u(\gamma(0), \gamma(1)) = \lim_{\varepsilon \rightarrow 0} \tilde{\Delta}_u(\gamma(\varepsilon), \gamma(1-\varepsilon)).$$

By the previous display, we may and will restrict our attention to the case when $\gamma(t) \notin A_u$ for every $t \in [0, 1]$. By compactness, the quantity $\delta := \inf_{(t, v) \in [0, 1] \times \partial A_u} \Delta(\gamma(t), v)$ is positive. Let $n > 1$ be an integer such that for every $0 \leq i \leq n-1$:

$$\Delta(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})) < \frac{\delta}{3}.$$

We are going to show that $\Delta(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})) \geq \tilde{\Delta}_u(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n}))$ for every $0 \leq i \leq n-1$. By the triangle inequality this will imply that $\Delta(\gamma) \geq \tilde{\Delta}_u(\gamma(0), \gamma(1))$. Fix $0 \leq i \leq n-1$ and consider $u_i, v_i \in \mathcal{T}_\infty$ such that $(\Pi(u_i), \Pi(v_i)) = (\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n}))$. Remark that we must have $u_i, v_i \in \mathcal{T}_\infty^{(u)}$ and recall that:

$$\Delta(u_i, v_i) = \inf_{u_i = u_{i,1}, u_{i,2}, \dots, u_{i,m} = v_i} \sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1})$$

where the infimum is over all choices of the integer $m \geq 1$ and all finite sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ with $(u_{i,1}, u_{i,m}) = (u_i, v_i)$. Since $\Delta(u_i, v_i) < \delta/3$ we can restrict the previous infimum to finite sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ with $(u_{i,1}, u_{i,m}) = (u_i, v_i)$ such that:

$$\sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1}) < \frac{\delta}{2}.$$

Consider such a sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ and remark that $\Delta(u_{i,1}, u_{i,k}) < \frac{\delta}{2}$ for every $1 \leq k \leq m$ by triangle inequality. This implies by the definition of δ that $u_{i,k} \in \mathcal{T}_\infty^{(u)}$ for every $1 \leq k \leq m$. We conclude from the definition of $\tilde{\Delta}_u$ that:

$$\sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1}) \geq \tilde{\Delta}_u(u_i, v_i).$$

Consequently, $\Delta(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})) \geq \tilde{\Delta}_u(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n}))$ for every $0 \leq i \leq n-1$ and thus $\Delta(\gamma) \geq \tilde{\Delta}_u(\gamma(0), \gamma(1))$.

• Case 2: We now assume that γ hits ∂A_u . Let $s := \inf\{r \in [0, 1] : \gamma(r) \in \partial A_u\}$. Without loss of generality, we may assume that $\gamma(s)$ belongs to the range of $\gamma_u^{(1)}$. Let s' be the largest element of $[0, 1]$ such that $\gamma(s')$ is in the range of $\gamma_u^{(1)}$. If $s' = 1$ or if $\gamma(t) \notin \partial A_u$ for every $s' < t \leq 1$, we have:

$$\Delta(\gamma) = \Delta(\gamma|_{[0,s]}) + \Delta(\gamma|_{[s,s']}) + \Delta(\gamma|_{[s',1]}) \geq \tilde{\Delta}_u(\gamma(0), \gamma(s)) + \Delta(\gamma|_{[s,s']}) + \tilde{\Delta}_u(\gamma(s'), \gamma(1))$$

since case 1 can be applied to $\gamma|_{[0,s]}$ and $\gamma|_{[s',1]}$. Moreover, since $\gamma_u^{(1)}$ is also a geodesic for $\tilde{\Delta}_u$, we have $\Delta(\gamma|_{[s,s']}) \geq \Delta(\gamma(s), \gamma(s')) = \tilde{\Delta}_u(\gamma(s), \gamma(s'))$ and by the triangle inequality we obtain $\Delta(\gamma) \geq \tilde{\Delta}_u(\gamma(0), \gamma(1))$.

It remains to consider the case where $s' < 1$ and $\{t \in (s', 1] : \gamma(t) \in \partial A_u\}$ is not empty. Let t be the smallest element of $[s', 1]$ such that $\gamma(t) \in \partial A_u$. Then $\gamma(t)$ belongs to the range of $\gamma_u^{(2)}$. Let t' be the largest element of $[t, 1]$ such that $\gamma(t')$ belongs to the range of $\gamma_u^{(2)}$. Then we get that:

$$\Delta(\gamma) = \Delta(\gamma|_{[0,s]}) + \Delta(\gamma|_{[s,s']}) + \Delta(\gamma|_{[s',t]}) + \Delta(\gamma|_{[t,t']}) + \Delta(\gamma|_{[t',1]}).$$

Since $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ are two geodesics paths for $\tilde{\Delta}_u$ we have:

$$\Delta(\gamma|_{[s,s']}) \geq \Delta(\gamma(s), \gamma(s')) = \tilde{\Delta}_u(\gamma(s), \gamma(s'))$$

and

$$\Delta(\gamma|_{[t,t']}) \geq \Delta(\gamma(t), \gamma(t')) = \tilde{\Delta}_u(\gamma(t), \gamma(t')).$$

On the other hand, $\gamma|_{[0,s]}$, $\gamma|_{[s',t]}$ and $\gamma|_{[t',1]}$ belong to case 1. Consequently, we obtain $\Delta(\gamma) \geq \tilde{\Delta}_u(\gamma(0), \gamma(1))$. \square

Let us deduce Proposition 5 from Lemma 7.

Proof of Proposition 5. Fix $\delta > 0$ and $r \geq 1$. We want to give a lower bound for $L_{r,r+\delta}$. By Lemma 4 it is enough to give a lower bound for $\Delta(\gamma)$ for every $\gamma \in \mathcal{A}$ taking values in $B_{r,r+\delta}^\circ$. Recall the notation of Section 3.2 and for every $u \in K_r^{r+2\delta}$ not belonging to the spine $[0, \infty)$ set:

$$M_u := \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v.$$

Let $u_1, \dots, u_{\tilde{N}_r^{r+2\delta}}$ be the elements of $K_r^{r+2\delta}$ that do not belong to the spine $[0, \infty)$ and such that $M_u > r-1/2$. By Lemma 7, for every $1 \leq k \leq \tilde{N}_r^{r+2\delta}$ the set A_{u_k} is homeomorphic to the closed unit disk. Since ∂B_r^\bullet and $\partial B_{r+\delta}^\bullet$ are injective cycles (see Section 2.4), it is straightforward to verify (using Jordan's theorem) that $A_{u_k} \cap B_{r,r+\delta}^\bullet$ is homeomorphic to the unit disk and its boundary is:

$$(A_{u_k} \cap \partial B_{r,r+\delta}^\bullet) \cup \gamma_{u_k}^{(1)}([\delta, 2\delta]) \cup \gamma_{u_k}^{(2)}([r - M_u, r + \delta - M_u]).$$

This implies that any separating cycle $\gamma \in \mathcal{A}$ taking values in $B_{r,r+\delta}^\circ$ has to connect $\gamma_{u_k}^{(1)}([\delta, 2\delta])$ and $\gamma_{u_k}^{(2)}([r - M_u, r + \delta - M_u])$ while staying in A_{u_k} see figure 3 for an illustration. In other words, for every $1 \leq k \leq \tilde{N}_r^{r+2\delta}$, there exist $t_k < t'_k$ such that $\gamma|_{[t_k, t'_k]}$ takes values in $A_{u_k} \cap B_{r,r+\delta}^\circ$, $\gamma(t_k) \in \gamma_{u_k}^{(1)}([\delta, 2\delta])$ and $\gamma(t'_k) \in \gamma_{u_k}^{(2)}([r - M_u, r + \delta - M_u])$. In particular:

$$\Delta(\gamma) \geq \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \Delta(\gamma|_{[t_k, t'_k]}) \geq \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \Delta_{A_{u_k}}(\gamma(t_k), \gamma(t'_k)).$$

Set

$$D_k := \inf \{ \Delta_{A_{u_k}}(x, y) : (x, y) \in \gamma_{u_k}^{(1)}([\delta, 2\delta]) \times \gamma_{u_k}^{(2)}([r - M_u, r + \delta - M_u]) \},$$

and note that $D_k > 0$ by property (F) (no point of $\gamma_{u_k}^{(1)}([\delta, 2\delta])$ can be identified with a point of $\gamma_{u_k}^{(2)}([r - M_u, r + \delta - M_u])$). With this notation we have $\Delta(\gamma) \geq \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k$ for every $\gamma \in \mathcal{A}$ taking values in $B_{r,r+\delta}^\circ$. Consequently, we obtain that:

$$L_{r,r+\delta} \geq \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k.$$

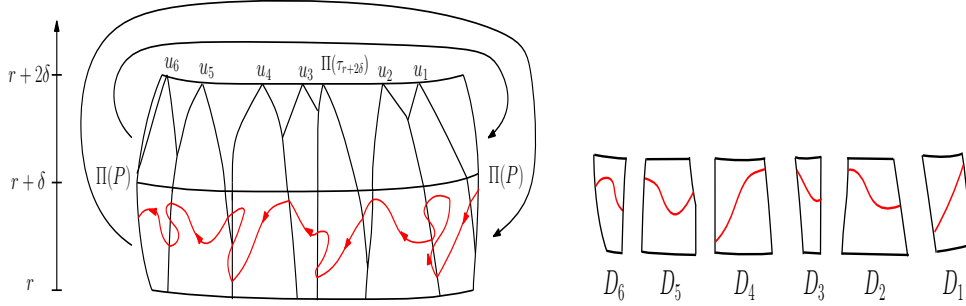


Figure 3: Illustration of the inequality $L_{r,r+\delta} \geq \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k$. The red path is an element of \mathcal{A} taking values in $B_{r,r+\delta}^\circ$ and here $\tilde{N}_r^{r+2\delta} = 6$.

