From Step Reinforced Random Walks to Noise Reinforced Lévy Processes

and

Excursion Theory for Markov Processes Indexed by Lévy Trees.

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The content of this thesis, written under the supervision of Jean Bertoin and Armand Riera from September 2020 to June 2023 at the University of Zürich, is composed by two independent and non-related parts. The first one is titled *From step reinforced random walks to noise reinforced Lévy processes* and falls into the broader setting of reinforcement of stochastic processes. It is composed by the two works [16, 84], the first one written in collaboration with Marco Bertenghi. The first work [16] has been published in *Journal of Statistical Physics*, while [84] has been accepted with revisions pending in *Electronic Journal of Probability.* The second part of this thesis is titled *Excursion theory for Markov processes indexed by Lévy trees*, and belongs to the broader framework of stochastic geometry. It is composed by the papers [82, 83], both written in collaboration with Armand Riera. The first work [82] has been accepted with revisions pending in *Probability Theory and Related Fields*, while [83] is still work in progress at an advanced stage.

[16] M. Bertenghi and A. Rosales-Ortiz. "Joint invariance principles for random walks with positively and negatively reinforced steps" (2022). *J. Stat. Phys*, 189.

[84] A. Rosales-Ortiz. "Noise reinforced Lévy processes: Lévy-Itô decomposition and applications" (2022). ArXiv: 2210.00564.

[82] A. Riera and A. Rosales-Ortiz. "The structure of the local time of Markov processes indexed by Lévy trees" (2022). ArXiv: 2205.04446.

[83] A. Riera and A. Rosales-Ortiz. "Excursion theory for Markov processes indexed by Lévy trees" (work in progress).

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Contents

Introduction

Chapter 1

Introduction to Part I

Reinforcement of stochastic processes has been a topic of active research for decades. Roughly speaking, one is interested in studying $\mathbb{N}$ or $\mathbb{R}_+$-indexed stochastic processes with memory, or in how introducing memory in a Markovian process affects its long range behaviour. This description is vague, and for instance memory can be introduced in multiple forms. Let us start by discussing some examples. Probably, the oldest family of process with reinforcement appearing in the literature are Pólya-Urn type processes. In short, one starts with an urn with a (possibly random) number of balls of different colours. Then, at each discrete time-step, one ball is drawn unifromly at random from the urn, and is replaced by a (possibly empty) collection of new balls. The composition of the balls that are added to the urn depend on the colour of the ball that was drawn, the replacement rule being fixed from the start. The evolution of the number of balls of each colour in the urn can be interpreted as a reinforcement process - note that despite the inherent memory of the model, it is a Markov chain. Next, we have the edge or vertex reinforced random walks on graphs [41, 78], where a discrete walk travels through the vertices of a graph and the probability of moving to an adjacent vertex depends on the number of visits by the walk to the latter, or to the connecting edge. In continuous time, we have for example the vertex-reinforced jump process [38], which consists in a continuous time process defined in a graph, which jumps from any state x to a neighbouring edge y at time t at a rate proportional to the occupation measure of state y up to time t. For a thorough overview of various reinforcement models we refer to the survey [79]. It is important to mention that a process modelling a dynamic where memory is involved might *a priori* still be Markovian, as it is the case in most of the examples we just mentioned. In this work, we shall be interested in yet another type of reinforcement called step reinforcement. Let us be more precise. We consider a $\mathbb{N}$-indexed process on the real numbers; at each time step and with some fixed probability p, the process chooses one of the preceding steps at random and repeats it, after applying a possibly random transformation to it. On the opposite case, the motion performs an independent step with some fixed law. The parameter p is often referred to as the memory or reinforcement parameter, and note that when $p = 0$, we simply have a random walk. In contrast, in general a step reinforced process is not a Markov process. We stress that in our setting, a past step is chosen *uniformly* at random, but other distributions - giving for instance higher probability to steps that have been recently performed or to the most ancient ones - have been addressed in related settings, see e.g. [49, 50]. Despite the fact that this reinforcement procedure is inherently discrete, we shall see how one can introduce these dynamics in the continuum through limit approximations. Let us mention that in contrast with the discrete setting, research on reinforcement of $\mathbb{R}_+$-indexed processes is, to this day, rather sparse. Let us now give a more precise overview on how this introduction is structured as well as the objects we shall be working with.

In this introduction, we shall give an overview of the main results of the works [16, 84] in the topic of reinforcement of stochastic processes, obtained under the supervision of Jean Bertoin and Armand Riera, and the first one in collaboration with Marco Bertenghi. In short, step reinforced random walks are a family of processes indexed by $\mathbb{N} = \{0, 1, 2, \dots\}$ that are obtained from random walks by preforming a reinforcement on its steps. At each fixed time, the step-reinforced random walk either, repeats with probability p one of the preceding steps chosen uniformly at random - after applying some possibly random transformation on the chosen step - or performs an independent step with fixed law μ with complementary probability $1 - p$. In this work, we shall start by investigating two classes of such processes, namely the so-called *noise reinforced random walk* and *counterbalanced random walk*. In [16], we obtain strong laws of large numbers and investigate the scaling limits of this two families of processes. In particular, we recover the results previously obtained in [21] for the noise-reinforced random walk by different methods. The reinforcement procedure is inherently discrete, and it is natural to wonder if in the continuum one can make sense of this notion. In this direction, in the work of Bertoin [19], it was established that the reinforced version of skeletons of Lévy processes converge, as the partition mesh tends to 0, towards a family of processes in the continuum baptised *noise reinforced Lévy processes*. In the second work [84] we investigate this family of time-indexed processes and strengthen the convergence results obtained in [19]. We shall see that noise reinforced Lévy processes share striking similarities with Lévy processes and notably, satisfy reinforced versions of the celebrated Lévy-Itô decomposition and Itô synthesis. Moreover, their jump measure is a point process that can be constructed from the jump measure of a Lévy process by a continuum analogue of the reinforcement algorithm we described for step-reinforced random walks. For this reason, we shall refer to this family of point measures as *noise reinforced Poisson point processes*. As we shall see, they play a central role in the development of [84].

The introduction of this first part is organised as follows: We start in Section 1.1 by introducing the family of $\mathbb{N}$-indexed processes that we shall be working with for the rest of this survey. Much of our analysis in both works [16, 84] relies on a remarkable martingale, and we include a brief discussion in order to give some background and general remarks on this process. We then present in Section 1.2 the results obtained in [16]. Sections 1.3 - 1.6 are devoted to the second work [84]. We start in Section 1.3 with some preliminary results on Yule-Simon processes and noise reinforced Lévy processes. We then introduce in Section 1.4 noise reinforced Poisson point processes and present the reinforced version of Lévy-Itô decomposition and Itô synthesis. In Section 1.5 we strengthen the convergence result obtain in [19, Theorem 3.1] and conclude in Section 1.6 with some applications.

1.1 Step reinforced random walks

Let us start by introducing the family of discrete processes that we shall be working with for the rest of this work.

The elephant random walk

The story begins with a process indexed by the non-negative integers, with memory and unitary increments known as *the elephant random walk* with memory parameter $q \in [0, 1]$. The elephant starts at time 1 in $\{-1, +1\}$ according to some fixed distribution and performs steps according to the following dynamics. At each $n \geqslant 2$ the elephant chooses a past step uniformly at random; with probability q this step is repeated in the n-th step, and with complementary probability $1-q$ the increment is repeated after changing its sign. In particular, when $q = 1/2$ the elephant follows a simple symmetric random walk. The name stems from the fact that, as the saying goes, elephants never forget. The study of this process has spiked the interest of probabilists in recent years, see e.g. [9, 11, 34, 33, 59] and references therein for general background, [13, 15] for results in multiple dimensions and [8, 12, 47, 60] for variations. One of the crucial features of the elephant random walk is that it presents a phase transition on the asymptotic of its fluctuations as $n \uparrow \infty$ at the critical parameter $q = 3/4$. Notably, when $q < 3/4$ the elephant random walk is diffusive, while for $q > 3/4$ it is super-diffusive. More precisely, for $q < 3/4$ when scaled by $\sqrt{n}$, the elephant random walk converges as $n \uparrow \infty$ in distribution to a centred Gaussian random variable. On the other hand for $q > 3/4$, when scaled by a factor of n^{2q-1}, the almost sure limit exists as $n \uparrow \infty$, the limit being a non-degenerate random variable. We refer to [9] for a detailed account. As we shall see, the scaling at the critical parameter $q = 3/4$ is of more complicated nature. In the first part of this work, we shall be interested in two natural processes generalising the dynamics of the elephant random walk. Namely, in the so-called noise-reinforced random walk and the counterbalanced random walk. Let us start by briefly introducing the former.

The noise reinforced random walk

Let $(X_n)_{n\in\mathbb{N}}$ be a collection of identically distributed random variables with same law as some fixed (non-degenerate) random variable $X \in L_2(\mathbb{P})$. We write $m := \mathbb{E}[X]$ and denote the variance of X by σ^2. We define the *noise-reinforced* version of the random walk $S_n = X_1 + \cdots + X_n$, $n \geqslant 1$ with $S_0 := 0$ by performing the following reinforcement procedure on its steps: first, we consider independent families of independent random variables $(U[n])_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ where for every n, $U[n]$ is uniformly distributed in $\{1,\dots,n\}$ while ε_n is distributed Bernoulli with parameter p. We define a family of random variables $(\hat{X}_n)_{n\in\mathbb{N}}$ by letting $\hat{X}_1 = X_1$ and for $n \geqslant 1$ we set recursively

$$\hat{X}_{n+1} = \begin{cases} X_{n+1} & \text{if } \varepsilon_{n+1} = 0, \\ \hat{X}_{U[n]} & \text{if } \varepsilon_{n+1} = 1. \end{cases}$$

We shall refer to the process $\hat{S}_n := \hat{X}_1 + \cdots + \hat{X}_n$ for $n \geqslant 1$, $\hat{S}_0 := 0$ as the *noise reinforced version* of (S_n) with reinforcement parameter $p \in [0,1]$. In other terms, for every $n \geqslant 2$ and with probability $1-p$, the step $\hat{X}_{n+1}$ is made of an independent copy with law X, and therefore shares the same increment as (S_n), while with complementary probability p one of the previous increments is chosen uniformly at random and repeated by the motion $(\hat{S}_n)$. In particular, if $p = 0$ we recover the random walk (S_n).

The connection with the elephant random walk is the following: as was noted by Kürsten [60], when X is distributed Rademacher, viz. $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$, then $(\hat{S}_n)$ is a version of the elephant random walk with parameter $q = (p+1)/2$, with first step distributed Rademacher. Observe that since $p \in [0,1]$, we can only cover the spectrum $q \in [1/2, 1]$.

The noise reinforced random walk has been subject of active research in recent years, see e.g. [21, 31, 18] to name a few, and its fluctuations present a phase transition at the critical parameter $p = 1/2$. Let us briefly give a non-exhaustive overview of some of its main properties. First, from the recursive reinforcement algorithm we infer that for any bounded measurable function

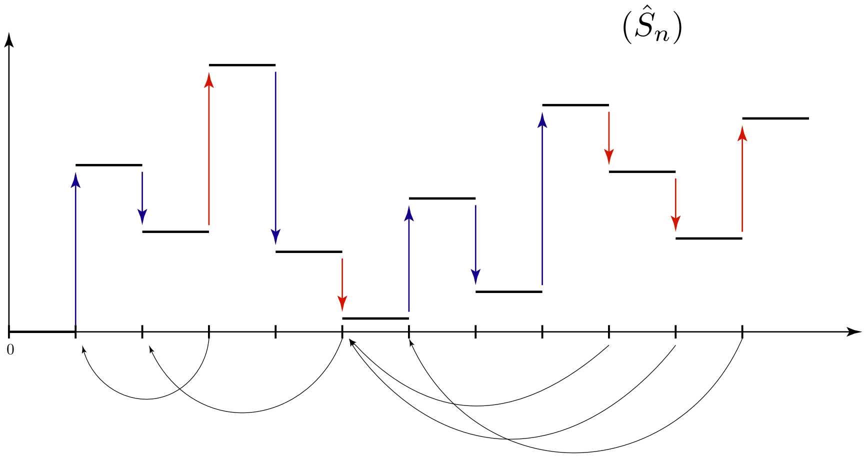


Figure 1.1: Sketch of a sample path of $(\widehat{S}_n)$. Steps marked in blue are innovations while in red are marked those issued from a reinforcement. The reinforced step is linked to the later by an arrow.

$f : \mathbb{R} \mapsto \mathbb{R}$ we have the recursion

$$\mathbb{E}[f(\hat{X}_{n+1})] = (1-p)\mathbb{E}[f(X_{n+1})] + \frac{p}{n}\sum_{j=1}^{n}\mathbb{E}[f(\hat{X}_j)].$$

Inductively, this gives that each $\hat{X}_n$ has same law as X and in particular we have that $\mathbb{E}[\hat{S}_n] = n\mathbb{E}[X]$ for $n \geqslant 0$. Beware however that the sequence $(\hat{X}_n)$ is not stationary. In [16] we establish that for any $p \in [0,1]$ the noise reinforced random walk is ballistic, and more precisely fulfils the following strong law of large numbers:

Theorem 1.1. (Law of large numbers) *Recalling the notation $m = \mathbb{E}[X]$, for any $p \in [0,1)$ we have the $L^2(\mathbb{P})$ and almost sure convergence:*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n} = m.$$

Note that when $p = 1$ the latter result can not hold, since for every $n \geqslant 1$ we have plainly $\hat{S}_n = nX_1$. As we already mentioned, the fluctuations however do present a phase transition at the critical parameter $p = 1/2$. Let us be more precise.

∘*Super-diffusive regime:* It was established in [21, Theorem 3.2] that when $p > 1/2$, the following limit holds in $L_2(\mathbb{P})$

$$\lim_{n\to\infty} \frac{\hat{S}_n - nm}{n^p} = L$$

where L is a non-degenerate random variable defined in terms of a martingale limit. The convergence was later proved to hold a.s. as well in [16]. Note that because of the strong nature of the convergence (and in contrast with the diffusive regime discussed below) it is of no interest to state a functional version of this convergence.

∘*Diffusive regime:* On the other hand, when $p \in (0,1/2)$ the fluctuations are always Gaussian and the scaling no longer depends on p. Namely, in [21] it was proved that for $p \in (0,1/2)$ the sequence of time-indexed processes $(\hat{S}_{\lfloor nt \rfloor})_{t\in\mathbb{R}_+}$ satisfies the following weak invariable principle:

$$\frac{\hat{S}_{\lfloor nt \rfloor} - \lfloor nt \rfloor m}{\sigma^2\sqrt{n}} \xrightarrow{\mathscr{L}} (\hat{B}_t)_{t\in\mathbb{R}_+} \tag{1.1}$$

the convergence holding weakly with respect to the Skorokhod topology, and where $\hat{B} := (\hat{B}_t)_{t\in\mathbb{R}_+}$ is a continuous centred Gaussian process with covariance structure

$$\mathbb{E}[\hat{B}_t\hat{B}_s] = \frac{t^p s^{1-p}}{1-2p} \quad \text{for } 0 \leqslant s \leqslant t \quad \text{and } p \in (0, 1/2).$$

We stress that the law of $\hat{B}$ does depend on the choice of p, and the dependence on the notation was dropped for sake of clarity.

The process $\hat{B}$ was baptised *the noise reinforced Brownian motion* with reinforcement parameter p. Observe that the law of $\hat{B}$ does not depend on the choice of X and therefore, the weak convergence result in (1.1) should be interpreted as the reinforced version of the classic Donsker invariance principle in our noise-reinforced setting. It readily follows from the identity in the last display that $\hat{B}$ admits the following representation in terms of a stochastic integral,

$$\hat{B}_t = t^p \int_0^t s^{-p} \mathrm{d}B^r_s, \quad t \geqslant 0,$$

where B^r is a standard Brownian motion. For a more detailed account on noise reinforced Brownian motion we refer to [21]. In the critical case $p = 1/2$ the fluctuations turn out to still be Gaussian, but do require of a different scaling. We shall come back to it in the sequel. The noise reinforced Brownian motion had already appeared as the scaling limit for diffusive regimes of the elephant random walk and other Polya urn related processes, see [9, 34], [15] for higher dimensional generalisations, as well as [7].

The counterbalanced random walk

We maintain the same notations as before and still assume that $X \in L_2(\mathbb{P})$. We stress that we shall use the same sequences of random variables $(X_n, \varepsilon_n, U[n])_{n\in\mathbb{N}}$ we used in the previous section to construct the noise reinforced version of (S_n). In the same vein as before, we define a sequence of random variables $(\check{X}_n)$ as follows: we set $\check{X}_1 = X_1$, and recursively for $n \geqslant 1$ we let

$$\check{X}_{n+1} = \begin{cases} X_{n+1} & \text{for } \varepsilon_{n+1} = 0, \\ -\check{X}_{U[n]} & \text{for } \varepsilon_{n+1} = 1. \end{cases}$$

The process $\check{S}_n := \check{X}_1 + \cdots + \check{X}_n$ for $n \geqslant 1$, $\check{S}_n := 0$, shall be referred to as the *counterbalanced random walk* with reinforcement parameter p. This process was recently introduced and studied by Bertoin in [22]. The name stems from the fact that in contrast with the noise reinforced random walk, when a past increment is repeated, its sign is changed and therefore compensates the chosen step. One of the motivations behind the definition of this process comes from the fact that if X is distributed Rademacher, the counterbalanced random walk with reinforcement parameter p is a version of the elephant random walk with memory parameter $q = (1-p)/2 \in [0, 1/2]$, with first step distributed Rademacher.

In contrast with the noise reinforced case, it no longer holds that the sequence $(\hat{X}_n)_{n\in\mathbb{N}}$ is identically distributed. However, since for $n \geqslant 1$ we have the recursion

$$\mathbb{E}[\check{S}_{n+1}] = (1-p)m + (1-p/n)\mathbb{E}[\check{S}_n], \qquad n \geqslant 1$$

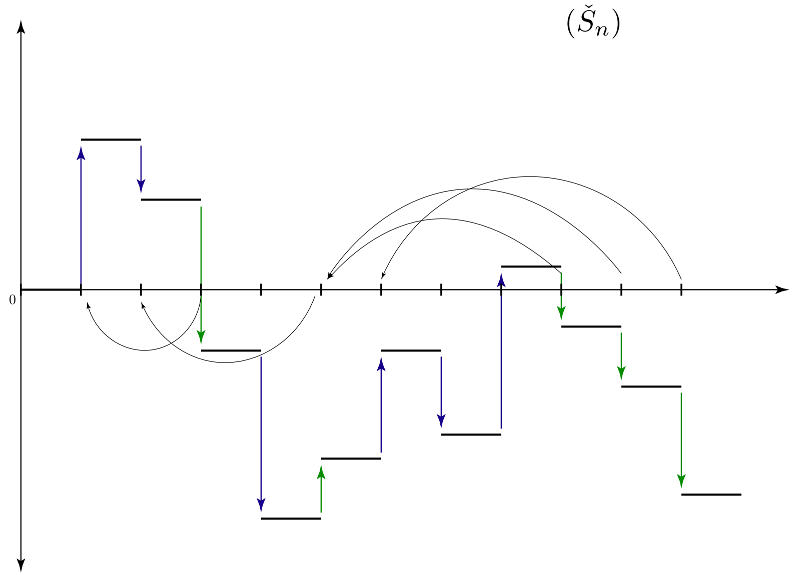


Figure 1.2: Sketch of a sample path of $(\check{S}_n)$. Steps marked in blue are innovations while in green are marked those issued from a counterbalanced. The counterbalanced step is linked to the later by an arrow. The same sample of random variables used in Figure 1.1 to construct $(\hat{S}_n)$ was used to construct $(\check{S}_n)$.

with initial condition $\mathbb{E}[\check{S}_1] = \mathbb{E}[\check{X}_1] = m$, it readily follows that $(\mathbb{E}[\check{S}_n] : n \geqslant 1)$ is still ballistic as $n \uparrow \infty$ and more precisely

$$\mathbb{E}[\check{S}_n] \sim \frac{1-p}{1+p} mn, \qquad \text{as } n \uparrow \infty.$$

It was established in [22, Proposition 1.1] that $(\check{S}_n)$, under the weaker assumption that $X \in L_1(\mathbb{P})$, satisfies a weak law of large numbers. We later proved in [16] by different methods that under our standing assumption $X \in L_2(\mathbb{P})$, the result can be strengthened:

Theorem 1.2. (Law of large numbers) *For any $p \in [0,1]$, we have the $L^2(\mathbb{P})$ and almost sure convergences:*

$$\lim_{n\to\infty} \frac{\check{S}_n}{n} = \frac{1-p}{1+p} m.$$

In contrast with the noise-reinforced case, for every $p \in [0,1)$ the fluctuations are always Gaussian. Namely, if we let $m_2 := \mathbb{E}[X^2]$, it was established in [22, Theorem 1.2] that the following weak convergence holds towards a centred Gaussian random variable:

$$\lim_{n\to\infty} \frac{\check{S}_n - \frac{1-p}{1+p} mn}{\sqrt{n}} \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, \frac{m_2 - \left(\frac{1-p}{1+p} m\right)^2}{1+2p}\right).$$

Observe that when $p = 1$, we have plainly $\check{S}_n = X_1 \check{S}'_n$ for $n \geqslant 1$ where $(\check{S}'_n)$ is a counterbalanced random walk with typical step distributed δ_1. If we further assume that the law of X is δ_1, [22, Corollary 2.4] yields that the convergence in the previous display still holds.

In the same spirit as before, for every $p \in [0,1)$ we let $\check{B} = (\check{B}_t)_{t \in \mathbb{R}_+}$ be the continuous centred Gaussian process with covariance structure given by

$$\mathbb{E}[\check{B}_t \check{B}_s] = \frac{1}{2p+1} \frac{s^{1+p}}{t^p}, \qquad \text{for } 0 \leqslant s \leqslant t. \tag{1.2}$$

We shall refer to $\check{B}$ as the *counterbalanced Brownian motion* with reinforcement parameter $p \in [0,1)$. Our terminology will be justified by the result of Section 1.2. It readily follows from (1.2) that the law of $\check{B}$ admits as well the following integral representation

$$\check{B}_t = t^{-p} \int_0^t s^p \mathrm{d}B^c_s, \qquad t \geqslant 0$$

in terms of a standard Brownian motion $B^c = (B^c_s)_{s\geqslant 0}$.

On a remarkable martingale

The results of both works [16, 84] rely on a martingale present both in the discrete setting and in the continuum. We think therefore worthwhile to explain its origins, its connections with branching processes and recall some of its basic properties for later use.

We start by defining two sequences $(\widehat{a}_n, n \geqslant 1)$, $(\check{a}_n, n \geqslant 1)$ of real numbers as follows: Let $\widehat{a}_1 = \check{a}_1 = 1$ and for every $n \in \{2, 3, \dots\}$, we set

$$\widehat{a}_n = \frac{\Gamma(n)}{\Gamma(n+p)}, \qquad \text{and} \qquad \check{a}_n = \frac{\Gamma(n)}{\Gamma(n-p)}$$

where the notation Γ stands for the standard Gamma function. It readily follows from our definitions that if X is centred or $p = 1$, the pair of processes $\hat{M} = (\widehat{a}_n \hat{S}_n : n \in \mathbb{N}^*)$ and $\check{M} = (\check{a}_n \check{S}_n : n \in \mathbb{N}^*)$ are square-integrable martingales. Further their respective predictable quadratic variation processes $\langle \hat{M} \rangle$, $\langle \check{M} \rangle$ are defined for all $n \geqslant 1$ by the relations

$$\langle \hat{M} \rangle_n = \sigma^2 + \sum_{k=2}^{n} \widehat{a}_k^2 \left((1-p)\sigma^2 - p^2 \left(\frac{\hat{S}_{k-1}}{k-1} \right)^2 + p \frac{\hat{V}_{k-1}}{k-1} \right) \tag{1.3}$$

and

$$\langle \check{M} \rangle_n = \sigma^2 + \sum_{k=2}^{n} \check{a}_k^2 \left((1-p)\sigma^2 - p^2 \left(\frac{\check{S}_{k-1}}{k-1} \right)^2 + p \frac{\hat{V}_{k-1}}{k-1} \right), \tag{1.4}$$

where $(\hat{V}_n)_{n\geqslant 1}$ is the process defined for every $n \geqslant 1$ as $\hat{V}_n = \hat{X}_1^2 + \cdots + \hat{X}_n^2$. This martingale had already made its appearance in the literature in different forms. In the setting of the elephant random walk, it was already at the heart of the analysis performed by Bercu in [11] and it was exploited as well in latter related works, see e.g. [13, 14]. Further, both $\hat{M}$ and $\check{M}$ have continuous time analogues that can be conjectured from the asymptotic behaviours $\widehat{a}_n \sim n^{-p}$, $\check{a}_n \sim n^p$ and the invariance principle (1.1). Namely, the processes $\hat{N} := (t^{-p} \hat{B}_t)_{t>0}$, $\check{N} := (t^p \check{B}_t)_{t\geqslant 0}$ are still martingales. Let us now briefly discuss the origins of $(\hat{M}, \check{M})$.

The following remark is from Bertoin [21] and our definitions are taken from [24]. We write $M_p(\mathbb{R})$ for the space of finite atomic measures in $\mathbb{R}$. We consider $\mathbf{Z} = (\mathbf{Z}_t(\mathrm{d}x))_{t\in\mathbb{R}_+}$ a $M_p(\mathbb{R}_+)$-valued particle system governed by the following dynamics: $\mathbf{Z}$ starts at time $t = 0$ with a collection of static particles $(x_1, \dots x_k) \in \mathbb{R}^k$ for some $k \geqslant 1$. Then, every particle dies at rate 1 and is replaced by a copy of itself and, either with an independent random variable with law X with probability $1-p$, or with a second copy of itself with probability p. In particular, the process keeping track of the number of particles alive at every time t, that we denote by $(\langle \mathbf{Z}_t, 1 \rangle : t \geqslant 0)$, is a standard Yule process.

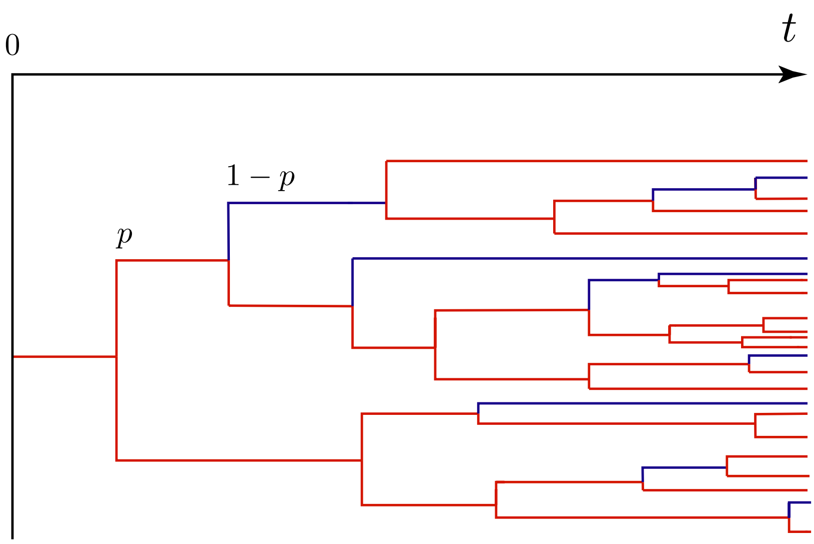


Figure 1.3: Every horizontal line represent the lifetime of a particle. At death, each particle is replaced by a copy of itself (in red) and, a second copy of itself is added (in red) with probability p while with probability $1-p$ an innovation is introduced (in blue).

Formally, $\mathbf{Z}$ is the time-continuous Feller process with infinitesimal generator defined at every $\mathbf{x} = \sum_{i=1}^{k} \delta_{x_i}$ by the relation:

$$\mathcal{A}f(\mathbf{x}) = \sum_{j=1}^{k} \int_{M_p(\mathbb{R}_+)} \big(\varphi(\mathbf{x}_j^* + \mathbf{y}) - \varphi(\mathbf{x})\big)\widehat{\Pi}(x_j, \mathrm{d}\mathbf{y}), \tag{1.5}$$

where $\mathbf{x}_j^* := \sum_{i \neq j} \delta_{x_i}$, while $\widehat{\Pi}(x, \mathrm{d}\mathbf{y})$ is the law of the random measure in $M_p(\mathbb{R}_+)$ defined as $(1 + 1_{\{\varepsilon_1 = 1\}})\delta_x + 1_{\{\varepsilon_1 = 0\}}\delta_X$. In particular $\mathbf{Z}$ is a so-called *Uchiyama's* process [24, Lemma 2.1]. If we write $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ for the identity function and for $n \geqslant 1$ we set $T_n := \inf\{t \geqslant 0 : \langle \mathbf{Z}_t, 1\rangle = n\}$, the key relation with our setting is that the process $(\langle \mathbf{Z}_{T_n}, \mathrm{Id}\rangle : n \geqslant 0)$ started from a single particle with law X, is a version of the noise reinforced random walk $(\hat{S}_n : n \geqslant 1)$ with parameter p and typical step distributed as X. Further, we have the identity in distribution

$$(\langle \mathbf{Z}_t, \mathrm{Id}\rangle : t \geqslant 0) \overset{(d)}{=} (\hat{S}_{Z_t} : t \geqslant 0)$$

where $Z = (Z_t : t \geqslant 0)$ is an independent standard Yule process. When X is centred or $p = 1$, it readily follows that the function $\mu \mapsto \langle \mu, \mathrm{Id}\rangle$ is an eigenfunction for the generator $\mathcal{A}$ with eigenvalue p, as can be checked from (1.5). It now follows from classic theory of Feller processes that the process

$$\hat{S}_{Z_t} - p\int_0^t \hat{S}_{Z_s}\,\mathrm{d}s, \qquad \text{for } t \geqslant 0,$$

is a martingale, and an integration by parts gives that $M_t = e^{-pt}\hat{S}_{Z_t}$ for $t \geqslant 0$ is on its turn a martingale. More precisely, recalling that $Z_t \sim e^t$ as $t \uparrow \infty$ as well as the asymptotic behaviour $\widehat{a}_n \sim n^{-p}$ as $n \uparrow \infty$, the discrete martingale $\hat{M}$ can be thought as M observed at a logarithmic time-scale. If one wishes to perform the analogous analysis for the counterbalanced random walk, it suffices to consider in (1.5) instead of $\widehat{\Pi}(x, \mathrm{d}\mathbf{y})$, the kernel $\check{\Pi}(x, \mathrm{d}\mathbf{y})$ defined for every $x \in \mathbb{R}$ as the law of $\delta_x + 1_{\{\varepsilon_1 = 1\}}\delta_{-x} + 1_{\{\varepsilon_1 = 0\}}\delta_X$.

1.2 The invariance principles

Making use of the martingales introduced in the previous section, we shall now investigate the fluctuations of $(\hat{S}_n)$ when $p \in [0, 1/2]$ and of $(\check{S}_n)$ for $p \in [0, 1]$. This will be achieved by establishing respective invariance principles. Recall that the processes $(S_n, \hat{S}_n, \check{S}_n)$ are coupled by construction, and therefore it is natural to investigate the joint scaling limit of the triplet. In this direction, we have the following result:

Theorem 1.3. *Fix $p \in [0, 1/2)$ and consider the triplet $(S_n, \hat{S}_n, \check{S}_n)$ consisting of the random walk (S_n) with its reinforced version and its counterbalanced version of parameter p. Assume further that X is centred. Then, the following weak convergence holds in the sense of Skorokhod as n tends to infinity,*

$$\left(\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}, \frac{1}{\sigma\sqrt{n}} \hat{S}_{\lfloor nt \rfloor}, \frac{1}{\sigma\sqrt{n}} \check{S}_{\lfloor nt \rfloor}\right)_{t \in \mathbb{R}_+} \xrightarrow{\mathscr{L}} \left(B_t, \hat{B}_t, \check{B}_t\right)_{t \in \mathbb{R}_+} \tag{1.6}$$

where $\left(B, \hat{B}, \check{B}\right)$ is a Gaussian process and B, $\hat{B}$, $\check{B}$ denote respectively a standard BM, a noise reinforced BM and a counterbalanced BM with covariances, $\mathbb{E}(B_s \check{B}_t) = t^{-p}(t \wedge s)^{p+1}(1-p)/(1+p)$, $\mathbb{E}(B_s \hat{B}_t) = t^p (t \wedge s)^{1-p}$, $\mathbb{E}(\hat{B}_t \check{B}_s) = t^p s^{-p}(t \wedge s)(1-p)/(1+p)$.

The restriction $p \in [0, 1/2)$ stems from the fact that as we already mentioned, the fluctuations of $(\hat{S}_n)$ are no longer Gaussian for $p \in (1/2, 1]$, while the scaling at the critical parameter $p = 1/2$ changes drastically - see Theorem 1.5 below. In contrast, the ones of $(\check{S}_n)$ are still Gaussian in those regimes and in [16] we established as well the corresponding scaling limits. To keep this presentation concise we skip the precise statement. The idea behind the proof of Theorem 1.6 is to establish instead the convergence of the martingales

$$\Big(\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}, \frac{\widehat{a}_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \check{S}_{\lfloor nt \rfloor}, \frac{\check{a}_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \hat{S}_{\lfloor nt \rfloor}\Big) \qquad \text{for } t \geqslant 0,$$

towards the triple $(B_t, t^{-p}\hat{B}_t, t^p \check{B}_t)$ for $t \geqslant 0$, by exploiting the explicit form of the quadratic variations (1.3), (1.4) in conjunction with the following martingale functional limit theorem taken from [54].

Theorem 1.4. [54, VIII-Theorem 3.11] *Assume $\boldsymbol{M} = (M^1, \ldots, M^d)$ is d-dimensional continuous Gaussian martingale with independent increments, and predicable covariance process $(\langle M^i, M^j \rangle)_{i,j \in \{1,\ldots,d\}}$. For each n, let $\boldsymbol{M}^n = (M^{n,1}, \ldots, M^{n,d})$ be a d-dimentional local martingale with uniformly bounded jumps $|\Delta \boldsymbol{M}^n| \leqslant K$ for some constant K. The following conditions are equivalent:*

(i) $\boldsymbol{M}^n \xrightarrow{\mathscr{L}} \boldsymbol{M}$ *in the sense of Skorokhod,*

(ii) *There exists some dense set $D \subset \mathbb{R}_+$ such that for each $t \in D$ and $i, j \in \{1, \ldots, d\}$, as $n \uparrow \infty$,*

$$\langle M^{n,i}, M^{n,j} \rangle_t \to \langle M^i, M^j \rangle_t \qquad \textit{in probability,}$$

and

$$\sup_{s \leqslant t} |\Delta \boldsymbol{M}^n_s| \to 0 \qquad \textit{in probability.}$$

The boundedness hypothesis on the jumps can be circumvented by a truncation argument. Observe that in Theorem 1.3 we assumed that X is centred. In the noise-reinforced case, this can be assumed without loss of generality since $\hat{S}_n - n\mathbb{E}(X)$ for $n \geqslant 0$ is still a noise reinforced random walk with typical step distributed $X - \mathbb{E}(X)$, but this is no longer the case for the counterbalanced random walk. However, one can expect that this restriction can be lifted by working with the martingale $\check{a}_n(\check{S}_n - \mathbb{E}(\check{S}_n))$ for $n \geqslant 0$ and by performing a straightforward adaptation of our arguments.

Lastly, our method extends as well to the critical regime $p = 1/2$, in which case we have the following:

Theorem 1.5. *Let $p = 1/2$ and suppose that $X \in L^2(\mathbb{P})$. Then, we have the weak convergence of the sequence of processes in the sense of Skorokhod as n tends to infinity*

$$\left(\frac{\hat{S}_{\lfloor n^t \rfloor} - n^t \mathbb{E}(X)}{\sigma \sqrt{\log(n)} n^{t/2}} \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathscr{L}} (B_t)_{t \in \mathbb{R}_+}$$

where $B = (B_t)_{t \geqslant 0}$ denotes a standard Brownian motion.

1.3 Noise reinforcement for Lévy processes

In order to motivate the upcoming sections let us start with an informal discussion. In the last section, we introduced the notion of a noise reinforced Brownian motion with reinforcement parameter p. It is therefore natural to ask if more generally, one can make sense of the notion of a *noise reinforced Lévy process* with reinforcement parameter p. In the following sections we shall see that this is indeed the case as long as we impose some restriction on the reinforcement parameter p - note that this was already the case for noise reinforced Brownian motion. Since noise reinforcement is an inherently discrete procedure, one needs to define such a process by a limiting procedure. More precisely, a noise reinforced Lévy processes is obtained by noise-reinforcing the n-skeleton of a Lévy process for a mesh of size $1/n$, and letting $n \uparrow \infty$. In the rest of this introduction we shall give an overview of the main results obtained in [84] concerning the study of noise reinforced Lévy processes. To this end, we shall start by recalling the main results of the seminal work [19] where this family of processes was introduced. We stress that in the work [84] we did not addressed other types of reinforcement and for instance we shall no longer work with the counterbalanced random walk.

The definition and study of noise reinforced Lévy processes is closely related to a family of heavy tailed distributions on the positive integers called the Yule-Simon distribution. We shall start by briefly introducing the later as well as its functional version.

From the Yule-Simon distribution to the Yule-Simon process

Imagine one writes a book recursively at random by iterating the following rule: we start by introducing a word and then, recursively, at each step and with some fixed probability, we either introduce a brand new word or we repeat one of the former ones. In the work [90], Simon was interested in studying the asymptotic frequencies of the number of words that had appeared exactly k times up to time n, say $\nu_k(n)$, as $n \uparrow \infty$. This model is of course closely related to the reinforcement algorithm we have introduced. Namely, in the setting of the noise-reinforced random walk, for every $1 \leqslant k \leqslant n$ the variable $\nu_k(n)$ counts the number of steps among $(X_j)_{j\in\mathbb{N}}$ that have been repeated exactly k times up to time n. It was established in [22, Lemma 3.1] the following convergence in probability of the asymptotic frequencies:

$$\lim_{n\to\infty} \frac{\nu_k(n)}{n} = \frac{(1-p)}{p} \mathrm{B}(k, 1+1/p) \tag{1.7}$$

where B stands for the Beta function. Now, the distribution on $\{1, 2, \dots\}$ defined by

$$p^{-1}\mathrm{B}(k, 1+1/p), \qquad \text{for } k \geqslant 1$$

is the so-called Yule-Simon distribution of parameter $1/p$. This result was greatly generalised in [19, Proposition 3.3] by proving a functional version of this convergence towards a counting process named the *Yule-Simon process*. Namely, if we consider a uniform random variable U in $[0,1]$ and a standard Yule process $Z = (Z(t))_{t\in\mathbb{R}_+}$, the Yule-Simon process $Y = (Y_t : t \geqslant 0)$ of parameter $1/p$ is the counting process defined by the relation:

$$Y_t = 1_{\{U\leqslant t\}} Z\big(p(\log(t) - \log(U))\big), \qquad \text{for } t \in [0,1].$$

Since the functional version established in [19, Proposition 3.3] will be key for the development of our theory, we shall now present the general statement. In order to explain the intricate relation between the Yule-Simon distribution, the Yule-Simon process and noise reinforced Lévy processes, for later use we sketch the main ideas behind its proof.

In this direction, for every $j \geqslant 1$ let us write $N_j(n)$ for the number of repetition up to time n of the variable X_j by the reinforcement algorithm. In particular, if $\varepsilon_j = 1$ we have plainly $N_j(n) = 0$ for every $n \geqslant 1$. Note that in particular, the noise reinforced random walk $(\hat{S}_n)$ can now be written in terms of this family of counting processes as follows:

$$\hat{S}_n = \sum_{j=1}^{n} N_j(n) X_j, \qquad \text{for } n \geqslant 1. \tag{1.8}$$

Now, [19, Proposition 3.3] states that for every continuous functional F on the Skorokhod space $\mathbb{D}([0,1],\mathbb{R})$ vanishing on the identically 0 trajectory, the following convergence holds in $L_2(\mathbb{P})$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F\big(N_j(\lfloor n\cdot \rfloor)\big) = (1-p)\mathbb{E}[F(Y)] \tag{1.9}$$

where Y is a Yule Simon process of parameter $1/p$. The convergence (1.7) can be recovered from the result (1.9) by considering the functional $F(\omega) = 1_{\{\omega_1 = k\}}\omega_1$ for $\omega \in \mathbb{D}([0,1],\mathbb{R})$. In particular, Y_1 is distributed Yule-Simon of parameter $1/p$, hence the name. The key behind (1.9) stems from the limiting behaviour of the scaled counting processes $(N_j(\lfloor nt \rfloor) : t \in [0,1])_{j\geqslant 1}$. Namely, if for every $n \geqslant 1$ we let $U[n]$ be an independent random variable uniformly distributed in $\{1, 2, \dots, n\}$, conditionally on $\varepsilon_{U[n]} = 0$, by [19, Lemma 3.6] we have the following weak convergence in the sense of Skorokhod:

$$\Big(N_{U[n]}\big(\lfloor nt \rfloor\big) : t \in [0,1]\Big) \xrightarrow{\mathscr{L}} \big(Y_t : t \in [0,1]\big).$$

Observe that the first jump of the process in the last display is uniformly distributed in $\{1, \dots, n\}$. Roughly speaking, in the continuum the first jump of Y - which is uniformly distributed in $[0,1]$

- corresponds to the time at which an innovation occurs and the subsequent jumps are repetitions of the latter. The second ingredient needed to define a noise reinforced Lévy process is, of course, a Lévy processes. We shall now briefly recall the main notions needed for our presentation.

Preliminaries on Lévy processes

Let ξ be a Lévy process and write Ψ for its characteristic exponent, viz. the function defined by the relation $\mathbb{E}[\exp(i\lambda\xi_t)] = \exp(t\psi(\lambda))$ for $\lambda \in \mathbb{R}$. Then, it is classic that Ψ is of Lévy-Khintchine the form

$$\Psi(\lambda) = ia\lambda - \frac{1}{2}q^2\lambda^2 + \int_{\mathbb{R}} \Lambda(\mathrm{d}x)\big(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\leqslant 1\}}\big)$$

for some $a, q \in \mathbb{R}_+$ and a Lévy measure $\Lambda(\mathrm{d}x)$ in $\mathbb{R}\backslash\{0\}$. Further, we denote by $\beta(\Lambda)$ the Blumenthal-Getoor (upper) index of the Lévy measure Λ, viz. the non-negative number $\beta(\Lambda) \in [0,2]$ defined as:

$$\beta(\Lambda) := \inf\Big\{b > 0 : \int_{|x|\leqslant 1} \Lambda(\mathrm{d}x)\,|x|^b < \infty\Big\}.$$

Now, the Blumenthal-Getoor index β of the Lévy process ξ is defined in terms of $\beta(\Lambda)$ by the relation:

$$\beta := \begin{cases} 2 & \text{if } q \neq 0, \\ \beta(\Lambda) & \text{if } q = 0. \end{cases}$$

One can think of $\beta(\Lambda)$ as measuring the regularity of the Lévy measure Λ. For instance, if Λ is finite we have plainly $\beta(\Lambda) = 0$ while if the Lévy process is of finite variation, in which case $\int \Lambda(\mathrm{d}x)(1 \wedge |x|) < \infty$, it holds that $\beta(\Lambda) \leqslant 1$. Finally, if the Lévy process is α-stable for some $0 < \alpha < 2$, we get that $\beta(\Lambda) = \alpha$. We are now in position to introduce our main object of interest.

Noise reinforced Lévy processes

Fix a Lévy process ξ with Lévy-Khintchine triplet (a, q^2, Λ). For every $n \geqslant 1$ and $k \geqslant 1$, we write $X_k^{(n)} := \xi_{k/n} - \xi_{(k-1)/n}$ for the k-th increment of ξ for a partition with mesh of size $1/n$. Then, $S_k^{(n)} := X_1^{(n)} + \cdots + X_k^{(n)}$ for $k \geqslant 1$ with $S_0^{(n)} = 0$ is a random walk that we refer as the skeleton of ξ for a mesh of size $1/n$. We shall write

$$\hat{S}_k^{(n)} := \hat{X}_1^{(n)} + \cdots + \hat{X}_k^{(n)}, \qquad \text{for } k \geqslant 1,$$

with $\hat{S}_0^{(n)} = 0$ for its noise reinforced version, for some reinforcement parameter $p \in (0,1)$ that we fix from now on. If we further assume that p fulfils the condition:

$$p \cdot \beta < 1, \tag{1.10}$$

in which case we say that p is admissible for the triplet (a, q^2, Λ), the main result in [19] states that the sequence of skeletons converges as $n \uparrow \infty$, in the sense of finite-dimensional distributions, towards a time-indexed process

$$\Big(\hat{S}^{(n)}_{\lfloor nt \rfloor}\Big)_{t\in\mathbb{R}_+} \xrightarrow{fdd} (\hat{\xi}_t)_{t\in\mathbb{R}_+} \tag{1.11}$$

named the noise reinforced Lévy process - or in short NRLP - with characteristics (a, q^2, Λ, p). In particular, if the starting Lévy process ξ is a Brownian motion, the corresponding noise reinforced Lévy process is a noise reinforced Brownian motion with reinforcement parameter p. We can now introduce our main object of study.

Proposition 1.6. *Let (a, q^2, Λ) be the triplet of a Lévy process of exponent Ψ, consider an admissible memory parameter $p \in (0,1)$ and consider a Yule Simon process Y with parameter $1/p$. There exists a process $\hat{\xi} = (\hat{\xi}_s)_{s\in\mathbb{R}_+}$ whose finite dimensional distributions satisfy that, for any $0 \leqslant s_1 < \cdots < s_k \leqslant t$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, we have*

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j\hat{\xi}_{s_j}\right\}\right] = \exp\left\{(1-p)t\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k}\lambda_i Y(s_j/t)\right)\right]\right\}, \tag{1.12}$$

where the right-hand side does not depend on the choice of t. The process $\hat{\xi}$ is called a noise reinforced Lévy process with characteristics (a, q^2, Λ, p).

From now on, when considering a NRLP with characteristics (a, q^2, Λ, p) it is implicitly assumed that p is admissible for the triplet (a, q^2, Λ) in the sense of (1.10). Let us briefly explain the reason behind condition (1.10). First, note that if $\beta \neq 2$ only $p < 1/2$ is admissible for the triplet, in agreement with Theorem 1.3. Indeed, in that case the Brownian component is non-null and by the scaling property of Brownian motion the corresponding sequence of reinforced skeletons for the Brownian component fall in the scope of Theorem 1.10. Further, if we suppose that $\beta = 0$, observe that we get a restriction on p only when $\beta(\Lambda) > 1$, in which case the so-called "martingale compensation on the jumps" of ξ is present. Roughly speaking, if p is too large, the reinforcement algorithm might break the compensation mechanism and no limiting object can be defined. For a more detailed discussion we refer to [19, Section 2].

In order to explain the presence of the Yule-Simon process on the characteristic function of the finite-dimensional distributions (1.12) of $\hat{\xi}$, we briefly sketch the proof of the finite-dimensional convergence (1.11). Our arguments are taken from the proof of [19, Theorem 3.1]. For every $n \geqslant 1$, we write $(N_j^{(n)}(k) : k \geqslant 1, j \geqslant 1)$ for the collection of counting processes associated to $(\hat{S}_k^{(n)})$ and observe that these are identically distributed. To establish (1.11) it suffices to prove that as $n \uparrow \infty$, the characteristic function of the finite-dimensional of $\big(\hat{S}_{\lfloor nt \rfloor} : t \in [0,1]\big)$ converge towards (1.12). Recalling the representation (1.8), the characteristic function of the finite dimensional distributions of $S^{(n)}_{\lfloor n\cdot \rfloor}$ at times $0 \leqslant s_1 < \cdots < s_k \leqslant 1$ writes

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j\hat{S}_{\lfloor ns_j\rfloor}\right\}\right] = \exp\left\{\frac{1}{n}\mathbb{E}\left[\sum_{\ell=1}^{n}\Psi\left(\sum_{j=1}^{k}\lambda_j N_\ell(\lfloor ns_j\rfloor)\right)\right]\right\}.$$

for arbitrary $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. One can then obtain, by making use of (1.9) for an obvious choice of functional F and a truncation argument, that the limit as $n \uparrow \infty$ in the last display is precisely the right-hand side in (1.12). For a detailed proof we refer to [19, Theorem 3.1].

The building blocks of NRLPs

A Lévy process with triplet (a, q^2, Λ) can be written as the sum of three independent processes

$$\xi_t = (at + qB_t) + \xi_t^{(2)} + \xi_t^{(3)}, \qquad \text{for } t \geqslant 0$$

where B is a standard Brownian motion, $\xi^{(2)}$ is a compound Poisson process with Lévy-Khintchine triplet $(0, 0, \Lambda(\mathrm{d}x)1_{|x|>1})$ while $\xi^{(3)}$ is a martingale with triplet $(0, 0, \Lambda(\mathrm{d}x)1_{|x|\leqslant 1})$, often colloquially referred to as a compensated sum of jumps. One can infer from formula (1.12) that the law of a NRLP $\hat{\xi}$ of characteristics (a, q^2, Λ, p) can be written in terms of tree independents NRLPs

$$\hat{\xi}_t \overset{\mathscr{L}}{=} (at + q\hat{B}_t) + \hat{\xi}^{(2)}_t + \hat{\xi}^{(3)}_t, \qquad \text{for } t \geqslant 0 \tag{1.13}$$

the identity holding in law, where $\hat{\xi}^{(2)}, \hat{\xi}^{(3)}$ are NRLPs with respective characteristics given by $(0, 0, \Lambda(\mathrm{d}x)1_{|x|>1}, p)$, $(0, 0, \Lambda(\mathrm{d}x)1_{|x|\leqslant 1}, p)$ while $\hat{B}$ is a noise-reinforced Brownian motion with reinforcement p if $q \neq 0$, and should read as null otherwise. More precisely, $\hat{\xi}^{(2)}$ is the noise reinforced version of the compound Poisson process $\xi^{(2)}$ while $\hat{\xi}^{(3)}$ is the noise reinforced version of the martingale $\xi^{(3)}$ (we stress however that the martingale property is not preserved by the reinforcement). This two processes admit representations on the interval $[0, 1]$ in terms of Poissonian sums of Yule-Simon processes, shedding some light on the jump structure of NRLPs. The two following constructions are taken from [19, Section 2].

◦ *The reinforced compound Poisson process.* Suppose that ξ is a compound Poisson process with Lévy measure $\Lambda(\mathrm{d}x)$ given by $cP_X(\mathrm{d}x)$, for some non-negative constant c and a probability measure $P_X(\mathrm{d}x)$. Observe that in that case, any reinforcement parameter $p \in (0, 1)$ is admissible. Denote the law of the Yule-Simon process of parameter $1/p$ in $\mathbb{D}([0, 1], \mathbb{R})$ by $\mathbb{Q}$ and consider $\mathcal{M} = \sum_{i\in\mathbb{N}} \delta_{(x_i,Y_i)}$ a Poisson random measure in $\mathbb{R}_+ \times \mathbb{D}([0, 1], \mathbb{R})$ with intensity $(1-p)\Lambda(\mathrm{d}x)\otimes\mathbb{Q}$. Then, as can be verified by making use of the exponential formula, the process

$$\hat{\xi}_t = \sum_i x_i Y_i(t), \qquad \text{for } 0 \leqslant t \leqslant 1,$$

has the law of the noise reinforced version of ξ and will be called a noise reinforced Poisson process. Observe from our description that the law of the jumps is still dictated by $P_X(\mathrm{d}x)$ and that the Yule-Simon process Y_i reinforces the jump x_i at every $0 \leqslant t \leqslant 1$ such that $\Delta Y_i(t) \neq 0$. Observe as well that the process in the last display is rcll. Getting back to the decomposition (1.13), it follows that the process $\hat{\xi}^{(2)}$ can (and will) be chosen rcll and with jumps of size larger than 1.

◦ *Compensated compound Poisson processes.* Let us now turn our attention to the reinforced version of the martingale $\xi^{(3)}$. To this end, with the same notations as before we consider a Poisson measure $\mathcal{M}$ with intensity $(1 - p)\Lambda(\mathrm{d}x) \otimes \mathbb{Q}$. For every $0 < a < 1$, we introduce the following finite-variation process:

$$\hat{\xi}_{(a,1)}(t) := \sum_i 1_{\{a<|x_i|\leqslant 1\}} x_i Y_i(t) - t \int_{\{a<|x|\leqslant 1\}} x\Lambda(\mathrm{d}x), \qquad \text{for } t \in [0, 1].$$

It readily follows from Campbell's formula that the process in the last display in centred, and by the exponential formula it has the law of the noise reinforced version of a Lévy process with triplet $(0, 0, 1_{\{a<|x|\leqslant 1\}}\Lambda(\mathrm{d}x))$. It was established in [19] that for any fixed reinforcement parameter p satisfying the condition $p < 1/\beta(\Lambda)$, for each fixed $t \in [0, 1]$ the limit

$$\lim_{a\downarrow 0} \hat{\xi}_{(a,1)}(t)$$

exists a.e. and in $L_1(\mathbb{P})$ and we denote it by $\hat{\xi}^{(3)}_t$. Moreover, $\hat{\xi}^{(3)} = (\hat{\xi}^{(3)}_t)_{t\in[0,1]}$ has the law of the noise reinforced version of $\xi^{(3)}$. This construction is reminiscent of the non-reinforced setting, where $\xi^{(3)}$ is a so-called compensated sum of jumps. In contrast with our previous case, once can no longer infer from this description if $\hat{\xi}^{(3)}$ posses a rcll modification. This shall be our first concern.

1.4 The reinforced Lévy-Itô decomposition and synthesis

The definition given in Proposition 1.6 gives no information on the trajectorial regularity of NRLPs. In this direction, let us state the following fundamental result:

Theorem 1.7. *A noise reinforced Lévy process $\hat{\xi}$ has a rcll modification, and we still denote it by $\hat{\xi}$.*

Let us be more precise. From our previous discussion, finding a rcll modification of a NRLP boils down to investigating the regularity of $\hat{\xi}^{(3)}$. The key observation now is that when a NRLP $\hat{\xi}$ is centred, the process

$$t^{-p}\hat{\xi}_t, \qquad \text{for } t > 0$$

defined as 0 at the origin, is a martingale and therefore posses an rcll modification. For a proof of this result we refer to [84, Proposition 3.2]. The martingale in the previous display is, roughly speaking, the continuous time analogue of the discrete remarkable martingale we introduced in Section 1.1.

Now that we have stated that a NRLP is a rcll process, we shall study the structure of its jump process $(\Delta\hat{\xi}_t)$. Since it will share striking similarities with the jump process of a Lévy process, we start by recalling some well known results on $(\Delta\xi_t)$. If ξ is a Lévy process with Lévy measure Λ, its jump measure

$$\mu(\mathrm{d}t, \mathrm{d}x) = \sum_s 1_{\{\Delta\xi_s \neq 0\}}\delta_{(s,\Delta\xi_s)}(\mathrm{d}t, \mathrm{d}x) \tag{1.14}$$

is a homogeneous Poisson point process (abbreviated PPP) with characteristic measure $\Lambda(\mathrm{d}x)$. Such a PPP can be constructed by decorating point process of jumps of Poisson processes, and it is classic that (1.14) is determined by the following two properties:

(i) For any Borelian A with $\Lambda(A) < \infty$, the counting process of jumps $\Delta\xi_s \in A$ occurring until time t, defined as
$$N_A(t) = \#\big\{(s, \Delta\xi_s) \in [0,t] \times A\big\}, \qquad t \geqslant 0,$$
is a Poisson process with rate $\Lambda(A)$.

(ii) If $A_1, \dots A_k$ are disjoint Borelians with $\Lambda(A_i) < \infty$ for all $i \in \{1, \dots, k\}$, the processes $N_{A_1}, \dots, N_{A_k}$ are independent.

In particular, from (i), it follows that $(N_A(t) - \Lambda(A)t)_{t\in\mathbb{R}_+}$ is a martingale.

Let us then turn our attention to the study of the jump measure

$$\hat{\mu}(\mathrm{d}t, \mathrm{d}x) := \sum_s 1_{\{\Delta\hat{\xi}_s \neq 0\}}\delta_{(s,\Delta\hat{\xi}_s)}(\mathrm{d}t, \mathrm{d}x).$$

To this end, we shall introduce a family of measures in $\mathbb{R}_+ \times \mathbb{R}$ parameterised by $(\Lambda(\mathrm{d}x), p)$ of independent interest under the name *noise reinforced Poisson point processes*, and abbreviated NRPPP. In analogy with the non-reinforced setting, NRPPPs are constructed by decorating the jumps of *reinforced* Poisson processes. To motivate the introduction of this family of measures we shall postpone their explicit construction and start by discussing the deep connections between NRPPPs and NRLPs.

The first main result of the section states that NRPPPs play the role of PPP in the reinforced setting:

Theorem 1.8. (Reinforced Lévy-Itô decomposition)
The jump measure $\hat{\mu}$ of $\hat{\xi}$ is a noise reinforced Poisson point process with characteristic measure $\Lambda(\mathrm{d}x)$ and reinforcement parameter p.

Moreover, if we denote by $(\mathscr{F}_t)$ the natural filtration of $\hat{\xi}$, Proposition 4.11 in [84] states that the predictable compensator $\hat{\mu}^{\mathrm{p}}$ of $\hat{\mu}$ is given by

$$\hat{\mu}^{\mathrm{p}}(\omega; \mathrm{d}t, \mathrm{d}x) = (1-p)\mathrm{d}t \otimes \Lambda(\mathrm{d}x) + p\frac{\mathrm{d}t}{t}\mathscr{E}_t(\omega; \mathrm{d}x)$$

where $\mathscr{E}_t(\mathrm{d}x) = \sum_{s<t} \delta_{\Delta\hat{\xi}_s}(\mathrm{d}x)$ is the empirical measure of jumps that occurred strictly before time t. In particular, if we take $p = 0$ in the last display, no reinforcement ocurs and we recover the compensator of a PPP with intensity $\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$.

Let us now turn our attention to the representation of $\hat{\xi}$ in terms of its jump measure, a result that we shall refer to as the reinforced Itô synthesis. In this direction, it might be worth recalling the precise statement of Itô synthesis in the setting of Lévy processes. Write μ for the jump measure of ξ and denote by $\mu^{(sc)}$ its so-called compensated measure of jumps. Then, Itô synthesis states that there exits a standard Brownian motion B, such that the following equality holds a.e.

$$\xi_t = at + qB_t + \int_{[0,t]\times[-1,1]^c} x\mu(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times[-1,1]} x\mu^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0 \tag{1.15}$$

with the convention that B is null if $q = 0$. The reinforced Lévy-Itô synthesis that we shall now state shows that an analogous result holds in the setting of NRLPs, where the Brownian motion is replaced by a noise reinforced Brownian motion $\hat{B}$ and the measure μ by the reinforced version $\hat{\mu}$. After properly introducing the "space-compensated" measure $\hat{\mu}^{(sc)}$, we prove:

Theorem 1.9. (Reinforced Itô's synthesis)
Let $\hat{\mu}$ be the jump measure of a NRLP $\hat{\xi}$ of characteristics (a, q^2, Λ, p). Then, a.s. we have

$$\hat{\xi}_t = at + q\hat{B}_t + \int_{[0,t]\times[-1,1]^c} x\hat{\mu}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times[-1,1]} x\hat{\mu}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0,$$

for some noise reinforced Brownian motion $\hat{B}$, with the convention that if $p \geqslant 1/2$ the process $\hat{B}$ is null. Moreover, the integrals in the previous display are NRLPs with respective characteristics $(0, 0, 1_{[-1,1]^c}\Lambda, p)$, $(0, 0, 1_{[-1,1]}\Lambda, p)$.

Now that we have stated the main results of this section and motivated the notion of a NRPPP, let us introduce this family of random measures. We shall then give a brief sketch on how on can prove Theorem 1.8.

The jumps of reinforced Poisson processes

Let $\hat{N}$ be the reinforced version of a Poisson process with intensity c for some reinforcement parameter p. With a slight abuse of notation, we still refer to the point measure $\hat{\mathscr{P}} := \mathrm{d}\hat{N}_s$ in $\mathbb{R}_+$ as a reinforced Poisson process, and we shall start by investigating the nature of this random measure in the non-negative real line.

The measure $\hat{\mathscr{P}}$ admits a simple representation in terms of a decorated Poisson process $\mathscr{P}$ with intensity $c\,(1-p)\mathrm{d}t$ in $\mathbb{R}_+$. Consider a point measure $D = \{0, T_1, T_2, \dots\}$ satisfying that the increments $(T_{k+1} - T_k : k \geqslant 0)$ are independent and exponentially distributed with respective parameter pk, with the convention $T_0 = 0$. In other terms, if we let Z be a standard Yule process, D has the law of the jump measure of $(Z_{pt})_{t\in\mathbb{R}_+}$ with an additional Dirac mass at 0. Then, by (4.5) in [84] the following identity holds in distribution:

$$\sum_{s\in\mathbb{R}_+} 1_{\{s:\Delta\hat{N}_s=1\}}\delta_s \stackrel{\mathscr{L}}{=} \sum_{u\in\mathscr{P}}\sum_{t\in D_u} \delta_{ue^t}.$$

From this description, the dynamics of the jumps of $\hat{N}$ can be described as follows: first, since $0 \in D_u$, for every $u \in \mathscr{P}$ the process $\hat{N}$ performs a jump at time u. Then, this jump is reinforced at the subsequent times $ue^{t_i^u}$ for every $t_i^u \in D_u$. Therefore, roughly speaking, the jumps of $\hat{N}$ consist in Poissonian jumps $u \in \mathscr{P}$ which – in analogy with the discrete setting – we refer to as innovations, and each u has attached to it a family $\{ue^t : t \in D_u, t \neq 0\}$ which should be interpreted as repetitions of the original u through time. The connection with the Yule-Simon process is the following: for fixed $u \in \mathscr{P}$, the jumps $\{ue^t : t \in D_u, t \neq 0\}$ are precisely the jumps of a Yule-Simon process but started at the Poissonian time u instead of a uniform time in $[0,1]$. We refer to [84, Proposition 4.2] for exponential formulas characterising the law of $\hat{\mathscr{P}}$.

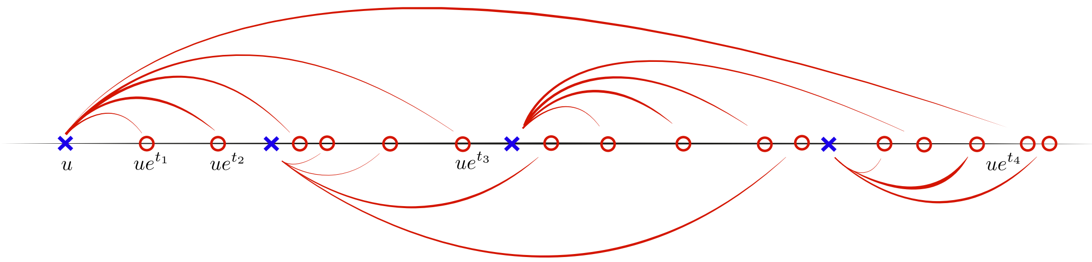


Figure 1.4: Sketch of the jumps of a noise reinforced Poisson process. We marked by x the jumps corresponding to innovations, while each linked o is a repetition of the former.

The noise reinforced Poisson point process with characteristic measure Λ and parameter p can now be constructed by decorating independent reinforced Poisson processes. We shall now describe this procedure.

The reinforced Poisson point process

Fix a Lévy measure $\Lambda(\mathrm{d}x)$ in $\mathbb{R}$ and consider a disjoint partition $(A_j)_{j\in\mathcal{I}}$ of $\mathbb{R}\backslash\{0\}$ satisfying that $\Lambda(A_j) < \infty$. Further, we fix some reinforcement parameter $p \in (0,1)$.

- *Step 1.* For every $j \in \mathcal{I}$ let $\mathscr{P}_j$ be independent Poisson processes in $\mathbb{R}_+$ with intensity $(1-p)\Lambda(A_j)$. Further, consider independent collections $(x_u : u \in \mathscr{P}_j)$ of i.i.d. random variables with law $\Lambda(\,\cdot \cap A_j)/\Lambda(A_j)$ and $(D_u : u \in \mathscr{P}_j)$ i.i.d. with same law as D. We set

$$\hat{\mathcal{N}}_j(\mathrm{d}s, \mathrm{d}x) := \sum_{u\in\mathscr{P}_j}\sum_{t\in D_u} \delta_{(ue^t, x_u)}.$$

Note that this corresponds to marking with the collection $(x_u : u \in \mathscr{P}_j)$ the reinforced Poisson process $\sum_{u\in\mathscr{P}_j}\sum_{t\in D_u}\delta_{ue^t}$ with intensity $\Lambda(A_j)$ and parameter p.

- *Step 2.* Set $\mathscr{P} := \sum_j \mathscr{P}_j$ and write $\hat{\mathcal{N}} := \sum_j \hat{\mathcal{N}}_j$ for the measure obtained by superposition of $(\hat{\mathcal{N}}_j : j \in I)$. To simplify notation we simply write:

$$\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) := \sum_{u\in\mathscr{P}}\sum_{t\in D_u} \delta_{(ue^t, x_u)}.$$

Definition 1.10. *The measure $\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x)$ is referred to as a noise reinforced Poisson point process with characteristic measure $\Lambda(\mathrm{d}x)$ and reinforcement parameter p.*

One can then characterise the law of $\hat{\mathcal{N}}$ by computing its exponential formulas, we refer to Proposition [84, Proposition 4.8] for a precise statement and to [84, Lemma 4.6] for some basic properties. Notably, NRPPPs admit a characterisation in the same spirit as the one stated for PPP after (1.14). More precisely, the following holds:

Proposition 1.11. *Let $\hat{\mathcal{N}}$ be a point process in $\mathbb{R}_+ \times \mathbb{R}$ and for any Borelian $A \subset \mathbb{R}$, set*

$$\hat{N}_A(t) := \hat{\mathcal{N}}([0,t] \times A), \quad t \geqslant 0.$$

Then, $\hat{\mathcal{N}}$ is a noise reinforced Poisson point process with characteristic measure Λ and reinforcement parameter p if and only if the two following conditions are satisfied:

(i) *For any Borelian A with $\Lambda(A) < \infty$, the process $\hat{N}_A$ is a noise reinforced Poisson process with rate $\Lambda(A)$ and reinforcement parameter p.*

(ii) *If $A_1, \dots A_k$ are disjoint Borelians with $\Lambda(A_i) < \infty$ for all $i \in \{1, \dots, k\}$, the processes $\hat{N}_{A_1}, \dots, \hat{N}_{A_k}$ are independent.*

Moreover, if $\Lambda(A) < \infty$ the process $(t^{-p}(\hat{N}_A(t) - t\Lambda(A)) : t > 0)$ is a martingale.

Now, Theorem 1.8 can be established by showing that $\hat{\mu}$ satisfies properties (i) and (ii) in the previous proposition. We refer to [84, Theorem 4.1] for a detailed proof of this result.

1.5 Weak convergence of skeletons

We shall make use of the notations introduced at the beginning of Section 1.3. We fix a Lévy process ξ, an admissible reinforcement parameter p for its triplet and we consider the pair

$$\left(S^{(n)}_{\lfloor nt \rfloor}, \hat{S}^{(n)}_{\lfloor nt \rfloor}\right)_{t \in \mathbb{R}_+}$$

composed by the skeleton of the Lévy process ξ paired with its noise-reinforced version. The first component converges point-wise, with respect to the Skorokhod topology, towards the starting Lévy process ξ and by the result of Bertoin [19, Theorem 3.1] we know that the second component converges, in the sense of finite-dimensional distributions, towards a NRLP with characteristics (a, q^2, Λ, p). Observe however that in contrast with the discrete setting, in the continuum we don't have *a priori* a natural coupling for ξ with its reinforced version. In this direction, let us start by introducing the law for the coupling that we shall work with.

Proposition 1.12. *There exists a pair $(\xi, \hat{\xi})$, where $\hat{\xi}$ has the law of a NRLP with characteristics (a, q^2, Λ, p), with law determined by the following: for all $k \geqslant 1$, $\lambda_1, \dots, \lambda_k$, $\beta_1, \dots \beta_k$ real numbers, and $0 < t_1 < \cdots < t_k \leqslant t$, we have*

$$\mathbb{E}\left[\exp\left\{i \sum_{j=1}^{k} \left(\lambda_j \xi_{t_j} + \beta_j \hat{\xi}_{t_j}\right)\right\}\right] =$$

$$\exp\left\{t \cdot p\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \lambda_j 1_{\{U \leqslant t_j/t\}}\right)\right] + t \cdot (1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} (\lambda_j 1_{\{Y(t_j/t) \geqslant 1\}} + \beta_i Y(t_j/t))\right)\right]\right\},$$

where U is a uniform random variable in $[0,1]$. A pair of processes with such distribution will always be denoted by $(\xi, \hat{\xi})$.

We stress that since noise-reinforcement is an inherently discrete procedure, one can not define naively this notion in the continuum without considering a discretisation of the process. Our definition for the joint law $(\xi, \hat{\xi})$ is justified by the following joint convergence, which is the main result of Section 5 in [84].

Theorem 1.13. *Let ξ be a Lévy process with characteristic triplet (a, q^2, Λ), fix $p \in (0, 1/2)$ an admissible memory parameter and for each n, let $(S_k^{(n)}, \hat{S}_k^{(n)})$ be the pair of the n-skeleton of ξ and its reinforced version. Then, there is weak convergence in $D^2(\mathbb{R}_+, \mathbb{R})$ as $n \uparrow \infty$*

$$\left(S_{\lfloor n \cdot \rfloor}^{(n)}, \hat{S}_{\lfloor n \cdot \rfloor}^{(n)} \right) \xrightarrow{\mathscr{L}} (\xi, \hat{\xi}).$$

This is achieved by proving separately tightness and finite dimensional convergence for the pair. We refer to Section 5 in [84] for a proof of this statement. We believe it is of particular interest to dwell further into the definition provided in Proposition 1.12 for the joint law $(\xi, \hat{\xi})$. To this end, we shall now sketch how one can construct a coupling $(\xi, \hat{\xi})$ with finite-dimensional distributions characterised by Proposition 1.12. Heuristically, as we shall see this is achieved by performing the reinforcement algorithm in the continuum to the starting Lévy process ξ. A precise description of the law $(\xi, \hat{\xi})$ will allow us to understand how the reinforcement of ξ affects its sample paths.

The joint law $(\xi, \hat{\xi})$

By Lévy-Itô synthesis (1.15), the Lévy process ξ can be written as $(a \cdot Id + q^2 B) + \xi^{(2)} + \xi^{(3)}$, where the processes $(\xi^{(2)}, \xi^{(3)})$ are independent of B and can be constructed from the jump measure μ. On the other hand, by the reinforced Lévy-Itô synthesis [Theorem 1.9] we can write the law of $\hat{\xi}$ as $(a \cdot Id + q^2 \hat{B}) + \hat{\xi}^{(2)} + \hat{\xi}^{(3)}$ where $(\hat{\xi}^{(2)}, \hat{\xi}^{(3)})$ are independent of $\hat{B}$ and can be constructed from $\hat{\mu}$. Therefore, in order to define the joint law $(\xi, \hat{\xi})$, it suffices to introduce the law of a Brownian motion paired with its reinforced version $(B, \hat{B})$, as well as the distribution of the pair $(\mu, \hat{\mu})$, where we denoted by $\hat{\mu}$ a NRPPP with parameters $(\Lambda(\mathrm{d}x), p)$.

∘ *The joint law* $(\mathcal{N}, \hat{\mathcal{N}})$. Denote the set of jump times of ξ by $\mathscr{I} := \{u \in \mathbb{R}_+ : \Delta\xi_u \neq 0\}$ and consider the jump measure

$$\mu(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{I}} \delta_{(u, \Delta\xi_u)}.$$

The construction of $\hat{\mu}$ that we shall now describe in terms of μ is reminiscent of the reinforcement algorithm in the discrete setting. Roughly speaking, in the continuum, the steps (X_n) are replaced by jumps $\Delta\xi_s$ of the Lévy process ξ. With probability $1 - p$, the jump-time and the respective jump is shared with its reinforced version $\hat{\xi}$ while with complementary probability p it is discarded by the reinforcement algorithm. The jumps that are not discarded by this procedure are then repeated at each jump time of an independent counting process that will be attached to it. The process of discarding jumps with probability p is traduced in a thinning of the jump measure of ξ. Formally, consider a family of Bernoulli random variables $(\varepsilon_u)_{u \in \mathscr{I}}$ with parameter $1-p$ as well as independent collections $(D_u : u \in \mathscr{I})$ with law D. Then, the measure defined by the relation

$$\hat{\mu}(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{I}} 1_{\{\varepsilon_u = 1\}} \sum_{t \in D_u} \delta_{(ue^t, \Delta\xi_u)}$$

is a NRPPP with parameters $(\Lambda(\mathrm{d}x), p)$ explicitly constructed in terms of μ.

∘ *The joint law* $(B, \hat{B})$. Observe that we do have a natural candidate for the law of the pair $(B, \hat{B})$. Namely, the law obtained in Theorem 1.3 when we investigated invariance principles for noise-reinforced random walks: for fixed $p \in (0, 1/2)$, we let $(B, \hat{B})$ be a pair of Gaussian processes with respective covariances given by

$$\mathbb{E}\left[B_t B_s\right] = (t \wedge s), \qquad \mathbb{E}[B_t \hat{B}_s] = (t \wedge s)^{1-p} s^p, \qquad \mathbb{E}[\hat{B}_t \hat{B}_s] = \frac{(t \vee s)^p (t \wedge s)^{1-p}}{1-2p}$$

for any $s, t \in \mathbb{R}_+$.

Finally, consider independent pairs $(B, \hat{B})$ and $(\mu, \hat{\mu})$. Now we can state:

Definition 1.14. *We call the noise reinforced Lévy process*

$$\hat{\xi}_t = at + q\hat{B}_t + \int_{[0,t]\times[-1,1]^c} x\hat{\mu}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times[-1,1]} x\hat{\mu}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0,$$

with characteristics (a, q^2, Λ, p) the noise reinforced version of ξ, the unicity of the pair only holding in distribution. Every time we consider a pair $(\xi, \hat{\xi})$, it will be implicitly assumed that $\hat{\xi}$ has been constructed by the procedure we just described in terms of ξ.

Now, from a rather long but straightforward computation one gets that the law of the pair $(\xi, \hat{\xi})$ satisfies the identity of Proposition 1.12.

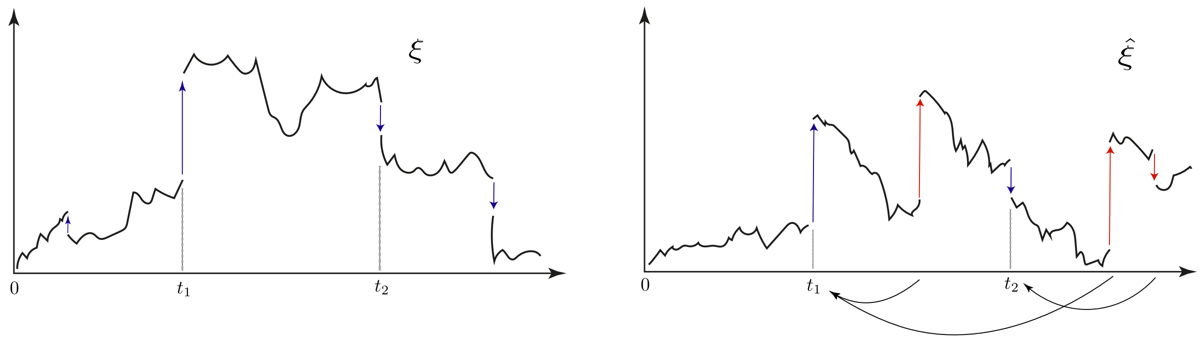


Figure 1.5: Sketch of a sample path of a Lévy process and its reinforced version. Jumps in blue correspond to jumps coming from the jump measure of ξ (innovations), and in red are marked the subsequent reinforcements. The jumps in blue in the path of ξ that are not present in the right-hand side have been deleted by the thinning.

1.6 Applications

We conclude this introduction by briefly presenting two applications developed in [84] addressing two very different aspects of NRLPs. We first start by studying the rates of growth of NRLPs at the origin, and we compare these with analogous results holding for Lévy processes. This part strongly relies on the structure of NRPPPs and Theorem 1.8. On the other hand, we shall identify the main features of NRLPs in the much broader setting of Infinitely Divisible Process, and study some of its properties.

Rates of growth at the origin

Let ξ be a Lévy process with characteristic triplet $(a, 0, \Lambda)$. Observe that in particular, since ξ has no Gaussian component, the Blumenthal-Getoor index of ξ is given by $\beta = \beta(\Lambda)$. We make the following assumptions on the characteristic exponent:

• If $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) = \infty$, we assume the characteristic exponent can be written as follows:

$$\Psi(\lambda) = \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\leqslant 1\}}\right)\Lambda(\mathrm{d}x).$$

Observe that in this case, we have $\beta(\Lambda) \in [1, 2]$.

• If $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) < \infty$, which can happen for $\beta(\Lambda) \in [0, 1]$, we suppose Ψ takes the following form:

$$\Psi(\lambda) = \int_{\mathbb{R}} \left(e^{i\lambda x} - 1\right)\Lambda(\mathrm{d}x).$$

This is, when the Lévy process has finite variation, we are supposing that it has no linear drift - the reason being that in that case the behaviour at 0 is dominated by the drift term. We will refer to these hypothesis as hypothesis (H). In [84], we proved that the behaviour at zero of a NRLP is twofold and dictated by the Blumenthal-Getoor index of the Lévy measure Λ.

Proposition 1.15. *Let ξ be a Lévy process with triplet (a, q^2, Λ) satisfying hypothesis* (H)*, and consider $\hat{\xi}$ its noise reinforced version for an admissible parameter p. Then, almost surely, we have*

$$\lim_{t\downarrow 0} t^{-\gamma}\hat{\xi}_t = 0, \qquad \text{if } \beta(\Lambda) < 1/\gamma,$$

while

$$\limsup_{t\downarrow 0} t^{-\gamma}|\hat{\xi}_t| = \infty, \qquad \text{if } \beta(\Lambda) > 1/\gamma.$$

It was established by Blumenthal and Getoor in [26] that under (H), the same result holds if we replace the NRLP $\hat{\xi}$ with the corresponding Lévy process ξ. Therefore, despite the reinforcement, the behaviour at the origin - in the sense of Proposition 1.15 - is left unchanged. Finer asymptotic analysis such as a law of iterated logarithm in the reinforced setting were however not addressed in [84]. We refer however to [21] for a law of the iterated logarithm for noise-reinforced Brownian motion. Let us now turn our attention to the second application discussed in [84].

Infinite divisibility

Let T be an arbitrary set. A process $X = (X_t)_{t\in T}$ is said to be infinitely divisible if for every $n \geqslant 1$, we can write the law of X as a sum of n independent and identically distributed copies of some other process. The theory of infinitely divisible processes has been subject of intensive research and general tools have been developed making their study possible. These have found remarkable applications in different fields, we refer to [86] for a general overview of the theory, several important examples and applications. One of the main results of the theory states that infinitely divisible processes are in bijection with so called functional triplets $(b, \Gamma, \bar{\nu})$, where $b \in \mathbb{R}^T$ is a path, $\Gamma : T \times T \to \mathbb{R}$ is a covariance function and $\bar{\nu}$ is a measure in the path space $\mathbb{R}^T$, often referred to as the *path Lévy measure*. The finite-dimensional distributions of an infinitely divisible process can be expressed in terms of the triplet $(b, \Gamma, \bar{\nu})$. In this direction, for every finite subset $I \subset T$ with $I = \{t_1, \dots t_n\}$ and a path $e \in \mathbb{R}^T$ we let $e_I := (e(t_1), \dots, e(t_n)))$. If we write Γ_I for the restriction of Γ to $I \times I$, it holds that

$$\mathbb{E}\left[\exp\left\{i\sum_{t\in I}\theta_t X_t\right\}\right] = \exp\left\{i\langle b_I, \theta\rangle - \frac{1}{2}\langle \theta\,\Gamma_I, \theta\rangle + \int_{\mathbb{R}^T}\left(e^{i\langle\theta, e_I\rangle} - 1 - i\langle\theta, [\![e_I]\!]\rangle\right)\bar{\nu}(\mathrm{d}e)\right\}.$$

Having an explicit representation for the triplet $(b, \Gamma, \bar{\nu})$ turn out to be essential for applications, since it unlocks much of the powerful machinery developed for the study of ID processes.

One can infer from (1.12) that a noise reinforced Lévy processes is infinitely divisible. In Section 6.2 of [84], we identify their functional triplet, an for instance we show that the path Lévy measure of a NRLP with characteristics (a, q^2, Λ, p) when restricted to the interval $[0, 1]$ is given by

$$\bar{\nu} := (1-p)(\Lambda \otimes \mathbb{Q}) \circ V^{-1},$$

where $V : \mathbb{R} \times D[0, 1] \to \mathbb{R}^{[0,1]}$ is the mapping defined by $V(x, y) := xy$, and $\mathbb{Q}$ is the law of the Yule-Simon process. As an application, making use of the so-called *Isomorphism theorem for infinitely divisible processes* we prove the following result:

Proposition 1.16. *Let $\hat{\xi}$ be a noise reinforced Lévy process with characteristics $(a, 0, \Lambda, p)$. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a bounded, continuous function with $f(x) = O(x^2)$ at 0. Then, we have*

$$\lim_{h\downarrow 0} h^{-1}\mathbb{E}\big[f\big(\hat{\xi}_h\big)\big] = p^{-1}(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\sum_{k=1}^{\infty} f(kx)\mathrm{B}(k, 1/p+1).$$

where in the last display we denoted by B *the beta function.*

The probability mass function appearing in the right-hand side of the last display is the Yule-Simon distribution.

Chapter 2

Introduction to Part II

The second part of this work is devoted on the one hand, at expanding upon the theory of (continuous) Markov processes indexed by Lévy trees developed in [43], and on the other hand, at developing an excursion theory for this class of tree-indexed processes. The purpose of this work is to present the recent development of an excursion theory for this family of tree-indexed processes, holding under rather general assumptions. More precisely, this introduction is devoted to giving an overview of the main results obtained in collaboration with Armand Riera and under the supervision of Jean Bertoin and Armand Riera in [82, 83]. Since the content of both works is technical and relies on a rather broad spectrum of topics, in this introduction we shall give an informal presentation with an emphasis on providing the heuristics behind the objects and results we present. This often comes however at the expense of some lack of precision in some of our statements. Let us start with an informal description of our objects of interest.

Informally, a Markov process indexed by a Lévy tree can be understood as follows: consider a Markov process started at the root of a tree $\mathcal{T}$; the motion moves through the geodesic paths of $\mathcal{T}$ away from the root and at each branching point, it splits in copies with same distribution that continue to evolve independently. In contrast with the time-indexed setting, this process is defined through two layers of randomness. Namely, now the indexing set $\mathcal{T}$ for the motion is random, and to be more precise consists in a Lévy-tree. It is important to mention that Markov processes indexed by Lévy trees are canonical probabilistic objects and for instance, are closely related to the theory of super-processes [43, Section 4.2]. More recently, Brownian motion indexed by the Brownian tree has been used as building block for the construction of the universal random metric space arising in random geometry called the Brownian map [65, 76], as well as in the construction of other related random surfaces [10, 72]. Establishing fine properties of such random surfaces therefore often crucially relies in a proper understanding of Brownian motion indexed by the Brownian tree and more generally, in the theory of Markov processes indexed by Lévy trees.

Let us now turn our attention to the development of an excursion theory for this class of tree-indexed processes. To motivate the forthcoming results, let us start by briefly recalling some well known aspects of excursion theory of time-indexed Markov processes. For a detailed overview of the theory, we refer to e.g. [17, 25].

Excursion theory for time-indexed Markov processes.
Excursion theory has been subject of active research for decades, and this short discussion only serves the purpose of recalling the facts needed for our exposition. For a detailed account we refer to e.g. [17, 25]. We consider $(\xi_t : t \geqslant 0)$ a time-indexed strong Markov process with rcll paths, taking values in a Polish space E. For $y \in E$ we write Π_y for its law started at y. Let us first recall some definitions. A point $x \in E$ is called regular if $\Pi_x(\inf\{t > 0 : \xi_t = x\} = 0) = 1$, and it

is called instantaneous if $\Pi_x(\inf\{t > 0 : \xi_t \neq x\} = 0) = 1$; note that by Blumenthal's $0-1$ law, these probabilities are either 1 or 0. Finally, x is said to be recurrent if for every $y \in E$, under Π_y the Markov process returns to x almost surely. Now, we assume that for some point $x \in E$ the following holds:

($\mathbf{H_1}$) The point x is regular, instantaneous and recurrent for ξ.

($\mathbf{H_2}$) The Markov process ξ does not spend time at x, viz.

$$\int_0^\infty \mathrm{d}t \, \mathbb{1}_{\{\xi_t = x\}} = 0, \qquad \Pi_x - \text{a.s.}$$

Under the first two conditions on x in ($\mathbf{H_1}$), the set

$$\mathscr{Z}^\circ = \{t \geqslant 0 : \xi_t = x\}$$

is prefect and nowhere dense, and therefore of fractal nature. Its study is delicate and relies on a remarkable continuous non-decreasing process $\mathcal{L} = (\mathcal{L}_t : t \geqslant 0)$, unique up to a multiplicative constant that we fix arbitrarily, and with Stieltjets measure $\mathrm{d}\mathcal{L}$ supported on the closure of $\mathscr{Z}^\circ$. Roughly speaking, at any time t, the variable $\mathcal{L}_t$ measures the number of visits of ξ at x, and $\mathcal{L}$ is known under the name *the local time of ξ at x*. This description is informal, and for instance under our standing hypothesis the number of visits of ξ to x is uncountable. The right inverse $\mathcal{L}^{-1}$ of $\mathcal{L}$ is a subordinator and hypothesis ($\mathbf{H_2}$) ensures that it possesses no drift. For instance, if ξ is a Brownian motion and $x = 0$, under Π_0 the process $\mathcal{L}^{-1}$ is a 1/2-stable subordinator. The recurrence hypothesis is assumed for convenience and our presentation holds up to some minor modifications if this assumption is dropped.

One of the key properties of $\mathcal{L}$ is that it can be used to index the excursion away from x of ξ and crucially, this indexing is compatible with the ordering induced by time. More precisely, let $(a_i, b_i)_{i\in\mathbb{N}}$ be the connected components of $\mathbb{R}_+\backslash\overline{\mathscr{Z}}^\circ$ and for every i, set $\xi^i := (\xi_{(a_i+t)\wedge b_i} : t \geqslant 0)$. Then, ξ^i is a continuous piece of path, taking the value x for $t \in \{0\} \cup [b_i - a_i, \infty)$ and satisfying that $\xi_t \neq x$ for every $t \in (0, b_i - a_i)$. We refer to ξ^i as the excursion away from x of ξ associated to the excursion interval (a_i, b_i). We let $C(\mathbb{R}_+, E)$ be the space of $\mathbb{R}_+$ indexed E-valued continuous functions endowed with the local uniform topology. We shall refer to the following point measure

$$\mathcal{E}^\circ := \sum_{i\in\mathbb{N}} \delta_{(\mathcal{L}_{a_i}, \xi^i)}$$

on $\mathbb{R}_+ \times C(\mathbb{R}_+, E)$ as the the excursion process of ξ. Then, it is a classic result on excursion theory that the measure $\mathcal{E}^\circ$ is a Poisson point measure with intensity $\mathrm{d}t\otimes\mathcal{N}$, where $\mathcal{N}$ is a sigma finite measure on $C(\mathbb{R}_+, E)$. We shall refer to $\mathcal{N}$ as the excursion measure of ξ. Moreover, the path ξ can be recovered from $\mathcal{E}^\circ$. In the special case when ξ is a Brownian motion, this result is due to Itô.

Let us mention that the study of the excursions away from x of ξ is only delicate when the point x is regular and instantaneous. Indeed, when x is not regular the set $\mathscr{Z}^0$ is discrete, while when x is not instantaneous, $\mathscr{Z}^0$ is a countable union of closed intervals. For a more detailed discussion, we refer to [17, Chapter IV].