To conclude we use the following claim:

(C): Conditionally on $(Z_{r+2\delta}, \tilde{N}_r^{r+2\delta})$, the variables $(D_k)_{1 \leq k \leq \tilde{N}_r^{r+2\delta}}$ are independent and identically distributed according to a distribution μ_δ that does not depend on r and is supported on $(0, \infty)$.

Before proving (C), let us explain why Proposition 5 follows.

We set $\chi_\delta := \int \mu_\delta(dx) e^{-x} \in (0, 1)$, and then we have

$$\begin{aligned} \Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) &\leq \Theta_0\left(\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k < 1 \mid Z_{r+2\delta}\right) \\ &\leq e \Theta_0\left(\exp\left(-\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k\right) \mid Z_{r+2\delta}\right) \\ &= e \Theta\left(\chi_\delta^{\tilde{N}_r^{r+2\delta}} \mid Z_{r+2\delta}\right) \end{aligned}$$

by our claim (C). By the special Markov property, conditionally on $Z_{r+2\delta}$, the variable $\tilde{N}_r^{r+2\delta}$ is distributed as a Poisson variable with parameter $Z_{r+2\delta} \mathbb{N}_{r+2\delta}(r - 1/2 < W_* \leq r)$. It follows that

$$\Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) \leq e \exp\left(-Z_{r+2\delta} \mathbb{N}_{r+2\delta}(r - 1/2 < W_* \leq r)(1 - \chi_\delta)\right)$$

and we obtain the desired result with

$$\alpha_\delta := \mathbb{N}_{r+2\delta}(r - 1/2 < W_* \leq r)(1 - \chi) = \mathbb{N}_{2\delta}(1/2 < W_* \leq 1)(1 - \chi_\delta).$$

Let us explain why (C) holds in order to complete the proof. Let $(\omega^i)_{i \in I_{r+2\delta} \cup J_{r+2\delta}}$ be the atoms of $\mathfrak{L}^{r+2\delta, \infty}$ and $\mathfrak{R}^{r+2\delta, \infty}$. Let $\tilde{I}_{r+2\delta} \cup \tilde{J}_{r+2\delta}$ be the set of indices $i \in I_{r+2\delta} \cup J_{r+2\delta}$ such that $\omega_*^i < r + 2\delta$. For every $i \in \tilde{I}_{r+2\delta} \cup \tilde{J}_{r+2\delta}$ we write $(\omega^{i,n})_{n \in \mathbb{N}}$ for the excursions of ω^i outside $r + 2\delta$. By construction:

$$\tilde{N}_r^{r+2\delta} := \#\{(i, n) \in (\tilde{I}_{r+2\delta} \cup \tilde{J}_{r+2\delta}) \times \mathbb{N} : r - 1/2 < \omega_*^{i,n} < r\}.$$

For every $1 \leq k \leq \tilde{N}_r^{r+2\delta}$, there exists a unique (i, n) with $r - 1/2 < \omega_*^{i,n} < r$ such that the labeled trees $\mathcal{T}_{\omega^{i,n}}$ and $\mathcal{T}_\infty^{(u_k)}$ can be identified and we write ω^k instead of $\omega^{i,n}$ to simplify notation. By the special Markov property, conditionally on $Z_{r+2\delta}$, the point measure:

$$\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \delta_{\omega^k}$$

is a Poisson measure with intensity:

$$Z_{r+2\delta} \mathbb{N}_{r+2\delta}\left(\cdot \cap \left\{r - \frac{1}{2} < W_* < r\right\}\right).$$

So conditionally on $(Z_{r+2\delta}, \tilde{N}_r^{r+2\delta})$, the sequence $(\omega^k - r + 1)_{1 \leq k \leq \tilde{N}_r^{r+2\delta}}$ is an i.i.d. sequence with common distribution $\mathbb{N}_\delta(\cdot \cap \{\frac{1}{2} < W_* < 1\})$. In particular, this distribution does not depend on r . Moreover $\tilde{\Delta}_{u_k}$ depends only on the labeled tree $\mathcal{T}_\infty^{(u_k)} = \mathcal{T}_{\omega^k}$ and the definition (24) shows that $\tilde{\Delta}_{u_k}$ is not affected if labels are shifted by $(-r + 1)$. So $\tilde{\Delta}_{u_k}$ is also a function of $\omega^k - r + 1$. Our claim (C) follows since by Lemma 7, we have $\Delta_{A_{u_k}} = \tilde{\Delta}_{u_k}$ for every $1 \leq k \leq \tilde{N}_r^{r+2\delta}$. \square

We conclude this section with the proof of part (ii) of Theorem 1.

Proof of Theorem 1 (ii). We want to show that there exists c_2 , such that for every $\varepsilon \geq 0$:

$$\Theta_0(L_1 < \varepsilon) \leq c_2 \varepsilon^2.$$

To do so fix $\varepsilon \in (0, 1/2)$ and remark that:

$$\{L_1 < \varepsilon\} \subset \bigcup_{m=\lfloor \frac{1}{\varepsilon} \rfloor - 1}^{\infty} \{L_{m\varepsilon, (m+3)\varepsilon} < \varepsilon\} \quad (26)$$

Let us explain why (26) holds. On the event $\{L_1 < \varepsilon\}$, let γ be a separating cycle in \check{B}_1° such that $\Delta(\gamma) < \varepsilon$. Since the sets $B_{(m+1)\varepsilon}^\bullet \setminus B_{m\varepsilon}^\bullet$, for $m \geq \lfloor \frac{1}{\varepsilon} \rfloor$, cover \check{B}_1° , we can find $m_0 \geq \lfloor \frac{1}{\varepsilon} \rfloor$ such that $\gamma(0) \in B_{(m_0+1)\varepsilon}^\bullet \setminus B_{m_0\varepsilon}^\bullet$. Then notice that $\Delta(\gamma(0), B_{(m_0-1)\varepsilon}^\bullet) \geq \varepsilon$ and $\Delta(\gamma(0), B_{(m_0+2)\varepsilon}^\bullet) \geq \varepsilon$. Since the length of γ is smaller than ε , it follows that the path γ stays inside $B_{(m_0-1)\varepsilon, (m_0+2)\varepsilon}^\circ$ and consequently $\Delta(\gamma) \leq L_{(m_0-1)\varepsilon, (m_0+2)\varepsilon}$.

(26) implies that:

$$\Theta_0(L_1 < \varepsilon) \leq \sum_{m=\lfloor \frac{1}{\varepsilon} \rfloor - 1}^{\infty} \Theta_0(L_{m\varepsilon, (m+3)\varepsilon} < \varepsilon) = \sum_{m=\lfloor \frac{1}{\varepsilon} \rfloor - 1}^{\infty} \Theta_0(L_{m, m+3} < 1)$$

where to obtain the right equality we use the scale invariance of \mathcal{M}_∞ . We now can apply Proposition 5 to obtain that there exists $\alpha > 0$ such that :

$$\Theta_0(L_1 < \varepsilon) \leq e \sum_{m=\lfloor \frac{1}{\varepsilon} \rfloor - 1}^{\infty} \Theta_0(\exp(-\alpha Z_{m+4})) = e \sum_{m=\lfloor \frac{1}{\varepsilon} \rfloor - 1}^{\infty} (1 + \frac{2}{3}\alpha(m+4)^2)^{-\frac{3}{2}}$$

where we used (13) in the last equality. The desired result follows. \square

3.4 Application to the infinite volume Brownian disk

The goal of this section is to extend Theorem 3 to random levels and then to derive some properties of injective cycles separating the boundary from infinity in the infinite volume Brownian disk. Let us recall the notation of Subsection 2.4.2, and in particular, the definition of the coding triple $(\tilde{X}^{(r)}, \tilde{\mathcal{L}}_r, \tilde{\mathfrak{R}}_r)$ for every $r \geq 0$. On the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$, for every $r \geq 0$, let \mathcal{F}_r be the σ -field generated by B_r^\bullet (view as a random variable with values in \mathbb{K} as explained in Section 2.4.1) and the class of all Θ_0 -negligible sets. The approximation property (15) implies that Z_r is \mathcal{F}_{r+} -measurable, for every $r \geq 0$. We write $\rho_r := \Pi(\tau_r)$ for every $r \geq 0$, where $(\tau_r)_{r \geq 0}$ is defined in (10).