In this work, we shall explain how one can obtain analogous results in the tree-indexed setting. Now, the role played by the set $\mathscr{Z}^\circ$ is taken over by the subset of points in $\mathcal{T}$ at which the spatial motion takes the value x, say $\mathscr{Z}$. Roughly speaking and in analogy with the time-indexed setting, the excursion components consist in the connected components of $\mathcal{T}\backslash\mathscr{Z}$, and the excursion associated to a connected component consists in the restriction of the motion to such component. Much of the effort in [82, 83] was devoted to, on the one hand, understanding the structure of the set $\mathscr{Z}$ and, on the other hand, studying the family of excursions away from x by developing an excursion theory. It is important to mention that when the random tree is the Brownian tree and the Markov process a standard Brownian motion, an excursion theory was developed by C. Abraham and J.-F. Le Gall in [1]. Our theory complements the results obtained in [1] by employing different methods and notably, we introduce a notion of local time at a recurrent, instantaneous point for the spatial motion when performing a clockwise exploration of the tree-indexed process. We shall explain the differences and similarities with our work in more detail in the sequel.

The introduction is organised as follows: After introducing the setting we shall be working with, in Section 2.1 we start with a brief introduction to Lévy trees. In Section 2.2 we introduce a remarkable time-indexed process called the *Lévy-Snake*, which is the process behind the formalism of tree-indexed Markov processes. The content of Section 2.2 includes an overview of the theory of exit local times and the special Markov property. The Lévy snake was defined on its current most general framework in [43] and was further studied on [S]. For instance, in [S] we strengthen the trajectorial regularity of the Lévy snake when the spatial motion is continuous and further developed the theory of exit local times under more general initial distributions for the Lévy snake. We then discuss in Sections 2.3 and 2.3 the extension of the classic excursion theory of time indexed processes to our tree-indexed setting. More precisely, in Section 2.3 we introduce an additive functional that shall play the role of the local time at a regular, instantaneous point x when performing a clockwise exploration of the tree-indexed process. We then explain how one can, making use of this additive functional, encode in a random tree the subset of points of $\mathcal{T}_H$ at which the motion takes the value x. For instance, we recover some of the results obtained in [66] by different methods. In Section 2.4, after defining the notion of an excursion away from x, we introduce the Poisson process of excursions and identify its intensity measure. Finally, in Section 2.5 and in analogy with the time-indexed setting, we address reconstruction questions in terms of the excursion process.

2.1 Lévy trees

A rooted $\mathbb{R}$-tree $(\mathcal{T}, d)$ is a compact metric space with a distinguished point, called the root, satisfying that for every pair $a, b \in \mathcal{T}$, there exists a unique geodesic path $[\![a, b]\!] \subset \mathcal{T}$ isometric to the interval $[0, d(a, b)]$ connecting the points a, b. The $\mathbb{R}-$trees we consider in this work are often canonically constructed from a continuous non-negative function.

Trees coded by continuous functions

Let us start by briefly describing how one can construct a tree out of a continuous non-negative function $e : \mathbb{R}_+ \to \mathbb{R}_+$ started at $e(0) = 0$. We write $\sigma_e \in [0, \infty]$ for its duration viz. $\sigma_e := \sup\{t \geqslant 0 : e(t) \neq 0\}$ and if $\sigma_e = \infty$ we shall use the convention $[0, \sigma_e] := [0, \infty)$. If for every

$s, t \in [0, \sigma_e]$, we set

$$d_e(s,t) := e(s) + e(t) - 2 \cdot \min_{[s\wedge t, s\vee t]} e,$$

the mapping d_e is a pseudo-distance on $[0, \sigma_e]$ and it induces an equivalence relation $\sim_e$ on $[0, \sigma_e]$, by setting $s \sim_e t$ if $d_e(s,t) = 0$. We write $\mathcal{T}_e := [0, \sigma_e]/\sim_e$ for the corresponding quotient space and we endow it with the distance d_e. Let $p_e : [0, \sigma_e] \to \mathcal{T}_e$ be the function mapping every element $s \in [0, \sigma_e]$ to its equivalence class in $\mathcal{T}_e$. The image of 0 under p_e is called the root of $\mathcal{T}_e$, and it will be denoted by $\varnothing$. If we write Vol_e for the pushforward measure of the Lebesgue measure on $[0, \sigma_e]$ by p_e, the triple $(\mathcal{T}_e, d_e, \mathrm{Vol}_e)$ is an $\mathbb{R}$-tree equipped with a volume measure.

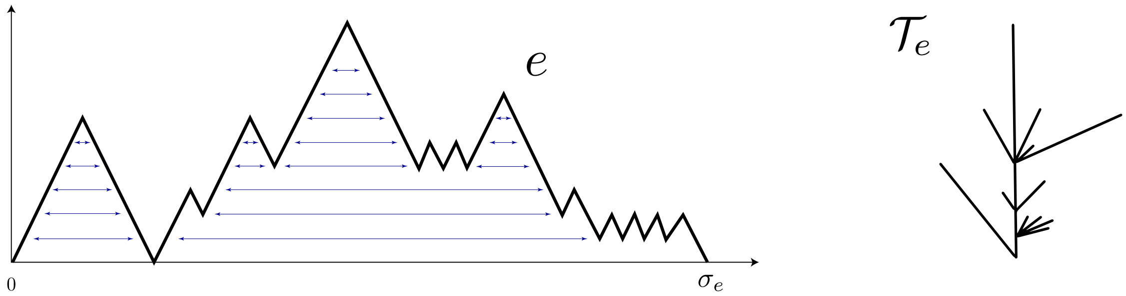


Figure 2.1: A non-negative continuous function e and the resulting tree $\mathcal{T}_e$.

The total mass of the measure Vol_e is $\mathrm{Vol}_e(\mathcal{T}_e) = \sigma_e$ and if $\sigma_e < \infty$, the tree $\mathcal{T}_e$ is compact. Note that the function e is well defined in the quotient space $\mathcal{T}_e$ and for every $a \in \mathcal{T}_e$, the value $e(a)$ is precisely the distance of a to the root $\varnothing$. The set $\mathbb{T}$ of rooted $\mathbb{R}$-trees equipped with a volume measure, considered up to isometry and equipped with the local Gromov-Haussdorf-Prokhorov metric is Polish, we refer to Section 3 on [3] for the precise statement and a detailed account.

Let us briefly discuss some geometric aspects of $\mathcal{T}_e$. First, the multiplicity of a point $a \in \mathcal{T}_e$ is defined as the (possibly infinite) number of connected components of $\mathcal{T}_e \backslash \{a\}$. The points with multiplicity 1 are called the *leaves* of $\mathcal{T}_e$, while the family of points with multiplicity 2 is referred to as *the skeleton*. Finally, every point with multiplicity $i \geqslant 3$ is called a *branching point* and for every $i \in \{3, \dots, \infty\}$ we write $\mathrm{Bp}_i(\mathcal{T}_e)$ for the collection of branching points with multiplicity i. One can interpret $\mathcal{T}_e$ as a genealogical tree, where every $a \in \mathcal{T}_e$ is an individual, and its multiplicity -1 corresponds to its number of children. The branching points with finite multiplicity should be interpreted as microscopic events on the scale of a population evolving through time at which one individual gives birth to a finite number of children, while the ones with infinite multiplicity correspond to macroscopic events at which the population increases dramatically. For every $a, b \in \mathcal{T}_e$, we shall say that a is an ancestor of b if $a \in [\![\varnothing, b]\!]$, and when this holds we write $b \succ a$. It is then natural to interpret the geodesic path $[\![\varnothing, b]\!]$ as the ancestral line of b, the distance to the root of b being the moment in time at which the individual b was alive. The first common ancestor $a \curlywedge b$ between a, b is the element of $\mathcal{T}_e$ defined by the relation $[\![\varnothing, a \curlywedge b]\!] = [\![\varnothing, a]\!] \cap [\![\varnothing, b]\!]$. We stress that the dependence on e when considering geodesic paths $[\![a, b]\!]$ as well as in the ancestral order $\succ$ is omitted to simplify notation.

The tree $\mathcal{T}_e$ comes naturally equipped with a temporal exploration. Namely, we refer to the mapping $(p_e(t) : 0 \leqslant t \leqslant \sigma_e)$ as the *clockwise exploration* of $\mathcal{T}_e$. Roughly speaking, $(p_e(t) : 0 \leqslant t \leqslant \sigma_e)$ starts at time $t = 0$ at the root, and then travels through $\mathcal{T}_e$ in clockwise order following its contour. If for $t \in [0, \sigma_e]$ we set $\mathcal{T}_e(t) := p_e([0, t])$, one can think of $\mathcal{T}_e(t)$ as the subset the tree that has been explored up to time t. On the other hand, the closure of each connected component of $\mathcal{T}_H \backslash \mathcal{T}_e(t)$ is a sub-tree of $\mathcal{T}_e$, say $\mathcal{T}_e^i(t)$, and we denote this family by $(\mathcal{T}_e^i(t) : i \in \mathcal{I})$. In $\mathcal{T}_e$, every $\mathcal{T}_e^i(t)$ is attached to the geodesic path $[\![\varnothing, p_H(t)]\!]$ at some height that we denote by h_i - see Figure 2.5 below. It is then natural to encode $\mathcal{T}_e \backslash \mathcal{T}_e(t)$, viz the subset of $\mathcal{T}_e$ that has yet to be explored, on a point measure $\sum_{i \in \mathcal{I}} \delta_{(h_i, \mathcal{T}_e^i(t))}$ on $\mathbb{R}_+ \times \mathbb{T}$. When working with some fixed t, the dependence on t is omitted to simplify notation.

In what follows, the coding functions we shall consider will be random. Before introducing formally the setting we shall work with, let us start with some motivations from the discreet setting.

Coding of plane trees and Galton-Watson trees.

We write $\mathbf{T}$ for the set of finite plane trees defined through the formalism of Neveu [77]. In particular, every $\mathrm{T} \in \mathbf{T}$ can be thought as a finite graph embedded in the plane with a distinguished point (that we call the root) and with no loops. For simplicity, we shall enumerate the set of vertices $v(\mathrm{T})$ of T in lexicographical order $\{0, 1, 2, \dots, |v(\mathrm{T})| - 1\}$, the root being labelled with 0 and $|v(\mathrm{T})|$ being the cardinally of $v(\mathrm{T})$. The distance between two elements $a, b \in v(\mathrm{T})$ is just the number of edges on the unique path $[\![a, b]\!]$ connecting a and b. The height $h(a)$ of a vertex a is defined as its distance to the root. Now, we define the height process $H = (H_n : 0 \leqslant n \leqslant |v(\mathrm{T})| - 1)$ of T by the relation $H_n := h(n),$ for $0 \leqslant n \leqslant |v(\mathrm{T})| - 1$.

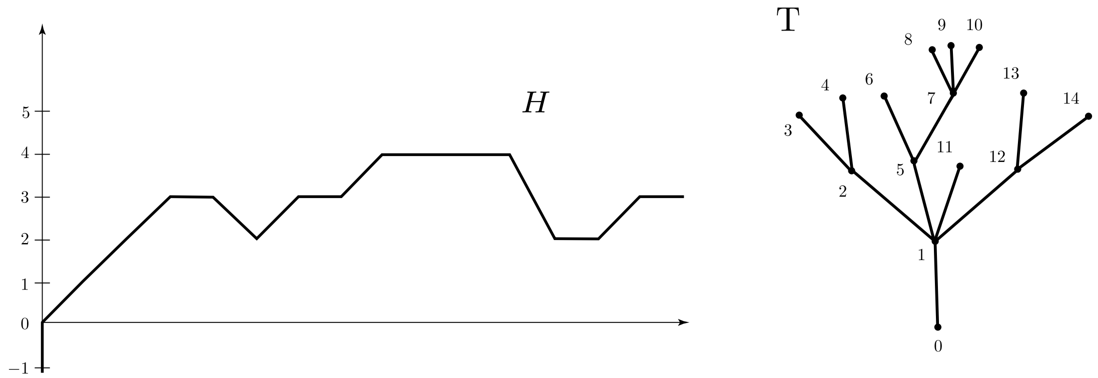


Figure 2.2: A plane tree T and the corresponding height function.

By definition, the height process H encodes the distances to the root when performing a clockwise exploration of T and note that one can clearly recover the tree T from H and vice-versa.

Let us now introduce another closely related functional of T encoding now the progeny of every individual in T. In this direction, note that the notion of multiplicity of a vertex $a \in v(\mathrm{T})$ still makes sense in this setting and we denote it by $k(a)$. The so-called Lukasiewicz walk associated to T is the $\{-1, 0, 1, 2, \dots\}$-valued process $(\mathcal{X}_n)_{n \in \mathbb{N}}$ defined recursively as follows: we set $\mathcal{X}_0 = 0$ and for $n \geqslant 0$, we let

$$\mathcal{X}_{n+1} := \begin{cases} \mathcal{X}_n + k(n+1) - 1, & \text{if } \mathcal{X}_n > -1, \\ -1 & \text{otherwise.} \end{cases}$$

We can think of the variable $\mathcal{X}_n$ as counting the number of vertices attached to the right of the geodesic path connecting the vertex n to the root. For every $0 \leqslant n \leqslant v(T)$, the increment $\mathcal{X}_{n+1} - \mathcal{X}_n + 1$ is precisely the number of children of the vertex n.

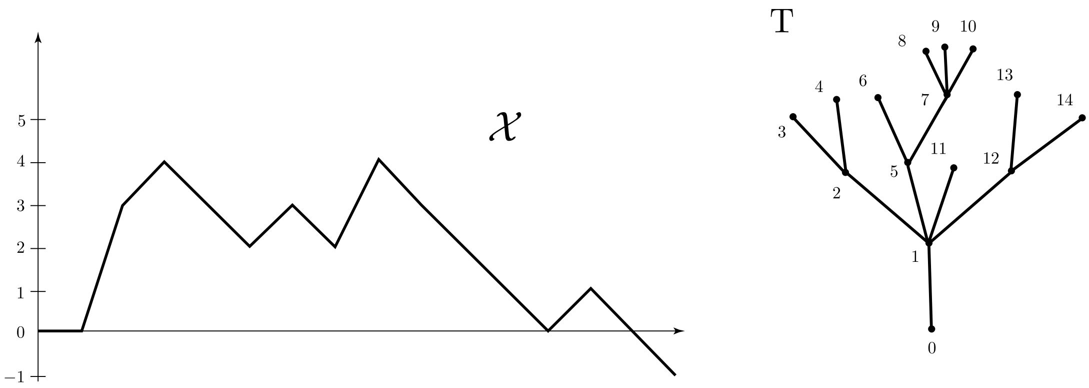


Figure 2.3: A plane tree T and the corresponding Lukasiewicz walk

An important feature of $\mathcal{X}$ is that the height of a vertex $n \in v(\mathrm{T})$ can be obtained by the relation:

$$h(n) = \#\big\{k \in \{0, 1, \dots, n-1\} : \mathcal{X}_k = \inf_{k \leqslant i \leqslant n} \mathcal{X}_i\big\}. \tag{2.1}$$

Therefore, one can recover as well the starting tree T from the path $\mathcal{X}$. If one is only interested in metric properties of T - as shall often be our case - it is clear that H is of greater use than $\mathcal{X}$. The main problem stems from the fact that, even in the simplest models of random plane trees, the height process is in general significantly more difficult to study than the Lukasiewicz walk. Let us be more precise.

We write $\mathrm{GW}(\mu)$ for the law of a Galton Watson tree on $\mathbf{T}$ with sub-critical spring distribution $\mu = (\mu(k) : k \geqslant 0)$. We recall that μ being sub-critical is defined through the condition $\sum_{k \geqslant 0} k\mu(k) \leqslant 1$, and that under this assumption the corresponding Galton-Watson tree is indeed a.s. finite. With a slight abuse of notation, under $\mathrm{GW}(\mu)$ we write T for the corresponding Galton-Watson tree and we maintain the notation $H, \mathcal{X}$ for its height process and the corresponding Lukasiewicz walk. If for every $n \geqslant 0$ we set $Y_n := |\{a \in v(T) : h(a) = n\}|$, the process (Y_n) is a Galton-Watson process describing the evolution of the population encoded by T through time. While in general the process H is not Markovian, the process $\mathcal{X}$ is of a significantly simpler nature. Namely, if for $k \geqslant -1$ we let $\mu'(k) := \mu(k+1)$, the process $\mathcal{X}$ is a simple random walk with step distribution $\mu' = (\mu'(k) : k \geqslant -1)$ started from 0 and stopped at its first hitting time of -1. In the continuum, the role played by the Lukasiewicz walk is (roughly) taken over by an excursion of a Lévy process X and the Galton-Watson process shall be replaced by a continuous-time branching process. As in the discrete setting, the height process in the continuum is a functional of X and the corresponding tree encoded by H is a Lévy tree. Defining formally H in terms of X is a technical task, but we shall see that its definition stems from the motivations we provided above. Further, since Lévy trees can be obtained as limit of scaled Galton-Watson trees, Lévy trees should be considered as their continuum analogue - see e.g. Section 2 of [43]. Let us now introduce the setting we shall be working with for the rest of this work.

Populations encoded by spectraly positive Lévy processes.
Let Y be a continuous state branching process with branching mechanism ψ started at some $x > 0$. One can think of Y as describing the evolution of a population through time and in particular, the stopping time $\inf\{t \geqslant 0 : Y_t = 0\}$ is interpreted (when finite) as the extinction of the corresponding population. We further assume that the branching process is (sub)critical, viz

that $\psi'(0+) \geqslant 0$ and that the following condition holds:

$$\int_1^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)} < \infty. \tag{2.2}$$

It is well known that these two conditions ensure the almost sure extinction of Y. Further, under our standing hypothesis, the branching mechanism is necessarily of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}x)(e^{-\lambda x} - 1 - x\lambda)$$

for some $\alpha, \beta \geqslant 0$ and a measure π on $(0,\infty)$ satisfying the condition $\int \pi(\mathrm{d}x)(x \wedge x^2) < \infty$. We shall be now interested in encoding the genealogy of the population associated to Y in a random tree. The almost sure extinction of Y will ensure that the corresponding genealogical tree is compact. More precisely, it has been argued in [43] that the genealogy of its population can be encoded in a ψ-Lévy tree. The later is an $\mathbb{R}$-tree, coded by a continuous functional of a spectrally positive Lévy process X with Laplace exponent ψ, known under the name the height process. Before providing a proper introduction, for later use we start with some general facts on the ψ-Lévy process X, that we define for concreteness under some probability P. First, recall that X and ψ are linked by the relation:

$$E[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)), \qquad \text{for } \lambda \geqslant 0$$

and note that since condition (2.2) holds if and only if $\beta > 0$ or $\int_{(0,\infty)} x\pi(\mathrm{d}x) = \infty$, the paths of X have infinite variation. Further, since we have $\psi'(0+) \geqslant 0$, it follows that X oscillates or drift towards $-\infty$ and in particular, the running infimum $I = \inf_{[0,t]} X$ drifts towards $-\infty$. Now, we shall write $X - I = (X_t - \inf_{[0,t]} X : t \geqslant 0)$ for the reflected Lévy process at its running infimum. It is classic that $X - I$ is a strong Markov process and that the point 0 is instantaneous and regular. Further, $-I$ is a local time at 0 for $X - I$ and we write N for the corresponding excursion measure. We shall now explain how one can construct from X a genealogical tree encoding the population of Y, by introducing the height process of X. The theory to achieve this was developed in the monograph [43] and we shall now present some of its elements.

The height and exploration processes.

As we already mentioned, in the continuum the role played by the Lukasiewicz walk is taken over by the Lévy process X. Now, the corresponding height H_t associated to some fixed $t \geqslant 0$ can be understood as follows: informally, under N and under P the variable H_t measures the size of the set

$$\{s \in [0,t] : \ X_{s-} \leqslant \inf_{s \leqslant r \leqslant t} X_r\}. \tag{2.3}$$

Observe that this is reminiscent of (2.1). One can make sense of this informal description by making use of local times and a time-reversal argument. Let us be more precise: first, for each $t \geqslant 0$, we consider the time-reversed process

$$\widehat{X}^{(t)}_s := X_t - X_{(t-s)-} \quad \text{and} \quad \widehat{S}^{(t)}_s := \sup_{[0,s]} \widehat{X}^{(t)}, \qquad \text{for } 0 \leqslant s \leqslant t$$

with the convention $\widehat{X}^{(t)}_t = X_t$. Then, it is well known that $(X_s : 0 \leqslant s \leqslant t)$ has the same distribution as the time-reversed process $(\widehat{X}^{(t)}_s : 0 \leqslant s \leqslant t)$. Further, the point 0 is instantaneous and regular for the strong Markov process $S - X = (\sup_{[0,t]} X - X_t : t \geqslant 0)$ and the restriction of the latter to $[0,t]$ is distributed as $\hat{S}^{(t)} - \hat{X}^{(t)}$. Now, for every $t \geqslant 0$ and for some fixed decreasing sequence (ε_k) converging to 0 we set

$$\Gamma_t(\hat{X}^{(t)}) := \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t \mathrm{d}s \, 1_{\{\sup_{[0,s]} \hat{X}^{(t)} - \hat{X}^{(t)}_s < \varepsilon_k\}},$$

the convergence holding a.s. Then, $\Gamma_t(\widehat{X}^{(t)})$ is the value of the local time at 0 of $\widehat{S}^{(t)} - \widehat{X}^{(t)}$ taken at time t and note that the set

$$\{s \in [0,t] : \widehat{S}^{(t)}_s - \widehat{X}^{(t)}_s = 0\}$$

is precisely the image of (2.3) under the mapping $s \mapsto t - s$. Now, for every $t \geqslant 0$, we can define the value of the height process at time t by the relation:

$$H_t := \Gamma_t(\widehat{X}^{(t)}).$$

Under our standing assumptions on ψ, Theorem 1.4.3 from [43] ensures that the non-negative process $H = (H_t : t \geqslant 0)$ possesses a continuous modification that we consider from now on and still denote by H. It will be crucial for our purposes to note that H is well defined under the excursion measure N. From the informal description given in (2.3), this should not come as a surprise since for each $t \geqslant 0$, the variable H_t only depends on the excursion of $X - I$ straddling t. Moreover, the excursion intervals away from 0 of H and $X - I$ coincide. We can now introduce:

Definition 2.1. *We define the law of the ψ-Lévy tree as the law on $\mathbb{T}$ of the $\mathbb{R}$-tree $\mathcal{T}_H$ under the excursion measure N, and the law of a ψ-Lévy forest as the law of the non-compact metric space $\mathcal{T}_H$ under P.*

The terminology stems from the fact that under N, $\mathcal{T}_H \backslash \varnothing$ posses a unique connected component while under P, the set $\mathcal{T}_H \backslash \varnothing$ is conformed by an infinite family of connected components. Moreover, the closure of each one of them is on its turn a compact tree. Namely, each excursion H^i of H away from 0 gives rise to a compact tree $\mathcal{T}_{H^i}$, and $\mathcal{T}_H$ is obtained by concatenating the family $(\mathcal{T}_{H^i} : i \in \mathbb{N})$ at their respective roots, following the order induced by the local time $-I$. Therefore, we can think of every excursion H^i as encoding the evolution of a sub-population with corresponding genealogical tree $\mathcal{T}_{H^i}$. In particular, under N the process H describes the evolution of a single sub-population up to the moment of its extinction. Let us mention that the connection with the starting ψ-CSBP Y can made through the Ray-Knight theorem [43, Theorem 1.4.1]; since it relies in the notion of the local time at a height $a \geqslant 0$, we shall not provide the details. When X is a Brownian motion, N is [1] the positive Itô excursion measure, H under N is a non-negative Brownian excursion and $\mathcal{T}_H$ is the so-called free *Brownian tree*. Further, under the conditioning $\sigma = 1$, $\mathcal{T}_H$ is the CRT or *Continuum Random Tree* [4].

In the same vein as Galton-Watson trees, Lévy trees satisfy a branching property. In this direction, recall that by construction, for every t the variable H_t is the distance of $p_H(t)$ from $\varnothing$ and we write $\mathcal{H}(\mathcal{T}_H) := \sup_{t\geqslant 0} H_t$ for the height of $\mathcal{T}_H$. For $h, \varepsilon > 0$, under N and on the event $\mathcal{H}(\mathcal{T}_H) > h + \varepsilon$, we let $\mathcal{T}^1, \dots, \mathcal{T}^K$ be the sub-trees of $\mathcal{T}_H$ rooted at height h and reaching a height $\mathcal{H}(\mathcal{T}^i) > \varepsilon$. Then, by Corollary 3.2 in [44], for $k \geqslant 1$ and conditionally on $K = k$, the

[1] Up to an unimportant factor 2.

sub-trees $\mathcal{T}^1, \ldots, \mathcal{T}^k$ are independent and distributed as $\mathcal{T}_H$ under $N(\,\cdot\,|\mathcal{H}(\mathcal{T}_H) > \varepsilon)$. Notably, a converse statement holds, we refer to Theorem 1.1 in [91].

Let us make a brief comment on how the geometry of $\mathcal{T}_H$ is influenced by the choice of ψ. First, it is important to mention that if $i \notin \{3, \infty\}$ the set $\mathrm{Bp}_i(\mathcal{T}_H)$ is a.e. empty independently of the choice of ψ. The set of branching points $\mathrm{Bp}_3(\mathcal{T}_H)$ corresponds to strict local minima of H, while $\mathrm{Bp}_\infty(\mathcal{T}_H)$ is in bijection with $\{s \geqslant 0 : \Delta X_s > 0\}$ by the mapping p_H. For instance, $\mathcal{T}_H$ only possesses points of infinite multiplicity if the Lévy measure of ψ is non-null. One can further argue that the "fractal size" of the branching point $p_H(t) \in \mathrm{Bp}_\infty(\mathcal{T}_H)$ for $t \in \{s \geqslant 0 : \Delta X_s > 0\}$ is precisely ΔX_t. This notion can be made rigorous by making use of the notion of *the local time*

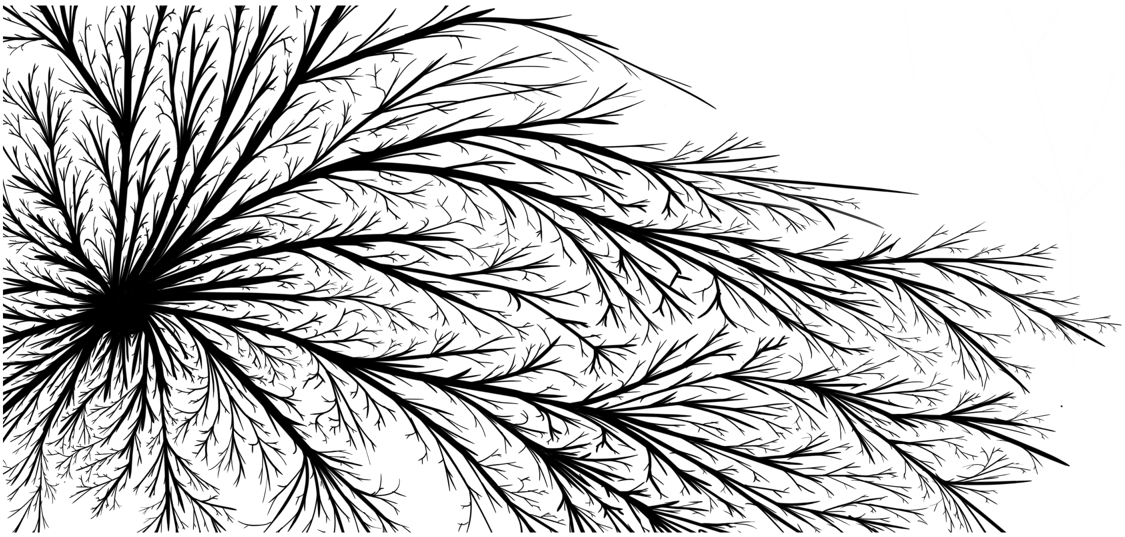


Figure 2.4: Zooming around an infinite multiplicity point associated to a large jump of the Lévy process.

at a branching point introduced below. For a thorough study of the geometry of Lévy trees we refer to [44].

As for Galton-Watson trees, the height process of a Lévy process is in general not Markovian, and this makes its study a rather difficult task. To this end, let us introduce another crucial process closely related to H called the *exploration process*. While H encodes the distances of $(p_H(t) : 0 \leqslant t \leqslant \sigma_H)$ to the root $\varnothing$ of $\mathcal{T}_H$, the exploration process further encodes at each t, the height of the sub-trees in $\mathcal{T}_H$ attached to the right of the geodesic path $[\![0, p_H(t)]\!]$; let us be more precise. For every $0 \leqslant s \leqslant t$, set $I_{s,t} := \inf_{[s,t]} X$ and write $\mathcal{M}_f(\mathbb{R}_+)$ for the space of finite function in $\mathbb{R}_+$ equipped with the weak topology. The exploration process is the $\mathcal{M}_f(\mathbb{R}_+)$ valued process denoted by $\rho = (\rho_t : t \geqslant 0)$ and defined, for each $t \geqslant 0$ under N and under P, by the relation

$$\rho_t(\mathrm{d}h) := \beta \mathbb{1}_{[0,H_t]}(h)\mathrm{d}h + \sum_{\substack{0<s\leqslant t \\ X_{s-}<I_{s,t}}} (I_{s,t} - X_{s-})\, \delta_{H_s}(\mathrm{d}h), \quad t \geqslant 0. \tag{2.4}$$

We denote by $(\mathcal{F}_t)$ the completed natural filtration of ρ. If for $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ we write $H(\mu) := \sup \operatorname{supp} \mu$ and $\langle \mu, 1 \rangle := \mu([0, \infty))$ for the total mass of μ, the pair $(H, X - I)$ and ρ are linked by the identities $H_t = H(\rho_t)$ and $\langle \rho_t, 1 \rangle = X_t - I_t$ for $t \geqslant 0$. The key now is that despite its technical definition, ρ is a right-continuous (with respect to the total variation distance of measures) strong Feller process [2].

For every fixed t, the variable ρ_t carries the following geometric information: let $[\![\varnothing, p_H(t)]\!]$ be the path connecting $p_H(t)$ to the root and consider the measure $\mathcal{M} = \sum_{j \in \mathcal{I}_t} \delta_{(h_j, \mathcal{T}_H^j)}$ on $\mathbb{R}_+ \times \mathbb{T}$, composed by the family of subtrees attached to the right of $[\![\varnothing, p_H(t)]\!]$ indexed by their respective heights. Then by the Markov property, conditionally on $\mathcal{F}_t$, the measure $\mathcal{M}$ is a Poisson point measure with intensity $\rho_t(\mathrm{d}h)N(\mathcal{T}_H \in \cdot)$. For example, if X is a Brownian motion, the measure ρ_t is simply given by $\mathbb{1}_{[0,H_t]}(h)\mathrm{d}h$ and therefore, conditionally on $\mathcal{F}_t$, the heights of the subtrees that are yet to be explored are uniformly spread and dense in $[0, H_t]$, each $\mathcal{T}_H^i$ being distributed as $\mathcal{T}_H$ under the positive Itô measure. To formalise the previous statements one needs to introduce the law of ρ started from an arbitrary $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, we refer to Chapter 1 in [43] for a detailed discussion.

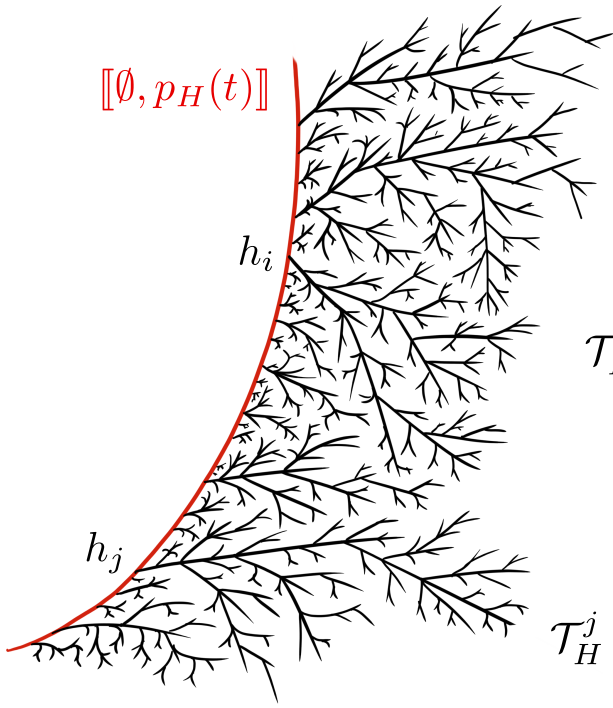


Figure 2.5: Sketch of the right spine seen from $p_H(t)$.

Let us close our discussion on the height process and the exploration process with an important result for the development of Section 2.5. We already mentioned that H is a functional of ρ since we can write $H = (H(\rho_t) : t \geqslant 0)$. Further, by construction ρ is a function of the Lévy process X. In the other direction, the null measure 0 is regular and instantaneous for ρ and $-I$ is a local time for ρ. Since for every $t \geqslant 0$ we have $\langle \rho_t, 1 \rangle = X_t - I_t$, it follows that X can be recovered from ρ as well. Notably, by Lemma 6.1 the Lévy process X and (therefore ρ) can be constructed from H, which yields that despite the fact that X, ρ and H encode different aspects of $\mathcal{T}_H$, they carry the same amount of information.

The local time at a branching point

We now introduce an important notion for the sequel, which is the *local time at a branching point* $b \in \mathrm{Bp}_\infty(\mathcal{T}_H)$. This is a continuous, non-decreasing process $\lambda^{\ell,b} = (\lambda^{\ell,b} : 0 \leqslant t \leqslant \sigma_H)$ measuring at every $t \geqslant 0$ the time spent by $(p_H(s) : s \geqslant 0)$ at b up to time t. We refer to $\lambda_\infty^{\ell,b}$ as the total mass of b. The content of this section has been adapted from [83].

Recall that the mapping p_H realises a bijection between the sets $\{t \geqslant 0 : \Delta X_t > 0\}$ and $\mathrm{Bp}_\infty(\mathcal{T}_H)$. For every jump-time $s \in \{t \geqslant 0 : \Delta X_t > 0\}$ we write $z(s) := \inf\{t \geqslant s : H_t < H_s\}$, the latter coinciding with $\inf\{t \geqslant s : X_t \leqslant X_{s-}\}$. We define the local time at the branching point $b := p_H(s)$ by the relation:

$$\lambda_t^{\ell,b} := X_s - I_{s,t}, \text{ for } t \in [s, z(s)],$$

with $\lambda_t^{\ell,b} = 0$ if $0 \leqslant t < s$ and $\lambda_t^{\ell,b} = \Delta X_t$ if $t > z(s)$. From our definitions, the process $\lambda^{\ell,b}$ is continuous and non-decreasing. Next, for $t \geqslant 0$ we let $\lambda_t^{r,b} := \Delta X_s - \lambda_t^{r,b}$ which in particular gives that

$$\lambda_t^{r,b} = I_{s,t} - X_{s-}, \text{ for } t \in [s, z(s)].$$

We shall now justify our terminology. Consider the connected components of $\{s \leqslant t \leqslant z(s) : H_t > \min_{[s,t]} H\}$ as well as the corresponding excursions of $(H_t : s \leqslant t \leqslant z(s))$ over its running infimum. Each excursion interval is mapped by p_H in a sub-tree $\mathcal{T}^i$ rooted at b. If $(u, v) \subset \mathbb{R}_+$ is an arbitrary interval, we set $n((u, v), \varepsilon)$ for the number of these excursions starting in (u, v) and reaching a height greater than ε. This corresponds roughly to the number of trees $\mathcal{T}^i$ rooted at b with height greater than ε and contained in the closure of $\mathcal{T}_H(v)\backslash\mathcal{T}_H(u)$. Finally, for $\varepsilon > 0$ we set $v(\varepsilon) := N(\sup_t H_t > \varepsilon)$. Lemma 6.2 states that, under P and N, a.e. for every $b \in \mathrm{Bp}_\infty(\mathcal{T}_H)$ and $t \geqslant 0$ we have the following convergences:

$$\lambda_t^{\ell,b} = \lim_{\varepsilon \to 0} \frac{n_b((t \wedge s, t \wedge z(s)), \varepsilon)}{v(\varepsilon)}, \quad \lambda_t^{r,b} = \lim_{\varepsilon \to 0} \frac{n_b((t \vee s, t \vee z(s)), \varepsilon)}{v(\varepsilon)}$$

where $s := p_H^{-1}(b) \cap \{t \geqslant 0 : \Delta X_t > 0\}$. Moreover, the family $((\lambda^{\ell,b}, \lambda^{r,b}), b \in \mathrm{Bp}_\infty(\mathcal{T}_H))$ can be obtained as a function of H. The approximation of the total mass for points with infinite multiplicity was already considered in [44], but it will be crucial for our purposes to have the refined version stated above. Observe that the exploration process ρ can be expressed in terms of the family $(\lambda^{r,b} : b \in \mathrm{Bp}(\mathcal{T}_H))$. Namely, for every $t \geqslant 0$, we have

$$\rho_t(\mathrm{d}h) := \beta \mathbb{1}_{[0,H_t]}(h)\mathrm{d}h + \sum_{\substack{0 < s \leqslant t \\ \Delta X_s > 0}} \lambda_t^{r,b(s)} \delta_{H_s}(\mathrm{d}h), \quad t \geqslant 0 \tag{2.5}$$

with the notation $b(s) := p_H(s)$. For fixed $t > 0$, the variable $\lambda_t^{r,b}$ thus encodes the "number" of sub-trees rooted at b that have yet to be explored after time t.

2.2 Markov processes indexed by Lévy trees

All the ingredients are in place to introduce the notion of a Markov process indexed by a Lévy tree. Since the formal definition of this process relies in the so-called Lévy Snake which is a rather technical time-indexed process, we start with an informal discussion. Let E be a Polish space and fix an arbitrary $y \in E$. For every $y \in E$ we let $(\xi_t : t \geqslant 0)$ under Π_y be a continuous time-indexed strong Markov process taking values in E and started at $\xi_0 = y$. We can define informally the Markov process ξ indexed by $\mathcal{T}_H$ and started from y as follows. We first start by sampling $\mathcal{T}_H$ under N or P. Then, conditionally on $\mathcal{T}_H$, we consider a spatial motion governed by Π_y and indexed by $\mathcal{T}_H$. The motion starts at the root $\varnothing \in \mathcal{T}_H$ at y and moves through $\mathcal{T}_H$ away from $\varnothing$ along the geodesic paths according to Π_y, with the condition that at each branching point of $\mathcal{T}_H$, it splits into independent copies with same law. This process is denoted by $(\xi_a)_{a \in \mathcal{T}_H}$. We stress that we are dealing with two layers of randomness: the branching structure of $(\xi_a)_{a \in \mathcal{T}_H}$ is determined by the choice of ψ while the spatial displacement is governed by Π_y.
Now, the corresponding Lévy snake is a time indexed process encoding both the branching structure and the labels of $(\xi_a)_{a \in \mathcal{T}_H}$. More precisely, for each t, write W_t for the finite E-valued finite path defined by the relation

$$W_t := \big(\xi_a : a \in [\![0, p_H(t)]\!]\big).$$

In other terms, W_t encodes the labels of the ancestral line $[\![\varnothing, p_H(t)]\!]$ of $p_H(t)$, and recall that ρ_t encodes precisely the right spine attached to $[\![\varnothing, p_H(t)]\!]$ in $\mathcal{T}_H$. The Lévy snake is the pair (ρ_t, W_t) for $t \geqslant 0$, when $\mathcal{T}_H$ is sampled under N or under P. We stress that this definition is

Figure 2.6: Sketch of the spatial positions of an $\mathbb{R}^2$-valued Markov process indexed by a Lévy tree. The underlying tree is not being plotted but can be recovered by looking at the genealogy of the spatial positions. The paths do not intersect in the picture only for sake of clarity.

purely heuristic, and for instance, the notion of a Markov process indexed by a Lévy tree is defined through the Lévy snake, and not the other way around. Let us now give a formal definition.

The ψ Lévy snake with spatial motion ξ.

The content of this section is taken from [43]. We write $\mathcal{W}_E$ for the space of finite E-valued continuous paths. More precisely, every element $\mathrm{w} \in \mathcal{W}_E$ is a continuous path $\mathrm{w} : [0, \zeta_{\mathrm{w}}] \to E$ defined in a compact interval for some finite $\zeta_{\mathrm{w}} \in [0, \infty)$, that we refer to as the lifetime of w. Further, we shall denote by $\widehat{\mathrm{w}}$ the tip of the path $\mathrm{w}(\zeta_{\mathrm{w}})$. With a slight abuse of notation, for $y \in E$ we still denote by y the element $\mathrm{w} \in \mathcal{W}_E$ with $\mathrm{w}(0) = y$ having null lifetime. If we equip the space $\mathcal{W}_E$ with the distance

$$d_{\mathcal{W}_E}(\mathrm{w}, \mathrm{w}') := |\zeta_{\mathrm{w}} - \zeta_{\mathrm{w}'}| + \sup_{r \geqslant 0} d_E\big(\mathrm{w}(r \wedge \zeta_{\mathrm{w}}), \mathrm{w}'(r \wedge \zeta_{\mathrm{w}'})\big)$$

the metric space $(\mathcal{W}_E, d_{\mathcal{W}_E})$ is Polish. We start by defining the notion of a snake driven by a continuous function, with spatial motion $\Pi := (\Pi_y)_{y \in E}$. This is a $\mathcal{W}_E$-valued time inhomogenous Markov process that was first introduced in [43]. To this end, fix a finite path $\mathrm{w} \in \mathcal{W}_E$ with $\mathrm{w}(0) = y$ for some fixed $y \in E$ and for every $0 \leqslant a \leqslant \zeta_{\mathrm{w}}$, $b \geqslant a$ we consider a probability kernel $R_{a,b}(\mathrm{w}, d\mathrm{w}')$ on $\mathcal{W}_E$ characterised by the following properties:

(i) $R_{a,b}(\mathrm{w}, d\mathrm{w}')$-a.s., $\mathrm{w}'(s) = \mathrm{w}(s)$ for every $s \in [0, a]$.

(ii) $R_{a,b}(\mathrm{w}, d\mathrm{w}')$-a.s., $\zeta_{\mathrm{w}'} = b$.

(iii) Under $R_{a,b}(\mathrm{w}, \mathrm{dw}')$, $(\mathrm{w}'(s+a))_{s\in[0,b-a]}$ is distributed as $(\xi_s)_{s\in[0,b-a]}$ under $\Pi_{\mathrm{w}(a)}$.

We next fix a continuous function $h : \mathbb{R}_+ \times \mathbb{R}_+$ with $h(0) = \zeta_{\mathrm{w}_0}$, that we refer as the driving function, and for $0 \leqslant s \leqslant t$ we let $m_h(s,t) := \min_{[s,t]} h$. If we write $W = (W_t : t \geqslant 0)$ for the canonical process in $\mathcal{W}_E^{\mathbb{R}_+}$, for every fixed $\mathrm{w}_0 \in \mathcal{W}_E$ we shall denote by $Q^h_{\mathrm{w}_0}$ the probability measure on $\mathcal{W}_E^{\mathbb{R}_+}$ characterised by the following relation:

$$\begin{aligned} &Q^h_{\mathrm{w}_0}\big(W_{s_0} \in A_0, W_{s_1} \in A_1, ..., W_{s_n} \in A_n\big) \\ &= \mathbb{1}_{\{\mathrm{w}_0\in A_0\}} \int_{A_1\times\cdots\times A_n} R_{m_h(s_0,s_1),h(s_1)}(\mathrm{w}_0, \mathrm{dw}_1)\dots R_{m_h(s_{n-1},s_n),h(s_n)}(\mathrm{w}_{n-1}, \mathrm{dw}_n) \end{aligned}$$

holding for every $0 = s_0 < s_1 < ... < s_n$ and $A_0, ..., A_n$ Borelian sets of $\mathcal{W}_E$. We refer to the canonical process W under $Q^h_{\mathrm{w}_0}$ as the snake driven by h with spatial motion Π started from w_0. Informally the dynamics of W under $Q^h_{\mathrm{w}_0}$ can be described as follows: at time 0, the path W_0 is precisely w_0; when h decreases, this path is erased from its tip while when h increases, it is extended by adding "little pieces" of trajectories of ξ at the tip. The term snake stems from the fact that, for every fixed $0 \leqslant s < t$, we have that $W_s(r) = W_t(r)$ for every $0 \leqslant r \leqslant m_h(s,t)$.

Now, we shall randomise the driving function h by considering instead the height process of a ψ-Lévy tree. Formally, denote by $\mathbf{P}_0$ the law of ρ under P on $\mathcal{M}_f(\mathbb{R}_+)$ where, with a slight abuse of notation, we write 0 for the identically null measure. We define a probability measure on $(\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)^{\mathbb{R}_+}$ by the relation:

$$\mathbb{P}_{0,y}(\mathrm{d}\rho, \mathrm{d}W) := \mathbf{P}_0(\mathrm{d}\rho) Q_y^{H(\rho)}(\mathrm{d}W).$$

The process (ρ, W) under $\mathbb{P}_{0,y}$ is referred to as the ψ-snake with spatial motion Π started from $(0, y)$. For a more general definition of the Lévy snake starting from any pair $(\mu, \mathrm{w}) \in \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, we refer to [43, Chapter 4.1]. Under appropriate assumptions on the pair (ψ, Π) that we shall assume from now on, the $\mathcal{W}_E$-valued process $(W_t : t \geqslant 0)$ possesses a continuous modification with respect to the metric $d_{\mathcal{W}_E}$ - we refer to Proposition 5.2 for the precise assumptions. With a slight abuse of notation, this modification shall still be denoted by W. For latter use, we gather some of its main properties:

- The pair $(\rho, W) := ((\rho_t, W_t) : t \geqslant 0)$ is a right-continuous $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$-valued strong Markov process.

- The point $(0, y)$ is instantaneous and regular for (ρ, W) and $-I = -\inf_{[0,t]} X$ can be taken as a local time. Further, the corresponding excursion measure $\mathbb{N}_y$ can be written as follows:

$$\mathbb{N}_y(\mathrm{d}\rho, \mathrm{d}W) = N(\mathrm{d}\rho) Q_y^{H(\rho)}(\mathrm{d}W).$$

- For each fixed t, under $\mathbb{P}_{0,y}$ and $\mathbb{N}_y$, the path W_t conditionally on H_t is distributed as $(\xi_h : 0 \leqslant h \leqslant H_t)$ under Π_y.

- The process W under $\mathbb{P}_{0,y}$ and under $\mathbb{N}_y$ satisfies the *snake property*: a.e. for every $0 \leqslant s < t$, we have that

$$W_s(r) = W_t(r), \qquad \text{for every } 0 \leqslant r \leqslant m_H(s,t)$$

The snake property entails in particular that a.e., under $\mathbb{P}_{0,y}$ and $\mathbb{N}_y$, if $H_s = H_t = m_H(s,t)$ the two tip of paths $\widehat{W}_t$, $\widehat{W}_s$ coincide. This gives that $\widehat{W} := (\widehat{W}_t : t \geqslant 0)$ is well defined in the quotient space $\mathcal{T}_H$ and with a slight abuse of notation we denote this process by $(\xi_a)_{a\in\mathcal{T}_H}$.

Remark: In the sequel, for sake of clarity but at the expense of some lack of rigour, we shall often write our explanations in terms of the tree indexed process $(\xi_a)_{a\in\mathcal{T}_H}$ instead of (ρ, W). This practice will however be avoided for our main results.

The study of geometric properties of $(\xi_a)_{a\in\mathcal{T}_H}$ often relies in understanding the law of (ρ, W) at typical times sampled with respect to random measures of different natures. These formulas are often colloquially known under the name *many-to-one* formulas. In this direction, under E^0, denote by $\mathcal{U} = (\mathcal{U}_t : t \geqslant 0)$ a subordinator with Laplace exponent $\psi(\lambda)/\lambda - \alpha$ for $\lambda \geqslant 0$. It might be worth recalling that the subordinator with Laplace exponent $\psi(\lambda)/\lambda$ is the so-called ladder height process of X, see e.g. Lemma 1.1.2 [43]. For every $a \geqslant 0$, let J_a be the element of $\mathcal{M}_f(\mathbb{R}_+)$ defined by the relation $J_a := 1_{[0,a]}(r)\, \mathrm{d}\mathcal{U}_r$. The following lemma is taken from [43].

Lemma 2.2. [43, Formula (4.2)] *For fixed $y \in E$ and for every non-negative measurable function Φ in $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, we have:*

$$\mathbb{N}_y\Big(\int_0^\sigma \mathrm{d}s\, \Phi(\rho_s, W_s)\Big) = \int_0^\infty \mathrm{d}a\, \exp\big(-\alpha a\big) \cdot E^0 \otimes \Pi_y\Big[\Phi\big(J_a, (\xi_s : s \leqslant a)\big)\Big].$$

For a more general statement describing the law of the right and left spines, we refer to Proposition 6.17.

The later formula should be interpreted as follows. Consider the pointed measure $\mathbb{N}_y^\bullet$ in $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E \times \mathbb{R}_+$ defined by the relation $\mathbb{N}_y^\bullet(F(\rho, W, U)) = \mathbb{N}_y\big(\int_0^\sigma \mathrm{d}s\, F(\rho, W, s)\big)$. Note that the law of (ρ, W) under $\mathbb{N}_y^\bullet$ is absolutely continuous with respect to $\mathbb{N}_y$. Under $\mathbb{N}_y^\bullet$, one should think of U as a point uniformly distributed in $[0, \sigma]$ conditionally on (ρ, W). Then, the law of the Lévy snake at a typical point (ρ_U, W_U) under $\mathbb{N}_y^\bullet$ is characterised by the following. Conditionally on H_U, the pair (ρ_U, W_U) is independent and the law of H_U is given by $\exp(-\alpha a)1_{\mathbb{R}_+}\mathrm{d}a$. Moreover, for every $a \geqslant 0$ and conditionally on $H_U = a$, the pair (ρ_U, W_U) is distributed as $\big(J_a, (\xi_s : s \leqslant a)\big)$ under $E^0 \otimes \Pi_y$.

The study of time indexed Markov processes relies crucially on the Markov property. However, the latter is intrinsically related to time, and therefore not suited for the tree-indexed process $(\xi_a)_{a\in\mathcal{T}_H}$. We shall now introduce the preliminary notions needed to state a *spatial* version of the Markov property satisfied by $(\xi_a)_{a\in\mathcal{T}_H}$.

Exit local times and the special Markov property.

Let us start presenting the technical background needed to state our first main result. Fix an open set D of the Polish space E as well as a point $y \in D$, and consider $(\xi_a)_{a\in\mathcal{T}_H}$ under $\mathbb{N}_y$. If under Π_y we let τ_D be the first exit time of the Markov process from D we further assume that

$$\Pi_y(\tau_D < \infty) > 0.$$

Let $\mathrm{Tr}(\mathcal{T}_H)$ be the subset of $\mathcal{T}_H$ conformed by points $a \in \mathcal{T}_H$ satisfying that, for every $b \in [\![\varnothing, a[\![$ we have $\xi_b \in D$. Then, $\mathrm{Tr}_D(\mathcal{T}_H)$ is a tree and we denote by $\mathrm{Tr}_D(\xi)$ the restricted process $(\xi_a : a \in \mathrm{Tr}_D(\mathcal{T}_H))$. Further, the closure of each connected component of $\mathcal{T}_H \backslash \mathrm{Tr}_D(\mathcal{T}_H)$ is as well a tree. We denote this family of sub-trees by $(C_i^D : i \in \mathcal{I})$ and we set $\xi_i^D := (\xi_a : a \in C_i^D)$ for its corresponding labels. If we let $\varnothing^i$ be the root of C_i^D, it follows that ξ_i^D is a tree-indexed process started at the point $\xi_i^D(\varnothing^i) \in D$, and we shall refer to it as a subtrajectory that exits the domain D. Due to the fractal nature of $\mathcal{T}_H$, the number of subtrajectories that exits the domain D is a.e. either 0 or (countably) infinite.

Figure 2.7: Sketch of the spatial positions of an $\mathbb{R}^2$-valued Markov process indexed by a Lévy tree $(\xi_a)_{a\in\mathcal{T}_H}$ started from some $y \in D$. We stress that a subtrajectory exiting the domain D might return into D, but any subsequent exit from D of the latter will not translate into an exit point from D for the spatial motion $(\xi_a)_{a\in\mathcal{T}_H}$.

The exploration $(p_H(t) : 0 \leqslant t \leqslant \sigma_H)$ induces a natural order on this family of subtrajectories. Namely, if for every $i \in \mathcal{I}$ we write $a_i := \inf\{t \geqslant 0 : p_H(t) \in C_i^D\}$ for the first time the exploration visits the component C_i^D, we can define a partial order on $(\xi_i^D : i \in \mathcal{I})$ by considering the order induced by the corresponding first visit times $(a_i : i \in \mathcal{I})$. Note however that this order can not be recovered solely from the family $(\xi_i^D : i \in \mathcal{I})$. To this end, we shall now explain how one can index the family of subtrajectories away from D by means of a continuous additive functional of the Lévy snake compatible with the ordering induced by the clockwise exploration. For every $t \geqslant 0$ recall the notation $\mathcal{T}_H(t) = p_H([0,t])$ and for every $i \in \mathcal{I}$, we set $C_i^D(t) = C_i^D$ if $C_i^D(t) \subset \mathcal{T}_H(t)$ while $C_i^D(t) := \varnothing$ otherwise. Then, if we write $B_\varepsilon^i(\varnothing^i)$ for the ball of radius ε centred at $\varnothing^i$ in $C_i^D(t)$, the limit

$$L_t^D := \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathrm{Vol}_H\Big(\bigcup_{i\in\mathcal{I}} C_i^D(t) \cap B_\varepsilon^i(\varnothing^i)\Big)$$

exists a.e. uniformly in compact intervals. The process $L^D = (L_t^D : t \geqslant 0)$ is a continuous, non-decreasing additive functional of (ρ, W) called the *exit local time from* D [43, Section 4.3]. The terminology can be justified by the fact that, roughly speaking, at each fixed t, the variable L_t^D measures the "number" of connected components of $\mathcal{T}_H(t)\backslash \mathrm{Tr}_D(\xi)$. Moreover, the Stieltjets measure $\mathrm{d}L^D$ is supported on the set:

$$\big\{t \geqslant 0 : \xi_{p_H(t)} \in \partial D\big\}.$$

The following first-moment formula taken from [43, Proposition 4.3.2] describes the law of (ρ, W) at a typical time taken with respect to the measure $\mathrm{d}L^D$.

Lemma 2.3. [43, Proposition 4.3.2] *For fixed $y \in E$ and for every non-negative measurable function Φ in $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, we have:*

$$\mathbb{N}_y\left(\int_0^\sigma \mathrm{d}L_s^D \ \Phi(\rho_s, W_s)\right) = E^0 \otimes \Pi_y\Big[\mathbb{1}_{\{\tau_D<\infty\}} \exp(-\alpha\tau_D)\Phi\big(J_{\tau_D}, (\xi_t : t \leqslant \tau_D)\big)\Big].$$

We can now turn our attention to the first main result in [82], which is a spacial version of the Markov property, known as the special Markov property. Roughly speaking, the special Markov property describes the law of the family $((L_{a_i}^D, \xi_i^D) : i \in \mathcal{I})$ conditionally on $\mathrm{Tr}_D(\xi)$. Let us start by providing some historical context. This result was originally introduced in [63, Section 2] in a weaker version for the Brownian motion indexed by the Brownian tree, and has played a fundamental role in its study, we refer to e.g. [63, 66, 70, 72]. More recently, a stronger version was proved in [66] still when the tree is the Brownian tree but holding for more general spatial motions. In this section we shall extend this result to an arbitrary Lévy tree. It is worth mentioning that the special Markov property is closely related to the one established by Dynkin in the setting of superprocesses [45] under very general assumptions, the crucial difference being that in the context of the Lévy snake, it keeps track of the genealogy and the respective labels of each individual.

Let us introduce the last pieces of notation needed to state formally this result. For $\mathrm{w} \in \mathcal{W}_E$ and with a slight abuse of notation, set $\tau_D(\mathrm{w}) := \inf\{t \geqslant 0 : \mathrm{w}(h) \notin D\}$ and consider the functional

$$V_t^D := \int_0^t \mathrm{d}s \, \mathbb{1}_{\{H_s \leqslant \tau_D(W_s)\}}, \qquad t \geqslant 0.$$

Note that the variable V_t^D measures the volume of $\mathrm{Tr}_D(\mathcal{T}_H) \cap \mathcal{T}_H(t)$ and write Γ^D for its right-inverse, viz. the right-continuous process defined for every $s \in [0, V_\infty^D)$ as $\Gamma_s^D := \inf\big\{t \geqslant 0 : V_t^D > s\big\}$. Further, we set

$$\big(\mathrm{Tr}_D(\rho), \mathrm{Tr}_D(W), \mathrm{Tr}_D(L^D)\big) := (\rho_{\Gamma_t^D}, W_{\Gamma_t^D}, L_{\Gamma_t^D}^D) \quad \text{ for } t \geqslant 0.$$

and we write $(\theta_r : 0 \leqslant r < L_\infty^D)$ for the right-inverse of $\mathrm{Tr}_D(L^D)$. The truncated process $(\mathrm{Tr}_D(\rho), \mathrm{Tr}_D(W))$ thus encodes the labelled tree $\mathrm{Tr}_D(\xi)$ and we shall denote by $\mathcal{F}_D$ its generated sigma-field. The key now is that Proposition 5.7 ensures that $\mathrm{Tr}_D(L^D)$ is $\mathcal{F}_D$-measurable. Since this last point is one of the main technical difficulties that need to be sorted to establish Theorem 2.4, we shall provide the outline of the proof.

Sketch of proof of Proposition 5.7. Let $(D_n : n \geqslant 1)$ be an increasing sequence of open domains containing y, and satisfying both $\overline{D}_n \subset D_{n+1}$ for every n and $\cup_n D_n = D$. To prove that $\mathrm{Tr}_D(L^D)$ is $\mathcal{F}_D$-measurable, it suffices to show that under $\mathbb{N}_y$, the sequence of $\mathcal{F}_D$-measurable processes $(\mathrm{Tr}_D(L^{D_n}) : n \geqslant 1)$ converges towards $\mathrm{Tr}_D(L^D)$ a.e. uniformly in compact intervals along some sub-sequence. Exploiting the fact that these are continuous additive functionals of (ρ, W), this problem can be reduced to establishing the a.e. convergence of the total mass $L_\sigma^{D_n} \to L_\sigma^D$ along some subsequence, as $n \uparrow \infty$. In the context of the Brownian motion indexed by the Brownian tree, this was proved by Le Gall in [63, Proposition 2.3] by establishing that the convergence holds in $L^2(\mathbb{N}_y)$. In the general framework of Lévy trees, this argument can be adapted, but one needs to make use of a truncation argument since *a priori* the convergence no longer holds in $L^2(\mathbb{N}_y)$. We refer to Section 5.3.1 for a detailed proof.

Finally, for every $i \in \mathcal{I}$ we let (ρ_i^D, W_i^D) be the snake trajectory encoding the labelled tree ξ_i^D - formally, one needs to make use of the notion of subtrajectory of a snake path, we refer to Section 5.2.3 for a precise definition. Now all the ingredients are in place to state:

Theorem 2.4. (Special Markov property) *Under $\mathbb{P}_{0,y}$ and $\mathbb{N}_y$, conditionally on $\mathcal{F}_D$, the point measure*

$$\sum_{i\in\mathcal{I}} \delta_{(L^D_{a_i}, \rho_i^D, W_i^D)}(\mathrm{d}\ell, \mathrm{d}\rho, \mathrm{d}W)$$

is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{M}_f(E) \times \mathcal{W}_E$ with intensity $\mathbb{1}_{[0,L^D_\sigma]}(\ell)\, \mathrm{d}\ell\, \mathbb{N}_{\mathrm{tr}_D(\widehat{W})_{\theta_\ell}}(\mathrm{d}\rho, \mathrm{d}W)$.

In the last statement, we denoted by $\mathrm{tr}_D(\widehat{W})_{\theta_\ell}$ the process $(\widehat{W}_{\Gamma^D_t} : t \geqslant 0)$ taken at time θ_ℓ.

2.3 The local time at x and the subordinate tree

Alongside the special Markov property, the main contribution of [82, 83] consists in the development of an excursion theory for Markov processes indexed by Lévy trees, holding under rather general conditions on the pair (ψ, ξ). In the following sections we shall present its main elements. As we already mentioned, we extend the theory developed in the previous work [1] from C. Abraham and J.-F. Le Gall in the setting of Brownian motion indexed by the Brownian tree, with the notable addition of a notion of local time, making these two approaches rather different. We shall be more precise in the sequel. It is worth mentioning that the work [1] has found numerous applications in Brownian geometry, see e.g. [67, 70] and we expect our results to have applications outside the scope of this work.

Up to this point, we have presented the theory of Markov processes indexed by Lévy trees under a very large degree of generality. However, as in the time-indexed case, to develop an excursion theory one needs to impose further restrictions to the class of spatial motions we consider. To this end, we shall henceforth assume that the Markov process ξ satisfies assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ for some point $x \in E$. We shall write $(\mathcal{L}_t)_{t\in\mathbb{R}_+}$ for its local time at x and we denote the corresponding excursion measure by $\mathcal{N}$.

The local time at x of $\widehat{W}$.

The content of this section is taken from Section 5.4.2. From now on, the motions we shall consider consists in pairs of the form,

$$\overline{\xi} := (\xi_t, \mathcal{L}_t), \qquad \text{for } t \geqslant 0.$$

We let $(\rho, W, \Lambda) = ((\rho_t, W_t, \Lambda_t) : t \geqslant 0)$ be the ψ-Lévy snake with spatial motion $\overline{\xi}$ and we write $(\overline{\xi}_a)_{a\in\mathcal{T}_H}$ for the corresponding tree-indexed process, where $\overline{\xi}_a = (\xi_a, \mathcal{L}_a)$ for $a \in \mathcal{T}_H$. To motivate the forthcoming results we start with an informal discussion. In the sequel, we will direct our efforts towards studying two key objects of interest. First, the role of the set $\mathscr{Z}^\circ$ is now taken over by the following random subset of $\mathcal{T}_H$:

$$\mathscr{Z} := \{a \in \mathcal{T}_H : \xi_a = x\}.$$

Observe that $\mathscr{Z}$ inherits a tree structure from $\mathcal{T}_H$ and therefore, is of a significantly more intricate nature than the subset of the real line $\mathscr{Z}^0$. On the other hand, we have of course the family of excursions of $(\xi_a)_{a\in\mathcal{T}_H}$ away from x. Roughly speaking, these consist on the restrictions of $(\xi_a)_{a\in\mathcal{T}_H}$ to the connected component of $\mathcal{T}_H\backslash\mathscr{Z}$. Note however that, in contrast with the time-indexed

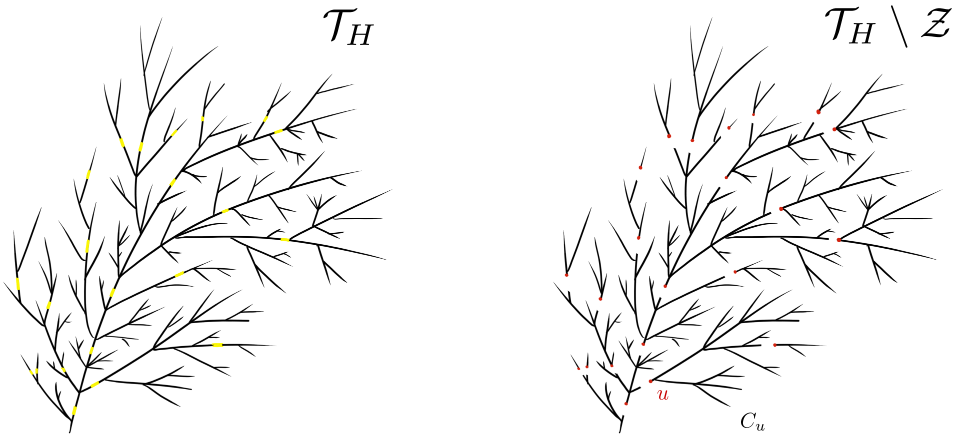


Figure 2.8: In the left hand side, sketch of $\mathcal{T}_H$ with the set $\mathscr{Z}$ coloured in yellow. In the right-hand side, the corresponding family of excursion components $(C_u)_{u\in D}$; each debut is marked with a red dot.

setting, we lack of a proper way to index this family of excursions. We shall start by addressing this concern and, to this end, we introduce a remarkable continuous additive functional of the Lévy snake (ρ, W, Λ). The construction we provide relies crucially on the theory of exit local times.

For every $r > 0$, let $\mathscr{L}^r = (\mathscr{L}_t^r : t \geqslant 0)$ be the exit local time of $(\bar{\xi}_a)_{a\in\mathcal{T}_H}$ from the domain $E \times [0, r)$. Proposition 6 in [82] states that the process defined by the relation

$$A_t := \int_{\mathbb{R}_+} \mathrm{d}r \mathscr{L}_t^r, \qquad \text{for } t \geqslant 0,$$

is an $\mathbb{R}_+$-valued continuous additive functional of the Lévy snake with Stieltjets measure $\mathrm{d}A$ supported on an explicit subset of $\{t \geqslant 0 : \widehat{W}_t = x\}$. For this reason and with some abuse of notation, we refer to $A = (A_t : t \geqslant 0)$ as the local time at x of $\widehat{W}$. The process A can be interpreted as well as the total variation of $\widehat{\Lambda} := (\widehat{\Lambda}_t : t \geqslant 0)$ in the following sense. Let $\mathcal{C}^*$ be the subset of $\mathbb{R}_+$ defined by the relation: $t \in \mathcal{C}^*$ if and only if for some open neighbourhood of t, $\widehat{\Lambda}$ is constant. Then we have the following characterisation for the support of the measure $\mathrm{d}A$.

Theorem 2.5. *Under $\mathbb{P}_{0,x,0}$ and $\mathbb{N}_{x,0}$ we have*

$$\text{supp } \mathrm{d}A = [0, \sigma]\backslash\mathcal{C}^*.$$

The support of $\mathrm{d}A$ can also be written in terms of the so-called *exit times from x* of (ρ, W, Λ), we refer to Definition 5.26 and Theorem 5.30 for a formal definition and the precise statement. For every $r > 0$, under $\Pi_{y,0}$ we let $\tau_r := \inf\{t \geqslant 0 : \mathcal{L}_r > r\}$. The following lemma describes the law of (ρ, W, Λ) at a typical time taken with respect to the measure $\mathrm{d}A$.

Lemma 2.6. *For $y \in E$ and for every non-negative measurable function Φ on $M_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}}$ we have*

$$\mathbb{N}_{y,0}\left(\int_0^\sigma \mathrm{d}A_s \;\Phi\left(\rho_s, W_s, \Lambda_s\right)\right) = \int_0^\infty \mathrm{d}r\;\; E^0 \otimes \Pi_{y,0}\big[\exp\big(-\alpha\tau_r\big)\cdot \Phi\big(J_{\tau_r}, (\xi_t, \mathcal{L}_t : t \leqslant \tau_r)\big)\big].$$

Let us briefly discuss some connections with related works. When $(\xi_a)_{a\in\mathcal{T}_H}$ is the Brownian motion indexed by the Brownian tree and $x = 0$, the additive functional A is closely related to the local time of the Brownian motion indexed by the Brownian tree and the so-called ISE (or integrated super-Brownian excursion) introduced by Aldous in [5]. First, the local time of the Brownian motion indexed by the Brownian tree is defined as the $\mathbb{R}_+$-valued continuous process $(\ell^y : y \in \mathbb{R})$ defined under $\mathbb{N}_0$ by the relation

$$\int_0^\sigma \mathrm{d}t\, F(\widehat{W}_t) = \int_{\mathbb{R}} \mathrm{d}y\, F(y)\cdot \ell^y.$$

In other terms, $(\ell^y : y \in \mathbb{R})$ is the density with respect to the Lebesgue measure of the occupation measure of $\widehat{W}$. The identity in the last display shares obvious similarities with the occupation times formula for local times of Brownian motion and for every $y \in \mathbb{R}$, one can think of ℓ^y as measuring the size of the set $\{a \in \mathcal{T}_H : \xi_a = y\}$. For this reason $(\ell^y : y \in \mathbb{R})$ is referred in the literature as the local time of Brownian motion indexed by the Brownian tree, but we stress that it should not be mistaken with our additive functional A. By [71, Proposition 3], the processes A and $(\ell^y : y \in \mathbb{R})$ are linked through the relation

$$A_\sigma = \ell^0.$$

Notably, it has been proved recently in [68] that the local time of Brownian motion indexed by the Brownian tree paired with its derivative $((\ell^y, \dot{\ell}^y) : y \geqslant 0)$ is a time-homogeneous Markov process. One can think of such result as a variant of the classic Ray-Knight theorems. The integrated super-Brownian excursion is the process $(\ell^y : y \in \mathbb{R})$ under the conditioning $\sigma = 1$. The ISE has been subject of active research in recent years [5, 29, 39] and appears as scaling limit of multiple functionals on discrete tree models, see e.g. [28, 32] and references therein.

The subordinate tree by the local time

The content of this section is taken from [82]. As we already mentioned, the set $\mathscr{Z}$ inherits a genealogical structure from $\mathcal{T}_H$. It is then natural to look for a way to encode the set $\mathscr{Z}$ in a tree. To this end, we shall make use of the notion of subordination of trees by continuous non-decreasing functions introduced in [66].

The mapping $(\mathcal{L}_a)_{a\in\mathcal{T}_H}$ is continuous and non decreasing with respect to the genealogical order, viz. if $a \prec b$, then $\mathcal{L}_a < \mathcal{L}_b$. If for every $a, b \in \mathcal{T}_H$ we set

$$d_{\mathcal{L}}(a,b) := \mathcal{L}_a + \mathcal{L}_b - 2\min_{[\![a,b]\!]} \mathcal{L},$$

then $d_{\mathcal{L}}$ is a pseudo-distance on $\mathcal{T}_H$ and it induces an equivalence relation on $\mathcal{T}_H$: namely, we write $a \sim_{\mathcal{L}} b$ if $\mathcal{L}$ is constant on $[\![a,b]\!]$. Then, the quotient space $\widetilde{\mathcal{T}} := (\mathcal{T}_H/\sim_{\mathcal{L}}, d_{\mathcal{L}})$ is again a compact $\mathbb{R}$-tree that we shall refer to as *the subordinate tree by the local time $\mathcal{L}$*. The terminology stems from the fact that $\widetilde{\mathcal{T}}$ is obtained from identifying the connected components of $\mathcal{T}_H$ where

$(\mathcal{L}_a)_{a\in\mathcal{T}_H}$ is constant, which correspond roughly to the connected components of $\mathcal{T}_H\backslash\mathscr{Z}$. This last description shows that $\widetilde{\mathcal{T}}$ encodes as well the genealogical structure of the excursions away from x of $(\xi_a)_{a\in\mathcal{T}_H}$, this fact will be key in the sequel. The function $(\mathcal{L}_a)_{a\in\mathcal{T}_H}$ is well defined in the quotient space $\widetilde{\mathcal{T}}$ and now, for every $a\in\widetilde{\mathcal{T}}$ the variable $\mathcal{L}_a$ is precisely the distance from a to the root of $\widetilde{\mathcal{T}}$. Notably, the random tree $\widetilde{\mathcal{T}}$ is again a Lévy tree. More precisely, write $E^* := E\backslash\{x\}$ and for $y\neq x$ set $u_\lambda(y) := \mathbb{N}_y(1-\exp(-\lambda L_\sigma^{E_*}))$.

Proposition 2.7. *The random tree $\widetilde{\mathcal{T}}$ under the measure $\mathbb{N}_{x,0}$ is a Lévy tree with Laplace exponent given by*

$$\widetilde{\psi}(\lambda) = \mathcal{N}\Big(\int_0^\sigma \mathrm{d}h\, \psi(u_\lambda(\xi_h))\Big).$$

When the tree is the Brownian tree, this result is due to Le Gall [66, Theorem 16].

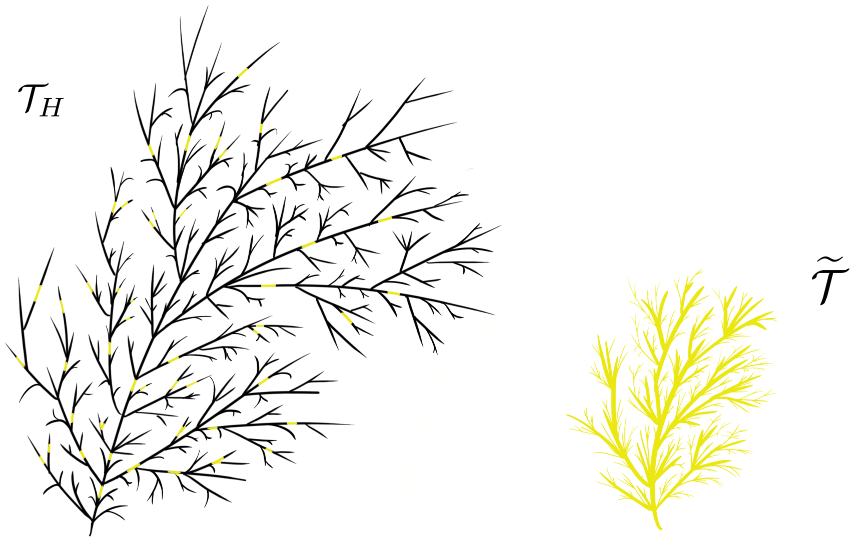


Figure 2.9: Subordination of $\mathcal{T}_H$ by the local time $\mathcal{L}$. In the left hand side, the set $\mathscr{Z}$ is coloured in yellow. Heuristically, the tree $\widetilde{\mathcal{T}}$ is obtained from $\mathcal{T}_H$ by identifying the black connected components in $\mathcal{T}_H$.

The fact that $\widetilde{\mathcal{T}}$ is a Lévy tree is a rather straightforward consequence of the special Markov property 2.4 and Theorem 1.1 in [91]. We refer to [66, Theorem 16] for a proof when the random tree is the Brownian tree. One can think of the subordinate tree $\widetilde{\mathcal{T}}$ as the geometric analogue of the inverse local time $(\mathcal{L}_t^{-1} : t \geqslant 0)$. It might be however worth noting that, if we further assume that $(\xi_a)_{a\in\mathcal{T}_H}$ is the Brownian motion indexed by the Brownian tree, the Laplace exponent $\widetilde{\psi}$ of the subordinate tree is the one of a 3/2-stable Lévy process (and therefore is less regular than the Brownian tree) while in contrast, as we already discussed $(\mathcal{L}_t^{-1} : t \geqslant 0)$ is a 1/2-stable subordinator. Note that the latter does not even fulfil our hypothesis on the Laplace exponent. It is natural to observe a drop in the regularity of the subordinate tree with respect to $\mathcal{T}_H$: identifying the excursion components of ξ away from x generates points of infinite multiplicity in $\widetilde{\mathcal{T}}$ as soon as the corresponding excursion returns to x. We shall come back to this point in the sequel. Finally, we mention that our results on subordination of trees with respect to the local time are closely related, in the terminology of Lévy snakes, to Theorem 4 in [23] stated in the setting of superprocesses – the main difference being that in our work we encode the associated genealogy.

Since $\widetilde{\mathcal{T}}$ is a Lévy tree constructed in terms of (ρ, W, Λ), it is natural to try to express in terms of (ρ, W, Λ), the corresponding height process $\widetilde{H}$, the Lévy process $\widetilde{X}$, or even the exploration process $\widetilde{\rho}$. In that regard, the additive functional A we introduced will play a central role. In this direction we conclude the section with the second main contribution of [82].

Theorem 2.8. *The following properties hold:*

(i) *Under $\mathbb{N}_{x,0}$, the subordinate tree $\widetilde{\mathcal{T}}$ is isometric to the tree coded by the continuous function $(\widehat{\Lambda}_{A_r^{-1}} : r \geqslant 0)$.*

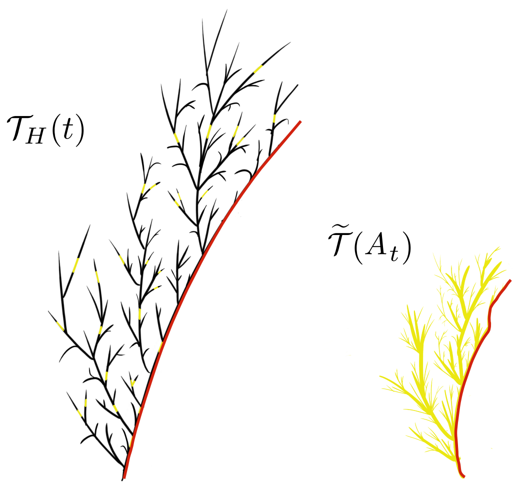
(ii) *Moreover, under $\mathbb{N}_{x,0}$ the process $\big(\widehat{\Lambda}_{A_r^{-1}} : r \geqslant 0\big)$ is distributed as the height process of a $\widetilde{\psi}$-Lévy tree.*

Since the arguments employed to establish (ii) might be of interest in latter works, let us sketch the proof of (ii). The later relies crucially on the notion of marked discrete trees embedded in an excursion, that we shall briefly present. Let $e : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative continuous function with finite lifetime $\sigma_e < \infty$ and write $\mathcal{T}_e$ for the corresponding tree coded by e. First, consider a single fixed point $t_1 \in [0, \sigma_e]$ and think of $p_H(t_1)$ as a mark on $\mathcal{T}_e$. Recall that the path $[\![0, p_e(t_1)]\!]$ is isometric to the interval $[0, e(t_1)]$ and in particular the distance to the root of $p_H(t_1)$ is $e(t_1)$. We shall write $\theta(e, t_1)$ for the discrete (ordered) tree with a single vertex with label $e(t_1)$. Further, if we consider two marks $0 < t_1 < t_2 < \sigma_e$, the geodesics paths $[\![\varnothing, p_H(t_1)]\!]$, $[\![\varnothing, p_H(t_2)]\!]$ are respectively isometric to $[0, e(t_1)], [0, e(t_2)]$ and share the ancestral line $[\![0, p_e(t_1) \curlywedge p_e(t_2)]\!]$, which on its turn is isometric to $[0, \min_{[t_1,t_2]} e]$. Therefore, the marks naturally induce a discrete ordered labelled tree $\theta(e, t_1, t_2)$, compatible with the order induced by e and encoding the height and the genealogy between $p_e(t_1)$ and $p_e(t_2)$. Namely, the tree $\theta(e, t_1, t_2)$ possess one leaf per mark, each one with respective labels $e(t_1)$, $e(t_2)$ and both are linked to a common ancestor with label $\min_{[t_1,t_2]} e$, playing the role of the root. This construction can be generalised inductively to an arbitrary finite number of marks $0 < t_1 < \cdots < t_m < \sigma_e$ for $m \geqslant 1$ and yields a discrete labelled ordered tree that we denote by $\theta(e, t_1, \dots, t_M)$. With a slight abuse of notation we write $\ell(e) := \{1, \dots, M\}$ for its set of leaves listed in chronological order; we refer to Section 5.5.1 for a precise definition. We stress that this definition differs slightly with the notion of marginals of trees introduced in [43].

Let us get back to the proof of Theorem 2.8. To simplify notation set $H^A := (\widehat{\Lambda}_{A_r^{-1}} : 0 \leqslant r \leqslant A_\sigma)$ and consider $\widetilde{H} = (\widetilde{H}_r : 0 \leqslant r \leqslant \tilde{\sigma})$ the height process of a $\widetilde{\psi}$-Lévy tree defined under the corresponding excursion measure $\widetilde{N}$. We write $\mathcal{T}_{H^A}$, $\mathcal{T}_{\widetilde{H}}$ for the corresponding trees coded respectively by H^A, $\widetilde{H}$ and consider Poissonian marks $\{\tau_1, \tau_2, \dots\}$ with rate λ in both $[0, A_\sigma]$ and $[0, \tilde{\sigma}]$. We write M, $\widetilde{M}$ for the number of marks falling in $[0, A_\sigma]$ and $[0, \tilde{\sigma}]$ respectively, and work under the probability measures $\mathbb{N}_{x,0}(\,\cdot\,|M \geqslant 1)$, $\widetilde{N}(\,\cdot\,|\widetilde{M} \geqslant 1)$. Then, Proposition 5.34 states that the discrete trees $\theta(H^A, t_1, \dots t_M)$, $\theta(\widetilde{H}, t_1, \dots t_{\widetilde{M}})$ have the same distribution. The law of $\theta(H^A, t_1, \dots t_M)$, $\theta(\widetilde{H}, t_1, \dots t_{\widetilde{M}})$ can be computed by exploiting the Markovian character of the Poisson marks, combined with the Markov property of the Lévy snake, the special Markov property 2.4 and Proposition 2.6. Noting that the labels on the leafs $\ell(H^A) = \{1, \dots, M\}$ and $\ell(\widetilde{H}) = \{1, \dots, \widetilde{M}\}$ are precisely the respective heights $\{H^A_{t_1}, \dots, H^A_{t_M}\}$ and $\{\widetilde{H}_{t_1}, \dots, \widetilde{H}_{t_{\widetilde{M}}}\}$, one can then conclude from an approximation argument that H^A under $\mathbb{N}_{x,0}(\,\cdot\,|M \geqslant 1)$ and $\widetilde{H}$ under $\widetilde{N}(\,\cdot\,|\widetilde{M} \geqslant 1)$ have the same law.

Figure 2.10: The tree embedded on $\widetilde{\mathcal{T}}$ generated by the Poisson marks $t_1, \ldots, t_n$.

From now on, H^A will henceforth be denoted by $\widetilde{H}$ and we still write $\widetilde{\mathcal{T}}$ for the Lévy tree coded by $\widetilde{H}$. By Lemma 6.1 we can construct from $\widetilde{H}$ a pair $(\widetilde{X}, \widetilde{\rho})$ where $\widetilde{X}$ is a $\widetilde{\psi}$-Lévy process and $\widetilde{\rho}$ is its associated exploration process. We shall provide in the sequel a more explicit construction of $(\widetilde{X}, \widetilde{\rho})$ in terms of (ρ, W, Λ), but it relies on the development of the excursion theory for $(\xi_a)_{a \in \mathcal{T}_H}$ that we shall now introduce. Finally, let us mention that the explicit construction of $\widetilde{\mathcal{T}}$ in terms of $\mathcal{T}_H$ and its labels yields that we can simultaneously explore both trees in clockwise order: more precisely, recalling that $\mathcal{T}_H(t)$ is the subset of $\mathcal{T}_H$ that has been explored up to time t, by (ii) at time t we have explored the subset $\widetilde{\mathcal{T}}(A_t)$ from $\widetilde{\mathcal{T}}$.



2.4 The excursion theory

This section is devoted to the study of the excursions away from x of $(\xi_a)_{a \in \mathcal{T}_H}$, the content is taken from [83]. We start with some definitions and first properties. In the context of Brownian motion indexed by the Brownian tree, these had already been established in [1].

Debuts and excursions away from x

Our first definition is taken from [1]. A point $u \in \mathcal{T}_H$ is called an *excursion debut* for $(\xi_a)_{a \in \mathcal{T}_H}$ if the following properties hold:

(i) We have $\xi_u = x$.

(ii) We can find a strict descendant v of u such that $\xi_a \neq x$ for every a in $]\!]u, v]\!]$.

We denote by D the collection of excursion debuts. For every $u \in D$, we write C_u for the subset of points $v \in \mathcal{T}_H$ fulfilling that $v \succ u$ with $\xi_a \neq x$ for every $a \in]\!]u, v[\![$. For latter use, we gather some basic properties:

- For every $u \in D$, set $C_u^0 := C_u \cap \{a \in \mathcal{T}_H : \xi_a \neq x\}$. Then, the family $(C_u^0)_{u \in D}$ are the connected components of the open set $\{a \in \mathcal{T}_H : \xi_a \neq x\}$ [Lemma 6.7].
- The local time $(\mathcal{L}_a)_{a \in \mathcal{T}_H}$ is constant on every C_u and we denote its value by ℓ_u. Moreover, if we consider some other $u' \in D$ with $u \neq u'$, we have $\ell_u \neq \ell_{u'}$ [Lemma 6.9].

Finally, we can now introduce:

Definition 2.9. *For every $u \in D$ we set $\xi^u := (\xi_a : a \in C_u)$. We refer to the family $(\xi^u : u \in D)$ as the excursions away from x of $(\xi_a)_{a \in \mathcal{T}_H}$.*

If follows from our definition that every excursion is again a tree indexed process and the first point above yields that the family $(\xi^u : u \in D)$ is countable. Further, the second point shows that we can make use of $(\mathcal{L}_a)_{a \in \mathcal{T}_H}$ to index the family of excursions, by considering the pairs $((\ell_u, \xi^u) : u \in D)$. This was the approach followed in [1] to introduce an excursion measure in the setting of the Brownian motion indexed by the Brownian tree. We shall however follow a different path exploiting the properties of the local time A.

The excursion measure

In this section we define an infinite measure that we shall refer to as the excursion measure for $(\xi_a)_{a \in \mathcal{T}_H}$, as well as a notion of fractal length for its boundary. The terminologies will be justified by the results of the next sub-section. Our definitions rely on several preliminary construction that we shall now introduce.

Let us start with an informal discussion. We start by defining a tree-indexed process as follows: let $\mathcal{T}_H$ be a ψ-Lévy tree under N. Instead of considering as spatial motion the Markov process with law Π_y, we shall consider as spatial motion an excursion of ξ under $\mathcal{N}$. We write $(\mathbf{e}_a)_{a \in \mathcal{T}_H}$ for the corresponding tree-indexed process. Observe that if $\mathbf{e}_a = x$ for some $a \neq \varnothing$, then for every $b \succ a$ it must hold that $\mathbf{e}_b = x$. The excursion measure we shall define encodes the law of $(\mathbf{e}_a)_{a \in \mathcal{T}_H}$ restricted to points in $\mathcal{T}_H$ satisfying that $\mathbf{e}_a \neq x$ for every $a \in [\![\varnothing, b[\![$.

Let us now formalise our previous discussion. The objects we shall introduce rely on several preliminary constructions that we shall now briefly introduce. For every $y \in E$, write $\Pi_y^\dagger$ for the law of ξ under Π_y stopped at its first hitting time of x. Fix an arbitrary continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with finite lifetime and satisfying $h(0) = 0$. For every $t_0 \geqslant 0$, we write $h \circ \theta_{t_0}$ for the shifted function $(h(t_0 + t) : t \geqslant 0)$ and for any $\mathrm{w} \in \mathcal{W}_E$ satisfying $h(t_0) = \zeta_{\mathrm{w}}$, we denote by $Q_{\mathrm{w}}^{h \circ \theta_{t_0}}$ the law of the snake driven by $h \circ \theta_{t_0}$ with spatial motion $\Pi^\dagger := (\Pi_y^\dagger)_{y \in E}$ started from w. Next, we write $\nu_{t_0}^h(\mathrm{dw})$ for the law of $(\xi_t : 0 \leqslant t \leqslant h(t_0))$ under the excursion measure $\mathcal{N}$ - note that $\nu_{t_0}^h$ is a sigma-finite measure on $\mathcal{W}_E$. Now, by Kolmogorov's theorem there exists a unique measure $Q_{\mathcal{N}}^h$ on $\mathcal{W}_E$, charging the subset of paths of $\mathcal{W}_E^{\mathbb{R}_+}$ taking the value x at time 0, characterised by the relation

$$Q_{\mathcal{N}}^h(W_{t_0} \in A_0, W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n) = \int_{\mathcal{W}_E} \nu_{t_0}^h(\mathrm{dw}) \mathbb{1}_{\{\mathrm{w} \in A_0\}} Q_{\mathrm{w}}^{h \circ \theta_{t_0}}(W_{t_1} \in A_1, \ldots, W_{t_n} \in A_n)$$

the latter holding for every $0 < t_0 < t_1 < \cdots < t_n$, and $A_0, \ldots, A_n$ Borelians in $\mathcal{W}_E$. Roughly speaking, one can think of W under $Q_{\mathcal{N}}^h$ as the snake driven by h with spatial motion $\mathcal{N}$. Finally, we set

$$\mathbf{N}_x^*(\mathrm{d}\rho, \mathrm{d}W) := N(\mathrm{d}\rho) Q_{\mathcal{N}}^{H(\rho)}(\mathrm{d}W).$$

Heuristically, (ρ, W) under $\mathbf{N}_x^*$ is the ψ-Lévy snake with spatial motion $\mathcal{N}$ - we stress that this description is informal since $\mathcal{N}$ is an infinite measure. It follows from our definitions that for every fixed t and conditionally on H_t, the law of $W_t = (W_t(r) : 0 \leqslant r \leqslant H_t)$ under $\mathbf{N}_x^*$, is the one of $(\xi_t : 0 \leqslant t \leqslant H_t)$ under $\mathcal{N}$. Under our assumptions, W has a continuous modification under $\mathbf{N}_x^*$ that we consider from now on and still denote by W. The process W satisfies the snake property, which gives that we can set $(\mathbf{e}_a)_{a \in \mathcal{T}_H}$ for the function $\widehat{W}$ under $\mathbf{N}_x^*$ in the quotient space $\mathcal{T}_H$.

Under $\mathbf{N}_x^*$, it still holds that (ρ, W) is a strong Markov process [Proposition 6.13] encoding the branching structure and labels of the tree indexed process $(\mathbf{e}_a)_{a \in \mathcal{T}_H}$. Note that the lifetime $\tau_x^*(W_t) := \inf\{h > 0 : W_t(h) = x\}$ of W_t might a priori be smaller than H_t, in which case we have plainly $W_t(h) = x$ for every $h \in [\tau_x^*(W_t), H_t]$. This leads us to consider the following trimmed sub-tree of $\mathcal{T}_H$

$$\mathrm{Tr}_*(\mathcal{T}_H) := \big\{ a \in \mathcal{T}_H : \mathbf{e}_b \neq x \text{ for every } b \in]\!]\varnothing, a[\![\big\}.$$

See Figure 2.11 below. Note that by definition, with the exception of the root, only the leafs of $\mathrm{Tr}_*(\mathcal{T}_H)$ might have x as label.

Figure 2.11: In the left hand side, the tree $\mathcal{T}_H$ with the set $\{a \in \mathcal{T}_H \backslash \varnothing : \xi_a = x\}$ coloured in red. On the right-hand side, the trimmed tree $\mathrm{Tr}_*(\mathcal{T}_H)$, obtained from removing the set $\{a \in \mathcal{T}_H \backslash \varnothing : \xi_a = x\}$ from $\mathcal{T}_H$.

Finally, under $\mathbf{N}_x^*$ we introduce

$$\mathrm{Tr}_*(\mathbf{e}) := (\mathbf{e}_a : a \in \mathrm{Tr}_*(\mathcal{T}_H)).$$

Making use of techniques stemming from the theory of exit local times, one can define a notion of measure for the boundary $\partial \mathrm{Tr}_*(\mathcal{T}_H) := \{a \in \mathrm{Tr}_*(\mathcal{T}_H) : \xi_a = x\}$. In this direction, for every $t \geqslant 0$ consider the family of connected components of $\mathcal{T}_H(t) \backslash \mathrm{Tr}_*(\mathcal{T}_H)$. The closure of each one of them is a tree, say C_i^*, and we denote this family by $(C_i^*(t))_{i \in \mathcal{J}}$. If we let $B_\varepsilon^i(\varnothing^i)$ be the ball in $C_i^*(t)$ of radius ε centred at the root $\varnothing^i$ of $C_i^*(t)$, the limit

$$L_t^* := \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathrm{Vol}_H \Big(\bigcup_{i \in \mathcal{J}} C_i^*(t) \cap B_\varepsilon(\varnothing^i) \Big)$$

exists uniformly in compact intervals in measure under $\mathbf{N}_x^* 1_V$, for any measurable V with finite mass. It follows from our definitions that the process $L^* := (L_t^* : t \geqslant 0)$ is continuous and non-decreasing. We stress that the theory of exit local times can not be directly applied to define L^* since we are not considering the exit time from an open set, and the law of the spatial motion is now an infinite measure. The Stieltjets measure $\mathrm{d}L^*$ is supported on the set $\{t \geqslant 0 : H_t = \tau_x^*(W_t)\}$ and the total mass L_∞^*, which is a.e. finite, can be interpreted as the length of $\partial \mathrm{Tr}_*(\mathcal{T}_H)$. For this reason, in the sequel we write $|\partial \mathrm{Tr}_*(\mathcal{T}_H)| := L_\sigma^*$. Note that $\mathrm{Tr}_*(\mathbf{e})$ is obtained from $(\mathbf{e}_a)_{a \in \mathcal{T}_H}$ by removing every tip of path that has returned to x before the end of its respective lifetime. For $t \geqslant 0$ we set $V_t^* := \int_0^t \mathrm{d}s\, 1_{\{H_s \leqslant \tau_x^*(W_s)\}}$ and write $(\Gamma_t^* : t \geqslant 0)$ for the right-inverse of $(V_t^* : t \geqslant 0)$. Informally, one can think of the truncated process $\mathrm{Tr}_*(\rho, W) := ((\rho_{\Gamma_t^*}, W_{\Gamma_t^*}) : t \geqslant 0)$ as the snake encoding the branching structure and labels of $\mathrm{Tr}_*(\mathbf{e})$.

Definition 2.10. *The law of $\mathrm{Tr}_*(\rho, W)$ under $\mathbf{N}_x^*$ is denoted by $\mathbb{N}_x^*$. We shall henceforth refer to $\mathbb{N}_x^*$ as the excursion measure of $(\xi_a)_{a \in \mathcal{T}_H}$ away from x.*

Figure 2.12: Sketch of an excursion away from x under $\mathbb{N}_x^*$; the distance of each label to x is plotted with respect to the vertical axis. In yellow are coloured the points at which the spatial motion returns to x. The "size" of the set $\{a \in \mathcal{T}_H \backslash \varnothing : \xi_a = x\}$ is measured by L_σ^*.

Moreover, under $\mathbb{N}_x^*$ the process (ρ, W) is still a right-continuous strong Markov process. The next proposition describes the law of (ρ, W) under the pointed version of $\mathbb{N}_x^*$ at a typical time in $[0, \sigma]$.

Proposition 2.11. *For every non-negative measurable function Φ on $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, we have*

$$\mathbb{N}_x^*\Big(\int_0^\sigma \mathrm{d}s\, \Phi(\rho_s, W_s) \Big) = E^0 \otimes \mathcal{N}\Big(\int_0^\sigma \mathrm{d}a \, \exp\big(-\alpha a \big) \cdot \Phi(J_a, (\xi_s : s \leqslant a)) \Big).$$

It will be crucial for our purposes to define as well the notion of boundary length under $\mathbb{N}_x^*$. In this direction, if under $\mathbf{N}_x^*$ we set $\mathrm{Tr}_*(L^*) := (L^*_{\Gamma_t^*} : t \geqslant 0)$, Proposition 6.35 states that $\mathrm{Tr}_*(L^*)$ is $\mathrm{Tr}_*(\rho, W)$-measurable and therefore well defined under $\mathbb{N}_x^*$. Under $\mathbb{N}_x^*$, we shall write

$$\big((\mathbf{e}_a)_{a \in \mathcal{T}_H}, \mathcal{T}_H, |\partial \mathcal{T}_H|, L^*\big) \text{ for } \big(\mathrm{Tr}_*(\mathbf{e}), \mathrm{Tr}_*(\mathcal{T}_H), |\partial \mathrm{Tr}_*(\mathcal{T}_H)|, \mathrm{Tr}_*(L^*)\big) \text{ under } \mathbf{N}_x^*.$$

We obtained as well a description for the law of (ρ, W) at a typical time taken with respect to the measure $\mathrm{d}L^*$.

Proposition 2.12. *For every non-negative measurable function Φ on $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, we have*

$$\mathbb{N}_x^*\Big(\int_0^\sigma \mathrm{d}L_s^*\, \Phi(\rho_s, W_s)\Big) = E^0 \otimes \mathcal{N}\big(\exp(-\alpha\sigma)\Phi(J_\sigma, (\xi_s : s \leqslant \sigma))\big).$$

For a more general version of Proposition 2.11 and 2.12 encoding as well the left and right spine we refer to Proposition 6.39 and Proposition 6.37. As an application of Proposition 2.12, we obtain the characteristic triplet of the Laplace exponent $\widetilde{\psi}$. If we set

$$\widetilde{\alpha} := \mathcal{N}(1 - \exp(-\alpha\sigma)), \quad \widetilde{\beta} := 0, \quad \widetilde{\pi}(\mathrm{d}x) := \mathbb{N}_x^*\big(L_\sigma^* \in \mathrm{d}x \cap (0, \infty)\big),$$

Corollaries 5.21 and 6.38 that the Lévy-Khintchine triplet of $\widetilde{\psi}$ is given by $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\pi})$.

The excursion process
In the last section we introduced a measure $\mathbb{N}_x^*$ that we baptised the excursion measure away from x of $(\xi_a)_{a\in\mathcal{T}_H}$. We shall now justify our choice of terminology. Under $\mathbb{N}_{x,0}$ and $\mathbb{P}_{0,x,0}$, for every fixed $u \in D$ let us write (ρ^u, W^u) for the Lévy snake encoding the sub-tree C_u and write $H^u := (H(\rho_t^u) : t \geqslant 0)$ for the corresponding height function - we refer to Definition 6.8 for a precise definition of (ρ^u, W^u). Let $g(u)$ be the first time at which the exploration $(p_H(t) : 0 \leqslant t \leqslant \sigma_H)$ visits the excursion component C_u, viz. $g(u) := \inf\{t \geqslant 0 : p_H(t) \in C_u\}$. We shall refer to the point measure

$$\mathcal{E} = \sum_{u\in D} \delta_{(A_{g(u)}, \rho^u, W^u)}$$

as the *excursion process* of $(\xi_a)_{a\in\mathcal{T}_H}$. It is important to note that by Lemma 6.27, for every $u \in D$ the point $g(u)$ belongs to supp $\mathrm{d}A$. Therefore, for every pair of debuts $u \neq u'$, we have $A_{g(u)} \neq A_{g(u')}$. Observe as well that the ordering induced by A is precisely the one induced by the clockwise exploration. Hence, at least at an heuristic level it sounds plausible that one could reconstruct the tree indexed process $(\xi_a)_{a\in\mathcal{T}_H}$ or more precisely the paths of (ρ, W), in terms of $\mathcal{E}$. We shall address this question in the next section.

We can now state the main contribution of [83].

Theorem 2.13. *Under $\mathbb{P}_{0,x,0}$, the measure*

$$\mathcal{E} := \sum_{u\in D} \delta_{(A_{g(u)}, \rho^u, W^u)}$$

is a Poisson measure on $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$ with intensity $\mathrm{d}t \otimes \mathbb{N}_x^*$.

The result as stated in [83] is in fact stronger, since it shows that $\mathcal{E}$ is a Poisson point process -or in short PPP- with respect to the (rather complicated) excursion filtration $(\mathcal{G}_t)_{t\geqslant 0}$. We refer to Section 6.6.2 for its definition and the general statement. Section 6.6.2 is entirely devoted to the proof of Theorem 2.13 and it is divided in two main steps: we first show that $\mathcal{E}$ is a PPP for some intensity measure $\mathrm{d}t \otimes \widehat{\mathbb{N}}_x^*$, and we then proceed to show the identity $\mathbb{N}_x^* = \widehat{\mathbb{N}}_x^*$. One of the main tools used to prove this last point are the so-called *spinal decompositions* of the Lévy snake under $\mathbb{N}_{x,0}$ and $\mathbb{N}_x^*$. This notion has had recurrent use in the setting of Branching random walks and in Brownian geometry, see e.g. [72] for a unified representation for non-compact Brownian surfaces. Spinal decompositions were extensively exploited in our work [83].

Under $\mathbb{N}_{x,0}$, the measure $\mathcal{E}$ is no longer a Poisson measure but we still have the following averaging formula.

Corollary 2.14. *For every non-negative measurable functions Φ and g, we have*

$$\mathbb{N}_{x,0}\Big(\sum_{u\in D} g(A_{g(u)}, \ell_u)\Phi(\rho^u, W^u)\Big) = \mathbb{N}_{x,0}\Big(\int_0^{A_\sigma} \mathrm{d}r \, g(r, \widetilde{H}_r)\Big)\mathbb{N}_x^*(\Phi).$$

The proof of this result strongly relies on the fact that $\mathcal{E}$ is a $(\mathcal{G}_t)_{t\geqslant 0}$-PPP. In the special case when $(\xi_a)_{a\in\mathcal{T}_H}$ is a Brownian motion indexed by the Brownian tree and $x = 0$, if one considers in the last display a function g that does not depend on $A_{g(u)}$ we recover [1, Theorem 1]. In particular, this yields that the excursion measure introduced by C. Abraham and J.-F. Le Gall in [1] and $\mathbb{N}_0^*$ coincide.

2.5 Reconstructions

We now turn our attention to the last part of this introduction. In this section we shall address reconstruction related questions in a rather general sense, but the objective of the section is to establish the following:

> **Claim:** *The Lévy snake (ρ, W), and therefore $(\xi_a)_{a\in\mathcal{T}_H}$, can be recovered from the excursion process $\mathcal{E}$.*

Recall from our discussion shortly after the introduction of the exploration process (2.4) that ρ can be recovered from H. Therefore, by the snake property it readily follows that to prove this claim it suffices to establish that the $\mathbb{R}_+ \times E$-valued process $(H, \widehat{W})$ can be constructed from $\mathcal{E}$. Let us give a more precise outline of the section. Recall from Theorem 2.8 that we already wrote the height process $\widetilde{H}$ of $\widetilde{\mathcal{T}}$ in terms of (ρ, W, Λ). Moreover, by Lemma 6.1 we can construct from $\widetilde{H}$ a pair $(\widetilde{X}, \widetilde{\rho})$ where $\widetilde{X}$ is a $\widetilde{\psi}$-Lévy process and $\widetilde{\rho}$ is its associated exploration process. In this section we describe explicit constructions, in terms of the excursion process $\mathcal{E}$, for the following random objects:

- The Lévy process $\widetilde{X}$ and its jump measure $\sum_{s\in\mathbb{R}_+} \delta_{(s,\Delta\widetilde{X}_s)}$.
- The local times $((\widetilde{\lambda}^{\ell,b}, \widetilde{\lambda}^{r,b}) : b \in \mathrm{Bp}_\infty(\mathcal{T}_H))$ at the branching points of $\widetilde{\mathcal{T}}$ and a *fortiori*, for the exploration process $\widetilde{\rho}$.
- The Lévy snake (ρ, W).

We shall obtain these representation in the order stated above since each construction relies on the preceding ones.

Let us write D_+ for the subsets of debuts $u \in D$ satisfying that $L_\sigma^*(\rho^u, W^u) > 0$. We stress that *a priori*, the contention $D_+ \subset D$ might be strict. For every $u \in D_+$ we consider the mapping

$$\widetilde{g} : u \mapsto A_{g(u)}$$

and we start with the following result:

Proposition 2.15. *The mapping* $\widetilde{g}$ *is a bijection between* D_+ *and* $\{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}$.

Since the branching points of $\widetilde{\mathcal{T}}$ and $\{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}$ are as well in bijection, this gives that D_+ and $\mathrm{Bp}_\infty(\widetilde{\mathcal{T}})$ are in one-to-one correspondence by the mapping $p_{\widetilde{H}} \circ \widetilde{g} : D_+ \to \mathrm{Bp}_\infty(\widetilde{\mathcal{T}})$.

The proof of this proposition relies in the fact that one can identify the jump-times of $\widetilde{X}$ in terms of its height process $\widetilde{H}$. We refer to Lemma 6.50 for a precise statement of this result. For $u \in D$, we introduce the time change:

$$\sigma_u(t) := \int_0^t \mathrm{d}s\, \mathbb{1}_{\{\widehat{\Lambda}_s = \ell_u\}}, \quad t \geqslant 0.$$

Since, as was discussed in Section 2.4, if u, u' are distinct debut points we have $\ell_u \neq \ell_{u'}$, the variable $\sigma_u(t)$ measures the amount of time spent by $(p_H(s) : 0 \leqslant s \leqslant \sigma_H)$ in C_u up to time t, or equivalently $\sigma_u(t) = \mathrm{Vol}_H(\mathcal{T}_H(t) \cap C_u)$. For every $u \in D_+$ and with a slight abuse of notation, we write $(\widetilde{\lambda}^{\ell,u}, \widetilde{\lambda}^{r,u})$ for the local times at the branching point $p_{\widetilde{H}} \circ \widetilde{g}(u)$ in $\widetilde{\mathcal{T}}$. Now we can state the key relationship between the local times at the branching points of $\widetilde{\mathcal{T}}$ and the family of processes $(L^*(\rho^u, W^u) : u \in D)$.

Proposition 2.16. $\mathbb{N}_{x,0}$-*a.e. for every* $u \in D_+$ *we have*

$$\widetilde{\lambda}^{\ell,u}_{A_t} = L^*_{\sigma_u(t)}(\rho^u, W^u), \quad \widetilde{\lambda}^{r,u}_{A_t} = L^*_\sigma(\rho^u, W^u) - L^*_{\sigma_u(t)}(\rho^u, W^u)$$

and in particular $\Delta\widetilde{X}_{A_{g(u)}} = L^*_\sigma(\rho^u, W^u)$.

In the last statement we used that for every $t \geqslant 0$, by definition of $\widetilde{\lambda}^{r,u}$ we have $\widetilde{\lambda}^{\ell,u}_t + \widetilde{\lambda}^{r,u}_t = \Delta\widetilde{X}_{A_{g(u)}}$. Recalling from (2.5) the relationship between the exploration process and the local times at the branching points of $\widetilde{\mathcal{T}}$, we obtain a representation for $\widetilde{\rho}$ at time A_t in terms of $(L^*(\rho^u, W^u) : u \in D)$. As a straight consequence of Propositions 2.15 and 2.16 we obtain the following representation for the jump measure of $\widetilde{X}$.

Theorem 2.17. *We have the identity,*

$$\sum_{u \in D_+} \delta_{(A_{g(u)},\, L^*_\sigma(\rho^u, W^u))} = \sum_{s \in \mathbb{R}_+} \delta_{(s, \Delta\widetilde{X}_s)}.$$

In other terms, the jump measure of $\widetilde{X}$ *is the push-forward of* $\mathcal{E}$ *under the mapping* $(A_{g(u)}, (\rho^u, W^u)) \mapsto (A_{g(u)}, L^*_\sigma(\rho^u, W^u))$.

Since the Lévy process $\widetilde{X}$ has no Brownian component, by Itô synthesis 1.15 we can recover the Lévy process $\widetilde{X}$ from the measure in the last display. Recall that the Lévy measure of $\widetilde{\psi}$ is given by $\widetilde{\pi}(\mathrm{d}z) = \mathbb{N}^*_x(L^*_\sigma \in \mathrm{d}z \cap (0, \infty))$. We let $(\mathbb{N}^{*,z}_x)_{z \in (0,\infty)}$ be the $\widetilde{\pi}$-a.e. unique family of measures in the Polish space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ characterised by the relation

$$\mathbb{N}^*_x(\mathrm{d}\rho, \mathrm{d}W \cap \{L^*_\sigma > 0\}) = \int_{(0,\infty)} \widetilde{\pi}(\mathrm{d}z) \mathbb{N}^{*,z}_x(\mathrm{d}\rho, \mathrm{d}W).$$

Recalling that $\widetilde{H}$ is a functional of $\widetilde{X}$, we obtain the following corollary:

Corollary 2.18. *For $u \in D_+$ we write $z_u := L^*_\sigma(\rho^u, W^u)$. For every non-negative measurable functions $g : \mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E \mapsto \mathbb{R}_+$ and $f : C(\mathbb{R}_+, \mathbb{R}_+) \mapsto \mathbb{R}_+$, we have*

$$\mathbb{N}_{x,0}\bigg(f(\widetilde{H}) \exp\Big(- \sum_{u\in D_+} g(A_{g(u)}, \xi^u)\Big)\bigg) = \mathbb{N}_{x,0}\bigg(f(\widetilde{H}) \prod_{u\in D_+} \mathbb{N}^{*,z_u}_x \Big(\exp\big(- g(A_{g(u)}, \cdot)\big)\Big)\bigg).$$

In other terms, conditionally on $\widetilde{H}$, the excursions $((\rho^u, W^u) : u \in D_+)$ are independent with respective laws $(\mathbb{N}^{*,z_u}_x)_{u\in D_+}$.

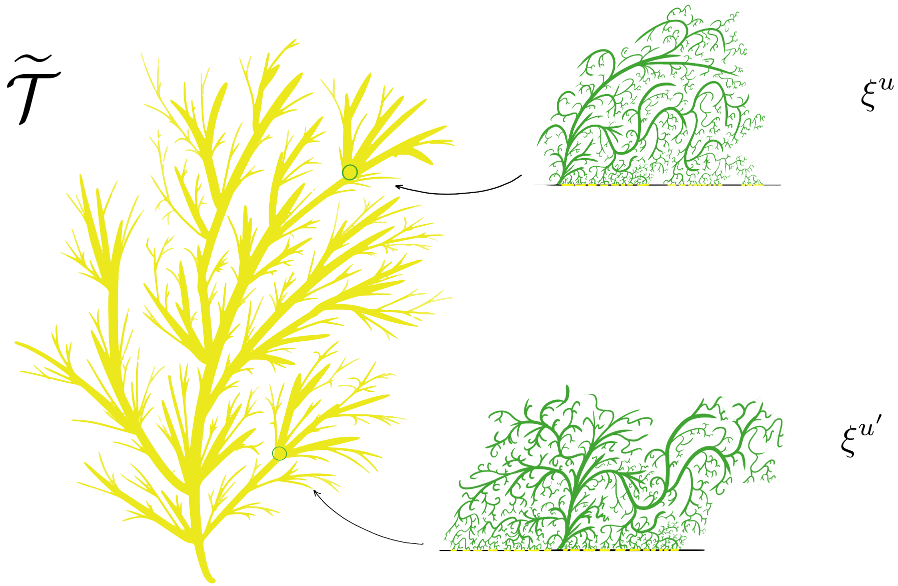


Figure 2.13: Sketch of the correspondence between the set of branching points with infinite multiplicity $\mathrm{Bp}_\infty(\widetilde{\mathcal{T}})$ and the excursions $(\xi^u)_{u\in D_+}$ away from x with non null measure L^*. For every such excursion ξ^u, the "size" of the set $\{a \in C_u : \xi^u_a = x\}$ is given by $L^*_{\sigma_u}(\rho^u, W^u)$ and coincides with the "size" of the corresponding branching point in $\widetilde{\mathcal{T}}$.

This result is closely related to [1, Theorem 40] in the setting of Brownian motion indexed by the Brownian tree. Let us finally address the reconstruction of $(H, \widehat{W})$ in terms of $\mathcal{E}$. Then, making use of the first moment formula from Lemma 2.2, one can deduce the following Lemma:

Lemma 2.19. *For every $t \geqslant 0$ such that $\widehat{W}_t \neq x$ we have*

$$\widehat{W}_t = \widehat{W}^u_{\sigma_u(t)} \text{ if } \widehat{\Lambda}_t = \ell_u, \quad \text{and} \quad H_t = \sum_{u\in D} H^u_{\sigma_u(t)}. \tag{2.6}$$

The condition $\widehat{\Lambda}_t = \ell_u$ in the last display holds if the exploration p_H is visiting the excursion component C_u at time t, viz. if $p_H(t) \in C_u$. It is however not clear if the representation (2.6) can be expressed solely in terms of $\mathcal{E}$. In this direction, we shall rely on the delicate connection between the connected components of $(\mathrm{supp}\ \mathrm{d}A)^c$ and the ones of $(\mathrm{supp}\ \mathrm{d}L^*(\rho^u, W^u))^c$, for $u \in D$. We write $(\alpha_i(u), \beta_i(u))_{i\in\mathcal{Q}_u}$ for the connected components of the complement of

$$\big(\mathrm{supp}\ \mathrm{d}L^*(\rho^u, W^u)\big)^c \setminus \{0, \sigma_{H^u}\}$$

with the convention that if $L^*_\sigma(\rho^u, W^u) = 0$ we let $\mathcal{Q}_u := \{0\}$ and $(\alpha_0(u), \beta_0(u)) := (0, \sigma_{H^u})$. We set $\mathcal{X} := \{(\alpha_i(u), \beta_i(u)) : u \in D,\ i \in \mathcal{Q}_u\}$ and let $\mathcal{X}'$ be the family of connected components of (supp $\mathrm{d}A)^c$. Next, consider the mapping q on $\mathcal{X}$ defined for every $(\alpha_i(u), \beta_i(u)) \in \mathcal{X}$ by the relation:

$$\mathrm{q}(\alpha_i(u), \beta_i(u)) := \big(\sigma_u^{-1}(\alpha_i(u)), \sigma_u^{-1}(\beta_i(u)-)\big)$$

where for $t \geqslant 0$, $\sigma_u^{-1}(t-)$ stands for the left limit of the right inverse σ_u^{-1} of the time change σ_u at time t. To simplify notation, the interval on the right-hand side is denoted by $(g(u,i), d(u,i))$.

Proposition 2.20. *The mapping* q *is a bijection between* $\mathcal{X}$ *and* $\mathcal{X}'$*.*

More precisely, the bijection q satisfies, for every $u \in D$ and $i \in \mathcal{Q}_u$, that

$$\big(\widehat{W}_{(g(u,i)+t)\wedge d(u,i)} : t \geqslant 0\big) = \big(\widehat{W}^u_{(\alpha_i(u)+t)\wedge\beta_i(u)} : t \geqslant 0\big).$$

Finally, let $\widehat{W}^{u,i} := (\widehat{W}^u_{(\alpha_i(u)+t)\wedge\beta(u)} : t \geqslant 0)$, $H^{u,i} := (H_{(\alpha_i(u)+t)\wedge\beta_i(u)} : t \geqslant 0)$ and we introduce the measure

$$\mathcal{E}' := \sum_{u\in D, i\in\mathcal{Q}_u} \delta_{(A_{g(u,i)}, H^{u,i}, \widehat{W}^{u,i})}.$$

The key now is that by Lemma 6.48, the measure $\mathcal{E}'$ can be constructed from $\mathcal{E}$. The proof of this result crucially relies in the fact $((\widetilde{\lambda}^{\ell,u}, \widetilde{\lambda}^{r,u}) : u \in D_+)$ is a function of $\mathcal{E}$ and Proposition 2.16. Finally, we show that (2.6) can be expressed in terms of $\mathcal{E}'$ which gives that the process $(H_t, \widehat{W}_t : t \geqslant 0)$ can be recovered from the excursion process $\mathcal{E}$. This last remark concludes the proof of our initial claim.

Part I

From step reinforced random walks to noise reinforced Lévy processes

Chapter 3

Joint invariance principles for random walks with positively and negatively reinforced steps

The content of this chapter is taken from the paper [16], written in collaboration with Marco Bertenghi, and has been published in the journal *Journal of Statistical Physics*.

Abstract. Given a random walk (S_n) with typical step distributed according to some fixed law and a fixed parameter $p \in (0,1)$, the associated positively step-reinforced random walk is a discrete-time process which performs at each step, with probability $1-p$, the same step as (S_n) while with probability p, it repeats one of the steps it performed previously chosen uniformly at random. The negatively step-reinforced random walk follows the same dynamics but when a step is repeated its sign is also changed. In this work, we shall prove functional limit theorems for the triplet of a random walk, coupled with its positive and negative reinforced versions when $p < 1/2$ and when the typical step is centred. The limiting process is Gaussian and admits a simple representation in terms of stochastic integrals,

$$\left(B(t),\, t^p \int_0^t s^{-p} \mathrm{d}B^r(s),\, t^{-p} \int_0^t s^p \mathrm{d}B^c(s)\right)_{t \in \mathbb{R}^+}$$

for properly correlated Brownian motions $B, B^r,\ B^c$. The processes in the second and third coordinate are called the noise reinforced Brownian motion (as named in [21]), and the noise counterbalanced Brownian motion of B. Different couplings are also considered, allowing us in some cases to drop the centredness hypothesis and to completely identify for all regimes $p \in (0,1)$ the limiting behaviour of step reinforced random walks. Our method exhausts a martingale approach in conjunction with the martingale functional CLT.

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3.1 Introduction

In short, the purpose of this work is to establish invariance principles for *random walks with step reinforcement*, a particular class of random walks with memory that has been of increasing interest in recent years. Historically, the so-called *elephant random walk* (ERW) has been an important and fundamental example of a step-reinforced random walk that was originally introduced in the physics literature by Schütz and Trimper [89] more than 15 years ago. We shall first recall the setting of the ERW in order to motivate the two types of reinforcement that we will work with.

The ERW is a one-dimensional discrete-time nearest neighbour random walk with infinite memory, in allusion to the traditional saying that *an elephant never forgets where it has been before*. It can be depicted as follows: Fix some $q \in (0,1)$, commonly referred to as the *memory parameter*, and suppose that an elephant makes an initial step in $\{-1, 1\}$ at time 1. After, at each time $n \geqslant 2$, the elephant selects uniformly at random a step from its past; with probability q, the elephant repeats the remembered step, whereas with complementary probability $1 - q$ it makes a step in the opposite direction. In particular, in the case $q = 1/2$, the elephant merely follows the path of a simple symmetric random walk. Notably, the ERW is a time-inhomogeneous Markov chain (although some works in the literature improperly assert its non-Markovian character). The ERW has generated a lot of interest in recent years, a non-exhaustive list of references (with further references therein) is [9], [11], [13], [15], [34], [33], [35], [48], [59], [60], see also [8], [12], [47] for variations. A striking feature that has been pointed at in those works, is that the long-time behaviour of the ERW exhibits a phase transition at some critical memory parameter. Functional limit theorems for the ERW were already proved by Baur and Bertoin in [9] by means of limit theorems for random urns. Indeed, the key observation is that the dynamics of the ERW can be expressed in terms of Pólya-type urn experiments and fall in the framework of the work of Janson [55]. For a strong invariance principle for the ERW, we refer to Coletti, Gava and Schütz in [33].

The framework of the ERW is however limited, and it is natural to look for generalisation of its dynamics that allow the typical step to have an arbitrary distribution on $\mathbb{R}$. In this work, we aim to study the more general framework of *step-reinforced random walks*. We shall discuss two such generalisations, called positive and negative step-reinforced random walks, the former generalising the ERW when $q \in (1/2, 1)$ while the later covers the spectrum $q \in [0, 1/2]$, in both cases when the typical step is Rademacher distributed. We start by introducing the former. For the rest of the work, X stands for a random variable that we assume belongs to $L^2(\mathbb{P})$, we denote by σ^2 its variance and by μ its law. Moreover, unless specified otherwise, (S_n) will always denote a random walk with typical step distributed as μ.

The noise reinforced random walk: A *(positive) step-reinforced random walk* or *noise reinforced random walk* is a generalisation of the ERW, where the distribution of a typical step of the walk is allowed to have an arbitrary distribution on $\mathbb{R}$, rather than just Rademacher. The impact of the reinforcement is still described in terms of a fixed parameter $p \in (0,1)$, that we also refer to as the *memory parameter* or the *reinforcement parameter*. We will work with different values of p but for readability purposes p does not explicitly appear in the notation or terminology used in this work.

Vaguely speaking, the dynamics are as follows: at each discrete time, with probability p a step reinforced random walk repeats one of its preceding steps chosen uniformly at random, and otherwise, with complementary probability $1-p$, it has an independent increment with a fixed but arbitrary distribution. More precisely, given an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $X_1, X_2, \ldots$ of i.i.d. copies of the random variable X with law μ, we define $\hat{X}_1, \hat{X}_2, \ldots$ recursively as follows: First, let $(\varepsilon_i : i \geqslant 2)$ be an independent sequence of Bernoulli random variables with parameter $p \in (0,1)$ and also consider $(U[i] : i \geqslant 2)$ an independent sequence where each $U[i]$ is uniformly distributed on $\{1, \ldots, i\}$. We set first $\hat{X}_1 = X_1$, and next for $i \geqslant 2$, we let

$$\hat{X}_i = \begin{cases} X_i, & \text{if } \varepsilon_i = 0, \\ \hat{X}_{U[i-1]}, & \text{if } \varepsilon_i = 1. \end{cases}$$

Finally, the sequence of the partial sums

$$\hat{S}_n := \hat{X}_1 + \cdots + \hat{X}_n, \quad n \in \mathbb{N},$$

is referred to as a *positive step-reinforced random walk*. We have from the definition of the sequence $(\hat{X}_i)$ that

$$\hat{X}_{n+1} = (1 - \varepsilon_{n+1}) X_{n+1} + \varepsilon_{n+1} \hat{X}_{U[n]}$$

which implies that for any bounded measurable $f : \mathbb{R} \mapsto \mathbb{R}^+$,

$$\mathbb{E}(f(\hat{X}_{n+1})) = (1-p)\mathbb{E}(f(X_{n+1})) + \frac{p}{n} \sum_{j=1}^{n} \mathbb{E}(f(\hat{X}_j))$$

and it follows by induction that each $\hat{X}_n$ has law μ. Beware however that the sequence $(\hat{X}_i)$ is not stationary. Notice that if $(\hat{S}_n)$ is not centred, it is often fruitful to reduce our analysis to the centred case by considering $(\hat{S}_n - n\mathbb{E}(X))$, which is a centred noise reinforced random walk with typical step distributed as $X - \mathbb{E}(X)$. Observe that in the degenerate case $p = 1$, the dynamics of the positive step-reinforced random walk become essentially deterministic. Indeed when $p = 1$ we have $\hat{S}_n = nX_1$ for all $n \geqslant 1$, in particular the only remaining randomness for this process stems from the random variable X_1.

In this setting, when μ is the Rademacher distribution, Kürsten [60] (see also [46]) pointed out that $\hat{S} = (\hat{S}_n)_{n \geqslant 1}$ is a version of the elephant random walk with memory parameter $q = (p+1)/2 \in (1/2, 1)$ in the present notation. The remaining range of the memory parameter can be obtained by a simple modification that we will address when we introduce random walks with *negatively reinforced steps*. When μ has a symmetric stable distribution, $\hat{S}$ is the so-called *shark random swim* which has been studied in depth by Businger [31]. More general versions when the distribution μ is infinitely divisible have been considered by Bertoin in [19], and we will briefly comment on this setting in a moment. Finally, when we replace the sequence of Bernoulli random variables (ε_n) by a deterministic sequence (r_n) with $r_n \in \{0,1\}$, the scaling exponents of the corresponding step reinforced random walks have been studied by Bertoin in [20].

In stark contrast to the ERW, the literature available on general step-reinforced random walks remains rather sparse. Quite recently, Bertoin [21] established an invariance principle for the step-reinforced random walk in the diffusive regime $p \in (0,1/2)$. Bertoin's work concerned a rather simple real-valued and centered Gaussian process $\hat{B} = (\hat{B}(t))_{t\geqslant 0}$ with covariance function given by

$$\mathbb{E}\left(\hat{B}(t)\hat{B}(s)\right) = \frac{t^p s^{1-p}}{1-2p} \quad \text{for } 0 \leqslant s \leqslant t \quad \text{and } p \in (0,1/2). \tag{3.1}$$

This process has notably appeared as the scaling limit for diffusive regimes of the ERW and other Polya urn related processes, see [9, 34], [15] for higher dimensional generalisations, and [7]. In [21] the process displayed in (3.1) is referred to as a *noise reinforced Brownian motion* and belongs to a larger class of reinforced processes recently introduced by Bertoin in [19] called *noise reinforced Lévy processes*. The noise reinforced Brownian motion plays, in the framework of noise reinforced Lévy processes, the same role as the standard Brownian motion in the context of Lévy processes. Moreover, just as the standard Brownian motion B corresponds to the integral of a white noise, $\hat{B}$ can be thought of as the integral of a reinforced version of the white noise, hence the name. More precisely, from (3.1) it readily follows that the law of $\hat{B}$ admits the following integral representation

$$\hat{B}(t) = t^p \int_0^t s^{-p} \mathrm{d}B^r(s), \quad t \geqslant 0,$$

where $B^r = (B^r_s)_{s\geqslant 0}$ is a standard Brownian motion, or equivalently, $\hat{B} = (\hat{B}(t))_{t\geqslant 0}$ has the same law as

$$\left(\frac{t^p}{\sqrt{1-2p}} B(t^{1-2p}) \right)_{t\geqslant 0}.$$

Some further properties of the noise reinforced Brownian motion can be found in [21], where the following functional limit theorem [21, Theorem 3.3] has been established: let $p \in (0,1/2)$ and suppose that $X \in L^2(\mathbb{P})$. Then, we have the weak convergence of the scaled sequence in the sense of Skorokhod as n tends to infinity

$$\left(\frac{\hat{S}(\lfloor nt \rfloor) - nt\mathbb{E}(X)}{\sqrt{\sigma^2 n}} \right)_{t\in\mathbb{R}^+} \Longrightarrow (\hat{B}(t))_{t\in\mathbb{R}^+} \tag{3.2}$$

where $(\hat{B}(t))_{t\geqslant 0}$ is a noise reinforced Brownian motion.

Our work generalises this result but our approach differs from [21] as we work with a discrete martingale introduced by Bercu [11] for the ERW and later generalised in [14] for step-reinforced random walks. The martingale we work with is a discrete-time stochastic process of the form $\widehat{a}_n \hat{S}_n$, where $(\widehat{a}_n)_{n\geqslant 0}$ is a properly defined sequence of positive real numbers of order n^{-p}. As we shall see, investigation of said martingale and in particular its quadratic variation process, in conjunction with the functional martingale CLT [92], yields an alternative proof of Theorem 3.3

in [21].

The counterbalanced random walk: Next we turn our attention to the second process of interest, called the *counterbalanced random walk* or *negative step-reinforced random walk*, introduced recently by Bertoin in [22]. Beware that p in our work always corresponds to the probability of a repetition event, while in [22] this happens with probability $1-p$. Similarly, we will consider a sequence of i.i.d. random variables $(X_n)_{n\in\mathbb{N}}$ with distribution μ on $\mathbb{R}$ and at each time step, the step performed by the walker will be, with probability $1-p \in (0,1)$, an independent step X_n from the previous ones while with complementary probability p, the new step is one of the previously performed steps, chosen uniformly at random, with its sign changed. This last action will be referred to as a *counterbalance* of the uniformly chosen step. In particular, when μ is the Rademacher distribution, we obtain an ERW with parameter $(1-p)/2 \in [0,1/2]$.

Formally, recall that $X_1, X_2, \ldots$ is a sequence of i.i.d. copies of X and $(\varepsilon_i : i \geqslant 2)$ is an independent sequence of Bernoulli random variables with parameter $p \in (0,1)$. We define the sequence of increments $\check{X}_1, \check{X}_2, \ldots$ recursively as follows (beware of the difference of notation between $\hat{X}$ and $\check{X}$): we set first $\check{X}_1 = X_1$, and next for $i \geqslant 2$, we let

$$\check{X}_i = \begin{cases} X_i, & \text{if } \varepsilon_i = 0, \\ -\check{X}_{U[i-1]} & \text{if } \varepsilon_i = 1 \end{cases},$$

where $U[i-1]$ denotes an independent uniform random variable in $\{1,\ldots,i-1\}$. Finally, the sequence of partial sums

$$\check{S}_n := \check{X}_1 + \cdots + \check{X}_n, \quad n \in \mathbb{N},$$

is referred to as a *counterbalanced random walk (or random walk with negatively reinforced steps).* Notice also that, in contrast with the positive step-reinforced random walk, when $p=1$ we still get a stochastic process, consisting of consecutive counterbalancing of the initial step X_1 while for $p=0$ we just get the dynamics of a random walk. For the positive reinforced random walk we already pointed out that the steps are identically distributed and hence are centred as soon as X is centred. On the other hand, for the negatively step-reinforced random walk, since

$$\check{X}_{n+1} = (1-\varepsilon_{n+1})X_{n+1} - \varepsilon_{n+1}\check{X}_{U[n]}$$

we clearly have

$$\mathbb{E}\left(\check{S}_{n+1}\right) = (1-p)m + (1-p/n)\mathbb{E}\left(\check{S}_n\right), \qquad n \geqslant 1 \tag{3.3}$$

with initial condition $\mathbb{E}(\check{S}_1) = \mathbb{E}(\check{X}_1) = m$. As was noted in [22], it follows from the previous recurrence that:

$$\mathbb{E}(\check{S}_n) \sim \left(\frac{(1-p)m}{1+p}\right) n \quad \text{as } n \uparrow \infty, \tag{3.4}$$

and note that the process $(\check{S}_n)$ is also centered if X is centred. Observe however that in stark contrast to the positive step-reinforced random walk, we cannot say that the typical step is centered without loss of generality: Indeed, since $n \mapsto \mathbb{E}(\check{X}_n)$ is no longer constant as soon as $m \neq 0$, due to the random swap of signs in the negative reinforcement algorithm, the centered process $(\check{S}_n - \mathbb{E}(\check{S}_n))$ is also no longer a counterbalanced random walk.

Turning our attention to its asymptotic behaviour, Proposition 1.1 in [22] shows that the behaviour of the counterbalanced random walk $\check{S}_n$ is ballistic. More precisely, denoting by $m =$

$\mathbb{E}(X)$ the mean of the typical step X, then for all $p \in [0,1]$ the process $(\check{S}_n)$ satisfies a law of large numbers:

$$\lim_{n\to\infty} \frac{\check{S}_n}{n} = \frac{(1-p)m}{1+p} \quad \text{in probability.}$$

Moreover, Theorem 1.2 in [22] shows that if we also assume that the second moment $m_2 = \mathbb{E}(X^2)$ is finite, then the fluctuations are Gaussian for all choices $p \in [0,1)$:

$$\frac{\check{S}_n - \frac{1-p}{1+p}mn}{\sqrt{n}} \Longrightarrow \mathcal{N}\left(0, \frac{m_2 - \left(\frac{1-p}{1+p}m\right)^2}{1+2p}\right).$$

In particular, when X is centred as will be our case, we simply get

$$\frac{\check{S}_n}{\sqrt{\sigma^2 n}} \Longrightarrow \mathcal{N}\left(0, (1+2p)^{-1}\right).$$

On the other hand, when $p = 1$ which corresponds to the purely counterbalanced case, and under the additional assumption that X follows the Rademacher distribution, then

$$\frac{1}{\sqrt{n}}\check{S}_n \Longrightarrow \mathcal{N}(0, 1/3).$$

The proofs of these results rely on remarkable connections with random recursive trees and even if these will not be needed in the present work, we encourage the interested reader to consult [22] for more details. In this article, we will establish a functional version of the asymptotic normality mentioned above under the additional assumption that $m = 0$, i.e. the typical step is centered. We recall that this assumption cannot be made without the loss of generality.

In the same spirit as in the noise-reinforced setting, we will call a *noise counterbalanced Brownian motion of parameter* $p \in [0,1)$ a Gaussian process $\check{B}$ with covariance given by

$$\mathbb{E}\left(\check{B}(t)\check{B}(s)\right) = \frac{1}{2p+1}\frac{s^{1+p}}{t^p} \quad \text{for } 0 \leqslant s \leqslant t \quad \text{and } p \in [0,1), \tag{3.5}$$

and it follows that the law of $\check{B}$ admits the following integral representation

$$\check{B}(t) = t^{-p}\int_0^t s^p \mathrm{d}B^c(s), \qquad t \geqslant 0 \tag{3.6}$$

in terms of a standard Brownian motion $B^c = (B^c(s))_{s\geqslant 0}$. Let us now state the main results of this work.

Law of large numbers for step-reinforced random walks: In order to establish our invariance principles, we shall need to investigate the asymptotic behaviour of step-reinforced random walks. In this direction, we establish in Section 3.2 the following result:

Theorem 3.1. (Law of large numbers) *For any $p \in (0,1)$, we have the $L^2(\mathbb{P})$ and almost sure convergences:*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n} = m \qquad \textit{and} \qquad \lim_{n\to\infty}\frac{\check{S}_n}{n} = \frac{1-p}{1+p}m. \tag{3.7}$$

Moreover, if $p = 1$, (3.7) still holds for the counterbalanced random walk.

Note that if $p = 1$, in the noise-reinforced case the result is clearly false, since we just have $\hat{S}_n = nX_1$ for $n \geqslant 1$ while in the counterbalanced case, we can write $\check{S}_n = X_1 \check{S}'_n$ for $n \geqslant 1$, where $\check{S}'$ is a counterbalanced random walk with same parameter and with typical step distributed δ_1. Theorem 3.1 will be proved by means of two remarkable martingales, denoted throughout this work by $\hat{M}$ and $\check{M}$, associated respectively to noise reinforced and counterbalanced random walks. These will be introduced and studied in Section 3.2 and will play a crucial role in this work. We stress that the second convergence in Theorem 3.1 was already established in [22] in probability by different methods.

The invariance principles: Before stating the functional versions of the results we just mentioned, notice that given a sample of i.i.d. random variables (X_n) with law μ, and an additional independent collection (ε_i), $(U[i])$ of Bernoulli random variables and uniform random variables respectively as before, we can construct from the same sample simultaneously to the associated random walk (S_n), the processes $(\hat{S}_n)$ and $(\check{S}_n)$, that we refer respectively as *the* positive step-reinforced version and *the* negative step-reinforced version of (S_n). It is then natural to compare the dynamics of the triplet $(S_n, \hat{S}_n, \check{S}_n)$, instead of individually working with $(\hat{S}_n)$ and $(\check{S}_n)$. When considering such a triplet, it will always be implicitly assumed that $(\hat{S}_n), (\check{S}_n)$ have been constructed in this special way from (S_n). In particular, we used the same sequence of uniform and Bernoulli random variables to define both reinforced versions. Now we have all the ingredients to state our first main result:

Theorem 3.2. *Fix $p \in [0, 1/2)$ and consider the triplet $(S_n, \hat{S}_n, \check{S}_n)$ consisting of the random walk (S_n) with its reinforced version and its counterbalanced version of parameter p. Assume further that X is centred. Then, the following weak convergence holds in the sense of Skorokhod as n tends to infinity,*

$$\left(\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}, \frac{1}{\sigma\sqrt{n}} \hat{S}_{\lfloor nt \rfloor}, \frac{1}{\sigma\sqrt{n}} \check{S}_{\lfloor nt \rfloor} \right)_{t \in \mathbb{R}^+} \implies \left(B(t), \hat{B}(t), \check{B}(t) \right)_{t \in \mathbb{R}^+} \tag{3.8}$$

where B, $\hat{B}$, $\check{B}$ denote respectively a standard BM, a noise reinforced BM and a counterbalanced BM with covariances, $\mathbb{E}(B(s)\check{B}(t)) = t^{-p}(t \wedge s)^{p+1}(1-p)/(1+p)$, $\mathbb{E}(B(s)\hat{B}(t)) = t^p (t \wedge s)^{1-p}$, $\mathbb{E}(\hat{B}(t)\check{B}(s)) = t^p s^{-p}(t \wedge s)(1-p)/(1+p)$.

Notice that in the case $p = 0$, i.e. when no reinforcement events occur, this is just Donsker's invariance principle since $(\check{S}_n), (\hat{S}_n)$ are just the random walk (S_n) and $\hat{B}$, $\check{B}$ are just B. Hence, from now on we will assume that $p > 0$. The process in the limit admits the following simple integral representation in terms of stochastic integrals

$$\left(B(t),\, t^p \int_0^t s^{-p} \mathrm{d}B^r(s),\, t^{-p} \int_0^t s^p \mathrm{d}B^c(s) \right)_{t \in \mathbb{R}^+} \tag{3.9}$$

where $B = (B(t))_{t \geqslant 0}$, $B^r = (B^r(t))_{t \geqslant 0}$, $B^c = (B^c(t))_{t \geqslant 0}$ denote three standard Brownian motions with covariance structure $\mathbb{E}(B(s)B^r(t)) = (1-p)(t \wedge s)$, $\mathbb{E}(B(s)B^c(t)) = (1-p)(t \wedge s)$, $\mathbb{E}(B^r(s)B^c(t)) = (t \wedge s)(1-p)/(1+p)$.

The restriction on the parameter $p \in (0, 1/2)$ comes from the fact that, as we will see, for the noise reinforced random walk only for such parameter the functional version works with this scaling, while the centred hypothesis is a restriction coming from the counterbalanced random walk. Now we point at some variants with less restrictive hypothesis, holding as long as we no longer consider the triplet. This allows us to drop some of the conditions we just mentioned, and the proofs will be embedded in the proof of Theorem 3.2. We start by removing the centred hypothesis when only working with the pair $(S_n, \hat{S}_n)$ in the diffusive regime $p \in [0, 1/2)$.

Theorem 3.3. *Let $p \in [0, 1/2)$ and suppose that $X \in L^2(\mathbb{P})$. Let (S_n) be a random walk with typical step distributed as X and denote by $(\hat{S}_n)$ its positive step reinforced version. Then, we have the weak joint convergence of the scaled sequence in the sense of Skorokhod as n tends to infinity towards a Gaussian process*

$$\left(\frac{S(\lfloor nt \rfloor) - nt\mathbb{E}(X)}{\sigma\sqrt{n}}, \frac{\hat{S}(\lfloor nt \rfloor) - nt\mathbb{E}(X)}{\sigma\sqrt{n}} \right)_{t \in \mathbb{R}^+} \Longrightarrow (B(t), \hat{B}(t))_{t \in \mathbb{R}^+} \tag{3.10}$$

where B is a Brownian motion, $\hat{B}$ is a noise reinforced Brownian motion with covariance $\mathbb{E}[B(s)\hat{B}(t)] = t^p(t \wedge s)^{1-p}$.

It follows that the limit process in (3.10) admits the integral representation

$$\left(B(t), t^p \int_0^t s^{-p} \mathrm{d}B^r(s) \right)_{t \in \mathbb{R}^+}$$

where $B = (B(t))_{t \geqslant 0}$ and $B^r = (B(t)^r)_{t \geqslant 0}$ denote two standard Brownian motions with convariances $\mathbb{E}(B(t)B^r(s)) = (1-p)(t \wedge s)$. This result extends Theorem 3.3 in [21] to the pair $(S, \hat{S})$. Notice that the factor $1-p$ in the correlation can be interpreted in terms of the definition of the noise reinforced random walk, since at each discrete time step, with probability $1-p$ the processes $\hat{S}$ and S share the same step X_n.

Turning our attention to the counterbalanced random walk, when only working with the pair $(S_n, \check{S}_n)$ we can extend the convergence to $p \in [0, 1)$, and is the content of the following result:

Theorem 3.4. *Let $p \in [0, 1)$ and suppose that $X \in L^2(\mathbb{P})$ is centred. If (S_n) is a random walk with typical step distributed as X and $(\check{S}_n)$ is its counterbalanced version of parameter p, then we have the weak convergence of the sequence of processes in the sense of Skorokhod as n tends to infinity*

$$\left(\frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}, \frac{1}{\sigma\sqrt{n}} \check{S}_{\lfloor nt \rfloor} \right)_{t \in \mathbb{R}^+} \Longrightarrow \big(B(t), \check{B}(t) \big)_{t \in \mathbb{R}^+} \tag{3.11}$$

where B is a Brownian motion and $\check{B}$ is a noise counterbalanced Brownian motion with covariance $\mathbb{E}[B(s)\check{B}(t)] = t^{-p}(t \wedge s)^{p+1}(1-p)/(1+p)$ and $\sigma^2 = \mathbb{E}[X^2]$. If $p = 1$ and X follows the Rademacher distribution, the result still holds and in particular B and $\check{B}$ are independent.

Moreover, the limit process in (3.11) admits the simple integral representation

$$\left(B(t), t^{-p} \int_0^t s^p \mathrm{d}B^c(s) \right)_{t \in \mathbb{R}^+}$$

where $B = (B(t))_{t \geqslant 0}$ and $B^c = (B^c(t))_{t \geqslant 0}$ denote two standard Brownian motions with convariances $\mathbb{E}(B(s)B^c(t)) = (1-p)(t \wedge s)$.

Finally, we turn back our attention to the noise reinforced setting when the parameter is $p = 1/2$. Our method allows us to establish an invariance principle for the step-reinforced random walk at criticality $p = 1/2$ but notice that in this case we do not establish a joint convergence, as the required scalings are no longer compatible.

Theorem 3.5. *Let $p = 1/2$ and suppose that $X \in L^2(\mathbb{P})$. Then, we have the weak convergence of the sequence of processes in the sense of Skorokhod as n tends to infinity*

$$\left(\frac{\hat{S}_{\lfloor n^t \rfloor} - n^t \mathbb{E}(X)}{\sigma \sqrt{\log(n)} n^{t/2}} \right)_{t \in \mathbb{R}^+} \Longrightarrow (B(t))_{t \in \mathbb{R}^+} \tag{3.12}$$

where $B = (B(t))_{t \geqslant 0}$ denotes a standard Brownian motion.

Our proofs rely on a version of the martingale Functional Central Limit Theorem (abreviated MFCLT), which we state for the reader's convenience. For more general versions, we refer to Chapter VIII in [54]. If $\mathbf{M} = (M^1, \ldots, M^d)$ is a real rcll d-dimentional process, we denote by $\Delta \mathbf{M}$ its jump process, which is the d-dimensional process null at 0 defined as $(M^1_t - M^1_{t-}, \ldots, M^d_t - M^d_{t-})_{t \in \mathbb{R}^+}$.

Theorem 3.6 (MFCLT, VIII-3.11 from [54])**.** *Assume $\mathbf{M} = (M^1, \ldots, M^d)$ is d-dimentional continuous Gaussian martingale with independent increments, and predicable covariance process $(\langle M^i, M^j \rangle)_{i,j \in \{1,\ldots,d\}}$. For each n, let $\mathbf{M}^n = (M^{n,1}, \ldots, M^{n,d})$ be a d-dimentional local martingale with uniformly bounded jumps $|\Delta \mathbf{M}^n| \leqslant K$ for some constant K. The following conditions are equivalent:*

(i) $\mathbf{M}^n \Rightarrow \mathbf{M}$ *in the sense of Skorokhod,*

(ii) *There exists some dense set $D \subset \mathbb{R}^+$ such that for each $t \in D$ and $i, j \in \{1, \ldots, d\}$, as $n \uparrow \infty$,*

$$\langle M^{n,i}, M^{n,j} \rangle_t \to \langle M^i, M^j \rangle_t \qquad \text{in probability,} \tag{3.13}$$

and

$$\sup_{s \leqslant t} |\Delta \mathbf{M}^n_s| \to 0 \qquad \text{in probability.} \tag{3.14}$$

The rest of this paper is organised as follows: In Section 3.2 we introduce two crucial martingales for our reasoning associated with step-reinforced random walks and investigate their properties. We derive maximal inequalities and asymptotic results for step reinforced random walks that will be needed in the sequel and establish Theorem 3.1. Section 3.3 is devoted to the proof of Theorem 3.2 under the additional assumption that the typical step X is bounded and in Section 3.4 we discuss how to relax this assumption to the general case of unbounded steps by a truncation argument. In the process, we will also deduce the proofs of Theorem 3.3 and Theorem 3.4. Finally, in Section 3.5 we address the proof of Theorem 3.5 and we shall again proceed in two stages. Since many arguments can be carried over from the previous sections, some details are skipped.

3.2 The martingales associated to a reinforced random walk and proof of Theorem 3.1

In this section we work under the additional assumption that the typical step $X \in L^2(\mathbb{P})$ is centred and recall that we denote by $\sigma^2 = \mathbb{E}(X^2)$ its variance. The centred hypothesis is maintained for Sections 3 and 4, but dropped in Section 5.

Recall that if $M = (M_n)_{n\geqslant 0}$ is a discrete-time real-valued and square integrable martingale with respect to a filtration $(\mathcal{F}_n)$, then its predicable variation process $\langle M\rangle$ is the process defined by $\langle M\rangle_0 = 0$ and for $n \geqslant 1$,

$$\langle M\rangle_n = \sum_{k=1}^{n} \mathbb{E}(\Delta M_k^2 \mid \mathcal{F}_{k-1}),$$

while if (Z_n) is another martingale, the predictable covariation of the pair $\langle M, Z\rangle$ is the process defined by $\langle M, Z\rangle_0 = 0$ and for $n \geqslant 1$,

$$\langle M, Z\rangle_n = \sum_{k=1}^{n} \mathbb{E}(\Delta M_k \Delta Z_k \mid \mathcal{F}_{k-1}).$$

We define two sequences $(\widehat{a}_n, n \geqslant 1)$, $(\check{a}_n, n \geqslant 1)$ as follows: Let $\widehat{a}_1 = \check{a}_1 = 1$ and for each $n \in \{2, 3, \ldots\}$, set

$$\widehat{a}_n = \prod_{k=1}^{n-1} \widehat{\gamma}_k^{-1} = \frac{\Gamma(n)}{\Gamma(n+p)}, \qquad \check{a}_n = \prod_{k=1}^{n-1} \check{\gamma}_k^{-1} = \frac{\Gamma(n)}{\Gamma(n-p)} \tag{3.15}$$

for respectively $\widehat{\gamma}_n = \frac{n+p}{n}$, $\check{\gamma}_n = \frac{n-p}{n}$ when $n \geqslant 2$.

Proposition 3.7. *The processes $\hat{M} = (\hat{M}_n)_{n\geqslant 0}$, $\check{M} = (\check{M}_n)_{n\geqslant 0}$ defined as $\hat{M}_0 = \check{M}_0 = 0$ and $\hat{M}_n = \widehat{a}_n \hat{S}_n$, $\check{M}_n = \check{a}_n \check{S}_n$ for $n \geqslant 1$ are centred square integrable martingales and we denote the natural filtration generated by the pair by $(\mathcal{F}_n)$, where $\mathcal{F}_0$ is the trivial sigma-field. Further, their respective predictable quadratic variation processes is given by $\langle \hat{M}\rangle_0 = \langle \check{M}\rangle_0 = 0$ and, for all $n \geqslant 1$*

$$\langle \hat{M}\rangle_n = \sigma^2 + \sum_{k=2}^{n} \widehat{a}_k^2 \left((1-p)\sigma^2 - p^2 \left(\frac{\hat{S}_{k-1}}{k-1} \right)^2 + p \frac{\hat{V}_{k-1}}{k-1} \right) \tag{3.16}$$

and

$$\langle \check{M}\rangle_n = \sigma^2 + \sum_{k=2}^{n} \check{a}_k^2 \left((1-p)\sigma^2 - p^2 \left(\frac{\check{S}_{k-1}}{k-1} \right)^2 + p \frac{\hat{V}_{k-1}}{k-1} \right) \tag{3.17}$$

where $(\hat{V}_n)_{n\geqslant 1}$ is the step-reinforced process given by $\hat{V}_n = \hat{X}_1^2 + \cdots + \hat{X}_n^2$ and the sums should be considered identical to zero for $n = 1$.

Proof. Starting with the positive-reinforced case, notice that for any $n \geqslant 1$ we have

$$\mathbb{E}\left(\hat{X}_{n+1} \mid \mathcal{F}_n \right) = (1-p)\mathbb{E}(X) + p \frac{\hat{X}_1 + \cdots + \hat{X}_n}{n} = p \frac{\hat{S}_n}{n}. \tag{3.18}$$

Hence, since $\hat{S}_{n+1} = \hat{S}_n + \hat{X}_{n+1}$, and $\widehat{\gamma}_n = (n+p)/n$,

$$\mathbb{E}(\hat{S}_{n+1} \mid \mathcal{F}_n) = \widehat{\gamma}_n \cdot \hat{S}_n \tag{3.19}$$

and therefore, we obtain

$$\mathbb{E}(\hat{M}_{n+1} \mid \mathcal{F}_n) = \widehat{a}_{n+1} \mathbb{E}(\hat{S}_{n+1} \mid \mathcal{F}_n) = \widehat{a}_{n+1} \widehat{\gamma}_n \check{S}_n = \widehat{a}_n \hat{S}_n = \hat{M}_n.$$

Moreover, as X is centred and the steps $(\hat{X}_k)$ are identically distributed by what was discussed in the introduction, we have

$$\mathbb{E}(\hat{M}_n) = \mathbb{E}(\hat{X}_1) = \mathbb{E}(X) = 0$$

and we conclude that $(\hat{M}_n)_{n\geqslant 0}$ is a martingale. Turning our attention to its quadratic variation, we have $\mathbb{E}(\hat{S}_n^2) \leqslant n^2\mathbb{E}(X^2) = n^2\sigma^2$ and hence, $\hat{M}_n$ is indeed square integrable and its predictable quadratic variation exists. Next, we observe that for $n \geqslant 1$ we have

$$\begin{aligned}\mathbb{E}(\hat{M}_{n+1}^2 - \hat{M}_n^2 \mid \mathcal{F}_n) &= \mathbb{E}((\hat{M}_{n+1} - \hat{M}_n)^2 \mid \mathcal{F}_n) \\ &= \widehat{a}_{n+1}^2 \mathbb{E}((\hat{X}_{n+1} - \mathbb{E}(\hat{X}_{n+1} \mid \mathcal{F}_n))^2 \mid \mathcal{F}_n) \\ &= \widehat{a}_{n+1}^2 \left(\mathbb{E}(\hat{X}_{n+1}^2 \mid \mathcal{F}_n) - (\mathbb{E}(\hat{X}_{n+1} \mid \mathcal{F}_n))^2\right) \\ &= \widehat{a}_{n+1}^2 \left(\mathbb{E}(\hat{X}_{n+1}^2 \mid \mathcal{F}_n) - \frac{p^2}{n^2}\hat{S}_n^2\right). \end{aligned} \tag{3.20}$$

Finally, as was pointed out in the proof of Lemma 3 in [19], and can be verified from the definition of the $\hat{X}_n$, it holds that

$$\mathbb{E}(\hat{X}_{n+1}^2 \mid \mathcal{F}_n) = p\frac{\hat{V}_n}{n} + (1-p)\sigma^2, \tag{3.21}$$

and hence we arrive at the formula (3.16).

For the negative-reinforced case, the proof follows very similar steps after minor modifications have been made. Since for $n \geqslant 1$,

$$\mathbb{E}(\check{X}_{n+1} \mid \mathcal{F}_n) = m(1-p) - p\frac{\check{X}_1 + \cdots + \check{X}_n}{n} = -\frac{p}{n}\check{S}_n \tag{3.22}$$

we now have

$$\mathbb{E}(\check{S}_{n+1} \mid \mathcal{F}_n) = \left(\frac{n-p}{n}\right)\check{S}_n = \check{\gamma}_n\check{S}_n, \tag{3.23}$$

and the martingale property for $(\check{M}_n)_{n\geqslant 0}$ follows. For the quadratic variation, the proof is the same after noticing that since clearly $\check{X}_k^2 = \hat{X}_k^2$, we can also write $\hat{V}_n = \check{X}_1^2 + \cdots + \check{X}_n^2$. □

We write for further use the following asymptotic behaviours: the first ones are related to the study of the positive-reinforced case and hold for $p \in (0, 1/2)$:

$$\lim_{n\to\infty} n^{2p-1}\sum_{k=1}^{n} \widehat{a}_k^2 = \frac{1}{1-2p}, \qquad \widehat{a}_n = \frac{\Gamma(n)}{\Gamma(n+p)} \sim n^{-p} \quad \text{as } n \uparrow \infty \tag{3.24}$$

while for $p = 1/2$ we have a change on the asymptotic behaviour in the series,

$$\lim_{n\to\infty} \frac{1}{\log(n)}\sum_{k=1}^{n} \widehat{a}_k^2 = 1, \qquad \widehat{a}_n = \frac{\Gamma(n)}{\Gamma(n+1/2)} \sim n^{-\frac{1}{2}} \quad \text{as } n \uparrow \infty \tag{3.25}$$

which is the reason behind the different scaling showing in Theorem 3.5. On the other hand, for the negatively-reinforced case we have for $p \in (0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n^{1+2p}}\sum_{k=1}^{n} \check{a}_k^2 = \frac{1}{1+2p}, \qquad \check{a}_n = \frac{\Gamma(n)}{\Gamma(n-p)} \sim n^{p} \quad \text{as } n \uparrow \infty. \tag{3.26}$$

The limits are derived from standard Gamma function asymptotic behaviour, and were already pointed out in Bercu [11].

Before turning our attention to the proof of Theorem 3.1, let us introduce a more general version of $\check{M}$ that will be needed in our analysis, when the steps of the counterbalanced random walk are not centred. In this direction set $Y_0 := 0$ and for $n \geqslant 1$, let

$$Y_n := \check{a}_n \check{S}_n - \check{a}_n \mathbb{E}(\check{S}_n). \tag{3.27}$$

It readily follows by (3.22) and the recursive formula (3.3) that for $n \geqslant 1$, we have

$$\begin{aligned}\mathbb{E}\left(Y_{n+1} \mid \mathcal{F}_n\right) &= \check{a}_{n+1}\left(m(1-p) + \check{\gamma}_n \check{S}_n - \mathbb{E}(\check{S}_{n+1})\right) \\ &= \check{a}_n \check{S}_n + \check{a}_{n+1}\left(m(1-p) - \mathbb{E}(\check{S}_{n+1})\right) = Y_n,\end{aligned}$$

and we deduce that $(Y_n)_{n\geqslant 1}$ is a centred a martingale – note that if $m = 0$, we have $Y = \check{M}$.

We shall now make use of $\hat{M}$, $\check{M}$ and Y to study the rate of growth of $\hat{S}$, $\check{S}$ and to establish Theorem 3.1. In this direction, the following lemma has already been observed in [14, 21] using a different technique, we present here a more elementary approach.

Lemma 3.8. *For every fixed $p \in (1/2, 1)$, the following convergence holds a.s. and in $L^2(\mathbb{P})$,*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n^p} = \hat{W}$$

where $\hat{W} \in L^2(\mathbb{P})$ is a non-degenerate random variable.

Proof. Thanks to Proposition 3.7 we know that $\hat{M}_n = \hat{a}_n \hat{S}_n$ is a martingale. Further, we obtain from (3.20) and the asymptotics $\widehat{a}_n \sim n^{-p}$ that, for some constant C large enough,

$$\mathbb{E}(|\hat{M}_n|^2) = \mathbb{E}(\langle \hat{M} \rangle_n) \leqslant \sigma^2 \sum_{k=1}^{n} \widehat{a}_k^2 \leqslant C \sum_{k=1}^{n} \frac{1}{k^{2p}},$$

for all $n \in \mathbb{N}$. Since $p > 1/2$, the latter series is summable and we conclude that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|\hat{M}_n|^2) < \infty.$$

By Doob's martingale convergence theorem there exists a non-degenerate random variable $\hat{W} \in L^2(\mathbb{P})$ such that $\hat{M}_n \to \hat{W}$ a.s. and in $L^2(\mathbb{P})$ as $n \to \infty$. Using the asymptotics $\widehat{a}_n \sim n^{-p}$ we conclude the proof. □

We now focus our attention on establishing the almost sure convergence of Theorem 3.1. We shall show in Corollary 3.11 below that both convergences also hold in $L^2(\mathbb{P})$. However, additional estimates are still needed to deduce the $L^2(\mathbb{P})$ convergence.

Proof of the a.s. convergences in Theorem 3.1. Let us start with the NRRW and in this direction recall that $E[\hat{X}_n] = m$ for all $n \geqslant 1$. First, since $\hat{S}(n) - n\mathbb{E}[X]$, for $n \geqslant 1$ is a NRRW with same parameter p and centered steps with law $X - \mathbb{E}[X]$, it would suffice to show that for centered X, we have $n^{-1}\hat{S}(n) = 0$. Considering first the case $p \in (0, 1/2]$, this can now be achieved by making use of Theorem 1.3.17 in [42] and the martingale $\hat{M}$ that we introduced in Proposition 3.7. More precisely, remark that for any $p \in (0, 1/2]$ and $\alpha > 0$, we have

$$n^{-\alpha}\mathbb{E}[\langle M\rangle_n] \leqslant n^{-\alpha}\sum_{k=1}^{n} \hat{a}_k^2,$$

where the asymptotic behaviour of $\sum_{k=1}^{n} \hat{a}_k^2$ as $n \uparrow \infty$ is dictated for $p \in (0, 1/2)$ and $p = 1/2$ respectively by (3.24) and (3.25). Now, it readily follows from these estimations that if $p \in (0, 1/2)$, for $\alpha := 2p - 1$ we have

$$\sup_n n^{-(1-2p)}\mathbb{E}(\langle \hat{M}_n\rangle) \leqslant \sigma^2 \sup_n n^{2p-1}\sum_{k=2}^{n} \hat{a}_k^2 < \infty \tag{3.28}$$

while if $p = 1/2$, for $\alpha := \varepsilon$ for any $\varepsilon > 0$ it holds that

$$\sup_n n^{-\varepsilon}\mathbb{E}(\langle \hat{M}_n\rangle) \leqslant \sigma^2 \sup_n n^{-\varepsilon}\sum_{k=2}^{n} \hat{a}_k^2 < \infty.$$

We deduce from Theorem 1.3.17 in [42], taking $\beta := \alpha$, that $n^{-(1-2p)}\hat{M}_n \to 0$ and $n^{-\varepsilon}\hat{M}_n \to 0$, the convergences holding a.s. Recalling that $a_n \sim n^{-p}$, we get from the definition of $\hat{M}$ that $n^{-1}\hat{S}_n \to 0$ almost surely. The case $p \in (1/2, 1)$ now easily follows from the convergence of Proposition 3.8.

The counterbalanced case will follow from Theorem 1.3.24 in [42]. In this direction, fix $p \in (0, 1]$, recall that the process (Y_n) defined in (3.27) is a martingale, and we claim that:

$$n^{-(1+p)}Y_n \to 0 \quad a.s. \tag{3.29}$$

Let us first explain why this yields the desired result. Recalling from (3.4) that $\mathbb{E}(\check{S}_n) \sim n(1-p)m/(1+p)$ and $\check{a}_n \sim n^p$ as $n \uparrow \infty$, it follows that

$$\lim_{n\to\infty} n^{-(1+p)}\check{a}_n\mathbb{E}(\check{S}_n) = \frac{(1-p)m}{1+p}, \qquad \text{which yields} \qquad \lim_{n\to\infty} n^{-(1+p)}\check{a}_n\check{S}_n = \frac{(1-p)m}{1+p} \quad a.s.$$

by definition of (Y_n). The second convergence in (3.7) now follows.

We now shall prove (3.29) for $p \in (0, 1]$. Let us recall from Theorem 1.3.17 in [42] that, if for any $0 < \alpha/2 < \beta$, we have

$$\sup_n n^{-\alpha}\mathbb{E}(\langle Y_n\rangle) < \infty, \tag{3.30}$$

then $n^{-\beta}Y_n \to 0$ a.s. In order to make use of this result, we start with some estimates for the angle bracket process $(\langle Y_n\rangle)$. In this direction, note that

$$\begin{aligned} Y_{n+1} - Y_n &= \check{a}_{n+1}\check{S}_{n+1} - \check{a}_n\check{S}_n - \check{a}_{n+1}\mathbb{E}(\check{S}_{n+1}) + \check{a}_n\mathbb{E}(\check{S}_n) \\ &= \check{a}_{n+1}(\check{S}_{n+1} - \gamma_n\check{S}_n) - \check{a}_{n+1}(\mathbb{E}(\check{S}_{n+1}) - \gamma_n\mathbb{E}(\check{S}_n)) \\ &= \check{a}_{n+1}\left(\check{X}_{n+1} + \frac{p}{n}\check{S}_n - \mathbb{E}\left(\check{X}_{n+1} + \frac{p}{n}\check{S}_n\right)\right). \end{aligned}$$

Therefore, recalling the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, by Jensen's inequality we have for some constant c that

$$\mathbb{E}((Y_{n+1} - Y_n)^2) \leqslant c \cdot \check{a}_{n+1}^2\left(\mathbb{E}(\check{X}_{n+1}^2) + \frac{1}{n^2}\mathbb{E}[(\check{S}_n)^2]\right) = c \cdot \check{a}_{n+1}^2\left(\sigma^2 + \frac{1}{n^2}\mathbb{E}[(\check{S}_n)^2]\right),$$

where in the last equality we used that $\check{X}_{n+1}^2 = \hat{X}_{n+1}^2$, with $\mathbb{E}(\hat{X}_{n+1}^2) = \sigma^2$. Set $\alpha := 1 + 2p$ and note that from our previous estimates, we get:

$$n^{-(1+2p)}\mathbb{E}(\langle Y_n \rangle) \leqslant c \cdot n^{-(1+2p)} \sum_{k=1}^{n} \check{a}_{k+1}^2 \left(\sigma^2 + \frac{1}{k^2}\mathbb{E}[(\check{S}_k)^2] \right).$$

Recalling the asymptotic behaviour (3.26) of the series $\sum_{k=1}^n a_k^2$, the estimate (3.30) will follow if we prove that $\sup_n \frac{1}{n^2}\mathbb{E}[(\check{S}_n)^2] < \infty$. In this direction, note that

$$n^{-2}(\check{S}_n)^2 = n^{-2} \left(\sum_{k=1}^{n} \check{X}_k \right)^2 \leqslant n^{-2} n \sum_{k=1}^{n} \check{X}_k^2 = n^{-1}\hat{V}_n.$$

By taking expectations on the last display, we infer the uniform bound:

$$n^{-2}\mathbb{E}[(\check{S}_n)^2] \leqslant n^{-1}\mathbb{E}[\hat{V}_n] = \sigma^2$$

and we deduce that (3.30) holds. Finally, since $(1 + 2p)/2 \leqslant 1 + p$, we can take $\beta = 1 + p$ to conclude that $n^{-(1+p)}Y_n \to 0$ almost surely.

□

When the typical step is centred, we can derive from our previous arguments sharper results:

Corollary 3.9. *Suppose that $p \in (0, 1/2)$ and that $E[X] = 0$. We have the almost sure convergences:*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n^{1-p}} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\check{S}_n}{n^{1-p}} = 0.$$

Proof. The first convergence has already been established during the proof of Theorem 3.1. Indeed, it is a consequence of the convergence $n^{-(1-2p)}\hat{M}_n \to 0$ and the asymptotic behaviour $a_n \sim n^{-p}$. For the counterbalanced case, we can proceed similarely, noticing that by (3.26), we now have:

$$\sup_n n^{-(1+2p)}\mathbb{E}(\langle \check{M}_n \rangle) \leqslant \sup_n \sigma^2 n^{-2p-1} \sum_{k=2}^{n} \check{a}_k^2 < \infty.$$

Let now $\alpha = 1 + 2p$ and $\beta = 1$. Then, $\alpha < 2\beta$ if and only if $p < 1/2$ and the conditions of Theorem 1.3.17 in [42] are again satisfied. It follows, as before, that $n^{-1}\check{M}_n = 0 \to 0$ a.s. and since $\check{a}_n \sim n^p$ as $n \to \infty$, the claim follows. □

We continue by investigating bounds for the second moments of the supremum process of the step-reinforced random walk $\hat{S}$ for all regimes. These bound will be needed for establishing the $L_2(\mathbb{P})$ convergences of Theorem 3.1.

Lemma 3.10. *For every $n \geqslant 1$, the following bounds hold for some numerical constant c:*

$$\sigma^{-2}\mathbb{E}\left(\sup_{k\leqslant n} |\hat{S}_k|^2 \right) \leqslant \begin{cases} cn, & \textit{if } p \in (0, 1/2) \\ cn\log n, & \textit{if } p = 1/2 \\ cn^{2p}, & \textit{if } p \in (1/2, 1). \end{cases}$$

Proof. We tackle each of the three cases $p \in (0, 1/2)$, $p = 1/2$ and $p \in (1/2, 1)$ individually:

(i) Let us first consider the case when $p \in (0, 1/2)$. We observe that by (3.20) and by (3.24)

$$\mathbb{E}(\hat{M}_n^2) = \mathbb{E}(\langle \hat{M} \rangle_n) \leqslant \sum_{k=1}^{n} \sigma^2 \widehat{a}_k^2 \sim \sigma^2 \frac{1}{1-2p} n^{1-2p}, \quad \text{as } n \to \infty.$$

Hence we obtain by Doob's inequality that

$$\mathbb{E}\left(\sup_{k \leqslant n} |\hat{M}_k|^2\right) \leqslant c_1 \sigma^2 n^{1-2p}$$

where $c_1 > 0$ is some constant. Since it evidently holds that

$$\mathbb{E}\left(\sup_{k \leqslant n} |\hat{S}_k|^2\right) \leqslant \frac{1}{\widehat{a}_n^2} \mathbb{E}\left(\sup_{k \leqslant n} |\hat{M}_k|^2\right),$$

it follows readily that

$$\mathbb{E}\left(\sup_{k \leqslant n} |\hat{S}_k|^2\right) \leqslant c_1 \sigma^2 \frac{n^{1-2p}}{\widehat{a}_n^2} \sim c_1 \sigma^2 n, \quad \text{as } n \to \infty.$$

By monotonicity, we conclude the proof for this case.

(ii) Let us now assume that $p = 1/2$, we then obtain by (3.25) and monotonicity that for all $n \geqslant 1$ we have

$$\mathbb{E}(\langle \hat{M} \rangle_n) \leqslant \sigma^2 \log n.$$

We conclude as in the previous case that this implies

$$\mathbb{E}\left(\sup_{k \leqslant n} |\hat{S}_k|^2\right) \leqslant c_2 \sigma^2 n \log n,$$

where $c_2 > 0$ is some constant.

(iii) Finally, let us consider the case $p > 1/2$. Here, we then have as $n \to \infty$

$$\sigma^2 \sum_{k=1}^{n} \widehat{a}_{k+1}^2 \leqslant C \sigma^2 \sum_{k=1}^{n} \frac{1}{k^{2p}} < \tilde{c}$$

for a constant C large enough and some finite constant $\tilde{c}$. This entails that $\mathbb{E}(\langle \hat{M} \rangle_n) \leqslant \sigma^2 \tilde{c}$ and we deduce as before the bound

$$\mathbb{E}\left(\sup_{k \leqslant n} |\hat{S}_k|^2\right) \leqslant c_3 \sigma^2 n^{2p},$$

where $c_3 > 0$ is some constant.

Thus we have established the desired bounds for all regimes. □

As an application of the maximal inequalities displayed in Lemma 3.10 for the noise reinforced random walk, we establish $L^2(\mathbb{P})$ convergence type results for all regimes $p \in (0, 1)$ and we deduce that the LLN stated in Theorem 3.1 also hold in $L^2(\mathbb{P})$.

Corollary 3.11. *We have the following convergences in the $L^2(\mathbb{P})$-sense.*

(i) *For $p \in (0, 1/2)$ we have*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n^{1-p}} = 0.$$

(ii) *For $p = 1/2$ we have*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{\sqrt{n}\log n} = 0.$$

(iii) *For $p \in (1/2, 1)$ we have*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n} = 0.$$

In particular, all the convergences in Theorem 3.1 also hold in $L^2(\mathbb{P})$.

Proof. Let $(f(n))$ be a sequence of positive numbers and notice that by Lemma 3.10, if as $n \uparrow \infty$

$$\begin{cases} \frac{1}{f^2(n)} n \to 0, & \text{if } p \in (0, 1/2) \\ \frac{1}{f^2(n)} (n \log n) \to 0, & \text{if } p = 1/2 \\ \frac{1}{f^2(n)} n^{2p} \to 0, & \text{if } p \in (1/2, 1) \end{cases},$$

then we have convergence in the L^2-sense to 0 of the sequence $(\hat{S}_n / f(n))$. Now, respectively for each one of the tree cases:

(i) We take $f(n) := n^{1-p}$ and observe that $n^{2p-1} \to 0$ as $n \to \infty$ since $p \in (0, 1/2)$.

(ii) We take $f(n) := \sqrt{n}\log n$, plainly $1/\log(n) \to 0$ as $n \to \infty$.

(iii) We take $f(n) := n$ and observe that $n^{2(p-1)} \to 0$ as $n \to \infty$ because $p < 1$.

This concludes the first part of the proof. Next, notice that (i), (ii), (iii) imply that for any $p \in (0, 1)$, we have $n^{-1}\hat{S}_n \to m$ in $L^2(\mathbb{P})$. Indeed, it suffices to notice once again that $\hat{S}_n - n\mathbb{E}[X]$ for $n \geqslant 1$ is a centered noise reinforced random walk. To deduce the convergence in the counterbalanced case, fix $p \in (0, 1)$ and remark that we can bound,

$$|n^{-1}\hat{S}_n| \leqslant n^{-1} \sum_{i=1}^{n} |\check{X}_i|,$$

where now, $\sum_{i=1}^{n} |\check{X}_i|$ for $n \geqslant 1$ is a noise reinforced random walk with typical step distributed $|X|$. In particular, it follows from the first part of the proof that $n^{-1} \sum_{i=1}^{n} |\check{X}_i| \to \mathbb{E}[|X|]$ in $L^2(\mathbb{P})$. Now, the desired convergence follows by the (generalized) dominated convergence theorem. □

This concludes the proof of Theorem 3.1 and we shall now turn our attention to the proof of the stated invariance principles.

3.3 Proof of Theorem 3.2, 3.3 and 3.4 when X is bounded.

Recall that in this section and Section 3.4 we work under the additional assumption that X is centred. As was discussed in the introduction, for positive step-reinforced random walks the centredness hypothesis can be assumed without loss of generality, but that is no longer the case for negative step-reinforced random walks. We are now in a position to prove Theorem 3.2 when X is bounded and in the process we will also establish Theorem 3.3 and Theorem 3.4. For that reason, in several statements we also consider $p \in [1/2, 1]$ when working with the counterbalanced random walk. Additionally, when we work with the counterbalanced random walk for $p = 1$, we assume as in Theorem 3.4 that X is Rademacher distributed, this will be recalled when necessary. Our approach relies on using the martingale introduced in Proposition 3.7 and applying the MFCLT 3.6. We will establish the general case for $X \in L^2(\mathbb{P})$ by a truncation argument, detailed in Section 3.4.

Now, the key is to notice that, since by (3.24) resp. (3.26) we have for any $t \geqslant 0$

$$\frac{\widehat{a}_{\lfloor nt \rfloor}}{n^{-p}} \sim t^{-p} \qquad \text{and} \qquad \frac{\check{a}_{\lfloor nt \rfloor}}{n^{p}} \sim t^{p} \qquad \text{as } n \uparrow \infty,$$

in order to get the convergence (3.8) it is enough to prove (except for a technical detail at the origin in the third coordinate that will be properly addressed), the convergence

$$\left(\frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}, \frac{1}{\sqrt{n}} \frac{\widehat{a}_{\lfloor nt \rfloor}}{n^{-p}} \hat{S}_{\lfloor nt \rfloor}, \frac{1}{\sqrt{n}} \frac{\check{a}_{\lfloor nt \rfloor}}{n^{p}} \check{S}_{\lfloor nt \rfloor} \right)_{t \in \mathbb{R}^+} \Longrightarrow \left(\sigma B_t, \sigma \int_0^t s^{-p} dB_s^r, \sigma \int_0^t s^p dB_s^c \right)_{t \in \mathbb{R}^+} \tag{3.31}$$

for Brownian motions B and B^r and B^c defined as in (3.9) where the sequence on the left-hand side is now composed by martingales. More precisely, for each $n \in \mathbb{N}$, the processes

$$\left(\hat{N}_t^{(n)} \right)_{t \in \mathbb{R}^+} := \left(\frac{1}{\sqrt{n}} \frac{\widehat{a}_{\lfloor nt \rfloor}}{n^{-p}} \hat{S}_{\lfloor nt \rfloor} \right)_{t \in \mathbb{R}^+}, \qquad \left(\check{N}_t^{(n)} \right)_{t \in \mathbb{R}^+} := \left(\frac{1}{\sqrt{n}} \frac{\check{a}_{\lfloor nt \rfloor}}{n^{p}} \check{S}_{\lfloor nt \rfloor} \right)_{t \in \mathbb{R}^+} \tag{3.32}$$

are just rescaled, continuous-time versions of the martingales we introduced in Proposition 3.7, multiplied by respective factors of $n^{p-1/2}$ and n^{-1-p}. We will also denote as $N^{(n)}$ the scaled random walk in the first coordinate and we proceed at establishing (3.31) by verifying that the conditions of the MFCLT 3.6 are satisfied. In that direction and recalling the condition (3.13), we start by investigating the asymptotic negligibility of the jumps:

Lemma 3.12 (Asymptotic negligibility of jumps)**.**

(i) *Fix $p \in (0, 1/2)$. For each $t > 0$, the following convergence holds almost surely:*

$$\sup_{s \leqslant t} |\Delta \hat{N}_s^{(n)}| \to 0 \qquad \textit{as } n \uparrow \infty.$$

(ii) *Fix $p \in (0, 1]$. For each $t > 0$, the following convergence holds almost surely:*

$$\sup_{s \leqslant t} |\Delta \check{N}_s^{(n)}| \to 0 \qquad \textit{as } n \uparrow \infty.$$

Proof. (i) Notice that

$$\begin{aligned}
\sup_{s\leqslant t}|\Delta\hat{N}^{(n)}_s| &\leqslant \frac{1}{\sqrt{n}}\sup_{k\leqslant\lfloor nt\rfloor}\left|n^p\widehat{a}_{k+1}\hat{S}_{k+1}-n^p\widehat{a}_k\hat{S}_k\right| \\
&= \frac{1}{n^{1/2-p}}\sup_{k\leqslant\lfloor nt\rfloor}\left|\widehat{a}_{k+1}\left(\hat{S}_{k+1}-\widehat{\gamma}_k\hat{S}_k\right)\right| \\
&= \frac{1}{n^{1/2-p}}\sup_{k\leqslant\lfloor nt\rfloor}\widehat{a}_{k+1}\left|\sum_{j=1}^{k}\hat{X}_j(1-\widehat{\gamma}_k)+\hat{X}_{k+1}\right| \\
&\leqslant \frac{\|X\|_\infty}{n^{1/2-p}}\sup_{k\leqslant\lfloor nt\rfloor}\widehat{a}_{k+1}(k|1-\widehat{\gamma}_k|+1),
\end{aligned}$$

where by hypothesis we have $\|X\|_\infty<\infty$. Now, since $\widehat{a}_k\sim k^{-p}$, we have $\sup_k\widehat{a}_k<\infty$ and we deduce recalling the definition $\widehat{\gamma}_k=(k+p)/k$ that:

$$\sup_{s\leqslant t}|\Delta\hat{N}^{(n)}_s|\leqslant\frac{C\|X\|_\infty}{n^{1/2-p}},$$

for some constant $C>0$ and (i) follows.
(ii) Similarly, since we also have $\Delta\check{M}_{k+1}=\check{a}_{k+1}(\check{S}_{k+1}-\check{\gamma}_k\check{S}_k)$, arguing as before we get:

$$\begin{aligned}
\sup_{s\leqslant t}|\Delta\check{N}^{(n)}_s| &= \frac{1}{n^{1/2+p}}\sup_{k\leqslant\lfloor nt\rfloor}\left|\check{a}_{k+1}\left(\check{S}_{k+1}-\check{\gamma}_k\check{S}_k\right)\right| \\
&= \frac{1}{n^{1/2+p}}\sup_{k\leqslant\lfloor nt\rfloor}\check{a}_{k+1}\left|\sum_{j=1}^{k}\check{X}_j(1-\check{\gamma}_k)+\check{X}_{k+1}\right| \\
&\leqslant \frac{\|X\|_\infty}{n^{1/2+p}}\sup_{k\leqslant\lfloor nt\rfloor}\check{a}_{k+1}(k|1-\check{\gamma}_k|+1) \\
&= \frac{\|X\|_\infty}{n^{1/2+p}}\sup_{k\leqslant\lfloor nt\rfloor}\check{a}_{k+1}(p+1),
\end{aligned}$$

since $\check{\gamma}_n=(n-p)/n$. Recalling from (3.26) the asymptotic behaviour $\check{a}_n\sim n^p$, we get that $\sup_{s\leqslant t}|\Delta\check{N}^{(n)}_s|\to 0$ pointwise for each t. □

Now we turn our attention to the joint convergence of the quadratic variation process, and this is the content of the following lemma:

Lemma 3.13 (Convergence of quadratic variations). *For each fixed $t\in\mathbb{R}^+$, the following convergences hold almost surely for $p\in(0,1/2)$, unless specified otherwise:*

(i) $\lim_{n\to\infty}\langle\hat{N}^{(n)},\hat{N}^{(n)}\rangle_t=\sigma^2\int_0^t s^{-2p}\mathrm{d}s.$

(ii) $\lim_{n\to\infty}\langle\check{N}^{(n)},\check{N}^{(n)}\rangle_t=\sigma^2\int_0^t s^{2p}\mathrm{d}s,$ *for $p\in(0,1]$.*

(iii) $\lim_{n\to\infty}\langle\hat{N}^{(n)},N^{(n)}\rangle_t=\sigma^2(1-p)\int_0^t s^{-p}\mathrm{d}s.$

(iv) $\lim_{n\to\infty} \langle \check{N}^{(n)}, N^{(n)} \rangle_t = \sigma^2(1-p)\int_0^t s^p \mathrm{d}s, \qquad$ *for* $p \in (0,1]$.