Theorem 4. *Let T be a stopping time of the filtration $(\mathcal{F}_{r+})_{r \geq 0}$ such that we have $0 < T < \infty$, Θ_0 -a.s. Then conditionally on $Z_T = z$, the coding triple $(\tilde{X}^{(T)}, \tilde{\mathcal{L}}_T, \tilde{\mathfrak{R}}_T)$ is distributed according to Θ_z and is independent of B_T^\bullet . Furthermore, the intrinsic distance $\check{\Delta}^{(T)}$ on \check{B}_T° has a unique continuous extension to \check{B}_T^\bullet . The space \check{B}_T^\bullet equipped with this continuous extension of $\check{\Delta}^{(T)}$, with the restriction of the volume measure and with the distinguished point ρ_T coincides (as an element of \mathbb{K}_∞) with the metric space associated with $(\tilde{X}^{(T)}, \tilde{\mathcal{L}}_T, \tilde{\mathfrak{R}}_T)$. In particular, conditionally on $Z_T = z$, the space \check{B}_T^\bullet is an infinite Brownian disk with perimeter z and is independent of B_T^\bullet .*

Proof. Let T be as in the statement of the theorem. Recall the notation $M(\mathcal{S})$ and the distance $d_{M(\mathcal{S})}$ introduced in Section 2.3. Let F_1 and F_2 be two bounded nonnegative measurable functions on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$. Assume that F_1 is \mathcal{F}_{T+} -measurable and that F_2 is continuous. We will show that

$$\Theta_0(F_1 \times F_2(X^{(T)}, \tilde{\mathcal{L}}_T, \tilde{\mathfrak{R}}_T)) = \Theta_0(F_1 \Theta_{Z_T}(F_2)). \quad (27)$$

Remark that (27) implies that, conditionally on $Z_T = z$, the coding triple $(\tilde{X}^{(T)}, \tilde{\mathfrak{L}}_T, \tilde{\mathfrak{R}}_T)$ is distributed according to Θ_z and is independent of B_T^\bullet (the hull B_T^\bullet is \mathcal{F}_{T+} measurable, since the process $t \mapsto B_t^\bullet$ is adapted to $(\mathcal{F}_{t+})_{t \geq 0}$ and T is a stopping time). In particular, $(\tilde{X}^{(T)}, \tilde{\mathfrak{L}}_T, \tilde{\mathfrak{R}}_T)$ will a.s. verify (H_2) . Then the different assertions of the theorem follow from Lemma 2. It remains to establish (27). For every integer $n \geq 1$ we have:

$$\Theta_0(F_1 \times F_2(X^{(\frac{[nT]}{n})}, \tilde{\mathfrak{L}}_{\frac{[nT]}{n}}, \tilde{\mathfrak{R}}_{\frac{[nT]}{n}})) = \sum_{k=0}^{\infty} \Theta_0(F_1 \mathbb{1}_{\frac{k}{n} \leq T < \frac{k+1}{n}} F_2(X^{(\frac{k+1}{n})}, \tilde{\mathfrak{L}}_{\frac{k+1}{n}}, \tilde{\mathfrak{R}}_{\frac{k+1}{n}})) \quad (28)$$

For every atom (ℓ, ω) of \mathfrak{R} or \mathfrak{L} such that $\ell > \tau_T$, an application of [1, Lemma 11] shows that $\text{tr}_{\frac{[nT]}{n}}(\omega) \rightarrow \text{tr}_T(\omega)$ as $n \rightarrow \infty$. Using also the fact that $r \rightarrow \tau_r$ is càdlàg, we easily obtain that $\tilde{\mathfrak{L}}_{\frac{[nT]}{n}} \rightarrow \tilde{\mathfrak{L}}_T$ and $\tilde{\mathfrak{R}}_{\frac{[nT]}{n}} \rightarrow \tilde{\mathfrak{R}}_T$ when $n \rightarrow \infty$, with respect to the topology on $M(\mathcal{S})$. Since F_2 is bounded and continuous, we can take the limit when n goes to ∞ to obtain:

$$\lim_{n \rightarrow \infty} \Theta_0(F_1 \times F_2(X^{(\frac{[nT]}{n})}, \tilde{\mathfrak{L}}_{\frac{[nT]}{n}}, \tilde{\mathfrak{R}}_{\frac{[nT]}{n}})) = \Theta_0(F_1 \times F_2(X^{(T)}, \tilde{\mathfrak{L}}_T, \tilde{\mathfrak{R}}_T)). \quad (29)$$

On the other hand, for every $k \geq 0$, $F_1 \mathbb{1}_{\frac{k}{n} \leq T < \frac{k+1}{n}}$ is $\mathcal{F}_{\frac{k+1}{n}}$ -measurable and is thus equal, Θ_0 -a.s., to a measurable function of $B_{\frac{k+1}{n}}^\bullet$. Hence we can apply the spatial Markov property of Theorem 3 to obtain:

$$\begin{aligned} \sum_{k=0}^{\infty} \Theta_0(F_1 \mathbb{1}_{\frac{k}{n} \leq T < \frac{k+1}{n}} F_2(X^{(\frac{k+1}{n})}, \tilde{\mathfrak{L}}_{\frac{k+1}{n}}, \tilde{\mathfrak{R}}_{\frac{k+1}{n}})) &= \sum_{k=0}^{\infty} \Theta_0(F_1 \mathbb{1}_{\frac{k}{n} \leq T < \frac{k+1}{n}} \Theta_{Z_{\frac{k+1}{n}}} (F_2)) \\ &= \Theta_0(F_1 \Theta_{Z_{\frac{[nT]}{n}}} (F_2)). \end{aligned}$$

Since the process Z is càdlàg, we have $Z_{\frac{[nT]}{n}} \rightarrow Z_T$ as $n \rightarrow \infty$. Moreover, the fact that F_2 is bounded and continuous and the scaling property of $(\Theta_l)_{l>0}$ imply that the mapping $l \mapsto \Theta_l(F_2)$ is also bounded and continuous. It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \Theta_0(F_1 \mathbb{1}_{\frac{k}{n} \leq T < \frac{k+1}{n}} F_2(X^{(\frac{k+1}{n})}, \tilde{\mathfrak{L}}_{\frac{k+1}{n}}, \tilde{\mathfrak{R}}_{\frac{k+1}{n}})) &= \lim_{n \rightarrow \infty} \Theta_0(F_1 \Theta_{Z_{\frac{[nT]}{n}}} (F_2)) \\ &= \Theta_0(F_1 \Theta_{Z_T} (F_2)). \end{aligned} \quad (30)$$

The identity (27) then follows by passing to the limit $n \rightarrow \infty$ in (28), using (29) and (30). \square

Let us state some direct consequences of Theorem 4. For every $z > 0$, set

$$T_z := \inf\{r \geq 0 : Z_r \geq z\} \quad (31)$$

which is a stopping time of the filtration $(\mathcal{F}_{r+})_{r \geq 0}$. Note that $0 < T_z < \infty$, Θ_0 -a.s. Moreover since Z does not have positive jumps we have $Z_{T_z} = z$. Applying Theorem 4 with $T = T_z$, we obtain the following result.

Corollary 1. *Let $z > 0$. Under Θ_0 , $(\check{B}_{T_z}^\bullet, \rho_{T_z}, \check{\Delta}^{T_z}, |\cdot|_{\check{\Delta}^{T_z}})$ is an infinite volume Brownian disk with perimeter z and is independent of $(B_{T_z}^\bullet, 0, \Delta^{T_z}, |\cdot|_{\Delta^{T_z}})$.*

The next goal is to extend the definition of process Z under Θ_z . It will be useful to consider the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions from \mathbb{R}_+ into \mathbb{R} . We write $(\xi_t)_{t \geq 0}$ for the canonical process on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$

and $(\mathcal{D}_t)_{t \geq 0}$ for the canonical filtration. We introduce a probability measure \mathbb{P} on $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}), \mathcal{D})$ such that under \mathbb{P} , the process ξ is distributed as a Lévy process without positive jumps with Laplace exponent $\psi(\lambda) := \sqrt{\frac{8}{3} \frac{\Gamma(\lambda + \frac{3}{2})}{\Gamma(\lambda)}}$, i.e.:

$$\mathbb{E}[\exp(\lambda \xi_1)] = \exp(\psi(\lambda)), \quad \forall \lambda > -\frac{3}{2},$$

where \mathbb{E} stands for the expectation with respect to \mathbb{P} . We refer to [4, Lemma 2.1] for the existence of this Lévy process. Since $\psi'(0+) > 0$, standard properties of Levy processes imply that ξ drifts to ∞ (see for example [3, Chapter VII]). We also introduce the time change:

$$\kappa(r) := \inf \left\{ s \geq 0 : \int_0^s \exp\left(\frac{1}{2} \xi_t\right) dt \geq r \right\}.$$

Theorem 24 in [18] states that the process Z under Θ_0 is a self-similar Markov process started at 0 with index $\frac{1}{2}$ and Laplace exponent ψ . In particular, the process $(Z_{T_z+t})_{t \geq 0}$ is distributed under Θ_0 as:

$$\left(z \exp\left(\xi_{\kappa(z - \frac{1}{2} r)}\right) \right)_{r \geq 0}$$

under \mathbb{P} . As a consequence of (15) and Corollary 1 we obtain:

Corollary 2. *Fix $z > 0$. Then, for every $r \geq 0$,*

$$Z_r := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} |\check{B}_r^\bullet \cap B_{r+\varepsilon}|$$

exists Θ_z -a.s. Moreover the process $(Z_r)_{r \geq 0}$ has a càdlàg modification under Θ_z , which is distributed as a $(\frac{1}{2}, \psi)$ -self-similar Markov process started at z , i.e.

$$(Z_r)_{r \geq 0} \stackrel{(d)}{=} \left(z \exp\left(\xi_{\kappa(z - \frac{1}{2} r)}\right) \right)_{r \geq 0} \tag{32}$$

where ξ is distributed according to \mathbb{P} .

Let $\mathcal{M}_\infty^{(z)}$ be the infinite volume Brownian disk with perimeter z defined under the probability measure Θ_z as explained in Section 2.4.2. We say that $\gamma : [s, t] \rightarrow \mathcal{M}_\infty^{(z)}$ is a separating cycle if it is an injective continuous cycle that does not hit the boundary of $\mathcal{M}_\infty^{(z)}$ and if any path connecting this boundary to ∞ has to cross the range of γ . We are going to use the "strong" spatial Markov property (Theorem 4) to study the separating cycles of the infinite volume Brownian disk. As in the previous sections, we consider $B_{r,s}^\circ = \text{Int}(B_s^\bullet \setminus B_r^\bullet)$ and

$$L_{r,s} = \inf \{ \Delta(g) : g : [0, 1] \rightarrow B_{r,s}^\circ \text{ separating cycle} \},$$

for every $0 \leq r < s$, and we will now study $L_{r,s}$ under Θ_z . To simplify notation we write $L_r := L_{r,\infty}$ for every $r \geq 0$. Note that L_0 is the infimum of lengths of paths separating the boundary of $\mathcal{M}_\infty^{(z)}$ from infinity. We have the following analog of Theorem 1 for the infinite volume Brownian disk:

Proposition 6. *Fix z a positive real number.*

(i) *We have*

$$\limsup_{u \rightarrow \infty} \frac{\log(\Theta_z(L_0 > u))}{u} \leq - \sup_{s > 1} \frac{1}{2(s-1)} \log\left(\frac{s^2}{2s-1}\right).$$

Consequently, $\Theta_z(L_0 > u)$ decreases at least exponentially fast when u goes to ∞ .

(ii) There exist two constants $0 < \tilde{c}_1 \leq \tilde{c}_2$ such that:

$$\forall \varepsilon \geq 0, \tilde{c}_1(1 \wedge \varepsilon^2) \leq \Theta_z(L_0 < \varepsilon) \leq \tilde{c}_2 \varepsilon^2.$$

Proof. By scaling, it is enough to consider $z = 1$. The spatial Markov property (Theorem 3) and again a scaling argument show that the distribution of $Z_1^{-\frac{1}{2}}L_1$ under Θ_0 coincides with the distribution of L_0 under Θ_1 and moreover $Z_1^{-\frac{1}{2}}L_1$ is independent of Z_1 under Θ_0 . We obtain:

$$\Theta_1(L_0 > u)\Theta_0(Z_1 > 1) = \Theta_0(L_1 > Z_1^{\frac{1}{2}}u, Z_1 > 1) \leq \Theta_0(L_1 > u).$$

Part (i) of the proposition then follows from Theorem 1 .

Let us prove (ii). The upper bound is a direct consequence of the beginning of the proof, noting that for every $\varepsilon \geq 0$:

$$\Theta_1(L_0 < \varepsilon)\Theta_0(Z_1 < 1) = \Theta_0(Z_1^{-\frac{1}{2}}L_1 < \varepsilon, Z_1 < 1) \leq \Theta_0(L_1 < \varepsilon)$$

so that, by Theorem 1 , if $\tilde{c}_2 := c_2/\Theta_0(Z_1 < 1) \in (0, \infty)$ we have:

$$\Theta_1(L_0 < \varepsilon) \leq \tilde{c}_2 \varepsilon^2 \tag{33}$$

for every $\varepsilon > 0$. We argue in a similar way to obtain the lower bound. Let $\varepsilon \leq 1$ and let $m_0 \geq 1$ be an integer. We have similarly:

$$\begin{aligned} \Theta_0(L_1 \leq \varepsilon) &= \Theta_0(Z_1^{\frac{1}{2}}Z_1^{-\frac{1}{2}}L_1 \leq \varepsilon) \\ &\leq \Theta_0(Z_1^{-\frac{1}{2}}L_1 \leq m_0\varepsilon, Z_1^{\frac{1}{2}} > \frac{1}{m_0}) + \sum_{m=m_0}^{\infty} \Theta_0\left(Z_1^{\frac{1}{2}} \in \left[\frac{1}{m+1}, \frac{1}{m}\right], Z_1^{-\frac{1}{2}}L_1 \leq (m+1)\varepsilon\right). \end{aligned}$$

Now we can use the first observations of the proof and (33) to get:

$$\begin{aligned} \Theta_0(L_1 \leq \varepsilon) &\leq \Theta_0(Z_1^{-\frac{1}{2}}L_1 \leq m_0\varepsilon) + \tilde{c}_2 \varepsilon^2 \sum_{m=m_0}^{\infty} (m+1)^2 \Theta_0(Z_1 \in [(m+1)^{-2}, m^{-2}]) \\ &= \Theta_1(L_0 \leq m_0\varepsilon) + \tilde{c}_2 \varepsilon^2 \sum_{m=m_0}^{\infty} (m+1)^2 \Theta_0(Z_1 \in [(m+1)^{-2}, m^{-2}]) \end{aligned}$$

Under Θ_0 , the density of Z_1 is $\frac{3^{\frac{3}{2}}}{\sqrt{2\pi}}\sqrt{x}\exp(-\frac{3}{2}x) dx$, so for every $0 < a < b$:

$$\Theta_0(Z_1 \in [a, b]) \leq \sqrt{\frac{6}{\pi}}(b^{\frac{3}{2}} - a^{\frac{3}{2}}).$$

Hence we can find a constant $c_3 > 0$, which does not depend on the choice of m_0 , such that

$$\Theta_0(L_1 \leq \varepsilon) \leq \Theta_1(L_0 \leq m_0\varepsilon) + c_3 \varepsilon^2 \sum_{m=m_0}^{\infty} \frac{1}{m^2}.$$

Then using Theorem 1 , we get for every $\varepsilon \in [0, 1]$:

$$(c_1 - c_3 \sum_{m=m_0}^{\infty} \frac{1}{m^2}) \varepsilon^2 \leq \Theta_1(L_0 \leq m_0 \varepsilon).$$

We obtain the lower bound in (ii) by choosing m_0 such that $\sum_{m=m_0}^{\infty} m^{-2} < \frac{c_1}{c_3}$. \square

Recall that $0 < \tilde{c}_1 < \tilde{c}_2$ are the constants appearing in Proposition 6. The end of this section is devoted to the proof of the following result which will be crucial for the proof of Theorem 2 (i). Before stating the result, observe that the definition (31) of T_r for $r > 0$ also makes sense under Θ_z by Corollary 2.

Proposition 7. *There exists $\tilde{c}_3 > 0$ such that, for every $r > 2\tilde{c}_2/\tilde{c}_1$ and $\varepsilon > 0$,*

$$\Theta_1(L_{0, T_{2r}} \leq \varepsilon) \geq \tilde{c}_3(1 \wedge \varepsilon^2).$$

The proof of Proposition 7 is based on the next lemma:

Lemma 8. *Let $z > 0$. Then, for every $A > z$ and $\beta < 3$ we have:*

$$\Theta_z(\sup_{[0, \varepsilon]} Z \geq A) = o(\varepsilon^\beta)$$

as $\varepsilon \rightarrow 0$.

Proof. By a scaling argument, it is enough to prove the lemma with $z = 1$. Fix $A > 1$. Introduce the stopping time $T := \inf\{t \geq 0 : \xi_t \geq \log(A)\}$ which is finite \mathbb{P} -a.s. By Corollary 2 for every $\varepsilon > 0$:

$$\Theta_1(\sup_{t \in [0, \varepsilon]} Z_t \geq A) = \mathbb{P}(T \leq \kappa(\varepsilon)) = \mathbb{P}(\int_0^T \exp(\frac{1}{2}\xi_r) dr \leq \varepsilon).$$

Let $\alpha \in (0, 1)$, we split $\mathbb{P}(\int_0^T \exp(\frac{1}{2}\xi_r) dr \leq \varepsilon)$ as follows:

$$\mathbb{P}(\int_0^T \exp(\frac{1}{2}\xi_r) dr \leq \varepsilon) \leq \mathbb{P}(T \leq \varepsilon^\alpha) + \mathbb{P}(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r) dr \leq \varepsilon) \quad (34)$$

and we study each term separately. We need to estimate $\mathbb{P}(T < \delta)$ for $\delta > 0$. As ξ is a Lévy process without positive jumps which drifts to ∞ , we have by standard properties of Lévy processes

$$\mathbb{E}[\exp(-\psi(\lambda)T)] = \exp(-\lambda \log(A))$$

for every $\lambda > 0$. See for example [3, Chapter VII] for a proof. Remark that there exists $c > 0$ such that, for every $\lambda > 1$, we have $\psi(\lambda) < c\lambda^{\frac{3}{2}}$ and that an application of Markov's inequality gives:

$$\mathbb{P}(T < \delta) = \mathbb{P}(-\psi(\lambda)T > -\psi(\lambda)\delta) \leq \exp(\psi(\lambda)\delta - \lambda \log(A)).$$

So taking $\lambda = \delta^{-\frac{2}{3}}$ in the previous bound we obtain:

$$\mathbb{P}(T < \delta) = O_{\delta \downarrow 0}(\exp(-\delta^{-\frac{2}{3}} \log(A))).$$

Consequently, for every $q > 0$, $\mathbb{P}(T < \delta) = o(\delta^q)$ as $\delta \downarrow 0$. Let us study the other term appearing in (34). Fix $\beta \in (0, 3)$. Again by using Markov's inequality we have:

$$\mathbb{P}\left(\int_0^{\varepsilon^\alpha} \exp\left(\frac{1}{2}\xi_r\right) dr \leq \varepsilon\right) \leq \varepsilon^\beta \mathbb{E}\left[\left(\int_0^{\varepsilon^\alpha} \exp\left(\frac{1}{2}\xi_r\right) dr\right)^{-\beta}\right].$$

But then an application of Jensen inequality gives

$$\begin{aligned} \mathbb{P}\left(\int_0^{\varepsilon^\alpha} \exp\left(\frac{1}{2}\xi_r\right) dr \leq \varepsilon\right) &\leq \varepsilon^{(1-\alpha)\beta-\alpha} \mathbb{E}\left[\int_0^{\varepsilon^\alpha} \exp\left(-\frac{\beta}{2}\xi_r\right) dr\right] \\ &= \varepsilon^{(1-\alpha)\beta-\alpha} \frac{\exp\left(\psi\left(-\frac{\beta}{2}\right)\varepsilon^\alpha\right) - 1}{\psi\left(-\frac{\beta}{2}\right)}. \end{aligned}$$

We obtain that $\mathbb{P}\left(\int_0^{\varepsilon^\alpha} \exp\left(\frac{1}{2}\xi_r\right) dr \leq \varepsilon\right) = O(\varepsilon^{(1-\alpha)\beta})$ as $\varepsilon \downarrow 0$. Since this is true for every $\beta \in (0, 3)$ and $\alpha \in (0, 1)$, the lemma follows. \square

Let us deduce Proposition 7 from Lemma 8.