(v) $\lim_{n\to\infty} \langle \hat{N}^{(n)}, \check{N}^{(n)} \rangle_t = t\sigma^2 \dfrac{1-p}{1+p}$.

where for the case $p = 1$ *in* (ii) *and* (iv) *we assume that* X *is distributed Rademacher.*

Lemma 3.13 provides the key asymptotic behaviour for the sequence of quadratic variations and its proof is rather long.

Proof. We tackle each item (i)–(v) individually, item (v) being the most arduous.

(i) For each $n \in \mathbb{N}$, we gather from (3.16) that the predictable quadratic variation of this martingale is given by for $t \geqslant 1/2$ by

$$\langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t = \frac{1}{n^{1-2p}}\left(\sigma^2 + (1-p)\sigma^2 \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k^2 - p^2 \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k^2 \left(\frac{\hat{S}_{k-1}}{k-1}\right)^2 + p \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k^2 \left(\frac{\hat{V}_{k-1}}{k-1}\right)\right),$$

with $\langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t = 0$ if $t < 1/n$. We will study separately the limit as $n \to \infty$ of the three nontrivial terms, as the first one evidently vanishes. To start with, it follows readily from (3.24) that

$$\lim_{n\to\infty} \frac{\sigma^2}{n^{1-2p}}(1-p) \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k^2 = \frac{\sigma^2}{1-2p} t^{1-2p}(1-p). \tag{3.33}$$

Now, we claim that the second term converges to zero:

$$\lim_{n\to\infty} \frac{1}{n^{1-2p}} p^2 \sum_{k=1}^{\lfloor nt \rfloor} \widehat{a}_k^2 \left(\frac{\hat{S}_{k-1}}{k-1}\right)^2 = 0 \quad \text{a.s.} \tag{3.34}$$

Indeed, by (3.24) it suffices to notice that by Proposition 3.1, we have

$$\lim_{k\to\infty} \frac{\hat{S}_k}{k} = 0 \quad \text{a.s.}$$

since we recall that by our standing assumptions X is centered. Finally, we claim that for the last term, the following limit holds:

$$\lim_{n\to\infty} \frac{1}{n^{1-2p}} p \sum_{k=1}^{\lfloor nt \rfloor} \widehat{a}_k^2 \frac{\hat{V}_{k-1}}{(k-1)} = \frac{\sigma^2}{1-2p} t^{1-2p} p \quad \text{a.s.} \tag{3.35}$$

In this direction, notice that $(\hat{V}_n)_{n\in\mathbb{N}}$ is the reinforced version of the (non-centered) random walk

$$V_n = X_1^2 + \cdots + X_n^2, \qquad n \in \mathbb{N}$$

with mean $\mathbb{E}(\hat{X}_i^2) = \mathbb{E}(X_i^2) = \sigma^2$. Hence, by Theorem 3.1, we have $n^{-1}\hat{V}_n \to \sigma^2$ as $n \uparrow \infty$ and (3.35) follows. Now, combining (3.33), (3.34) and (3.35) we conclude that

$$\lim_{n\to\infty} \langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t = \frac{\sigma^2}{1-2p} t^{1-2p} \quad \text{a.s.}$$

(ii) By (3.17), we have

$$\langle \check{N}^{(n)}, \check{N}^{(n)} \rangle_t = \frac{1}{n^{1+2p}} \left(\sigma^2 + \sum_{k=2}^{\lfloor nt \rfloor} \check{a}_k^2 \left((1-p)\sigma^2 - p^2 \left(\frac{\check{S}_{k-1}}{k-1} \right)^2 + p \frac{\hat{V}_{k-1}}{k-1} \right) \right)$$

and we shall now study the convergence of the normalised series in the previous expression. First, by (3.26), the first term converges towards

$$\lim_{n\to\infty} \sigma^2 \frac{(1-p)}{n^{1+2p}} \sum_{k=2}^{\lfloor nt \rfloor} \check{a}_k^2 = \sigma^2 \frac{1-p}{1+2p} t^{1+2p} = \sigma^2 (1-p) \int_0^t s^{2p} \mathrm{d}s.$$

Turning our attention to the second term, we recall from Proposition 3.1 that $(\check{S}_n)$ satisfies a law of large numbers:

$$\lim_{n\to\infty} \frac{1}{n} \check{S}_n = \frac{(1-p)}{1+p} m = 0 \qquad \text{a.s.}$$

This paired with the asymptotic behaviour of the series (3.26) yields:

$$\lim_{n\to\infty} \frac{1}{n^{1+2p}} \sum_{k=2}^{\lfloor nt \rfloor} \check{a}_k^2 \left(\frac{\check{S}_{k-1}}{k-1} \right)^2 = 0 \qquad \text{a.s. for every } t \geqslant 0.$$

Finally, assuming first that $p < 1$, we can proceed as in (3.35) to deduce from (3.26) that

$$\lim_{n\to\infty} \frac{p}{n^{1+2p}} \sum_{k=2}^{\lfloor nt \rfloor} \check{a}_k^2 \left(\frac{\hat{V}_{k-1}}{k-1} \right) = \sigma^2 \frac{p}{1+2p} t^{1+2p} = \sigma^2 p \int_0^t s^{2p} \mathrm{d}s \quad \text{a.s.} \tag{3.36}$$

If $p = 1$, by hypothesis X takes its values in $\{-1, 1\}$ and $\check{V}_{k-1} = k-1$, yielding that the previously established limit (3.36) still holds. Notice however that if we allowed X to take arbitrary values, we can no longer proceed as we just did since in that case, $\hat{V}_n$ is a straight line with random slope:

$$\check{V}_n = n\check{X}_1^2.$$

Putting all pieces together, we obtain (ii).

(iii) Recalling that $\hat{X}_k = X_k \mathbf{1}_{\{\varepsilon_k = 0\}} + \hat{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k = 1\}}$, and from independence of X_k, ε_k and $U[k-1]$ from $\mathcal{F}_{k-1}$, we get for $k \geqslant 2$

$$\begin{aligned}
\mathbb{E}(\Delta \hat{M}_k X_k \mid \mathcal{F}_{k-1}) &= \mathbb{E}\left((\hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1}) + \hat{X}_k \widehat{a}_k) X_k \mid \mathcal{F}_{k-1} \right) \\
&= \widehat{a}_k \mathbb{E}\left(\left(X_k \mathbf{1}_{\{\varepsilon_k = 0\}} + \hat{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k = 1\}} \right) X_k \mid \mathcal{F}_{k-1} \right) \\
&= \widehat{a}_k (1-p) \mathbb{E}(X^2) + \sum_{j=1}^{k-1} \mathbb{E}\left(X_k X_j \mathbf{1}_{\{U[k-1]=j, \varepsilon_k = 1\}} \mid \mathcal{F}_{k-1} \right) \\
&= \widehat{a}_k (1-p) \sigma^2
\end{aligned}$$

since the steps are centered, while for $k = 1$ we simply get $\mathbb{E}(\hat{M}_1 X_1) = \sigma^2$. From here, we deduce

$$\langle \hat{N}^{(n)}, N^{(n)} \rangle_t = n^{p-1} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\Delta M_k X_k \mid \mathcal{F}_{k-1}) = \sigma^2 (1-p) n^{p-1} \left((1-p)^{-1} + \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k \right)$$

and from the convergence

$$\lim_{n\to\infty} n^{p-1} \sum_{k=2}^{n} \widehat{a}_k = (1-p)^{-1}$$

we conclude:

$$\lim_{n\to\infty} \langle \hat{N}^{(n)}, N^{(n)} \rangle_t = \sigma^2 (1-p) \lim_{n\to\infty} n^{p-1} \sum_{k=2}^{\lfloor nt \rfloor} \widehat{a}_k = t^{1-p} = \sigma^2 (1-p) \int_0^t s^{-p} \mathrm{d}s.$$

(iv) Recalling that in the counterbalanced case $\check{X}_k = X_k \mathbf{1}_{\{\varepsilon_k = 1\}} - \check{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k = 0\}}$, we deduce from similar arguments as in the reinforced case that for $k \geqslant 2$ we have,

$$\begin{aligned}
\mathbb{E}(\Delta \check{M}_k X_k \mid \mathcal{F}_{k-1}) &= \mathbb{E}\left((\check{S}_{k-1}(\check{a}_k - \check{a}_{k-1}) + \check{X}_k \check{a}_k) X_k \mid \mathcal{F}_{k-1} \right) \\
&= \check{a}_k \mathbb{E}\left(\left(X_k \mathbf{1}_{\{\varepsilon_k = 1\}} - \check{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k = 0\}} \right) X_k \mid \mathcal{F}_{k-1} \right) \\
&= \check{a}_k \cdot (1-p) \mathbb{E}(X^2) - \sum_{j=1}^{k-1} \mathbb{E}\left(X_k X_j \mathbf{1}_{\{U[k-1] = j, \varepsilon_k = 0\}} \mid \mathcal{F}_{k-1} \right) \\
&= \check{a}_k \cdot (1-p) \sigma^2.
\end{aligned}$$

Notice that if $p = 1$ the argument still holds and hence the above quantity is null for all $k \geqslant 2$. Since if $k = 1$ we simply have $\mathbb{E}[\Delta \check{M}_1 X_1] = \sigma^2$, it follows that for $t \geqslant 1/n$,

$$\langle \check{N}^{(n)}, N^{(n)} \rangle_t = n^{-(1+p)} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\Delta \check{M}_k X_k \mid \mathcal{F}_{k-1}) = \sigma^2 (1-p) \cdot n^{-(1+p)} \left((1-p)^{-1} + \sum_{k=2}^{\lfloor nt \rfloor} \check{a}_k \right)$$

and from the convergence

$$\lim_{n\to\infty} n^{-(1+p)} \sum_{k=1}^{n} \check{a}_k = (1+p)^{-1}$$

we conclude

$$\sigma^{-2} \lim_{n\to\infty} \langle \check{N}^{(n)}, N^{(n)} \rangle_t = (1-p) \lim_{n\to\infty} n^{-(1+p)} \sum_{k=1}^{\lfloor nt \rfloor} \check{a}_k = \frac{1-p}{(1+p)} t^{1+p} = (1-p) \int_0^t s^p \mathrm{d}s.$$

Finally if $p = 1$, we clearly have $\lim_{n\to\infty} \langle \check{N}^{(n)}, N^{(n)} \rangle_t = 0$.

(v) Notice that

$$\begin{aligned}
\mathbb{E}(\Delta \check{M}_k \Delta \hat{M}_k \mid \mathcal{F}_{k-1}) &= \mathbb{E}\left((\hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1}) + \hat{X}_k \widehat{a}_k)(\check{S}_{k-1}(\check{a}_k - \check{a}_{k-1}) + \check{X}_k \check{a}_k) \mid \mathcal{F}_{k-1} \right) \\
&= \hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1}) \check{S}_{k-1}(\check{a}_k - \check{a}_{k-1}) + \hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1}) \mathbb{E}(\check{X}_k | \mathcal{F}_{k-1}) \check{a}_k \\
&\quad + \check{S}_{k-1}(\check{a}_k - \check{a}_{k-1}) \mathbb{E}(\hat{X}_k \mid \mathcal{F}_{k-1}) \widehat{a}_k + \mathbb{E}(\check{X}_k \hat{X}_k \mid \mathcal{F}_{k-1}) \widehat{a}_k \check{a}_k \\
&=: P_k^{(a)} + P_k^{(b)} + P_k^{(c)} + P_k^{(d)}
\end{aligned}$$

where the notation was assigned in order of appearance. We write,

$$\langle \check{N}^n, \hat{N}^n \rangle_t = n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \left(P_k^{(a)} + P_k^{(b)} + P_k^{(c)} + P_k^{(d)} \right)$$

and study the asymptotic behaviour of these four terms individually. In that direction, we recall from (3.18) and (3.22) the identities $\mathbb{E}(\hat{X}_k \mid \mathcal{F}_{k-1}) = p\hat{S}_{k-1}/(k-1)$, $\mathbb{E}(\check{X}_k \mid \mathcal{F}_{k-1}) = -p\check{S}_{k-1}/(k-1)$ as well as the asymptotic behaviour $(\widehat{a}_k - \widehat{a}_{k-1}) \sim -pk^{-(p+1)}$ and $(\check{a}_k - \check{a}_{k-1}) \sim pk^{p-1}$.

- We first show that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} P_k^{(c)} = 0 \quad \text{a.s..}$$

From the identities and asymptotic estimates we just recalled, we have

$$\check{S}_{k-1}(\check{a}_k - \check{a}_{k-1})\mathbb{E}(\hat{X}_k \mid \mathcal{F}_{k-1})\widehat{a}_k = \check{S}_{k-1}(\check{a}_k - \check{a}_{k-1})p\frac{\hat{S}_{k-1}}{k-1}\widehat{a}_k \sim \frac{\check{S}_{k-1}}{k}k^p p^2 \frac{\hat{S}_{k-1}}{k-1}\widehat{a}_k$$

and since $\widehat{a}_k \sim k^{-p}$, we have for some constant C large enough that

$$n^{-1}\left|\sum_{k=1}^{\lfloor nt \rfloor} P_k^{(c)}\right| \leqslant n^{-1} C \sum_{k=1}^{\lfloor nt \rfloor}\left|\frac{\check{S}_{k-1}}{k}\frac{\hat{S}_{k-1}}{k-1}\right|.$$

However, this converges a.s. towards 0 as $n \uparrow \infty$ by Proposition 3.1.

- Next, since

$$\hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1})\mathbb{E}(\check{X}_k \mid \mathcal{F}_{k-1})\check{a}_k = -\hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1})p\frac{\check{S}_{k-1}}{k-1}\check{a}_k \sim \frac{\hat{S}_{k-1}}{k}k^{-p}p^2\frac{\check{S}_{k-1}}{k-1}\check{a}_k$$

we can follow exactly the same line of reasoning in order to establish

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} P_k^{(b)} = 0 \quad \text{a.s..}$$

- Since

$$(\widehat{a}_k - \widehat{a}_{k-1})(\check{a}_k - \check{a}_{k-1}) \sim -p^2 k^{-2},$$

we deduce that

$$\hat{S}_{k-1}(\widehat{a}_k - \widehat{a}_{k-1})\check{S}_{k-1}(\check{a}_k - \check{a}_{k-1}) \sim \hat{S}_{k-1}\check{S}_{k-1}(-p^2)k^{-2}$$

and we conclude as before that we have:

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} P_k^{(a)} = 0 \quad \text{a.s..}$$

- Finally, since by definition

$$\hat{X}_k = X_k \mathbf{1}_{\{\varepsilon_k=0\}} + \hat{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k=1\}}, \qquad \check{X}_k = X_k \mathbf{1}_{\{\varepsilon_k=0\}} - \check{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k=1\}}$$

we have

$$\begin{aligned}\hat{a}_k \check{a}_k E(\check{X}_k \hat{X}_k \mid \mathcal{F}_{k-1}) &= \hat{a}_k \check{a}_k E(X_k^2 \mathbf{1}_{\{\varepsilon_k=0\}} \mid \mathcal{F}_{k-1}) - \hat{a}_k \check{a}_k E(\check{X}_{U[k-1]} \hat{X}_{U[k-1]} \mathbf{1}_{\{\varepsilon_k=1\}} \mid \mathcal{F}_{k-1}) \\ &= \hat{a}_k \check{a}_k (1-p)\sigma^2 - \hat{a}_k \check{a}_k \sum_{j=1}^{k-1} E(\check{X}_j \hat{X}_j \mathbf{1}_{\{\varepsilon_k=1, U[k-1]=j\}} \mid \mathcal{F}_{k-1}).\end{aligned}$$

Since on one hand, $\check{X}_j, \hat{X}_j$ for $j < k$ are $\mathcal{F}_{k-1}$ measurable while $\varepsilon_k, U[k-1]$ are independent of $\mathcal{F}_{k-1}$, denoting as $\check{G}$ the counterbalanced random walk made from the i.i.d. sequence $X_1^2, X_2^2, \ldots$ from the same instance of the reinforcement algorithm, we deduce

$$P_k^{(d)} = \hat{a}_k \check{a}_k \left((1-p)\sigma^2 - \frac{1}{k-1} p \sum_{j=1}^{k-1} \check{X}_j \hat{X}_j \right) = \hat{a}_k \check{a}_k \left((1-p)\sigma^2 - p \frac{\check{G}(k-1)}{k-1} \right)$$

and since $\hat{a}_k \check{a}_k \to 1$ as $k \to \infty$, the problem boils down to studying the convergence as $n \uparrow \infty$ of

$$\frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \left((1-p)\sigma^2 - p \frac{\check{G}(k-1)}{k-1} \right).$$

The first term obviously converges towards $t(1-p)\sigma^2$ and we turn our attention to the second one. Now, by Theorem 3.1 applied to $\check{G}$ we get:

$$\lim_{n\to\infty} \frac{p}{n} \sum_{k=1}^{\lfloor nt \rfloor} \frac{\check{G}(k-1)}{k} = p\sigma^2 t \frac{1-p}{1+p} \qquad \text{a.s.}$$

and we conclude that the following convergence holds almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} P_k^{(d)} = t(1-p)\sigma^2 - p\sigma^2 t \frac{1-p}{1+p}.$$

Bringing all our calculations above together we deduce the following almost sure convergence:

$$\lim_{n\to\infty} \langle \hat{N}^n, \check{N}^n \rangle_t = \sigma^2(1-p)t - p\sigma^2(1-p)(1+p)^{-1}t.$$

This conclues the proof of the lemma. □

With this, we conclude the proof of Theorem 3.2 when X is bounded with an appeal to Lemma 3.12, Lemma 3.13 and the MFCLT (Theorem 3.6).

3.4 Reduction to the case of bounded steps.

In this section, we shall only assume that the typical step $X \in L^2(\mathbb{P})$ of the step-reinforced random walk $\hat{S}$ is centred and no longer that it is bounded. We shall complete the proof of Theorem 3.2 by means of the truncation argument reminiscent to the one of Section 4.3 in [21].

3.4.1 Preliminaries

The reduction argument relies on the following lemma taken from [54], that we state for the reader's convenience:

Lemma 3.14 (Lemma 3.31 in Chapter VI of [54]).
Let (Z^n) be a sequence of d-dimensional rcll (càdlàg) processes and suppose that

$$\forall N > 0, \quad \forall \varepsilon > 0 \qquad \lim_{n\to\infty} \mathbb{P}\left(\sup_{s\leqslant N} |Z^n_s| > \varepsilon\right) = 0.$$

If (Y^n) is another sequence of d-dimensional rcll processes with $Y^n \Rightarrow Y$ in the sense of Skorokhod, then $Y^n + Z^n \Rightarrow Y$ in the sense of Skorokhod.

Finally, we will need the following lemma concerning convergence on metric spaces:

Lemma 3.15. *Let (E, d) be a metric space and consider $(a_n^{(m)} : m, n \in \mathbb{N})$ a family of sequences, with $a_n^{(m)} \in E$ for all $n, m \in \mathbb{N}$. Suppose further that the following conditions are satisfied:*

(i) *For each fixed m, $a_n^{(m)} \to a_\infty^{(m)}$ as $n \uparrow \infty$ for some element $a_\infty^{(m)} \in E$.*

(ii) *$a_\infty^{(m)} \to a_\infty^{(\infty)}$ as $m \uparrow \infty$, for some $a_\infty^{(\infty)} \in E$.*

Then, there exists a non-decreasing subsequence $(b(n))_n$ with $b(n) \to \infty$ as $n \uparrow \infty$, for which the following convergence holds:

$$a_n^{(b(n))} \to a_\infty^{(\infty)} \qquad \text{as } n \uparrow \infty.$$

Proof. Since the sequence $(a_\infty^{(m)})_m$ converges, we can find an increasing subsequence $m_1 \leqslant m_2 \leqslant \dots$ satisfying

$$d(a_\infty^{(m_k)}, a_\infty^{(m_{k+1})}) \leqslant 2^{-k} \qquad \text{for each } k \in \mathbb{N}.$$

Moreover, since for each fixed m_k the corresponding sequence $(a_n^{(m_k)})_n$ converges, there exists a strictly increasing sequence $(n_k)_k$ satisfying that, for each k,

$$d(a_i^{(m_k)}, a_\infty^{(m_k)}) \leqslant 2^{-k} \qquad \text{for all } i \geqslant n_k.$$

Now, we set for $n < n_1$, $b(n) := m_1$ and for $k \geqslant 1$, $b(n) := m_k$ if $n_k \leqslant n < n_{k+1}$ and we claim $(a_n^{b(n)})_n$ is the desired sequence. Indeed, it suffices to observe that for $n_k \leqslant n < n_{k+1}$,

$$d(a_n^{(b(n))}, a_\infty^{(\infty)}) = d(a_n^{(m_k)}, a_\infty) \leqslant d(a_n^{(m_k)}, a_\infty^{(m_k)}) + d(a_\infty^{(m_k)}, a_\infty) \leqslant 2^{-k} + \sum_{i=k}^{\infty} 2^{-i}.$$

□

3.4.2 Reduction argument

Recall that we are assuming that the typical step is centred. During the course of this section we will use that the truncated versions of the counterbalanced and noise reinforced random walks are still counterbalanced (resp. noise reinforced) random walks.

Indeed, notice that if $(\check{S}_n)$ and $(\hat{S}_n)$ have been built from the i.i.d. sequence $(X_n)_{n\geqslant 1}$ by means of the negative-reinforcement and positive-reinforcement algorithms described in the introduction, splitting each X_i for $i \in \mathbb{N}$ as

$$X_i = X_i^{\leqslant K} + X_i^{>K}$$

where respectively,

$$X_i^{\leqslant K} := X_i \mathbf{1}_{\{|X_i| \leqslant K\}} - \mathbb{E}\left(X_i \mathbf{1}_{\{|X_i| \leqslant K\}}\right)$$
$$X_i^{>K} := X_i \mathbf{1}_{\{|X_i| > K\}} - \mathbb{E}(X_i \mathbf{1}_{\{|X_i| > K\}}),$$

yields a natural decompositions for $(\check{S}_n)$ and $(\hat{S}_n)$ in terms of two counterbalanced (reps. noise reinforced) random walks:

$$\check{S}_n = \check{S}_n^{\leqslant K} + \check{S}_n^{>K}, \qquad \hat{S}_n = \hat{S}_n^{\leqslant K} + \hat{S}_n^{>K}$$

where now $(\check{S}_n^{\leqslant K})$, $(\check{S}_n^{>K})$ are counterbalanced versions with typical step centred and distributed respectively as

$$X^{\leqslant K} := X \mathbf{1}_{\{|X| \leqslant K\}} - \mathbb{E}\left(X \mathbf{1}_{\{|X| \leqslant K\}}\right) \tag{3.37}$$

and

$$X^{>K} := X \mathbf{1}_{\{|X| > K\}} - \mathbb{E}(X \mathbf{1}_{\{|X| > K\}}), \tag{3.38}$$

an analogue statement holding in the reinforced case for $(\hat{S}_n^{\leqslant K})$, $(\hat{S}_n^{>K})$. Moreover, $X^{\leqslant K}$ is centred with variance σ_K^2 and $\sigma_K^2 \to \sigma^2$ as $K \nearrow \infty$ while the variance of $X^{>K}$ that we denote by η_K^2, converges towards zero as $K \uparrow \infty$. We will also write the respective truncated random walk as

$$S_n^{\leqslant K} = X_1^{\leqslant K} + \cdots + X_n^{\leqslant K} \qquad S_n^{>K} = X_1^{>K} + \cdots + X_n^{>K} \qquad n \geqslant 1.$$

Notice that $(S^{\leqslant K})$, $(\hat{S}_n^{\leqslant K})$ and $(\check{S}_n^{\leqslant K})$ have now *bounded* steps, allowing us to apply the result established in Section 3.3 to this triplet.

Remark 3.16. *We point out that while $(\hat{S}_n^{\leqslant K})$ can be simply obtained by considering the NRRW made from the steps $X_i 1_{\{|X_i| \leqslant K\}}$, $i \geqslant 1$ and substracting $n\mathbb{E}(X 1_{\{X \leqslant K\}})$ at the n-th step for each $n \geqslant 1$, and hence yielding a NRRW with steps given by*

$$\hat{X}_i 1_{\{|\hat{X}_i| \leqslant K\}} - \mathbb{E}(X 1_{\{|X| \leqslant K\}}),$$

for the counterbalanced case we need to subtract the counterbalanced random walk issued from the constants $\mathbb{E}(X_i 1_{\{|X_i| \leqslant K\}})$, $i \geqslant 1$, which in contrast with the reinforced case, is a process on its own right because of the sign swap.

For each k, write as $N^{n,K}$ $\hat{N}^{n,K}$ and $\check{N}^{n,K}$ the corresponding martingales as defined in (3.31) relative to $S^{\leqslant K}$, $\hat{S}^{\leqslant K}$ and $\check{S}^{\leqslant K}$ respectively. An application of Theorem 3.2 in the bounded case yields for every K, that

$$\left(N_t^{n,\leqslant K}, \hat{N}_t^{\leqslant K,n}, \check{N}_t^{\leqslant K,n}\right)_{t \in \mathbb{R}^+} \Longrightarrow \left(\sigma_K B,\, \sigma_K \int_0^t s^{-p} dB_s^r,\, \sigma_K \int_0^t s^p dB_s^c\right).$$

However recalling the asymptotic behaviour $n^p \widehat{a}_{\lfloor nt \rfloor} \sim t^{-p}$ as $n \to \infty$ and the definition of $N^{n,\leqslant K}$, we deduce that

$$\left(\frac{S^{\leqslant K}(\lfloor nt \rfloor)}{\sqrt{n}}, \frac{\hat{S}^{\leqslant K}(\lfloor nt \rfloor)}{\sqrt{n}}, \check{N}_t^{\leqslant K,n}\right)_{t \in \mathbb{R}^+} \Longrightarrow \left(\sigma_K B,\, \sigma_K t^p \int_0^t s^{-p} dB_s^r,\, \sigma_K \int_0^t s^p dB_s^c\right).$$

Since as $K \uparrow \infty$, the right hand side converges weekly towards $(\sigma B_t, \sigma t^p \int_0^t s^{-p} dB_s^r, \sigma \int_0^t s^p dB_s^c)$ and the convergence in distribution is metrisable, by Lemma 3.15 there exists a slowly increasing sequence converging towards infinity that we denote as $(K(n) : n \geqslant 1)$, satisfying that, as $n \uparrow \infty$,

$$\left(\frac{S^{\leqslant K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \frac{\hat{S}^{\leqslant K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \check{N}_t^{\leqslant K(n),n} \right)_{t \in \mathbb{R}^+} \implies \left(\sigma B, \sigma t^p \int_0^t s^{-p} dB_s, \sigma \int_0^t s^p dB_s \right).$$

On the other hand, for each n we can clearly decompose

$$\left(\frac{S(\lfloor nt \rfloor)}{\sqrt{n}}, \frac{\hat{S}(\lfloor nt \rfloor)}{\sqrt{n}}, \check{N}^n \right) = \left(\frac{S^{\leqslant K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \frac{\hat{S}^{\leqslant K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \check{N}_t^{\leqslant K(n),n} \right) + \left(\frac{S^{> K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \frac{\hat{S}^{> K(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, \check{N}_t^{> K(n),n} \right),$$

and in order to apply Lemma 3.14 we need the following lemma:

Lemma 3.17. *For any sequence $(K(n) : n \geqslant 1)$ increasing towards infinity the following limits hold:*

(i) $\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leqslant nt} \left| S_k^{>K(n)} \right|^2 \right) = 0.$

(ii) $\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leqslant nt} \left| \hat{S}_k^{>K(n)} \right|^2 \right) = 0,$ *for $p \in (0, 1/2)$.*

(iii) $\lim_{n\to\infty} \mathbb{P} \left(\sup_{s \leqslant T} \left| \check{N}_s^{n,>K(n)} \right|^2 \geqslant \varepsilon \right) = 0,$ *for every $\varepsilon > 0$ and $p \in (0, 1)$.*

Proof. Recall that we denoted by η_K^2 the variance of $X^{>K}$.

(i) By Doob's inequality and independence of the steps we inmediatly get that

$$\frac{1}{n} \mathbb{E} \left(\sup_{k \leqslant nt} |S_k^{>K(n)}|^2 \right) \leqslant \frac{4}{n} \eta_{K(n)} \lfloor nt \rfloor$$

which converges towards 0 as $n \uparrow \infty$.

(ii) From Lemma 3.10 for $0 < p < 1/2$ we deduce that for any $t > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leqslant nt} |\hat{S}^{>K(n)}(k)|^2 \right) \leqslant c_1 \lim_{n\to\infty} \eta_{K(n)}^2 t = 0, \tag{3.39}$$

proving the claim.

(iii) Doob's maximal inequality yields

$$\mathbb{P} \left(\sup_{s \leqslant T} |\check{N}_s^{n,>K(n)}| \geqslant \varepsilon \right) \leqslant \varepsilon^{-2} \mathbb{E} \left(\langle \check{N}^{n,>K(n)}, \check{N}^{n,>K(n)} \rangle_T \right),$$

and if we denote by $\hat{V}^{>(n)}$ the sum of squared steps associated to $(S^{>K(n)})$, notice that

$$\langle \check{N}^{n,>K(n)}, \check{N}^{n,>K(n)} \rangle_T \leqslant \frac{1}{n^{1+2p}} \left(\eta_{K(n)}^2 + \sum_{k=2}^{\lfloor nT \rfloor} \check{a}_k^2 \left((1-p)\eta_{K(n)}^2 + p \frac{\hat{V}_{k-1}^{>K(n)}}{k-1} \right) \right).$$

Recalling that $\mathbb{E}(\hat{V}_{k-1}^{>K(n)}) = (k-1)\eta_{K(n)}^2$, this yields the bound

$$\mathbb{P}\left(\sup_{s\leqslant T}|\check{N}_s^{n,K(n)}| \geqslant \varepsilon\right) \leqslant \varepsilon^{-2}\eta_{K(n)}^2 \frac{1}{n^{1+2p}}\left(1+\sum_{k=2}^{\lfloor nT\rfloor}\check{a}_k^2\right).$$

Since on the one hand we have $\eta_{K(n)}^2 \to 0$ as $n\uparrow\infty$ while on the other by (3.26) it holds that

$$\limsup_{n\uparrow\infty} n^{-(1+2p)}\sum_{k=2}^{\lfloor nT\rfloor}\check{a}_k^2 < \infty,$$

the desired convergence follows.

This concludes the proof of the lemma. □

Now, recalling the definition of $\check{N}^n$, we deduce from Lemma 3.14 that as $n\uparrow\infty$,

$$\left(\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor}, \frac{1}{\sqrt{n}}\hat{S}_{\lfloor nt\rfloor}, \frac{1}{\sigma\sqrt{n}}\frac{b_{\lfloor nt\rfloor}}{n^p}\check{S}_{\lfloor nt\rfloor}\right)_{t\in\mathbb{R}^+} \implies \left(B_t, t^p\int_0^t s^{-p}dB_s^r, \int_0^t s^p dB_s^c\right)_{t\in\mathbb{R}^+}$$

and since $b_{\lfloor nt\rfloor}/n^p \sim t^p$, we conclude that for any $\delta>0$, the desired convergence

$$\left(\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor}, \frac{1}{\sqrt{n}}\hat{S}_{\lfloor nt\rfloor}, \frac{1}{\sigma\sqrt{n}}\check{S}_{\lfloor nt\rfloor}\right)_{t\in[\delta,\infty)} \implies \left(B_t, t^p\int_0^t s^{-p}dB_s,\, t^{-p}\int_0^t s^p dB_s\right)_{t\in[\delta,\infty)},$$

holds away for the origin (this restriction is due to the fact that t^{-p} is unbounded on any neighbourhood of 0). In order to get the convergence on $\mathbb{R}^+$ and finally prove the claimed convergence in Theorem 3.2, we proceed as follows: We will only work with the third coordinate, as it is the only one presenting the difficulty. The argument is readily adapted to the triplet. Assume without loss of generality that $\sigma^2=1$, fix $\delta>0$ and consider the partition of $[0,\delta]$, with points $\{\delta 2^{-i} : i=0,1,2,\dots\}$. Since the sequence $(\check{a}_k)$ is increasing we obtain,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}} > \varepsilon\right) &\leqslant \mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{\check{a}_{\lfloor ns\rfloor}}{\check{a}_{\lfloor n2^{-(i+1)}\delta\rfloor}}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}} > \varepsilon\right)\\ &\leqslant \mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\check{a}_{\lfloor ns\rfloor}|\check{S}_{\lfloor ns\rfloor}| > \varepsilon\cdot\check{a}_{\lfloor n2^{-(i+1)}\delta\rfloor}\sqrt{n}\right)\\ &= \frac{1}{\varepsilon^2\cdot n\check{a}^2_{\lfloor 2^{-(i+1)}\delta n\rfloor}}\mathbb{E}\left(\sup_{s\leqslant 2^{-i}\delta}|\check{a}_{\lfloor ns\rfloor}\check{S}_{\lfloor ns\rfloor}|^2\right).\end{aligned}$$

Denoting as usual by $(\check{M}_n)$ the martingale $(\check{a}_n\check{S}_n)_{n\geqslant 0}$, notice that by (3.17), the remark that follows, and (3.26),

$$\mathbb{E}(\check{M}_n^2) = \mathbb{E}(\langle\check{M},\check{M}\rangle_n) \leqslant c\sum_{k=1}^n\check{a}_k^2 \leqslant cn^{1+2p}$$

for some constant c that might change from one inequality to the other. We deduce by Doob's inequality

$$\mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}}>\varepsilon\right)\leqslant c\left(2^{-i}\delta n\right)^{1+2p}\frac{1}{\varepsilon^2\cdot n\check{a}^2_{\lfloor 2^{-(i+1)}\delta n\rfloor}}$$
$$=c2^{-i(1+2p)}\delta^{1+2p}\frac{n^{2p}}{\check{a}^2_{\lfloor 2^{-(i+1)}\delta n\rfloor}}$$

which, recalling the asymptotic behaviour $\check{a}_n\sim n^p$, yields for some constant c that might differ from one line to the other:

$$\sup_n\mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}}>\varepsilon\right)\leqslant c2^{-i(1+2p)}\delta^{1+2p}\cdot 2^{2p(i+1)}\delta^{-2p}$$
$$=c\cdot 2^{-i}\delta.$$

From the previous estimate, we deduce the uniform bound

$$\mathbb{P}\left(\sup_{s\in[0,\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}}>\varepsilon\right)\leqslant\sum_{i=0}^{\infty}\mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}}>\varepsilon\right)$$
$$\leqslant\sum_{i=0}^{\infty}\sup_n\mathbb{P}\left(\sup_{s\in[2^{-(i+1)}\delta,2^{-i}\delta]}\frac{|\check{S}_{\lfloor ns\rfloor}|}{\sqrt{n}}>\varepsilon\right)\leqslant K\cdot\delta. \tag{3.40}$$

Finally, write $X^{(n)}=(\frac{1}{\sqrt{n}}\check{S}_{\lfloor nt\rfloor})_{t\in\mathbb{R}^+}$. Since for any $\delta>0$ we have $(X^{(n)}_t)_{t\geqslant\delta}\Rightarrow(\check{B}_t)_{t\geqslant\delta}$ as $n\uparrow\infty$ and of course $(\check{B}_{t+\delta})_{t\in\mathbb{R}^+}\Rightarrow(\check{B}_t)_{t\in\mathbb{R}^+}$ as $\delta\downarrow 0$, we deduce that there exists some decreasing sequence $(\delta(n))\downarrow 0$ such that

$$\left(X^{(n)}_{s+\delta(n)}\right)_{s\in\mathbb{R}^+}\Rightarrow\check{B}\qquad\text{as } n\uparrow\infty$$

while by (3.40),

$$\sup_{s\in[0,\delta(n)]}X^{(n)}_s\to 0\qquad\text{in probability.}$$

This establishes that the convergence $\left(\frac{1}{\sqrt{n}}\check{S}_{\lfloor nt\rfloor}\right)_{t\in\mathbb{R}^+}\Rightarrow\check{B}$ holds on $\mathbb{R}^+$ and with this, we conclude our proof of Theorem 3.2.

Remark 3.18. *In the process of proving Theorem 3.2 in Section 3.3 and 3.4, we showed also that if we no longer consider the noise-reinforced random walk, we can extend the convergence of the pair to $p\in(0,1)$,*

$$\left(\frac{1}{\sigma\sqrt{n}}S_{\lfloor nt\rfloor},\frac{1}{\sigma\sqrt{n}}\check{S}_{\lfloor nt\rfloor}\right)_{t\in\mathbb{R}^+}\Longrightarrow\left(B_t,\int_0^t s^p dB^c_s\right)_{t\in\mathbb{R}^+} \tag{3.41}$$

where as usual B^c, B are two Brownian motions with $\langle B,B^c\rangle_t=(1-p)t$, and that the result still holds if $p=1$ if we assume X follows the Rademacher distribution, in which case the processes are independent. This is precisely the content of Theorem 3.4. Finally, Theorem 3.3 also follows by recalling that $\hat{S}_n-n\mathbb{E}(X)$ is a centred positive step-reinforced random walk and hence falls in our framework.

3.5 The critical regime for the positive-reinforced case: proof of Theorem 3.5

In this last section we turn our attention to the critical regime $p = 1/2$ for the noise reinforced case and prove the invariance principle with our martingale approach. The arguments are very similar and rely on exploiting the martingale defined in Proposition 3.7, the MFCLT and a truncation argument. The main difference comes from the fact that, for $p = 1/2$, the asymptotic behaviour of $\sum_{k=1}^n \widehat{a}_k^2$ is no longer the one claimed in (3.24). Namely, as we pointed out previously,

$$\lim_{n\to\infty} \frac{1}{\log(n)} \sum_{k=1}^n \widehat{a}_k^2 = 1$$

and the different scaling that we will use makes impossible to couple the convergence with the random walk or the counterbalanced random walk. Once again, we start with a law of large numbers-type result:

Lemma 3.19. *Suppose $\|X\|_\infty < \infty$. We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{\hat{S}_n}{\sqrt{n}\log n} = 0 \quad a.s.$$

and fortiori we have $\lim_{n\to\infty} n^{-1}\hat{S}_n = 0 \quad a.s.$

Proof. The proof of this statement follows along the same lines as the proof of Lemma 3.9. Since $p = 1/2$ we have now, that as $n \to \infty$,

$$\nu_n := \sum_{k=1}^n \widehat{a}_{k+1}^2 \sim K' \cdot \log n,$$

where K' is a positive constant. That is, ν_n increases slowly to infinity with a logarithmic speed. We obtain again from Theorem 1.3.24 in [42] that

$$\frac{\hat{M}_n^2}{\log n} = O(\log\log n) \quad \text{a.s.}$$

Hence, as $\hat{M}_n = \widehat{a}_n \hat{S}_n$, the above readily implies that

$$\widehat{a}_n^2 \frac{\hat{S}_n^2}{\log n} = O(\log\log n) \quad \text{a.s.}$$

Further, we deduce from (3.25) that for $p = 1/2$, $\lim_{n\to\infty} \widehat{a}_n^2 \cdot n = 1$ and hence we deduce that

$$\frac{\hat{S}_n^2}{n\log n} = O(\log\log n) \quad \text{a.s.}$$

which immediately implies the claim. □

We now prove the invariance principle under the assumption of boundedness for X.

Proof of Theorem 3.5 when $\|X\|_\infty < \infty$. The proof relies on similar ideas to the ones used in the proof of Theorem 3.3. Recalling that,

$$\widehat{a}_k \sim k^{-p} = k^{-1/2} \qquad \text{as } k \to \infty$$

from the substitution $k = \lfloor n^t \rfloor$, we deduce that

$$\widehat{a}_{\lfloor n^t \rfloor} \sim \frac{1}{n^{t/2}}.$$

Then, the limit (3.12) can equivalently be shown by establishing the desired convergence towards $B = (B_t)_{t \geqslant 0}$ for the following sequence of martingales:

$$\left(\frac{\widehat{a}_{\lfloor n^t \rfloor}}{\sqrt{\log(n)}} \hat{S}_{\lfloor n^t \rfloor} \right)_{t \in \mathbb{R}^+} \Longrightarrow (\sigma B_t)_{t \in \mathbb{R}^+}.$$

Once again, we denote

$$(\hat{N}^n_t)_{t \in \mathbb{R}^+} = \left(\frac{\widehat{a}_{\lfloor n^t \rfloor}}{\sqrt{\log(n)}} \hat{S}_{\lfloor n^t \rfloor} \right)_{t \in \mathbb{R}^+}$$

and deduce as before that for each $n \in \mathbb{N}$, the predictable quadratic variation of $\hat{N}^n$ is given by

$$\langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t = \frac{1}{\log(n)} \left(\sigma^2 + \sigma^2 (1-p) \sum_{k=2}^{\lfloor n^t \rfloor} \widehat{a}_k^2 - p^2 \sum_{k=2}^{\lfloor n^t \rfloor} \widehat{a}_k^2 \left(\frac{\hat{S}_{k-1}}{k-1} \right)^2 + p \sum_{k=2}^{\lfloor n^t \rfloor} \widehat{a}_k^2 \left(\frac{\hat{V}_{k-1}}{k-1} \right) \right). \tag{3.42}$$

By the MFCLT, in order to prove our claim it suffices to show that

$$\lim_{n \to \infty} \langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t = \sigma^2 t \qquad \text{a.s.}$$

and that $\sup_t |\Delta \hat{N}^{(n)}_t| \to 0$ in probability as $n \to \infty$. Since $\|X\|_\infty < \infty$, this last requirement follows from very similar arguments to the ones we used in the proof of Theorem 3.3. On the other hand, since $\log(\lfloor n^t \rfloor)/\log(n) \to t$ as $n \to \infty$ and by (3.25), the first nontrivial term of (3.42) satisfies the following convergence

$$\lim_{n \to \infty} \frac{1}{\log(n)} (1-p) \sum_{k=1}^{\lfloor n^t \rfloor} \widehat{a}_k^2 = t(1-p).$$

By the same arguments we used in the proof of Theorem 3.3 but using the law of large numbers for the critical regime (Lemma 3.19), we obtain that the second term in (3.42) converges to zero while for the last term,

$$\lim_{n \to \infty} \frac{1}{l \log(n)} p \sum_{k=1}^{\lfloor n^t \rfloor} \widehat{a}_k^2 \frac{\hat{V}_k}{k} = t \cdot p\sigma^2 \qquad \text{a.s.}$$

It follows that $\langle \hat{N}^{(n)}, \hat{N}^{(n)} \rangle_t \to t\sigma^2$ for each t as $n \to \infty$, which proves the desired result under the additional assumption that $\|X\|_\infty < \infty$. □

Now we establish the general case by means of the usual reduction argument. We will not be as detailed as before, since the ideas are exactly the same. We do still assume without loss of generality that the steps are centred.

Proof of Theorem 3.5, general case. Maintaining the notation introduced for the truncated reinforced random walks of Section 3.4 as well as for the respective variances η_K and σ_K for $K > 0$, Theorem 3.5 in the bounded step case shows for each $K > 0$ the convergences in distribution as n tends to infinity in the sense of Skorokhod,

$$\left(\frac{\hat{S}^{\leqslant K}(\lfloor n^t \rfloor)}{\sqrt{\log(n) n^t}} \right)_{t \in \mathbb{R}^+} \Longrightarrow (\sigma_K B(t))_{t \in \mathbb{R}^+} \tag{3.43}$$

and from $\lim_{K \to \infty} \sigma_K = \sigma$, it follows readily from (3.43) and the same arguments as before that as n tends to infinity,

$$\left(\frac{\hat{S}^{\leqslant K(n))}(\lfloor n^t \rfloor)}{\sqrt{\log(n) n^t}} \right)_{t \in \mathbb{R}^+} \Longrightarrow (\sigma B(t))_{t \in \mathbb{R}^+}$$

for some increasing sequence $(K(n))_{n \geqslant 0}$ of positive real numbers converging towards infinity. On the other hand, from Lemma 3.10 for $p = 1/2$ we deduce that

$$\lim_{n \to \infty} \frac{1}{n^t \log(n)} \mathbb{E} \left(\sup_{k \leqslant n^t} |\hat{S}^{> b(n)}(k)|^2 \right) \leqslant c_2 \lim_{n \to \infty} \eta_{K(n)}^2 t = 0$$

and from here we can proceed as we did in the previous section. With this, we conclude the proof of Theorem 3.5. □

Chapter 4

Noise Reinforced Lévy Processes: Lévy-Itô Decomposition and Applications

THE CONTENT OF THIS CHAPTER IS TAKEN FROM THE PAPER [84], WHICH HAS BEEN ACCEPTED FOR PUBLICATION, WITH REVISIONS PENDING, IN THE JOURNAL *Electronic Journal of Probability*.

Abstract. A step reinforced random walk is a discrete time process with memory such that at each time step, with fixed probability $p \in (0,1)$, it repeats a previously performed step chosen uniformly at random while with complementary probability $1-p$, it performs an independent step with fixed law. In the continuum, the main result of Bertoin in [19] states that the random walk constructed from the discrete-time skeleton of a Lévy process for a time partition of mesh-size $1/n$ converges, as $n \uparrow \infty$ in the sense of finite dimensional distributions, to a process $\hat{\xi}$ referred to as a noise reinforced Lévy process. Our first main result states that a noise reinforced Lévy processes has rcll paths and satisfies a *noise reinforced* Lévy Itô decomposition in terms of the *noise reinforced* Poisson point process of its jumps. We introduce the joint distribution of a Lévy process and its reinforced version $(\xi, \hat{\xi})$ and show that the pair, conformed by the skeleton of the Lévy process and its step reinforced version, converge towards $(\xi, \hat{\xi})$ as the mesh size tend to 0. As an application, we analyse the rate of growth of $\hat{\xi}$ at the origin and identify its main features as an infinitely divisible process.

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4.1 Introduction

The Lévy-Itô decomposition is one of the main tools for the study of Lévy processes. In short, any real Lévy process ξ has rcll sample paths and its jump process induces a Poisson random measure – called the jump measure $\mathcal{N}$ of ξ – whose intensity is described by its Lévy measure Λ. Moreover, it states that ξ can be written as the sum of tree process

$$\xi_t = \xi_t^{(1)} + \xi_t^{(2)} + \xi_t^{(3)}, \qquad t \geqslant 0,$$

of radically different nature. More precisely, the continuous part of ξ is given by $\xi^{(1)} = (at + qB_t : t \geqslant 0)$ for a Brownian motion B and reals a, q, while $\xi^{(2)}$ is a compound Poisson process with jump-sizes greater than 1 and $\xi^{(3)}$ is a purely discontinuous martingale with jump-sizes smaller than 1. Moreover, the processes $\xi^{(2)}$, $\xi^{(3)}$ can be reconstructed from the jump measure $\mathcal{N}$. It is well known that $\mathcal{N}$ is characterised by the two following properties: for any Borel A with $\Lambda(A) < \infty$, the counting process of jumps $\Delta\xi_s \in A$ that we denote by N_A is a Poisson process with rate $\Lambda(A)$, and for any disjoint Borel sets $A_1, \ldots, A_k$ with $\Lambda(A_i) < \infty$, the corresponding Poisson processes $N_{A_1}, \ldots, N_{A_k}$ are independent. We refer to e.g. [17, 61, 88] for a complete account on the theory of Lévy processes.

In this work, we shall give an analogous description for *noise reinforced Lévy processes* (abbreviated NRLPs). This family of processes has been recently introduced by Bertoin in [19] and correspond to weak limits of step reinforced random walks of skeletons of Lévy process.

In order to be more precise, let us briefly recall the connection between these discrete objects and our continuous time setting. Fix a Lévy process ξ and denote, for each fixed n, by $X_k^{(n)} := \xi_{k/n} - \xi_{(k-1)/n}$ the k-th increment of ξ for a partition of size $1/n$ of the real line. The process $S_k^{(n)} := X_1^{(n)} + \cdots + X_k^{(n)} = \xi_{k/n}$ for $k \geqslant 1$ is a random walk, also called the n-skeleton of ξ. Now, fix a real number $p \in (0,1)$ that we call the reinforcement or memory parameter and let $\hat{S}_1^{(n)} := X_1^{(n)}$. Then, define recursively $\hat{S}_k^{(n)}$ for $k \geqslant 2$ according to the following rule: for each $k \geqslant 2$, set $\hat{S}_k^{(n)} := \hat{S}_{k-1}^{(n)} + \hat{X}_k^{(n)}$ where, with probability $1-p$, the step $\hat{X}_k^{(n)}$ is the increment $X_k^{(n)}$ with law $\xi_{1/n}$ – and hence independent from the previously performed steps – while with probability p, $\hat{X}_k^{(n)}$ is an increment chosen uniformly at random from the previous ones $\hat{X}_1^{(n)}, \dots, \hat{X}_k^{(n)}$. When the former occurs, the step is called an innovation, while in the latter case it is referred to as a reinforcement. The process $(\hat{S}_k^{(n)})$ is called the step-reinforced version of $(S_k^{(n)})$. It was shown in [19] that, under appropriate assumptions on the memory parameter p, we have the following convergence *in the sense of finite dimensional distributions* as the mesh-size tends to 0

$$(\hat{S}^{(n)}_{\lfloor nt \rfloor})_{t\in[0,1]} \xrightarrow{f.d.d.} (\hat{\xi}_t)_{t\in[0,1]}, \tag{4.1}$$

towards a process $\hat{\xi}$ identified in [19] and called a noise reinforced Lévy process. It should be noted that the process $\hat{\xi}$ constructed in [19] is a priori not even rcll, and this will be one of our first concerns.

We are now in position to briefly state the main results of this work. First, we shall prove the existence of a rcll modification for $\hat{\xi}$. In particular, this allow us to consider the jump process $(\Delta\hat{\xi}_s)$; a proper understanding of its nature will be crucial for this work. In this direction, we introduce a new family of random measures in $\mathbb{R}^+ \times \mathbb{R}$ of independent interest under the name *noise reinforced* Poisson point processes (abbreviated NRPPPs) and we study its basic properties. This lead us towards our first main result, which is a version of the Lévy-Itô decomposition in the reinforced setting. More precisely, we show that the jump measure of $\hat{\xi}$ is a NRPPP and that $\hat{\xi}$ can be written as

$$\hat{\xi}_t = \hat{\xi}_t^{(1)} + \hat{\xi}_t^{(2)} + \hat{\xi}_t^{(3)}, \quad t \geqslant 0,$$

where now, $\hat{\xi}^{(1)} = (at + q\hat{B}_t : t \geqslant 0)$ for a continuous Gaussian process $\hat{B}$, the process $\hat{\xi}^{(2)}$ is a reinforced compound Poisson process with jump-sizes greater than one, while $\hat{\xi}^{(3)}$ is a purely discontinuous semimartingale. The continuous Gaussian process $\hat{B}$ is the so-called noise reinforced Brownian motion, a Gaussian process introduced in [21] with law singular with respect to B, and arising as the universal scaling limit of noise reinforced random walks when the law of the typical step is in $L_2(\mathbb{P})$ – and hence plays the role of Brownian motion in the reinforced setting, see also [16] for related results. Needless to say that if the starting Lévy process ξ is a Brownian motion, the limit $\hat{\xi}$ obtained in (4.1) is a noise reinforced Brownian motion. As in the non-reinforced case, $\hat{\xi}^{(2)}$ and $\hat{\xi}^{(3)}$ can be recovered from the jump measure $\hat{\mathcal{N}}$, but in contrast, they are not Markovian. The terminology used for the jump measure of $\hat{\xi}$ is justified by the following remarkable property: for any Borel A with $\Lambda(A) < \infty$, the counting process of jumps $\Delta\hat{\xi}_s \in A$ that we denote by $\hat{N}_A$ is a reinforced Poisson process and, more precisely, it has the law of the noise reinforced version of N_A (hence, the terminology $\hat{N}_A$ is consistent). Moreover, for any disjoint Borel sets $A_1, \dots, A_k$ with $\Lambda(A_i) < \infty$, the corresponding $\hat{N}_{A_1}, \dots, \hat{N}_{A_k}$ are independent noise reinforced Poisson processes. Informally, the reinforcement induces memory on the jumps of $\hat{\xi}$, and these are repeated at the jump times of an independent counting process. When working on the unit interval, this counting process is the so-called *Yule-Simon* process.

The second main result of this work consists in defining pathwise, the noise reinforced version $\hat{\xi}$ of the Lévy process ξ. We always denote such a pair by $(\xi, \hat{\xi})$. This is mainly achieved by transforming the jump measure of ξ into a NRPPP, by a procedure that can be interpreted as the continuous time analogue of the reinforcement algorithm we described for random walks. More precisely, the steps $X_k^{(n)}$ of the n-skeleton are replaced by the jumps $\Delta\xi_s$ of the Lévy process; each jump of ξ is shared with its reinforced version $\hat{\xi}$ with probability $1-p$, while with probability p it is discarded and remains independent of $\hat{\xi}$. We then proceed to justify our construction by showing that the skeleton of ξ and its reinforced version $(S^{(n)}_{\lfloor n\cdot \rfloor}, \hat{S}^{(n)}_{\lfloor n\cdot \rfloor})$ converge weakly towards $(\xi, \hat{\xi})$, strengthening (4.1) considerably.

Section 4.6 is devoted to applications: on the one hand, in Section 4.6.1 we study the rates of growth at the origin of $\hat{\xi}$ and prove that well know results established by Blumenthal and Getoor in [26] for Lévy processes still hold for NRLPs. On the other hand, in Section 4.6.2 we analyse NRLPs under the scope of infinitely divisible processes in the sense of [86]. We shall give a proper description of $\hat{\xi}$ in terms of the usual terminology of infinitely divisible processes, as well as an application, by making use of the so-called Isomorphism theorem for infinitely divisible processes.

Let us mention that in the discrete setting, reinforcement of processes and models has been subject of active research for a long time, see for instance the survey by Pemantle [79] as well as e.g. [22, 15, 9, 74, 11, 6] and references therein for related work. However, reinforcement of time-continuous stochastic processes, which is the topic of this work, remains a rather unexplored subject.

The rest of the work is organised as follows: in Section 4.2 we recall the basic building blocs needed for the construction of NRLPs and recall the main results that will be needed. Notably, we give a brief overview of the features of the Yule-Simon process and present some important examples of NRLPs. In Section 4.3 we show that a NRLP has a rcll modification. In Section 4.4 we construct NRPPPs, study their main properties of interest, and in Section 4.4.3 we prove that the jump measure of a NRLP is a NRPPP – a result that we refer to as the "reinforced Lévy-Itô decomposition". In Section 4.5 we show that the pair conformed by the n-skeleton of a Lévy process and its reinforced version converge in distribution, as the mesh size tends to 0, towards $(\xi, \hat{\xi})$. To achieve this, first we start by proving in Section 4.5.1 that a NRLP can be reconstructed from its jump measure – a result that we refer to as the "reinforced Lévy Itô synthesis". Making use of this result in Section 4.5.2 we define the joint law $(\xi, \hat{\xi})$ and in Section 4.5.3 we establish our convergence result. Finally, Section 4.6 is devoted to applications. Particular attention is given through this work at comparing, when possible and pertinent, our results for NRLPs to the classical ones for Lévy processes.

4.2 Preliminaries

4.2.1 Yule-Simon processes

In this section, we recall several results from [19] concerning Yule-Simon processes needed for defining NRLPs. These results will be used frequently in this work and are re-stated for ease of reading.

A Yule-Simon process on the interval $[0,1]$ is a counting process, started from 0, with first jump time uniformly distributed in $[0,1]$, and behaving afterwards as a (deterministically) time-changed standard Yule process. More precisely, for fixed $p \in (0,1)$, if U is a uniform random variable in $[0,1]$ and Z a standard Yule process,

$$Y(t) := \mathbb{1}_{\{U \leqslant t\}} Z_{p(\ln(t)-\ln(U))}, \quad t \in [0,1], \tag{4.2}$$

is a Yule-Simon process with parameter $1/p$. Its law in $D[0,1]$, the space of $\mathbb{R}$-valued rcll functions in the unit interval endowed with the Skorokhod topology, will be denoted by $\mathbb{Q}$. It readily follows from the definition that this is a time-inhomogeneaous Markov process, with time-dependent birth rates given at time t by $\lambda_0(t) = 1/(1-t)$ and $\lambda_k(t) = pk/t$ for $k \in \{1,2,\dots\}$. Remark as well that we have $\mathbb{P}(Y(t) \geqslant 1) = t$. In our work, only $p \in (0,1)$ will be used, and it always corresponds to the reinforcement parameter. The Yule-Simon process with parameter $1/p$ is closely related to the *Yule-Simon distribution* with parameter $1/p$, i.e. the probability measure supported on $\{1,2,\dots\}$ with probability mass function given in terms of the Beta function $\mathrm{B}(x,y)$ by

$$p^{-1}\mathrm{B}(k, 1/p+1) = p^{-1}\int_0^1 u^p(1-u)^{k-1}\mathrm{d}u, \qquad \text{for } k \geqslant 1. \tag{4.3}$$

The relation with the Yule process is simply that $Y(1)$ is distributed Yule-Simon with parameter $1/p$. In this work, we refer to $p \in (0,1)$ as a reinforcement or memory parameter, for reasons that will be explained shortly. In the following lemma we state for further use the conditional self-similarity property of the Yule-Simon process, a key feature that will be used frequently.

Lemma 4.1. [19, Corollary 2.3]
Let Y be a Yule-Simon process with parameter $1/p$ and fix $t \in (0,1]$. Then, the process $(Y(rt))_{r\in[0,1]}$ conditionally on $\{Y(t) \geqslant 1\}$ has the same distribution $\mathbb{Q}$ as Y.

In particular, conditionally on $\{Y(t) \geqslant 1\}$, $Y(t)$ is distributed Yule-Simon with parameter $1/p$ and it follows that for every $t \in [0,1]$, $Y(t)$ has finite moments only of order $r < 1/p$. Moreover, by the previous lemma and the Markov property of the standard Yule process Z, we deduce that if Y is a Yule-Simon process with parameter $1/p$ with $p \in (0,1)$ and $k \geqslant 1$, we have

$$\mathbb{E}[Y(t)] = (1-p)^{-1}t \qquad \text{and} \qquad \mathbb{E}[Y(t)|Y(s)=k] = k(t/s)^p \quad \text{for any } 0 < s \leqslant t \leqslant 1, \tag{4.4}$$

while if $1/p > 2$,

$$\mathbb{E}[Y(s)Y(t)] = \frac{1}{(1-p)(1-2p)} s^{1-p}t^p. \tag{4.5}$$

More details on these statements can be found in Section 2 of [19].

4.2.2 Noise reinforced Lévy processes

Now, we turn our attention to the main ingredients involved in the construction of NRLPs. For the rest of the section, fix a real valued Lévy process ξ of characteristic triplet (a, q^2, Λ), where Λ is the Lévy measure, and recall that its characteristic exponent $\Psi(\lambda) := \log \mathbb{E}\left[e^{i\lambda\xi_1}\right]$ is given by the Lévy-Khintchine formula

$$\Psi(\lambda) = ia\lambda - \frac{q^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - ix\lambda \mathbb{1}_{\{|x|\leqslant 1\}}\right)\Lambda(\mathrm{d}x). \tag{4.6}$$

The constraints on the reinforcement parameter p are given in terms of the following two indices introduced by Blumenthal and Getoor: the Blumenthal-Getoor (upper) index $\beta(\Lambda)$ of the Lévy measure Λ is defined as

$$\beta(\Lambda) := \inf\big\{r > 0 : \int_{[0,1]} |x|^r \Lambda(\mathrm{d}x) < \infty\big\}, \tag{4.7}$$

while the Blumenthal-Getoor index β of the Lévy process ξ is defined by the relation

$$\beta := \begin{cases} \beta(\Lambda) & \text{if } q^2 = 0 \\ 2 & \text{if } q^2 \neq 0. \end{cases} \tag{4.8}$$

When ξ has no Gaussian component, we have $\beta = \beta(\Lambda)$ and both notations will be used indifferently. We say that a memory parameter $p \in (0,1)$ is admissible for the triplet (a, q^2, Λ) if $p\beta < 1$. Now, fix p an admissible memory parameter for ξ. If $(S_k^{(n)})$ is the n-skeleton of the Lévy process ξ, the sequence of reinforced versions with parameter p,

$$(\hat{S}^{(n)}_{\lfloor nt \rfloor})_{t\in[0,1]}, \qquad n \geqslant 1,$$

converge in the sense of finite dimensional distributions, as the mesh-size tends to 0, towards a process whose law was identified in [19] and called the noise reinforced Lévy process $\hat{\xi}$ of characteristics (a, q^2, Λ, p). In the sequel, when considering a NRLP with parameter p, it will be implicitly assumed that p is admissible for the corresponding triplet. For instance, when working with a memory parameter $p \geqslant 1/2$ it is implicitly assumed that $q = 0$. It was shown in [19, Corollary 2.11] that the finite-dimensional distributions of $\hat{\xi}$ can be expressed in terms of the Yule-Simon process Y with parameter $1/p$ and the characteristic exponent Ψ as follows:

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\lambda_i\hat{\xi}_{s_i}\right\}\right] = \exp\left\{(1-p)\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i)\right)\right]\right\}, \tag{4.9}$$

for $0 < s_1 < \cdots < s_k \leqslant 1$. Now we turn our attention at defining NRLPs in $\mathbb{R}^+$. Notice that the construction given in the unit interval in [19] can not be directly extended to the real line since it relies on Poissonian sums of Yule-Simon processes, and these are only defined on the unit interval.

Proposition 4.2. (NRLPs in $\mathbb{R}^+$)
Let (a, q^2, Λ) be the triplet of a Lévy process of exponent Ψ and consider an admissible memory parameter $p \in (0,1)$. There exists a process $\hat{\xi} = (\hat{\xi}_s)_{s\in\mathbb{R}^+}$ whose finite dimensional distributions satisfy that, for any $0 < s_1 < \cdots < s_k \leqslant t$,

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\lambda_i\hat{\xi}_{s_i}\right\}\right] = \exp\left\{(1-p)t\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/t)\right)\right]\right\}, \tag{4.10}$$

where the right-hand side does not depend on the choice of t. The process $\hat{\xi}$ is called a noise reinforced Lévy process with characteristics (a, q^2, Λ, p).

Proof. First, let us show that the right-hand side of (4.10) does not depend on t. To prove this, pick another arbitrary $T > t$ and write $r_i = s_i/t \in [0,1]$. From conditioning on $\{Y_{t/T} \geqslant 1\}$, an event with probability t/T, by Lemma 4.1 we get

$$\begin{aligned} T\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/T)\right)\right] &= t(T/t)\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(r_i\cdot(t/T))\right)\right] \\ &= t\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(r_i\cdot(t/T))\right)\Big| Y(t/T)\geqslant 1\right] \\ &= t\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/t)\right)\right], \end{aligned} \tag{4.11}$$

proving our claim, and where in the second equality we used that $\Psi(0)=0$. Now, let us establish the existence of a process with finite-dimensional distributions characterised by (4.10). Remark that by Kolmogorov's consistency theorem, it suffices to show that for arbitrary $1\leqslant S<T$, there exists processes $\hat{X}^S=(\hat{X}^S_t)_{t\in[0,S]}$, $\hat{X}^T:=(\hat{X}^T_t)_{t\in[0,T]}$ with finite dimensional distributions characterised by the identity (4.10) for (s_i) in $[0,S]$, $t=S$ and (s_i) in $[0,T]$, $t=T$ respectively – and hence satisfying that $(\hat{X}^T_t)_{t\in[0,S]}\stackrel{\mathscr{L}}{=}(\hat{X}^S_t)_{t\in[0,S]}$. Write $\hat{\xi}^S=(\hat{\xi}^S_t)_{t\in[0,1]}$ the reinforced version of the Lévy process $(\xi_{tS})_{t\in[0,1]}$, remark that the latter has characteristic exponent $S\Psi$, and set $(\hat{X}^S_t)_{t\in[0,S]}:=(\hat{\xi}^S_{t/S})_{t\in[0,S]}$. From the identity (4.9), we deduce that, for any $0<s_1<\cdots<s_k$ in the interval $[0,S]$, we have:

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\lambda_i\hat{X}^S(s_i)\right\}\right]=\exp\left\{(1-p)S\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/S)\right)\right]\right\}. \tag{4.12}$$

In particular $\hat{X}^S$ restricted to the interval $[0,1]$ has the same distribution as $(\hat{\xi}_t)_{t\in[0,1]}$ by the first part of the proof and (4.9). If we consider the restriction of $(\hat{X}^T)_{t\in[0,T]}$ to the interval $[0,S]$, we obtain similarly and by applying (4.11) that, for any $0<s_1<\cdots<s_k\leqslant S$,

$$\begin{aligned} \mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\lambda_i\hat{X}^T(s_i)\right\}\right] &= \exp\left\{(1-p)T\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/T)\right)\right]\right\} \\ &= \exp\left\{(1-p)S\mathbb{E}\left[\Psi\left(\sum_{i=1}^{k}\lambda_i Y(s_i/S)\right)\right]\right\}, \end{aligned}$$

and it follows that $\hat{X}^T$ restricted to $[0,S]$ has the same distribution as $\hat{X}^S$. Since this holds for any $1\leqslant S<T$, we deduce by Kolmogorov's consistency theorem the existence of a process satisfying for any $0<s_1<\cdots<s_k\leqslant t$, the identity (4.10). In particular, from taking the value $t=1$, it follows that this process satisfies that its restriction to $[0,1]$ has the same law as $\hat{\xi}$ by (4.9). $\square$

For later use, notice from (4.10) that for any fixed $t\in\mathbb{R}^+$, we have the following equality in law

$$(\hat{\xi}_{st})_{s\in[0,1]}\stackrel{\mathscr{L}}{=}(\widehat{\xi_{\cdot t}})_{s\in[0,1]} \tag{4.13}$$

where the right-hand side stands for the noise-reinforced version of the Lévy process $(\xi_{st})_{s\in[0,1]}$. In particular, $(\hat{\xi}_{st})_{s\in[0,1]}$ is the NRLP associated to the exponent $t\Psi$ with same reinforcement parameter.

4.2.3 Building blocks: noise reinforced Brownian motion and noise reinforced compound Poisson process

The characteristic exponent Ψ can be naturally decomposed in tree terms,

$$\Psi(\lambda) = \big(ia\lambda - q\frac{\lambda^2}{2}\big) + \Phi^{(2)}(\lambda) + \Phi^{(3)}(\lambda), \tag{4.14}$$

where respectively, we write

$$\Phi^{(2)}(\lambda) := \int_{\{|x|\geqslant 1\}} \left(e^{i\lambda x} - 1\right) \Lambda(\mathrm{d}x) \quad \text{and} \quad \Phi^{(3)}(\lambda) := \int_{\{|x|<1\}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \Lambda(\mathrm{d}x).$$

This decomposition yields that the Lévy process ξ can be written as the sum of tree independent Lévy process of radically different nature. Namely, we have $\xi_t = (at + qB_t) + \xi_t^{(2)} + \xi_t^{(3)}$, for $t \geqslant 0$, where B is a Brownian motion, $\xi^{(2)}$ is a compound Poisson process with exponent $\Phi^{(2)}$ and $\xi^{(3)}$ is the so-called compensated sum of jumps with characteristic exponent $\Phi^{(3)}$. In the reinforced setting, it readily follows from the identity (4.10) that an analogous decomposition holds for NRLPs. More precisely, the NRLP $\hat{\xi}$ of characteristics (a, q^2, Λ, p) can be written as a sum of three independent NRLPs,

$$\hat{\xi}_t \overset{\mathscr{L}}{=} (at + q\hat{B}_t) + \hat{\xi}_t^{(2)} + \hat{\xi}_t^{(3)}, \quad t \geqslant 0, \tag{4.15}$$

the equality holding in law, and where we denoted respectively by $\hat{B}$, $\hat{\xi}^{(2)}$, $\hat{\xi}^{(3)}$, independent reinforced versions of the Lévy processes B, $\xi^{(2)}$, $\xi^{(3)}$. Notice that their respective characteristics are given by $(a, q^2, 0, p)$, $(0, 0, \mathbb{1}_{(-1,1)^c}\Lambda, p)$ and $(0, 0, \mathbb{1}_{(-1,1)}\Lambda, p)$. Let us now give a brief description of these three building blocks separately:

∘ *Noise reinforced Brownian motion:* Assume $p < 1/2$, consider a Brownian motion B and set $\xi := B$. In that case, we simply have $\Psi(\lambda) = -\lambda^2/2$ and we write $\hat{B}$ for the corresponding noise reinforced Lévy process $\hat{\xi}$. The process $\hat{B}$ is the so-called noise reinforced Brownian motion (abbreviated NRBM) with reinforcement parameter p, a centred Gaussian process with covariance given by:

$$\mathbb{E}\left[\hat{B}_t\hat{B}_s\right] = \frac{(t \vee s)^p (t \wedge s)^{1-p}}{1-2p}. \tag{4.16}$$

Indeed, recalling (4.5), observe first that for any $0 \leqslant t, s < T$ the covariance (4.16) can be written in terms the Yule-Simon process Y with parameter $1/p$ as follows:

$$\mathbb{E}\left[\hat{B}_t\hat{B}_s\right] = (1-p)T \cdot \mathbb{E}\left[Y(t/T)Y(s/T)\right]. \tag{4.17}$$

It is now straightforward to deduce from (4.10) with $\Psi(\lambda) = -\lambda^2/2$ that the noise reinforced version of B corresponds to the Gaussian process with covariance (4.16). The noise reinforced Brownian motion admits a simple representation as a Wiener integral. More precisely, the process

$$t^p \int_0^t s^{-p} \mathrm{d}B_s, \qquad t \geqslant 0, \tag{4.18}$$

has the law of a noise reinforced Brownian motion with parameter p. Remark that when $p = 0$, there is no reinforcement and we recover a Brownian motion in (4.18). As was already mentioned, noise reinforced Brownian motion plays the role of Brownian motion in the reinforced setting, since it is the scaling limit of noise reinforced random walks under mild assumptions on the law of the typical step. We refer to [21, 16] for a detailed discussion.

○ *Noise reinforced compound Poisson process:* If ξ is a compound Poisson process with rate $c > 0$ and jumps with law P_X, then its Lévy measure is just $\Lambda(\mathrm{d}x) = cP_X(\mathrm{d}x)$, and any $p \in (0,1)$ is admissible. When working in $[0,1]$, the noise reinforced compound Poisson process $\hat{\xi}$ admits a simple representation in terms of Poissonian sums of Yule-Simon processes. In this direction, let $\mathbb{Q}$ be the law of the Yule Simon process with parameter $1/p$ and consider a Poisson random measure $\mathcal{M}$ in $\mathbb{R}^+ \times D[0,1]$ with intensity $(1-p)\Lambda \otimes \mathbb{Q}$. If we denote its atoms by (x_i, Y_i), the process

$$\hat{\xi}_t = \sum_i x_i Y_i(t), \quad t \in [0,1], \tag{4.19}$$

has the law of the noise reinforced version of ξ with reinforcement parameter p – as can be easily verified by Campbell's formula and was already established in [19, Corollary 2.11]. Notice that (4.19) is a finite variation process and its jump sizes are dictated by $P_X(\mathrm{d}x)$. Getting back to (4.15), it readily follows form our discussion that the NRLP $\hat{\xi}^{(2)}$ associated with the exponent $\Phi^{(2)}$ is a reinforced compound Poisson process and its jumps-sizes are greater than one. Finally, notice that if $P_X = \delta_1$, the Lévy process ξ is just a Poisson process with rate c and we deduce from the last display a simple representation for the reinforced Poisson process $\hat{N}$ in $[0,1]$. Observe that it is a counting process, since the atoms x_i are then identically equal to 1.

○ *Noise reinforced compensated compound Poisson process:* Let us now introduce properly $\hat{\xi}^{(3)}$, viz. the noise reinforced version of the compensated martingale $\xi^{(3)}$. When working in $[0,1]$, this process also admits a representation in terms of random series of Yule-Simon processes. In this direction, consider $\mathcal{M} := \sum_i \delta_{(x_i, Y_i)}$ a Poisson random measure with intensity $(1-p)\Lambda \otimes \mathbb{Q}$ and for each $a \in [0,1]$, set

$$\hat{\xi}^{(3)}_{a,1}(t) := \sum_i \mathbb{1}_{\{a \leqslant |x_i| < 1\}} x_i Y_i(t) - t \int_{\{a \leqslant |x| < 1\}} x \Lambda(\mathrm{d}x), \quad t \in [0,1]. \tag{4.20}$$

In the terminology of [19, Section 2], the process $\hat{\xi}^{(3)}_{a,1}$ is a Yule-Simon compensated series [1], and note that $\mathbb{E}[\hat{\xi}^{(3)}_{a,1}(t)] = 0$ for every $t \in [0,1]$. Moreover, the following family indexed by $a \in (0,1)$,

$$\hat{\xi}^{(3)}_{a,1}(t), \qquad \text{for } t \in [0,1], \tag{4.21}$$

is a collection of NRLPs with memory parameter p, Lévy measure $\mathbb{1}_{\{a \leqslant |x| < 1\}} \Lambda(\mathrm{d}x)$ and the corresponding exponent writes:

$$\Phi^{(3)}_a(\lambda) := \int_{\{a \leqslant |x| < 1\}} \left(e^{i\lambda x} - 1 - i\lambda x \right) \Lambda(\mathrm{d}x).$$

Notice that for each $a > 0$, the process $\hat{\xi}^{(3)}_{a,1}$ is rcll and with jump-sizes in $[a,1]$. Now, the process defined at each fixed t as the pointwise and $L_1(\mathbb{P})$-limit

$$\hat{\xi}^{(3)}_t := \lim_{a \downarrow 0} \hat{\xi}^{(3)}_{a,1}(t), \tag{4.22}$$

[1] The notation used in [19] for $\hat{\xi}^{(2)}_t$ and $\hat{\xi}^{(3)}_t$ is respectively $\Sigma_{1,\infty}(t)$ and $\Sigma^{(c)}_{0,1}(t)$. These are respectively referred to as Yule-Simon series and compensated Yule-Simon series.

is a NRLP with characteristics $(0, 0, \mathbb{1}_{\{|x|<1\}}\Lambda)$. In contrast with $\xi^{(3)}$, the noise reinforced version $\hat{\xi}^{(3)}$ is no longer a martingale, we shall discuss this point in the next section in detail. For latter use, we point out from [19, Section 2] that the convergence in the previous display also holds in $L_r(\mathbb{P})$, for r chosen according to

$$r \in (\beta(\Lambda) \vee 1, 1/p), \text{ if } 1/p \leqslant 2 \quad \text{ and } \quad r = 2, \text{ if } 1/p > 2. \tag{4.23}$$

In particular, we have $\hat{\xi}_t^{(3)} \in L_r(\mathbb{P})$ and $\mathbb{E}[\hat{\xi}_t^{(3)}] = 0$ for every t. We refer to [19] for a complete account on this construction and for a proof of the convergence in (4.22). The convergence in (4.22) will be strengthen in the sequel, by showing that it holds uniformly in $[0, 1]$. At this point, we have introduced the main ingredients needed for this work.

4.3 Trajectorial regularity

The purpose of this short section is to establish the following regularity theorem:

Theorem 4.3. *A noise reinforced Lévy process $\hat{\xi}$ has a rcll modification, that we still denote by $\hat{\xi}$. Moreover, if for $\varepsilon \in (0, 1)$, $\hat{\xi}_{0,\varepsilon}^{(3)}$ denotes a NRLP with characteristics $(0, 0, \mathbb{1}_{\{|x|<\varepsilon\}}\Lambda, p)$, then for any $t > 0$ we have:*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\sup_{s \leqslant t} |\hat{\xi}_{0,\varepsilon}^{(3)}(s)|\right] = 0. \tag{4.24}$$

Before proving this result, let us explain the role of (4.24). Working in $[0, 1]$ and with the construction (4.22) for $\hat{\xi}^{(3)}$, remark that for any $\varepsilon \in (0, 1)$ we can write $\hat{\xi}^{(3)} = \hat{\xi}_{0,\varepsilon}^{(3)} + \hat{\xi}_{\varepsilon,1}^{(3)}$, where $|\Delta\hat{\xi}_{\varepsilon,1}^{(3)}(t)| \geqslant \varepsilon$ for every jump-time $t \in [0, 1]$ by construction. Now, the convergence (4.24) shows that in fact, the jumps of $\hat{\xi}$ of size greater than ε are precisely the jumps of $\hat{\xi}^{(2)} + \hat{\xi}_{\varepsilon,1}^{(3)}$. Hence, when working in $[0, 1]$, the jumps of $\hat{\xi}^{(3)}$ are precisely the jumps of the weighted Yule-Simon processes $x_i Y_i(t)$ – heuristically, this is the continuous-time analogue of the dynamics described for the noise reinforced random walk. This fact will be used in Section 4.4.3. Moreover, (4.24) allow us to improve the convergence stated in (4.22) towards $\hat{\xi}^{(3)}$. Namely, it follows that for some subsequence (a_n) with $a_n \downarrow 0$ as $n \uparrow \infty$, the convergence

$$\lim_{n\to\infty} (\hat{\xi}_{a_n,1}^{(3)}(s))_{s\in[0,1]} = (\hat{\xi}_s^{(3)})_{s\in[0,1]},$$

holds a.s. uniformly in $[0, 1]$. Remark that the convergence in the previous display was only stated when working in $[0, 1]$ since, so far, the only explicit construction of NRLPs is the one in the unit interval we recalled from [19]. In Section 4.5.1 we shall address this point.

The rest of the section is devoted to the proof of Theorem 4.3. Recalling the building blocks introduced in Section 4.2.3 and the identity in distribution (4.15), $\hat{\xi}^{(2)}$ is a reinforced compound Poisson process and hence has finite variation rcll trajectories, while $\hat{B}$ is continuous. It is then clear that the only difficulty consists in establishing the regularity of the process $\hat{\xi}^{(3)}$ and we rely on a remarkable martingale associated with centred NRLPs, that we now introduce. This martingale will play a key role in this work.

Proposition 4.4. *Consider a Lévy process ξ with characteristic exponent Ψ satisfying $\Psi'(0) = 0$ and Lévy measure fulfilling the integrability condition $\int_{\{|x|\geqslant 1\}} x\Lambda(\mathrm{d}x) < \infty$. Then, the process $M = (M_t)_{t\in\mathbb{R}^+}$ defined as $M_0 = 0$ and for $t > 0$, as $M_t = t^{-p}\hat{\xi}_t$, is a martingale. Consequently, M has a rcll modification.*

Proof. Recall from (4.15) that in that case, $\hat{\xi}$ can be written as a sum of two independent processes $\hat{\xi} = q\hat{B} + \hat{\xi}^{(3)}$, where $\hat{B}$ is a noise reinforced Brownian motion. Recalling the representation (4.18) for $\hat{B}$, it follows that $(t^{-p}\hat{B}_t)_{t\in\mathbb{R}^+}$ is a continuous martingale and we assume therefore that $q = 0$.

Turning our attention to $\hat{\xi}^{(3)}$, notice that M_t is in $L_r(\mathbb{P})$ for r chosen according to (4.23) and that $E[M_t] = 0$ since, as we discussed after (4.23), we have $\hat{\xi}_t^{(3)} \in L_r(\mathbb{P})$, $\mathbb{E}[\hat{\xi}_t^{(3)}] = 0$. Now, it remains to show that $(M_t)_{t\in(0,1]}$ satisfies the martingale property. In this direction it is enough to check that for any $0 < t_0 < \cdots < t_k < t$ and $\lambda_1, \ldots, \lambda_{k-1} \in \mathbb{R}$, we have

$$\mathbb{E}\left[t_k^{-p}\hat{\xi}^{(3)}_{t_k} \exp\left\{ i\sum_{i=1}^{k-1}\lambda_i \hat{\xi}^{(3)}_{t_i}\right\}\right] = \mathbb{E}\left[t_{k-1}^{-p}\hat{\xi}^{(3)}_{t_{k-1}} \exp\left\{ i\sum_{i=1}^{k-1}\lambda_i \hat{\xi}^{(3)}_{t_i}\right\}\right]. \tag{4.25}$$

On the one hand, under our standing assumptions, the left-hand side of (4.25) corresponds to the derivative at $\lambda_k = 0$ of (4.10) multiplied by $-it_k^{-p}$ and hence equals:

$$-it(1-p)\exp\left\{ t(1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k-1}\lambda_j Y(t_j/t)\right)\right]\right\}\cdot \mathbb{E}\left[H\left(Y(s)\,:\, s \leqslant t_{k-1}/t\right)Y(t_k/t)\right]t_k^{-p},$$

for H defined as

$$H\left(Y(s)\,:\, s\leqslant t_{k-1}/t\right) := \Psi'\left(\sum_{j=1}^{k-1}\lambda_i Y(t_j/t)\right).$$

Remark that this is a $\sigma(Y(s) : s \leqslant t_{k-1}/t)$-measurable random variable. On the other hand, the right-hand side of (4.25) corresponds to the derivative with respect to λ_{k-1} of (4.10) multiplied by $-it_{k-1}^p$ for $\lambda_k = 0$ and similarly, we deduce that the right-hand side of (4.25) writes:

$$-it(1-p)\exp\left\{ t(1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k-1}\lambda_j Y(t_j/t)\right)\right]\right\}\cdot \mathbb{E}\left[H\left(Y(s)\,:\, s \leqslant t_{k-1}/t\right)Y(t_{k-1}/t)\right]t_{k-1}^{-p}.$$

Now, it only remains to show that:

$$\mathbb{E}\left[H(Y(s)\,:\, s\leqslant t_{k-1}/t)Y(t_k/t)\right]t_k^{-p} = \mathbb{E}\left[H\left(Y(s)\,:\, s\leqslant t_{k-1}/t\right)Y(t_{k-1}/t)\right]t_{k-1}^{-p}. \tag{4.26}$$

Notice that since $\Psi'(0) = 0$ and Y is increasing, $H\left(Y(s) : s \leqslant t_{k-1}/t\right)$ vanishes if $Y(t_{k-1}/t) = 0$. This allows us to restrict the terms inside the expectations in (4.26) to $\{Y(t_{k-1}/t) \geqslant 1\}$ and to apply the Markov property (4.4) at time t_{k-1}/t to get:

$$\begin{aligned}
&\mathbb{E}\left[H(Y(r) : r\leqslant t_{k-1}/t)Y(t_k/t)\right]t_k^{-p}\\
&= \sum_{j=1}^{\infty}\mathbb{E}\left[H(Y(r) : r\leqslant t_{k-1}/t)\mathbb{E}\left[Y(t_k/t)|Y(t_{k-1}/t)=j\right]\mathbb{1}_{\{Y(t_{k-1}/t)=j\}}\right]t_k^{-p}\\
&= \sum_{j=1}^{\infty}\mathbb{E}\left[H(Y(r) : r\leqslant t_{k-1}/t)\cdot j(t_k/t_{k-1})^p\mathbb{1}_{\{Y(t_{k-1}/t)=j\}}\right]t_k^{-p}\\
&= \mathbb{E}\left[H(Y(r) : r\leqslant t_{k-1}/t)Y(t_{k-1}/t)\right]t_{k-1}^{-p},
\end{aligned} \tag{4.27}$$

proving the claim. □

Let us now conclude the proof of Theorem 4.3.

Proof of Theorem 4.3. The first assertion is now a consequence of the following simple observation: denoting by $\overline{M}$ the rcll modification of the martingale $M = (t^{-p}\hat{\xi}_t^{(3)})_{t\in\mathbb{R}^+}$, it is then clear that the process $\hat{J}^{(3)} := t^p \overline{M}_t$, for $t \geqslant 0$, is a rcll modification of $\hat{\xi}^{(3)}$. Notice by intergrating by parts that consequently, the process $\hat{\xi}^{(3)}$ is a semimartingale, this will be needed in Section 4.4.3. To prove the second claim, remark that by the observation right after (4.13), it suffices to work on the time interval $[0,1]$. Moreover, by Proposition 4.4, for each $\varepsilon > 0$, the process

$$M^{(\varepsilon)} := (s^{-p}\hat{\xi}^{(3)}_{0,\varepsilon}(s))_{s\in(0,1]}$$

with $M_0^{(\varepsilon)} = 0$, is a $L_r(\mathbb{P})$ rcll martingale in $[0,1]$, for r chosen according to (4.23). Since $r > 1$, by Doob's inequality at time $t = 1$ we have

$$\mathbb{E}\left[\sup_{s\leqslant t} |\hat{\xi}^{(3)}_{0,\varepsilon}(s)|^r\right] \leqslant \mathbb{E}\left[\sup_{s\leqslant t} \left|s^{-p}\hat{\xi}^{(3)}_{0,\varepsilon}(s)\right|^r\right] \leqslant C_r \mathbb{E}\left[|\hat{\xi}^{(3)}_{0,\varepsilon}(1)|^r\right],$$

for some constant $C_r > 0$, and it remains to show that the right-hand side converges to 0 as $\varepsilon \downarrow 0$. However, this is a consequence of (4.22). More precisely, recalling the construction detailed in (4.20), note that $\hat{\xi}^{(3)}_t - \hat{\xi}^{(3)}_{\varepsilon,1}(t)$ has the same distribution as $\hat{\xi}^{(3)}_{0,\varepsilon}(t)$ for every $t \in [0,1]$ and $\varepsilon > 0$. Since the convergence (4.22) still holds in $L_r(\mathbb{P})$, the result follows by taking the limit as $\varepsilon \downarrow 0$. □

Now that we have established that a NRLP is a rcll process, in the next section we study the structure of its jump process $(\Delta\hat{\xi}_t)$. Since it will share striking similarities with the jump process of a Lévy process, before concluding the section we recall well known results on $(\Delta\xi_t)$. Namely, if ξ is a Lévy process with Lévy measure Λ, its jump measure

$$\mu(\mathrm{d}t, \mathrm{d}x) = \sum_s \mathbb{1}_{\{\Delta\xi_s \neq 0\}} \delta_{(s,\Delta\xi_s)}(\mathrm{d}t, \mathrm{d}x), \tag{4.28}$$

is a homogeneous Poisson point process (abbreviated PPP) with characteristic measure $\Lambda(\mathrm{d}x)$. Such a PPP can be constructed by decorating the point process of jumps of a Poisson process, and it is classic that (4.28) is determined by the following two properties:

(i) For any Borelian A with $\Lambda(A) < \infty$, the counting process of jumps $\Delta\xi_s \in A$ occurring until time t, defined as

$$N_A(t) = \#\{(s, \Delta\xi_s) \in [0,t] \times A\}, \qquad t \geqslant 0,$$

is a Poisson process with rate $\Lambda(A)$.

(ii) If $A_1, \dots A_k$ are disjoint Borelians with $\Lambda(A_i) < \infty$ for all $i \in \{1, \dots, k\}$, the processes $N_{A_1}, \dots, N_{A_k}$ are independent.

In particular, from (i), it follows that $(N_A(t) - \Lambda(A)t)_{t\in\mathbb{R}^+}$ is a martingale.

4.4 Reinforced Lévy-Itô decomposition

This section is devoted to the study of the jump process $(\Delta\hat{\xi}_s)_{s\in\mathbb{R}^+}$ and the associated jump measure in $\mathbb{R}^+ \times \mathbb{R}$, viz.

$$\hat{\mu}(\mathrm{d}t, \mathrm{d}x) := \sum_s \mathbb{1}_{\{\Delta\hat{\xi}_s \neq 0\}} \delta_{(s,\Delta\hat{\xi}_s)}(\mathrm{d}t, \mathrm{d}x). \tag{4.29}$$

In this direction, we shall introduce in Definition 4.9 below a family of random measures in $\mathbb{R}^+ \times \mathbb{R}$ under the name *noise reinforced Poisson point processes* – abbreviated NRPPPs – that will play the analogous role of PPPs for the jump measure of Lévy processes. Each element of this family of measures is parametrized by a sigma finite measure Λ in $\mathbb{R}$, that we refer to as its characteristic measure, and a positive value $p \in (0,1)$, that we call its reinforcement parameter. The construction of NRPPPs consists essentially in the reinforced version of the one of PPPs. More precisely, we shall construct them by decorating the point process of jumps of a *reinforced* Poisson process. The main result of this section is the reinforced version of the celebrated Lévy-Itô decomposition:

Theorem 4.5. (Reinforced Lévy-Itô decomposition)
The jump measure $\hat{\mu}$ of $\hat{\xi}$ is a noise reinforced Poisson point process with characteristic measure $\Lambda(\mathrm{d}x)$ and reinforcement parameter p.