Proof of Proposition 7. Fix $r > 2\tilde{c}_2/\tilde{c}_1 \geq 2$. Let γ be a path separating the boundary of $\mathcal{M}_\infty^{(1)}$ from infinity. If γ does not stay inside $B_{T_{2r}}^\bullet$ then it has to stay outside $B_{T_r}^\bullet$ or to connect $B_{T_r}^\bullet$ and $\check{B}_{T_{2r}}^\bullet$. Since the distance between $B_{T_r}^\bullet$ and $\check{B}_{T_{2r}}^\bullet$ is $T_{2r} - T_r$ we have:

$$L_0 \geq L_{0,T_{2r}} \wedge L_{T_r} \wedge (T_{2r} - T_r) \quad \Theta_1\text{-a.s.}$$

Consequently, for every $\varepsilon > 0$:

$$\Theta_1(L_0 \leq \varepsilon) \leq \Theta_1(L_{0,T_{2r}} \leq \varepsilon) + \Theta_1(L_{T_r} \leq \varepsilon) + \Theta_1(T_{2r} - T_r \leq \varepsilon).$$

By Theorem 4 and Corollary 1, the distribution of L_{T_r} under Θ_1 is the distribution of L_0 under Θ_r . Using a scaling argument, we obtain:

$$\Theta_1(L_{T_r} \leq \varepsilon) = \Theta_r(L_0 \leq \varepsilon) = \Theta_1\left(L_0 \leq \frac{\varepsilon}{\sqrt{r}}\right) \leq \tilde{c}_2 \frac{\varepsilon^2}{r}$$

and

$$\Theta_1(T_{2r} - T_r \leq \varepsilon) = \Theta_r(T_{2r} \leq \varepsilon) = \Theta_1\left(T_2 \leq \frac{\varepsilon}{\sqrt{r}}\right) = \Theta_1\left(\sup_{t \in [0, \frac{\varepsilon}{\sqrt{r}}]} Z_t \geq 2\right) = \underset{\varepsilon \downarrow 0}{o}(\varepsilon^2)$$

where the last equality comes from Lemma 8 (taking $A = 2$). We finally derive that:

$$\Theta_1(L_{0,T_{2r}} \leq \varepsilon) \geq \Theta_1(L_0 \leq \varepsilon) - \tilde{c}_2 \frac{\varepsilon^2}{r} + o(\varepsilon^2) \geq \tilde{c}_1(1 \wedge \varepsilon^2) - \tilde{c}_2 \frac{\varepsilon^2}{r} + o(\varepsilon^2)$$

where in the second line we use Proposition 6. Since $r > 2\tilde{c}_2/\tilde{c}_1$ we obtain the desired result. \square

4 Isoperimetric inequalities

4.1 Preliminary results on the volume of the hulls

This section is devoted to preliminary results about the volume of hulls. This will simplify some arguments in the derivation of Theorem 2. We are going to use the following result [9, Theorem 1.4]

$$\begin{aligned} \Theta_0(\exp(-\lambda|B_r^\bullet|) \mid Z_r = l) &= r^3 (2\lambda)^{\frac{3}{4}} \frac{\cosh((2\lambda)^{\frac{1}{4}} r)}{\sinh^3((2\lambda)^{\frac{1}{4}} r)} \\ &\quad \cdot \exp\left(-l\left(\sqrt{\frac{\lambda}{2}}(3 \coth^2((2\lambda)^{\frac{1}{4}} r) - 2) - \frac{3}{2r^2}\right)\right) \end{aligned} \quad (35)$$

for every $\lambda > 0$. In particular, using (13), we obtain that for every $\lambda \geq 0$:

$$\Theta_0(\exp(-\lambda|B_r^\bullet|)) = 3^{\frac{3}{2}} \cosh((2\lambda)^{\frac{1}{4}} r) (\cosh^2((2\lambda)^{\frac{1}{4}} r) + 2)^{-\frac{3}{2}} \quad (36)$$

(this formula also appears in [9]).

Corollary 3. *There exists a constant $C > 0$ such that for every $z > 0$ and $r > 0$:*

$$\Theta_z(|B_r^\bullet|) \leq C(r + \sqrt{z})^4.$$

Proof. Fix $z > 0$ and $r > 0$. First remark that, under $\Theta_0(\cdot \mid T_z \leq \sqrt{z})$, the hull $B_{T_z+r}^\bullet$ is contained in $B_{\sqrt{z}+r}^\bullet$. So an application of Corollary 1 gives:

$$\begin{aligned} \Theta_0(|B_{r+\sqrt{z}}^\bullet|) &\geq \Theta_0(|B_{r+\sqrt{z}}^\bullet| \mathbb{1}_{T_z < \sqrt{z}}) \\ &\geq \Theta_0((|B_{T_z+r}^\bullet| - |B_{T_z}^\bullet|) \mathbb{1}_{T_z < \sqrt{z}}) \\ &= \Theta_z(|B_r^\bullet|) \Theta_0(T_z < \sqrt{z}). \end{aligned}$$

On the other hand, we have $\{Z_{\sqrt{z}} > z\} \subset \{T_z \leq \sqrt{z}\}$. By scaling we also have $\Theta_0(Z_{\sqrt{z}} > z) = \Theta_0(Z_1 > 1) > 0$ and $\Theta_0(|B_{r+\sqrt{z}}^\bullet|) = (r + \sqrt{z})^4 \Theta_0(|B_1^\bullet|)$. Finally, it is easy to deduce from (36) that $\Theta_0(|B_1^\bullet|)$ is finite. This gives the statement of the corollary with $C = \Theta_0(|B_1^\bullet|)/\Theta_0(Z_1 > 1)$. \square

We now give two lemmas that will be useful to control the fluctuations of the volume of hulls in the Brownian plane.

Lemma 9. *For every $\beta \in \mathbb{R}$, we have $\Theta_0(|B_1^\bullet|^\beta) < \infty$ if and only if $\beta < \frac{3}{2}$.*

Proof. To simplify notation write $F(\lambda) := \Theta_0(\exp(-\lambda|B_1^\bullet|))$ for every $\lambda \geq 0$. Remark that for every $\lambda > 0$, we have $F''(\lambda) = \Theta_0(|B_1^\bullet|^2 \exp(-\lambda|B_1^\bullet|))$. If $\alpha > 0$, we have:

$$\int_{\mathbb{R}_+} \lambda^{\alpha-1} F''(\lambda) d\lambda = \Theta_0\left(|B_1^\bullet|^2 \int_{\mathbb{R}_+} \lambda^{\alpha-1} \exp(-\lambda|B_1^\bullet|) d\lambda\right) = \Gamma(\alpha) \Theta_0(|B_1^\bullet|^{2-\alpha}).$$

From the explicit expression of F given in (36) we get:

$$F''(\lambda) = \frac{1}{9\sqrt{2}} \lambda^{-\frac{1}{2}} + O(1)$$

as $\lambda \downarrow 0$, and

$$F''(\lambda) = O(\exp(-2(2\lambda)^{\frac{1}{4}}))$$

as $\lambda \uparrow \infty$. So for $\alpha = \frac{1}{2}$, $\Theta_0(|B_1^\bullet|^{\frac{3}{2}}) = \int_{\mathbb{R}_+} \lambda^{-\frac{1}{2}} F''(\lambda) d\lambda = \infty$ and for every $\alpha > \frac{1}{2}$, $\Theta_0(|B_1^\bullet|^{2-\alpha}) < \infty$. \square

We conclude this section with the following consequence of Lemma 9:

Lemma 10. *For every $\beta_1 > 0$ and $\beta_2 > 2/3$ we have Θ_0 -a.s.*

$$\inf_{r>0} \frac{|B_r^\bullet|}{r^4(1+|\log(r)|)^{-\beta_1}} > 0$$

and

$$\sup_{r>0} \frac{|B_r^\bullet|}{r^4(1+|\log(r)|)^{\beta_2}} < \infty.$$

Proof. Fix β_1 and β_2 as in the statement. By Lemma 9, the quantities $\Theta_0(|B_1^\bullet|^{-1/\beta_1})$ and $\Theta_0(|B_1^\bullet|^{1/\beta_2})$ are finite. This implies by the scaling invariance of \mathcal{M}_∞ :

$$\sum_{m \in \mathbb{Z}} \Theta_0(|B_{2^m}^\bullet|^{-1} > |m|^{\beta_1} 2^{-4m}) \leq 2 \sum_{m=0}^{\infty} \Theta_0(|B_1^\bullet|^{-\frac{1}{\beta_1}} > m) < \infty$$

and

$$\sum_{m \in \mathbb{Z}} \Theta_0(|B_{2^m}^\bullet| > |m|^{\beta_2} 2^{4m}) \leq 2 \sum_{m=0}^{\infty} \Theta_0(|B_1^\bullet|^{\frac{1}{\beta_2}} > m) < \infty.$$

The result then follows by the Borel-Cantelli lemma. \square

4.2 Proof of Part (i) of Theorem 2

The goal of this section is to prove the following slightly more precise form of Theorem 2 (ii).