The rest of the section is organised as follows: In Section 4.4.1 we restrict our study to the jump process of reinforced Poisson processes. In Section 4.4.2, we construct NRPPPs by decorating the jump process of reinforced Poisson processes and then study its basic properties. For instance, in Proposition 4.12 we prove a characterisation in the same vein as the one holding for PPPs, recalled at the end of Section 4.3. Finally, in Section 4.4.3 we prove Theorem 4.5 and in Proposition 4.15 we identify the predictable compensator of $\hat{\mu}$.

4.4.1 The jumps of noise reinforced Poisson processes

Let us start by introducing the basic building block of this section.

○ *Noise reinforced Poisson process:* When ξ is a Poisson process N with rate c, any reinforcement parameter $p \in (0,1)$ is admissible and recall from the discussion following (4.19) that $\hat{N}$ is a counting process. Moreover, the corresponding noise reinforced Poisson process (abbreviated NRPP) with rate p has finite dimensional distributions characterised, for any $0 < s_1 < \cdots < s_k \leqslant t$ and $\lambda_j \in \mathbb{R}$, by the identity

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\lambda_i \hat{N}_{s_i}\right\}\right] = \exp\left\{(1-p)ct\mathbb{E}\left[\left(\exp\left\{i\sum_{i=1}^{k}\lambda_i Y(s_i/t)\right\}-1\right)\right]\right\}. \tag{4.30}$$

A Poisson process with rate c has associated to it the random measure $\mathrm{d}N_s$, also called its point process of jumps. This is a Poisson random measure in $\mathbb{R}^+$ with intensity $c\mathrm{d}t$ and it has a natural reinforced counterpart: namely, the random measure $\mathrm{d}\hat{N}_s$, that we shall now study in detail.

To do so, we start by introducing some standard notation for point processes. We shall identify discrete random sets $D = \{t_1, t_2, \dots\} \subset \mathbb{R}$ with counting measures $\sum_{t\in D}\delta_t$ and for $f : \mathbb{R} \mapsto \mathbb{R}$, we use the notation $\langle D, f\rangle$ for $\sum_{t\in D} f(t)$. The collection of counting measures in $\mathbb{R}$ is denoted by $\mathcal{M}_c$. We will make use of the following two basic transformations: for $x \in \mathbb{R}$, we denote by $\mathcal{T}_x D$ the translated point process $\{t + x : t \in D\}$ and for $f : \mathbb{R} \mapsto \mathbb{R}$, we write $D \circ f^{-1}$ the push-forwarded point process $\{f(t) : t \in D\}$.

Now, consider an increasing sequence of random times $0 = T_0 < T_1 < T_2 < \dots$, such that the increments $(T_n - T_{n-1} : n \geqslant 1)$ are independent and for any $n \geqslant 1$, $T_n - T_{n-1}$ is exponentially distributed with parameter pn. Write $D := \{0, T_1, T_2, \dots\}$ the point process associated to this family and we denote its law in $\mathcal{M}_c$ by $\mathbb{D}(\mathrm{d}\mu)$. From these ingredients, we define a decorated measure as follows: first, consider $\mathscr{E}$ a Poisson point process with intensity $c(1-p)e^t\mathrm{d}t$ in $\mathbb{R}$ and, for each atom $u \in \mathscr{E}$, let D_u be an independent copy of D. Then, we set

$$\mathcal{L}(\mathrm{d}s) := \sum_{u\in\mathscr{E}} \sum_{t\in D_u} \delta_{u+t} = \sum_{u\in\mathscr{E}} \mathcal{T}_u D_u. \tag{4.31}$$

Remark that if (Z_t) is a standard Yule process started from 1, D has the same law as the point process induced by the jump-times of (Z_{tp}), with a Dirac mass at 0. The next proposition shows that the law of the point process of jumps of a noise reinforced Poisson process with rate c is precisely $\mathcal{L} \circ \exp^{-1}$, the pushforward of $\mathcal{L}$ by the exponential function.

Proposition 4.6. *The following properties hold:*

(i) *Let $\hat{N}$ be a noise-reinforced Poisson process with rate c and write $\hat{\mathscr{P}} := \mathrm{d}\hat{N}_s$ the point process of its jump-times in $\mathbb{R}^+$. Then, we have the equality in distribution $\hat{\mathscr{P}} \overset{\mathscr{L}}{=} \mathcal{L} \circ \exp^{-1}$. We will still refer to $\hat{\mathscr{P}}$ as a reinforced Poisson process with rate c and reinforcement parameter p.*

(ii) *If Y is a Yule-Simon process with parameter $1/p$, for any $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ we have*

$$-\log \mathbb{E}\left[\exp\left\{ -\langle \hat{\mathscr{P}}, \mathbb{1}_{(0,t]} f\rangle \right\}\right] = tc \cdot (1-p)\mathbb{E}\left[1 - e^{-\int_0^1 f(st)\mathrm{d}Y(s)}\right]. \tag{4.32}$$

In particular, from (4.31) and (i) we deduce the following identity in distribution: if $\mathscr{P}$ is a Poisson process in $\mathbb{R}_+$ with intensity $c(1-p)\mathrm{d}t$, we have

$$\hat{\mathscr{P}} = \sum_{s\in\mathbb{R}^+} \mathbb{1}_{\{s:\Delta\hat{N}_s=1\}}\delta_s \overset{\mathscr{L}}{=} \sum_{u\in\mathscr{P}} \sum_{t\in D_u} \delta_{ue^t}. \tag{4.33}$$

Roughly speaking, the jumps of $\hat{N}$ consist in Poissonian jumps $u \in \mathscr{P}$ which – in analogy with the discrete setting – we refer to as innovations, and each u has attached to it a family $\{ue^t : t \in D_u,\ t \neq 0\}$ which should be interpreted as repetitions of the original u through time.

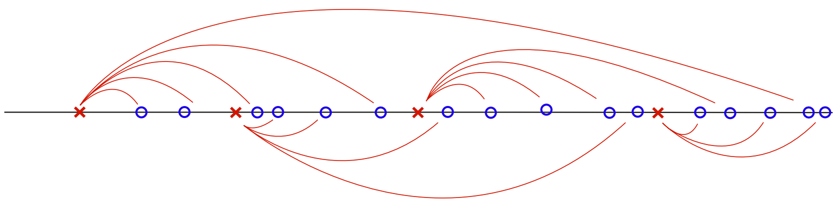

Figure 4.1: Sketch of the jumps of a noise reinforced Poisson process. We marked by x the jumps corresponding to innovations, while each linked o is a repetition of the former.

Notice that the time at which u occurs affects the rate of the subsequent repetitions, slowing the rate down as u grows. This is closely related to what happens to the rate at which a step is repeated in a step reinforced random walk, depending on its first time of appearance. For later use, remark that for fixed $u \in \mathbb{R}^+$, the atoms of $\sum_{t\in D}\delta_{ue^t}$ are distributed as the jump times of the counting process

$$\mathbb{1}_{\{u\leqslant s\}} Z_{p(\ln(s)-\ln(u))}, \qquad s \geqslant 0. \tag{4.34}$$

Proof. To establish the identity in distribution stated in (i), we compute the respective Laplace functional of both random measures. Starting with $\hat{\mathscr{P}}$, fix $t \geqslant 0$ and recall from the identity in distribution (4.13) that $(\hat{N}_{ts})_{s\in[0,1]}$ has the same law as a noise reinforced Poisson process with same reinforcement parameter p and rate tc, say $(\hat{N}_s^{(t)})_{s\in[0,1]}$. This NRLP is defined in $[0,1]$ and hence admits a simple representation in terms of Poisson random measures: by (4.19), if $\sum_i \delta_{Y_i}$ is a Poisson random measure in $D[0,1]$ with intensity $tc(1-p)\mathbb{Q}$, the process $(\sum_i Y_i(s) : s \in [0,1])$ has the same distribution as $(\hat{N}_s^{(t)})_{s\in[0,1]}$. In particular, we have

$$\int_0^t f(s)\mathrm{d}\hat{N}_s = \int_0^1 f(st)\mathrm{d}\hat{N}_{st} \overset{\mathscr{L}}{=} \sum_i \int_0^1 f(st)\mathrm{d}Y_i(s).$$

Putting everything together, we deduce (4.32) by making use of the Laplace formula for integrals with respect to Poisson random measures – we invite the reader to compare (4.32) with the identity (4.10) for the finite-dimensional distributions of NRLPs – and it remains to show that the Laplace functional of $\mathcal{L} \circ \exp^{-1}$ coincide with this expression.

In this direction, recall the observation made in (4.34) and denote by $\mathbf{Z}$ the law of the standard Yule process Z. It follows that the law of $\langle \mathcal{L} \circ \exp^{-1}, 1_{(0,t]} f \rangle$ can be expressed in terms of the Poisson random measure $\mathcal{M} := \sum_i \delta_{(u_i, Z^{(i)})}$ in $\mathbb{R}^+ \times D[0,1]$, with intensity $c(1-p)\mathrm{d}t \otimes \mathbf{Z}$, by considering the functional

$$\sum_i \int_{(0,t]} f(s)\,\mathrm{d}\left(\mathbb{1}_{\{u_i \leqslant s\}} Z^{(i)}_{p(\ln(s)-\ln(u_i))}\right),$$

where the integrals in the previous expression are respectively with respect to the Stieltjes measure associated to the counting process $s \mapsto \mathbb{1}_{\{u_i \leqslant s\}} Z^{(i)}_{p(\ln(s)-\ln(u_i))}$. It now follows also by the exponential formula that

$$\begin{aligned}
-\log \mathbb{E}\left[e^{-\langle \mathcal{L}\circ\exp^{-1}, 1_{\{\cdot\leqslant t\}} f\rangle}\right] &= (1-p)c\int_{\mathbb{R}} \mathrm{d}u\, \mathbb{E}\left[1 - \exp\left\{ - \int_0^t f(s)\mathrm{d}\left(\mathbb{1}_{\{u\leqslant s\}} Z_{p(\ln(s)-\ln(u))}\right) \right\}\right] \\
&= (1-p)c\int_0^t \mathrm{d}u\, \mathbb{E}\left[1 - \exp\left\{ - \int_0^t f(s)\mathrm{d}\left(\mathbb{1}_{\{u\leqslant s\}} Z_{p(\ln(s)-\ln(u))}\right) \right\}\right] \\
&= t(1-p)c\mathbf{Z}^\bullet\left(1 - \exp\left\{ \int_0^1 f(st)\mathrm{d}\left(\mathbb{1}_{\{u\leqslant st\}} Z_{p(\ln(st)-\ln(u))}\right) \right\} \middle| u \leqslant t\right)
\end{aligned} \tag{4.35}$$

where we denoted in the last line by $\mathbf{Z}^\bullet(\,\cdot\,|u \leqslant t)$ the integral in $\mathbb{R}^+ \times D[0,\infty)$ with respect to the probability measure

$$\mathbf{Z}^\bullet(\,\cdot\,|u \leqslant t) := \frac{\mathbb{1}_{\{u\leqslant t\}}}{t}\mathrm{d}u\,\mathbf{Z}(\mathrm{d}Z).$$

Now, we deduce by Lemma 4.47 - (ii) in the Appendix that (4.35) is precisely (4.32). ☐

Finally, for later use we state the following equivalent expression for the Laplace functional associated to the random measure $\mathcal{L} \circ \exp^{-1}$.

Lemma 4.7. *For any measurable $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have*

$$-\log \mathbb{E}\left[\exp\left\{ - \langle \mathcal{L} \circ \exp^{-1}, f\rangle\right\}\right] = (1-p)c\int_0^\infty \mathrm{d}u \int_{\mathcal{M}_c} 1 - e^{-\langle T_{\log(u)}\mu, f\circ\exp\rangle}\mathbb{D}(\mathrm{d}\mu). \tag{4.36}$$

Proof. The proof follows from the equality $\langle \mathcal{L} \circ \exp^{-1}, f \rangle = \langle \mathcal{L}, f \circ \exp \rangle$ and the identity:

$$-\log \mathbb{E}\left[\exp\left\{-\langle \mathcal{L}, h\rangle\right\}\right] = (1-p)c \int_0^\infty \mathrm{d}u \int_{\mathcal{M}_c} 1 - e^{-\langle \mathcal{T}_{\log(u)}\mu, h\rangle} \mathbb{D}(\mathrm{d}\mu),$$

holding for any measurable $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$. The proof of the later is just a straightforward consequence of (4.31) and the exponential formula for Poisson random measures. □

Remark 4.8. *Notice from (4.33) that the reinforced Poisson process with rate c can be interpreted as a Yule-Simon process with immigration: this is, a process modelling the evolution of a population where new independent immigrants arrive according to a Poisson point process with intensity* $(1-p)c \cdot \mathrm{d}t$ *and reproduce according to a time changed Yule process, independent of the rest.*

4.4.2 Construction of noise reinforced Poisson point processes by decoration

This section is devoted to the construction of *noise reinforced Poisson point processes* and to establishing their first properties. From here, we fix $p \in (0,1)$.

- *Step 1:* Suppose first that $0 < \Lambda(\mathbb{R}) < \infty$. With the same notation of Section 4.4.1, denote by $\mathscr{E}$ a Poisson random measure in $\mathbb{R}$ with intensity $\Lambda(\mathbb{R})(1-p)e^t\mathrm{d}t$ and consider the Poisson point process $\Sigma_{u \in \mathscr{E}} \delta_{(u, x_u)}$ in $\mathbb{R} \times \mathbb{R}$ with intensity $(1-p)e^t\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$. Now, for each $u \in \mathscr{E}$, consider an independent copy D_u of D and set

$$\mathcal{L}^x(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{E}} \sum_{t \in D_u} \delta_{(u+t, x_u)}. \tag{4.37}$$

This is just the point process $\mathcal{L}$ from (4.31) with $c := \Lambda(\mathbb{R})$, marked by a collection of i.i.d. random variables with law $\Lambda(\mathrm{d}x)/\Lambda(\mathbb{R})$. Formula (4.37) defines a random measure in $\mathbb{R} \times \mathbb{R}$ and if we consider its push forward by $(t, x) \mapsto (\exp(t), x)$, that we denote as $\hat{\mathcal{N}} := \mathcal{L}^x \circ (\exp, \mathrm{Id})^{-1}$, we obtain the measure in $\mathbb{R}^+ \times \mathbb{R}$ given by

$$\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{P}} \sum_{t \in D_u} \delta_{(ue^t, x_u)}, \tag{4.38}$$

where $\mathscr{P} := \mathscr{E} \circ \exp^{-1}$ is a Poisson point process in $\mathbb{R}_+$ with intensity $\Lambda(\mathbb{R})(1-p)\mathrm{d}t$. We refer to the measure in the previous display as a NRPPP with (finite) characteristic measure Λ and reinforcement parameter p.

- *Step 2:* If we no longer assume $\Lambda(\mathbb{R}) < \infty$, we proceed by superposition. More precisely, let $(A_j)_{j \in \mathcal{I}}$ be a disjoint partition of $\mathbb{R}\backslash\{0\}$ such that $\Lambda(A_j) < \infty$. Consider a collection of independent NRPPPs $(\hat{\mathcal{N}}_j(\mathrm{d}s, \mathrm{d}x) : j \in \mathcal{I})$ with respective characteristic measures $(\Lambda(\,\cdot \cap A_j) : j \in \mathcal{I})$ constructed as in (4.38), respectively in terms of:
  - independent Poisson random measures $\sum_{u \in \mathscr{P}_j} \delta_{(u, x_u)}$ with intensities $(1-p)\mathrm{d}t \otimes \Lambda(\,\cdot \cap A_j)$.
  - independent collections $(D_u)_{u \in \mathscr{P}_j}$ of i.i.d. copies of D.

Finally, set $\mathscr{P} := \sum_j \mathscr{P}_j$. Now we are in position to introduce NRPPPs with sigma-finite characteristic measures:

Definition 4.9. (Noise Reinforced Poisson Point Process - NRPPP)
The random measure $\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) := \sum_{j\in\mathcal{I}} \hat{\mathcal{N}}_j(\mathrm{d}s, \mathrm{d}x)$ *is called a reinforced Poisson point process with reinforcement (or memory) parameter* p *and characteristic measure* Λ*. Moreover,* $\hat{\mathcal{N}}$ *writes*

$$\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) = \sum_{u\in\mathscr{P}} \sum_{t\in D_u} \delta_{(ue^t, x_u)}. \tag{4.39}$$

From the identity in the previous display and recalling that the first element of D is just 0, the measure $\hat{\mathcal{N}}$ naturally decomposes as $\hat{\mathcal{N}} = \mathcal{N}' + \mathcal{N}''$, where $\mathcal{N}'$ is a PPP with intensity $(1-p)\mathrm{d}t \otimes \Lambda$. Moreover, the following properties readily follow from our construction:

Lemma 4.10. *Let* $\hat{\mathcal{N}}$ *be a NRPPP with characteristic measure* Λ *and reinforcement parameter* p*.*

(i) *If* $A \in \mathscr{B}(\mathbb{R})$*, the restriction* $\mathbb{1}_A(x)\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x)$ *is a NRPPP with characteristic measure* $\mathbb{1}_A\Lambda$ *and parameter* p*.*

(ii) *If* $A_1, A_2 \in \mathscr{B}(\mathbb{R})$ *are disjoints, then* $\mathbb{1}_{A_1}(x)\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x)$*,* $\mathbb{1}_{A_2}(x)\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x)$ *are independent.*

(iii) *If* $\hat{\mathcal{N}}_1$*,* $\hat{\mathcal{N}}_2$ *are independent NRPPPs with respective characteristic measures* Λ_1*,* Λ_2 *and same reinforcement parameter* p*, then* $\hat{\mathcal{N}}_1 + \hat{\mathcal{N}}_2$ *is a NRPPP with characteristic measure* $\Lambda_1 + \Lambda_2$ *and parameter* p*.*

The following lemma shows that the intensity measure of a NRPPP with characteristic measure Λ and parameter p, coincides with the one of a PPP with characteristic measure Λ.

Lemma 4.11. *Let* $\hat{\mathcal{N}}$ *be a NRPPP with characteristic measure* Λ *and reinforcement parameter* p*. For any measurable* $f : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$*, we have* $\mathbb{E}[\langle f, \hat{\mathcal{N}}\rangle] = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \Lambda(\mathrm{d}x) f(s,x)$*.*

Proof. Suppose first that $\Lambda(\mathbb{R}) < \infty$ and recall from (4.34) that for fixed $u \in \mathbb{R}_+$, the atoms of the measure $\sum_{t\in D} \delta_{ue^t}$ are precisely the jumps of the time-changed Yule process (4.34). Hence, if $\sum_{u\in\mathscr{P}} \delta_{(u,x_u)}$ is a Poisson random measure with intensity $(1-p)\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$ and $(Z^{(u)})_{u\in\mathscr{P}}$ is an independent collection with law $\mathbf{Z}$, it is then clear from our construction in the finite case (4.38) that we can write

$$\mathbb{E}\left[\hat{\mathcal{N}}(0,T] \times A\right] = \mathbb{E}\left[\sum_{u\in\mathscr{P}} \mathbb{1}_{\{u\leqslant T\}} Z^{(u)}_{\{p(\ln(T)-\ln(u))\}} \mathbb{1}_{\{x_u\in A\}}\right],$$

where the random measure $\sum_{u\in\mathscr{P}} \delta_{(u,x_u,Z^{(u)})}$ is Poisson with intensity $(1-p)\mathrm{d}t \otimes \Lambda \otimes \mathbf{Z}$. Consequently, recalling that $E[Z_t] = e^t$, by Campbell's formula we obtain that

$$\mathbb{E}\left[\hat{\mathcal{N}}(0,T] \times A\right] = T \cdot \Lambda(A),$$

and we deduce that the intensity measure of $\hat{\mathcal{N}}$ is given by $\mathrm{d}t \otimes \Lambda$. When $\Lambda(\mathbb{R}) = \infty$, we can proceed by superposition. □

We now identify the law of $\hat{\mathcal{N}}$ by computing its exponential functionals.

Proposition 4.12. *Let* $\hat{\mathcal{N}}$ *be a NRPPP with characteristic measure* Λ *and reinforcement parameter* p*.*

(i) *For every measurable* $f : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ *and* $t \geqslant 0$ *we have*

$$\mathbb{E}\left[\exp\left\{-\int_{(0,t]\times\mathbb{R}} f(s,x)\hat{\mathcal{N}}(\mathrm{d}s,\mathrm{d}x)\right\}\right] \tag{4.40}$$
$$= \exp\left\{-t(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[1-\exp\left(-\int_0^1 f(st,x)\mathrm{d}Y(s)\right)\right]\right\}.$$

(ii) *If we no longer assume that* f *is non-negative, under the condition* $\int_0^t \mathrm{d}s \int_{\mathbb{R}} \Lambda(\mathrm{d}x)|f(s,x)| < \infty$ *we have:*

$$\mathbb{E}\left[\exp\left\{i\int_{(0,t]\times\mathbb{R}} f(s,x)\hat{\mathcal{N}}(\mathrm{d}s,\mathrm{d}x)\right\}\right] \tag{4.41}$$
$$= \exp\left\{t(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[\exp\left(i\int_0^1 f(st,x)\mathrm{d}Y(s)\right)-1\right]\right\}.$$

Proof. (i) We start by considering the finite case $\Lambda(\mathbb{R}) < \infty$ and we make use of the notations introduced in (4.37); for instance, recall that $\langle \hat{\mathcal{N}}, f\rangle = \langle \mathcal{L}^x, f\circ(\exp, \mathrm{Id})\rangle$. We start showing the result for f of the form $f(s,x) = h(s)g(x)$, for non-negatives $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $g : \mathbb{R} \mapsto \mathbb{R}^+$, in which case we can write

$$\langle \mathcal{L}^x, (h\circ\exp)g\rangle = \sum_{u\in\mathscr{E}}\sum_{t\in D_u} h\circ\exp(u+t)g(x_u) = \sum_{u\in\mathscr{E}} g(x_u)\langle \mathcal{T}_u D_u, h\circ\exp\rangle. \tag{4.42}$$

Now, we deduce from the formula for the Laplace transform of Poisson integrals and a change of variable that

$$-\log\mathbb{E}\left[e^{-\langle\mathcal{L}^x,(h\circ\exp)g\rangle}\right] = (1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\int_{\mathbb{R}^+}\mathrm{d}u\int_{\mathcal{M}_c} 1-e^{-g(x)\langle\mathcal{T}_{\log(u)}\mu,h\circ\exp\rangle}\mathbb{D}(\mathrm{d}\mu).$$

If we now replace h by $h\mathbb{1}_{\{\cdot\leqslant t\}}$, making use of the equivalent identities (4.36) and (4.32), we obtain that the previous display writes:

$$t\cdot(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[1-e^{-g(x)\int_0^1 h(st)\mathrm{d}Y(s)}\right],$$

proving the claim. Now, still under the hypothesis $\Lambda(\mathbb{R}^+) < \infty$, fix arbitrary $\alpha_{i,j} \in \mathbb{R}^+$, consider $0 = t_1 < \cdots < t_{k+1} < t$ as well as disjoint subsets $A_1, \ldots, A_n$ of $\mathbb{R}^+$. Further, suppose that f is of the form

$$f(s,x) := \sum_{j=1}^n\sum_{i=1}^k \alpha_{i,j}\mathbb{1}_{(t_i,t_{i+1}]}(s)\mathbb{1}_{A_j}(x) \qquad \text{and write} \quad g_j(s,x) := \sum_{i=1}^k \alpha_{i,j}\mathbb{1}_{(t_i,t_{i+1}]}(s)\mathbb{1}_{A_j}(x). \tag{4.43}$$

Recall from Lemma 4.10 that the restrictions $\mathbb{1}_{A_1}\hat{\mathcal{N}}, \ldots, \mathbb{1}_{A_n}\hat{\mathcal{N}}$ are independent NRPPPs with respective characteristic measures $\Lambda(\cdot\cap A_i)$. By independence and applying the previous case to each g_j, we deduce that

$$\mathbb{E}\left[\exp\left\{-\langle\hat{\mathcal{N}},\mathbb{1}_{\{\cdot\leqslant t\}}f\rangle\right\}\right]$$
$$=\prod_{j=1}^{n}\mathbb{E}\left[\exp\left\{-\langle\hat{\mathcal{N}},\mathbb{1}_{\{\cdot\leqslant t\}}g_j\rangle\right\}\right]$$
$$=\prod_{j=1}^{n}\exp\left\{t(1-p)\Lambda(A_j)\mathbb{E}\left[1-\exp\left\{-\int_0^1\sum_{i=1}^{k}\alpha_{i,j}\mathbb{1}_{(t_i<st\leqslant t_{i+1}]}(s)\mathrm{d}Y(s)\right\}\right]\right\}$$
$$=\exp\left\{t(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[1-\exp\left\{-\int_0^1 f(st,x)\mathrm{d}Y(s)\right\}\right]\right\},$$

and once again we recover (4.40). Finally, if f is non-negative and bounded with support in $[0,t]\times\mathbb{R}$, it can be approximated by a bounded sequence of functions (f_n) of the form (4.43), the convergence holding $\mathrm{d}t\Lambda(\mathrm{d}x)$ a.e. For each n, we have

$$\mathbb{E}\left[\exp\left\{-\langle\hat{\mathcal{N}},f_n\rangle\right\}\right]=\exp\left\{t(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[1-\exp\left\{-\int_0^1 f_n(st,x)\mathrm{d}Y(s)\right\}\right]\right\},\quad(4.44)$$

and by Lipschitz-continuity, it follows that

$$\mathbb{E}\left[|\exp\{-\langle\hat{\mathcal{N}},f\rangle\}-\exp\{-\langle\hat{\mathcal{N}},f_n\rangle\}|\right]\leqslant\mathbb{E}\left[\int_{[0,t]\times\mathbb{R}}|f(s,x)-f_n(s,x)|\hat{\mathcal{N}}(\mathrm{d}s,\mathrm{d}x)\right]$$
$$=\int_0^t\mathrm{d}s\int_{\mathbb{R}}\Lambda(\mathrm{d}x)|f(s,x)-f_n(s,x)|\to 0\ \text{ as } n\uparrow\infty.$$

In the last equality we used Lemma 4.11. From the same arguments we also obtain that

$$\int\Lambda(\mathrm{d}x)\mathbb{E}\left[|e^{-\int_0^1 f(st,x)\mathrm{d}Y(s)}-e^{-\int_0^1 f_n(st,x)\mathrm{d}Y(s)}|\right]\leqslant\int\Lambda(\mathrm{d}x)\mathbb{E}\left[\int_0^1|f(st,x)-f_n(st,x)|\mathrm{d}Y(s)\right]$$
$$=(1-p)^{-1}\int_0^1\mathrm{d}s\int_{\mathbb{R}}\Lambda(\mathrm{d}x)|f(st,x)-f_n(st,x)|$$

which converges to 0 as $n\uparrow\infty$. Now, we deduce from taking the limit as $n\uparrow\infty$ in (4.44) that the identity (4.40) also holds for f.

If we suppose that $\Lambda(\mathbb{R})=\infty$, the proof follows by superposition. Namely, with the same notation used for constructing (4.39), the random measures $(\hat{\mathcal{N}}_j)_{j\in\mathcal{I}}$ are independent NRPPPs with respective finite characteristic measures $\Lambda(\cdot\cap A_j)$ and by definition we have $\hat{\mathcal{N}}=\sum_j\hat{\mathcal{N}}_j$. From the formula for the Laplace transform we just proved in the finite case and independence it follows that

$$\mathbb{E}\left[e^{-\langle\hat{\mathcal{N}},f\mathbb{1}_{\{\cdot\leqslant t\}}\rangle}\right]=\prod_{j\in\mathcal{I}}\mathbb{E}\left[e^{-\langle\hat{\mathcal{N}}_j,f\mathbb{1}_{\{\cdot\leqslant t\}}\rangle}\right]$$
$$=\prod_{j\in\mathcal{I}}\exp\left\{-t\cdot(1-p)\int_{A_j}\Lambda(\mathrm{d}x)\mathbb{E}\left[\left(1-e^{-\int_0^1 f(st,x)\mathrm{d}Y(s)}\right)\right]\right\},$$

proving (i). Now (ii) follows from similar arguments, by making use of the formula for the characteristic function for Poissonian integrals and the inequality $|e^{ib}-e^{ia}|\leqslant|a-b|$ for $a,b\in\mathbb{R}$, we omit the details. □

The following result is the reinforced analogue of the well known characterisation result for Poisson point processes. The arguments we use are similar to the ones in the non-reinforced case.

Proposition 4.13. *Let $\hat{\mathcal{N}}$ be a point process in $\mathbb{R}^+ \times \mathbb{R}$ and for any Borelian $A \subset \mathbb{R}$, set*

$$\hat{N}_A(t) := \hat{\mathcal{N}}([0,t] \times A), \quad t \geqslant 0.$$

Then, $\hat{\mathcal{N}}$ is a noise reinforced Poisson point process with characteristic measure Λ and parameter p if and only if the two following conditions are satisfied:

(i) *For any Borelian A with $\Lambda(A) < \infty$, the process $\hat{N}_A$ is a noise reinforced Poisson process with rate $\Lambda(A)$ and reinforcement parameter p.*

(ii) *If $A_1, \dots A_k$ are disjoint Borelians with $\Lambda(A_i) < \infty$ for all $i \in \{1, \dots, k\}$, the processes $\hat{N}_{A_1}, \dots, \hat{N}_{A_k}$ are independent.*

Proof. First, let us prove that NRPPP do satisfy (i) and (ii). Remark that (ii) is just a consequence of Lemma 4.10 - (ii) and we focus on (i). Fix A as in (i) as well as times $0 < t_1 < \cdots < t_k \leqslant t$, and we proceed by computing the characteristic function of the finite dimensional distributions of $\hat{N}_A$. This can now be done by considering the function $f(s,x) := \sum_{i=1}^k \lambda_i \mathbb{1}_{\{s \leqslant t_i\}} \mathbb{1}_A(x)$ and applying the exponential formula (4.41), yielding

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{i \sum_{i=1}^k \lambda_i \hat{N}_A(t_i)\right\}\right] \\
&= \exp\left\{t(1-p)\int_{\mathbb{R}} \Lambda(\mathrm{d}x)\mathbb{E}\left[\exp\left(i\sum_{i=1}^k \int_0^1 \lambda_i \mathbb{1}_{\{st \leqslant t_i\}} \mathbb{1}_A(x) \mathrm{d}Y(s)\right) - 1\right]\right\} \\
&= \exp\left\{t(1-p)\Lambda(A)\mathbb{E}\left[\left(\exp\left\{i\sum_{i=1}^k \lambda_i Y(t_i/t)\right\} - 1\right)\right]\right\}.
\end{aligned}$$

Recalling the identity (4.30), we deduce that $\hat{N}_A$ is a noise reinforced Poisson process with rate $\Lambda(A)$ and reinforcement p.

Now, we argue that if $\hat{\mathcal{N}}$ is a random measure satisfying (i) and (ii), then it is a NRPPP. We will establish this claim by showing that $\hat{\mathcal{N}}$ satisfies the exponential formula (4.41). First, observe that (i) implies that $\mathbb{E}[\hat{N}_A(t)] = t\Lambda(A)$, for example by making use of Lemma 4.11 and the fact that if $\hat{\mathcal{M}}$ is a NRPPP with characteristic measure Λ and parameter p, then $(\hat{\mathcal{M}}([0,t] \times A) : t \geqslant 0)$ is a reinforced Poisson process with rate $\Lambda(A)$ and parameter p. We deduce by a monotone class argument that $\hat{\mathcal{N}}$ satisfies, for any measurable $f : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$, the identity:

$$\mathbb{E}\left[\int_{[0,t]\times\mathbb{R}} f(s,x)\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x)\right] = \int_0^t \mathrm{d}s \int \Lambda(\mathrm{d}x) f(s,x). \tag{4.45}$$

Still for A as in (i) and for an arbitrary collection of times $0 = t_1 < t_2 < \cdots < t_{k+1} < t$, we set

$$g(s,x) := \sum_{i=1}^k \alpha_i \mathbb{1}_{(t_i, t_{i+1}]}(s) \mathbb{1}_A(x). \tag{4.46}$$

Since by hypothesis $(\hat{N}_A(t))_{t\in\mathbb{R}^+}$ is a NRPP with rate $\Lambda(A)$, by the formula (4.30) for the characteristic function of the finite dimensional distributions of reinforced Poisson processes, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{i\langle\hat{\mathcal{N}}, \mathbb{1}_{\{\cdot\leqslant t\}}g\rangle\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{i\sum_{i=1}^{k}\alpha_i(N_A(t_{i+1}) - N_A(t_i))\right\}\right] \\
&= \exp\left\{t(1-p)\Lambda(A)\mathbb{E}\left[\exp\left(i\sum_{i=1}^{k}\alpha_i(Y(t_{i+1}/t) - Y(t_i/t))\right) - 1\right]\right\} \\
&= \exp\left\{t(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[\exp\left(i\int_0^1\sum_{i=1}^{k}\alpha_i\mathbb{1}_{\{t_i<st\leqslant t_{i+1}\}}\mathbb{1}_A(x)\mathrm{d}Y(s)\right) - 1\right]\right\}.
\end{aligned}$$

Remark that this is precisely the identity (ii) of Proposition 4.12 for our choice of g. Making use of the independence hypothesis of $\hat{N}_{A_1}, \dots \hat{N}_{A_k}$ for disjoints $A_1, \dots, A_k$ with $\Lambda(A_i) < \infty$, we can also show that the identity holds for f as in (4.43) for such collection of sets. Now, if f is non-negative, bounded and supported on $[0,t]\times A$ with $\Lambda(A) < \infty$, making use of (4.45), we can proceed as in (4.44) for the proof of Proposition 4.12, approximating f by a bounded sequence of the form (4.43), and show that the exponential formula (4.41) still holds. The general case follows by sigma finiteness of Λ and we deduce that $\hat{\mathcal{N}}$ is a NRPPP with the desired parameters. □

4.4.3 Proof of Theorem 4.5 and compensator of the jump measure

Let us now establish Theorem 4.5. Remark that paired with Proposition 4.13, it entails that the role of the counting process of jumps $\Delta\hat{\xi}_s \in A$ for fixed $A \in \mathscr{B}(\mathbb{R})$ is played precisely by noise-reinforced Poisson processes, in analogy with the non-reinforced setting.

Proof of Theorem 4.5. The result will follow as soon as we establish (i) and (ii) of Proposition 4.13 for

$$\hat{N}_A(t) := \#\{(s, \Delta\hat{\xi}_s) \in [0,t]\times A\}, \qquad t \geqslant 0, \tag{4.47}$$

where A is an arbitrary Borelian satisfying $\Lambda(A) < \infty$. By the identity in distribution (4.13), we can restrict our arguments to the unit interval and hence we can make use of the explicit construction of NRLPs in $[0,1]$ that we recalled in Section 4.2.3, in terms of Yule-Simon series.

Denote by $\mathcal{M} := \sum_i \delta_{(x_i,Y_i)}$ the Poisson random measure with intensity $(1-p)\Lambda\otimes\mathbb{Q}$ and recall the discussion following Theorem 4.3. If (x_i, Y_i) is an atom of $\mathcal{M}$, then at time $U_i = \inf\{t \geqslant 0 : Y_i(t) = 1\}$, the process $\hat{\xi}$ performs the jump x_i for the first time, i.e. $\Delta\hat{\xi}_{U_i} = x_i$ and this precise jump x_i is repeated in the interval $[0,1]$ at each jump time of Y_i. It follows that for any $f : \mathbb{R} \mapsto \mathbb{R}^+$ we have:

$$\sum_{s\leqslant t} f(\Delta\hat{\xi}_s) = \sum_i f(x_i)Y_i(t), \tag{4.48}$$

and in particular, we get:

$$\hat{N}_A(t) = \sum_i \mathbb{1}_{\{x_i\in A\}}Y_i(t).$$

Hence, by the independence property of Poisson random measures, the processes $\hat{N}_{A_1}, \dots, \hat{N}_{A_n}$ are independent as soon as $A_i \cap A_j = \varnothing$ for all $i \neq j$. Now, if we fix $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, $0 \leqslant t_1 <$

$\cdots < t_k \leqslant 1$, we deduce from the formula for the characteristic function for Poisson integrals the equality:

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j\hat{N}_A(t_j)\right\}\right] = \exp\left\{(1-p)\Lambda(A)\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j Y(t_j)\right\}-1\right]\right\}.$$

Comparing with (4.30), we get that the right-hand side in the previous display is precisely the characteristic function of the finite dimensional distributions at times $t_1, \dots, t_k$ of a reinforced Poisson process with rate $\Lambda(A)$ and parameter p. □

Recalling the explicit construction of NRPPPs from Definition 4.9, we stress that Theorem 4.5 formalises the idea that the jumps of NRLPs are jumps that are repeated through time, similarly to the dynamics of noise reinforced random walks – we refer to the beginning of Section 4.5.2 for a brief introduction to the later. Our terminology and notation for the reinforced measure $\hat{\mu}$ can now be justified by the following: if μ is the jump measure of ξ, the counting process $(\mu([0,t]\times A) : t \geqslant 0)$ is a Poisson process with rate $\Lambda(A)$ while $(\hat{\mu}([0,t]\times A) : t \geqslant 0)$ is a reinforced Poisson process with rate $\Lambda(A)$. Said otherwise, the following identity holds in distribution:

$$\hat{\mu}([0,\cdot\,]\times A) \overset{\mathscr{L}}{=} \widehat{\mu([0,\cdot\,]\times A)}. \tag{4.49}$$

Now that the key result of the section has been established, we continue our study of the jump process of NRLPs. In this direction, we start by briefly recalling notions of semi-martingale theory that will be needed. Let X be a semimartingale defined on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$. Its jump measure μ_X is an integer valued random measure on $(\mathbb{R}^+ \times \mathbb{R}, \mathscr{B}(\mathbb{R}^+)\otimes\mathscr{B}(\mathbb{R}))$, in the sense of [54, Chapter II-1.13]. Denote the predictable sigma-field on $\Omega\times\mathbb{R}^+$ by $\mathscr{P}\mathrm{r}$. If H is a $\mathscr{P}\mathrm{r}\otimes\mathscr{B}(\mathbb{R}^+)$-measurable function, we simply write $H * \mu_X$ for the process defined at each $t \in \mathbb{R}^+$ as

$$(H*\mu_X)_t(\omega) := \int_{(0,t]\times\mathbb{R}} \mu_X(\omega; \mathrm{d}s, \mathrm{d}x) H_s(\omega; x), \qquad \text{if} \quad \int_{(0,t]\times\mathbb{R}} \mu_X(\omega; \mathrm{d}s, \mathrm{d}x)|H_s(\omega;x)| < \infty, \tag{4.50}$$

and ∞ otherwise. Both notations for the integral will be used indifferently. Further, we denote by $\mathscr{A}^+$ the class of increasing, adapted rcll finite-variation processes (A_t), with $A_0 = 0$ such that $\mathbb{E}[A_\infty] < \infty$, and by $\mathscr{A}^+_{loc}$ its localisation class. The jump measure μ_X posses a predictable compensator, this is, a random measure $\mu_X^{\mathtt{p}}$ on $(\mathbb{R}^+\times\mathbb{R}, \mathscr{B}(\mathbb{R}^+)\otimes\mathscr{B}(\mathbb{R}))$ unique up to a $\mathbb{P}$-null set, characterised by being the unique predictable random measure (in the sense of [54, Chapter II-1.6]) satisfying that for any non-negative $H \in \mathscr{P}\mathrm{r}\otimes\mathscr{B}(\mathbb{R})$, the equality

$$\mathbb{E}\left[(H*\mu_X)_\infty\right] = \mathbb{E}\left[(H*\mu_X^{\mathtt{p}})_\infty\right]$$

holds. Equivalently, for any $H \in \mathscr{P}\mathrm{r}\otimes\mathscr{B}(\mathbb{R})$ such that $|H|*\mu_X \in \mathscr{A}^+_{loc}$, the process $|H|*\mu_X^{\mathtt{p}}$ belongs to $\mathscr{A}^+_{loc}$ and $H*\mu_X^{\mathtt{p}}$ is the predictable compensator of $H*\mu_X$. Said otherwise, $H*\mu_X - H*\mu_X^{\mathtt{p}}$ is a local martingale.

Recall that by Proposition 4.4, the process $\hat{\xi}$ is a semimartingale. Hence, we can consider $\hat{\mu}^{\mathtt{p}}$, the predictable compensator of its jump measure $\hat{\mu}$, and our purpose is to identify explicitly $\hat{\mu}^{\mathtt{p}}$. In contrast, it might be worth mentioning that if ξ is a Lévy process with Lévy measure Λ, the compensator of its jump measure μ is just the deterministic measure $\mu^{\mathtt{p}} = \mathrm{d}t\otimes\Lambda(\mathrm{d}x)$. The first step consists in observing the following:

Lemma 4.14. *Let $A \in \mathscr{B}(\mathbb{R})$ be a Borel set that doesn't intersect some open neighbourhood of the origin. If we denote by $(\mathscr{F}_t^A)$ the natural filtration of $\hat{N}_A$, then the process $M_A = (M_A(t))_{t\in\mathbb{R}^+}$ defined as $M_A(0) = 0$ and*

$$M_A(t) = t^{-p}\left(\hat{N}_A(t) - t\Lambda(A)\right), \qquad t \geqslant 0,$$

is a finite variation $(\mathscr{F}_t^A)$-martingale.

Remark that this is just a special case of Proposition 4.4 for a Lévy measure of the form $\Lambda(A)\delta_1$ with $q = 0$. Now we can state:

Proposition 4.15. (Compensation formula)
Denote by $(\mathscr{F}_t)$ the natural filtration of $\hat{\xi}$ and by $\hat{\mu}$ its jump measure. The predictable compensator $\hat{\mu}^{\mathrm{p}}$ of $\hat{\mu}$ is given by

$$\hat{\mu}^{\mathrm{p}}(\omega; \mathrm{d}t, \mathrm{d}x) = (1-p)\mathrm{d}t \otimes \Lambda(\mathrm{d}x) + p\frac{\mathrm{d}t}{t}\mathscr{E}_t(\omega; \mathrm{d}x), \tag{4.51}$$

where $\mathscr{E}_t(\mathrm{d}x) = \sum_{s<t}\delta_{\Delta\hat{\xi}_s}(\mathrm{d}x)$ is the empirical measure of jumps that occurred strictly before time t.

Consequently, for any predictable process $H \in \mathscr{P}\mathrm{r} \otimes \mathscr{B}(\mathbb{R})$ such that $|H| * \hat{\mu} \in \mathscr{A}_{loc}^+$, we have $|H| * \hat{\mu}^{\mathrm{p}} \in \mathscr{A}_{loc}^+$ and the following process is a local martingale:

$$M_t = \sum_{s\leqslant t} H_s(\cdot\,, \Delta\hat{\xi}_s) - (1-p)\int_0^t \mathrm{d}s \int_{\mathbb{R}} \Lambda(\mathrm{d}x) H_s(\cdot\,, x) - \int_0^t \sum_{r<s} H_s(\cdot\,, \Delta\hat{\xi}_r)\frac{p}{s}\mathrm{d}s, \quad t \geqslant 0. \tag{4.52}$$

The first compensating term appearing in (4.52) is compensating innovations, i.e. atoms appearing for the first time, while the second one should be interpreted as the compensator of the memory part of $\hat{\mu}$. Notice that Proposition 4.15 holds if $p = 0$. Indeed, in that case $\hat{\xi}$ is a Lévy process and its jump process μ is the Poisson point process (4.28). The compensator (4.51) is just the deterministic compensator $\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$ for the Poisson point processes with characteristic measure Λ and in (4.52) we recover the celebrated compensation formula, see e.g. [17, Chapter 1]. Remark that since the intensity of both μ and $\hat{\mu}$ is $\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$, we have, for both X a Lévy process and its associated NRLP, the equality $\mathbb{E}\left[\sum_{s\leqslant t} f(s, \Delta X_s)\right] = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \Lambda(\mathrm{d}x) f(s,x)$ for any $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}^+$. When $X := \xi$, by the compensation formula, this identity holds also if we replace f by a non-negative predictable process $H \in \mathscr{P}\mathrm{r} \otimes \mathscr{B}(\mathbb{R})$, viz.

$$\mathbb{E}\left[\sum_{s\leqslant t} H_s(\cdot, \Delta\xi_s)\right] = \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{\mathbb{R}} \Lambda(\mathrm{d}x) H_s(\cdot, x)\right]. \tag{4.53}$$

However, we point out that if we replace in (4.53) the Lévy process by its reinforced version $\hat{\xi}$, the identity no longer holds. Indeed, if such formula was satisfied, the exact same proof for the exponential formula of PPPs of XII-1.12 in [81] would hold in our reinforced setting, and since random measures are characterised by their Laplace functional, this would lead us to the conclusion that the law of $\hat{\mu}$ coincides with the law of μ.

Proof. (i) In order to establish (4.51), by (i) of Theorem II-1.8 of [54], it suffices to show that for any nonnegative predictable process $H \in \mathscr{P}\mathrm{r} \otimes \mathscr{B}(\mathbb{R})$,

$$\mathbb{E}\left[(H * \hat{\mu})_\infty\right] = \mathbb{E}\left[(H * \hat{\mu}^{\mathrm{p}})_\infty\right], \tag{4.54}$$

and the first step consists in showing the result for deterministic $H_s(\omega, x) = \mathbb{1}_B(x)$ for $B \in \mathscr{B}(\mathbb{R})$. Maintaining the notation introduced in Lemma 4.5 for the process $\hat{N}_B$, consider B an arbitrary interval not containing a neighbourhood of the origin as well as the associated martingale,

$$M_B(t) = t^{-p}\hat{N}_B^{(c)}(t) = t^{-p}\left(\hat{N}_B(t) - t\Lambda(B)\right).$$

Integrating by parts, we get

$$t^p M_B(t) = \int_0^t s^p \mathrm{d}M_B(s) + \int_0^t pM_B(s)s^{p-1}\mathrm{d}s,$$

and consequently,

$$\begin{aligned}\hat{N}_B(t) - t\Lambda(B) &= \int_0^t s^p \mathrm{d}M_B(s) + \int_0^t pM_B(s)s^{p-1}\mathrm{d}s \\ &= \int_0^t s^p \mathrm{d}M_B(s) + \int_0^t \left(\hat{N}_B(s) - s\Lambda(B)\right) ps^{-1}\mathrm{d}s.\end{aligned}$$

Said otherwise,

$$\hat{N}_B(t) - t(1-p)\Lambda(B) - \int_0^t \hat{N}_B(s)ps^{-1}\mathrm{d}s = \int_0^t s^p \mathrm{d}M_B(s),$$

is a martingale. Since $(N_B(\omega; s))_{s\in\mathbb{R}^+}$ and $(N_B(\omega; s-))_{s\in\mathbb{R}^+}$ differ in a set of null Lebesgue measure, the equality still holds replacing $\int_0^t \hat{N}_B(s)ps^{-1}\mathrm{d}s$ by $\int_0^t \hat{N}_B(s-)ps^{-1}\mathrm{d}s$ and we obtain precisely (4.52) for $H_s(\omega, x) = \mathbb{1}_B(x)$. Now we can proceed as in the proof of II-2.21 from [54]. Concretely, pick any positive Borel-measurable deterministic function $h = h(x)$, $x \in \mathbb{R}$ such that $h * \hat{\mu} - h * \hat{\mu}^{\mathrm{p}}$ is a local martingale and let T be an arbitrary stopping time. With the same terminology as in I.1.22 of [54] denote by $[\![0, T]\!]$ the subset of $\Omega \times \mathbb{R}^+$ defined by

$$[\![0, T]\!] = \{(\omega, s) : 0 \leqslant s \leqslant T(\omega)\}.$$

In particular, $(h * \hat{\mu})^T = \mathbb{1}_{[\![0,T]\!]} h * \hat{\mu}$ where the process $\mathbb{1}_{[\![0,T]\!]}$ is predictable (since left continuous) and moreover, by Theorem I 2.2 of [54], the sigma field generated by the collection

$$\{A \times \{0\} \text{ where } A \in \mathscr{F}_0, \text{ and } [\![0, T]\!] \text{ where } T \text{ is any } (\mathscr{F}_t)\text{-stopping time }\}$$

is precisely the predictable sigma field $\mathscr{P}$r. Then, if (T_n) is a localising sequence for the local martingale $h * \hat{\mu} - h * \hat{\mu}^{\mathrm{p}}$, it follows from Doob's stopping theorem that for each n,

$$\mathbb{E}\left[(h * \hat{\mu})_\infty^{T\wedge T_n}\right] = \mathbb{E}\left[(h * \hat{\mu}^{\mathrm{p}})_\infty^{T\wedge T_n}\right].$$

Consequently, taking the limit as $n \uparrow \infty$, we deduce by monotone convergence that

$$\mathbb{E}\left[(\mathbb{1}_{[\![0,T]\!]} h * \hat{\mu})_\infty\right] = \mathbb{E}\left[(\mathbb{1}_{[\![0,T]\!]} h * \hat{\mu}^{\mathrm{p}})_\infty\right]$$

which in turn implies that (4.54) holds for any predictable process $H = 1_B 1_{[\![0,T]\!]}$ where B is any closed interval not containing the origin and T an arbitrary stopping time. Now the claim follows by a monotone class argument. □

We close our discussion on the jump process of NRLPs with the property at the heart of the infinite divisibility of $\hat{\xi}$ as a stochastic process, a topic that will be studied in Section 4.6.2. We claim that, for $A \in \mathscr{B}(\mathbb{R})$ with $\Lambda(A) < \infty$ the point process of jumps

$$\nu_A(\mathrm{d}s) = \sum_s \mathbb{1}_{\{\Delta\hat{\xi}_s \in A\}} \delta_s, \tag{4.55}$$

is an infinitely divisible point process. More precisely, the measure ν_A is a reinforced Poisson point process $\hat{\mathscr{P}}$ with rate $\Lambda(A)$ in $\mathbb{R}^+$ and if we consider n independent copies $\nu_A^1, \dots, \nu_A^n$ of the reinforced Poisson process (4.55) but with rate $n^{-1}\Lambda(A)$, we have the equality in distribution

$$\nu_A \overset{\mathscr{L}}{=} \nu_A^1 + \cdots + \nu_A^n. \tag{4.56}$$

To see this, consider $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ a positive function with support in $[0, t]$, and observe that

$$\langle f, \nu_A \rangle = \sum_{s \leqslant t} \mathbb{1}_A(\Delta\xi_s) f(s).$$

Now the claim follows by computing the Laplace functional of ν_A, ν_A^i respectively, by applying the exponential formula (4.40) and from comparing with (4.32). For a more detailed discussion on infinitely divisible point processes we refer to page 5 of [73].

4.5 Weak convergence of the pair of skeletons

Before stating the first result of the section, let us briefly recall the statement of the Lévy-Itô synthesis for Lévy processes: a Lévy process ξ with triplet (a, q^2, Λ) can be written as $\xi = \xi^{(1)} + \xi^{(2)} + \xi^{(3)}$, where $\xi^{(1)} = (at + qB_t : t \geqslant 0)$ is a Brownian motion with drift while $\xi^{(2)} + \xi^{(3)}$ is a purely discontinuous process that can be explicitly built from the jump measure μ defined in (4.28). More precisely, if we denote by $\mu^{(sc)}$ the compensated measure of jumps $\mu^{(sc)} = \mu - \mathrm{d}t\Lambda(\mathrm{d}x)$, we can write

$$\xi_t = at + qB_t + \int_{[0,t]\times(-1,1)^c} x\mu(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times(-1,1)} x\mu^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0. \tag{4.57}$$

The reinforced Lévy-Itô synthesis, which is the first main result of the section, states that the analogous result holds for NRLPs where now, the PPP μ in (4.57) has been replaced by the reinforced version $\hat{\mu}$, and the Brownian motion B by its reinforced version $\hat{B}$ (if $p < 1/2$). More precisely, after properly defining the "space-compensated" measure $\hat{\mu}^{(sc)}$, we prove:

Theorem 4.16. (Reinforced Itô's synthesis)
Let $\hat{\mu}$ be the jump measure of a NRLP $\hat{\xi}$ of characteristics (a, q^2, Λ, p). Then, a.s. we have

$$\hat{\xi}_t = at + q\hat{B}_t + \int_{[0,t]\times(-1,1)^c} x\hat{\mu}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times(-1,1)} x\hat{\mu}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0,$$

for some noise reinforced Brownian motion $\hat{B}$, with the convention that if $p \geqslant 1/2$ the process $\hat{B}$ is null. Moreover, the integrals in the previous display are NRLPs with respective characteristics $(0, 0, \mathbb{1}_{(-1,1)^c}\Lambda, p)$, $(0, 0, \mathbb{1}_{(-1,1)}\Lambda, p)$.

Remark 4.17. *Beware of the notation, $\hat{\mu}^{(sc)}$ stands for the space-compensated jump measure $\hat{\mu}$ and should not be confused with the time-compensated measure $(\mu - \mu^p)$ in the sense of [54, Chapter II-1.27]. For instance, we stress that $\hat{\xi}^{(3)}$ is not a local martingale. Remark that for Lévy processes, the time and space compensation of its jump measure coincide, since the compensating measure is the same.*

After proving this result, we start settling the ground for the main result of the section. First, making use of Theorem 4.16, we define the joint law, of a Lévy process and its reinforced version, by introducing an appropriate coupling $(\xi, \hat{\xi})$. We then characterise its law by computing the characteristic function of its finite dimensional distributions:

Proposition 4.18. *There exists a pair $(\xi, \hat{\xi})$, where $\hat{\xi}$ has the law of a NRLP with characteristics (a, q^2, Λ, p), with law determined by the following: for all $k \geqslant 1$, $\lambda_1, \dots, \lambda_k$, $\beta_1, \dots \beta_k$ real numbers, and $0 < t_1 < \dots < t_k \leqslant t$, we have*

$$\mathbb{E}\left[\exp\left\{i \sum_{j=1}^{k} \left(\lambda_j \xi_{t_j} + \beta_j \hat{\xi}_{t_j}\right)\right\}\right] =$$

$$\exp\left\{t \cdot p\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \lambda_j \mathbb{1}_{\{U \leqslant t_j/t\}}\right)\right] + t \cdot (1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} (\lambda_j \mathbb{1}_{\{Y(t_j/t) \geqslant 1\}} + \beta_i Y(t_j/t))\right)\right]\right\}, \tag{4.58}$$

where U is a uniform random variable in $[0,1]$. A pair of processes with such distribution will always be denoted by $(\xi, \hat{\xi})$.

Now, we connect the distribution of the pair $(\xi, \hat{\xi})$ with the discrete setting. In this direction, consider the Lévy process ξ and for each fixed $n \in \mathbb{N}$ we set

$$X_k^{(n)} := \Delta^{(n)} \xi_k = \xi_{k/n} - \xi_{(k-1)/n}, \qquad \text{for } k \geqslant 1. \tag{4.59}$$

For each n, the sequence $(X_k^{(n)})$ is identically distributed with law $\xi_{1/n}$ and the random walk $S_k^{(n)} = X_1^{(n)} + \dots + X_k^{(n)}$ for $k \geqslant 1$, $S_0^{(n)} = 0$ built from these increments for a mesh of length $1/n$ is referred to as the $n-$skeleton of the Lévy process ξ. This process consists in the positions of ξ observed at discrete time intervals and, if we write $D(\mathbb{R}_+)$ for the space of $\mathbb{R}_+$ indexed rcll functions into $\mathbb{R}$ with the Skorokhod topology, we have $S_{\lfloor n \cdot \rfloor}^{(n)} \overset{D(\mathbb{R}_+)}{\to} \xi$ as $n \uparrow \infty$. Now, fix a memory parameter $p \in (0,1)$ and for each n, consider the associated noise reinforced random walk $(\hat{S}_k^{(n)})$ with parameter p built from the same collection of increments:

$$\hat{S}_k^{(n)} := \hat{X}_1^{(n)} + \dots + \hat{X}_k^{(n)}, \qquad \text{for } k \geqslant 1, \tag{4.60}$$

where we set $\hat{S}_0^{(n)} := 0$. For a detailed account on the noise reinforced random walk, we refer to the beginning of Section 4.5.2. The main result in [19] states that $\hat{S}_{\lfloor n \cdot \rfloor} \overset{f.d.d.}{\to} \hat{\xi}$, the convergence holding in the sense of finite-dimensional distributions, and we shall now strength this result. To simplify notation, write $D^2(\mathbb{R}_+)$ the product space $D(\mathbb{R}_+) \times D(\mathbb{R}_+)$ endowed with the product topology. Now we can state the main result of the section:

Theorem 4.19. *Let ξ be a Lévy process with characteristic triplet (a, q^2, Λ), fix $p \in (0, 1/2)$ an admissible memory parameter and for each n, let $(S_k^{(n)}, \hat{S}_k^{(n)})$ be the pair of the n-skeleton of ξ and its reinforced version. Then, there is weak convergence in $D^2(\mathbb{R}_+)$ as $n \uparrow \infty$*

$$\left(S_{\lfloor n \cdot \rfloor}^{(n)}, \hat{S}_{\lfloor n \cdot \rfloor}^{(n)} \right) \xrightarrow{\mathscr{L}} (\xi, \hat{\xi}), \tag{4.61}$$

where $(\xi, \hat{\xi})$ is a pair of processes with law (4.58).

The section is organised as follows: In Section 4.5.1, after introducing the (space) compensated integral with respect to NRPPPs, we shall establish Theorem 4.16. Making use of this result, in Section 4.5.2 we define the joint law of a Lévy process and its reinforced version $(\xi, \hat{\xi})$. More precisely, by Lévy-Itô Synthesis and its reinforced version of Theorem 4.16, it will suffice to define the joint law of $(\mu, \hat{\mu})$ and $(B, \hat{B})$. This is respectively the content of the construction detailed in 4.5.1 and Definition 4.23. The construction of $\hat{\mu}$ is done explicitly in terms of the jump measure of ξ by a procedure that should be interpreted as the continuous-time reinforcement analogue of the reinforcement algorithm for random walks. We then introduce the joint law $(\xi, \hat{\xi})$ in Definition 4.25 and prove Proposition 4.18. Finally, Section 4.5.3 is devoted to the proof of Theorem 4.19.

4.5.1 Proof of Theorem 4.16

Let us start by introducing the (space)-compensated integral with respect to NRPPPs. Recall the identity of Lemma 4.11 for the intensity measure of NRPPPs and for fixed $t \in \mathbb{R}$, let $f : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ be a measurable function satisfying, for all $0 < a < b$, the integrability condition

$$\int_{(0,t] \times \{a \leqslant |x| < b\}} |f(s,x)| \mathrm{d}s \Lambda(\mathrm{d}x) < \infty.$$

Next, we set

$$\int_{[0,t] \times \{a \leqslant |x| < b\}} f(s,x) \hat{\mathcal{N}}^{(\mathrm{sc})}(\mathrm{d}s, \mathrm{d}x)$$
$$:= \int_{[0,t] \times \{a \leqslant |x| < b\}} f(s,x) \hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) - \int_{(0,t] \times \{a \leqslant |x| < b\}} f(s,x) \mathrm{d}s \Lambda(\mathrm{d}x).$$

This is a centred random variable and if we denote it by $\Sigma_{a,b}^{(c)}(f,t)$, from Proposition 4.9 - (ii) we deduce that $(\Sigma_{e^{-r},b}^{(c)}(f,t))_{r \in [-\log(b), \infty)}$ has independent increments, and hence is a martingale. When the limit of this martingale exists, we will write

$$\int_{[0,t] \times (-b,b)} f(s,x) \hat{\mathcal{N}}^{(\mathrm{sc})}(\mathrm{d}s, \mathrm{d}x) := \lim_{r \uparrow \infty} \int_{[0,t] \times \{e^{-r} \leqslant |x| < b\}} f(s,x) \hat{\mathcal{N}}^{(\mathrm{sc})}(\mathrm{d}s, \mathrm{d}x). \tag{4.62}$$

Recall that the characteristics of a NRLP are being considered with respect to the cutoff function $x \mathbb{1}_{\{|x|<1\}}$ as well as the notation $f * \hat{\mathcal{N}}$ from (4.50). The following lemma shows that the sums of atoms of NRPPPs are precisely purely discontinuous NRLPs:

Lemma 4.20. *Fix a Lévy measure Λ, a parameter $p \in (0, 1)$ such that $\beta(\Lambda)p < 1$ and let $\hat{\mathcal{N}}$ be a NRPPP with characteristic measure Λ and reinforcement parameter p.*

(i) *For any $0 < a < b$, the process $\mathbb{1}_{\{a \leqslant |x| < b\}} x * \hat{\mathcal{N}}$ is a noise reinforced compound Poisson process with characteristics $(\Lambda(\mathbb{1}_{\{a \leqslant |x| < 1\}} x), 0, \mathbb{1}_{\{a \leqslant |x| < b\}} \Lambda, p)$.*

(ii) *For each $t \in \mathbb{R}^+$ the compensated integral*

$$\int_{[0,t]\times(-1,1)} x\hat{\mathcal{N}}^{(sc)}(\mathrm{d}s,\mathrm{d}x) := \lim_{r\uparrow\infty}\int_{[0,t]\times\{e^{-r}\leqslant|x|<1\}} x\hat{\mathcal{N}}^{(sc)}(\mathrm{d}s,\mathrm{d}x) \tag{4.63}$$

exists. The process $\mathbb{1}_{(-1,1)}x * \hat{\mathcal{N}}^{(sc)}$ is a NRLP with characteristics $(0,0,\mathbb{1}_{(-1,1)}\Lambda,p)$ and hence has a rcll modification. Moreover, the convergence (4.63) holds towards its rcll modification uniformly in compact intervals for some subsequence (r_n), and we shall consider it and denote it in the same way without further comments.

Proof. (i) If we consider $\hat{\xi}$ a reinforced compound Poisson process with such characteristics and $\hat{\mu}$ is its jump measure, it is a pure jump process and we can write it as the sum of its jumps. Our claim can now be proved directly from the identity $\hat{\xi} = (x * \hat{\mu}) \overset{\mathscr{L}}{=} (\mathbb{1}_{(a\leqslant|x|<b)}x * \hat{\mathcal{N}})$, since by Proposition 4.10 - (i), the restriction $\mathbb{1}_{(a\leqslant|x|<b)}\hat{\mathcal{N}}$ has the same distribution as $\hat{\mu}$. Alternatively, this can be established by means of the exponential formulas we obtained in Proposition 4.12, by fixing $0 < t_1 < \cdots < t_k < t$ and computing the characteristic function of the finite-dimensional distributions at times $t_1,\ldots,t_k$ of $\mathbb{1}_{\{a\leqslant|x|<b\}}x * \hat{\mathcal{N}}$, noticing that for $f(s,x) := \left(\sum_{j=1}^k \lambda_j \mathbb{1}_{\{s\leqslant t_j\}}\right) x\mathbb{1}_{\{a\leqslant|x|<b\}}$ we have

$$\sum_{j=1}^k \lambda_j(\mathbb{1}_{\{a\leqslant|x|<b\}}x * \hat{\mathcal{N}})_{t_j} = \int_{[0,t]\times\mathbb{R}} f(s,x)\hat{\mathcal{N}}(\mathrm{d}s,\mathrm{d}x).$$

The claim follows by comparing with the identity for the characteristic function of the finite-dimensional distributions (4.13) of $\hat{\xi}$.

(ii) Recall the notation introduced before (4.62) for the martingale $(\Sigma^{(c)}_{e^{-r},1}(f,t))_{r\geqslant0}$. In our case, we have $f(s,x) = x$ and we just write $(\Sigma^{(c)}_{e^{-r},1}(t))_{r\geqslant0}$. The fact that the martingale $(\Sigma^{(c)}_{e^{-r},1}(t))_{r\geqslant0}$ converges as $r \uparrow \infty$ and that the limit is a NRLP with characteristics $(0,0,\mathbb{1}_{(-1,1)}\Lambda)$ can be achieved by similar arguments as in [19] after a couple of observations. Starting with the former, recall the definition of $\hat{\mathcal{N}}$ from (4.39), and remark that for each $r > 0$ we have

$$\int_{[0,t]\times\mathbb{R}} \mathbb{1}_{\{e^{-r}\leqslant|x|<1\}}x\hat{\mathcal{N}}(\mathrm{d}s,\mathrm{d}x) = \sum_{u\in\mathscr{P}} \mathbb{1}_{\{u\leqslant t\}}\mathbb{1}_{\{e^{-r}\leqslant|x_u|<1\}}x_u \cdot \#\big\{\{ue^s : s\in D_u\}\cap[0,t]\big\}.$$

From the discussion right after Proposition 4.9, we infer that if we we consider $(Z^u)_{u\in\mathscr{P}}$ an independent collection of independent, standard Yule processes, the family $\{ue^s : s \in D_u\}$ has the same distribution as the collection of jump times of the counting process $\mathbb{1}_{\{u\leqslant t\}}Z^u_{p(\ln(t)-\ln(u))}$, $t \geqslant 0$. Hence the previous display can also be written as

$$\sum_{u\in\mathscr{P}} \mathbb{1}_{\{e^{-r}\leqslant|x_u|<1\}}x_u\mathbb{1}_{\{u\leqslant t\}}Z^u_{p(\ln(t)-\ln(u))},$$

and now the proof of the convergence as $r \uparrow \infty$ of $(\Sigma^{(c)}_{e^{-r},1}(t))_{r\geqslant0}$ follows by the same arguments as in [19, Lemma 2.6]. Alternatively, one can make use of (4.13) to restrict our arguments to the interval $[0,1]$ and apply [19, Lemma 2.6]. Next, to see that the process $\mathbb{1}_{(-1,1)}x * \hat{\mathcal{N}}^{(sc)}$ defines a NRLP with characteristics $(0,0,\mathbb{1}_{(-1,1)}\Lambda)$, fix $0 < t_1 < \cdots < t_k < t$ and for $\varepsilon > 0$, $\lambda \in \mathbb{R}$ set

$$\Phi^{(3)}_{\varepsilon,1}(\lambda) = \int_{\{\varepsilon\leqslant|x|<1\}} \left(e^{i\lambda x} - 1 - i\lambda x\right)\Lambda(\mathrm{d}x).$$

Recalling the formula (4.41) for the characteristic function of integrals with respect to NRPPPs, we deduce from considering the function $f(s,x) := \left(\sum_{j=1}^k \lambda_j \mathbb{1}_{\{s \leqslant t_j\}}\right) x \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}}$ that we have

$$\mathbb{E}\left[\exp\left\{i \sum_{j=1}^k \lambda_j (\mathbb{1}_{\{\varepsilon \leqslant |x|<1\}} x * \hat{\mathcal{N}}^{(sc)})_{t_j})\right\}\right] = \exp\left\{t(1-p)\mathbb{E}\left[\Phi^{(c)}_{\varepsilon,1}\left(\sum_{j=1}^k \lambda_j Y(t_j/t)\right)\right]\right\}.$$

Now we can apply the exact same reasoning as in the proof of Corollary 2.8 in [19] by writing $s_j = t_j/t \in [0,1]$ and taking the limit as $\varepsilon \downarrow 0$. The uniform convergence in compact intervals towards the rcll modification of $\mathbb{1}_{(-1,1)} x * \hat{\mathcal{N}}^{(sc)}$ follows from the second statement of Theorem 4.3, since for every $\varepsilon \in (0,1)$, the process

$$\int_{[0,t]\times\{0 \leqslant |x| < \varepsilon\}} x \hat{\mathcal{N}}^{(\mathrm{sc})}(\mathrm{d}s, \mathrm{d}x), \quad t \geqslant 0,$$

is a NRLP with characteristics $(0, 0, \mathbb{1}_{\{|x|<\varepsilon\}}\Lambda)$. □

It immediately follows from the previous lemma that if $\hat{\mathcal{N}}$ is a NRPPP with characteristic measure Λ, parameter p and, if $p < 1/2$, we consider $\hat{W}$ an independent NRBM with same parameter, then

$$\hat{X}_t = at + q\hat{W}_t + \int_{[0,t]\times(-1,1)^c} x\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]\times(-1,1)} x\hat{\mathcal{N}}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0, \tag{4.64}$$

defines a NRLP with characteristics (a, q^2, Λ, p). To obtain the a.s. statement of Theorem 4.16 we still need a short argument.

Proof of Theorem 4.16. The result will be deduced from the equality in distribution $\hat{\xi} \overset{\mathscr{L}}{=} \hat{X}$ for $\hat{X}$ defined as in (4.64) with same characteristics as $\hat{\xi}$. In this direction, wlog we assume $p < 1/2$, $q = 1$ and we set

$$\hat{\xi}^{\leqslant 1} := \hat{\xi}_t - \sum_{s \leqslant t} \mathbb{1}_{\{|\Delta \hat{\xi}_s| \geqslant 1\}} \Delta \hat{\xi}_s \quad \text{ and } \quad \hat{\xi}^{<\varepsilon}_t := \hat{\xi}^{\leqslant 1}_t - \Big(\sum_{s \leqslant t} \mathbb{1}_{\{\varepsilon \leqslant |\Delta \xi_s| < 1\}} \Delta \hat{\xi}_s - t\int_{\{\varepsilon \leqslant |x| < 1\}} x\Lambda(\mathrm{d}x)\Big).$$

Notice that for every $\varepsilon > 0$, we can write

$$\hat{\xi}_t = \hat{\xi}^{<\varepsilon}_t + \Big(\sum_{s \leqslant t} \mathbb{1}_{\{\varepsilon \leqslant |\Delta \xi_s| < 1\}} \Delta \hat{\xi}_s - t\int_{\{\varepsilon \leqslant |x| < 1\}} x\Lambda(\mathrm{d}x)\Big) + \hat{\xi}^{(2)}_t. \tag{4.65}$$

Since $\hat{\mu}$ is a reinforced PPP, by Lemma 4.20 the process (4.65) converges uniformly in compact intervals for some subsequence (ε_n) as $\varepsilon_n \downarrow 0$ towards $\hat{C} + \hat{\xi}^{(2)} + \hat{\xi}^{(3)}$, for some process $\hat{C} := \hat{\xi} - \hat{\xi}^{(2)} - \hat{\xi}^{(3)}$ continuous by construction. Since $\hat{\mu}$ is a reinforced PPP, by the independence properties of its restriction we know that $\hat{\xi}^{(2)}, \hat{\xi}^{(3)}$ are independent. Hence, it remains to show that $(\hat{\xi}^{(2)}, \hat{\xi}^{(3)})$ is independent of $\hat{C}$ and that $\hat{C} - at =: \hat{B}$ is a NRBM. Fix arbitrary $0 < u < v \leqslant \infty$ and maintain the notation for $\hat{W}, \hat{\mathcal{N}}$ used in the representation (4.64). Since $\hat{\mathcal{N}}$ is the clearly the jump measure of $\hat{X}$, we have the equality in distribution:

$$\Big(\hat{\xi}, \sum_{s \leqslant \cdot} \mathbb{1}_{\{u \leqslant |\Delta \hat{\xi}_s| < v\}} \Delta \hat{\xi}_s\Big) \overset{\mathscr{L}}{=} \left(\hat{X}, \mathbb{1}_{\{u \leqslant |x| < v\}} x * \hat{\mathcal{N}}\right). \tag{4.66}$$

Moreover, since $\hat{W}$ is independent of $\hat{\mathcal{N}}$, from the independence of restrictions of NRPPP and (4.66) we deduce that $\mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \hat{\mu}^{(sc)} + \mathbb{1}_{\{1 \leqslant |x|\}} x * \hat{\mu}$ and $\hat{\xi} - \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \hat{\mu}^{(sc)} - \mathbb{1}_{\{1 \leqslant |x|\}} x * \hat{\mu}$ are independent, the later having the same distribution as $at + \hat{W}_t + \mathbb{1}_{(-\varepsilon,\varepsilon)} x * \hat{\mathcal{N}}$. Now the claim follows by taking the limit as $\varepsilon \downarrow 0$. □

4.5.2 The joint law $(\xi, \hat{\xi})$ of a Lévy process and its reinforced version

In this section we construct explicitly, for an arbitrary fixed Lévy process ξ, the process $\hat{\xi}$ in terms of ξ that will be referred to as the noise reinforced version of ξ. This will yield a definition for the joint law $(\xi, \hat{\xi})$. Our construction will be justified by the weak convergence of Theorem 4.19. Let us start by recalling the discrete setting, since our construction is essentially the continuous-time analogue of the dynamics that we now describe.

○ *The noise reinforced random walk.* Given a collection of identically distributed random variables (X_n) with law X, denote by $S_n := X_1 + \cdots + X_n,$ for $n \geqslant 1$ the corresponding random walk. We construct, simultaneously to (S_n), a noise reinforced version using the same sample of random variables and performing the reinforcement algorithm at each discrete time step. In this direction, consider (ε_n) and $(U[n])$ independent sequences of Bernoulli random variables with parameter $p \in (0,1)$ and uniform random variables on $\{1, \dots, n\}$ respectively. Set $\hat{X}_1 := X_1$ and, for $n \geqslant 1$, define

$$\hat{X}_{n+1} := X_{n+1} \mathbb{1}_{\{\varepsilon_{n+1}=0\}} + \hat{X}_{U[n]} \mathbb{1}_{\{\varepsilon_{n+1}=1\}}.$$

Finally, we denote the corresponding partial sums by $\hat{S}_n := \hat{X}_1 + \cdots + \hat{X}_n$, $n \geqslant 1$. The process $(\hat{S}_n)$ is the so-called noise reinforced random walk with memory parameter p, and we refer to this particular construction of $(\hat{S}_n)$ as the noise reinforced version of (S_n). The process $(\hat{S}_n)$ can be written in terms of the individual contributions made by each one of the steps. In this direction, let us introduce a counting process keeping track of the number of times each step X_k is repeated up to time n. Since if the law of X has atoms, we have $\mathbb{P}(X_1 = X_2) > 0$, and we need to perform a slight modification to our algorithm. Namely, for each $n \geqslant 1$ we write $X'_n := (X_n, n)$ and we perform the reinforcement algorithm to the pairs (X'_n). This yields a sequence that, with a slight abuse of notation, we denote by $(\hat{X}'_n)$. If for every $k, n \geqslant 1$ we set:

$$N_k(n) := \#\{1 \leqslant i \leqslant n : X'_k = \hat{X}'_i\}, \tag{4.67}$$

we can write:

$$\hat{S}_n = \sum_{k=1}^{\infty} N_k(n) X_k, \qquad \text{for } n \geqslant 1. \tag{4.68}$$

For convenience, we always set $S_0 = 0 = \hat{S}_0$, and when working with pairs of the form $(S, \hat{S})$ it will always be implicitly assumed that the noise reinforced version has been constructed by the algorithm we described. For instance, it is clear that at each discrete time step n, with probability $1-p$, S_n and $\hat{S}_n$ share the same increment, while with complementary probability p, they perform different steps.

Roughly speaking, in the continuum, the steps (X_n) are replaced by jumps $\Delta\xi_s$ of the Lévy process ξ. With probability $1-p$, the jump is shared with its reinforced version $\hat{\xi}$ while with complementary probability p it is discarded and remains independent of $\hat{\xi}$. The jumps that are not discarded by this procedure are then repeated at each jump time of an independent counting process that will be attached to it. The process of discarding jumps with probability p is traduced in a thinning of the jump measure of ξ. Let us now give a formal description of this heuristic discussion.

Construction of the pair $(\mathcal{N}, \hat{\mathcal{N}})$

For the rest of the section, we fix a Lévy process ξ with non-trivial Lévy measure Λ, denote the set of its jump times by $\mathscr{I} := \{u \in \mathbb{R}^+ : \Delta\xi_u \neq 0\}$ and let

$$\mathcal{N}(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{I}} \delta_{(u, \Delta\xi_u)},$$

be its jump measure. By the Lévy-Itô decomposition, this is a PPP with characteristic measure Λ and we can write $\xi = \xi^{(1)} + J$, where $\xi^{(1)}$ is a continuous process while J is a process that can be explicitly recovered from $\mathcal{N}$, as we recalled in (4.57).

If $\hat{\xi}$ has the law of its reinforced version, by Theorem 4.16 it can also be written as $\hat{\xi} = \hat{\xi}^{(1)} + \hat{J}$, where $\hat{J}$ is a functional of a NRPPP $\hat{\mathcal{N}}$ with characteristic measure Λ. Hence, the main step for defining the law of the pair $(J, \hat{J})$ consists in appropriately defining $(\mathcal{N}, \hat{\mathcal{N}})$. However, recalling the construction of NRPPPs by superposition detailed before Definition 4.9, this can be achieved as follows: first, set $A_0 := \{1 \leqslant |x|\}$ and for each $j \geqslant 1$, let $A_j := \{1/(j+1) \leqslant |x| < 1/j\}$. Next, for $j \geqslant 0$ consider the point process

$$\mathscr{I}_j := \{u \in \mathbb{R}^+ : \Delta\xi_u \in A_j\},$$

remark that $\mathscr{I}_j$ is a PPP with intensity $\Lambda(A_j)\mathrm{d}t$ and write $\mathscr{I} := \cup_j \mathscr{I}_j$. Maintaining the notation of Section 4.4, consider $(D_u)_{u \in \mathscr{I}}$ a collection of i.i.d. copies of D and for each $j \geqslant 0$ we set

$$\mathcal{N}_j(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{I}_j} \sum_{t \in D_u} \delta_{(ue^t, \Delta\xi_u)}.$$

The measure $\mathcal{N}_j$ is a NRPPP with characteristic measure $(1-p)^{-1}\Lambda(\,\cdot \cap A_j)$, and we can now proceed as in Section 4.4.2 to construct the following NRPPP with parameter p by superposition of $(\mathcal{N}_j)_{j \geqslant 1}$,

$$\sum_{u \in \mathscr{I}} \sum_{t \in D_u} \delta_{(ue^t, \Delta\xi_u)}. \tag{4.69}$$

Notice however that its characteristic measure is $(1-p)^{-1}\Lambda$. In this direction, we consider a sequence of independent Bernoulli random variables $(\varepsilon_u)_{u \in \mathscr{I}}$ with parameter $1-p$ and apply a thinning:

$$\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) := \sum_{u \in \mathscr{I}} \mathbb{1}_{\{\varepsilon_u = 1\}} \sum_{t \in D_u} \delta_{(ue^t, \Delta\xi_u)}. \tag{4.70}$$

Now, $\hat{\mathcal{N}}$ is a NRPPP with characteristic measure Λ and reinforcement parameter p built explicitly from the jump process of ξ. From the construction, if a jump $\Delta\xi_u$ occurs at time u, with probability $1-p$ it is kept and repeated at each ue^t for $t \in D_u$, while with complementary probability p, it is discarded and remains independent of $\hat{\mathcal{N}}$. From now on, we always consider the pair $(\mathcal{N}, \hat{\mathcal{N}})$ constructed by this procedure. Then, by definition of $\mathcal{N}$ we can write

$$J_t = \xi_t^{(2)} + \xi_t^{(3)} = \int_{[0,t] \times \{|x| \geqslant 1\}} x\mathcal{N}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t] \times (-1,1)} x\mathcal{N}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0,$$

while on the other hand, by Theorem 4.16 the process defined as

$$\hat{J}_t = \hat{\xi}_t^{(2)} + \hat{\xi}_t^{(3)} := \int_{[0,t] \times \{|x| \geqslant 1\}} x\hat{\mathcal{N}}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t] \times (-1,1)} x\hat{\mathcal{N}}^{(sc)}(\mathrm{d}s, \mathrm{d}x), \qquad t \geqslant 0, \tag{4.71}$$

is a NRLP with characteristics $(0, 0, \Lambda, p)$. From our construction, the random measures $\mathcal{N}$, $\hat{\mathcal{N}}$ can be encoded in terms of a single Poisson random measure $\sum_{u \in \mathscr{I}} \delta_{(u, \Delta \xi_u, D_u, \varepsilon_u)}$, allowing us to compute explicitly the characteristic function of the finite dimensional distributions of $(\xi^{(2)}, \hat{\xi}^{(2)})$ and $(\xi^{(3)}, \hat{\xi}^{(3)})$. In this direction, for $\lambda \in \mathbb{R}$ recall the notation

$$\Phi^{(2)}(\lambda) := \int_{\{|x| \geqslant 1\}} (e^{i\lambda x} - 1)\Lambda(\mathrm{d}x). \tag{4.72}$$

Lemma 4.21. *For all $k \geqslant 1$, let $\lambda_1, \dots, \lambda_k$ and $\beta_1, \dots \beta_k$ be real numbers and fix times $0 < t_1 < \dots < t_k < t$. Then, we have*

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j\big(\xi^{(2)}_{t_j} + \beta_j\hat{\xi}^{(2)}_{t_j}\big)\right\}\right] =$$
$$\exp\left\{tp\mathbb{E}\left[\Phi^{(2)}\left(\sum_{j=1}^{k}\lambda_j \mathbb{1}_{\{Y(t_j/t)\geqslant 1\}}\right)\right] + t(1-p)\mathbb{E}\left[\Phi^{(2)}\left(\sum_{j=1}^{k}(\lambda_j \mathbb{1}_{\{Y(t_j/t)\geqslant 1\}} + \beta_i Y(t_j/t))\right)\right]\right\}, \tag{4.73}$$

where we denote by Y a Yule-Simon process with parameter $1/p$.

Let us briefly comment on this expression. The first exponential term in (4.73) corresponds to the characteristic function of the finite dimensional distributions of a Lévy process with law $(\xi^{(2)}_{pt})_{t \in \mathbb{R}^+}$, viz.

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\lambda_j\xi^{(2)}_{pt_j}\right\}\right] = \exp\left\{tp\mathbb{E}\left[\Phi^{(1)}\Big(\sum_{j=1}^{k}\lambda_j\mathbb{1}_{\{U\leqslant t_j/t\}}\Big)\right]\right\},$$

where U is a uniform random variable in $[0,1]$ (recall that the first jump time of a Yule-Simon process is uniformly distributed in $[0,1]$). More precisely, this Lévy process is built from the discarded jumps $\sum_u \mathbb{1}_{\{\varepsilon_u = 0\}}\delta_{(u,\Delta\xi_u)}$ and consequently is independent of $\hat{\xi}^{(2)}$ and $\sum_u \mathbb{1}_{\{\varepsilon_u=1\}}\delta_{(u,\Delta\xi_u)}$, which explains the form of the identity (4.73).

Proof. We can assume that $t_k < 1$ by working with $t_1/t < \dots < t_k/t$ and with the pair $(\xi_{st}, \hat{\xi}_{st})_{s\in[0,1]}$, which now has Lévy measure $t\Lambda$. Now, the proof follows by a rather long but straightforward application of the formula for the characteristic function of integrals with respect to Poisson random measures. □

We now turn our attention to the characteristic function of the finite dimensional distributions of $(\xi^{(3)}, \hat{\xi}^{(3)})$. In this direction, for $\lambda \in \mathbb{R}$, recall the notation

$$\Phi^{(3)}(\lambda) := \int_{\{|x|<1\}} (e^{i\lambda x} - 1 - i\lambda x)\Lambda(\mathrm{d}x). \tag{4.74}$$

Lemma 4.22. *For all $k \geqslant 1$, let $\lambda_1, \dots, \lambda_k$ and $\beta_1, \dots \beta_k$ be real numbers and fix times $0 < t_1 < \cdots < t_k < t$. Then, we have*

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\left(\lambda_j \xi^{(3)}_{t_j} + \beta_j \hat{\xi}^{(3)}_{t_j}\right)\right\}\right] =$$
$$\exp\left\{tp\mathbb{E}\left[\Phi^{(3)}\left(\sum_{j=1}^{k}\lambda_j \mathbb{1}_{\{Y(t_j/t)\geqslant 1\}}\right)\right] + t(1-p)\mathbb{E}\left[\Phi^{(3)}\left(\sum_{j=1}^{k}(\lambda_j \mathbb{1}_{\{Y(t_j/t)\geqslant 1\}} + \beta_i Y(t_j/t))\right)\right]\right\}, \tag{4.75}$$

where we denote by Y a Yule-Simon process with parameter $1/p$.

Proof. By the usual scaling argument we can suppose that $t_k < 1 = t$. Now, the proof is similar to the one of Corollary 2.8 in [19]. In this direction, notice that the processes $\xi^{(3)} = \mathbb{1}_{(-1,1)} x * \mathcal{N}^{(sc)}$ and $\hat{\xi}^{(3)} = \mathbb{1}_{(-1,1)} x * \hat{\mathcal{N}}^{(sc)}$ are respectively the limit as $\varepsilon \downarrow 0$ of

$$\xi^{(3)}_{\varepsilon,1} := \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \mathcal{N} - \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \mathrm{d}t \otimes \Lambda, \tag{4.76}$$

$$\hat{\xi}^{(3)}_{\varepsilon,1} := \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \hat{\mathcal{N}} - \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \mathrm{d}t \otimes \Lambda, \tag{4.77}$$

the convergence holding uniformly in compact intervals. The characteristic function of the finite-dimensional distributions of the pair $(\mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \mathcal{N}, \mathbb{1}_{\{\varepsilon \leqslant |x| < 1\}} x * \hat{\mathcal{N}})$ can be computed by the same arguments as in Lemma 4.21 and we obtain for each $0 < \varepsilon < 1$ that

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}(\lambda_j \xi^{(3)}_{\varepsilon,1}(t_j) + \beta_j \hat{\xi}^{(3)}_{\varepsilon,1}(t_j))\right\}\right]$$
$$= \exp\left\{p\mathbb{E}\left[\Phi^{(3)}_{\varepsilon}\left(\sum_{j=1}^{k}\lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}}\right)\right] + (1-p)\mathbb{E}\left[\Phi^{(3)}_{\varepsilon}\left(\sum_{j=1}^{k}\left(\lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}} + \beta_i Y(t_j)\right)\right)\right]\right\}. \tag{4.78}$$

In order to establish that this expression converges as $\varepsilon \downarrow 0$ towards (4.75), we recall that since $|e^{ix} - 1 - ix|$ is $O(|x^2|)$ as $|x| \downarrow 0$ and $O(|x|)$ as $|x| \uparrow \infty$, for any $r \in (\beta(\Lambda) \vee 1, 1/p \wedge 2)$ if $\beta(\Lambda) < 2$ and $r = 2$ if $\beta(\Lambda) = 2$, we have

$$C := \sup_{x \in \mathbb{R}} |x|^{-r} |e^{ix} - 1 - ix| < \infty.$$

It follows that for all $0 < \varepsilon < 1$, $\lambda \in \mathbb{R}$, we can bound

$$|\Phi^{(3)}_{\varepsilon}(\lambda)| \leqslant \int_{\{|x|<1\}} |e^{i\lambda x} - 1 - i\lambda x| \Lambda(\mathrm{d}x) \leqslant C|\lambda|^r \int_{\{|x|<1\}} |x|^r \Lambda(\mathrm{d}x).$$

Moreover, by the remark following Lemma 4.1, the random variable $Y(t) \in L_r(\mathbb{P})$ for any $r < 1/p$ and it follows that the term

$$\sum_{j=1}^{k}\left(\lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}} + \beta_i Y(t_j)\right),$$

is in $L_r(\mathbb{P})$. Hence, by dominated convergence, (4.78) converges towards (4.75) as $\varepsilon \downarrow 0$. On the other hand, since $(\xi^{(3)}_{\varepsilon,1}(t_j), \hat{\xi}^{(3)}_{\varepsilon,1}(t_j)) \to (\xi^{(3)}_{t_j}, \hat{\xi}^{(3)}_{t_j})$ as $\varepsilon \downarrow 0$, we obtain the desired result. □

The distribution of $(B, \hat{B})$ and proof of Proposition 4.18

The last ingredient needed to define the joint distribution of $(\xi, \hat{\xi})$ is the joint distribution of a Brownian motion B and its reinforced version $\hat{B}$, that we denote as $(B, \hat{B})$. Recall from [21] that $\hat{B}$ has the same law as the solution to the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{p}{t} X_t \mathrm{d}t, \tag{4.79}$$

and that X can be written explicitly in terms of the stochastic integral (4.18) with respect to the driving Brownian motion B. We also recall from (4.17) that for $0 < s, t < T$ the covariance of $\hat{B}$ can be expressed in terms of the Yule Simon process as follows:

$$\mathbb{E}\left[\hat{B}_t \hat{B}_s\right] = \frac{(t \vee s)^p (t \wedge s)^{1-p}}{1-2p} = T(1-p)\mathbb{E}\left[Y(t/T)Y(s/T)\right], \tag{4.80}$$

and for later use, we observe that

$$(t \wedge s)^{1-p} s^p = T(1-p)\mathbb{E}\left[\mathbb{1}_{\{Y(t/T) \geqslant 1\}} Y(s/T)\right]. \tag{4.81}$$

We stress that the right-hand side in the previous display do not depend on the choice of T. The proof of this identity is a consequence of the representation (4.2) of Y in terms of a standard Yule process and an independent uniform random variable.

Definition 4.23. *Let $(B, \hat{B})$ be a pair of Gaussian processes and fix a parameter $0 < p < 1/2$. We say that the pair $(B, \hat{B})$ has the law of a Brownian motion with its reinforced version if the respective covariances are given by*

$$\mathbb{E}\left[B_t B_s\right] = (t \wedge s), \qquad \mathbb{E}\left[B_t \hat{B}_s\right] = (t \wedge s)^{1-p} s^p, \qquad \mathbb{E}\left[\hat{B}_t \hat{B}_s\right] = \frac{(t \vee s)^p (t \wedge s)^{1-p}}{1-2p}, \tag{4.82}$$

for any $s, t \in \mathbb{R}^+$.

Let us briefly explain where this definition comes from: for fixed p, by [16, Theorem 1.1] the law of the pair $(B, \hat{B})$ is universal, in the sense that it is the weak joint scaling limit of random walks paired with its reinforced version with parameter p for $p < 1/2$, when the typical step is in $L_2(\mathbb{P})$. For more details, we refer to [21, 16].

Given a fixed Brownian motion B, it is clear that we can not expect to have an explicit construction of the reinforced version $\hat{B}$ in terms of B similar to the one performed for $(J, \hat{J})$. However, we can make use of the SDE (4.79) to get an explicit construct of $(B, \hat{B})$ with the right covariance structure. This can be easily achieved as follows: first, let W be an independent copy of B; if we set

$$\beta_t := (1-p)B_t + \sqrt{1-(1-p)^2} W_t, \tag{4.83}$$

then, B and β are two Brownian motions with $\mathbb{E}\left[B_t \beta_s\right] = (1-p)(t \wedge s)$. If we let $\hat{B}$ be the solution to the SDE,

$$\mathrm{d}\hat{B}_t = \mathrm{d}\beta_t + \frac{p}{t} \hat{B}_t \mathrm{d}t, \tag{4.84}$$

$\hat{B}$ has the law of a noise reinforced Brownian motion with reinforcement parameter p, and can be written explicitly as $\hat{B}_t = t^p \int_0^t s^{-p} \mathrm{d}\beta_s$. Moreover, it readily follows that the covariance of the pair of Gaussian processes $(B, \hat{B})$ satisfies (4.82). The decorrelation applied for constructing β is playing the role of the thinning in the construction of $(J, \hat{J})$.

Finally, we will need for the proof of Proposition 4.18 the following representation of the characteristic function of the finite-dimensional distributions of the pair $(B, \hat{B})$ in terms of the Yule-Simon process:

Lemma 4.24. *Let $(B, \hat{B})$ be a Brownian motion with its reinforced version for a memory parameter $p < 1/2$. For all $k \geqslant 1$, $\lambda_1, \ldots, \lambda_k, \beta_1, \ldots \beta_k$ real numbers and $0 < t_1 < \cdots < t_k < t$, we have*

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\left(\lambda_j B_{t_j} + \beta_j \hat{B}_{t_j}\right)\right\}\right] =$$

$$\exp\left\{-tp\mathbb{E}\left[\frac{q^2}{2}\left(\sum_{j=1}^{k}\lambda_j \mathbb{1}_{\{Y(\frac{t_j}{t})\geqslant 1\}}\right)^2\right] - t(1-p)\mathbb{E}\left[\frac{q^2}{2}\left(\sum_{j=1}^{k}\left(\lambda_j \mathbb{1}_{\{Y(\frac{t_j}{t})\geqslant 1\}} + \beta_j Y\left(\frac{t_j}{t}\right)\right)\right)^2\right]\right\}. \tag{4.85}$$

Proof. Since NRBM satisfies the same scaling property of Brownian Motion (see page 3 of [21]), from (4.82) we deduce $(B_{tc}, \hat{B}_{tc})_{t\in\mathbb{R}^+} \stackrel{\mathscr{L}}{=} (c^{1/2}B_t, c^{1/2}\hat{B}_t)_{t\in\mathbb{R}^+}$. Hence, as usual we can suppose that $t_k < 1$ and we take $t := 1$. To simplify notation we also suppose that $q = 1$. Now, the left hand side of (4.85) writes

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\left(\lambda_j B_{t_j} + \beta_j \hat{B}_{t_j}\right)\right\}\right]$$

$$= \exp\left\{-\frac{1}{2}\sum_{i,j}\lambda_i\lambda_j Cov(B_{t_i}, B_{t_j}) - \frac{1}{2}\sum_{i,j}\beta_i\beta_j Cov(\hat{B}_{t_i}, \hat{B}_{t_j}) - \sum_{i,j}\lambda_i\beta_j Cov(B_{t_i}, \hat{B}_{t_j})\right\}$$

$$= \exp\left\{-\frac{1}{2}\mathbb{E}\left[\left(\sum_{j=1}^{k}\lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}}\right)^2 + (1-p)\left(\sum_{j=1}^{k}\beta_j Y(t_j)\right)^2 + 2(1-p)\sum_{i,j}\lambda_i\beta_j \mathbb{1}_{\{Y(t_i)\geqslant 1\}}Y(t_j)\right]\right\}$$

where we used respectively for each one of the covariances in order of appearance that: the first jump time of a Yule-Simon process is uniformly distributed, (4.80) and (4.81). However, this is precisely the right hand side of (4.85). □

Now that all the ingredients have been introduced, we define the law of $(\xi, \hat{\xi})$.

∂ *Recipe for reinforcing Lévy processes:* consider a starting Lévy process ξ with triplet (a, q^2, Λ) and denote by $\xi_t = at + qB_t + J_t$ for $t \geqslant 0$ its Lévy Itô decomposition, where B and J are respectively a Brownian motion and a Lévy process with triplet $(0, 0, \Lambda)$. Further, fix $p \in (0, 1)$ an admissible parameter for the triplet, denote the jump measure of ξ by $\mathcal{N} = \sum \delta_{(u, \Delta\xi_u)}$ and consider the NRPPP $\hat{\mathcal{N}}$ with characteristic measure Λ and reinforcement parameter p as constructed in (4.70) in terms of $\mathcal{N}$. Denote by $\hat{J} := \mathbb{1}_{(-1,1)}x * \hat{\mathcal{N}}^{(sc)} + \mathbb{1}_{(-1,1)^c}x * \hat{\mathcal{N}}$ the corresponding NRLP of characteristics $(0, 0, \Lambda, p)$ and finally, consider a NRBM $\hat{B}$ independent of $(J, \hat{J})$, such that $(B, \hat{B})$ has the law of a Brownian motion with its reinforced version – for example by proceeding as in (4.84).

Definition 4.25. *We call the noise reinforced Lévy process* $\hat{\xi}_t := at + q\hat{B}_t + \hat{J}_t$ *for* $t \geqslant 0$ *of characteristics* (a, q^2, Λ, p) *the noise reinforced version of* ξ*, the unicity only holding in distribution. From now on, every time we consider a pair* $(\xi, \hat{\xi})$*, it will be implicitly assumed that* $\hat{\xi}$ *has been constructed by the procedure we just described in terms of* ξ*.*

Let us now conclude the proof of Proposition 4.18.

Proof of Proposition 4.18. If Ψ is the characteristic exponent of ξ, we can write

$$\Psi(\lambda) = ia\lambda - \frac{1}{2}q^2\lambda^2 + \Phi^{(2)}(\lambda) + \Phi^{(3)}(\lambda),$$

for $\Phi^{(2)}$, $\Phi^{(3)}$ defined respectively by (4.72) and (4.74). Recalling the independence between the pairs $(B, \hat{B})$, $(\xi^{(2)}, \hat{\xi}^{(2)})$, $(\xi^{(3)}, \hat{\xi}^{(3)})$, the proof of Proposition 4.18 now follows from Lemmas 4.21, 4.22, 4.24 and the previous decomposition for the characteristic exponent Ψ. □

From the construction of $(\mathcal{N}, \hat{\mathcal{N}})$, we can sketch a sample path of $(\xi, \hat{\xi})$, where the jumps that are not appearing on the path of $\hat{\xi}$ are precisely the ones deleted by the thinning:

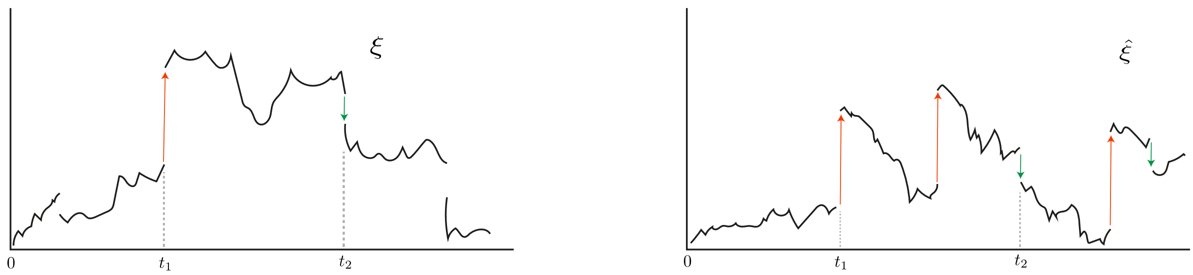

Figure 4.2: Sample path of a Lévy process and its reinforced version.

4.5.3 Proof of Theorem 4.19

Let us outline the proof of Theorem 4.19. First, by (4.13), it suffices to prove the convergence in $[0, 1]$ and we therefore work with $\xi = (\xi_t)_{t \in [0,1]}$. Next, since we are working in $D^2[0, 1]$, it suffices to establish tightness coordinate-wise to obtain tightness for the sequence of pairs. The first coordinate in (4.61) converges a.s. towards ξ in $D[0, 1]$ (and in particular is tight) and hence it remains to establish tightness for the sequence of reinforced n-skeletons. This is the content of Section 4.5.3 and more precisely, of Proposition 4.28. This is achieved by means of the celebrated Aldous tightness criterion and our arguments rely on the discrete counterpart of the remarkable martingale from Proposition 4.4. This discrete martingale is introduced in Lemma 4.26 and we recall from [16, 15] its main features. This is the content of Section 4.5.3. Finally, the joint convergence in the sense of finite-dimensional distributions towards $(\xi, \hat{\xi})$ is proved in Proposition 4.31, by establishing the convergence of the corresponding characteristic functions.

The martingale associated with a noise reinforced random walk

◦ *The elephant random walk and its associated martingale.* Let us start with some historical context. In [11], Bercu was interested in establishing asymptotic convergence results for a particular random walk with memory, called the elephant random walk. This process is defined as follows: for a fixed $q \in (0, 1)$ that we still call the reinforcement parameter, we set $\mathcal{E}_0 := 0$ and let Y_1 be a random variable with $Y_1 \in \{-1, 1\}$. Then, the position of our elephant at time $n = 1$ is given by $\mathcal{E}_1 = Y_1$ and for $n \geqslant 2$, it is defined recursively by the relation $\mathcal{E}_{n+1} := \mathcal{E}_n + Y_{n+1}$, for Y_{n+1} constructed by selecting uniformly at random one of the previous increments $\{Y_1 \dots Y_n\}$, and changing its sign with probability $1 - q$. The analysis of Bercu relies on a martingale associated to the elephant random walk, defined as $M_1 = \mathcal{E}_1$ and for $n \geqslant 2$, as

$$M_n := \hat{a}_n \mathcal{E}_n, \qquad \text{for } \hat{a}_n := \frac{\Gamma(n)\Gamma(2q)}{\Gamma(n + 2q - 1)}, \tag{4.86}$$

and where Γ stands for the Euler-Gamma function. This martingale had already made its appearance in the literature in Coletti, Gava, Schütz [34]. As was pointed out by Kürsten [60], the key is that when $q \in [1/2, 1)$, the elephant random walk is a version of the noise reinforced random walk when the typical step X has distribution $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$ with memory parameter $p = 2q - 1$.

Getting back to our setting, we maintain the notation introduced at the beginning of Section 4.5.2 for the noise reinforced random walk for a memory parameter $p \in (0, 1)$. Our first observation is that the martingale (4.86) associated to the elephant random walk is still a martingale in our setting – we stress that the reinforcement parameter q in [11] corresponds to the parameter $p = 2q - 1$ in our context. This martingale plays a fundamental role in our reasoning, and also played a central role in [16, 15]. More precisely, let $a_1 := 1$ and for $n \geqslant 2$ we set

$$a_n := \frac{\Gamma(n)}{\Gamma(n + p)} = \prod_{k=1}^{n-1} \gamma_k^{-1}, \tag{4.87}$$

for $\gamma_n := \frac{n+p}{n}$. We write $\mathscr{F}_n := \sigma(\hat{X}_1, \dots, \hat{X}_n)$ the filtration generated by the reinforced steps. The following lemma is taken from [16].

Lemma 4.26. [16, Proposition 2.1] *Suppose that the typical step X is centred and in $L_2(\mathbb{P})$. Then, the process M defined by $M_0 = 0$ and $M_n = a_n \hat{S}_n$ for $n \geqslant 1$ is a square-integrable martingale with respect to the filtration $(\mathscr{F}_n)$.*

In order to establish tightness for our sequence of reinforced skeletons, we will make use of the explicit form of the predictable quadratic variation $\langle M, M \rangle$ of this martingale, which is the process defined as $\langle M, M \rangle_0 = 0$ and

$$\langle M, M \rangle_n = \sum_{k=1}^{n} \mathbb{E}\left[(\Delta M_k)^2 \,|\mathscr{F}_{k-1} \right], \qquad n \geqslant 1.$$

In this direction, we introduce:

$$\hat{V}_n := \hat{X}_1^2 + \cdots + \hat{X}_n^2, \qquad n \geqslant 1,$$

with $\hat{V}_0 = 0$. The following lemma is also taken from [16] and was the main tool for establishing the invariance principles proven in that work.

Lemma 4.27. [16, Proposition 2.1] *The predictable quadratic variation process $\langle M, M \rangle$ is given by $\langle M, M \rangle_0 = 0$ and for $n \geqslant 1$,*

$$\langle M, M \rangle_n = \sigma^2 + \sum_{k=2}^{n} a_k^2 \left((1-p)\sigma^2 - p^2 \frac{\hat{S}_{k-1}^2}{(k-1)^2} + p \frac{\hat{V}_{k-1}}{k-1} \right), \tag{4.88}$$

where the sum should be considered null for $n = 1$.

Proof of tightness

We stress that the f.d.d. convergence of the sequence of reinforced skeletons towards a NRLP $\hat{\xi}$ of characteristics (a, q^2, Λ, p) was already established in Theorem 3.1 of [19].

Proposition 4.28. *Let $p < 1/2$ be an admissible memory parameter for the triplet (a, q^2, Λ). Then, the sequence of laws associated to the reinforced skeletons*

$$\big\{ \big(\hat{S}^{(n)}_{\lfloor nt \rfloor}\big)_{t \in [0,1]} : n \in \mathbb{N} \big\} \quad \textit{is tight in } D[0,1]. \tag{4.89}$$

Therefore, the convergence $\big(\hat{S}^{(n)}_{\lfloor nt \rfloor}\big)_{t\in[0,1]} \xrightarrow{\mathscr{L}} (\hat{\xi}_t)_{t\in[0,1]}$ holds in $D[0,1]$.

The reason behind the restriction $p < 1/2$ and why we don't expect our proof to work for $p \geqslant 1/2$ is explained in Remark 4.30, at the end of the proof.

Proof of Proposition 4.28 for centred ξ with compactly supported Lévy measure.

Until further notice, we restrict our reasoning to the case when ξ is a centred Lévy process, with Lévy measure Λ concentrated in $[-K, K]$ for some $K > 0$, and without loss of generality we suppose that $K = 1$. In consequence, ξ has finite moments of any order and we set $\sigma_n^2 := \mathbb{E}[\xi_{1/n}^2] = \mathbb{E}[(\hat{X}_1^{(n)})^2]$. Notice that under our standing hypothesis, $\hat{\xi}$ writes

$$t^p \int_0^t s^{-p} \mathrm{d}B_s + \hat{\xi}^{(3)}_t, \qquad t \in [0,1],$$

for some Brownian motion B independent of $\hat{\xi}^{(3)}$. Further, remark that under our restrictions, the family of discrete skeletons $(\hat{S}^{(n)}_k)$, for $n \in \mathbb{N}$, have typical steps centred and in $L_2(\mathbb{P})$. Consequently, we can make use of Lemma 4.26.

Next, to establish Proposition 4.28 under our current restrictions, we claim that it would be enough to show that the following convergence holds,

$$\Big(n^p a_{\lfloor nt \rfloor} \hat{S}^{(n)}_{\lfloor nt \rfloor} \Big)_{t\in[0,1]} \xrightarrow{\mathscr{L}} \big(t^{-p} \hat{\xi}_t \big)_{t \in [0,1]}, \tag{4.90}$$

where now, the sequence on the left-hand side of (4.90) is a sequence of martingales, while the process on the right hand side is the martingale introduced in Proposition 4.4, viz.

$$N_t := \int_0^t s^{-p} \mathrm{d}B_s + t^{-p} \hat{\xi}^{(3)}_t, \qquad t \in [0,1].$$

Indeed, for each n, let M^n be the continuous time version of martingale of Lemma 4.26 associated with the n-reinforced skeleton $(\hat{S}^{(n)}_k)_{k\in\mathbb{N}}$, i.e.

$$M^n_{\lfloor nt \rfloor} = a_{\lfloor nt \rfloor} \hat{S}^{(n)}_{\lfloor nt \rfloor}, \quad t \geqslant 0,$$

and remark that by Lemma 4.27, the predictable quadratic variation of $M^n_{\lfloor n\cdot \rfloor}$ is given by

$$\langle M^n, M^n \rangle_{\lfloor nt \rfloor} = \mathbb{1}_{\{1/n \leqslant t\}} \sigma_n^2 + \sum_{k=2}^{\lfloor nt \rfloor} a_k^2 \left((1-p)\sigma_n^2 - p^2 \frac{(\hat{S}^{(n)}_{k-1})^2}{(k-1)^2} + p \frac{\hat{V}^{(n)}_{k-1}}{k-1} \right).$$

It follows that for each $n \in \mathbb{N}$, the following process

$$N_t^n := n^p M_{\lfloor nt \rfloor}^n = n^p a_{\lfloor nt \rfloor} \hat{S}_{\lfloor nt \rfloor}^{(n)}, \quad t \geqslant 0,$$

is also a martingale, and its predictable quadratic variation writes:

$$\langle N^n, N^n \rangle_t = n^{2p} \langle M^n, M^n \rangle_{\lfloor nt \rfloor} = \mathbb{1}_{\{1/n \leqslant t\}} n^{2p} \sigma_n^2 + n^{2p} \sum_{k=2}^{\lfloor nt \rfloor} a_k^2 \left((1-p)\sigma_n^2 - p^2 \frac{(\hat{S}_{k-1}^{(n)})^2}{(k-1)^2} + p \frac{\hat{V}_{k-1}^{(n)}}{k-1} \right). \tag{4.91}$$

Moreover, by Stirling's formula, we have

$$a_n = \frac{\Gamma(n)}{\Gamma(n+p)} \sim n^{-p}, \quad \text{as } n \uparrow \infty,$$

which gives:

$$n^p a_{\lfloor nt \rfloor} \sim t^{-p}, \quad \text{as } n \uparrow \infty.$$

This yields the claimed equivalence between (4.89) and (4.90) under our current restrictions for ξ. For technical reasons, we shall prove first that the convergence of the martingales (N^n) towards N holds in the interval $[\varepsilon, 1]$, for any $\varepsilon > 0$. This leads us to the following lemma:

Lemma 4.29. *For any $\varepsilon > 0$, the sequence $(N_t^n)_{t \in [\varepsilon,1]}$ for $n \in \mathbb{N}$ is tight.*

Proof. We denote by $(\mathscr{F}_t^n)$ the natural filtration of N^n. By Aldous's tightness criterion (see for e.g. Kallenberg [57] Theorem 16.11), it is enough to show that for any sequence (τ_n) of (bounded) $(\mathscr{F}_t^n)$-stopping times in $[\varepsilon, 1]$ and any sequence of positive real numbers (h_n) converging to 0, we have

$$\lim_{n \uparrow \infty} |N_{\tau_n + h_n}^n - N_{\tau_n}^n| = 0, \quad \text{in probability.}$$

By Rebolledo's Theorem (see e.g. Theorem 2.3.2 in Joffre and Metivier [56]) it's enough to show that the sequence of associated predictable quadratic variations $(\langle N^n, N^n \rangle)$ satisfies Aldous's tightness criterion, i.e. that

$$\lim_{n \uparrow \infty} \langle N^n, N^n \rangle_{\tau_n + h_n} - \langle N^n, N^n \rangle_{\tau_n} = 0, \quad \text{in probability.}$$

In this direction, by (4.91), we have

$$\begin{aligned} \langle N^n, N^n \rangle_{\tau_n + h_n} - \langle N^n, N^n \rangle_{\tau_n} &= n^{2p} \sum_{k=\lfloor n\tau_n \rfloor + 1}^{\lfloor n(\tau_n + h_n) \rfloor} a_k^2 \left((1-p)\sigma_n^2 - p^2 \frac{(\hat{S}_{k-1}^{(n)})^2}{(k-1)^2} + p \frac{\hat{V}_{k-1}^{(n)}}{k-1} \right) \\ &\leqslant (1-p) n^{2p} \sum_{k=\lfloor n\tau_n \rfloor + 1}^{\lfloor n(\tau_n + h_n) \rfloor} a_k^2 \sigma_n^2 + p \cdot n^{2p} \sum_{k=\lfloor n\tau_n \rfloor + 1}^{\lfloor n(\tau_n + h_n) \rfloor} a_k^2 \frac{\hat{V}_{k-1}^{(n)}}{k-1}, \end{aligned} \tag{4.92}$$

and it remains to show that both terms in the right hand side converge to 0 in probability as $n \uparrow \infty$. The key now is in the asymptotic behaviour of the series $\sum_{k=1}^n a_k^2$. As was already pointed out in [11], for $p \in (0, 1/2)$, we have

$$\lim_{n \uparrow \infty} n^{2p-1} \sum_{k=1}^n a_k^2 = \frac{1}{1-2p}. \tag{4.93}$$

Furthermore, since the Lévy measure of ξ is compactly supported, it holds that

$$\sigma_n^2 = \mathbb{E}\left[(\hat{X}_1^{(n)})^2\right] = \mathbb{E}\left[\xi_{1/n}^2\right] = O(1/n), \quad \text{as } n \uparrow \infty. \tag{4.94}$$

Now, from (4.93) and (4.94) it follows that

$$\lim_{n\uparrow\infty}(1-p)n^{2p-1}\sum_{k=\lfloor n\tau_n\rfloor+1}^{\lfloor n(\tau_n+h_n)\rfloor} a_k^2 = 0 \quad \text{a.s.}$$

and a fortiori in probability, which entails that the first term in (4.92) converges in probability to 0

as $n \uparrow \infty$. In order to show that the second term in (4.92) also converges in probability to 0, we need to proceed more carefully. First, since $\tau_n \in [\varepsilon, 1]$, we can bound the second term in (4.92) by

$$n^{2p}\sum_{k=\lfloor n\tau_n\rfloor+1}^{\lfloor n(\tau_n+h_n)\rfloor} a_k^2 \frac{\hat{V}_{k-1}^{(n)}}{k-1} \leqslant n^{2p}\frac{\sup_{\lfloor n\varepsilon\rfloor\leqslant k\leqslant n}\hat{V}_k^{(n)}}{\lfloor n\varepsilon\rfloor}\sum_{k=\lfloor n\tau_n\rfloor+1}^{\lfloor n(\tau_n+h_n)\rfloor} a_k^2.$$

Next, since $\frac{n^{2p}}{\lfloor n\varepsilon\rfloor} \sim n^{2p-1}\varepsilon^{-1}$, in order to proceed as before we need to show that

$$\sup_{\lfloor n\varepsilon\rfloor\leqslant k\leqslant n}\hat{V}_k^{(n)} = O(1), \quad \text{in probability as } n \uparrow \infty,$$

i.e. that the sequence is stochastically bounded. To do so we proceed as follows: for each n, notice that the process

$$\hat{V}_k^{(n)} = (\hat{X}_1^{(n)})^2 + \cdots + (\hat{X}_k^{(n)})^2, \quad k \geqslant 1,$$

is the reinforced version of the random walk

$$V_k^{(n)} = (X_1^{(n)})^2 + \cdots + (X_k^{(n)})^2, \quad k \geqslant 1,$$

where $(\hat{X}_i^{(n)})^2$ are i.i.d. variables with law $\xi_{1/n}^2$. In order to have a centred noise reinforced random walk, for $k \geqslant 1$ set $\hat{Y}_k^{(n)} := (\hat{X}_k^{(n)})^2 - \mathbb{E}[\xi_{1/n}^2]$ and we introduce:

$$\begin{aligned}\hat{W}_k^{(n)} &:= \hat{V}_k^{(n)} - k\mathbb{E}\left[\xi_{1/n}^2\right] \\ &= \left((\hat{X}_1^{(n)})^2 - \mathbb{E}\left[\xi_{1/n}^2\right]\right) + \cdots + \left((\hat{X}_k^{(n)})^2 - \mathbb{E}\left[\xi_{1/n}^2\right]\right) = \hat{Y}_1^{(n)} + \cdots + \hat{Y}_k^{(n)},\end{aligned}$$

Now, the process $(\hat{W}_k^{(n)})_{k\in\mathbb{N}}$ is the noise reinforced version of the centred random walk defined for $k \geqslant 1$ as

$$\begin{aligned}W_k^{(n)} &= V_k^{(n)} - k\mathbb{E}\left[\xi_{1/n}^2\right] \\ &= \left((X_1^{(n)})^2 - \mathbb{E}\left[\xi_{1/n}^2\right]\right) + \cdots + \left((X_k^{(n)})^2 - \mathbb{E}\left[\xi_{1/n}^2\right]\right) = Y_1^{(n)} + \cdots + Y_k^{(n)},\end{aligned}$$

where $(Y_i^{(n)})_{i\in\mathbb{N}}$ are i.i.d. with law $\xi_{1/n}^2 - \mathbb{E}[\xi_{1/n}^2]$. We can now apply Corollary 4.3 from [21] to $W^{(n)}$: recalling that $\sigma_n^2 = \mathbb{E}[\xi_{1/n}^2] = O(1/n)$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{k\leqslant n}\left(\hat{V}_k^{(n)}\right)^2\right] = \mathbb{E}\left[\sup_{k\leqslant n}\left(\hat{W}_k^{(n)} + k\sigma_n^2\right)^2\right] &\leqslant C\mathbb{E}\left[\sup_{k\leqslant n}\left(\hat{W}_k^{(n)}\right)^2\right] + O(1) \\ &\leqslant C'\mathbb{E}\left[\left(\xi_{1/n}^2 - \mathbb{E}\left[\xi_{1/n}^2\right]\right)^2\right]n + O(1).\end{aligned}$$

Once again, since ξ has compactly supported Lévy measure, $\mathbb{E}[\xi_{1/n}^2]$ and $\mathbb{E}[\xi_{1/n}^4]$ are both $O(1/n)$ as $n \uparrow \infty$ and we deduce that

$$\mathbb{E}\left[\sup_{k\leqslant n}\left(\hat{V}_k^{(n)}\right)^2\right] = O(1), \qquad \text{as } n \uparrow \infty.$$

Hence, by Markov's inequality the sequence $(\sup_{k\leqslant n} \hat{V}_k^{(n)})_n$ is $O(1)$ in probability and we can conclude as before by bounding as follows for $L > 0$:

$$\mathbb{P}\left(n^{2p}\frac{\sup_{\lfloor n\varepsilon\rfloor\leqslant k\leqslant n} V_k^{(n)}}{\lfloor n\varepsilon\rfloor}\sum_{k=\lfloor n\tau_n\rfloor+1}^{\lfloor n(\tau_n+h_n)\rfloor} a_k^2 > \eta\right)$$
$$\leqslant \mathbb{P}\left(\sup_{k\leqslant n}\hat{V}_k^{(n)} > L\right) + \mathbb{P}\left(L\frac{n^{2p}}{\lfloor n\varepsilon\rfloor}\sum_{k=\lfloor n\tau_n\rfloor+1}^{\lfloor n(\tau_n+h_n)\rfloor} a_k^2 > \eta\right).$$

□

We shall now conclude the proof of Proportion 4.28 under our standing assumptions, and in this direction recall our discussion prior to Lemma 4.29. To extend the convergence to the interval $[0,1]$ we shall use a truncation argument similar to the one employed in Section 4.3 of [21]. For each $\varepsilon > 0$, we have $(N_t^n)_{t\in[\varepsilon,1]} \xrightarrow{\mathscr{L}} (N_t)_{t\in[\varepsilon,1]}$ and since $(N_{t+\varepsilon})_{t\in[0,1]} \to (N_t)_{t\in[0,1]}$ by right continuity (extending $N_{\cdot+\varepsilon}$ for $t \in [1-\varepsilon, 1]$ identically as the constant N_1), we deduce by metrisability of the weak convergence that there exists some sequence $(\varepsilon(n))_{n\in\mathbb{N}}$, converging to 0 slowly enough as $n \uparrow \infty$ such that $(N_t^n)_{t\in[\varepsilon(n),1]} \xrightarrow{\mathscr{L}} (N_t)_{t\in[0,1]}$ and we only need to show:

$$\sup_{s\leqslant\varepsilon(n)} n^p a_{\lfloor ns\rfloor}\hat{S}_{\lfloor ns\rfloor}^{(n)} \to 0, \qquad \text{in probability as } n \uparrow \infty. \tag{4.95}$$

In this direction, notice the inequality

$$\langle N^n, N^n\rangle_s \leqslant n^{2p}\sigma_n^2 + n^{2p}\sum_{k=2}^{\lfloor ns\rfloor} a_k^2\left((1-p)\sigma_n^2 + p\frac{\hat{V}_{k-1}^{(n)}}{k-1}\right).$$

Since $\mathbb{E}[\hat{V}_k^{(n)}] = k\sigma_n^2$, an application of Doob's inequality and the previous display yield that, for any $\delta > 0$, we have

$$\mathbb{P}\left(\sup_{s\leqslant\varepsilon(n)}|n^p a_{\lfloor ns\rfloor}\hat{S}_{\lfloor ns\rfloor}^{(n)}| \geqslant \delta\right) \leqslant \delta^{-2}\mathbb{E}\left[\langle N^{(n)}, N^{(n)}\rangle_{\varepsilon(n)}\right]$$
$$\leqslant \delta^{-2}n^{2p}\sigma_n^2 + \delta^{-2}\mathbb{E}\left[n^{2p}\sum_{k=2}^{\lfloor nt\rfloor} a_k^2\left((1-p)\sigma_n^2 + p\frac{\hat{V}_{k-1}^{(n)}}{k-1}\right)\right]$$
$$\leqslant \delta^{-2}\sigma_n^2 n^{2p}\sum_{n=2}^{\lfloor n\varepsilon(n)\rfloor} a_k^2.$$

From the asymptotics,

$$\lim_{n\uparrow\infty} n^{2p-1}\sum_{k=1}^{n} a_k^2 = \frac{1}{1-2p}, \qquad \text{and} \qquad \sigma_n^2 = O(1/n),$$

we deduce that, as $n \uparrow \infty$, the convergence (4.95) holds and we can conclude by an application of Lemma 3.31 - VI from Jacod and Shiryaev [54].

Remark 4.30. Before proceeding, we point out that our proof no longer works for $p \geqslant 1/2$: indeed, one might notice that the change in the asymptotic behaviour of the series $\sum_{k=1}^{n} a_k^2$ for $p \geqslant 1/2$ makes the preceding reasoning unfruitful. Let us be more precise: these series possess three different asymptotic regimes depending on p and are the reason behind the different regimes appearing in the behaviour of the Elephant random walk, see e.g. [11]. More generally, they are behind the three regimes appearing in the invariance principles [21, 16]. When $p \geqslant 1/2$, there is no Brownian component and the martingale $t^{-p}\hat{\xi}^{(3)}$ is no longer in $L_2(\mathbb{P})$ because $Y(t) \in L_q(\mathbb{P})$ for $q < 1/p$. Since N^n is converging weakly towards $t^{-p}\hat{\xi}_t^{(3)}$ by (4.90), working with the sequence of quadratic variations $\langle N^n, N^n \rangle$ might not be the right approach to obtain tightness.

Proof of Proposition 4.28, general case.

Let us start by introducing some notation. First, if $\hat{\mathcal{N}}$ is the jump measure of $\hat{\xi}$, we will shorten our notation for the compensated integrals and simply write $\hat{\xi}_{u,v}^{(3)}(t) := (\mathbb{1}_{\{u \leqslant |x| < v\}} x * \hat{\mathcal{N}}^{(sc)})_t$, for $0 \leqslant u < v$. Hence, for $K > 1$, we can write

$$\hat{\xi}_{0,K}^{(3)}(t) = \hat{\xi}_t^{(3)} + \hat{\xi}_{1,K}^{(3)}(t), \quad t \in [0,1].$$

It will also be convenient to introduce the following notation for the sums of jumps: for fixed $0 < a < b$, we write

$$\Sigma_{a,b}(t) := \sum_{s \leqslant t} \mathbb{1}_{\{a \leqslant |\Delta\hat{\xi}_s| < b\}} \Delta\hat{\xi}_s = \sum_i \mathbb{1}_{x_i \in [a,b)} x_i Y_i(t), \quad \text{for } t \in [0,1], \tag{4.96}$$

so that in particular we have $\hat{\xi}^{(2)} = \Sigma_{1,\infty}$. Next, if ξ can be decomposed into $\xi = L^{(1)} + L^{(2)}$, for independent Lévy processes $L^{(1)}, L^{(2)}$, we denote its reinforced skeleton by $\hat{S}^{(n)}(\xi) = (\hat{S}_{[nt]}^{(n)}(\xi))_{t \in [0,1]}$ and we write:

$$\hat{S}^{(n)}(\xi) = \hat{S}^{(n)}(L^{(1)}) + \hat{S}^{(n)}(L^{(2)}),$$

for the decomposition that is naturally induced. More precisely, the two noise reinforced random walks in the right-hand side of the previous display are made with the same sequence of Bernoulli random variables as $\hat{S}(\xi)$, and just result from decomposing each increment as

$$\Delta^{(n)}\xi_i = +\Delta^{(n)} L_i^{(1)} + \Delta^{(n)} L_i^{(2)}.$$

Now, we proceed by lifting progressively our restriction imposed in 4.5.3 as follows:

Step 1: First, if ξ satisfies that $\xi = M^{\leqslant K}$ where $M^{\leqslant K}$ is the sum of a Brownian motion with diffusion q and a compensated martingale with jumps smaller than K, by 4.5.3 the following convergence holds in distribution:

$$\hat{S}^{(n)}\left(M^{\leqslant K}\right) \xrightarrow{\mathscr{L}} q\hat{B} + \hat{\xi}_{0,K}^{(3)}, \quad \text{as } n \uparrow \infty. \tag{\textit{End of Step 1}}$$

Step 2: If b is a deterministic constant, let $b \cdot Id := (bt : t \geqslant 0)$ and suppose now that ξ can be written as $\xi = b \cdot Id + M^{\leqslant K}$. Then, we can write

$$\hat{S}^{(n)}(\xi) = \hat{S}^{(n)}(b \cdot Id) + \hat{S}^{(n)}(M^{\leqslant K}),$$

where the sequence of processes $(\hat{S}^{(n)}(b\cdot Id) : n \geqslant 1)$ is deterministic and converges uniformly to the continuous function $b\cdot Id$. Indeed, notice that the reinforcement doesn't affect the drift term since $\hat{S}^{(n)}(b\cdot Id)_t = b\lfloor nt \rfloor / n$. We deduce from [54, Lemma 3.33] that, as $n\uparrow\infty$, we still have

$$\hat{S}^{(n)}\left(b\cdot Id + M^{\leqslant K}\right) = \hat{S}^{(n)}(b\cdot Id) + \hat{S}^{(n)}\left(M^{\leqslant K}\right) \xrightarrow{\mathscr{L}} b\cdot Id + q\hat{B} + \hat{\xi}^{(3)}_{0,K}. \tag{4.97}$$

(*End of Step 2*)

From here, we work with the Lévy process ξ with triplet (a, q^2, Λ), with Lévy-Itô decomposition given by:

$$\xi = a\cdot Id + M^{\leqslant 1} + \xi^{(3)},$$

and we denote its jump measure by $\mathcal{N}$ – in particular, we have $\xi^{(3)} = \mathbb{1}_{(-1,1)^c} x * \mathcal{N}$. For any $K > 1$, we can rearrange the triplet by compensating and modifying appropriately the drift coefficient, in such a way that we have:

$$\xi = b_K Id + M^{\leqslant K} + \xi^{\geqslant K},$$

where $\xi^{\geqslant K} := \mathbb{1}_{(-K,K)^c} x * \mathcal{N}$. Before moving to *Step 3*, let us make the two following remarks.

• First, notice that for each fixed n, $S^{(n)}\left(\xi^{\geqslant K}\right) \xrightarrow{\mathbb{P}} 0$ uniformly in probability as $K\uparrow\infty$. Indeed, we have

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |\hat{S}^{(n)}_{\lfloor nt \rfloor}\left(\xi^{\geqslant K}\right)| > \varepsilon\Big) \leqslant \mathbb{P}\left(\Delta\xi^{\geqslant K}_t \neq 0 \text{ for some } t\in[0,1]\right),$$

where the right-hand side can be written in terms of the jump process $\mathcal{N}$ of ξ as

$$\mathbb{P}\Big(\mathcal{N}\big(\{(t,x)\in[0,1]\times\mathbb{R} : |x|\geqslant K\}\big)\geqslant 1\Big) = 1 - e^{-((-\infty,K]\cup[K,\infty))}. \tag{4.98}$$

The right-hand side in the previous display converges to 0 as $K\uparrow\infty$ and notice that the bound does not depend on n.

• Let $\hat{\xi}$ be the noise reinforced Lévy process of characteristics (a, q^2, Λ, p) and write its jump measure by $\hat{\mathcal{N}}$. Again, we can rewrite $\hat{\xi}$, by compensating appropriately and modifying the drift coefficient, as follows:

$$\hat{\xi} = b_K Id + q\hat{B} + \hat{\xi}^{(3)}_{0,K} + \Sigma_{K,\infty}.$$

Arguing as before, we have the uniform convergence in probability $b_K Id + q\hat{B} + \hat{\xi}^{(3)}_{0,K} \xrightarrow{\mathbb{P}} \hat{\xi}$ as $K\uparrow\infty$, since, by the description of $\hat{\mathcal{N}}$ given in Definition 4.9, we have

$$\mathbb{P}\Big(\sup_{t\in[0,1]} \big|\Sigma_{K,\infty}(t)\big| \geqslant \varepsilon\Big) \leqslant \mathbb{P}\left(\hat{\mathcal{N}}\big(\{(t,x)\in[0,1]\times\mathbb{R} : |x|\geqslant K\}\big)\geqslant 1\right) = 1 - e^{-(1-p)\Lambda((-\infty,K]\cup[K,\infty))}.$$

Step 3: To conclude, for $K > 1$, we write respectively the Lévy process and the corresponding NRLP without their jumps of size greater than K as

$$\xi^{\leqslant K} := b_K Id + M^{\leqslant K}, \qquad \text{and} \qquad \hat{\xi}^{\leqslant K} := b_K Id + q\hat{B} + \hat{\xi}^{(3)}_{0,K}.$$

In (4.97), we already proved that for each fixed K, we have

$$\hat{S}^{(n)}(\xi^{\leqslant K}) \xrightarrow{\mathscr{L}} \hat{\xi}^{\leqslant K}, \text{ as } n\uparrow\infty, \qquad \text{while by our second remark, it holds that} \qquad \hat{\xi}^{\leqslant K} \xrightarrow{\mathscr{L}} \hat{\xi}, \text{ as } K\uparrow\infty.$$

Since the convergence in distribution is metrisable, there exists an increasing sequence $(K(n) : n \geqslant 1)$ converging to infinity slowly enough as $n \uparrow \infty$, such that

$$\hat{S}^{(n)}\big(\xi^{\leqslant K(n)}\big) \xrightarrow{\mathscr{L}} \hat{\xi}, \qquad \text{as } n \uparrow \infty.$$

Moreover, we can write

$$\hat{S}^{(n)}\left(\xi\right) = \hat{S}^{(n)}\big(\xi^{\leqslant K(n)}\big) + \hat{S}^{(n)}\big(\xi^{\geqslant K(n)}\big),$$

where for each $\varepsilon > 0$, by (4.98) we have:

$$\lim_{n\uparrow\infty} \mathbb{P}\Big(\sup_{t\in[0,1]} |\hat{S}^{(n)}_{\lfloor nt\rfloor}\left(\xi^{\geqslant K(n)}\right)| > \varepsilon\Big) \leqslant \lim_{n\uparrow\infty} 1 - e^{-\Lambda((-\infty,K(n)]\cup[K(n),\infty))} = 0.$$

We can now apply [54, Lemma 3.31, Chapter VI] to deduce that the convergence $\hat{S}^{(n)}(\xi) \xrightarrow{\mathscr{L}} \hat{\xi}$ holds.

(*End of Step 3*)

With this last result we conclude the proof of Proposition 4.28.

Convergence of finite-dimensional distributions

We maintain the notation and setting introduced at the beginning of Section 4.5.

Proposition 4.31. *Let ξ be a Lévy process of characteristic triplet (a, q^2, Λ) and denote its characteristic exponent by Ψ. Fix $p \in (0,1)$ an admissible memory parameter, and for each n, let $(S^{(n)}_k, \hat{S}^{(n)}_k)$ be the sequence of n-skeletons and its corresponding reinforced versions as defined in (4.60). Then, there is the weak convergence in the sense of finite-dimensional distributions,*

$$\Big((S^{(n)}_{\lfloor nt\rfloor})_{t\in[0,1]}, (\hat{S}^{(n)}_{\lfloor nt\rfloor})_{t\in[0,1]} \Big) \xrightarrow{f.d.d.} \Big((\xi_t)_{t\in[0,1]}, (\hat{\xi}_t)_{t\in[0,1]} \Big), \tag{4.99}$$

where we denoted by $(\xi, \hat{\xi})$ a pair of processes with law characterised by (4.58).

Remark that since the convergence is in the sense of finite dimensional distributions, the restriction $p < 1/2$ is dropped. Our proof will rely on two results taken respectively from [19] and [26]. We state them without proof for ease of reading:

Corollary 3.7 of [19] *Let F be a continuous functional on counting functions such that $F(0) = 0$ where, with a slight abuse of notation we still write 0 for the identically 0 trajectory. Further, suppose that there exists $c > 0$ and $1 \leqslant \gamma < 1/p$ such that $|F(\omega)| \leqslant c\omega(1)^\gamma$ for every counting function $\omega : [0,1] \to \mathbb{N}$. Then, if Y is a Yule-Simon process with parameter $1/p$, the following convergence holds in $L_1(\mathbb{P})$:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F\left(N_j(\lfloor n\cdot\rfloor)\right) = (1-p)\mathbb{E}\left[F(Y)\right]. \tag{4.100}$$

The second result concerns the asymptotic behaviour of Ψ.

Lemma 3.1 of [26] *The asymptotic behaviour of the characteristic exponent Ψ as $|z| \uparrow \infty$ is given by:*

$$|\Psi(z)| = \begin{cases} o(|z|^{2+\eta}) & \text{when } q \neq 0 \\ o(|z|^{\beta(\Lambda)+\eta}) & \text{when } q = 0 \text{ and } \int_{|x|\leqslant 1} |x|\Lambda(\mathrm{d}x) = \infty \\ o(|z|^{1+\eta}) & \text{when } q = 0 \text{ and } \int_{|x|\leqslant 1} |x|\Lambda(\mathrm{d}x) < \infty. \end{cases}$$

Now we have all the ingredients needed for the proof of Proposition 4.31.

Proof. We fix $k \geqslant 1$, $0 < \lambda_1 < \cdots < \lambda_k \leqslant 1$, and let $\beta_1, \dots \beta_k$ be real numbers. In order to establish the finite dimensional convergence, it suffices to show that

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\left(\lambda_j S_{\lfloor nt_j\rfloor} + \beta_j \hat{S}_{\lfloor nt_j\rfloor}\right)\right\}\right], \tag{4.101}$$

converges as $n \uparrow \infty$ towards (4.58). In this direction, for each n, we write $(N_\ell^{(n)}(k))_{k\geqslant1,\ell\geqslant1}$ the counting process of repetitions of $\hat{S}^{(n)}$ introduced in (4.67). Recalling the identity (4.68), we can write,

$$\hat{S}^{(n)}_{\lfloor nt\rfloor} = \sum_{\ell=1}^{n} N_\ell^{(n)}(\lfloor nt\rfloor) X_\ell^{(n)} \qquad \text{and} \qquad S^{(n)}_{\lfloor nt\rfloor} = \sum_{\ell=1}^{n} \mathbb{1}_{\{\ell\leqslant\lfloor nt\rfloor\}} X_\ell^{(n)},$$

with $\mathbb{E}[e^{i\lambda X_\ell^{(n)}}] = e^{\frac{1}{n}\Psi(\lambda)}$ for every ℓ. Then, by independence of the counting processes $(N_\ell^{(n)}(k))_{k\geqslant1,\ell\geqslant1}$ from the sequence $(X_\ell^{(n)})_{\ell\geqslant0}$, the characteristic function (4.101) can be written as follows

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k}\left(\lambda_j S_{\lfloor nt_j\rfloor} + \beta_j \hat{S}_{\lfloor nt_j\rfloor}\right)\right\}\right]\\
&= \mathbb{E}\left[\exp\left\{i\sum_{\ell=1}^{n}\left(\sum_{j=1}^{k}\left(\lambda_j N_\ell^{(n)}(\lfloor nt_j\rfloor) + \beta_j \mathbb{1}_{\{\ell\leqslant\lfloor nt_j\rfloor\}}\right)\right)X_\ell^{(n)}\right\}\right]\\
&= \mathbb{E}\left[\exp\left\{-\frac{1}{n}\sum_{\ell=1}^{n}\Psi\left(\sum_{j=1}^{k}\lambda_j N_\ell^{(n)}(\lfloor nt_j\rfloor) + \beta_j \mathbb{1}_{\{\ell\leqslant\lfloor nt_j\rfloor\}}\right)\right\}\right].
\end{aligned}$$

Remark that since the law of $(N_\ell^{(n)}(k))_{k\geqslant1,\ell\geqslant1}$ doesn't depend on n, we can drop the up-script (n) in the last display. Next, recall that $N_\ell(\lfloor nt\rfloor) = 0$ for all $t \in [0,1]$ if $\varepsilon_\ell = 1$ while on the other hand, if $\varepsilon_\ell = 1$, $N_\ell(\lfloor ns\rfloor) = 0$ for $\lfloor ns\rfloor < l$, and $N_\ell(\lfloor ns\rfloor) \geqslant 1$ if $\lfloor ns\rfloor \geqslant l$. Hence, we have:

$$\mathbb{1}_{\{\ell\leqslant\lfloor ns\rfloor\}} = \mathbb{1}_{\left\{N_\ell\lfloor ns\rfloor\geqslant1\right\}}, \qquad \text{on } \{\varepsilon_\ell = 0\}.$$

By the previous observations, we can write:

$$\begin{aligned}
&\frac{1}{n}\sum_{\ell=1}^{n}\Psi\left(\sum_{j=1}^{k}\lambda_j N_\ell(\lfloor nt_j\rfloor) + \beta_j \mathbb{1}_{\{\ell\leqslant\lfloor nt_j\rfloor\}}\right) \qquad (4.102)\\
&= \frac{1}{n}\sum_{\ell=1}^{n}\Psi\left(\sum_{j=1}^{k}\lambda_j N_\ell(\lfloor nt_j\rfloor) + \beta_j \mathbb{1}_{\left\{N_\ell\lfloor nt_j\rfloor\geqslant1\right\}}\right)\mathbb{1}_{\{\varepsilon_\ell=0\}} + \frac{1}{n}\sum_{\ell=1}^{n}\Psi\left(\sum_{j=1}^{k}\beta_j \mathbb{1}_{\{\ell\leqslant\lfloor nt_j\rfloor\}}\right)\mathbb{1}_{\{\varepsilon_\ell=1\}}.
\end{aligned}$$

Now, let us establish the convergence in probability of both terms in the previous display separately. Starting with the first one, we introduce the functional $F : D[0,1] \to \mathbb{C}$ defined as follows:

$$F(\omega) := \Psi\left(\sum_{j=1}^{k}\lambda_j\omega(t_j) + \beta_j \mathbb{1}_{\{\omega(t_j)\in[1,\infty]\}}\right), \tag{4.103}$$

for $\omega : [0,1] \to \mathbb{N}$ a generic counting function. This is a $\mathbb{Q}$ - a.s. continuous functional, since $\omega \mapsto \mathbb{1}_{\{\omega(s)\in[1,\infty]\}}$ can be written as $\omega \mapsto \omega(s) \wedge 1$, which is a composition of a $\mathbb{Q}$-a.s. continuous functional with the continuous mapping $x \mapsto x \wedge 1$. Moreover, we have $F(0) = 0$, and notice that we can bound:

$$\left| \sum_{j=1}^{k} \lambda_j \omega(t_j) + \beta_j \mathbb{1}_{\{\omega(t_j) \in [1,\infty]\}} \right| \leqslant w(1) \left(\sum_{j=1}^{k} |\lambda_j| + |\beta_j| \right),$$

by monotonicity of ω and the inequality $\mathbb{1}_{\{\omega(s)\in[1,\infty]\}} \leqslant \omega(s)$. Now, by Lemma 3.1 of [26], we deduce that F satisfies the hypothesis of Corollary 3.7 from [19], since

$$|F(\omega)| \leqslant |\omega(1)|^\gamma K \left(\sum_{j=1}^{k} |\lambda_j| + |\beta_j| \right)^\gamma, \quad \text{with} \quad \begin{cases} \gamma \in (2, 1/p), & \text{if } q \neq 0 \\ \gamma \in (1, 1/p), & \text{if } q = 0, \end{cases} \tag{4.104}$$

for a constant K that only depends on $\beta(\Lambda)$ and q. From an application of Corollary 3.7 of [19], we obtain the following convergence:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \Psi\left(\sum_{j=1}^{k} \lambda_j N_\ell(\lfloor nt_j \rfloor) + \beta_j \mathbb{1}_{\left\{N_\ell \lfloor nt_j \rfloor \geqslant 1\right\}} \right) \mathbb{1}_{\{\varepsilon_\ell = 0\}} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \Psi\left(\sum_{j=1}^{k} \lambda_j N_\ell(\lfloor nt_j \rfloor) + \beta_j \mathbb{1}_{\left\{N_\ell \lfloor nt_j \rfloor \geqslant 1\right\}} \right) \\ &= (1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \lambda_j Y(t_j) + \beta_j \mathbb{1}_{\left\{Y(t_j)\geqslant 1\right\}} \right) \right]. \end{aligned} \tag{4.105}$$

Turning our attention to the second term, similarly, we claim that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \Psi\left(\sum_{j=1}^{k} \beta_j \mathbb{1}_{\{\ell \leqslant \lfloor nt_j \rfloor\}} \right) \mathbb{1}_{\{\varepsilon_\ell = 1\}} = p\mathbb{E}\left[\Psi\left(\sum_{j=1}^{n} \beta_j \mathbb{1}_{\left\{U \leqslant t_j\right\}} \right) \right]. \tag{4.106}$$

Indeed, if for each n we denote by $u(n)$ a uniform random variable on $\{1, \ldots, n\}$ independent of the i.i.d. sequence $(\varepsilon_n)_n$ of Bernoulli with parameter p, we have

$$\begin{aligned} \mathbb{E}\left[\frac{1}{n} \sum_{\ell=1}^{n} \Psi\left(\sum_{j=1}^{k} \beta_j \mathbb{1}_{\{\ell \leqslant \lfloor nt_j \rfloor\}} \right) \mathbb{1}_{\{\varepsilon_l = 1\}} \right] &= \mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \beta_j \mathbb{1}_{\{u(n) \leqslant \lfloor nt_j \rfloor\}} \right) \mathbb{1}_{\{\varepsilon_{u(n)} = 1\}} \right] \\ &= \mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \beta_j \mathbb{1}_{\{u(n) \leqslant \lfloor nt_j \rfloor\}} \right) \right] p, \end{aligned} \tag{4.107}$$

since $\varepsilon_{u(n)}$ is independent of $u(n)$ for each n. Further, since $u(n)/n$ converges in law towards a uniform random variable in $[0, 1]$, the sequence of step processes $(\mathbb{1}_{\{u(n) \leqslant \lfloor n \cdot \rfloor\}})_{n\in\mathbb{N}}$ converges weekly towards $\mathbb{1}_{\{U \leqslant \cdot\}}$. Consequently, as $n \uparrow \infty$, (4.107) converges towards

$$p\,\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \beta_j \mathbb{1}_{\{U \leqslant t_j\}} \right) \right],$$

where we recall that $\mathbb{1}_{\{U \leqslant t\}}$ has the same distribution as $\mathbb{1}_{\{Y(t)\geqslant 1\}}$ by the description (4.2). Finally, recall the identity of Proposition 4.18 for characteristic function of the finite dimensional distributions of the pair $(\xi, \hat{\xi})$. It follows from (4.102) and the limits (4.105), (4.106) that as $n \uparrow \infty$, we have the convergence towards the characteristic function of the finite-dimensional distributions of $(\xi, \hat{\xi})$,

$$\lim_{n\uparrow\infty} \mathbb{E}\left[\exp\left\{i \sum_{j=1}^{k} \left(\lambda_j S_{\lfloor nt_j \rfloor} + \beta_j \hat{S}_{\lfloor nt_j \rfloor}\right)\right\}\right]$$
$$= \exp\left\{p\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}}\right)\right] + (1-p)\mathbb{E}\left[\Psi\left(\sum_{j=1}^{k} \lambda_j \mathbb{1}_{\{Y(t_j)\geqslant 1\}} + \beta_i Y(s_i)\right)\right]\right\}.$$

□

This result paired with the tightness established in Proposition 4.28 proves Theorem 4.19.

4.6 Applications

We conclude this work with three sections devoted to applications.

4.6.1 Rates of growth at the origin

In this section we turn our attention to the trajectorial behaviour of noise reinforced Lévy processes at the origin. In this direction let us start by recalling a well known result established by Blumenthal and Getoor [26] for Lévy processes. Let ξ be a Lévy process with characteristic triplet (a, q^2, Λ) with no Gaussian component, viz. $q = 0$; in particular $\beta(\Lambda) = \beta$. Further, we make the following hypothesis:

• If $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) = \infty$, the characteristic exponent can be written as follows:

$$\Psi(\lambda) = \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{|x|\leqslant 1\}}\right) \Lambda(\mathrm{d}x).$$

Observe that in this case, we have $\beta(\Lambda) \in [1, 2]$.

• If $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) < \infty$, which can happen for $\beta(\Lambda) \in [0, 1]$, we suppose Ψ takes the following form:

$$\Psi(\lambda) = \int_{\mathbb{R}} \left(e^{i\lambda x} - 1\right) \Lambda(\mathrm{d}x).$$

This is, when $\int_{[0,1]} |x|\Lambda(\mathrm{d}x) < \infty$ we are supposing that the Lévy process has no linear drift, the reason being that in that case the behaviour at 0 is dominated by the drift term. We insist in the fact that when $\beta(\Lambda) = 1$ the integral $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x)$ can be finite or infinite.

We will be working for the rest of the section under these hypothesis, and we will refer to them as hypothesis (H). It was established by Blumenthal and Getoor in [26] that under (H), the behaviour at zero of a Lévy process is dictated by the Blumenthal-Getoor index of the Lévy measure Λ. More precisely, almost surely, we have:

$$\lim_{t\downarrow 0} t^{-\gamma}\xi_t = 0, \quad \text{if } \beta(\Lambda) < 1/\gamma \qquad \text{and} \qquad \limsup_{t\downarrow 0} t^{-\gamma}|\xi_t| = \infty, \quad \text{if } \beta(\Lambda) > 1/\gamma.$$

We will show that the same result still holds if we replace the Lévy process ξ by its noise reinforced version. Concretely, the main result of the section is the following:

Proposition 4.32. *Let ξ be a Lévy process with triplet (a, q^2, Λ) satisfying hypothesis* (H)*, and consider $\hat{\xi}$ its noise reinforced version for an admissible parameter p. Then, almost surely, we have*

$$\lim_{t \downarrow 0} t^{-\gamma} \hat{\xi}_t = 0, \qquad \text{if } \beta(\Lambda) < 1/\gamma, \tag{4.108}$$

while

$$\limsup_{t \downarrow 0} t^{-\gamma} |\hat{\xi}_t| = \infty, \qquad \text{if } \beta(\Lambda) > 1/\gamma. \tag{4.109}$$

The rest of the section is devoted to the proof of Proposition 4.32 and it is achieved in several steps. We start by proving the second statement (4.109), in Lemma 4.33 we prove (4.108) for $\beta(\Lambda) \geqslant 1$, $\int_{|x| \leqslant 1} |x| \Lambda(\mathrm{d}x) = \infty$ and the case $\beta(\Lambda) \leqslant 1$, $\int_{|x| \leqslant 1} |x| \Lambda(\mathrm{d}x) < \infty$ is treated separately in Lemma 4.35.

Proof of (4.108). It suffice to prove that for some $r > 0$ and $\varepsilon > 0$ a.s. there exists a sequence of jumps occurring in $[0, \varepsilon]$ at times, that we denote by (t_i), satisfying

$$|\Delta \hat{\xi}_{t_i}| > t_i^{\gamma - r}.$$

Now, recall from the discussion following (4.39) that the jump measure $\hat{\mathcal{N}}$ of $\hat{\xi}$ dominates a Poisson point process with intensity $(1-p)(\mathrm{d}u \otimes \Lambda)$, say $\mathcal{N}'$. If we denote the atoms of $\mathcal{N}'$ by (u_i, x_i), we deduce that

$$\#\{(u_i, x_i) \in \mathcal{N}' : u_i \in [0, \varepsilon] \text{ and } |x_i| > 2 u_i^{\gamma - r}\},$$

is distributed Poisson with parameter

$$(1-p)\mathrm{d}u \otimes \Lambda \left((u, x) \in [0, \varepsilon] \times \mathbb{R} : |x|^{1/(\gamma - r)} > 2 \cdot u \right) = \int_{\mathbb{R}} \left(2^{-1} |x|^{1/(\gamma - r)} \wedge \varepsilon \right) \Lambda(\mathrm{d}x)(1-p). \tag{4.110}$$

Now, take $r > 0$ small enough such that the inequality $1/(\gamma - r) < \beta(\Lambda)$ still holds. For such a choice of r, the integral (4.110) is infinite by definition of the index $\beta(\Lambda)$ and the claim follows. □

Now we focus on showing that $\lim_{t \downarrow 0} t^{-\gamma} |\hat{\xi}_t| = 0$ for $\gamma \in (0, 1/\beta(\Lambda))$. In this direction, let us start introducing some notation and with some preliminary remarks. First, notice that since we are interested in the behaviour of $\hat{\xi}$ at the origin, we can rely on the original construction in [19] in terms of Poissonian sums of Yule-Simon processes that we recalled in Section 4.2.3. Next, under (H), $\hat{\xi}$ can be written either as a sum of a compensated integral $\hat{\xi}^{(3)}$ and a reinforced compound Poisson process $\hat{\xi}^{(2)}$ viz.

$$\hat{\xi} = \hat{\xi}^{(3)} + \hat{\xi}^{(2)}, \qquad \text{if } \beta(\Lambda) > 1, \tag{4.111}$$

or as an absolutely convergent series of jumps,

$$\hat{\xi} = \sum_{s \leqslant t} \Delta \hat{\xi}_s, \ t \in [0, 1], \qquad \text{if } \beta(\Lambda) < 1. \tag{4.112}$$

We stress that if $\beta(\Lambda) = 1$, $\hat{\xi}$ takes the form (4.111) or (4.112) depending respectively on if $\int_{\{|x| \leqslant 1\}} |x| \Lambda(\mathrm{d}x)$ is infinite or not, and remark that γ can be strictly larger than one only when $\beta(\Lambda) < 1$. Since $\hat{\xi}^{(2)}$ is a finite sum of weighted Yule processes and $\hat{\xi}_0^{(2)} = 0$, independently of the value of $\beta(\Lambda)$ it holds that $\lim_{t \downarrow 0} t^{-\gamma} \hat{\xi}_t^{(2)} = 0$ and we can consequently restrict our study of (4.111) resp. (4.112) to the case where $\hat{\xi} = \hat{\xi}^{(3)}$ resp. $\hat{\xi}$ has Lévy measure concentrated in $[0, 1]$ – and hence is a reinforced, driftless subordinator.

Lemma 4.33. *Suppose that $\beta(\Lambda) \geqslant 1$ and let $\gamma \in (0, 1 \wedge 1/\beta(\Lambda))$. Then,*

$$\lim_{T\downarrow 0} \mathbb{E}\left[\sup_{s\leqslant T} s^{-\gamma}|\hat{\xi}_s^{(3)}|\right] = 0.$$

In particular, if $\beta(\Lambda) \geqslant 1$ with $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) = \infty$, we have $\lim_{t\downarrow 0} t^{-\gamma}|\hat{\xi}_t^{(3)}| = 0$ a.s.

Proof. Recall from Proposition 4.4 that $(t^{-p}\hat{\xi}_t^{(3)})_{t\in[0,1]}$ is a martingale. We start by fixing $s < u$ two times in $[0, 1]$ and notice that for any $r \in (\beta(\Lambda), 1/p \wedge 2)$ (or $r = 2$ if $\beta(\Lambda) = 2$), by Doob's inequality in $L_r(\mathbb{P})$ we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[s,u]} t^{-\gamma}|\hat{\xi}_t^{(3)}|\right] &\leqslant s^{-(\gamma-p)}\mathbb{E}\left[\sup_{t\in[s,u]} t^{-p}|\hat{\xi}_t^{(3)}|\right] \\ &\leqslant s^{-(\gamma-p)}\mathbb{E}\left[\sup_{t\in[s,u]} t^{-p\cdot r}|\hat{\xi}_t^{(3)}|^r\right]^{1/r} \leqslant c\cdot s^{-(\gamma-p)}u^{-p}\mathbb{E}\left[|\hat{\xi}_u^{(3)}|^r\right]^{1/r}, \end{aligned} \tag{4.113}$$

for some constant c. In order to bound the expectation on the right hand side, we recall from the proof of Lemma 2.6 in Bertoin [19] that the following bound holds [2] for some constant C large enough:

$$\mathbb{E}\left[|\hat{\xi}_u^{(3)}|^r\right]^{1/r} \leqslant C\mathbb{E}\left[\sum_j Y_j(u)^r|x_j|^r\right]^{1/r}. \tag{4.114}$$

Next, by Campbell's formula we have

$$\mathbb{E}\left[\sum_j Y_j(u)^r|x_j|^r\right] = \mathbb{E}\left[Y(u)^r\right]\int_{\{|x|<1\}} |x|^r\Lambda(\mathrm{d}x) < \infty, \tag{4.115}$$

and remark that $\mathbb{E}\left[Y(u)^r\right] = u \cdot \mathbb{E}\left[\eta^r\right]$ where η stands for a Yule-Simon random variable with parameter $1/p$. It now follows that we can bound $\mathbb{E}\left[|\hat{\xi}_u^{(3)}|^r\right]^{1/r} \leqslant K \cdot u^{1/r}$ for a positive constant K depending only on r. This observation paired with the bound we obtained in (4.113), yields:

$$\mathbb{E}\left[\sup_{t\in[s,u]} t^{-\gamma}|\hat{\xi}_t^{(3)}|\right] \leqslant s^{-(\gamma-p)}u^{-p+1/r}\cdot K, \tag{4.116}$$

for a finite constant K that only depends on the choice of r. Now, set $t_0 := 1$, $t_n := 2^{-n}$, for $n \geqslant 1$ and fix $N \in \mathbb{N}$. Applying the bound (4.116) to each interval $[2^{-(n+1)}, 2^{-n}]$, we get:

$$\mathbb{E}\left[\sup_{t\leqslant t_N} t^{-\gamma}|\hat{\xi}_t^{(3)}|\right] \leqslant \sum_{n\geqslant N} \mathbb{E}\left[\sup_{t\in[t_{n+1},t_n]} t^{-\gamma}|\hat{\xi}_t^{(3)}|\right] \leqslant 2^{\gamma-p}\sum_{n\geqslant N} 2^{n(\gamma-1/r)}, \tag{4.117}$$

and to conclude it suffices to show that, for an appropriate choice of r, the inequality $\gamma - 1/r < 0$ is satisfied. Since $r \in (\beta(\Lambda), 1/p\wedge 2)$, we can always choose ε small enough such that $r := \beta(\Lambda)+\varepsilon$ belongs to $(\beta(\Lambda), 1/p \wedge 2)$ and $\gamma < 1/(\beta(\Lambda) + \varepsilon)$, since we recall that $\gamma < 1/\beta(\Lambda)$. For such a particular choice of r, the series (4.117) converge and we obtain the desired result. In particular, this proves the statement of Proposition 4.32 when $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) = \infty$, which is when $\hat{\xi} = \hat{\xi}^{(3)}$. □

[2]The bound was first established for non-atomic Lévy measures Λ, but it was later shown that a similar bound holds if Λ has atoms by an approximation argument.

The statement (4.108) of Proposition 4.32 is incomplete only when the Lévy measure fulfils the integrability condition $\int_{\{|x|\leqslant 1\}}|x|\Lambda(\mathrm{d}x)<\infty$. Recalling the discussion prior to Lemma 4.33, we henceforth assume that the Lévy process is a driftless subordinator with jumps smaller than one, say (T_t), and we denote by $(\hat{T}_t)$ the corresponding reinforced version for a memory parameter $p\in(0,1)$. It is then convenient to work with its Laplace transform at time $t\in[0,1]$,

$$\mathbb{E}\left[e^{-\lambda\hat{T}_t}\right]=\exp\Big(-\mathbb{E}\left[\Phi(Y(t)\lambda)\right]\Big),\qquad \text{for } \lambda\geqslant 0,$$

for $\Phi(\lambda):=(1-p)\int_{\mathbb{R}^+}\left(1-e^{-x\lambda}\right)\Lambda(\mathrm{d}x)$ and Y is a Yule-Simon process with parameter $1/p$. The following result from [26] will be needed and we state it for the reader's convenience:

Theorem 4.34. [Blumenthal, Getoor][26] *If $\Phi(\lambda)$ is the Laplace exponent of a driftless subordinator with Lévy measure Λ, then for any $\varepsilon>0$,*

$$\Phi(\lambda)=o(\lambda^{\beta(\Lambda)+\varepsilon}),\qquad \textit{as } \lambda\uparrow\infty.$$

Let $\varepsilon>0$, fix $\lambda>0$ and observe from Theorem 4.34 that for $t\in(0,1)$, there exists positive constants K and R such that

$$\Phi(\eta\lambda t^{-\gamma})\leqslant\begin{cases}K & \text{if } \eta\lambda t^{-\gamma}\leqslant R,\\ \left(\lambda\eta t^{-\gamma}\right)^{\beta(\Lambda)+\varepsilon} & \text{if } \eta\lambda t^{-\gamma}>R.\end{cases}$$

Consequently, for $t\in(0,1)$ the following bound holds:

$$t\Phi(\eta\lambda t^{-\gamma})\leqslant t\left(K+\left(\lambda\eta t^{-\gamma}\right)^{\beta(\Lambda)+\varepsilon}\right)=tK+(\lambda\eta)^{\beta(\Lambda)+\varepsilon}t^{1-\gamma\beta(\Lambda)-\gamma\varepsilon}. \tag{4.118}$$

Lemma 4.35. *Let $\hat{T}$ be a reinforced subordinator of memory parameter p and Lévy measure Λ. Then, for any $\gamma\in\mathbb{R}^+$ such that $\gamma<1/\beta(\Lambda)$,*

$$\lim_{t\downarrow 0}t^{-\gamma}\hat{T}_t=0\quad a.s.$$

The proof relies on the same techniques used for subordinators, see [17, Proposition 10 - III.4].

Proof. Consider $t\in[0,1]$ and fix $a>0$. An application of Markov's inequality for $g(r)=1-e^{-r}$ and the inequality $g(r)\leqslant r$ for $r\geqslant 0$ yield

$$\begin{aligned}\mathbb{P}\big(\hat{T}_t>a\big)&\leqslant(1-e^{-1})^{-1}\left(1-\exp\left\{-\mathbb{E}\left[\Phi\left(a^{-1}Y(t)\right)\right]\right\}\right)\\&\leqslant(1-e^{-1})^{-1}\mathbb{E}\left[\Phi\left(a^{-1}Y(t)\right)\right].\end{aligned}$$

Since $\Phi(0)=0$ and $Y(t)$ conditioned to $Y(t)\geqslant 1$ follows the Yule-Simon distribution with parameter $1/p$, for a constant C we deduce the bound:

$$\mathbb{P}\big(\hat{T}_t>a\big)\leqslant Ct\mathbb{E}\left[\Phi\left(\eta/a\right)\right], \tag{4.119}$$

where we denoted by η a Yule Simon random variable with parameter $1/p$. Now, let h be an increasing function with $\lim_{t\downarrow 0}h(t)=0$, and consider $a=h(2^{-n})$, $t=2^{-(n-1)}$. Then, by (4.119) and from summing over $n\in\mathbb{N}$, we deduce

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\hat{T}_{2^{-(n-1)}}>h(2^{-n})\right)\leqslant 2C\mathbb{E}\left[\sum_{n=1}^{\infty}2^{-n}\Phi\left(\eta/h(2^{-n})\right)\right]. \tag{4.120}$$

In order to apply a Borel-Cantelli argument, we specialise in our case of interest: we set $h(t) := t^\gamma$ and we show that the right hand side of (4.120) is finite. From the first inequality in (4.118) with $\lambda = 1$, we get

$$\sum_{n=1}^{\infty} 2^{-n}\Phi\left(\eta/h(2^{-n})\right) \leqslant K\sum_{n=1}^{\infty} 2^{-n} + \eta^{\beta(\Lambda)+\varepsilon}\sum_{n=1}^{\infty}(2^{-n})^{1-\gamma\beta(\Lambda)-\gamma\varepsilon}.$$

For ε small enough, we have both $\eta^{\beta(\Lambda)+\varepsilon} \in L_1(\mathbb{P})$ (since η is in $L_q(\mathbb{P})$ for any $q < 1/p$ and $\beta(\Lambda) < 1/p$) and $1 - \gamma\beta(\Lambda) - \gamma\varepsilon > 0$, by our standing assumption $1 > \gamma\beta(\Lambda)$. Consequently, we have

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\hat{T}_{2^{-(n-1)}} > (2^{-n})^\gamma\right) < \infty,$$

which entails by Borel-Cantelli that $\hat{T}_{2^{-(n-1)}} < (2^{-n})^\gamma$ holds for all n large enough, a.s. From a monotony argument, it follows that a.s. $\hat{T}_t < t^\gamma$ for all t small enough and in consequence $\limsup_{t\downarrow 0} t^{-\gamma}\hat{T}_t \leqslant 1$. If we now take $h(t) = \delta t^\gamma$ for $\delta \in (0,1)$, by the same reasoning we obtain $\limsup_{t\downarrow 0} t^{-\gamma}\hat{T}_t \leqslant \delta$ which leads to the desired result. □

Finally, our proof of Proposition 4.32 is complete.

4.6.2 Noise reinforced Lévy processes as infinitely divisible processes

As was already mentioned in Section 4.4.3, NRLPs are *infinitely divisible processes* – abbreviated ID processes. In this final section, we study their properties under this new scope. In this direction, we start by giving a brief overview of the theory; our exposition mainly follows Rosinksi [86] and Chapter 3 of Samorodnitsky [87]. Then, we identify the features of NRLPs in this setting and more precisely, we identify the functional triplet of NRLPs, in the sense of ID processes. The objective here is hence to put Lévy processes and their NRLPs counterparts in the context of ID processes and compare then through this new lens. As an application, making use of the Isomorphism Theorem for ID processes [86, Theorem 4.4] we establish the following result:

Proposition 4.36. *Let $\hat{\xi}$ be a noise reinforced Lévy process with characteristics $(a, 0, \Lambda, p)$. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a bounded, continuous function with $f(x) = O(x^2)$ at 0. Then, we have*

$$\lim_{h\downarrow 0} h^{-1}\mathbb{E}\left[f\left(\hat{\xi}_h\right)\right] = p^{-1}(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\sum_{k=1}^{\infty} f(kx)\mathrm{B}(k, 1/p+1).$$

Note that the probability distribution appearing in the previous display is the Yule-Simon distribution (4.3). For an analogous result in the setting of Lévy processes, we refer to [86, Proposition 4.13] and we shall use in our proof similar type of arguments. To simplify notation, for the rest of the section we work with NRLPs in $[0,1]$, but our exposition can be adapted to $\mathbb{R}^+$ with some slight changes. Hence, we can make use of the construction of NRLPs from [19] in terms of Poissonian Yule-Simon series that we recalled at the end of Section 4.3. This construction will be used for the rest of the section.

Preliminaries on infinitely divisible processes

Let us introduce some standard notation mostly taken from [86]. For T a nonempty set, we denote by $\mathbb{R}^T$ the set of $\mathbb{R}$-valued functions indexed by $t \in T$. If $S \subset T$ is an arbitrary subset and $e = (e(t))_{t \in T} \in \mathbb{R}^T$, we write e_S for the restriction of e to S. Further, let π_S be the canonical projection $\pi_S : \mathbb{R}^T \to \mathbb{R}^S$ from $\mathbb{R}^T$ into $\mathbb{R}^S$, viz. the function defined as $\pi_S(e) := e_S$. For finite subsets of T of the form $I := \{t_1, \ldots, t_k\} \subset T$, the space $\mathbb{R}^I$ is identified with $\mathbb{R}^k$ and we write:

$$e_I = (e(t_1), \ldots, e(t_k)) \in \mathbb{R}^I.$$

As usual, the space $\mathbb{R}^T$ is equipped with the cylindrical sigma field $\mathscr{B}^T := \sigma(\pi_t \,:\, t \in T)$ generated by the projection mappings. For any arbitrary $S \subset T$, we denote by 0_S the 0 element of $\mathbb{R}^S$ and we write $\mathscr{B}_0^S := \{A \in \mathscr{B}^S : 0_S \notin A\}$. Consequently,

$$\pi_S^{-1}(0_S) = \{e \in \mathbb{R}^T : e(t) = 0 \text{ for all } t \in S\} =: \mathbf{0}_S \subset \mathbb{R}^T.$$

Notice however that this subset is not $\mathscr{B}^T$ measurable when S is uncountable. Finally, for $x \in \mathbb{R}$ we set $[\![x]\!] := x \mathbb{1}_{\{|x| \leqslant 1\}}$ and if $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, the term $[\![x]\!]$ should be interpreted component-wise, viz. $[\![x]\!] := ([\![x_1]\!], \ldots, [\![x_k]\!])$. Now let us start with the following definition:

Definition 4.37. *An $\mathbb{R}$-valued stochastic process $X = (X_t)_{t \in T}$ is said to be infinitely divisible (in law) if for any $n \in \mathbb{N}$, there exist independent and identically distributed processes $Y^{(n,1)}, \ldots Y^{(n,n)}$ such that*

$$X \stackrel{\mathscr{L}}{=} Y^{(n,1)} + \ldots Y^{(n,n)}.$$

When $T = \{1\}$ is a singleton, this is just the definition of a real valued infinitely-divisible random variable, in which case, the characteristic function of X_1 takes the Lévy-Khintchine form:

$$\mathbb{E}\left[e^{i\theta X_1}\right] = \exp\left\{i\theta b - \frac{q^2}{2}\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta [\![x]\!]\right) \nu(\mathrm{d}x)\right\},$$

for $q, b \in \mathbb{R}$, ν a Lévy measure. Further, it is well known that the set of infinitely divisible random variables and distributions of Lévy processes are in bijection and it is clear that if X is a Lévy process with characteristic exponent as in the previous display, we have

$$X \stackrel{\mathscr{L}}{=} Y^{(n,1)} + \cdots + Y^{(n,n)},$$

where for each $i \in \{1, \ldots, n\}$, $Y^{(n,i)}$ is an independent copy of a Lévy process with characteristic triplet $(b/n, q/n, \nu/n)$. Said otherwise, Lévy processes are infinitely divisible processes. Moreover, from the formula for the characteristic function of Proposition 4.2, it is clear that NRLPs are in turn infinitely divisible.

Now, recall that a Gaussian process $X = (X_t)_{t \in T}$ is a T-indexed process satisfying that, for any $I = \{t_1, \ldots, t_k\} \subset T$, the vector $X_I = (X_{t_1}, \ldots, X_{t_k})$ is Gaussian. In the sequel we also assume that the Gaussian processes we work with are centred. Gaussian processes are characterised by their covariance function, in the sense that the law of X is completely determined by the semi-definite positive function $\Gamma : T \times T \to \mathbb{R}$ defined by

$$\Gamma(t, s) := \mathbb{E}\left[X_t X_s\right], \qquad \text{for } t, s \in T. \tag{4.121}$$

The following characterisation of infinitely divisible stochastic processes shows that they are the natural generalisation of Gaussian processes:

Proposition 4.38. [Proposition 3.1.3][87] *An $\mathbb{R}$-valued stochastic process $X = (X_t)_{t \in T}$ is infinitely divisible if and only if for any finite collection of indices $I = \{t_1, \ldots t_k\} \subset T$, the random vector $X_I = (X_{t_1}, \ldots X_{t_k})$ is infinitely divisible.*

Hence, if X is an infinitely divisible process, by Lévy-Kintchine representation and the previous proposition, for every $I = \{t_1, \dots, t_k\}$ there exists: an $\mathbb{R}^k$-valued measure $\nu_I(\mathrm{d}x)$ verifying

$$\int_{\mathbb{R}^k} 1 \wedge |x|^2 \nu_I(\mathrm{d}x) < \infty, \qquad \text{and} \qquad \nu_I(\{0_I\}) = 0,$$

a semi-definite positive $I \times I$ matrix Γ_I and an $\mathbb{R}^k$ vector, that we denote as $b(I)$, satisfying for every $\theta \in \mathbb{R}^I$ the identity:

$$\mathbb{E}\left[\exp\left\{\sum_{t\in I}\theta_t X_t\right\}\right] = \exp\left\{i\langle b(I),\theta\rangle - \frac{1}{2}\langle\theta\,\Gamma_I,\theta\rangle + \int_{\mathbb{R}^I}\left(e^{i\langle\theta,x\rangle} - 1 - i\langle\theta,[\![x]\!]\rangle\right)\nu_I(\mathrm{d}x)\right\}. \tag{4.122}$$

It is possible to show that one can recover the collection of triplets $((b(I), \Gamma_I, \nu_I) : I \subset T,\ |I| < \infty)$ from a so called functional triplet $(b, \Gamma, \bar{\nu})$, consisting in a path $b \in \mathbb{R}^T$, a covariance function $\Gamma : T \times T \to \mathbb{R}$ and a path-valued measure $\bar{\nu}$ defined in $\mathbb{R}^T$, satisfying for any finite $I \subset T$, that

$$b(I) = \pi_I(b) \qquad \Gamma_I = \Gamma\big|_I \qquad \nu_I(\mathrm{d}x) = \bar{\nu}\circ\pi_I^{-1}(\mathrm{d}x) \quad \text{in } \mathscr{B}_0^I,$$

where $\bar{\nu}$ satisfies some regularity and integrability conditions that we now introduce:

Definition 4.39. *A measure $\bar{\nu}$ on $\mathbb{R}^T$ is called a path Lévy measure if it satisfies the following two conditions:*

(i) $\int_{\mathbb{R}^T} |e(t)|^2 \wedge 1\, \bar{\nu}(\mathrm{d}e) < \infty$ *for all $t \in T$.*

(ii) *For every $A \in \mathscr{B}^T$, there exists a countable subset $T_A \subset T$ such that $\bar{\nu}(A) = \bar{\nu}(A \backslash \pi_{T_A}^{-1}(0_{T_A}))$.*

Moreover, we consider the following third condition:

(iii) *There exists a countable subset $T_0 \subset T$ such that $\bar{\nu}(\pi_{T_0}^{-1}(0_{T_0})) = 0$.*

Then, (iii) is a stronger statement than (ii) and it has been shown that a path Lévy measure is σ-finite if and only if (iii) holds – see e.g. [86]. Condition (ii) states roughly speaking that $\bar{\nu}$ "does not charge the origin". As we already mentioned, in general 0_T is not measurable and hence we can not state this condition as in the finite-dimensional case of Lévy measures. One of the main results of the theory states that infinitely divisible processes are in bijection with functional triplets $(b, \Gamma, \bar{\nu})$, we refer to [86] for the proof:

Theorem 4.40. *For every infinitely divisible stochastic process $X = (X_t)_{t\in T}$ there exists a unique generating triplet $(b, \Gamma, \bar{\nu})$ consisting of a path $b \in \mathbb{R}^T$, a covariance function Γ in $T \times T$ and a path Lévy measure $\bar{\nu}$ in $\mathbb{R}^T$ such that for any finite $I \subset \hat{T}$,*

$$\mathbb{E}\left[\exp\left\{i\sum_{t\in I}\theta_t X_t\right\}\right] = \exp\left\{i\langle b_I,\theta\rangle - \frac{1}{2}\langle\theta\,\Gamma_I,\theta\rangle + \int_{\mathbb{R}^T}\left(e^{i\langle\theta,e_I\rangle} - 1 - i\langle\theta,[\![e_I]\!]\rangle\right)\bar{\nu}(\mathrm{d}e)\right\}. \tag{4.123}$$

Conversely, for every generating triplet $(b, \Gamma, \bar{\nu})$ there exists an infinitely divisible process satisfying (4.123).

Maintaining the notation of Theorem 4.40, it follows in particular that the law of any ID process X can be written as a sum of two independent processes $X \overset{\mathscr{L}}{=} G + P$, where G is Gaussian with covariance Γ and P is a so-called Poissonian ID process. When the equality $X = G + P$ holds almost surely, we call respectively G and P the Gaussian part and the Poissonian part of X. Let us conclude our presentation with the following notion that will be of use:

Definition 4.41. *A process $V = (V_t)_{t \in T}$ defined in a measure space $(S, \mathscr{S}, n)$ is called a representant of a path Lévy measure $\bar{\nu}$ if for any finite $I \subset T$, we have*

$$n\left(s \in S \, : \, V_I(s) \in B\right) = \bar{\nu}_I(B), \qquad \textit{for every } B \in \mathscr{B}_0^I.$$

The representation is called exact if $n \circ V^{-1} = \bar{\nu}$.

This is, if V is only a representant, the measure $\nu \circ V^{-1}$ might not be a Lévy measure since it might "charge the origin". In the situations we will be interested in the representations will always be exact, and we only enunciate the weaker definition to write the results we need in their full generality. Representants allow to build explicitly Poissonian ID processes in terms of Poisson random measure, for more details we refer to [86], see also our brief discussion before the proof of Proposition 4.36 below.

The characteristic triplet of a NRLP

We can now start investigating Lévy processes and their reinforced counterparts as ID processes, and we start with a basic analysis of the former. More precisely, we identify the path Lévy measure of Lévy processes as well as an exact representant. These results are known [86, Example 2.23] and the statements are only included to contrast with the analogous results for NRLPs – see Lemma 4.44 below.

Lemma 4.42. *The following assertions hold:*

(i) *Let ξ be a Lévy process with characteristic triplet (a, q, Λ). The path Lévy measure $\bar{\nu}$ of ξ is given by,*
$$\bar{\nu}(\mathrm{d}e) := (\mathrm{d}t \otimes \Lambda) \circ V^{-1}(\mathrm{d}e),$$
where we denoted by V the mapping $V : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^{\mathbb{R}^+}$ defined as $V(s,x) := x \mathbb{1}_{\{s \leqslant \cdot\}}$.

(ii) *Consider a measure Λ on $\mathbb{R}$ with $\Lambda(0) = 0$ and let V be defined as in* (i)*. Then, the condition $\int 1 \wedge |x|^2 \Lambda(\mathrm{d}x) < \infty$ holds if and only if $\bar{\nu} := (\mathrm{d}t \otimes \Lambda) \circ V^{-1}$ is a path Lévy measure in $\mathbb{R}^{\mathbb{R}^+}$. Moreover, if the later holds, the path Lévy measure $\bar{\nu}$ is σ-finite.*

In particular, from (i) we get that V is an exact representant of $\bar{\nu}$, on $(S, \mathscr{S}, n) := (\mathbb{R}^+ \times \mathbb{R}, \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{B}(\mathbb{R}), \mathrm{d}t \otimes \Lambda)$. We now turn our attention to noise reinforced Lévy processes and we start with the following technical lemma:

Lemma 4.43. *Let $\hat{\xi}$ be an NRLP of characteristic triplet $(a, 0, \Lambda, p)$ and let $T = [0,1]$. Then, for any $t \in T$, we have*

$$\mathbb{E}\left[\int_{\mathbb{R}} \left| [\![Y(t)x]\!] - Y(t)[\![x]\!] \right| \Lambda(dx)\right] < \infty. \tag{4.124}$$

Proof. First, recalling that $[\![x]\!] = x\mathbb{1}_{\{|x|\leqslant 1\}}$, we can write

$$\mathbb{E}\left[\left|[\![Y(t)x]\!] - Y(t)[\![x]\!]\right|\right] = \mathbb{E}\left[\left|Y(t)[\![x]\!]\right|\mathbb{1}_{\{|xY(t)|>1\}}\right] + \mathbb{E}\left[\left|Y(t)x - Y(t)[\![x]\!]\right|\mathbb{1}_{\{|xY(t)|\leqslant 1\}}\right]. \quad (4.125)$$

Remark that since Y takes values in $\{0, 1, 2, \dots\}$, the second term in the right-hand side vanishes. On the other hand, for any $q \in (\beta(\Lambda) \vee 1, 1/p)$, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{E}\left[\left|Y(t)[\![x]\!]\right|\mathbb{1}_{\{|x|Y(t)>1\}}\right]\Lambda(dx) &= \int_{\{|x|\leqslant 1\}} \mathbb{E}\left[Y(t)\mathbb{1}_{\{|x|Y(t)>1\}}\right]|x|\Lambda(dx) \\ &\leqslant \mathbb{E}\left[Y^q(t)\right]^{1/q} \int_{\{|x|\leqslant 1\}} \mathbb{P}\left(Y(t) > 1/|x|\right)^{(q-1)/q}|x|\Lambda(dx),\end{aligned} \quad (4.126)$$

where we recall that $Y \in L^q(\mathbb{P})$ for any $q < 1/p$. To conclude, recall the asymptotic behaviour from (10) in [19],

$$\mathbb{P}\left(Y(t) > 1/|x|\right) \sim t\Gamma(1/p+1)|x|^{1/p}, \quad \text{as } x \downarrow 0.$$

It now follows that we can take q close enough to $1/p$ such that the integral in (4.126) is finite and we deduce (4.124). □

Now, we identify the path Lévy measure of NRLPs.