Proposition 8. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ be a positive nondecreasing function such that $\sum_{m \in \mathbb{N}} f(m)^{-2} = \infty$. Then,*

$$\inf_{\substack{A \in \mathcal{K} \\ A \subset B_1^\bullet}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0, \quad \Theta_0\text{-a.s.} \quad (37)$$

and

$$\inf_{\substack{A \in \mathcal{K} \\ B_1^\bullet \subset A}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0, \quad \Theta_0\text{-a.s.} \quad (38)$$

Proof. Fix $r > 2\tilde{c}_2/\tilde{c}_1 \geq 2$, where \tilde{c}_1 and \tilde{c}_2 are as in Proposition 6, and let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ be a positive nondecreasing function such that $\sum_{m \in \mathbb{N}} f(m)^{-2} = \infty$. We give a detailed proof of the (38) since (37) can be obtained, *mutatis mutandis*, by the same method. Recall the notation $T_r := \inf\{t \geq 0 : Z_t \geq r\}$. Since Z does not have positive jumps, we have $Z_{T_r} = r$. For every $n \geq 1$, a separating cycle taking values in $B_{T_r, n, T_{r, n+1}}^\bullet$ bounds a Jordan domain $A \in \mathcal{K}$ such that $B_{T_r, n}^\bullet \subset A$ and for n large enough we also have $B_1^\bullet \subset A$. Hence,

$$\inf_{\substack{A \in \mathcal{K} \\ B_1^\bullet \subset A}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) \leq \liminf_{n \rightarrow \infty} \frac{L_{T_r, 2n+1, T_r, 2n+2}}{|B_{T_r, 2n+1}^\bullet|^{\frac{1}{4}}} f(\log(|B_{T_r, 2n+2}^\bullet|)), \quad \Theta_0\text{-a.s.}$$

So to obtain (38) it is enough to show that:

$$\liminf_{n \rightarrow \infty} \frac{L_{T_r, 2n+1, T_r, 2n+2}}{|B_{T_r, 2n+1}^\bullet|^{\frac{1}{4}}} f(\log(|B_{T_r, 2n+2}^\bullet|)) = 0, \quad \Theta_0\text{-a.s.} \quad (39)$$

Let us study the growth of the sequence $(T_{r^n})_{n \in \mathbb{N}}$. First note that:

$$\Theta_0(T_1 \geq u) \leq \Theta_0(Z_u \leq 1) = \Theta_0(Z_1 \leq u^{-2}) = \frac{3^{\frac{3}{2}}}{\sqrt{2\pi}} \int_0^{u^{-2}} \sqrt{x} \exp(-\frac{3}{2}x) dx \leq \sqrt{\frac{6}{\pi}} u^{-3}$$

where in the first equality we apply the scaling invariance of the Brownian plane and in the second one we use the density of Z_1 (see (13)). In particular, we have $\Theta_0(T_1) < \infty$. So by scaling invariance we obtain:

$$\sum_{n \in \mathbb{N}^*} \Theta_0(T_{r^n} > nr^{\frac{n}{2}}) = \sum_{n \in \mathbb{N}^*} \Theta_0(T_1 > n) < \infty.$$

The Borel-Cantelli lemma then implies that $\limsup_{n \rightarrow \infty} (nr^{n/2})^{-1} T_{r^n} \leq 1$, Θ_0 -a.s. Since $\lim_{s \rightarrow \infty} T_s = \infty$, Θ_0 -a.s., Lemma 10 gives:

$$\limsup_{n \rightarrow \infty} \frac{\log(|B_{T_{r^{2n+2}}}^\bullet|)}{4 \log(T_{r^{2n+2}})} \leq 1, \quad \Theta_0\text{-a.s.}$$

and then we deduce:

$$\limsup_{n \rightarrow \infty} \frac{\log(|B_{T_{r^{2n+2}}}^\bullet|)}{2n \log(r)} \leq 1, \quad \Theta_0\text{-a.s.}$$

Fix $h > 2 \log(r)$. As f is nondecreasing we have:

$$\liminf_{n \rightarrow \infty} \frac{L_{T_{r^{2n+1}}, T_{r^{2n+2}}}}{|B_{T_{r^{2n+2}}}^\bullet|^{\frac{1}{4}}} f(\log(|B_{T_{r^{2n+2}}}^\bullet|)) \leq \liminf_{n \rightarrow \infty} \frac{L_{T_{r^{2n+1}}, T_{r^{2n+2}}}}{|B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet|^{\frac{1}{4}}} f(hn). \quad (40)$$

We will use the Borel-Cantelli lemma to conclude. By Theorem 4, under Θ_0 :

$$\left(\frac{1}{r^{4n}} |B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet|, \frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}}, T_{r^{2n+2}}} \right)_{n \in \mathbb{N}}$$

is an i.i.d. sequence of random variables and for every $n \geq 0$ the variables:

$$\frac{1}{r^{4n}} |B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet|, \quad \text{and} \quad \frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}}, T_{r^{2n+2}}}$$

are independent. Moreover $\frac{1}{r^{4n}} |B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet|$ (resp. $\frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}}, T_{r^{2n+2}}}$) is distributed under Θ_0 as $B_{T_r}^\bullet$ (resp. L_{0, T_r}) under Θ_1 . Fix $\delta > 0$, such that $\Theta_1(B_{T_r}^\bullet > \delta) > 0$ and let $\varepsilon > 0$. By the previous remark we have:

$$\begin{aligned} \sum_{n=0}^{\infty} \Theta_0 \left(|B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet| > \delta r^{4n}, L_{T_{r^{2n+1}}, T_{r^{2n+2}}} < \frac{\varepsilon}{f(hn)} r^{n+\frac{1}{2}} \right) \\ = \sum_{n=0}^{\infty} \Theta_1(|B_{T_r}^\bullet| > \delta) \Theta_1(L_{0, T_r} < \frac{\varepsilon}{f(hn)}). \end{aligned}$$

By Proposition 7 the right-hand side of the last display is greater than

$$\tilde{c}_3 \Theta_1(|B_{T_r}^\bullet| > \delta) \sum_{n=0}^{\infty} \left(\frac{\varepsilon^2}{f(hn)^2} \wedge 1 \right)$$

which is infinite since $\sum_{m \in \mathbb{N}} f(m)^{-2} = \infty$. The Borel-Cantelli lemma then implies that:

$$\liminf_{n \rightarrow \infty} \frac{L_{T_{r^{2n+1}}, T_{r^{2n+2}}}}{|B_{T_{r^{2n}}, T_{r^{2n+1}}}^\bullet|^{\frac{1}{4}}} f(hn) \leq \varepsilon \delta^{-\frac{1}{4}} \sqrt{r}, \quad \Theta_0\text{-a.s.}$$

This holds for every $\varepsilon > 0$, which together with (40) gives (39). \square

4.3 Proof of Part (ii) of Theorem 2

We need to show that for any positive nondecreasing function f , the condition $\sum_{m \in \mathbb{N}} f(m)^{-2} < \infty$ implies

$$\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0, \Theta_0\text{-a.s.}$$

We begin with a technical lemma.

Lemma 11. *Let $\beta \in [0, 1)$. There exists a constant $C_\beta > 0$, which only depends on β , such that for every $r > 0$ and $\varepsilon > 0$:*

$$\Theta_0(\mathbb{1}_{L_1 < \varepsilon} |B_r^\bullet|^\beta) \leq C_\beta r^{4\beta} \varepsilon^2.$$

The reason for taking $\beta < 1$ is just technical and one can extend the result to $\beta < \frac{3}{2}$ but the proof will be more tedious. At an intuitive level, Lemma 11 states that if we know that there exists a small cycle separating the hull of radius 1 then the expected volume of the hull of radius r stays at most of order r^4 (with a uniform control).

Proof. Fix $\beta \in (0, 1)$ and let $r > 0$ and $\varepsilon > 0$.

To simplify notation, set $m := \frac{1}{\varepsilon}$, $q := \frac{1}{\beta}$ and $p := \frac{q}{q-1} = \frac{1}{1-\beta}$. By the scaling property of \mathcal{M}_∞ :

$$\Theta_0(\mathbb{1}_{L_1 < \varepsilon} |B_r^\bullet|^\beta) = \frac{1}{m^{4\beta}} \Theta_0(\mathbb{1}_{L_m < 1} |B_{mr}^\bullet|^\beta).$$

If $L_m < 1$, there is a separating cycle of length smaller than 1 that is contained in \check{B}_m° , and necessarily this separating cycle is contained in $B_{m+k, m+k+2}^\bullet$ for some integer $k \geq 0$. Hence,

$$\Theta_0(\mathbb{1}_{L_m < 1} |B_{mr}^\bullet|^\beta) \leq \sum_{k=0}^{\infty} \Theta_0(\mathbb{1}_{L_{m+k, m+k+2} < 1} |B_{mr}^\bullet|^\beta).$$

Applying the conditional version of the Hölder inequality with respect to Z_{m+k+4} we obtain:

$$\Theta_0(\mathbb{1}_{L_{m+k, m+k+2} < 1} |B_{mr}^\bullet|^\beta) \leq \Theta_0\left(\Theta_0(L_{m+k, m+k+2} < 1 \mid Z_{m+k+4})^{\frac{1}{p}} \Theta_0(|B_{mr}^\bullet| \mid Z_{m+k+4})^{\frac{1}{q}}\right).$$

By Proposition 5, there exists $\alpha_1 > 0$ such that

$$\Theta_0(L_{m+k, m+k+2} < 1 \mid Z_{m+k+4}) \leq e \cdot \exp(-\alpha_1 Z_{m+k+4})$$

for every $k \geq 0$ and thus we get :

$$\Theta_0(\mathbb{1}_{L_m < 1} |B_{mr}^\bullet|^\beta) \leq e \sum_{k=0}^{\infty} \Theta_0\left(\exp(-\alpha Z_{m+k+4}) \Theta_0(|B_{mr}^\bullet| \mid Z_{m+k+4})^{\frac{1}{q}}\right) \quad (41)$$

where $\alpha := \alpha_1/p$. Then again by the Hölder inequality:

$$\begin{aligned} & \Theta_0\left(\exp(-\alpha Z_{m+k+4}) \Theta_0(|B_{mr}^\bullet| \mid Z_{m+k+4})^{\frac{1}{q}}\right) \\ &= \Theta_0\left(\exp\left(-\frac{\alpha}{p} Z_{m+k+4}\right) \exp\left(-\frac{\alpha}{q} Z_{m+k+4}\right) \Theta_0(|B_{mr}^\bullet| \mid Z_{m+k+4})^{\frac{1}{q}}\right) \\ &\leq \Theta_0\left(\exp(-\alpha Z_{m+k+4})\right)^{\frac{1}{p}} \Theta_0\left(\exp(-\alpha Z_{m+k+4}) |B_{mr}^\bullet|\right)^{\frac{1}{q}}. \end{aligned} \quad (42)$$

By (13), Z_{m+k+4} follows the Gamma distribution with parameter $\frac{3}{2}$ and mean $(m+k+4)^2$. So we can find $d_1(\alpha) > 0$ independent of m and k such that:

$$\Theta_0(\exp(-\alpha Z_{m+k+4})) \leq \frac{d_1(\alpha)}{(m+k+4)^3}, \quad \Theta_0(Z_{m+k+4} \exp(-\alpha Z_{m+k+4})) \leq \frac{d_1(\alpha)}{(m+k+4)^3}. \quad (43)$$

Moreover, noting that $|\partial B_{m+k+4}^\bullet| = 0$, Θ_0 -a.s. , we obtain:

$$\begin{aligned} \Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{mr}^\bullet |) &= \Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{(m+k+4) \wedge mr}^\bullet |) \\ &\quad + \Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{m+k+4, mr}^\bullet |) \end{aligned}$$

where by convention $|B_{s_1, s_2}^\bullet| = 0$ if $s_2 \leq s_1$.

Let $0 < s_1 \leq s_2$. We observe that $|B_{s_1}^\bullet|$ is independent of Z_{s_2} conditionally on Z_{s_1} . This follows from the special Markov property and the spine independence property, using the fact that $|B_{s_1}^\bullet|$ is determined by the excursions of ω below s_1 for all atoms of \mathfrak{L} and \mathfrak{R} such that $t \geq \tau_{s_1}$, and by the atoms (t, ω) such that $t \leq \tau_{s_1}$. Thanks to this conditional independence property, we have

$$\Theta_0(|B_{s_1}^\bullet| | Z_{s_2}) = \Theta_0(\Theta_0(|B_{s_1}^\bullet| | Z_{s_1}) | Z_{s_2}). \quad (44)$$

By differentiating the right-hand side of (35) at $\lambda = 0$ we get

$$\Theta_0(|B_{s_1}^\bullet| | Z_{s_1}) = \frac{2}{15} s_1^4 + \frac{1}{5} s_1^2 Z_{s_1}$$

similarly we have from (14):

$$\Theta_0(Z_{s_1} | Z_{s_2}) = \frac{s_1^3}{s_2^3} Z_{s_2} + \frac{s_2 - s_1}{s_2} s_1^2.$$

So by (44), for every $0 < s_1 \leq s_2$

$$\Theta_0(|B_{s_1}^\bullet| | Z_{s_2}) = \frac{1}{3} s_1^4 - \frac{s_1^5}{5s_2} + \frac{s_1^5}{5s_2^3} Z_{s_2}.$$

Taking $s_1 = (m+k+4) \wedge (mr)$ and $s_2 = m+k+4$, we deduce from the last two formulas and (43) that there exists $d_2(\alpha) > 0$ independent of m and k such that:

$$\Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{(m+k+4) \wedge mr}^\bullet |) \leq \frac{d_2(\alpha)}{(m+k+4)^3} r^4 m^4.$$

Suppose that $m+k+4 < mr$. Then by the spatial Markov property and Corollary 3,

$$\begin{aligned} \Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{m+k+4, mr}^\bullet |) &= \Theta_0\left(\exp(-\alpha Z_{m+k+4}) \Theta_{Z_{m+k+4}}(|B_{mr-m-k-4}^\bullet|)\right) \\ &\leq C' \Theta_0(\exp(-\alpha Z_{m+k+4}) (m^4 r^4 + Z_{m+k+4}^2)). \end{aligned}$$

where $C' = 16C$, if C is the constant appearing in Corollary 3. Using again the distribution of Z_{m+k+4} , we get that there exists a constant $d_3(\alpha) > 0$ independent of m and k such that, if $m+k+4 < mr$,

$$\Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{m+k+4, mr}^\bullet |) \leq \frac{d_3(\alpha)}{(m+k+4)^3} r^4 m^4.$$

Summarizing, we get in both cases $m+k+4 < mr$ and $m+k+4 \geq mr$,

$$\Theta_0(\exp(-\alpha Z_{m+k+4}) | B_{mr}^\bullet |) \leq \frac{d_2(\alpha) + d_3(\alpha)}{(m+k+4)^3} r^4 m^4.$$

Coming back to (41) and (42), using (43) once again, and recalling that $q = 1/\beta$ and $m = 1/\varepsilon$,

we can find a constant $d(\alpha) > 0$ such that:

$$\begin{aligned}\Theta_0(\mathbb{1}_{L_1 < \varepsilon} |B_r^\bullet|^\beta) &= \frac{1}{m^{4\beta}} \Theta_0(\mathbb{1}_{L_m < 1} |B_{mr}^\bullet|^\beta) \\ &\leq \frac{c}{m^{4\beta}} \sum_{k=0}^{\infty} \Theta_0(\exp(-\alpha Z_{m+k+4}))^{\frac{1}{p}} \Theta_0(\exp(-\alpha Z_{m+k+4}) |B_{mr}^\bullet|)^{\frac{1}{q}} \\ &\leq \sum_{k=0}^{\infty} \frac{d(\alpha)}{(m+k+4)^3} r^{4\beta}\end{aligned}$$

and the lemma follows since $m = \varepsilon^{-1}$. \square

We now use Lemma 11 to prove that for any nondecreasing positive function f :

$$\sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} < \infty \implies \inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0, \Theta_0\text{-a.s.} \quad (45)$$

Proof. Fix a nondecreasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that $\sum_{m \in \mathbb{N}} f(m)^{-2} < \infty$. We begin by showing that for every $A \in \mathcal{K}$ we have:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) \geq \left(\inf_{m \in \mathbb{Z}} \frac{2^{m-1}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|) \right) \wedge \left(\inf_{m \in \mathbb{Z}} \frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|) \right). \quad (46)$$

Let $A \in \mathcal{K}$ and let m be the unique element of \mathbb{Z} such that:

$$|B_{2^m}^\bullet| < |A| \leq |B_{2^{m+1}}^\bullet|.$$

We divide the proof of (46) in two cases:

- Case 1: Assume that ∂A intersects $\partial B_{2^{m-1}}^\bullet$. As $|B_{2^m}^\bullet| < |A|$, there exists $x \in A \setminus B_{2^m}^\bullet$. Consider a path p_∞ connecting x to ∞ that does not hit $B_{2^m}^\bullet$ and let y be the last point of p_∞ that belongs to A . By construction we have $y \in \partial A$ and $y \notin B_{2^m}^\bullet$. Finally let $z \in \partial A \cap \partial B_{2^{m-1}}^\bullet$. Since ∂A connects y and z we have

$$\Delta(\partial A) \geq \Delta(y, z) \geq 2^{m-1}.$$

This implies that:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) \geq \frac{2^{m-1}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|).$$

- Case 2: Assume ∂A does not intersect $\partial B_{2^{m-1}}^\bullet$. Since $|A| > |B_{2^{m-1}}^\bullet|$, the set A is not contained in $B_{2^{m-1}}^\bullet$. This implies that ∂A separates $B_{2^{m-1}}^\bullet$ from infinity, and consequently $\Delta(\partial A) \geq L_{2^{m-1}}$. It follows that:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) \geq \frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|)$$

and this completes the proof of (46).

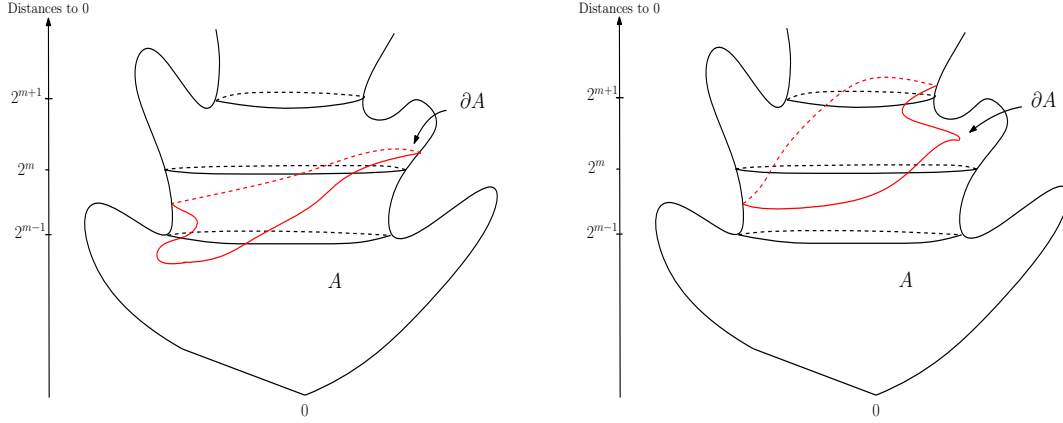


Figure 4: Illustration of (46). In red we represent the boundary of A . On the left we are in case 1 and we have $\Delta(\partial A) \geq 2^{m-1}$. On the right we are in case 2 and we have $\Delta(\partial A) \geq L_{2^{m-1}}$.