Lemma 4.44. *The following assertions hold:*

(i) *Let $\hat{\xi}$ be a NRLP with characteristic triplet (a, q^2, Λ, p). The path Lévy measure $\bar{\nu}$ of $\hat{\xi}$ is given by,*

$$\bar{\nu} := (1-p)(\Lambda \otimes \mathbb{Q}) \circ V^{-1},$$

where $V : D[0,1] \times \mathbb{R} \to \mathbb{R}^{[0,1]}$ is defined by $V(x,y) := xy$.

(ii) *Let $(a, 0, \Lambda)$ be the characteristic triplet of a Lévy Process and let V be defined as in* (i)*. Then, if a memory parameter $p \in (0,1)$ is admissible for the triplet $(a, 0, \Lambda)$, the measure $\bar{\nu} := (\Lambda \otimes \mathbb{Q}) \circ V^{-1}$ is a σ-finite path Lévy measure in $\mathbb{R}^{[0,1]}$. On the other hand, if $1/p < \beta(\Lambda)$, then the integrability condition 4.39 -* (i) *fails.*

In particular, from (i) we get that V is an exact representant of $\bar{\nu}$, in $(S, \mathscr{S}, n) = (D[0,1] \times \mathbb{R}, \mathscr{B}(D[0,1]) \otimes \mathscr{B}(\mathbb{R}), \mathbb{Q} \otimes \Lambda(1-p))$. On other hand, (ii) gives a natural interpretation in the terminology of ID processes for the admissibility of p for Λ.

Proof. To identify the Lévy measure, let us write the characteristic function of the finite dimensional distributions of $\hat{\xi}$ in the form (4.123) and to simplify notation, we suppose that $a, q = 0$. In this direction, consider a finite $I \subset T$, $\theta = (\theta_{t_1}, \dots, \theta_{t_k}) \in \mathbb{R}^I$, and denote by $y = (y(t))_{t\in[0,1]}$ an arbitrary counting function. Recall the formula for the finite dimensional distributions of $\hat{\xi}$ from Proposition 4.2, for $t = 1$. It now follows by Lemma 4.43 and the triangle inequality that we have:

$$\int_{\mathbb{R}} \Lambda(\mathrm{d}x)\mathbb{E}\left[\left|e^{i\langle\theta, Y_I\rangle x} - 1 - i\langle\theta, [\![xY_I]\!]\rangle\right|\right] < \infty. \quad (4.127)$$

Now, we can write

$$\mathbb{E}\left[\exp\left\{i\sum_{t\in I}\theta_t\hat{\xi}_t\right\}\right] = \exp\left\{(1-p)\int_{\mathbb{R}}\Lambda(\mathrm{d}x)\mathbb{E}\left[e^{\langle\theta,Y_I\rangle x} - 1 - i\langle\theta, Y_I\rangle[\![x]\!]\right]\right\}$$
$$= \exp\left\{(1-p)\int_{\mathbb{R}\times D[0,1]}\left(e^{\langle\theta,(xy)_I\rangle} - 1 - i\langle\theta,[\![xy_I]\!]\rangle\right)\Lambda\otimes\mathbb{Q}(\mathrm{d}x,\mathrm{d}y)\right.$$
$$\left. + i\int_{\mathbb{R}\times D[0,1]}\langle\theta,[\![xy_I]\!]\rangle - \langle\theta,y_I\rangle[\![x]\!]\Lambda\otimes\mathbb{Q}(\mathrm{d}x,\mathrm{d}y)(1-p)\right\},$$

where all the terms in the previous expression are well defined by Lemma 4.43 and (4.127). Since $(xy)_I = \pi_I(V(x,y))$ and $\bar{\nu} = (1-p)(\Lambda\otimes Q)\circ V^{-1}$, we obtain for a clear choice of b_I that

$$\mathbb{E}\left[\exp\left\{\sum_{t\in I}\theta_t\hat{\xi}_t\right\}\right] = \exp\left\{\int_{\mathbb{R}^T}\left(e^{\langle\theta,e_I\rangle} - 1 - i\langle\theta,[\![e_I]\!]\rangle\right)\bar{\nu}(\mathrm{d}e) + \langle\theta,b_I\rangle\right\}.$$

Next, notice that condition (iii) of Definition 4.39 is satisfied by $(\Lambda\otimes\mathbb{Q})\circ V^{-1}$. Indeed, if we let $T_0 := \{1\}$ and $\mathbf{0}_{T_0} := \{e\in\mathbb{R}^T : e(1) = 0\}$, recalling that $Y(1)\geqslant 1$ a.s., we deduce that

$$\Lambda\otimes\mathbb{Q}\left((x,y) : xy\in\{\mathbf{0}_{T_0}\}\right) = \Lambda(\{0\}) = 0.$$

To conclude, let us show that $\bar{\nu}$ satisfies the integrability condition (i) of Definition 4.39 if p is an admissible memory parameter for Λ, viz. if $\beta(\Lambda) < 1/p$, while when $\beta(\Lambda) > 1/p$, the condition fails. By definition of $\bar{\nu}$, we have

$$\int_{\mathbb{R}^T}\left(|e(t)|^2\wedge 1\right)\bar{\nu}(\mathrm{d}e) = \int_{\mathbb{R}}\mathbb{E}\left[|xY_t|^2\wedge 1\right]\Lambda(\mathrm{d}x), \tag{4.128}$$

and write:

$$\mathbb{E}\left[|xY_t|^2\wedge 1\right] = |x|^2\mathbb{E}\left[|Y_t|^2\mathbb{1}_{\{Y_t\leqslant 1/|x|\}}\right] + \mathbb{P}\left(Y_t > 1/|x|\right). \tag{4.129}$$

Now, recalling from (10) of [19] the asymptotic behaviour,

$$\mathbb{P}\left(Y_t > 1/|x|\right) \sim t\Gamma(p^{-1}+1)|x|^{1/p}, \qquad \text{as } |x|\downarrow 0,$$

it follows that if $\beta(\Lambda) < 1/p$, the term $\mathbb{P}\left(Y_t > 1/|x|\right)$ is integrable with respect to Λ and infinite if $\beta(\Lambda) > 1/p$. Let us now show that the same holds for

$$\int_{[0,1]}|x|^2\mathbb{E}\left[|Y_t|^2\mathbb{1}_{\{Y_t\leqslant 1/|x|\}}\right]\Lambda(\mathrm{d}x). \tag{4.130}$$

Recalling Lemma 4.1, we get:

$$\mathbb{E}\left[|Y_t|^2\mathbb{1}_{\{Y_t\leqslant 1/|x|\}}\right] = \sum_{n=1}^{\lfloor 1/|x|\rfloor} n^2\mathbb{P}\left(Y_t = n\right) = tp^{-1}\sum_{n=1}^{\lfloor 1/|x|\rfloor} n^2\mathrm{B}\left(n, p^{-1}+1\right),$$

where we denoted by B the Beta function. Now, from the asymptotic behaviour

$$\mathrm{B}(n,p^{-1}+1) \sim n^{-(1+p)/p}\Gamma(p^{-1}+1), \qquad \text{as } n\uparrow\infty,$$

it follows that (4.130) is finite if $\beta(\Lambda) < 1/p$ and infinite if $\beta(\Lambda) > 1/p$. □

Let us state the two last result that we need for the proof of Proposition 4.36. First, the Poissonian part of ID processes consists, roughly speaking, in Poissonian sums of i.i.d. trajectories – for instance, remark that for NRLPs those trajectories are the weighted Yule-Simon processes – for more examples see e.g. [86, Section 3]. More precisely, let $X = (X_t)_{t\in T}$ be an infinitely divisible process with characteristic triplet $(b, \Sigma, \bar{\nu})$ and suppose that $V = (V_t)_{t\in T}$ is a representant of $\bar{\nu}$ defined on a σ-finite measure space $(S, \mathscr{S}, n)$. To simplify notation, set $\chi(u) := \mathbb{1}_{\{|u|\leqslant 1\}}$ and consider $\mathcal{M}$ a Poisson random measure in $(S, \mathscr{S})$ with intensity n. Then, the following process has the same distribution as X,

$$b_t + G_t + \int_S V_t(s)\big(\mathcal{M}(\mathrm{d}s) - \chi(V_t(s))n(\mathrm{d}s)\big), \quad t \in T, \tag{4.131}$$

where $G = (G_t)_{t\in T}$ is an independent Gaussian process with covariance Σ. The integration in the previous display should read as a compensated integral, and for a detailed statement we refer to [86, Proposition 3.1]. For example, notice that if X is a Lévy process, $\mathcal{M}$ is Poisson with intensity $\mathrm{d}t \otimes \Lambda(\mathrm{d}x)$ and replacing V by $x\mathbb{1}_{\{s\leqslant \cdot\}}$ yields a Lévy-Itô representation. Finally, we give one of the statements that we use of the Isomorphism Theorem of infinitely divisible processes needed for our proof.

Theorem 4.45. [Isomorphism Theorem][86, 4.4] *Let $X = (X_t)_{t\in T}$ be an infinitely divisible process given by* (4.131)*. Choose an arbitrary measurable function* $\mathrm{q} : S \mapsto \mathbb{R}^+$ *such that* $\int_S \mathrm{q}(s)n(\mathrm{d}s) = 1$ *and set* $N(\mathrm{q}) := \int_S \mathrm{q}(s)\mathcal{M}(\mathrm{d}s)$*. Then, for any measurable functional* $F : \mathbb{R}^T \mapsto \mathbb{R}$*, we have*

$$\mathbb{E}\left[F((X_t)_{t\in T})\mathbb{1}_{\{N(\mathrm{q})>0\}}\right] = \int_S \mathbb{E}\left[F\left((X_t + V_t(s))_{t\in T}\right)\left(N(\mathrm{q}) + \mathrm{q}(s)\right)^{-1}\right]\mathrm{q}(s)n(\mathrm{d}s).$$

This allows for instance to study the law of X under different conditionings, for appropriate elections of q. This will be used in our reasoning below. Now, let us conclude the proof of Proposition 4.36.

Proof of Proposition 4.36. To simplify notation, we will perform a slight abuse of notation by writing Λ instead of $(1-p)\Lambda$. We start by fixing $\delta \in (0, 1)$ small enough such that $m = \Lambda(|x| > \delta) > 0$. Now, let $h > 0$ and as usual, write $y = (y(t))_{t\in[0,1]}$ for a generic counting trajectory in $D[0, 1]$. Recall the result of Lemma 4.44 and consider a Poisson random measure $\mathcal{M} = \sum \delta_{(x_i,Y_i)}$ with intensity $\Lambda \otimes \mathbb{Q}$. Next, we set

$$\mathrm{q}(y, x) := \frac{1}{mh}\mathbb{1}_{\{|x|\geqslant\delta\}}\mathbb{1}_{\{y(h)\geqslant 1\}},$$

and take

$$N(\mathrm{q}) = \frac{1}{mh}\int_{D\times\{|x|\geqslant\delta\}}\mathbb{1}_{\{y(h)\geqslant 1\}}\mathcal{M}(\mathrm{d}x, \mathrm{d}y) =: \frac{1}{mh}S_h.$$

Then, from the definition of S_h we have

$$S_h = \#\{(x_i, Y_i) : |x_i| \geqslant \delta \text{ and } Y_i(h) \geqslant 1\} \leqslant \#\{|\Delta\hat{\xi}_s| \geqslant \delta \text{ for } s \leqslant h\},$$

where the inequality stands since a jump x_i is repeated at each jump time of its respective Y_i, and consequently might be repeated multiple times in $[0, h]$. However, we do have $\#\{|\Delta\hat{\xi}_s| \geqslant$

δ for $s \leqslant h\} = 0$ when $S_h = 0$. Finally, we consider the functional $F(e) := f(e(h))$ for $e \in D[0,1]$. An application of Theorem 4.45 yields:

$$\mathbb{E}\left[f(\hat{\xi}_h)\mathbb{1}_{\{S_h>0\}}\right] = \int_{D\times\{|x|\geqslant\delta\}} \mathbb{E}\left[f\big(\hat{\xi}_h + xy(h)\big)\frac{1}{S_h+1}\right]\mathbb{1}_{\{y(h)\geqslant 1\}}\mathbb{Q}(dy)\Lambda(\mathrm{d}x)$$
$$= \int_{D\times\{|x|\geqslant\delta\}} G_h(xy(h))\mathbb{1}_{\{y(h)\geqslant 1\}}\mathbb{Q}(\mathrm{d}y)\Lambda(\mathrm{d}x),$$

for $G_h(z) = \mathbb{E}\left[f(\hat{\xi}_h + z)\frac{1}{S_h+1}\right]$ and notice that $\lim_{h\downarrow 0} G_h(z) = f(z)$ by right-continuity – remark that the previous display can be interpreted as the law of $\hat{\xi}_h$ conditioned at having at least one jump before time h of size greater than δ. If we let η be a random variable distributed Yule-Simon with parameter $1/p$ under $\mathbb{P}$, this entails that we can write:

$$\frac{1}{h}\mathbb{E}\left[f(\hat{\xi}_h)\right] = \frac{1}{h}\mathbb{E}\left[f(\hat{\xi}_h)\mathbb{1}_{\{S_h=0\}}\right] + \frac{1}{h}\mathbb{E}\left[f(\hat{\xi}_h)\mathbb{1}_{\{S_h>0\}}\right]$$
$$= \frac{1}{h}\mathbb{E}\left[f(\hat{\xi}_h)\mathbb{1}_{\{S_h=0\}}\right] + \int_{\{|x|\geqslant\delta\}} \mathbb{E}\left[G_h(x\eta)\right]\Lambda(\mathrm{d}x),$$

where in the last equality we used that the law of $y(h)$ under

$$\frac{\mathbb{1}_{\{y(h)\geqslant 1\}}}{\mathbb{Q}(y(h)\geqslant 1)}\mathbb{Q}(\mathrm{d}y),$$

is the Yule-Simon distribution with parameter $1/p$ by Lemma 4.1 and that $\mathbb{Q}(y(h) \geqslant 1) = h$. Consequently, we deduce that

$$\left|h^{-1}\mathbb{E}\big[f(\hat{\xi}_h)\big] - \int_{\mathbb{R}} \mathbb{E}\left[f(x\eta)\right]\Lambda(\mathrm{d}x)\right|$$
$$\leqslant h^{-1}\mathbb{E}\left[|f(\hat{\xi}_h)|\mathbb{1}_{\{S_h=0\}}\right] + \int_{\{|x|\leqslant\delta\}} \mathbb{E}\big[|f(x\eta)|\big]\Lambda(\mathrm{d}x) + \left|\int_{\{|x|\geqslant\delta\}} \mathbb{E}\left[G_h(x\eta)\right] - \mathbb{E}\left[f(x\eta)\right]\Lambda(\mathrm{d}x)\right|$$
$$=: K_1(h,\delta) + K_2(\delta) + K_3(h,\delta).$$

Now, we study the limit as $h \downarrow 0$ of these three terms separately and we start with $K_1(h,\delta)$. Recall the notation introduced in 4.5.3 for the compensated integrals as well as $\Sigma_{\delta,\infty} := \mathbb{1}_{(-\delta,\delta)^c}x * \hat{\mathcal{N}}$ for the process obtained by adding jumps of size greater that $\delta > 0$. Recall that on $\{S_h = 0\}$, the process $\hat{\xi}$ doesn't have jumps of seize greater than δ before time h. It now follows that, restricted to $\{S_h = 0\}$, the following equality holds:

$$\hat{\xi}_h = \hat{\xi}_h - \sum_i x_i Y_i(h)\mathbb{1}_{\{|x_i|\geqslant\delta\}} = a\cdot h + \hat{\xi}^{(3)}_{0,1}(h) + \Sigma_{1,\infty}(h) - \Sigma_{\delta,\infty}(h) = \hat{\xi}^{(3)}_{0,\delta}(h) - c_\delta\cdot h, \qquad (4.132)$$

for $c_\delta := -a + (1-p)^{-1}\int_{\{\delta\leqslant|x|\leqslant 1\}} x\Lambda(\mathrm{d}x)$ and denote the right hand side of (4.132) by $\hat{\xi}^\delta_h$. Now let us first consider the case $\beta(\Lambda) < 2$. Since f is bounded and $O(|x|^2)$ at the origin, for any $q \in (\beta(\Lambda)\vee 1, 1/p\wedge 2)$ satisfying $q < r$ we can bound $|f(x)| \leqslant C|x|^q$ for all $x \in \mathbb{R}$, for some constant C large enough. Then, for a constant C' that only depends on q we have

$$K_1(h,\delta) = h^{-1}\mathbb{E}\left[|f(\hat{\xi}_h)|\mathbb{1}_{\{S_h=0\}}\right]$$
$$\leqslant Ch^{-1}\mathbb{E}\left[|\hat{\xi}^\delta_h|^q\right] \leqslant C'h^{-1}\mathbb{E}\left[|\hat{\xi}^{(3)}_{0,\delta}(h)|^q\right] + C'h^{q-1}|c^\delta|^q.$$

Now, arguing as in (4.114), (4.115), recall that for $q \in (\beta(\Lambda) \vee 1, 1/p \wedge 2)$ we have the following bound for the compensated sum of Yule-Simon processes:

$$\mathbb{E}\left[|\hat{\xi}^{(3)}_{0,\delta}(h)|^q\right] \leqslant \mathbb{E}\left[Y(h)^q\right] \int_{\{|x|\leqslant\delta\}} |x|^q \Lambda(\mathrm{d}x) = h \cdot \mathbb{E}\left[\eta^q\right] \int_{\{|x|\leqslant\delta\}} |x|^q \Lambda(\mathrm{d}x) < \infty. \tag{4.133}$$

Since $q - 1 > 0$, we have $\limsup_{h\downarrow 0} K_1(h,\delta) \leqslant \mathbb{E}\left[\eta^q\right] \int_{\{|x|\leqslant\delta\}} |x|^q \Lambda(\mathrm{d}x)$ which can be made arbitrarily small for an appropriate choice of δ. Remark that the same reasoning applies for $K_2(\delta)$, by making use once again of the bound $|f(x)| \leqslant C|x|^q$. Finally, since for any choice of δ, $K_3(h,\delta) \downarrow 0$ as $h \downarrow 0$, we obtain the desired result.

If $\beta(\Lambda) = 2$, we set $q = 2$ and once again recall from page 9 of Bertoin [19] that the inequality (4.133) still holds. In this case, since $p\beta(\Lambda) < 1$, p must be smaller than $1/2$ and consequently $\mathbb{E}\left[\eta^2\right] < \infty$, while of course $\int_{\{|x|\leqslant\delta\}} |x|^2 \Lambda(\mathrm{d}x) < \infty$ by definition of a Lévy measure. We can then proceed as before. □

4.6.3 Convergence towards reinforced α-stable Lévy process

Before closing the section, we establish a complementary result that is well known in the setting of Lévy processes and exploits the explicit form of the finite-dimensional distributions (4.10). We start with some necessary background on the theory of convergence towards stable distributions and Lévy processes. We say that a sequence of iid random variables is in the domain of attraction of an α stable distribution for $\alpha \in (0,2)$ if for some sequence $a_n = n^{1/\alpha} h(n)$ with $h(n)$ slowly varying at infinity in the sense of Karamata, the following sequence of normalised sums converges weakly

$$\frac{X_1 + \cdots + X_n}{a_n} \xrightarrow{\mathscr{L}} Y \tag{4.134}$$

towards a non-degenerate random variable. For simplicity, we exclude the more delicate case of $\alpha = 1$ and from now on, α belongs to $(0,1) \cup (1,2)$. In that case, Y is an α-stable random variable and its characteristic exponent Ψ_α can be written as

$$\Psi_\alpha(u) = c|u|^\alpha \left(1 - i\beta \frac{u}{|u|} \tan(\pi\alpha 1/2)\right)$$

for some constants c and β. If we denote by $\varphi(u)$ the characteristic function of X_1, since $\varphi(0) = 1$ and φ is continuous, a branch of the logarithm $\log \varphi(u) =: \Psi(u)$ with $\Psi(0) = 0$ is defined in a neighbourhood of the origin for $|u| < r$, for r small enough. The condition of X_1 being in the domain of attraction of Y can then be equivalently phrased by asking Ψ to be of the form

$$\Psi(u) = i\gamma u - c|u|^\alpha \tilde{h}(u) \left(1 - i\beta \frac{u}{|u|} \tan(\alpha\pi 1/2)\right) \tag{4.135}$$

in some neighbourhood of the origin, for $\tilde{h}(u)$ some slow varying function as $u \downarrow 0$ (see for e.g. Theorem 2.6.5 in Ibragimov and Linnik [52]). Since we will work in the case without centring, γ is null when $\alpha \in (1,2)$. Further, when the scaling constants a_n are $n^{1/\alpha}$, we say that X_1 is in the normal domain of attraction of a stable law Y and in that case, $\tilde{h}(u)$ is just constant. The condition (4.134) can be then written, for n large enough, as

$$\lim_{n\to\infty} n\Psi(u/n^{1/\alpha}) = \Psi_\alpha(u) \qquad \text{for all } u \in \mathbb{R}.$$

Notice that in fact, since $\tilde{h}$ in (4.135) is constant, the stronger convergence holds:

$$\lim_{t\to\infty} t\Psi(u/t^{1/\alpha}) = \Psi_\alpha(u) \qquad \text{for all } u \in \mathbb{R},$$

which entails that

$$\Psi(u) = O(u^\alpha) \qquad \text{as } u \downarrow 0. \tag{4.136}$$

Now we specialise in our case of interest: suppose that X_1 is infinitely divisible and denote by Ψ its corresponding characteristic exponent. In order to establish weak convergence, in the sense of Skorokhod, of a sequence of Lévy processes (ξ^n) towards another Lévy process ξ, it is enough to show that the sequence of random variables (ξ_1^n) converges weakly towards ξ_1 (see for e.g. Jacod and Shiryaev [54] VII, Corollary 3.6). For instance, if ξ is a Lévy process and $X_1 = \xi_1$ is in the domain of normal attraction of an α-stable law Y and if we denote by $Y^{(\alpha)}$ a Lévyprocess with $Y_1^{(\alpha)} \stackrel{\mathscr{L}}{=} Y$ then

$$\left(n^{-1/\alpha}\xi_{nt}\right)_{t\in\mathbb{R}^+} \stackrel{\mathscr{L}}{\to} Y^\alpha \qquad \text{as } n \uparrow \infty.$$

The conditions under which ξ_1 is in the domain of attraction of a stable law Y can be expressed explicitly in terms of the characteristic triplet of ξ, see for example proposition 1 in Rosenbaum and Tankov [85]. Our following result shows that in the context of noise reinforced Lévy process, if ξ_1 is in the domain of attraction of a stable law Y and p is an admissible memory parameter for the Blumenthal-Getoor index of the Lévy measure Λ, that we denote as usual as $\beta(\Lambda)$, then we also have weak convergence of the sequence of rescaled reinforced processes $(n^{-1/\alpha}\hat{\xi}_{nt})_{t\in\mathbb{R}^+}$ towards the corresponding reinforced alpha-stable Lévy process $\hat{Y}^{(\alpha)}$ in the sense of finite dimensional distributions, as long as $\alpha \leqslant \beta(\Lambda)$. Now we state our result:

Proposition 4.46. *Consider ξ a Lévy process and suppose that ξ_1 is in the domain of normal attraction of an α-stable distribution Y, for $\alpha \leqslant \beta(\Lambda)$. We denote by Y^α a Lévy process with $Y_1^\alpha \stackrel{\mathscr{L}}{=} Y$. If $\hat{\xi}$ is the reinforced version of ξ for an admissible memory parameter p, then p is an admissible memory parameter for Y^α and*

$$\left(n^{-1/\alpha}\hat{\xi}_{nt}\right)_{t\in\mathbb{R}^+} \stackrel{fdd}{\to} \hat{Y}^\alpha \qquad \text{as } n \uparrow \infty$$

where $\hat{Y}^\alpha$ stands for the noise reinforced version of Y^α with memory parameter p.

Proof. Recall that by definition of an admissible memory parameter, we have $\beta(\Lambda)p < 1$. The first assertion then follows by noticing that the Blumenthal-Getoor index of the Lévy measure Λ_α of an α-stable Lévy process is $\beta(\Lambda_\alpha) = \alpha$ and in consequence $\beta(\Lambda_\alpha)p < 1$. We consider times $0 \leqslant t_1 < \cdots < t_k \in \mathbb{R}^+$ and we fix t with $t_k \leqslant t$. By Proposition 4.10, since $nt_j < nt$ for all j,

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^k \lambda_i n^{-1/\alpha}\hat{\xi}_{nt_i}\right\}\right] = \exp\left\{nt(1-p)\mathbb{E}\left[\Psi\left(n^{1/\alpha}\sum_{i=1}^k \lambda_i Y(t_i/t)\right)\right]\right\}$$

while the finite dimensional distribution of the α-stable reinforced process $\hat{Y}^\alpha$ are given by

$$\mathbb{E}\left[\exp\left\{i\sum_{i=1}^k \lambda_i \hat{Y}_{t_i}^{(\alpha)}\right\}\right] = \exp\left\{t(1-p)\mathbb{E}\left[\Psi_\alpha\left(\sum_{i=1}^k \lambda_i Y(t_i/t)\right)\right]\right\}.$$

If we denote by $\mathbb{Q}(dy)$ the law of a Yule-Simon process in $D[0,1]$, $y = (y(t))_{t\in[0,1]}$ a generic counting trajectory and we set $f(y) = \sum_{i=1}^{k} \lambda_i y(t_i/t)$, by hypothesis we have the pointwise convergence

$$n\Psi(f(y)/n^{1/\alpha}) \to \Psi_\alpha(f(y)) \qquad \text{as } n \uparrow \infty \tag{4.137}$$

and the result will be established by showing that

$$\mathbb{Q}\left(n\Psi(f(y)/n^{1/\alpha})\right) \to \mathbb{Q}\left(\Psi_\alpha(f(y))\right) \qquad \text{as } n \uparrow \infty. \tag{4.138}$$

This convergence will follow from the asymptotic behaviour of Ψ at the origin and at infinity. First, from (4.136), we deduce that for all $|u| < \varepsilon$ for ε small enough and for some positive constant C, $|\Phi(u)| \leqslant C|u|^\alpha$ and in consequence

$$n|\Psi(f(y)/n^{1/\alpha})\mathbb{1}_{\{|f(y)/n^{1/\alpha}|\leqslant\varepsilon\}}| \leqslant C|f(y)|^\alpha.$$

Since $\alpha p < 1$, $Y(t)$ has moments for order α and we conclude from (4.137) that

$$\lim_{n\uparrow\infty} \mathbb{Q}\left(n\Psi(f(y)/n^{1/\alpha})\mathbb{1}_{\{|f(y)/n^{1/\alpha}|\leqslant\varepsilon\}}\right) = \mathbb{Q}\left(\Psi_\alpha(f(y))\right). \tag{4.139}$$

On the other hand, we recall from Blumenthal and Getoor [26] that, since the Gaussian component is null, the asymptotic behaviour at infinity of Ψ is dictated by the integrablity of the Lévy measure at 0 as follows: for any $\delta > 0$,

$$|\Psi(u)| \begin{cases} o(|u|^{\beta(\Lambda)+\delta}) & \text{when } \int_{|x|\leqslant 1} |x|\Lambda(\mathrm{d}x) = \infty \\ o(|u|^{1+\delta}) & \text{when } \int_{|x|\leqslant 1} |x|\Lambda(\mathrm{d}x) < \infty. \end{cases}$$

If we first suppose that $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) = \infty$, since Ψ is bounded on any neighbourhood of the origin, we deduce that for a constant C' large enough

$$n|\Psi(f(y)/n^{1/\alpha})|\mathbb{1}_{\{|f(y)/n^{1/\alpha}|>\varepsilon\}} \leqslant C'|f(y)|^{\beta(\Lambda)+\delta} n^{1-\beta(\Lambda)/\alpha-\delta/\alpha} \tag{4.140}$$

where $\beta(\Lambda)/\alpha \geqslant 1$. Once again, since $\beta(\Lambda)p < 1$ and $Y(t) \in L_q(\mathbb{P})$ for any $q < 1/p$, for δ small enough $|f(y)|^{\beta(\Lambda)+\delta}$ is in $L_1(\mathbb{Q})$ and

$$\lim_{n\to\infty} \mathbb{Q}\left(n\Psi(f(y)/n^{1/\alpha})\mathbb{1}_{\{|f(y)/n^{1/\alpha}|>\varepsilon\}}\right) = 0. \tag{4.141}$$

Finally, from (4.141) and (4.139) we deduce the desired limit (4.138). If we now suppose that $\int_{\{|x|\leqslant 1\}} |x|\Lambda(\mathrm{d}x) < \infty$, the bound established in (4.140) is in this case

$$n|\Psi(f(y)/n^{1/\alpha})|\mathbb{1}_{\{|f(y)/n^{1/\alpha}|>\varepsilon\}} \leqslant C'|f(y)|^{1+\delta} n^{1-1/\alpha-\delta/\alpha}$$

and since $\beta(\Lambda) \leqslant 1$ under the stronger integrability condition $\int 1 \wedge |x|\Lambda(\mathrm{d}x) < \infty$, we can proceed as we did before. □

4.7 Appendix

This short section is devoted to proving a technical identity needed for the proof of Lemma 4.6. The proof was omitted from the main discussion for readability purposes.

Fix a Lévy measure Λ in $\mathbb{R}$, $p \in (0,1)$ and denote the law of the standard Yule process $Z = (Z(t))_{t\in\mathbb{R}^+}$ started at $Z_0 = 1$ by $\mathbf{Z}$. We write $D[0,\infty)$ the space of $\mathbb{R}^+$ indexed, $\mathbb{R}$-valued rcll functions. Since $\mathbf{Z}$ is supported on the subset of $D[0,\infty)$ of counting functions, $z = (z_t)_{t\in\mathbb{R}^+}$ in the sequel stands for a generic counting function. Moreover, if $F : \mathbb{R}^+ \times D[0,\infty) \mapsto \mathbb{R}^+$ is a measurable function, we write $\mathbf{Z}^\bullet$ for the measure in $\mathbb{R}^+ \times D[0,\infty)$ defined as $\mathbf{Z}^\bullet(F) := \int_{\mathbb{R}^+} \mathrm{d}u\, \mathbb{E}[F(u,Z)]$. Roughly speaking, the objective is to describe the law of the following "process":

$$(u,z) \mapsto \left(\mathbb{1}_{\{u\leqslant t\}} z_{p(\ln(t)-\ln(u))} : t \in \mathbb{R}^+\right) \quad \in D[0,\infty) \tag{4.142}$$

defined on the measure space $(\mathbb{R} \times D[0,\infty), \mathbf{Z}^\bullet)$, under different restrictions of the measure $\mathbf{Z}^\bullet$. In this direction, for $T > 0$ we write

$$\mathbf{Z}^\bullet(\cdot\,|u \leqslant T) := \frac{\mathbb{1}_{\{u\leqslant T\}}}{T}\mathrm{d}u\, \mathbf{Z}(\mathrm{d}z),$$

which is now a probability measure on $\mathbb{R}^+ \times D[0,\infty)$. The main properties of interest are stated in the following lemma, and shares obvious similarities with Lemma 4.1.

Lemma 4.47. *The following properties hold:*

(i) *For each fixed $t > 0$, the random variable*

$$(u,z) \mapsto \mathbb{1}_{\{u\leqslant t\}} z_{p\ln(t/u)} \quad \textit{under} \quad \mathbf{Z}^\bullet(\cdot\,|u \leqslant t),$$

has the same distribution as the Yule Simon random variable η with parameter $1/p$.

(ii) *For every $T > 0$, the process*

$$(u,z) \mapsto \left(\mathbb{1}_{\{u\leqslant Tt\}} z_{p\ln(Tt/u)} : t \in [0,1]\right) \quad \textit{under} \quad \mathbf{Z}^\bullet(\cdot\,|u \leqslant T),$$

has the same law as the Yule-Simon process $(Y(t))_{t\in[0,1]}$ with parameter $1/p$.

Notice that the conditioning $\{u \leqslant t\}$ is playing the exact same role as the conditioning on $\{Y(t) \geqslant 1\}$ in Lemma 4.1. Heuristically, (4.142) is then a Yule-Simon process started at a time chosen according to $\mathrm{d}u$ in $\mathbb{R}^+$.

Proof. (i) Since for each fixed t, $\mathrm{d}u \otimes \mathbf{Z}(u \leqslant t) = t$, for every bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbf{Z}^\bullet\left(f\left(\mathbb{1}_{\{u\leqslant t\}} z_{p\ln(t/u)}\right) |u \leqslant t\right) = t^{-1}\int_0^t \mathrm{d}u\, \mathbb{E}\Big[f\Big(Z\big(p(\ln(t) - \ln(u))\big)\Big)\Big], \tag{4.143}$$

where we denoted by Z a standard Yule process. Since Z_r is distributed geometric with parameter e^{-r}, it follows from the change of variable $y = (u/t)^p$ and (4.3) that (4.143) equals

$$p^{-1}\sum_{k\geqslant 1} f(k)\mathrm{B}(k, 1 + 1/p),$$

and the claim follows from (4.3).

(ii) In order to show the second claim, we fix an arbitrary collection of bounded measurable functions $(f_i)_{i\leqslant k}$ with $f_i : \mathbb{R} \mapsto \mathbb{R}$, and an increasing sequence of times $0 \leqslant t_1 < \cdots < t_k \leqslant 1$, and notice that

$$\mathbf{Z}^\bullet\left(\prod_{i=1}^k f_i\left(\mathbb{1}_{\{u\leqslant Tt_i\}} z_{p\ln(Tt_i/u)}\right) |u \leqslant T\right) = \int_0^1 \mathrm{d}u\, \mathbb{E}\left[\prod_{i=1}^k f_i\left(\mathbb{1}_{\{u\leqslant t_i\}} Z\left(p(\ln(t_i) - \ln(u))\right)\right)\right].$$

The claim now follows from the description (4.2) by independence between U and Z. □

Part II

Excursion theory for Markov processes indexed by Lévy trees.

Chapter 5

The structure of the local time of Markov processes indexed by Lévy trees

The content of this chapter is taken from the paper [82], written in collaboration with Armand Riera, and has been accepted for publication, with revisions pending, in the journal *Probability Theory and Related Fields*.

Abstract. We construct an additive functional playing the role of the local time – at a fixed point x – for Markov processes indexed by Lévy trees. We start by proving that Markov processes indexed by Lévy trees satisfy a special Markov property which can be thought as a spatial version of the classical Markov property. Then, we construct our additive functional by an approximation procedure and we characterize the support of its Lebesgue-Stieltjes measure. We also give an equivalent construction in terms of a special family of exit local times. Finally, combining these results, we show that the points at which the Markov process takes the value x encode a new Lévy tree and we construct explicitly its height process. In particular, we recover a recent result of Le Gall concerning the subordinate tree of the Brownian tree where the subordination function is given by the past maximum process of Brownian motion indexed by the Brownian tree.

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5.1 Introduction

Excursion theory plays a fundamental role in the study of $\mathbb{R}_+$–indexed Markov processes dating back to Itô's work [53]. The purpose of this theory is to describe the evolution of a Markov process between visits to a fixed point in the state space. To be more precise, consider a Polish space E, a strong E-valued continuous Markov process ξ and fix a point $x \in E$, regular and instantaneous for ξ. The paths of ξ can be decomposed in excursions away from x, where an excursion is a piece of path of random length, starting and ending at x, such that in between ξ stays away from x. Formally, they consist of the restrictions of ξ to the connected components of $\mathbb{R}_+ \backslash \{t \in \mathbb{R}_+ : \xi_t = x\}$. In order to keep track of the ordering induced by the time, the family of excursions is indexed by means of a remarkable additive functional of ξ, called its local time at x, and denoted throughout this work by $\mathcal{L}$. It is well known that $\mathcal{L}$ is a continuous process with Lebesgue-Stieltjes measure supported on the random set:

$$\big\{t \in \mathbb{R}_+ : \xi_t = x\big\}, \tag{5.1}$$

and that the trajectories of ξ can be recovered from the family of indexed excursions by gluing them together, taking into account the time spent by ξ at x. For technical reasons, we will also assume that the point x is recurrent for ξ. We stress that excursion theory holds under broader assumptions on the Markov process ξ, and we refer to e.g. [17, Chapter VI] and [25] for a complete account.

The purpose of this work is to set the first milestone towards introducing an excursion theory for Markov processes indexed by random trees. The random trees that we consider are the so-called *Lévy trees*. This family is canonical, in the sense that Lévy trees are scaling limits of Galton-Watson trees [43, Chapter 2] and are characterized by a branching property in the same vein as their discrete counterparts [66, 91]. At this point, let us mention that Markov processes indexed by Lévy trees are fundamental objects in probability theory – for instance, they are intimately linked to the theory of superprocesses [43, 64]. More recently, Brownian motion indexed by the Brownian tree has been used as the essential building block in the construction of the universal model of random geometry called the Brownian map [65, 76], as well as in the construction of other related random surfaces [10, 72]. We also stress that Brownian motion indexed by a stable tree is also a universal object, due to the fact that it arises as scaling limit of discrete models [75]. For the sake of completeness, we shall start with a brief and informal account of our objects of interest.

A Lévy tree can be encoded by a continuous $\mathbb{R}_+$-valued process $H = (H_t)$ called its *height process*; and for this reason we denote the associated tree by $\mathcal{T}_H$. Roughly speaking, the tree $\mathcal{T}_H$ has a root and H encodes the distances to it when the tree is explored in "clockwise order". Under appropriate assumptions, we consider the pair consisting of the Markov process ξ and its local time $\mathcal{L}$, indexed by a Lévy tree $\mathcal{T}_H$. With a slight abuse of notation, this process will be denoted in the rest of this work by:

$$\big((\xi_\upsilon, \mathcal{L}_\upsilon) : \upsilon \in \mathcal{T}_H\big). \tag{5.2}$$

In short, this process can be thought as a random motion defined on top of $\mathcal{T}_H$ and following the law of $((\xi_t, \mathcal{L}_t) : t \in \mathbb{R}_+)$, but splitting at every branching point of $\mathcal{T}_H$ into independent copies. The role played by $\{t \in \mathbb{R}_+ : \xi_t = x\}$ is taken over in this setting by the following random subset of $\mathcal{T}_H$:

$$\mathscr{Z} := \{\upsilon \in \mathcal{T}_H : \xi_\upsilon = x\}. \tag{5.3}$$

The definition of the excursions of $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$ away from x should then be clear at an intuitive level – since it suffices to consider the restrictions of $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$ to the connected components of $\mathcal{T}_H \backslash \mathscr{Z}$. Notice however that we lack a proper indexing for this family of excursions that would allow to recover the whole path, as in classical excursion theory. Moreover, one can expect the gluing of these excursions to be more delicate in our setting, since in the time-indexed case the extremities of an excursion consist of only two points, while in the present case, the extremities are subsets of $\mathcal{T}_H$ of significantly more intricate nature. In the same vein, since the set $\mathscr{Z}$ is a subset of $\mathcal{T}_H$, it inherits its tree structure and therefore it possesses richer spatial properties than the subset of the real line (5.1). More precisely, we consider the equivalence relation $\sim_\mathcal{L}$ on $\mathcal{T}_H$ which identifies the components of $\mathcal{T}_H$ where $(\mathcal{L}_\upsilon)_{\upsilon \in \mathcal{T}_H}$ stays constant. The resulting quotient space $\mathcal{T}_H^\mathcal{L} := \mathcal{T}_H / \sim_\mathcal{L}$ is also a tree, encoding the set $\mathscr{Z}$ and endowing it with an additional tree structure. In the terminology of [66], the tree $\mathcal{T}_H^\mathcal{L}$ is the so-called *subordinate tree* of $\mathcal{T}_H$ by $\mathcal{L}$. Since each component of $\mathcal{T}_H$ where $\mathcal{L}$ stays constant is naturally identified with an excursion of ξ away from x, a proper understanding of $\mathcal{T}_H^\mathcal{L}$ is crucial to develop an excursion theory for $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$. This work is devoted to both:

1. Introducing a continuous process suitable to index the excursion of $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$ away from x;

2. Studying the structure of the random set $\mathscr{Z}$.

As we shall explain, both questions are intimately related and, as we mentioned before, they lay the foundations for the development of an excursion theory for $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$. In the case of Brownian motion indexed by the Brownian tree, an excursion theory has already been developed in [1] and has turned out to have multiple applications in Brownian geometry, see e.g. [67, 70].

However, we stress that in [1] the excursions are not indexed and, in particular, a reconstruction of the Brownian motion indexed by the Brownian tree in terms of its excursions is still out of reach. Let us now present the general framework of this work.

In order to formally define the tree indexed process (5.2), we rely on the theory of Lévy snakes and we shall now give a brief account. The theory of Lévy snakes has mainly been developed in the monograph of Duquesne and Le Gall [43], and a detailed presentation of the results that we need is given in Section 5.2. The process (5.2) is built from two layers of randomness. First, as we already mentioned, the family of random trees that we work with are called Lévy trees. If ψ is the Laplace exponent of a spectrally positive Lévy process X, under appropriate assumptions on ψ, one can define the height process H as a functional of X. In order to explain how $\mathcal{T}_H$ is encoded by H, we work under the excursion measure of X above its running infimum and we write σ for the duration of an excursion. The relation:

$$d_H(s,t) := H_s + H_t - 2 \cdot \inf_{s\wedge t \leqslant u \leqslant s \vee t} H_u, \qquad \text{for all } (s,t) \in [0,\sigma]^2,$$

defines a pseudo-distance on $[0,\sigma]$, and the associated equivalence relation $\sim_H$ is defined by setting $s \sim_H t$ if and only if $d_H(s,t) = 0$. The pointed metric space $\mathcal{T}_H := ([0,\sigma_H]/\sim_H, d_H, 0)$ is a Lévy tree[1], where for simplicity we keep the notation 0 for the equivalence class of 0. We also write $p_H : [0,\sigma] \mapsto \mathcal{T}_H$ for the canonical projection on $\mathcal{T}_H$ and we refer to Section 5.2.2 for more details about this encoding. The point 0 is called the root of $\mathcal{T}_H$ and, by construction, the height process encodes the distances to it. We stress that the distribution of $\mathcal{T}_H$ is characterized by the exponent ψ, and we say that $\mathcal{T}_H$ is a ψ-Lévy tree. One of the main technical difficulties of this work is that, except when X is a Brownian motion with drift, the process H is not Markovian and we will need to introduce a measure-valued process – called the exploration process – which heuristically, carries the information needed to make H Markovian. This process will be denoted throughout this work by $\rho = (\rho_t : t \geqslant 0)$ and its nature has a crucial impact on the geometry of $\mathcal{T}_H$. For instance, ρ allows to characterize the multiplicity and genealogy of points of $\mathcal{T}_H$. More precisely, recall that the multiplicity of a point υ in $\mathcal{T}_H$ is defined as the number of connected components of $\mathcal{T}_H \backslash \{\upsilon\}$. For $i \in \mathbb{N}^* \cup \{\infty\}$, we write $\text{Multi}_i(\mathcal{T}_H)$ for the set of points of $\mathcal{T}_H$ of multiplicity i, and the points of multiplicity strictly larger than 2 are called *branching points*. For instance, if X does not have jumps, the measures $(\rho_t : t \geqslant 0)$ are atomless and all branching points have multiplicity 3. In contrast, as soon as the Lévy measure of X is non-null, the measures $(\rho_t : t \geqslant 0)$ have atoms and the set $\text{Multi}_\infty(\mathcal{T}_H)$ is non-empty. We also refer to [69] for the construction of the exploration process. The second layer of randomness consists in defining, given $\mathcal{T}_H$, a spatial motion indexed by $\mathcal{T}_H$ that roughly speaking behaves like the Markov process $(\xi_t)_{t\in\mathbb{R}_+}$ – when restricted to an injective path connecting the root of $\mathcal{T}_H$ to a leaf. This informal description can be formalized by making use of the theory of random snakes [43, Section 5]. More precisely, one can define a process $(W_s, \Lambda_s : s \in [0,\sigma])$ taking values in the collection of finite $E \times \mathbb{R}_+$–valued continuous paths, each (W_s, Λ_s) having lifetime H_s and such that, for each $s \in \mathbb{R}_+$ and conditionally on H_s, the path (W_s, Λ_s) has the same distribution as $(\xi_s, \mathcal{L}_s : s \in [0, H_s])$. The second main property of (W, Λ) is that it satisfies the *snake property*, viz.

$$\big(W_t(H_t), \Lambda_t(H_t)\big) = \big(W_s(H_s), \Lambda_s(H_s)\big), \quad \text{for every } s \sim_H t.$$

[1] More precisely, since the duration σ is random, $\mathcal{T}_H$ is referred to as a *free* Lévy tree.

For simplicity, from now on, we will write $(\widehat{W}_t, \widehat{\Lambda}_t) := (W_t(H_t), \Lambda_t(H_t))$ for the tip of (W_t, Λ_t). By the snake property, it follows that the process $(\widehat{W}_t, \widehat{\Lambda}_t : t \in [0, \sigma])$ is well defined in the quotient space $\mathcal{T}_H$, and hence it defines a random function indexed by $\mathcal{T}_H$ which will be denoted by (5.2). The triplet (ρ, W, Λ) is the so-called ψ-Lévy snake with spatial motion $(\xi, \mathcal{L})$, a Markov process that will be extensively studied throughout this work.

Let us now present the statements of our main results. These are stated under the excursion measure of (ρ, W, Λ), but let us mention that we will obtain similar results under the underlying probability measure. By construction, the study of $\mathscr{X}$ is closely related to the understanding of the random set:

$$\{t \in [0, \sigma] : \widehat{W}_t = x\}, \tag{5.4}$$

since $\mathscr{X}$ is precisely its image under the canonical projection p_H on $\mathcal{T}_H$. However, note that these two sets are of radically different natures. As in classical excursion theory for Markov processes, we shall start by constructing an additive functional $A = (A_t)_{t\in[0,\sigma]}$ of the Lévy snake (ρ, W, Λ) with suitable properties and Lebesgue-Stieltjes measure $\mathrm{d}A$ supported on (5.4). The first main result of this work is obtained in Section 5.4 and is divided in two parts:

(i) The construction of the additive functional A [Proposition 5.22];

(ii) The characterization of the support of $\mathrm{d}A$ [Theorem 5.30].

See also Theorem 5.15 for an equivalent formulation of (ii) in the terminology of the tree indexed process $(\xi_v)_{v\in\mathcal{T}_H}$. Recalling our initial discussion, the process $(A_t)_{t\in\mathbb{R}_+}$ is the natural candidate to index the excursions away from x of $(\xi_v)_{v\in\mathcal{T}_H}$. We are not yet in position in this introduction to formally state the content of (i) and (ii), but we can give a general description. Our construction of $(A_t)_{t\in\mathbb{R}_+}$ relies on the so-called *exit local times* of the Lévy snake (ρ, W, Λ). More precisely, if we consider the family of domains $\{E \times [0, r) : r \in (0, \infty)\}$, for each fixed $r > 0$, there exists an additive functional of (ρ, W, Λ) that heuristically measures, at every $t \geqslant 0$, the number of connected components of $\mathcal{T}_H \backslash \{v \in \mathcal{T}_H : \mathcal{L}_v \leqslant r\}$ visited up to time t. This description is informal and we refer to Section 5.3 for details. We establish in Section 5.4.1 that the corresponding family of exit local times possesses a jointly measurable version $(\mathscr{L}_t^r : t \geqslant 0, r > 0)$, and in Section 5.4.2 we define our continuous additive A by setting:

$$A_t := \int_0^\infty \mathrm{d}r\, \mathscr{L}_t^r, \quad t \geqslant 0.$$

After establishing that there is no branching point with label x, we give in Section 5.4.3 a precise characterization of the support of the measure $\mathrm{d}A$. Formally, we prove that:

$$\text{supp } \mathrm{d}A = \overline{\big\{t \in [0, \sigma] : \xi_{p_H(t)} = x,\, p_H(t) \in \text{Multi}_2(\mathcal{T}_H) \cup \{0\}\big\}}.$$

We also show in Theorem 5.30 that, equivalently, the support of $\mathrm{d}A$ is the complement of the constancy intervals of $(\widehat{\Lambda}_t : t \geqslant 0)$. In particular, if we denote the right inverse of A by $(A_t^{-1} : t \geqslant 0)$, the relation:

$$H_t^A := \widehat{\Lambda}_{A_t^{-1}}, \qquad t \geqslant 0,$$

defines a continuous non-negative process that plays a crucial role in the second part of our work.

In Section 5.5, we turn our attention to the study of $\mathscr{X}$ or, equivalently, to the structure of the subordinate tree $\mathcal{T}_H^{\mathcal{L}}$. Even if this is an object of very different nature, our analysis relies deeply on the results and the machinery developed in Section 5.4. The second main result of this work consists in showing that the process H^A satisfies the following properties:

(i') It encodes the subordinate tree $\mathcal{T}_H^{\mathcal{L}}$ [Theorem 5.31 (i)];

(ii') It is the height function of a Lévy tree, with an exponent $\widetilde{\psi}$ that we identify [Theorem 5.31 (ii)].

In particular, this shows that $\mathcal{T}_H^{\mathcal{L}}$ is a Lévy tree with exponent $\widetilde{\psi}$. We stress that a continuous function can fulfill (i') without satisfying (ii'), and it is remarkable that H^A follows the exploration order of a Lévy tree. We also mention that the previous two points were established – although with a different construction of the height process H^A – in [66, Theorem 1] for the subordination of the Brownian tree by the running maximum of the Brownian motion indexed by the Brownian tree[2]. These approaches are complementary, since the techniques employed in [66] rely on a discrete approximation of the height function, while we shall argue directly in the continuum. We also mention that one of the strengths of our method is that it gives an explicit definition of H^A which is suitable for computations. This point is crucial in order to study the excursions of $(\xi_v)_{v\in\mathcal{T}_H}$ from x. Our result shows that the height function of the subordinate tree $\mathcal{T}_H^{\mathcal{L}}$ can be constructed in terms of functionals of (ρ, W, Λ), and that A^{-1} defines an exploration of $\mathcal{T}_H^{\mathcal{L}}$ compatible with the order induced by H. Property (i') will be a consequence of our previous results (i), (ii) and Section 5.5 is mainly devoted to the proof of (ii'). The main difficulty to establish (ii') comes from the fact that, as we already mentioned, the height process of a Lévy tree is not always Markovian. To circumvent this difficulty, the proof of (ii') relies on the computation of the so-called marginals of the tree associated with H^A. In particular, it makes use of all the machinery developed in previous sections as well as standard properties of Poisson random measures.

Let us now close the presentation of our work with a result of independent interest which is used extensively throughout this paper. In Section 5.3, we state and prove the so-called *Special Markov property* of the Lévy snake. This section is independent of the setting of Sections 5.4 and 5.5, and we work with an arbitrary (ψ, ξ)-Lévy snake under general assumptions on the pair (ψ, ξ). Roughly speaking, the special Markov property is a spatial version of the classical Markov property for time-indexed Markov processes. The precise statement is the content of Theorem 5.10, see also Corollary 5.12. This result was established in [66, Theorem 20] for continuous Markov processes indexed by the Brownian tree, and a particular case was proved for the first time in [64]. Our result is a generalisation of [66, Theorem 20] holding in the broader setting of continuous Markov processes indexed by ψ-Lévy trees. The special Markov property of the Brownian motion indexed by the Brownian tree has already played a crucial role in multiple contexts, see for instance [37, 70, 72] and we expect this result to be useful outside the scope of this work. We also mention that the special Markov property of the Lévy snake is closely related to the one established by Dynkin in the context of superprocesses, see [45, Theorem 1.6]. However, we stress that the formulation in terms of the Lévy snake, although less general, gives additional and crucial information for our purposes. In particular, it takes into account the genealogy induced by the Lévy tree, and hence it caries geometrical information.

We conclude this introduction non-exhaustive summary of related works. First, as we already mentioned, we extend to the general framework of Markov processes indexed by Lévy snakes the

[2]When considering the process $(\xi_v, \mathcal{L}_v)_{v\in\mathcal{T}_H}$ indexed by the Brownian tree, the fact that the subordinate tree $\mathcal{T}_H^{\mathcal{L}}$ is a $\widetilde{\psi}$–Lévy tree is also proved in [66, Theorem 16] but the construction of its height process is lacking.

work of Le Gall on subordination in the case of the Brownian motion indexed by the Brownian tree [66]. Moreover, our results on subordination of trees with respect to the local time are closely related, in the terminology of Lévy snakes, to Theorem 4 in [23] stated in the setting of superprocesses – the main difference being that in our work we encode the associated genealogy. For instance, we recover [23, Theorem 4] in a more precise form in our case of interest. We also note that we expect our results to be useful beyond the scope of this work, for instance in Brownian geometry. Finally, in the case of Brownian motion indexed by the Brownian tree and when $x = 0$, our functional A is closely related to the so-called *integrated super-Brownian excursion* [5] – a random measure arising in multiple limit theorems for discrete probability models, but also in the theory of interacting particle systems [30, 36] and in a variety of models of statistical physics [40, 51]. More precisely, the total mass A_∞ is the density of the integrated super-Brownian excursion at 0, see [71, Proposition 3]. In particular, we hope that our construction of the functional A will be useful to obtain new explicit computations regarding the integrated super-Brownian excursion and to generalize these computations to related models.

The work is organised as follows: Section 2 gives an overview of the theory of Lévy trees and snakes. In Section 3, we state and prove the special Markov property for Lévy snakes and we explore some of its consequences. This section is independent of the rest of the work but is key for the development of Section 4 and 5. The preliminary results needed for its proof are covered in Section 5.3.1, and mainly concern approximation results for exit local times. Section 4 is devoted to first, constructing in Section 4.2 the additive functional A [Proposition 5.22], and afterwards to the characterization of the support of the measure $\mathrm{d}A$ [Theorem 5.30] in Section 4.3. We shall give two equivalent descriptions for the support of $\mathrm{d}A$, one in terms of the pair (H, W), and a second one only depending on Λ. The latter will be needed in Section 5.5 and we expect the former to be useful to develop an excursion theory – we plan to pursue this goal in future works. The preliminary results needed for our constructions are covered in Section 4.1. Finally, in Section 5, after recalling preliminary results on subordination of trees by continuous functions, we explore the tree structure of the set $\{v \in \mathcal{T}_H : \xi_v = x\}$ by considering the subordinate tree of $\mathcal{T}_H$ with respect to the local time $\mathcal{L}$. The main result of the section is stated in Theorem 5.31, and consists in proving (i') and (ii').

5.2 Preliminaries

5.2.1 The height process and the exploration process

Let us start by introducing the class of Lévy processes that we will consider throughout this work. We set X a Lévy process on $\mathbb{R}_+$, and we denote its law started from 0 by P. It will be convenient to assume that X is the canonical process on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ of rcll (right–continuous with left limits) real-valued paths equipped with the probability measure P. We denote the canonical filtration by $(\mathcal{G}_t : t \geqslant 0)$, completed as usual by the class of P– negligible sets of $\mathcal{G}_\infty = \bigvee_{t \geqslant 0} \mathcal{G}_t$. We henceforth assume that X verifies P-a.s. the following properties:

- (A1) X does not have negative jumps;
- (A2) The paths of X are of infinite variation;

- (A3) X does not drift to $+\infty$.

Since X has no negative jumps the mapping $\lambda \mapsto \mathbb{E}[\exp(-\lambda X_1)]$ is well defined in $\mathbb{R}_+$ and we denote the Laplace exponent of X by ψ, viz. the function defined by:

$$\mathbb{E}[\exp(-\lambda X_1)] = \exp(\psi(\lambda)), \qquad \text{for all } \lambda \geqslant 0.$$

The function ψ can be written in the Lévy-Khintchine form:

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}x)\ (\exp(-\lambda x) - 1 + \lambda x \mathbb{1}_{\{x \leqslant 1\}}),$$

where $\alpha_0 \in \mathbb{R}$, $\beta \in \mathbb{R}_+$ and π is a sigma-finite measure on $\mathbb{R}_+^*$ satisfying $\int_{(0,\infty)} \pi(\mathrm{d}x)(1 \wedge x^2) < \infty$. Moreover, it is well known that condition (A2) holds if and only if we have:

$$\beta \neq 0 \qquad \text{or} \qquad \int_{(0,1)} \pi(\mathrm{d}x)\ x = \infty.$$

The Laplace exponent ψ is infinitely differentiable and strictly convex in $(0,\infty)$ (see e.g. Chapter 8 in [61]). Since X does not drift towards ∞ one has $-\psi'(0+) = E[X_1] \leqslant 0$ which, in turn, implies that X oscillates, or drifts towards $-\infty$ and that X_t has a finite first moment for any t. In terms of the Lévy measure, this ensures that the additional integrability condition $\int_{(1,\infty)} \pi(\mathrm{d}x)\ x < \infty$ holds. Consequently, ψ can and will be supposed to be of the following form:

$$\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}x)(\exp(-\lambda x) - 1 + \lambda x),$$

where now π satisfies $\int_{(0,\infty)} \pi(\mathrm{d}x)(x \wedge x^2) < \infty$ and $\alpha, \beta \in \mathbb{R}_+$ since $\alpha = \psi'(0+)$. From now on, we will denote the infimum of X by I and remark that, under our current hypothesis, 0 is regular and instantaneous for the Markov process $X - I = (X_t - \inf_{[0,t]} X_s : t \geqslant 0)$. Moreover, it is standard to see that P –a.s., the Lebesgue measure of $\{t \in \mathbb{R}_+ : X_t = I_t\}$ is null. The process $-I$ is a local time of $X - I$ and we denote the associated excursion measure from 0 by N. To simplify notation, we write σ_e for the lifetime of an excursion e. Finally, we impose the following additional assumption on ψ:

$$\int_1^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)} < \infty. \tag{A4}$$

From now on, we will be working under (A1) – (A4).

Let us now briefly discuss the main implications of our assumptions. The condition (A4) is twofold: on the one hand, it ensures that $\lim_{\lambda\to\infty} \lambda^{-1}\psi(\lambda) = \infty$ which implies that X has paths of infinite variation [17, VII-5] (the redundancy in our hypothesis is on purpose for ease of reading). On the other hand, under our hypothesis (A1) – (A3), it is well known that there exists a continuous state branching process with branching mechanism $\psi(\lambda)$ (abbreviated ψ-CSBP) and that (A4) is equivalent to its a.s. extinction. The ψ-Lévy tree can be interpreted as the genealogical tree of this branching process and is defined in terms of a fundamental functional of X, called the height process, that we now introduce.

The height and exploration processes. Let us turn our attention to the so-called height process – the main ingredient needed to define Lévy trees. Our presentation follows [43, Chapter 1] and we start by introducing some standard notation. For every $0 < s \leqslant t$, we set

$$I_{s,t} := \inf_{s \leqslant u \leqslant t} X_u,$$

the infimum of X in $[s,t]$ and remark that when $s = 0$ we have $I_t = I_{0,t}$. Moreover, since X drifts towards $-\infty$ or oscillates, we must have $I_t \to -\infty$ when $t \uparrow \infty$. By [43, Lemma 1.2.1], for every fixed $t \geqslant 0$, the limit:

$$H_t := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[0,t]} \mathrm{d}r \; \mathbb{1}_{\{X_r < I_{r,t} + \varepsilon\}} \tag{5.5}$$

exists in probability. Roughly speaking, for every fixed t, the quantity H_t measures the size of the set:

$$\{r \leqslant t : \; X_{r-} \leqslant I_{r,t}\},$$

and we refer to $H = (H_t : t \geqslant 0)$ as the height process of X. By [43, Theorem 1.4.3], condition (A4) ensures that H possesses a continuous modification that we consider from now on and that we still denote by H.

The process H will be the building block to define Lévy trees. However, H is not Markovian as soon as $\pi \neq 0$ and we will need to introduce a process – called the exploration process – which roughly speaking carries the needed information to make H Markovian. More precisely, the exploration process is a Markov process and we will write H as a functional of it. In this direction, we write $\mathcal{M}_f(\mathbb{R}_+)$ for the set of finite measures on $\mathbb{R}_+$ equipped with the topology of weak convergence and with a slight abuse of notation we write 0 for the null measure on $\mathbb{R}_+$. The exploration process $\rho = (\rho_t : t \geqslant 0)$ is the random measure defined as:

$$\langle \rho_t, f \rangle := \int_{[0,t]} \mathrm{d}_s I_{s,t} \, f(H_s), \quad t \geqslant 0, \tag{5.6}$$

where $\mathrm{d}_s I_{s,t}$ stands for the measure associated with the non-decreasing function $s \mapsto I_{s,t}$. Equivalently, ρ can be defined as:

$$\rho_t(\mathrm{d}r) := \beta \mathbb{1}_{[0,H_t]}(r)\mathrm{d}r + \sum_{\substack{0 < s \leqslant t \\ X_{s-} < I_{s,t}}} (I_{s,t} - X_{s-}) \, \delta_{H_s}(\mathrm{d}r), \quad t \geqslant 0, \tag{5.7}$$

and remark that (5.6) implies that

$$\langle \rho_t, 1 \rangle = I_{t,t} - I_{0,t} = X_t - I_t, \quad t \geqslant 0.$$

In particular, ρ_t takes values in $\mathcal{M}_f(\mathbb{R}_+)$. By [43, Proposition 1.2.3], the process $(\rho_t : t \geqslant 0)$ is an $\mathcal{M}_f(\mathbb{R}_+)$-valued rcll strong Markov process, and we briefly recall some of its main properties for later use. For every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, we write $\mathrm{supp}(\mu)$ for the topological support of μ and we set $H(\mu) := \sup \mathrm{supp}(\mu)$ with the convention $H(0) = 0$.
The following properties hold:

(i) Almost surely, for every $t \geqslant 0$, we have $\mathrm{supp}\; \rho_t = [0, H_t]$ if $\rho_t \neq 0$.

(ii) The process $t \mapsto \rho_t$ is rcll with respect to the total variation distance.

(iii) Almost surely, the following sets are equal:

$$\{t \geqslant 0 : \rho_t = 0\} = \{t \geqslant 0 : X_t - I_t = 0\} = \{t \geqslant 0 : H_t = 0\}. \tag{5.8}$$

In particular, note that we have $(H(\rho_t))_{t\geqslant 0} = (H_t)_{t\geqslant 0}$ and that point (ii) implies that the excursion intervals away from 0 of $X - I$, H and ρ coincide. Moreover, since $I_t \to -\infty$ when $t \uparrow \infty$, the excursion intervals have finite length and by [43, Lemma 1.3.2] and the monotonicity of $t \mapsto I_t$ we have:

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\sup_{s\in[0,t]} \Big|\frac{1}{\varepsilon} \int_0^s \mathrm{d}u\, \mathbb{1}_{\{H_u<\varepsilon\}} + I_s\Big|\Big] = 0, \qquad \text{for every } t \geqslant 0. \tag{5.9}$$

By the previous display, $-I$ can be thought as the local time of H at 0.

The Markov process ρ in our previous definition starts at $\rho_0 = 0$ and, in order to make use of the Markov property, we need to recall how to define it's distribution starting from an arbitrary measure $\mu \in \mathcal{M}_f(\mathbb{R}_+)$. In this direction, we will need to introduce the following two operations:

Pruning. For every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and $0 \leqslant a < \langle \mu, 1\rangle$, we set $\kappa_a\mu$ the unique measure on $\mathbb{R}_+$ such that for every $r \geqslant 0$:

$$\kappa_a\mu([0,r]) := \mu([0,r]) \wedge (\langle \mu, 1\rangle - a).$$

If $a \geqslant \langle \mu, 1\rangle$ we simply set $\kappa_a\mu := 0$. The operation $\mu \mapsto \kappa_a\mu$ corresponds to a pruning operation "from the right" and note that, for every $a > 0$ and $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, the measure $\kappa_a\mu$ has compact support. In particular, one has $H(\kappa_a\mu) < \infty$ for every $a > 0$, even for μ with unbounded support.

Concatenation. Consider $\mu, \nu \in \mathcal{M}_f(\mathbb{R}_+)$ such that $H(\mu) < \infty$. The concatenation of the measure μ with ν is again an element of $\mathcal{M}_f(\mathbb{R}_+)$, denoted by $[\mu, \nu]$ and defined by the relation:

$$\langle [\mu, \nu], f\rangle := \int \mu(\mathrm{d}r) f(r) + \int \nu(\mathrm{d}r) f(H(\mu) + r).$$

Finally, for every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, the exploration process started from μ is denoted by ρ^μ and defined as:

$$\rho^\mu_t := [\kappa_{-I_t}\mu, \rho_t], \quad t > 0, \tag{5.10}$$

with the convention $\rho^\mu_0 := \mu$. In this definition we used the fact that, P-a.s., $I_t < 0$ for every $t > 0$, since we are not imposing the condition $H(\mu) < \infty$ on μ. Remark that the process $\langle \rho^\mu, 1\rangle := (\langle \rho^\mu_t, 1\rangle : t \geqslant 0)$ has the same distribution as X started from $\langle \mu, 1\rangle$, this fact will be used frequently.

For later use we also need to introduce the dual process of ρ, this is, the $\mathcal{M}_f(\mathbb{R}_+)$-valued process $(\eta_t : t \geqslant 0)$ defined by the formula

$$\eta_t(\mathrm{d}r) := \beta \mathbb{1}_{[0,H_t]}(r)\mathrm{d}r + \sum_{\substack{0<s\leqslant t\\ X_{s-}<I_{s,t}}} (X_s - I_{s,t})\, \delta_{H_s}(\mathrm{d}r), \quad t \geqslant 0. \tag{5.11}$$

This process will be only needed for some computations and the terminology will be justified by the identity (5.13) below. Moreover, η is rcll with respect to the total variation distance and the pair (ρ, η) is a Markov process. We refer to [43, Section 3.1] for a complete account on $(\eta_t : t \geqslant 0)$.

Before concluding this section, it will be crucial for our purposes to define the height process and the exploration process under the excursion measure N of $X - I$. In this direction, if for an arbitrary fixed r we set $g = \sup\{s \leqslant r : X_s - I_s = 0\}$ and $d = \inf\{s \geqslant r : X_s - I_s = 0\}$, it is straightforward to see that $(H_t : t \in [g, d])$ can be written in terms of a functional of the excursion of $X - I$ that straddles r, say $e_j = (X_{(g+t)\wedge d} - I_g : t \geqslant 0)$, and this functional does not depend on the choice of r. Informally, from the initial definition (5.5) this should not come as a surprise since the integral (5.5) for $t \in [g, d]$ vanishes on $[0, g]$, we refer to the discussion appearing before Lemma 1.2.4 in [43] for more details. We denote this functional by $H(e_j)$ and it satisfies that P–a.s., $H_t = H_{t-g}(e_j)$ for every $t \in [g, d]$. Furthermore, if we denote the connected components of $\{t \geqslant 0 : X_t - I_t = 0\}$ by $\big((a_i, b_i) : i \in \mathbb{N}\big)$ and the corresponding excursions by $(e_i : i \in \mathbb{N})$, then we have $H_{(a_i+t)\wedge b_i} = H_t(e_i)$, for all $t \geqslant 0$. By considering the first excursion e of $X - I$ with duration $\sigma_e > \varepsilon$ for every $\varepsilon > 0$, it follows that the functional $H(e)$ in $D(\mathbb{R}_+, \mathbb{R})$ under $N(\mathrm{d}e \,|\sigma_e > \varepsilon)$ is well defined, and hence it is also well defined under the excursion measure N.

Turning now our attention to the exploration process and its dual, observe that for $t \in [a_i, b_i]$ the mass of the atoms in (5.7) and (5.11) only depend on the corresponding excursion e_i. We deduce by our previous considerations on H that we can also write $\rho_{(a_i+t)\wedge b_i} = \rho_t(e_i)$ and $\eta_{(a_i+t)\wedge b_i} = \eta_t(e_i)$, for all $t \geqslant 0$, where the functionals $\rho(e)$, $\eta(e)$ are still defined by (5.7) and (5.11) respectively, but replacing X by e_i and H by $H(e_i)$ – translated in time appropriately. By the same arguments as before, we deduce that $\rho(e)$ and $\eta(e)$ under $N(\mathrm{d}e)$ are well defined $\mathcal{M}_f(\mathbb{R}_+)$-valued functionals. From now on, when working under N, the dependency on e is omitted from H, ρ and η. Remark that under N, we still have $H(\rho_t) = H_t$ and $\langle \rho_t, 1\rangle = X_t$, for every $t \geqslant 0$, where now X is an excursion of the reflected process. By excursion theory for the reflected Lévy process $X - I$ we deduce that the random measure in $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)$ defined as

$$\sum_{i\in\mathbb{N}} \delta_{(-I_{a_i}, \rho_{(a_i+\cdot)\wedge b_i}, \eta_{(a_i+\cdot)\wedge b_i})} \tag{5.12}$$

is a Poisson point measure with intensity $\mathbb{1}_{\ell\geqslant 0}\mathrm{d}\ell\, N(\mathrm{d}\rho, \mathrm{d}\eta)$. Finally, we recall for later use the equality in distribution under N:

$$\big((\rho_t, \eta_t) : t \geqslant 0\big) \overset{(d)}{=} \big((\eta_{(\sigma-t)-}, \rho_{(\sigma-t)-}) : t \geqslant 0\big), \tag{5.13}$$

and we refer to [43, Corollary 3.1.6] for a proof. This identity is the reason why η is called the dual process of ρ.

5.2.2 Trees coded by excursions and Lévy trees

The height process H under N is the main ingredient needed to define Lévy trees, one of the central objects studied in this work. Before giving a formal definition, we shall briefly recall some standard notation and notions related to (deterministic) pointed $\mathbb{R}$-trees.

Real trees. In the same vein as the construction of planar (discrete) trees in terms of their contour functions, there exists a canonical construction of pointed $\mathbb{R}$-trees in terms of positive continuous functions. In order to be more precise, we introduce some notation. Let $e : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function, set σ_e the functional $\sigma_e := \inf\{t > 0 : e(s) = 0, \text{ for every } s \geqslant t\}$ with the usual convention $\inf\{\varnothing\} := \infty$. In particular, when $e(0) = 0$, $\sigma_e < \infty$ and $e(s) > 0$ for all $s \in (0, \sigma_e)$, the function e is called an excursion with lifetime σ_e. Note that these notations are compatible with the ones introduced in the previous section. For convenience, we take $[0, \sigma_e] := [0, \infty)$ if $\sigma_e = \infty$. For every $s, t \in [0, \sigma_e]$ with $s \leqslant t$ set

$$m_e(s,t) := \inf_{s \leqslant u \leqslant t} e(u),$$

and consider the pseudo-distance on $[0, \sigma_e]$ defined by:

$$d_e(s,t) := e(s) + e(t) - 2 \cdot m_e(s \wedge t, s \vee t), \qquad \text{for all } (s,t) \in [0, \sigma_e]^2.$$

The pseudo-distance d_e induces an equivalence relation $\sim_e$ in $[0, \sigma_e]$ according to the following simple rule: for every $(s,t) \in [0, \sigma_e]^2$ we write $s \sim_e t$ if and only if $d_e(s,t) = 0$, and we keep the notation 0 for the equivalency class of the real number 0. The pointed metric space $\mathcal{T}_e := ([0, \sigma_e]/\sim_e, d_e, 0)$ is an $\mathbb{R}$-tree, called the tree encoded by e and we denote its canonical projection by $p_e : [0, \sigma_e] \to \mathcal{T}_e$. We stress that if $\sigma_e < \infty$, then $\mathcal{T}_e$ is a compact $\mathbb{R}-$tree.

Let us now give some standard properties and notations. We recall that in an $\mathbb{R}$-tree there is only one continuous injective path connecting any two points $u, v \in \mathcal{T}_e$, and we denote its image in $\mathcal{T}_e$ by $[u, v]_{\mathcal{T}_e}$. We say that u is an ancestor of v if $u \in [0, v]_{\mathcal{T}_e}$ and we write $u \preceq_{\mathcal{T}_e} v$. One can check directly from the definition that we have $u \preceq_{\mathcal{T}_e} v$ if and only if there exists $(s,t) \in [0, \sigma_e]^2$ such that $(p_e(s), p_e(t)) = (u, v)$ and $e(s) = m_e(s \wedge t, s \vee t)$. In other words, we have:

$$[0, v]_{\mathcal{T}_e} = p_e\big(\{s \in [0, \sigma_e] : \ e(s) = m_e(s \wedge t, s \vee t)\}\big),$$

where t is any preimage of v by p_e. To simplify notation, we write $u \curlywedge_{\mathcal{T}_e} v$ for the unique element on the tree verifying $[0, u \curlywedge_{\mathcal{T}_e} v]_{\mathcal{T}_e} = [0, u]_{\mathcal{T}_e} \cap [0, v]_{\mathcal{T}_e}$. The element $u \curlywedge_{\mathcal{T}_e} v$ is known as the common ancestor of u and v. Finally, if $u \in \mathcal{T}_e$, the number of connected components of $\mathcal{T}_e \backslash \{u\}$ is called the multiplicity of u. For every $i \in \mathbb{N}^* \cup \{\infty\}$, we will denote the set of points $u \in \mathcal{T}_e$ of multiplicity equal to i by $\mathrm{Mult}_i(\mathcal{T})$. The points of multiplicity larger than 2 are called *branching points*, and the points of multiplicity 1 are called *leaves*.

Lévy trees. We are now in position to introduce:

Definition 5.1. *The random metric space $\mathcal{T}_H$ under the excursion measure N is the (free) ψ-Lévy tree.*

The term free refers to the fact that the lifetime of H is not fixed under N and it will be omitted from now on. Note that the metric space $\mathcal{T}_H$ can be considered under P without any modifications. Since, under P, we have $\sigma_H = \infty$, the tree $\mathcal{T}_H$ stands for the space $(\mathbb{R}_+/\sim_H, d_H, 0)$, and in particular it is no longer a compact space. The rest of the properties however remain valid and we will use the same notations indifferently under P and N. Moreover, since the point 0 is recurrent for the process $X - I$, it is also recurrent for H by point (ii) of the previous section. This gives a natural interpretation of $\mathcal{T}_H$ as the concatenation at the root of infinitely many trees $\mathcal{T}_{H^i}$, where $(H^i)_{i \in \mathbb{N}} = (H(e_i))_{i \in \mathbb{N}}$ are the excursions of H away from 0, and where the concatenation follows the order induced by the local time $-I$. For this reason, we will say that $\mathcal{T}_H$ under P is a ψ-forest (made of ψ-Lévy trees). In particular, remark that under P (resp. N), the root $p_H(0)$ is a branching point of multiplicity ∞ (resp. a leaf).

Before concluding the discussion on $\mathbb{R}$-trees, we recall that, under P or N, $\text{Mult}_i(\mathcal{T}_H) = \varnothing$ for every $i \notin \{1, 2, 3, \infty\}$. Moreover, we have $\text{Mult}_\infty(\mathcal{T}_H)\backslash\{p_H(0)\} = \varnothing$ if and only if $\pi = 0$ or, equivalently, if X does not have jumps. More precisely, p_H realizes a bijection between $\{t \geqslant 0 : \Delta X_t > 0\}$ and $\text{Mult}_\infty(\mathcal{T}_H)\backslash\{p_H(0)\}$.

5.2.3 The Lévy snake

In this section, we give a short introduction to the so-called Lévy snake, a path-valued Markov process that allows to formalize the notion of a "Markov process indexed by a Lévy tree". We follow the presentation of [43, Chapter 4]. However, beware that in this work we consider continuous paths defined in closed intervals, and hence our framework differs slightly with the one considered in [43, Chapter 4][3].

Snakes driven by continuous functions. Fix a Polish space E equipped with a distance d_E inducing its topology and we let $\mathcal{W}_E$ be the set of E-valued killed continuous functions. Each $\mathrm{w} \in \mathcal{W}_E$ is a continuous path $\mathrm{w} : [0, \zeta_\mathrm{w}] \to E$, defined in a compact interval $[0, \zeta_\mathrm{w}]$. The functional $\zeta_\mathrm{w} \in [0, \infty)$ is called the lifetime of w and it will be convenient to denote the endpoint of w by $\widehat{\mathrm{w}} := \mathrm{w}(\zeta_\mathrm{w})$. Further, we write $\mathcal{W}_{E,x} := \{\mathrm{w} \in \mathcal{W}_E : \mathrm{w}(0) = x\}$ for the subcollection of paths in $\mathcal{W}_E$ starting at x, and we identify the trivial element of $\mathcal{W}_x$ with zero lifetime with the point x. We equip $\mathcal{W}_E$ with the distance

$$d_{\mathcal{W}_E}(\mathrm{w}, \mathrm{w}') := |\zeta_\mathrm{w} - \zeta_{\mathrm{w}'}| + \sup_{r \geqslant 0} d_E\big(\mathrm{w}(r \wedge \zeta_\mathrm{w}), \mathrm{w}'(r \wedge \zeta_{\mathrm{w}'})\big),$$

and it is straightforward to check that $(\mathcal{W}_E, d_{\mathcal{W}_E})$ is a Polish space. Let us insist that the notation e is exclusively used for continuous $\mathbb{R}_+$-valued functions defined in $\mathbb{R}_+$, and w is reserved for E-valued continuous paths defined in compact intervals $[0, \zeta_\mathrm{w}]$, viz. for the elements of $\mathcal{W}_E$.

We will now endow $\mathcal{W}_E^{\mathbb{R}_+}$ with a probability measure. In this direction, consider an E-valued Markov process $\xi = (\xi_t : t \geqslant 0)$ with continuous sample paths. For every $x \in E$, let Π_x denote the distribution of ξ started at x and also assume that ξ is time-homogeneous (it is implicitly assumed in our definition that the mapping $x \mapsto \Pi_x$ is measurable). Now, fix a deterministic continuous function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$. The first step towards defining the Lévy snake consists in introducing a $\mathcal{W}_E$-valued process referred as *the snake driven by h with spatial motion ξ*. In this direction, we also fix a point $x \in E$ and a path $\mathrm{w} \in \mathcal{W}_{E,x}$. For every a, b such that $0 \leqslant a \leqslant \zeta_\mathrm{w}$ and $b \geqslant a$, there exists a unique probability measure $R_{a,b}(\mathrm{w}, \mathrm{dw}')$ on $\mathcal{W}_{E,x}$ satisfying the following properties:

(i) $R_{a,b}(\mathrm{w}, \mathrm{dw}')$-a.s., $\mathrm{w}'(s) = \mathrm{w}(s)$ for every $s \in [0, a]$.

(ii) $R_{a,b}(\mathrm{w}, \mathrm{dw}')$-a.s., $\zeta_{\mathrm{w}'} = b$.

(iii) Under $R_{a,b}(\mathrm{w}, \mathrm{dw}')$, $(\mathrm{w}'(s + a))_{s \in [0, b-a]}$ is distributed as $(\xi_s)_{s \in [0, b-a]}$ under $\Pi_{\mathrm{w}(a)}$.

Denoting the canonical process on $\mathcal{W}_E^{\mathbb{R}_+}$ by $(W_s)_{s \geqslant 0}$, it is easy to see by Kolmogorov's extension theorem that, for every $\mathrm{w}_0 \in \mathcal{W}_{E,x}$ with $\zeta_{\mathrm{w}_0} = h(0)$, there exists a unique probability measure

[3]The paths considered in [43, Section 4.1] are rcll and defined in intervals of the form $[0, \zeta)$, for $\zeta \in (0, \infty)$.

$Q^h_{\mathrm{w}_0}$ on $\mathcal{W}_E^{\mathbb{R}_+}$ satisfying that

$$Q^h_{\mathrm{w}_0}\big(W_{s_0} \in A_0, W_{s_1} \in A_1, ..., W_{s_n} \in A_n\big)$$
$$= \mathbb{1}_{\{\mathrm{w}_0 \in A_0\}} \int_{A_1 \times A_2 \times \cdots \times A_n} R_{m_h(s_0,s_1),h(s_1)}(\mathrm{w}_0, \mathrm{dw}_1) R_{m_h(s_1,s_2),h(s_2)}(\mathrm{w}_1, \mathrm{dw}_2) ... R_{m_h(s_{n-1},s_n),h(s_n)}(\mathrm{w}_{n-1}, \mathrm{dw}_n).$$

for every $0 = s_0 \leqslant s_1 \leqslant ... \leqslant s_n$ and $A_0, ..., A_n$ Borelian sets of $\mathcal{W}_E$. The canonical process W in $\mathcal{W}_E^{\mathbb{R}_+}$ under $Q^h_{\mathrm{w}_0}$ is called the snake driven by h with spatial motion ξ started from w_0. The value $W_s = (W_s(t) : t \in [0, h(s)])$ of the Lévy snake at time s coincides with w_0 for $0 \leqslant t \leqslant m_h(0, s)$ while for $m_h(0, s) \leqslant t \leqslant h(s)$, it is distributed as the Markov process ξ started at $\mathrm{w}_0(m_h(0, s))$ and stopped at time $h(s) - m_h(0, s)$. Furthermore, informally, when h decreases, the path is erased from its tip and, when h increases, the path is extended by adding "little pieces" of trajectories of ξ at the tip. The term snake refers to the fact that, the definition of $Q^h_{\mathrm{w}_0}$ entails that for every $s < s'$ we have:

$$W_s(r) = W_{s'}(r), \quad r \in [0, m_h(s, s')], \quad Q^h_{\mathrm{w}_0}\text{–a.s.} \tag{5.14}$$

Note however that this property only holds for fixed s, s' $Q^h_{\mathrm{w}_0}$-a.s. A priori, under $Q^h_{\mathrm{w}_0}$, the process W does not have a continuous modification with respect to the metric $d_{\mathcal{W}_E}$, but it will be crucial for our work to find suitable conditions guaranteeing the existence of such modification. This question will be addressed in the following proposition. We start by introducing some notation. First recall the convention $[a, \infty] := [a, \infty)$ for $a < \infty$. Next, consider a $\mathcal{J}$– indexed family $a_i, b_i \in \mathbb{R}_+ \cup \{\infty\}$, $\mathcal{J} \subset \mathbb{N}$, with $a_i < b_i$ and suppose that the intervals $([a_i, b_i], i \in \mathcal{J})$ are disjoint. A continuous function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be locally r-Hölder continuous in $([a_i, b_i], i \in \mathcal{J})$ if, for every $n \in \mathbb{N}$, there exists a constant C_n satisfying that $|h(s) - h(t)| \leqslant C_n |s - t|^r$, for every $i \in \mathcal{J}$ and $s, t \in [a_i, b_i] \cap [0, n]$. We insist on the fact that the constant C_n does not depend on the index i.

Proposition 5.2. *Suppose that there exists a constant $C_\Pi > 0$ and two positive numbers $p, q > 0$ such that, for every $x \in E$ and $t \geqslant 0$, we have:*

$$\Pi_x\Big(\sup_{0 \leqslant u \leqslant t} d_E(\xi_u, x)^p\Big) \leqslant C_\Pi \cdot t^q. \tag{5.15}$$

Further, consider a continuous function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and denote by $((a_i, b_i) : \ i \in \mathcal{J})$ the excursion intervals above its running infimum. If h is locally r-Hölder continuous in $([a_i, b_i] : i \in \mathcal{J})$ with $qr > 1$ then, for every $\mathrm{w} \in \mathcal{W}_E$ with $\zeta_{\mathrm{w}} = h(0)$, the process W has a continuous modification under Q^h_{w}.

Proof. With the notation introduced in the statement of the proposition, we fix a continuous driving function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ locally r-Hölder continuous in $([a_i, b_i] : \ i \in \mathcal{J})$, an initial condition $\mathrm{w} \in \mathcal{W}_E$ with $\zeta_{\mathrm{w}} = h(0)$, and we consider an arbitrary $n \in \mathbb{N}$. By definition, for any $s, t \in [a_i, b_i] \cap [0, n]$, we have $|h(s) - h(t)| \leqslant C_n \cdot |s - t|^r$ for a constant C_n that does not depend on i. Next, we consider W, the snake driven by h under $Q^h_{\mathrm{w}}(\mathrm{d}W)$. The first step of the proof consists in showing that the process $(W_s : s \in \bigcup_{i \in \mathcal{J}} [a_i, b_i])$ has a locally Hölder-continuous modification on $\big([a_i, b_i] : \ i \in \mathcal{J}\big)$. In this direction, we remark that the definition of $d_{\mathcal{W}_E}$ gives:

$$Q^h_{\mathrm{w}}\big(d_{\mathcal{W}_E}(W_s, W_t)^p\big) \leqslant 2^p \cdot Q^h_{\mathrm{w}}\Big(\sup_{m_h(s,t) \leqslant u} d_E\big(W_s(u \wedge h(s)), W_t(u \wedge h(t))\big)^p\Big) + 2^p \cdot |h(s) - h(t)|^p,$$

for every $s, t \in [a_i, b_i] \cap [0, n]$. Next, note that the first term on the right hand side can be bounded above by:

$$
\begin{aligned}
&Q^h_{\mathrm{w}}\Big(\sup_{m_h(s,t)\leqslant u} d_E\big(W_s(u \wedge h(s)), W_t(u \wedge h(t))\big)^p \Big) \\
&\leqslant 2^p \cdot Q^h_{\mathrm{w}}\Big(\sup_{m_h(s,t)\leqslant u} d_E\big(W_s(u \wedge h(s)), W_s(m_h(s,t))\big)^p \Big) \\
&\qquad\qquad + 2^p \cdot Q^h_{\mathrm{w}}\Big(\sup_{m_h(s,t)\leqslant u} d_E\big(W_t(m_h(s,t)), W_t(u \wedge h(t))\big)^p \Big) \\
&\leqslant 2^p \cdot Q^h_{\mathrm{w}}\Big(\Pi_{W_s(m_h(s,t))}\big(\sup_{u\leqslant h(s)-m_h(s,t)} d_E(\xi_u, \xi_0)^p \big) \Big) \\
+ &\qquad\qquad 2^p \cdot Q^h_{\mathrm{w}}\Big(\Pi_{W_t(m_h(s,t))}\big(\sup_{u\leqslant h(t)-m_h(s,t)} d_E(\xi_0, \xi_u)^p \big) \Big) \\
&\leqslant 2^p C_\Pi \cdot \Big(\big|h(s) - m_h(s,t)\big|^q + \big|h(t) - m_h(s,t)\big|^q \Big),
\end{aligned}
$$

where in the second inequality we applied the Markov property at time $m_h(s,t)$, and in the last one we used the upper bound (5.15). By our assumptions on h we derive that, for every $n > 0$, there exists a constant C'_n such that:

$$
Q^h_{\mathrm{w}}\left(d_{\mathcal{W}_E}(W_s, W_t)^p \right) \leqslant C'_n \cdot \big(|t-s|^{qr} + |t-s|^{pr} \big), \qquad \text{for any } s, t \in [a_i, b_i] \cap [0, n],
$$

and we stress that the constant C'_n does not depend on i. Recall that $qr > 1$ and note that we can assume as well that $pr > 1$, since by replacing the distance $d_{\mathcal{W}_E}$ by $1 \wedge d_{\mathcal{W}_E}$, we can take p as large as wanted. Now, fix $r_0 \in (0, (qr-1)/p)$. We deduce by a standard Borel-Cantelli argument, similar to the proof of Kolmogorov's lemma, that there exists a modification of $(W_s : s \in [0,n] \cap \bigcup_{i\in\mathcal{J}} [a_i, b_i])$, say $(W^*_s : s \in [0,n] \cap \bigcup_{i\in\mathcal{J}} [a_i, b_i])$, satisfying that $Q^h_{\mathrm{w}}-$ a.s., for every $i \in \mathcal{J}$

$$
d_{\mathcal{W}_E}(W^*_s, W^*_t) \leqslant K_n |s-t|^{r_0}, \qquad \text{for every } s, t \in [a_i, b_i] \cap [0, n], \tag{5.16}
$$

where the (random) quantity K_n does not depend on i. Set $\mathcal{V} := \mathbb{R}_+ \backslash \bigcup_{i\in\mathcal{J}} [a_i, b_i]$ and remark that if $t \in \mathcal{V}$, then $h(t) = \inf\{h(u) : u \in [0,t]\}$. For every $t \in \mathcal{V}$, we set $W^*_t := (\mathrm{w}(u) : u \in [0, h(t)])$ and we consider the process $(W^*_t : t \in [0,n])$. Notice that by the very construction of W^*, we have $Q^h_{\mathrm{w}}(W_t = W^*_t) = 1$ for every $t \in [0,n]$, which shows that W^* is a modification of W in $[0,n]$.