Thanks to (46), the proof of (45) will be complete if we can verify that:

$$\inf_{m \in \mathbb{Z}} \frac{2^{m-1}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|) > 0, \quad \Theta_0\text{-a.s.} \quad (47)$$

and

$$\inf_{m \in \mathbb{Z}} \frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|) > 0, \quad \Theta_0\text{-a.s.} \quad (48)$$

Let us start by proving (47). By Lemma 10, Θ_0 -a.s., there is a positive integer M such that for every $m \in \mathbb{Z}$ with $|m| \geq M$:

$$\frac{1}{|m|} 2^{4m} \leq |B_{2^m}^\bullet| \leq |m| 2^{4m} \quad (49)$$

In particular, we have:

$$\inf_{\substack{m \in \mathbb{Z} \\ |m| > M}} \frac{2^{m-1}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} f(|\log(|B_{2^m}^\bullet|)|) \geq \frac{1}{4} \inf_{\substack{m \in \mathbb{Z} \\ |m| > M}} \frac{f(|4 \log(2)m - \log(|m|)|)}{(|m| + 1)^{\frac{1}{4}}} \quad \Theta_0\text{-a.s.}$$

On the other hand, by the Cauchy-Schwarz inequality:

$$\frac{1}{f(n)} \leq \frac{1}{n} \sum_{k=1}^n \frac{1}{f(k)} \leq \frac{1}{n^{\frac{1}{2}}} \left(\sum_{k=1}^n \frac{1}{f(k)^2} \right)^{\frac{1}{2}} \leq \frac{1}{n^{\frac{1}{2}}} \left(\sum_{k=1}^{\infty} \frac{1}{f(k)^2} \right)^{\frac{1}{2}}.$$

Consequently $\inf_{n \in \mathbb{N}} n^{-\frac{1}{2}} f(n) > 0$ and we obtain (47). Actually here it will be enough to have $\inf_{n \in \mathbb{N}} n^{-\frac{1}{4}} f(n) > 0$.

Let us prove (48). By (49), if we can verify that

$$\sum_{m \in \mathbb{Z}} \Theta_0 \left(\frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)} \right) < \infty$$

we will conclude by an application of the Borel-Cantelli lemma. Fix $\beta \in (\frac{3}{4}, 1)$. For every $m \in \mathbb{Z}$, by scaling we have:

$$\Theta_0\left(\frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\right) = \Theta_0\left(\frac{L_1}{|B_4^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\right).$$

Consequently, we get :

$$\begin{aligned} \Theta_0\left(\frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\right) &\leq \Theta_0\left(L_1 < \frac{1}{f(|m|)}, |B_4^\bullet| \leq 1\right) \\ &\quad + \sum_{n=1}^{\infty} \Theta_0(|B_4^\bullet|^{\frac{1}{4}} \in [n, n+1], L_1 < \frac{n+1}{f(|m|)}). \end{aligned}$$

Now remark that the right term of the above display is bounded above by

$$\Theta_0\left(L_1 < \frac{1}{f(|m|)}\right) + \sum_{n=1}^{\infty} \Theta_0(|B_4^\bullet|^\beta \geq n^{4\beta}, L_1 < \frac{n+1}{f(|m|)}).$$

We deduce by an application of Markov inequality that

$$\Theta_0\left(\frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\right) \leq \Theta_0\left(L_1 < \frac{1}{f(|m|)}\right) + \sum_{n=1}^{\infty} \frac{1}{n^{4\beta}} \Theta_0(|B_4^\bullet|^\beta \mathbb{1}_{L_1 < \frac{n+1}{f(|m|)}})$$

Lemma 11 and Theorem 1 imply that there exist two constants $c_2 \in (0, \infty)$ and $C \in (0, \infty)$ such that:

$$\sum_{m \in \mathbb{Z}} \Theta_0\left(\frac{L_{2^{m-1}}}{|B_{2^{m+1}}^\bullet|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\right) \leq 2c_2 \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} + C \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^{4\beta}} \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

This completes the proof. \square

We observe that the same proof will work mutatis mutandis if we replace $|A|$ by $\Delta(\partial A)$ inside the logarithm in theorem 2.

Recall that $\mathcal{M}_\infty^{(z)}$ stands for an infinite volume Brownian disk with perimeter z . With a slight abuse of terminology, we call Jordan domain of $\mathcal{M}_\infty^{(z)}$ the closure of the bounded component of the complement of an injective cycle of $\mathcal{M}_\infty^{(z)}$. As a direct consequence of Corollary 1, Proposition 8 (i) and Theorem 2 we obtain:

Corollary 4.

Fix $z > 0$. Let $\mathcal{M}_\infty^{(z)}$ be the infinite volume Brownian disk with perimeter z defined under the probability measure Θ_z . Consider the collection $\mathcal{K}^{(z)}$ of all Jordan domains of $\mathcal{M}_\infty^{(z)}$ whose interior contains the boundary of $\mathcal{M}_\infty^{(z)}$. For any nondecreasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$

(i) We have

$$\inf_{A \in \mathcal{K}^{(z)}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0, \Theta_z\text{-a.s.}, \text{ if } \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} = \infty.$$

(ii) We have

$$\inf_{A \in \mathcal{K}^{(z)}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0, \Theta_z\text{-a.s.}, \text{ if } \sum_{m \in \mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

Appendix: Proof of Lemma 1

This appendix is devoted to the proof of Lemma 1, which relies on [18, Proposition 8]. We use the notation of Subsection 2.4.1.

PROOF OF LEMMA 1.

First fix $0 < r_1 < r_2 < \infty$. The lemma will follow if we prove that, Θ_0 -a.s. , for every $r_1 \leq r \leq r_2$, we have:

$$Z_r = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|.$$

In order to prove this, we introduce the event $A(s) := \{Z_{r_2}^{s,\infty} = 0\}$, for every $s > r_2$. In particular, under the event $A(s)$, we have $Z_r = Z_r^{r,s}$ for every $r \leq r_2$.

Moreover, by Proposition 1 (taking the limit when $t \rightarrow \infty$) we also have:

$$\Theta(A(s)) = \left(\frac{s-r}{s}\right)^3,$$

which converges to 1 when $s \rightarrow \infty$. Consequently, to obtain the desired result it is sufficient to show that, for every $s > r_2$, under $A(s)$ we have:

$$Z_r^{r,s} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|,$$

for every $r \in [r_1, r_2]$. Let us now introduce, for every $r \in \mathbb{R}$ and $\omega \in \mathcal{S}$ with $\omega_0 > r$, the quantity:

$$\mathcal{Z}_r^\varepsilon(\omega) := \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbb{1}_{\text{hit}_r(\omega_s)=\infty, \widehat{\omega}_s < r+\varepsilon}.$$

In particular, note that $Z_r(\omega) = \liminf_{\varepsilon \rightarrow 0} \mathcal{Z}_r^\varepsilon(\omega)$. We set:

$$Z_r^{r,s}(\varepsilon) := \int \mathcal{Z}_r^\varepsilon(\omega) \mathfrak{R}^{r,s}(d\ell d\omega) + \int \mathcal{Z}_r^\varepsilon(\omega) \mathfrak{L}^{r,s}(d\ell d\omega).$$

Under $A(s)$, all the labels appearing after the point τ_s of the spine are greater than r_2 . This implies that, under $A(s)$, the quantity $\varepsilon^2 Z_r^{r,s}(\varepsilon)$ is exactly $|\check{B}_r^\circ \cap B_{r+\varepsilon}|$ (since $|\cdot|$ is the pushforward of Lebesgue measure under $\Pi \circ \mathcal{E}$). To conclude, we are going to use [18, Proposition 8], which states that, for every $s > 0$ and $\beta > 0$, we have

$$\sup_{r \in (-\infty, s-\beta]} |\mathcal{Z}_r^\varepsilon - Z_r| \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \mathbb{N}_s\text{-a.e.} \quad (50)$$

To translate (50) in terms of $Z_r^{r,s}$ and $Z_r^{r,s}(\varepsilon)$, we recall that $(X_{(\tau_s-\ell)\vee 0})_{\ell \geq 0}$ is a Bessel process of dimension -5 started from s , and that, conditionally on $(X_{(\tau_s-\ell)\vee 0})_{\ell \geq 0}$, the measures $\mathfrak{R}^{0,s}$ and $\mathfrak{L}^{0,s}$ are two independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity:

$$2\mathbb{1}_{[0,\tau_s]}(\ell) d\ell \mathbb{N}_{X_\ell}(d\omega \cap \{\omega_* > 0\}).$$

In particular, the distribution $(X_{(\tau_s-\ell)\vee 0})_{0 \leq \ell \leq \tau_s - \tau_{r_1}}$ is absolute continuous with respect to the distribution of a Brownian motion started from s and stopped when it hits r_1 . We can now apply [17, Proposition 2] to deduce that the distribution of $(Z_r^{r,s}, Z_r^{r,s}(\varepsilon))_{r \in [r_1, r_2]}$ is absolute continuous with respect to the distribution $(Z_{r-r_1}, Z_{r-r_1}^\varepsilon)_{r \in [r_1, r_2]}$ under \mathbb{N}_{s-r_1} . Consequently (50) gives that:

$$\sup_{r \in [r_1, r_2]} |Z_r^{r,s}(\varepsilon) - Z_r^{r,s}| \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \Theta_0\text{-a.s.}$$

which completes the proof. \square

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