Let us now show that W^* is continuous on $[0,n]$. The continuity for $t \in [0,n] \cap \bigcup_{i\in\mathcal{J}} (a_i, b_i)$ follows by (5.16) and we henceforth fix $t \in [0,n] \backslash \bigcup_{i\in\mathcal{J}} (a_i, b_i)$. In particular, we have $h(t) = \inf\{h(u) : u \in [0,t]\}$. On one hand, if $(s_k : k \geqslant 1)$ is a sequence with $s_k \to t$ as $k \uparrow \infty$, the continuity of w and h ensures that $(\mathrm{w}(u) : u \in [0, h(s_k)]) \to W^*_t$ with respect to $d_{\mathcal{W}_E}$. Consequently, if the subsequence $(s_k : k \geqslant 1)$ takes values in $[0,n] \backslash \bigcup_{i\in\mathcal{J}} (a_i, b_i)$, it holds that:

$$
\lim_{k\to\infty} d_{\mathcal{W}_E}(W^*_{s_k}, W^*_t) = \lim_{k\to\infty} d_{\mathcal{W}_E}\Big(\big(\mathrm{w}(u) : u \in [0, h(s_k)]\big), W^*_t \Big) = 0.
$$

On the other hand, for every $s \in [a_j, b_j] \cap [0,n]$ for some $j \in \mathcal{J}$ with $s < t$, we have

$$
\begin{aligned}
d_{\mathcal{W}_E}(W^*_s, W^*_t) &\leqslant d_{\mathcal{W}_E}(W^*_s, W^*_{b_j}) + d_{\mathcal{W}_E}\big(W^*_{b_j}, W^*_t\big) \\
&\leqslant K_n |s-t|^{r_0} + d_{\mathcal{W}_E}\Big(\big(\mathrm{w}(u \wedge \zeta_{\mathrm{w}}) : u \in [0, h(s)]\big), W^*_t \Big),
\end{aligned}
$$

which goes to 0 as $s \uparrow t$ since $W^*_t = \big(\mathrm{w}(u) : u \in [0, h(t)]\big)$. The case $s > t$ can be treated similarly by replacing b_i with a_i and it follows that, for any subsequence $(s_k : \ k \geqslant 1)$ with $s_k \to t$, we have $d_{\mathcal{W}_E}(W^*_{s_k}, W^*_t) \to 0$. Consequently, W^* is continuous on $[0, n]$. Since this holds for any n, we can define a continuous modification of W in $\mathbb{R}_+$. □

Under the conditions of Proposition 5.2, the measure Q^h_{w} can be defined in the Skorokhod space of $\mathcal{W}_E$-valued right-continuous paths $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_E)$ and, with a slight abuse of notation, we still denote it by Q^h_{w}. From now on, we shall work under these conditions and Q^h_{w} will always be considered as a measure in $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_E)$. In particular, remark that if we write W for the canonical process in $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_E)$, then W is Q^h_{w}–a.s. continuous. Finally, we point out that the regularity of W was partially addressed in the proof of [43, Proposition 4.4.1], for initial conditions of the form x with $x \in E$, when working with paths w defined in the half open interval $[0, \zeta_{\mathrm{w}})$.

The Lévy snake with spatial motion ξ. The driving function h of the random snake that we have considered so far was deterministic, and the next step consists in randomising h. We write $\mathcal{M}^0_f$ for the subset of $\mathcal{M}_f(\mathbb{R}_+)$ defined as

$$\mathcal{M}^0_f := \big\{\mu \in \mathcal{M}_f(\mathbb{R}_+) : \ H(\mu) < \infty \ \text{ and supp } \mu = [0, H(\mu)]\big\} \cup \{0\},$$

and we introduce Θ the collection of pairs $(\mu, \mathrm{w}) \in \mathcal{M}^0_f \times \mathcal{W}_E$ such that $H(\mu) = \zeta_{\mathrm{w}}$. Fix a Laplace exponent ψ satisfying (A1) – (A4), and set

$$\Upsilon := \sup\big\{r \geqslant 0 : \lim_{\lambda \to \infty} \lambda^{-r}\psi(\lambda) = \infty\big\}. \tag{5.17}$$

In particular, by the convexity of ψ we must have $\Upsilon \geqslant 1$. For every $\mu \in \mathcal{M}^0_f$, write $\mathbf{P}_\mu$ for the distribution of the exploration process started from μ in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$ – the space of right-continuous $\mathcal{M}_f(\mathbb{R}_+)$-valued paths. With a slight abuse of notation we denote the canonical process in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$ by ρ and observe that, by Definition 5.10, the process ρ under $\mathbf{P}_\mu$ takes values in $\mathcal{M}^0_f$. Notice that $H(\rho)$ under $\mathbf{P}_\mu$ is continuous since $\mu \in \mathcal{M}^0_f$. We can now state the hypothesis we will be working with.

In the rest of this work, we will always assume that:

Hypothesis $(\mathbf{H}_0)$. There exists a constant $C_\Pi > 0$ and two positive numbers $p, q > 0$ such that,
for every $x \in E$ and $t \geqslant 0$, we have:

$$\Pi_x\Big(\sup_{0 \leqslant u \leqslant t} d_E(\xi_u, x)^p\Big) \leqslant C_\Pi \cdot t^q, \qquad \text{and} \qquad q \cdot (1 - \Upsilon^{-1}) > 1. \tag{$\mathbf{H}_0$}$$

For instance, it can be checked that condition $(\mathbf{H}_0)$ is fulfilled if the Lévy tree has exponent $\psi(\lambda) = \lambda^\alpha$ for $\alpha \in (1, 2]$ and ξ is a Brownian motion. Let us discuss the implications of $(\mathbf{H}_0)$. Under $\mathbf{P}_\mu$, denote the excursion intervals of H above its running infimum by (α_i, β_i). Recall from (5.10) that $(\rho^\mu_t := [k_{-I_t}\mu, \rho_t] : \ t \geqslant 0)$, under $\mathbf{P}_0$, is distributed according to $\mathbf{P}_\mu$, and note that $H_t(\rho^\mu) = H(k_{-I_t}\mu) + H(\rho_t)$, for $t \geqslant 0$. By [43, Theorem 1.4.4], under $\mathbf{P}_0$ the process $H(\rho)$ is locally Hölder continuous of exponent m for any $m \in (0, 1 - \Upsilon^{-1})$. In particular, this holds for some $m := r$ verifying $qr > 1$ by the second condition in $(\mathbf{H}_0)$. Since $\big(H(k_{-I_t}\mu) : t \geqslant 0\big)$ is constant on each excursion interval (α_i, β_i) and $(H(\rho_t) : t \geqslant 0)$ is locally r-Hölder continuous, we deduce that $H(\rho^\mu)$ is locally r-Hölder continuous on $([\alpha_i, \beta_i] : i \in \mathbb{N})$. Said otherwise, $\mathbf{P}_\mu$-a.s., the paths of $H(\rho)$ satisfy the conditions of Proposition 5.2 and we will henceforth assume that the condition is satisfied for every path, and not only outside of a negligible set.

Finally, consider the canonical process (ρ, W) in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, the space of $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$-valued, right continuous paths. By our previous discussion we deduce that we can define a probability measure in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ by setting

$$\mathbb{P}_{\mu,\mathrm{w}}(\mathrm{d}\rho,\, \mathrm{d}W) := \mathbf{P}_\mu(\mathrm{d}\rho)\, Q_{\mathrm{w}}^{H(\rho)}(\mathrm{d}W),$$

for every $(\mu, \mathrm{w}) \in \Theta$. The process (ρ, W) under $\mathbb{P}_{\mu,\mathrm{w}}$ is called the ψ-Lévy snake with spatial motion ξ started from (μ, w). We denote its canonical filtration by $(\mathcal{F}_t : t \geqslant 0)$ and observe that by construction, $\mathbb{P}_{\mu,\mathrm{w}}$–a.s., W has continuous paths. Now, the proof of [43, Theorem 4.1.2] applies without any change to our framework and gives that the process $((\rho, W), (\mathbb{P}_{\mu,\mathrm{w}} : (\mu, \mathrm{w}) \in \Theta))$ is a strong Markov process with respect to the filtration $(\mathcal{F}_{t+})$. It should be noted that assumption $(\mathbf{H}_0)$ is the same as the one appearing in [43, Proposition 4.4.1], for paths defined in $[0, \zeta_{\mathrm{w}})$ and started from $x \in E$. In the particular case $\psi(\lambda) = \lambda^2/2$, the path regularity of W was already addressed in [62, Theorem 1.1].

Let us conclude our discussion concerning regularity issues by introducing the notion of *snake paths*, which summarises the regularity properties of (ρ, W) as well as some related notation that will be used throughout this work. Recall that $\mathcal{M}_f(\mathbb{R}_+)$, equipped with the topology of weak convergence, is a Polish space [58, Lemma 4.5]. We denote systematically the elements of the path space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ by:

$$(\mathsf{\rho}, \omega) = \big((\mathsf{\rho}_s, \omega_s) : s \in \mathbb{R}_+\big),$$

and by definition, we have $(\rho_s(\mathsf{\rho}), W_s(\omega)) = (\mathsf{\rho}_s, \omega_s)$ for $s \in \mathbb{R}_+$. For each fixed s, ω_s is an element of $\mathcal{W}_E$ with lifetime ζ_{ω_s}, and the $\mathbb{R}_+$-valued process $\zeta(\omega) := (\zeta_{\omega_s} : s \geqslant 0)$ is called the lifetime process of ω. We will occasionally use the notation $\zeta_s(\omega)$ instead of ζ_{ω_s}, and in such cases we will drop the dependence on ω if there is no risk of confusion.

Definition 5.3. *A snake path started from $(\mu, \mathrm{w}) \in \Theta$ is an element $(\mathsf{\rho}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ such that the mapping $s \mapsto \omega_s$ is continuous, and satisfying the following properties:*

(i) $(\mathsf{\rho}_0, \omega_0) = (\mu, \mathrm{w})$.

(ii) $(\mathsf{\rho}_s, \omega_s) \in \Theta$ *for all* $s \geqslant 0$*, in particular* $H(\rho) = \zeta(\omega)$.

(iii) ω *satisfies the snake property: for any* $0 \leqslant s \leqslant s'$,

$$\omega_s(t) = \omega_{s'}(t) \quad \text{for all} \quad 0 \leqslant t \leqslant \inf_{[s,s']} \zeta(\omega).$$

A continuous $\mathcal{W}_E$-valued path ω satisfying (iii) is called a snake trajectory. We point out that this notion had already been introduced in the context of the Brownian snake [1, Definition 6]. However, in the Brownian case the process W is Markovian and there is no need of working with pairs $(\mathsf{\rho}, \omega)$ – this is the reason why we have to introduce the notion of snake paths. We denote the collection of snake paths started from $(\mu, \mathrm{w}) \in \Theta$ by $\mathcal{S}_{\mu,\mathrm{w}}$ and simply write $\mathcal{S}_x$ instead of $\mathcal{S}_{0,x}$. Finally, we set:

$$\mathcal{S} := \bigcup_{(\mu,\mathrm{w}) \in \Theta} \mathcal{S}_{\mu,\mathrm{w}}.$$

For any given $(\mathsf{\rho}, \omega) \in \mathcal{S}$, we denote indifferently its duration by

$$\sigma_{H(\mathsf{\rho})} = \sigma(\omega) = \sup\{t \geqslant 0 : \zeta_{\omega_t} \neq 0\}.$$

Remark that, by continuity and the definition of Q^h_{w}, the process $((\rho, W), (\mathbb{P}_{\mu,\mathrm{w}} : (\mu, \mathrm{w}) \in \Theta))$ takes values in $\mathcal{S}$ – it satisfies the snake property by (5.14) and the continuity of W. Said otherwise, $\mathbb{P}_{\mu,\mathrm{w}}$-a.s., we have

$$\zeta_s = H(\rho_s), \quad \text{for every } s \geqslant 0,$$

and for any $t \leqslant t'$

$$W_t(s) = W_{t'}(s), \quad \text{for all } s \leqslant m_H(t, t').$$

We stress that when working on $\mathcal{S}$ the equivalent notations ζ_s, $H(\rho_s)$ and H_s will be used indifferently. The snake property implies that, for every $t, t' \geqslant 0$ such that $p_H(t) = p_H(t')$, we have $W_t = W_{t'}$. In particular, for such times it holds that $\widehat{W}_t = \widehat{W}_{t'}$ and hence $(\widehat{W}_t : t \geqslant 0)$ can be defined in the quotient space $\mathcal{T}_H$. More precisely, under $\mathbb{P}_{\mu,\mathrm{w}}$, the function defined with a slight abuse of notation for all $\upsilon \in \mathcal{T}_H$ as

$$\xi_\upsilon := \widehat{W}_t, \quad \text{where } t \text{ is any element of } p_H^{-1}(\upsilon),$$

is well defined and leads us to the notion of tree indexed processes. When $(\mu, \mathrm{w}) = (0, x)$, the process $(\xi_\upsilon)_{\upsilon \in \mathcal{T}_H}$ is known as the Markov process ξ indexed by the tree $\mathcal{T}_H$ and started from x.[4] In this work, we will need to consider the restriction of (ρ, W) to different intervals and therefore, it will be convenient to introduce a formal notion of subtrajectories.

Subtrajectories. Fix $s < t$ such that $H_s = H_t$ and $H_r > H_s$ for all $r \in (s, t)$. The subtrajectory of (ρ, W) in $[s, t]$ is the process taking values in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, denoted by $(\rho'_r, W'_r)_{r \in [0, t-s]}$ and defined as follows: for every $r \in [0, t-s]$, set

$$\langle \rho'_r, f \rangle := \int \rho_{r+s}(\mathrm{d}h) f(h - H_s) \mathbb{1}_{\{h > H_s\}} \qquad \text{and} \qquad W'_r(\cdot) := W_{s+r}(H_s + \cdot).$$

In particular, we have

$$\zeta(W'_r) = H_{s+r} - H_s = H(\rho'_r), \quad \text{for all } r \in [0, t-s].$$

Remark that if (ρ, W) is a snake path, then the subtrajectory (ρ', W') is also in $\mathcal{S}$. Informally, W' encodes the labels $(\xi_\upsilon : \upsilon \in p_H([s, t]))$.

5.2.4 Excursion measures of the Lévy snake

Fix $x \in E$ and consider the Lévy snake (ρ, W) under $\mathbb{P}_{0,x}$. By (5.8), the measure 0 is a regular recurrent point for the Markov process ρ, which implies that $(0, x)$ is on its turn regular and recurrent for the Markov process (ρ, W). Moreover, $(-I_t : t \geqslant 0)$ is a local time at 0 for ρ and hence it is a local time at $(0, x)$ for (ρ, W). We set $\mathbb{N}_x$ the excursion measure of (ρ, W) away from $(0, x)$ associated with the local time $-I$. We stress that $\mathbb{N}_x$ is a measure in the canonical space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$. By excursion theory of the Markov process (ρ, W), if $\{(\alpha_i, \beta_i) : i \in \mathcal{I}\}$

[4] With the terminology introduced in [1, Definition 7], the pair of processes $(H, \widehat{W})$ is called a treelike-path.

stands for the excursion intervals of (ρ, W) and (ρ^i, W^i) are the corresponding subtrajectories then, under $\mathbb{P}_{0,x}$, the measure

$$\sum_{i\in\mathcal{I}} \delta_{(-I_{\alpha_i},\rho^i,W^i)}, \tag{5.18}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\infty)}(\ell)\mathrm{d}\ell\, \mathbb{N}_x(\mathrm{d}\rho,\, \mathrm{d}\omega)$. Recalling the interpretation of the restrictions $\mathbb{N}_x(\,\cdot\,|\sigma > \varepsilon)$ as the law of the first excursion with length greater than ε, it follows that under $\mathbb{N}_x$, W satisfies the snake property and $(\rho, W) \in \mathcal{S}$. In particular, we can still make use of the definition of subtrajectories and $(\xi_v)_{v\in\mathcal{T}_H}$ under the excursion measure $\mathbb{N}_x$, and for simplicity we will use the same notation.

By the previous discussion, it is straightforward to verify that

$$\mathbb{N}_x(\mathrm{d}\rho,\, \mathrm{d}\eta,\, \mathrm{d}W) = N(\mathrm{d}\rho,\, \mathrm{d}\eta)\, Q_x^{H(\rho)}(\mathrm{d}W). \tag{5.19}$$

Said otherwise, under $\mathbb{N}_x$:

- The distribution of (ρ, η) is $N(\mathrm{d}\rho,\, \mathrm{d}\eta)$;
- The conditional distribution of W knowing (ρ, η) is $Q_x^{H(\rho)}$.

Remark that by construction and (5.13), under $\mathbb{N}_x$ we have

$$\big((\rho_t, \eta_t, W_t) : t \in [0, \sigma]\big) \overset{(d)}{=} \big((\eta_{(\sigma-t)-}, \rho_{(\sigma-t)-}, W_{\sigma-t}) : t \in [0, \sigma]\big), \tag{5.20}$$

where we used that by continuity, we have $W_{\sigma-t} = W_{(\sigma-t)-}$ for every $t \in [0, \sigma]$.

When starting from an arbitrary $(\mu, \mathrm{w}) \in \Theta$, the following variant of (5.18) will be used frequently in our computations: let $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$ be the distribution of (ρ, W) killed at time $\sigma := \inf\{t \geqslant 0 : H(\rho_s) = 0 \text{ for every } s \geqslant t\}$. For instance, it will be worth noting that by (5.10), the process $\langle\rho, 1\rangle$ is a Lévy process started from $\langle\mu, 1\rangle$ and stopped when reaching 0. Write $\big((\alpha_i, \beta_i) : i \in \mathbb{N}\big)$ for the excursion intervals over the running infimum of $\langle\rho, 1\rangle$ under $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$ and denote the corresponding subtrajectory associated with $[\alpha_i, \beta_i]$ by (ρ^i, W^i). If for $t \geqslant 0$ we write $I_t := \inf_{s\leqslant t}\langle\rho_s, 1\rangle - \langle\mu, 1\rangle$, the measure

$$\sum_{i\in\mathbb{N}} \delta_{(-I_{\alpha_i},\rho^i,W^i)}, \tag{5.21}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\langle\mu,1\rangle]}(u)\, \mathrm{d}u\, \mathbb{N}_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{d}\rho, \mathrm{d}W)$. Moreover, if $h_i := H_{\alpha_i} = H_{\beta_i}$, by (5.10) we have $h_i = H(\kappa_{-I_{\alpha_i}}\mu)$ and since the image measure of $\mathbb{1}_{[0,\langle\mu,1\rangle]}(u)\, \mathrm{d}u$ under the mapping $u \mapsto H(\kappa_u\mu)$ is precisely μ, we deduce that under $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$ the measure

$$\sum_{i\in\mathbb{N}} \delta_{(h_i,\rho^i,W^i)} \tag{5.22}$$

is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\mathrm{w}(h)}(\mathrm{d}\rho, \mathrm{d}W)$. We refer to [43, Lemma 4.2.4] for additional details.

We close this section by recalling a many-to-one formula that will be used frequently to obtain explicit computations. We start with some preliminary notations: consider a 2-dimensional subordinator $(U^{(1)}, U^{(2)})$ defined in some auxiliary probability space $(\Omega_0, \mathcal{F}_0, P^0)$ with Laplace exponent given by

$$-\log E^0\Big[\exp\big(-\lambda_1 U_1^{(1)} - \lambda_2 U_1^{(2)}\big)\Big] := \begin{cases} \frac{\psi(\lambda_1)-\psi(\lambda_2)}{\lambda_1-\lambda_2} - \alpha & \text{if } \lambda_1 \neq \lambda_2 \\ \psi'(\lambda_1) - \alpha & \text{if } \lambda_1 = \lambda_2, \end{cases} \tag{5.23}$$

where E^0 stands for the expectation taken with respect to P^0. Notice that in particular $U^{(1)}$ and $U^{(2)}$ are subordinators with Laplace exponent $\lambda \mapsto \psi(\lambda)/\lambda - \alpha$. Let $(J_a, \check{J}_a)$ be the pair or random measures defined by

$$(J_a, \check{J}_a) := \big(\mathbb{1}_{[0,a]}(t)\ \mathrm{d}U_t^{(1)}, \mathbb{1}_{[0,a]}(t)\ \mathrm{d}U_t^{(2)}\big),$$

with the convention $(J_\infty, \check{J}_\infty) := (\mathbb{1}_{[0,\infty)}(t)\ \mathrm{d}U_t^{(1)}, \mathbb{1}_{[0,\infty)}(t)\ \mathrm{d}U_t^{(2)})$. The following many-to-one equation will play a central role in all this work:

Lemma 5.4. *For every $x \in E$ and every non-negative measurable functional Φ taking values in $\mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E$, we have:*

$$\mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}s\, \Phi\big(\rho_s, \eta_s, W_s\big)\Big) = \int_0^\infty \mathrm{d}a\ \exp\big(-\alpha a\big) \cdot E^0 \otimes \Pi_x\Big(\Phi\big(J_a, \check{J}_a, (\xi_t : t \leqslant a)\big)\Big). \tag{5.24}$$

Proof. First, remark that we have

$$\mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}s\, \Phi\big(\rho_s, \eta_s, W_s\big)\Big) = \int_0^\infty \mathrm{d}s\, \mathbb{N}_x\Big(\mathbb{1}_{\{s<\sigma_H\}}\, \Phi\big(\rho_s, \eta_s, W_s\big)\Big).$$

Next, we use (5.19) to write the previous display in the form:

$$\int_0^\infty \mathrm{d}s\, \mathbb{N}_x\Big(\mathbb{1}_{\{s<\sigma_H\}}\, \Pi_x\Big[\Phi\big(\rho_s, \eta_s, (\xi_r : r \leqslant H(\rho_s))\big)\Big]\Big) = N\Big(\int_0^\sigma \mathrm{d}s\, \Pi_x\Big[\Phi\Big(\rho_s, \eta_s, \big(\xi_r : r \leqslant H(\rho_s)\big)\Big)\Big]\Big).$$

Since now $\Pi_x\big[\Phi\big(\rho_s, \eta_s, (\xi_r : r \leqslant H(\rho_s))\big)\big]$ is a functional of (ρ_s, η_s), it suffices to establish (5.24) for a functional only depending on the pair (ρ_s, η_s). However, this is precisely formula (18) in [44]. □

5.3 Special Markov property

In this section we state and prove the (strong) special Markov property for the Lévy snake. This result was originally introduced in [63, Section 2] in the special case of the Brownian motion indexed by the Brownian tree, viz. when the Lévy exponent of the tree is of the form $\psi(\lambda) = \beta\lambda^2$ and the spatial motion ξ is a Brownian motion. This result plays a fundamental role in the study of Brownian motion indexed by the Brownian tree, see for example [63, 66, 70, 72]. More recently, a stronger version was proved in [66] still for $\psi(\lambda) = \beta\lambda^2$ but holding for more general spatial motions ξ. In this section we extend this result to an arbitrary exponent ψ of a Lévy tree. Even if we follow a similar strategy to the one introduced in [66], general Lévy trees are significantly less regular than the Brownian tree – in particular the height process H is not Markovian. The arguments need to be carefully reworked and for instance, the existence of points with infinite multiplicity hinder considerably the proof.

We start by introducing some standard notation that will be used in the rest of the section and recalling the preliminaries needed for our purpose. Fix $x \in E$ and for an arbitrary open subset $D \subset E$ containing x and $\mathrm{w} \in \mathcal{W}_{E,x}$, set

$$\tau_D(\mathrm{w}) := \inf\big\{t \in [0, \zeta_{\mathrm{w}}] :\ \mathrm{w}(t) \notin D\big\},$$

with the usual convention $\inf\{\varnothing\} = \infty$. Similarly, we will write $\tau_D(\xi) := \inf\{t \geqslant 0 : \ \xi_t \notin D\}$ for the exit time from D of the spatial motion ξ. When considering the later, the dependency on ξ is usually dropped when there is no risk of confusion. In the rest of the section, we will always assume that:

$$\Pi_x(\tau_D < \infty) > 0. \tag{$\mathbf{H_1}$}$$

The special Markov property is roughly speaking a spatial version of the Markov property. In order to state it, we need to properly define the notion of paths "inside D" and "excursions outside D", as well as a notion of measurability with respect to the information generated by the trajectories staying inside of D. Section 5.3.1 is devoted to the study of paths inside D and to a fundamental functional of the Lévy snake, called the exit local time. The study of the excursions outside D is postponed to Section 5.3.2.

5.3.1 The exit local time

Let us begin by introducing some useful operations and notation.

Truncation. We start by defining the *truncation* of a path $(\rho, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E,x})$ to D – we stress that we have $\omega_s(0) = x$ for every $s \geqslant 0$. In this direction, define the functional

$$V_t^D(\rho, \omega) := \int_0^t \mathrm{d}s \, \mathbb{1}_{\{\zeta_{\omega_s} \leqslant \tau_D(\omega_s)\}}, \qquad t \geqslant 0, \tag{5.25}$$

measuring the amount of time spent by ω without leaving D up to time t. Let us be more precise: at time s, we will say that ω_s doesn't leave D (or stays in D) if $\omega_s([0, \zeta_s)) \subset D$ (notice that $\widehat{\omega}_s$ might be in ∂D) and on the other hand, we say that the trajectory exits D if $\omega_s([0, \zeta_s)) \cap D^c \neq \varnothing$. Observe that a trajectory $(\omega_s(t) : t \in [0, \zeta_s])$ might exit the domain D and return to it before the lifetime ζ_s, but such a trajectory will not be accounted by V^D. Write $\mathcal{Y}_D(\rho, \omega) := V^D_{\sigma(\omega)}(\rho, \omega)$ for the total amount of time spent in D, and for every $s \in [0, \mathcal{Y}_D(\rho, \omega))$ set

$$\Gamma_s^D(\rho, \omega) := \inf\big\{t \geqslant 0 : V_t^D(\rho, \omega) > s\big\},$$

with the convention $\Gamma_s^D(\rho, \omega) := \sigma(\omega)$, if $s \geqslant \mathcal{Y}_D(\rho, \omega)$. The truncation of (ρ, ω) to D is the element of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E,x})$ with lifetime $\mathcal{Y}_D(\rho, \omega)$ defined as follows:

$$\mathrm{tr}_D\big(\rho, \omega\big) := \big(\rho_{\Gamma_s^D(\rho,\omega)}, \omega_{\Gamma_s^D(\rho,\omega)}\big)_{s \in \mathbb{R}_+}.$$

Indeed, observe that the trajectory $(\rho_{\Gamma^D}, \omega_{\Gamma^D})$ is rcll since ρ, ω and Γ^D are rcll. For simplicity, we set $\mathrm{tr}_D(\omega) = (\omega_{\Gamma_s^D(\omega)})_{s \in \mathbb{R}_+}$ and we write $\mathrm{tr}_D(\widehat{\omega})$ for $\widehat{\omega}_{\Gamma^D}$. Roughly speaking, $\mathrm{tr}_D(\omega)$ removes the trajectories ω_s from ω leaving D, glues the remaining endpoints, and hence encodes the trajectories ω_s that stay in D. Let us stress that when (ρ, ω) is an element of $\mathcal{S}_x$, the truncation $\mathrm{tr}_D(\rho, \omega)$ is still in $\mathcal{S}_x$ since $\mathrm{tr}_D(\omega)$ is a snake trajectory taking values in $D \cup \partial D$ by [1, Proposition 10], and condition (ii) in Definition 5.3 is clearly satisfied. Recall that (ρ, W) stands for the canonical process in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E,x})$, and that it takes values in $\mathcal{S}_x$ under $\mathbb{P}_{\mu,\mathrm{w}}$ for $(\mu, \mathrm{w}) \in \Theta$ or under $\mathbb{N}_y$ for $y \in E$. We will also need to introduce the sigma field

$$\mathcal{F}^D := \sigma\big(\mathrm{tr}_D(\rho, W)_s : s \geqslant 0\big) \tag{5.26}$$

in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, which roughly speaking, contains the information generated by the trajectories that stay in D. The following technical lemma will be often useful. It states that, under $\mathbb{N}_x$, when a trajectory W_s exits the domain D, then the measure ρ_s does not have an atom at level $\tau_D(W_s)$. More precisely:

Lemma 5.5. *Let D be an arbitrary open subset $D \subset E$ containing x. Then, $\mathbb{N}_x$–a.e.*

$$\rho_s(\{\tau_D(W_s)\}) = 0, \qquad \textit{for all } s \geqslant 0.$$

Proof. First, remark that the many-to-one formula (5.24) gives:

$$\begin{aligned}&\mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}s \, \mathbb{1}_{\{\tau_D(W_s)<\infty\}} \rho_s(\{\tau_D(W_s)\})\Big) \\ &= \int_0^\infty \mathrm{d}a \, \exp(-\alpha a) E^0 \otimes \Pi_x \Big(\mathbb{1}_{\{\tau_D((\xi_u : u \leqslant a))<\infty\}} J_a(\{\tau_D(\xi_u : u \leqslant a)\})\Big),\end{aligned}$$

which vanishes by the independence between ξ and J_a. This shows that $\mathbb{N}_x$–a.e., the Lebesgue measure of the set $\{s \in [0,\sigma] : \rho_s(\{\tau_D(W_s)\}) \neq 0\}$ is null and now we claim that this implies that $\mathbb{N}_x$–a.e. $\rho_s(\{\tau_D(W_s)\}) = 0$ for all $s \geqslant 0$. We argue by contradiction to prove this claim. Suppose that for some $s > 0$, we have $\rho_s(\{\tau_D(W_s)\}) > 0$. In this case, recalling that the exploration process ρ is rcll with respect to the total variation distance, we must have

$$\lim_{\varepsilon \downarrow 0} \big|\rho_s(\{\tau_D(W_s)\}) - \rho_{s+\varepsilon}(\{\tau_D(W_s)\})\big| \leqslant \lim_{\varepsilon \downarrow 0} \sup_{A \in \mathcal{B}(\mathbb{R})} \big|\rho_s(A) - \rho_{s+\varepsilon}(A)\big| = 0.$$

We infer that for some $\delta > 0$, it holds that $\rho_u(\{\tau_D(W_s)\}) > 0$ for all $u \in [s, s+\delta)$. In particular, we have $H_u \geqslant H_s$ for all $u \in [s, s+\delta)$. By the snake property, we deduce that, for every $u \in [s, s+\delta)$, $\tau_D(W_s) = \tau_D(W_u)$ and consequently:

$$\rho_u(\{\tau_D(W_u)\}) = \rho_u(\{\tau_D(W_s)\}) > 0.$$

However, this is in contradiction with the first part of the proof and the desired result follows. □

Exit local time. As in classical excursion theory, we will need to properly index the excursions outside D but we will also ask the indexing to be compatible with the order induced by H. To achieve it, we will make use of the *exit local time* from D. We briefly recall its definition and main properties and we refer to [43, Section 4.3] for a more detailed account. By Propositions 4.3.1 and 4.3.2 in [43], under $\mathbb{N}_x$ and $\mathbb{P}_{0,x}$, the limit

$$L_s^D := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathrm{d}r \mathbb{1}_{\{\tau_D(W_r) < H_r < \tau_D(W_r) + \varepsilon\}}, \tag{5.27}$$

exists for every $s \geqslant 0$, where the convergence holds uniformly in compact intervals in $L_1(\mathbb{P}_{0,x})$ and $L_1(\mathbb{N}_x)$. This defines a continuous non-decreasing process L^D called the exit local time from D of (ρ, W). We insist that, under $\mathbb{N}_x$ and $\mathbb{P}_{0,x}$, the process (ρ, W) takes values in $\mathcal{S}_x$ which yields that $H_s = \zeta_s$ for every $s \geqslant 0$. We also recall the first moment formula:

$$\mathbb{N}_x \bigg(\int_0^\sigma \mathrm{d}L_s^D \; \Phi(\rho_s, \eta_s, W_s) \bigg) = E^0 \otimes \Pi_x \bigg(\mathbb{1}_{\{\tau_D < \infty\}} \exp(-\alpha \tau_D) \Phi\big(J_{\tau_D}, \check{J}_{\tau_D}, (\xi_t : t \leqslant \tau_D)\big) \bigg), \tag{5.28}$$

see [43, Proposition 4.3.2] for a proof of this identity. In particular, remark that we have

$$\operatorname{supp} \mathrm{d}L_s^D \subseteq \{s \geqslant 0 : \; \tau_D(W_s) = H_s\}, \quad \mathbb{N}_x\text{–a.e.}$$

We stress that L^D is constant at every interval at which W_s stays in D and in each connected component of

$$\{s \geqslant 0 : \ \tau_D(W_s) < H_s\}.$$

We call such a connected component an excursion interval from D. This family of intervals will be studied in detail in the next section. The process L^D is not measurable with respect to $\mathcal{F}^D$, the informal reason being that it contains the information on the lengths of the excursions from D. However, as we are going to show in Proposition 5.7, the time-changed process

$$\widetilde{L}^D := \big(L^D_{\Gamma^D_s}\big)_{s\in\mathbb{R}_+}$$

is $\mathcal{F}^D$-measurable – notice that we removed precisely from L^D by means of the time change the constancy intervals generated by excursions from D. This measurability property will be crucial for the proof of the special Markov property and the rest of this section is devoted to its proof.

First remark that we have only defined the exit local time under the measures $\mathbb{P}_{0,x}$ and $\mathbb{N}_x$ for $x \in D$. In order to be able to apply the Markov property, we need to extend the definition to more general initial conditions $(\mu, \mathrm{w}) \in \Theta$. This construction will also be essential for the results of Section 5.4. The precise statement is given in the following proposition:

Proposition 5.6. *Fix $(\mu, \mathrm{w}) \in \Theta$ such that $\mathrm{w}(0) \in D$ and suppose that $\mu(\{\tau_D(\mathrm{w})\}) = 0$. Then, under $\mathbb{P}_{\mu,\mathrm{w}}$ there exists a continuous, non-decreasing process L^D with associated Lebesgue-Stieltjes measure $\mathrm{d}L^D$ supported on $\{t \in \mathbb{R}_+ : \widehat{W}_t \in \partial D\}$, such that, for every $t \geqslant 0$*

$$L^D_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s)<H_s<\tau_D(W_s)+\varepsilon\}}, \tag{5.29}$$

where the convergence holds uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\mathrm{w}})$. Moreover:

(i) *Under $\mathbb{P}_{\mu,\mathrm{w}}$, if $\tau_D(\mathrm{w}) < \infty$, we have $L^D_t = 0$ for every $t \leqslant \inf\{s \geqslant 0 : H_s < \tau_D(\mathrm{w})\}$.*

(ii) *Under $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$, recall the definition of the random point measure $\sum_{i\in\mathbb{N}} \delta_{(h_i,\rho^i,W^i)}$ defined in (5.22). Then we have:*

$$L^D_\infty(\rho, W) = \sum_{h_i<\tau_D(\mathrm{w})} L^D_\infty(\rho^i, W^i), \quad \mathbb{P}^\dagger_{\mu,\mathrm{w}}\text{-}a.s. \tag{5.30}$$

Proof. Let us start with preliminary remarks and introducing some needed notation. Fix $(\mu, \mathrm{w}) \in \Theta$ with $\mathrm{w}(0) \in D$ satisfying $\mu(\{\tau_D(\mathrm{w})\}) = 0$. We write

$$T_r := \inf\{t \geqslant 0 : \ H_t = r\}, \text{ for every } r \geqslant 0, \qquad \text{and} \qquad T_0^+ := \inf\{t \geqslant 0 : \ \langle \rho_t, 1\rangle = 0\}.$$

By (5.27) and the strong Markov property, we already know that $\varepsilon^{-1}\int_{T_0^+}^{T_0^+ + t} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s)<H_s<\tau_D(W_s)+\varepsilon\}}$ converges as $\varepsilon \downarrow 0$ uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\mathrm{w}})$ towards a non-decreasing continuous process supported on $\{t \geqslant T_0^+ : \widehat{W}_t \in \partial D\}$. Consequently, it suffices to prove the proposition under $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$. In this direction, we set

$$I(t, \varepsilon) := \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s)<H_s<\tau_D(W_s)+\varepsilon\}},$$

for every $\varepsilon > 0$. Recall now that under $\mathbb{P}^{\dagger}_{\mu,\mathrm{w}}$, the process $\langle \rho, 1\rangle$ is a killed Lévy process started at $\langle \mu, 1\rangle$ and stopped at its first hitting time of 0. Write $((\alpha_i, \beta_i) : \ i \in \mathbb{N})$, for the excursion intervals of $\langle \rho, 1\rangle$ over its running infimum, and let (ρ^i, W^i) be the subtrajectory associated with the excursion interval $[\alpha_i, \beta_i]$. To simplify notation, we also set $h_i := H(\alpha_i)$ and recall from (5.22) that the measure $\mathcal{M} := \sum_{i\in\mathbb{N}} \delta_{(h_i,\rho^i,W^i)}$ is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\mathrm{w}(h)}(\mathrm{d}\rho, \mathrm{d}W)$.

We suppose first that $\tau_D(\mathrm{w}) \geqslant \zeta_{\mathrm{w}}$. We shall prove that the collection $\big(I(t,\varepsilon), t \geqslant 0\big)$ for $\varepsilon > 0$ is Cauchy in $L_1(\mathbb{P}^{\dagger}_{\mu,\mathrm{w}})$ uniformly in compact intervals as $\varepsilon \downarrow 0$, viz.

$$\lim_{\delta,\varepsilon\to 0} \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\big[\sup_{s\leqslant t} |I(s,\varepsilon) - I(s,\delta)|\big] = 0. \tag{5.31}$$

This implies directly the existence of L^D defined as in (5.29) as well as point (i). We shall then deduce (ii), and the remaining case $\tau_D(\mathrm{w}) < \zeta_{\mathrm{w}}$ is treated afterwards. Let us proceed with the proof of (5.31). Since the Lebesgue measure of $\{t \in [0,\sigma] : \langle \rho_t, 1\rangle = \inf_{s\leqslant t}\langle \rho_s, 1\rangle\}$ is null, we can write

$$I(t,\varepsilon) = \frac{1}{\varepsilon}\sum_{i\in\mathbb{N}} \int_{\alpha_i\wedge t}^{\beta_i\wedge t} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s)<H_s<\tau_D(W_s)+\varepsilon\}},$$

which yields the following upper bound:

$$\begin{aligned}
&\mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\big[\sup_{s\leqslant t} |I(s,\varepsilon) - I(s,\delta)|\big] \\
&\leqslant \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\Big[\sum_{i\in\mathbb{N}} \sup_{s\leqslant t} \Big|\frac{1}{\varepsilon}\int_{\alpha_i\wedge s}^{\beta_i\wedge s} \mathrm{d}u\, \mathbb{1}_{\{\tau_D(W_u)<H_u<\tau_D(W_u)+\varepsilon\}} - \frac{1}{\delta}\int_{\alpha_i\wedge s}^{\beta_i\wedge s} \mathrm{d}u\, \mathbb{1}_{\{\tau_D(W_u)<H_u<\tau_D(W_u)+\delta\}}\Big|\Big] \\
&\leqslant \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\Big[\sum_{i\in\mathbb{N}} \sup_{s\leqslant \sigma(W^i)} \Big|\frac{1}{\varepsilon}\int_{0}^{s\wedge t} \mathrm{d}u\, \mathbb{1}_{\{\tau_D(W^i_u)<H(\rho^i_u)<\tau_D(W^i_u)+\varepsilon\}} - \frac{1}{\delta}\int_{0}^{s\wedge t} \mathrm{d}u\, \mathbb{1}_{\{\tau_D(W^i_u)<H(\rho^i_u)<\tau_D(W^i_u)+\delta\}}\Big|\Big].
\end{aligned}$$

Since $\mu(\{\tau_D(\mathrm{w})\}) = 0$, the last display is given by

$$\int_{[0,\tau_D(\mathrm{w}))} \mu(\mathrm{d}h)\, \mathbb{N}_{\mathrm{w}(h)}\Big(\sup_{s\leqslant t} |I(s,\varepsilon) - I(s,\delta)|\Big). \tag{5.32}$$

Let us now show that (5.32) converges towards 0 when $\varepsilon, \delta \downarrow 0$. Since for every $h \in [0, \tau_D(\mathrm{w}))$ we have $\mathrm{w}(h) \in D$, the term inside the integral in (5.32) converges towards 0 as $\varepsilon, \delta \downarrow 0$ by the approximation of exit local times under the excursion measure given in (5.27). Knowing that μ is a finite measure, it suffices to show that the term,

$$\mathbb{N}_{\mathrm{w}(h)}\Big(\sup_{s\leqslant t} |I(s,\varepsilon) - I(s,\delta)|\Big),$$

can be bounded uniformly in ε, δ. However, still under $\mathbb{N}_{\mathrm{w}(h)}$, we have the simple upper bound:

$$\sup_{s\leqslant t} |I(s,\varepsilon) - I(s,\delta)| \leqslant I(\sigma,\varepsilon) + I(\sigma,\delta),$$

and by the many-to-one formula (5.24), we deduce that

$$\mathbb{N}_{\mathrm{w}(h)}\big(I(\sigma,\varepsilon)\big) = \varepsilon^{-1} E^0 \otimes \Pi_{\mathrm{w}(h)}\Big[\int_0^\infty \mathrm{d}a\, \exp(-\alpha a)\mathbb{1}_{\{\tau_D(\xi)<H(J_a)<\tau_D(\xi)+\varepsilon\}}\Big] \leqslant 1,$$

for every $\varepsilon > 0$, where to obtain the previous inequality we use that $H(J_a) = a$. In particular, we have $\mathbb{N}_{\mathrm{w}(h)}\big(I(\sigma,\varepsilon) + I(\sigma,\delta)\big) \leqslant 2$ and (5.31) follows. Still under our assumption $\tau_D(\mathrm{w}) \geqslant \zeta_{\mathrm{w}}$ we now turn our attention to (5.30). We know that for any $(h_i, W^i, \rho^i) \in \mathcal{M}$ we have the limit in probability:

$$L^D_{\sigma_i}(\rho^i, W^i) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{a_i}^{b_i} \mathrm{d}s \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}}.$$

It then follows from our definitions that for every $r > 0$,

$$L^D_\sigma - L^D_{T_{\zeta_{\mathrm{w}}-r}} = \sum_{h_i \leqslant \zeta_{\mathrm{w}}-r} L^D_{\sigma_i}(\rho^i, W^i),$$

observing that the number of non-zero terms on the right-hand side is finite. By taking the limit as $r \downarrow 0$, we deduce (5.30) by monotonicity.

Let us now assume that $\tau_D(\mathrm{w}) < \zeta_{\mathrm{w}}$. To simplify notation, set $a := \tau_D(\mathrm{w})$ and notice that

$$(\rho_{T_a}, W_{T_a}) = \big(\mu \mathbb{1}_{[0,\tau_D(\mathrm{w})]}, (\mathrm{w}(h) : h \in [0, \tau_D(\mathrm{w})])\big),$$

where we recall that $\mu(\{\tau_D(\mathrm{w})\}) = 0$. By our previous discussion and the strong Markov property, we deduce that $(I(t,\varepsilon) - I(T_a,\varepsilon) : t \geqslant T_a)$ converges as $\varepsilon \downarrow 0$ uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\mathrm{w}})$ towards a continuous process. To conclude our proof, it suffices to show that:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\int_0^{T_a} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}} \Big] = 0.$$

To obtain the previous display, write

$$\int_0^{T_a} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}} = \sum_{h_i \geqslant a} \int_{\alpha_i}^{\beta_i} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}},$$

where we have $h_i \neq a$ for every $i \in \mathbb{N}$, since $\mu(\{a\}) = 0$. Moreover, for every i with $h_i > a$ notice that $\tau_D(W_s) = a$. This implies:

$$\int_0^{T_a} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}} \leqslant \sum_{a \leqslant h_i \leqslant a+\varepsilon} \int_0^{\sigma(W^i)} \mathrm{d}s\, \mathbb{1}_{\{0 < H(\rho^i_s) < \varepsilon\}},$$

and we can now use that $\mathcal{M}$ is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\mathrm{w}(h)}(\mathrm{d}\rho, \mathrm{d}W)$ to obtain:

$$\mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\int_0^{T_a} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}} \Big] \leqslant \mu([a, a+\varepsilon]) N\Big(\int_0^{\sigma} \mathrm{d}s \mathbb{1}_{\{0 \leqslant H(\rho_s) < \varepsilon\}} \Big). \tag{5.33}$$

Finally, by the many-to-one formula (5.24), the previous display is equal to $\varepsilon \cdot \mu([a, a+\varepsilon])$, giving:

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\int_0^{T_a} \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s)+\varepsilon\}} \Big] = \mu(\{a\}) = 0,$$

where in the last equality we use that $\mu \in \Theta$ which ensures that $\mu(\{a\}) = 0$. □

Now that we have defined the exit local time under more general initial conditions, let us turn our attention to the measurabliliy properties of $\widetilde{L}^D$. From now on, when working under $\mathbb{P}_{0,x}$ or $\mathbb{N}_x$, the sigma field $\mathcal{F}^D$ should be completed with the $\mathbb{P}_{0,x}$-negligible and $\mathbb{N}_x$-negligible sets respectively – for simplicity we use the same notation.

Proposition 5.7. *Under $\mathbb{P}_{0,x}$ and $\mathbb{N}_x$, the process $\widetilde{L}^D$ is $\mathcal{F}^D$-measurable.*

In particular, the proposition implies that, under $\mathbb{N}_x$, the total mass $L^D_\sigma = \widetilde{L}^D_\infty$ is $\mathcal{F}^D$-measurable. The proof will mainly rely on the two following technical lemmas.

Lemma 5.8. *Consider an open subset $D \subset E$ containing x. Fix an arbitrary $(\mu, \mathrm{w}) \in \Theta$ with $\mathrm{w}(0) = x$ and satisfying $\mu(\{\tau_D(\mathrm{w})\}) = 0$ if $\tau_D(\mathrm{w}) < \infty$. Then, for every $K > 0$, we have:*

$$\mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\int_0^\sigma \mathrm{d}L^D_s \mathbb{1}_{\{\langle \rho_s, 1\rangle \leqslant K\}}\Big] = \int_0^{\mu([0,\tau_D(\mathrm{w})))} \mathrm{d}u \ E^0 \otimes \Pi_{\mathrm{w}(H(\kappa_{\langle \mu,1\rangle - u}\mu))}\Big(\mathbb{1}_{\{\tau_D<\infty\}} \exp(-\alpha\tau_D)\mathbb{1}_{\{\langle J_{\tau_D},1\rangle \leqslant K-u\}}\Big).$$

Proof. Recall that, under $\mathbb{P}^\dagger_{\mu,\mathrm{w}}$, the process $\langle \rho, 1\rangle$ is a Lévy process started at $\langle \mu, 1\rangle$ and stopped at its first hitting time of 0. As usual, write $\{(\alpha_i, \beta_i) : i \in \mathbb{N}\}$ for the excursion intervals of $\langle \rho, 1\rangle - \langle \mu, 1\rangle$ over its running infimum, that we still denote by I. We write (ρ^i, W^i) for the subtrajectory associated with $[\alpha_i, \beta_i]$. As explained in (5.18), the measure:

$$\sum_{i\in\mathbb{N}} \delta_{(-I_{\alpha_i}, \rho^i, W^i)},$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\langle\mu,1\rangle]}(u)\mathrm{d}u\,\mathbb{N}_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{d}\rho, \mathrm{d}W)$. Furthermore, for every $i \in \mathbb{N}$, we have $H(\kappa_{-I_{\alpha_i}}\mu) = H_{\alpha_i} = H_{\beta_i}$ and to simplify notation we denote this quantity by h_i. Next, we notice that, by Proposition 5.6, we have $\int_0^\sigma \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle - \langle\mu,1\rangle = I_s\}} = 0$ and $L^D_t = 0$, for every $t \leqslant \inf\{s \geqslant 0 : H_s < \tau_D(\mathrm{w})\}$. From our previous observations, we get:

$$\begin{aligned}\int_0^\sigma \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}} &= \sum_{h_i<\tau_D(\mathrm{w})} \int_{\alpha_i}^{\beta_i} \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}} \\ &= \sum_{H(\kappa_{-I_{\alpha_i}}\mu)<\tau_D(\mathrm{w})} \int_0^{\beta_i-\alpha_i} \mathrm{d}L^D_s(\rho^i, W^i)\mathbb{1}_{\{\langle\rho^i_s,1\rangle\leqslant K-\langle\mu,1\rangle - I_{\alpha_i}\}},\end{aligned}$$

where we used in the second identity that $\langle \rho_{s+\alpha_i}, 1\rangle = \langle \rho^i_s, 1\rangle + \langle \rho_{\alpha_i}, 1\rangle = \langle \rho^i_s, 1\rangle + I_{\alpha_i} + \langle \mu, 1\rangle$, for every $s \in [0, \beta_i - \alpha_i]$. This implies that:

$$\begin{aligned}\mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\sum_{H(\kappa_{-I_{\alpha_i}}\mu)<\tau_D(\mathrm{w})} \int_0^{\beta_i-\alpha_i} \mathrm{d}L^D_s(\rho^i, W^i)\mathbb{1}_{\{\langle\rho^i_s,1\rangle\leqslant K-\langle\mu,1\rangle-I_{\alpha_i}\}}\Big]& \\ = \int_{\mu([\tau_D(\mathrm{w}),\infty))}^{\langle\mu,1\rangle} \mathrm{d}u\,\mathbb{N}_{\mathrm{w}(H(\kappa_u\mu))}\Big(\int_0^\sigma \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K-\langle\mu,1\rangle+u\}}\Big)&,\end{aligned}$$

and the desired result now follows by performing the change of variable $u \longleftrightarrow \langle \mu, 1\rangle - u$ and applying the many-to-one formula (5.28). □

Lemma 5.9. *Consider an increasing sequence of open subsets $(D_n : n \geqslant 1)$ containing x, such that $\cup_n D_n = D$ and $\overline{D_n} \subset D$. There exists a subsequence $(n_k : k \geqslant 0)$ converging towards infinity, such that*

$$\lim_{k\to\infty} \sup_{s\in[0,\sigma]} |L^{D_{n_k}}_s - L^D_s| = 0, \qquad \mathbb{N}_x\text{–}a.e. \tag{5.34}$$

Proof. The proof of this lemma will be achieved by using similar techniques as in [63, Proposition 2.3] in the Brownian setting. We start by showing that, for a suitable subsequence, the total mass $L_\sigma^{D_n}$ converges towards L_σ^D, $\mathbb{N}_x$-a.e. The uniform convergence will then be deduced by standard techniques. Notice however that in [63], this is mainly done by establishing an $L_2(\mathbb{N}_x)$ convergence of $L_\sigma^{D_n}$ towards L_σ^D, and that we do not have a priori moments of order 2 in our setting. In order to overcome this difficulty, we need to localize the tree by the use of a truncation argument. We start by showing that, for any fixed $K > 0$, we have:

$$\lim_{n\to\infty} \int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s, 1\rangle \leqslant K\}} = \int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s, 1\rangle \leqslant K\}}, \quad \text{in } L_2(\mathbb{N}_x). \tag{5.35}$$

In this direction, we write $\mathbb{N}_x\Big(\big|\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} - \int_0^\sigma \mathrm{d}L_s^{D_n}\mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big|^2\Big)$ in the following form

$$\begin{aligned}\mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)^2\Big) + \mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)^2\Big)&\\ - 2\mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)\cdot\big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)\Big)&, \end{aligned}\tag{5.36}$$

and the proof of (5.35) will follow by computing each term separately and by taking the limit as $n \uparrow \infty$. First, we remark that

$$\big(\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)^2 = 2\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \int_s^\sigma \mathrm{d}L_u^D \, \mathbb{1}_{\{\langle \rho_u,1\rangle\leqslant K\}},$$

and the idea now is to apply the Markov property. For convenience, we let Θ_D be the subset of Θ of all the pairs (μ, w) satisfying the condition $\mu(\{\tau_D(\mathrm{w})\}) = 0$ when $\tau_D(\mathrm{w}) < \infty$, and we define Θ_{D_n} similarly replacing D by D_n. Notice that by Lemma 5.5, we have, $\mathbb{N}_x$–a.e., $(\rho_t, W_t) \in \Theta_D \cap (\cap_{n\geqslant 1}\Theta_{D_n})$ for every $t \geqslant 0$. For $(\mu, \mathrm{w}) \in \Theta_D$, we set

$$\begin{aligned}\phi_D(\mu,\mathrm{w}) &:= \mathbb{E}^\dagger_{\mu,\mathrm{w}}\Big[\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\Big]\\ &= \int_0^{\mu([0,\tau_D(\mathrm{w}))} \mathrm{d}u \; E^0 \otimes \Pi_{\mathrm{w}(H(\kappa_{\langle \mu,1\rangle - u}\mu))}\Big(\mathbb{1}_{\{\tau_D<\infty\}}\exp(-\alpha\tau_D)\mathbb{1}_{\{\langle J_{\tau_D},1\rangle\leqslant K-u\}}\Big),\end{aligned}$$

where in the second equality we used Lemma 5.8. Note that the dependence of ϕ_D on K is being omitted to simplify the notation. By our previous discussion, an application of the Markov property gives:

$$\begin{aligned}\mathbb{N}_x\left(\left(\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\right)^2\right) &= 2\mathbb{N}_x\left(\int_0^\sigma \mathrm{d}L_s^D \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\phi_D(\rho_s,W_s)\right)\\ &= 2E^0\otimes\Pi_x\Big(\mathbb{1}_{\{\tau_D<\infty\}}\exp(-\alpha\tau_D)\mathbb{1}_{\{\langle J_{\tau_D},1\rangle\leqslant K\}}\phi_D(J_{\tau_D},\xi^{\tau_D})\Big),\end{aligned}\tag{5.37}$$

where to simplify notation, we write $\xi^{\tau_D} := (\xi_t : 0 \leqslant t \leqslant \tau_D)$. Observe that $(J_{\tau_D}, \xi^{\tau_D}) \in \Theta_D$ since by independence, we have $\mathbb{1}_{\{\tau_D<\infty\}}J_{\tau_D}(\{\tau_D\}) = 0$, $P^0\otimes\Pi_x$–a.s. Replacing D by D_n, we also have $(J_{\tau_{D_n}}, \xi^{\tau_{D_n}}) \in \Theta_{D_n}$ and we obtain

$$\mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n}\mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}}\big)^2\Big) = 2E^0\otimes\Pi_x\Big(\mathbb{1}_{\{\tau_{D_n}<\infty\}}\exp(-\alpha\tau_{D_n})\mathbb{1}_{\{\langle J_{\tau_{D_n}},1\rangle\leqslant K\}}\phi_{D_n}(J_{\tau_{D_n}},\xi^{\tau_{D_n}})\Big),\tag{5.38}$$

where for $(\mu, \mathrm{w}) \in \Theta_{D_n}$, we write

$$\phi_{D_n}(\mu, \mathrm{w}) = \int_0^{\mu([0,\tau_{D_n}(\mathrm{w}))} \mathrm{d}u \; E^0 \otimes \Pi_{\mathrm{w}(H(\kappa_{\langle \mu,1\rangle - u}\mu))} \Big(\mathbb{1}_{\{\tau_{D_n}<\infty\}} \exp(-\alpha\tau_{D_n}) \mathbb{1}_{\{\langle J_{\tau_{D_n}},1\rangle \leqslant K-u\}} \Big).$$

Our goal now is to take the limit in (5.38) as $n \uparrow \infty$ and to show that this limit is precisely (5.37). In this direction, we remark that under $\{\langle J_{\tau_{D_n}}, 1\rangle \leqslant K\}$, we have the trivial bound $\phi_{D_n}(J_{\tau_{D_n}}, \xi^{\tau_{D_n}}) \leqslant K$. Thanks to the dominated convergence theorem, it is then enough to show that, $P^0 \otimes \Pi_x$-a.s., the following convergence holds:

$$\lim_{n\to\infty} \mathbb{1}_{\{\tau_{D_n}<\infty\}} e^{-\alpha\tau_{D_n}} \mathbb{1}_{\{\langle J_{\tau_{D_n}},1\rangle\leqslant K\}} \phi_{D_n}(J_{\tau_{D_n}}, \xi^{\tau_{D_n}}) = \mathbb{1}_{\{\tau_D<\infty\}} e^{-\alpha\tau_D} \mathbb{1}_{\{\langle J_{\tau_D},1\rangle\leqslant K\}} \phi_D(J_{\tau_D}, \xi^{\tau_D}).$$

In order to prove it, we start noticing that we always have $\tau_{D_n} \uparrow \tau_D$ as $n \to \infty$. In particular, since $\langle J_\infty, 1\rangle = \infty$, we see that the limit in the previous display is 0 under $\{\tau_D = \infty\}$. Let us focus now on the event $\{\tau_D < \infty\}$. First remark that

$$\kappa_{\langle J_{\tau_{D_n}},1\rangle - u} J_{\tau_{D_n}} = \kappa_{\langle J_{\tau_D},1\rangle - u} J_{\tau_D},$$

for every $u \leqslant \langle J\tau_{D_n}, 1\rangle$. This combined with the independence between J and ξ ensures that, under $\{\tau_D < \infty\}$, the quantities $\langle J\tau_{D_n}, 1\rangle$ and $\phi_{D_n}(J_{\tau_{D_n}}, \xi^{\tau_{D_n}})$ convergence respectively to $\langle J\tau_D, 1\rangle$ and $\phi_D(J_{\tau_D}, \xi^{\tau_D})$, giving the desired convergence under $\{\tau_D < \infty\}$. Consequently, we get:

$$\lim_{n\to\infty} \mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big)^2 \Big) = \mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big)^2 \Big).$$

Turning our attention to the cross-term, we can apply similar steps and the Markov property as before to obtain

$$\begin{aligned}
&\mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big) \cdot \big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big) \Big) \\
&= \mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \int_s^\sigma \mathrm{d}L_u^{D} \mathbb{1}_{\{\langle \rho_u,1\rangle\leqslant K\}} \Big) + \mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \int_s^\sigma \mathrm{d}L_u^{D_n} \mathbb{1}_{\{\langle \rho_u,1\rangle\leqslant K\}} \Big) \\
&= E^0 \otimes \Pi_x \Big(\mathbb{1}_{\{\tau_{D_n}<\infty\}} \exp(-\alpha\tau_{D_n}) \mathbb{1}_{\{\langle J_{\tau_{D_n}},1\rangle\leqslant K\}} \phi_D(J_{\tau_{D_n}}, \xi^{\tau_{D_n}}) \Big) \\
&\qquad\qquad + E^0 \otimes \Pi_x \Big(\mathbb{1}_{\{\tau_D<\infty\}} \exp(-\alpha\tau_D) \mathbb{1}_{\{\langle J_{\tau_D},1\rangle\leqslant K\}} \phi_{D_n}(J_{\tau_D}, \xi^{\tau_D}) \Big),
\end{aligned}$$

and using the same method as before we get:

$$\lim_{n\to\infty} \mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big) \cdot \big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big) \Big) = \mathbb{N}_x\Big(\big(\int_0^\sigma \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \big)^2 \Big).$$

Taking the limit as $n \uparrow \infty$ in (5.36) we deduce the claimed $L_2(\mathbb{N}_x)$ convergence (5.35). Now that the convergence of the truncated total mass has been established, to derive the statement of the proposition we proceed as follows. First, we introduce the processes

$$A_t^n := \int_0^t \mathrm{d}L_s^{D_n} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}} \quad \text{and} \quad A_t := \int_0^t \mathrm{d}L_s^{D} \mathbb{1}_{\{\langle \rho_s,1\rangle\leqslant K\}},$$

which are continuous additive functionals of the Markov process (ρ, W). Then using the Markov property, we get

$$\mathbb{N}_x\left(A_\infty^n | \mathcal{F}_s\right) = A_{s\wedge\sigma}^n + \phi_{D_n}(\rho_{s\wedge\sigma}, W_{s\wedge\sigma}) \quad \text{and} \quad \mathbb{N}_x\left(A_\infty | \mathcal{F}_s\right) = A_{s\wedge\sigma} + \phi_D(\rho_{s\wedge\sigma}, W_{s\wedge\sigma}), \quad (5.39)$$

since $\phi_{D_n}(\mu, \mathrm{w}) = \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}[A^n_\infty]$, $\phi_D(\mu, \mathrm{w}) = \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}[A_\infty]$ and $\phi_{D_n}(\rho_\sigma, W_\sigma) = \phi_D(\rho_\sigma, W_\sigma) = 0$, $\mathbb{N}_x$-a.e. To simplify notation, we denote respectively by $M^n_s = \mathbb{N}_x(A^n_\infty|\mathcal{F}_s)$ and $M_s = \mathbb{N}_x(A_\infty|\mathcal{F}_s)$ for $s \geqslant 0$ the martingales in (5.39). Next, we apply Doob's inequality to derive:

$$\mathbb{N}_x\big(\sup_{s>0}|M^n_s - M_s| > \delta\big) \leqslant \delta^{-2}\mathbb{N}_x\big(|A^n_\sigma - A_\sigma|^2\big). \tag{5.40}$$

Indeed, even if $\mathbb{N}_x$ is not a finite measure, we can argue as follows: fix $a > 0$ and observe that $(M_{a+t})_{t\geqslant 0}$, $(M^n_{a+t})_{t\geqslant 0}$ under $\mathbb{N}_x(\,\cdot\,|\sigma > a)$ are uniformly integrable martingales, from which we obtain

$$\mathbb{N}_x\big(\sup_{s\geqslant a}|M^n_s - M_s| > \delta \;\big|\; \sigma > a\big) \leqslant \delta^{-2}\mathbb{N}_x\big(|A^n_\sigma - A_\sigma|^2 \;\big|\; \sigma > a\big),$$

and we deduce (5.40) by multiplying both sides by $\mathbb{N}_x(\sigma > a)$ and by taking the limit as $a \downarrow 0$ – using monotone convergence.

By (5.35), the right-hand side of (5.40) converges towards 0 as $n \uparrow \infty$ and we deduce that

$$\lim_{k\to\infty}\sup_{s>0}|M^{n_k}_s - M_s| = 0, \quad \mathbb{N}_x \text{ –a.e.}$$

for a suitable subsequence $(n_k : \; k \geqslant 1)$ increasing towards infinity. Since $\lim_{n\to\infty}\phi_{D_n}(\rho_s, W_s) = \phi_D(\rho_s, W_s)$, we obtain that $\mathbb{N}_x$-a.e., for every $t \geqslant 0$, $\int_0^t \mathrm{d}L^{D_{n_k}}_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}} \to \int_0^t \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}}$ as $k \to \infty$. By continuity, monotonicity and the fact that $\sigma < \infty$ $\mathbb{N}_x$–a.e., we can apply Dini's theorem to get:

$$\lim_{k\to\infty}\sup_{t>0}\Big|\int_0^t \mathrm{d}L^{D_{n_k}}_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}} - \int_0^t \mathrm{d}L^D_s \mathbb{1}_{\{\langle\rho_s,1\rangle\leqslant K\}}\Big| = 0, \qquad \mathbb{N}_x\text{– a.e.}$$

Consequently, we deduce that on the event $\{\sup_{s\geqslant 0}\langle\rho_s, 1\rangle \leqslant K\} = \{\sup X \leqslant K\}$, the $\mathbb{N}_x$-a.e. uniform convergence (5.34) holds under a subsequence (n_k), which depends on K. Since this holds for arbitrary K, we can use a diagonal argument to find a deterministic subsequence that we still denote by $(n_k : k \geqslant 1)$ converging towards infinity such that

$$\lim_{k\to\infty}\sup_{t\in[0,\sigma]}|L^{D_{n_k}}_t - L^D_t| = 0, \qquad \mathbb{N}_x\text{– a.e.}$$

□

We are now in position to prove that the process $\widetilde{L}^D$ is $\mathcal{F}^D$-measurable.

Proof of Proposition 5.7. Until further notice, we argue under $\mathbb{P}_{0,x}$. By (5.27) and monotonicity, a diagonal argument gives that we can find a subsequence $(\varepsilon_k : \; k \geqslant 1)$, with $\varepsilon_k \downarrow 0$ as $k \to \infty$, such that:

$$L^{D_n}_{\Gamma^D_s} = \lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_0^{\Gamma^D_s} \mathrm{d}r \mathbb{1}_{\{\tau_{D_n}(W_r)<H_r<\tau_{D_n}(W_r)+\varepsilon_k\}},$$

for every $n \geqslant 1$ and $s \geqslant 0$. Our goal is now to show that:

$$L^{D_n}_{\Gamma^D_s} = \lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_0^s \mathrm{d}r \mathbb{1}_{\{\tau_{D_n}(W_{\Gamma^D_r})<H_{\Gamma^D_r}<\tau_{D_n}(W_{\Gamma^D_r})+\varepsilon_k\}}, \tag{5.41}$$

which will imply that $(L^{D_n}_{\Gamma^D_s})_{s\geqslant 0}$ is $\mathcal{F}^D$-measurable for every $n\in\mathbb{N}$. In order to establish (5.41) we argue for ω fixed and observe that for k large enough, we have:

$$\mathbb{1}_{\{\tau_{D_n}(W_r)<H_r<\tau_{D_n}(W_r)+\varepsilon_k\}} = \mathbb{1}_{\{\tau_{D_n}(W_r)<H_r<\tau_{D_n}(W_r)+\varepsilon_k\}}\mathbb{1}_{\{H_r\leqslant\tau_D(W_r)\}}, \qquad \text{for all } r\in[0,\Gamma^D_s].$$

To see it, remark that if the previous display did not hold, by a compactness argument and continuity we would have $\tau_{D_n}(W_{r_0}) = \tau_D(W_{r_0}) \leqslant H_r$ for some r_0 in $[0,\Gamma^D_s]$. This gives a contradiction since $\overline{D}_n\subset D$ and $(W_{r_0}(t))_{t\in[0,H_{r_0}]}$ is continuous. Recalling the notation V^D given in (5.25), we deduce that

$$\begin{aligned}
L^{D_n}_{\Gamma^D_s} &= \lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_0^{\Gamma^D_s}\mathrm{d}r\,\mathbb{1}_{\{\tau_{D_n}(W_r)<H_r<\tau_{D_n}(W_r)+\varepsilon_k\}}\\
&= \lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_0^{\Gamma^D_s}\mathrm{d}V^D_r\,\mathbb{1}_{\{\tau_{D_n}(W_r)<H_r<\tau_{D_n}(W_r)+\varepsilon_k\}} = \lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_0^{s}\mathrm{d}r\,\mathbb{1}_{\{\tau_{D_n}(W_{\Gamma^D_r})<H_{\Gamma^D_r}<\tau_{D_n}(W_{\Gamma^D_r})+\varepsilon_k\}},
\end{aligned}$$

giving us (5.41). The same arguments can be applied under $\mathbb{N}_x$ and, to complete the proof of the proposition, it suffices to show that for every $t\geqslant 0$

$$\lim_{n\to\infty}\sup_{s\in[0,t]}|L^{D_n}_{\Gamma^D_s}-L^{D}_{\Gamma^D_s}| = 0, \qquad \text{under } \mathbb{P}_{0,x} \text{ and } \mathbb{N}_x, \tag{5.42}$$

at least along a suitable subsequence. However, note that when working under $\mathbb{N}_x$, this convergence follows by Lemma 5.9. Now, the result under $\mathbb{P}_{0,x}$ is a standard consequence of excursion theory. More precisely, recall that $-I$ is the local time of (ρ,W) at $(0,x)$ and, for fixed $r>0$, set $T_r := \inf\{t\geqslant 0:\ -I_t>r\}$. If we let $T_D := \inf\{t\geqslant 0:\tau_D(W_t)<\infty\}$, by continuity there exists a finite number of excursions (ρ^i,W^i) of (ρ,W) in $[0,T_r]$ satisfying $T_D(W^i)<\infty$, and their distribution is $\mathbb{N}_{x,0}(\,\cdot\,|T_D<\infty)$. Since $T_r\uparrow\infty$, the approximation (5.42) under $\mathbb{P}_{x,0}$ now follows from the result under $\mathbb{N}_{x,0}$. This completes the proof of Proposition 5.7. □

5.3.2 Proof of special Markov property

Now that we have already studied the trajectories staying in D, we turn our attention to the complementary side of the picture and we start by introducing formally the notion of excursions from D.

Excursions from D. Observe that (5.24) and assumption $(\mathbf{H_1})$ imply that

$$\mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}s\ \mathbb{1}_{\{\tau_D(W_s)<\zeta_s\}}>0\Big)>0.$$

Hence, the set $\{s\in[0,\sigma]:\tau_D(W_s)<\zeta_s\}$ is non-empty with non null measure under $\mathbb{N}_x$ and $\mathbb{P}_{0,x}$. If we define

$$\gamma^D_s := \big(\zeta_s-\tau_D(W_s)\big)_+, \qquad s\geqslant 0,$$

it is straightforward to show by the snake property and the continuity of ζ that γ^D is continuous. Set

$$\sigma^D_t := \inf\Big\{s\geqslant 0:\int_0^s\mathrm{d}r\,\mathbb{1}_{\{\gamma^D_r>0\}}>t\Big\},$$

and consider the process $(\rho^D_t)_{t\geqslant 0}$ taking values in $\mathcal{M}_f(\mathbb{R}_+)$ defined by:

$$\langle\rho^D_t,f\rangle := \int\rho_{\sigma^D_t}(\mathrm{d}h)f\big(h-\tau_D(W_{\sigma^D_t})\big)\mathbb{1}_{\{h>\tau_D(W_{\sigma^D_t})\}}. \tag{5.43}$$

Then, by Proposition 4.3.1 in [43], ρ^D and ρ have the same distribution under $\mathbb{P}_{0,x}$. In particular, $\langle \rho^D, 1\rangle$ has the same law as the reflected Lévy process $X - I$ and we denote its local time at 0 by $(\ell^D(s) : s \geqslant 0)$. Moreover, it is shown in [43, Section 4.3] that the process L^D is related to the local time ℓ^D by the identity:

$$L_t^D = \ell^D \left(\int_0^t \mathrm{d}s \, \mathbb{1}_{\{\gamma_s^D > 0\}} \right). \tag{5.44}$$

The proof of Proposition 4.3.1 in [43] shows that ρ^D can be obtained as limit of functions which are independent of $\mathcal{F}^D$, implying that ρ^D is on its turn independent of $\mathcal{F}^D$. Now, denote the connected components of the open set

$$\big\{t \geqslant 0 : \tau_D(W_t) < \zeta_t\big\} = \big\{t \geqslant 0 : \gamma_t^D > 0\big\},$$

by $\big((a_i, b_i) : i \in \mathcal{I}\big)$, where $\mathcal{I}$ is an indexing set that might be empty. By construction, for any $s \in (a_i, b_i)$, the trajectory W_s is a trajectory leaving D. Remark that $H_{a_i} = H_{b_i} < H_r$ for every $r \in (a_i, b_i)$ and let (ρ^i, W^i) be the subtrajectory of (ρ, W) associated with $[a_i, b_i]$ as defined in Section 5.2.3. Observe that in our setting, (ρ^i, W^i) is defined for each $s \in [0, b_i - a_i]$ and for any measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$\langle \rho_s^i, f\rangle = \int \rho_{a_i+s}(\mathrm{d}h) f(h - \tau_D(W_{a_i})) \mathbb{1}_{\{h > \tau_D(W_{a_i})\}}$$

and

$$W_s^i = W_{(a_i+s)\wedge b_i}(t + \tau_D(W_{a_i})) \quad \text{for } t \in [0, \zeta_{(a_i+s)\wedge b_i} - \tau_D(W_{a_i})],$$

with respective lifetime process given by

$$\zeta_s^i = \zeta_{(a_i+s)\wedge b_i} - \tau_D(W_{a_i}),$$

where $\tau_D(W_s) = \tau_D(W_{a_i}) = \zeta_{a_i}$. We say that (ρ^i, W^i) is an *excursion* of (ρ, W) from D. Observe that $W_s^i(0) = W_{a_i}^i(0)$ for all $s \in [a_i, b_i]$ by the snake property and that we have $W_{a_i}^i(0) \in \partial D$. This is the point of ∂D used by the subtrajectory W^i to escape from D.

In order to state the special Markov property we need to introduce one last notation. Let θ be the right inverse of $\widetilde{L}^D$, viz. the $\mathcal{F}^D$-measurable function defined as

$$\theta_r := \inf\big\{s \geqslant 0 \, : L_{\Gamma_s^D}^D > r\big\}, \quad \text{for all } r \in [0, L_\sigma^D).$$

Recall that we are considering some fixed $x \in D$, the notation $((\rho^i, W^i) : i \in \mathcal{I})$ for the excursions outside D, and that we are working under the hypothesis $(\mathbf{H_1})$. We are now going to state and prove the special Markov property under $\mathbb{P}_{0,x}$, and we will deduce by standard arguments a version under the excursion measure $\mathbb{N}_x$. Under $\mathbb{P}_{0,x}$ we use the same notation as under $\mathbb{N}_x$, but observing that $\sigma_H = \infty$ and noticing that $\mathbb{P}_{0,x}$-a.s., we have $\mathcal{Y}_D = \int_0^\infty \mathrm{d}s \mathbb{1}_{\{H_s \leqslant \tau_D(W_s)\}} = \infty$ and $L_\infty^D = \infty$. In particular, this implies that Γ_s^D and θ_s are finite for every $s < \infty$.

Theorem 5.10 (Special Markov property)**.** *Under $\mathbb{P}_{0,x}$, conditionally on $\mathcal{F}^D$, the point measure*

$$\sum_{i\in\mathcal{I}} \delta_{(L_{a_i}^D, \rho^i, W^i)}(\mathrm{d}\ell, \, \mathrm{d}\rho, \, \mathrm{d}\omega)$$

is a Poisson point process with intensity

$$\mathbb{1}_{[0,\infty)}(\ell) \, \mathrm{d}\ell \, \mathbb{N}_{\mathrm{tr}_D(\widehat{W})_{\theta_\ell}}(\mathrm{d}\rho, \, \mathrm{d}\omega).$$

Recall that we have established in Proposition 5.7 that $\widetilde{L}^D$ is $\mathcal{F}^D$-measurable. It might also be worth observing that if $F = F(\mathsf{\rho}, \omega)$ is a measurable function, when integrating with respect to the intensity measure $\mathbb{1}_{[0,\infty)}(\ell)\mathrm{d}\ell\mathbb{N}_{\mathrm{tr}_D(\widehat{W})_{\theta_\ell}}(\mathrm{d}\rho, \mathrm{d}\omega)$ we can re-write the expression in the following more tractable form:

$$\int_0^\infty \mathrm{d}\ell\, \mathbb{N}_{\mathrm{tr}_D\widehat{(W)}_{\theta_\ell}}(F) = \int_0^\infty \mathrm{d}\widetilde{L}^D_s\, \mathbb{N}_{\mathrm{tr}_D(\widehat{W})_s}(F) = \int_0^\infty \mathrm{d}L^D_s\, \mathbb{N}_{\widehat{W}_s}(F)$$

where in the last equality, we applied a change of variable for Lebesgue-Stieltjes integrals using the fact that L^D is constant on the excursion intervals $[\Gamma^D_{s-}, \Gamma^D_s]$ when $\Gamma^D_{s-} < \Gamma^D_s$. Let us now prove Theorem 5.10.

Proof. In this proof, we work with (ρ, W) under $\mathbb{P}_{0,x}$. Let us start with some preliminary constructions and remarks. First, we introduce the $\mathcal{S}_x$-valued process (ρ, W^*) defined at each $t \geqslant 0$ as

$$\big(\rho_t, W^*_t(s)\big) = \Big(\rho_t(\mathrm{d}h), W_t\big(s \wedge \tau_D(W_t)\big)\Big), \qquad \text{for } s \in [0, \zeta_{W_t}],$$

and let $\mathcal{F}^D_*$ be its generated sigma-field on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E,x})$. The snake (ρ, W^*) can be interpreted as the Lévy snake associated with (ψ, ξ^*), where ξ^* is the stopped Markov process $(\xi^*_t :\ t \geqslant 0) = (\xi_{t\wedge\tau_D(\xi)} :\ t \geqslant 0)$. Since, for every $t \geqslant 0$,

$$\big(\zeta_{W_t} - \tau_D(W_t)\big)_+ = \big(\zeta_{W^*_t} - \tau_D(W^*_t)\big)_+,$$

we derive that the process $\gamma^D_t = (\zeta_{W_t} - \tau_D(W_t))_+$ is $\mathcal{F}^D_*$−measurable. Consequently, we have $\mathcal{F}^D \subset \mathcal{F}^D_*$ since V^D – the functional measuring the time spent in D defined in (5.25) – is $\mathcal{F}^D_*$-measurable and by definition $\mathrm{tr_D}(\rho, \mathrm{W}) = \mathrm{tr_D}(\rho, \mathrm{W}^*)$. Recalling that $\big((a_i, b_i) : i \in \mathcal{I}\big)$ stands for the connected components of the open set

$$\{t \geqslant 0 :\ \tau_D(W_t) < \zeta_{W_t}\} = \{t \geqslant 0 : \gamma^D_t > 0\},$$

we deduce by the previous discussion and the identity $\tau_D(W_{a_i}) = \zeta_{a_i}$, that the variables

$$\widehat{W}_{a_i} = \widehat{W}^*_{a_i},\ \zeta^i = \zeta_{(a_i+\cdot)\wedge b_i} - \zeta_{a_i} \text{ and a fortiori } \rho^i \text{ are } \mathcal{F}^D_* - \text{ measurable.}$$

Informally, $\mathcal{F}^D_*$ encodes the information of the trajectories staying in D and the tree structure. We claim that conditionally on $\mathcal{F}^D_*$, the excursions $(W^i : i \in \mathbb{N})$ are independent, and that the conditional distribution of W^i is $\mathbb{Q}^{\zeta^i}_{\widehat{W}^*_{a_i}}$, where we recall from Section 5.2.3 that we denote the distribution of the snake driven by h started at x by $\mathbb{Q}^h_x$.

In order to prove this claim, consider a collection of snake trajectories $\big(W^{i,\prime} : i \in \mathcal{I}\big)$ such that, conditionally on (ρ, W^*), they are independent and each one is respectively distributed according to the measure $\mathbb{Q}^{\zeta^i}_{\widehat{W}^*_{a_i}}$. Next let W' be the process defined as follows: for every t such that $\gamma^D_t = 0$ set $W'_t = W^*_t$, and if $\gamma^D_t > 0$ we set:

$$W'_t(s) = \begin{cases} W^*_t(s) & \text{if } s \in [0, \tau_D(W^*_t)] \\ W^{i,\prime}_{t-a_i}\big(s - \tau_D(W^*_t)\big) & \text{if } s \in [\tau_D(W^*_t), \zeta(W^*_t)], \end{cases}$$

where i is the unique index such that $t \in (a_i, b_i)$. By construction, (ρ, W') is in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E,x})$ and a straightforward computation of its finite marginals shows that its distribution is $\mathbb{P}_{0,x}$, proving our claim.

Notice that (5.27) implies that L^D is constant on the intervals $[\Gamma^D_{s-}, \Gamma^D_s]$ when $\Gamma^D_{s-} < \Gamma^D_s$. Hence, $L^D_s = \widetilde{L}^D_{V^D_s}$ for all $s \geqslant 0$ and in particular $L^D_{a_i} = \widetilde{L}^D_{V^D_{a_i}}$, the latter being $\mathcal{F}^D_*$-measurable. Consider now U a bounded $\mathcal{F}^D$-measurable random variable, and remark that to obtain the desired result, it is enough to show that:

$$\mathbb{E}_{0,x}\Big[U \exp(-\sum_{i \in \mathcal{I}} F\big(L^D_{a_i}, \rho^i, W^i)\big)\Big] = \mathbb{E}_{0,x}\Big[U \exp\Big(-\int_0^\infty \mathrm{d}\ell\, \mathbb{N}_{\mathrm{tr}_D\widehat{(W)}_{\theta_\ell}}\big(1 - \exp(-F(\ell, \rho, W))\big)\Big)\Big],$$

for every non-negative measurable function F in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$. In order to prove this identity, we start by projecting the left term on $\mathcal{F}^D_*$: by the previous discussion and recalling that $\mathcal{F}^D \subset \mathcal{F}^D_*$, we get

$$\mathbb{E}_{0,x}\Big[U \exp(-\sum_{i \in \mathcal{I}} F\big(L^D_{a_i}, \rho^i, W^i)\big)\Big] = \mathbb{E}_{0,x}\Big[U \prod_{i \in \mathcal{I}} \mathbb{Q}^{\zeta^i}_{\widehat{W}^*_{a_i}}\big(\exp(-F(L^D_{a_i}, \rho^i, W)\big)\Big].$$

Moreover, it is straightforward to see that

$$\widehat{W}^*_{a_i} = \widehat{W}_{a_i} = \mathrm{tr}_D\widehat{(W)}_{\theta_{L^D_{a_i}}},$$

we omit the details of this identity since the argument used in (23) of [66, Theorem 20] for the Brownian snake applies directly to our framework. Consequently, we have:

$$\mathbb{E}_{0,x}\Big[U \prod_{i \in \mathcal{I}} \mathbb{Q}^{\zeta^i}_{\widehat{W}^*_{a_i}}\big(\exp(-F(L^D_{a_i}, \rho^i, W)\big)\Big] = \mathbb{E}_{0,x}\Big[U \prod_{i \in \mathcal{I}} \mathbb{Q}^{\zeta^i}_{\mathrm{tr}_D\widehat{(W)}_{\theta_{L^D_{a_i}}}}\big(\exp(-F(L^D_{a_i}, \rho^i, W)\big)\Big].$$

Now, we need to take the projection on $\mathcal{F}^D$. Recalling that $H(\rho^i) = \zeta^i$, observe that for every $i \in \mathcal{I}$,

$$\mathbb{Q}^{\zeta^i}_{\mathrm{tr}_D\widehat{(W)}_{\theta_{L^D_{a_i}}}}\big(\exp(-F(L^D_{a_i}, \rho^i, W)\big)$$

is a measurable function of the pair $(L^D_{a_i}, \rho^i)$ and the process $(\mathrm{tr}_D(W)_{\theta_r} : r \geqslant 0)$, the latter being $\mathcal{F}^D$-measurable. We are going to conclude by showing that the point measure

$$\sum_{i \in \mathcal{I}} \delta_{(L^D_{a_i}, \rho^i)}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\infty)}(\ell)\mathrm{d}\ell\, N(\mathrm{d}\rho)$ independent of $\mathcal{F}^D$. Remark that once this has been established, an application of the exponential formula for functionals of Poisson random measures yields

$$\mathbb{E}_{0,x}\Big[U \prod_{i \in \mathbb{N}} \mathbb{Q}^{\zeta^i}_{\mathrm{tr}_D\widehat{(W)}_{\theta_{L^D_{a_i}}}}\big(\exp(-F(L^D_{a_i}, \rho^i, W)\big)\Big] = \mathbb{E}_{0,x}\Big[U \exp\Big(-\int_0^\infty \mathrm{d}\ell \mathbb{N}_{\mathrm{tr}_D\widehat{(W)}_{\theta_\ell}}\big(1-\exp(-F(\ell, \rho, W))\big)\Big)\Big]$$

giving the desired result. In this direction, recall the definition of ρ^D given in (5.43), and that ℓ^D stands for the local time of ρ^D at 0. We denote the connected component of the open set $\{t \geqslant 0 : \langle \rho^D_t, 1\rangle \neq 0\} = \{t \geqslant 0 : H(\rho^D_t) > 0\}$ by $\big((c_j, d_j) : j \in \mathbb{N}\big)$ – the latter equality holding since $\rho_t^D(\{0\}) = 0$ – and observe that these are precisely the excursion intervals of $\langle \rho^D, 1\rangle$ from 0. It follows by (5.12) and the discussion before the proof that

$$\sum_{j \in \mathbb{N}} \delta_{(\ell^D(c_j),\, \rho^D_{(c_j+\cdot)\wedge d_j})}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\infty)}(\ell)\mathrm{d}\ell\, N(\mathrm{d}\rho)$ and observe that this measure is independent of $\mathcal{F}^D$ – since ρ^D is independent of $\mathcal{F}^D$. Furthermore, by (5.44) we have:

$$L^D_{\sigma^D_s} = \ell^D\Big(\int_0^{\sigma_s^D} \mathrm{d}r\, \mathbb{1}_{\{\gamma_r^D > 0\}}\Big) = \ell^D(s),$$

for every $s \geqslant 0$. It is now straightforward to deduce from our last observations that:

$$\big\{(L^D_{a_i}, \rho^i) : i \in \mathcal{I}\big\} = \big\{(\ell^D(c_j), \rho^D_{(c_j+\cdot)\wedge d_j}) : j \in \mathbb{N}\big\},$$

concluding the proof.

□

Setting $T_D = \inf\{t \geqslant 0 : \tau_D(W_t) < \infty\}$, we infer from our previous result a version of the special Markov property holding under the probability measure

$$\mathbb{N}_x^D := \mathbb{N}_x(\,\cdot\,| T_D < \infty).$$

Observe that $\mathbb{N}_x(T_D < \infty)$ is finite: if this quantity was infinite, by excursion theory, the process (ρ, W) under $\mathbb{P}_{0,x}$ would have infinitely many excursions exiting D on compact intervals, contradicting the continuity of its paths. Finally, note that (ρ, W) under $\mathbb{N}_x^D$ has the distribution of the first excursion exiting the domain D. As a straightforward consequence of Theorem 5.10, this observation allows us to deduce:

Theorem 5.11. *Under $\mathbb{N}_x^D$ and conditionally on $\mathcal{F}^D$, the point measure:*

$$\sum_{i \in \mathcal{I}} \delta_{(L^D_{a_i}, \rho^i, W^i)}(\mathrm{d}\ell,\, \mathrm{d}\rho,\, \mathrm{d}\omega)$$

is a Poisson point process with intensity

$$\mathbb{1}_{[0, L^D_\sigma]}(\ell)\, \mathrm{d}\ell\, \mathbb{N}_{\mathrm{tr}_D(\widehat{W})_{\theta_\ell}}(\mathrm{d}\rho,\, \mathrm{d}\omega).$$

Recall that the measure $\mathrm{d}L^D_s$ is supported on $\{s \geqslant 0 : \widehat{W}_s \in \partial D\}$ and consider a measurable function $g : \partial D \to \mathbb{R}_+$. Under $\mathbb{N}_x$, we define the *exit measure from D*, denoted by $\mathcal{Z}^D$ as:

$$\langle \mathcal{Z}^D, g\rangle := \int_0^\sigma \mathrm{d}L^D_s g(\widehat{W}_s).$$

The total mass of $\mathcal{Z}^D$ is L^D_σ and, in particular, $\mathcal{Z}^D$ is non-null only in $\{T_D < \infty\}$. Again by a standard change of variable, we get

$$\langle \mathcal{Z}^D, g\rangle = \int_0^\sigma \mathrm{d}\widetilde{L}^D_s g(\mathrm{tr}_D(\widehat{W}_s)) = \int_0^{L^D_\sigma} \mathrm{d}\ell\, g(\mathrm{tr}_D \widehat{W}_{\theta_\ell}), \qquad \mathbb{N}_x\text{–a.e.}$$

and this implies that $\mathcal{Z}^D$ is $\mathcal{F}^D$-measurable since $L^D_\sigma \in \mathcal{F}^D$ by Proposition 5.7. In this work, we shall frequently make use of the following simpler version of the special Markov property. By Theorem 5.11, we have

Corollary 5.12. *Under $\mathbb{N}_x^D$ and conditionally on $\mathcal{F}^D$, the point measure*

$$\sum_{i\in\mathcal{I}} \delta_{(\rho^i, W^i)}(\mathrm{d}\rho,\, \mathrm{d}\omega) \tag{5.45}$$

is a Poisson random measure with intensity $\int \mathcal{Z}^D(\mathrm{d}y)\mathbb{N}_y(\mathrm{d}\rho,\, \mathrm{d}\omega)$.

Let us close this section by recalling some well-known properties of $\mathcal{Z}^D$ that will be needed, and by introducing some useful notations. Remark by (5.28) that, for any measurable $g : \partial D \mapsto \mathbb{R}_+$ and for every $y \in D$, we have

$$\mathbb{N}_y\big(\langle \mathcal{Z}^D, g\rangle\big) = \Pi_y\Big(\mathbb{1}_{\{\tau_D<\infty\}} \exp(-\alpha\tau_D) g(\xi_{\tau_D})\Big),$$

and for such g, we set:

$$u_g^D(y) := \mathbb{N}_y\big(1 - \exp(-\langle \mathcal{Z}^D, g\rangle)\big), \qquad \text{for all } y \in D. \tag{5.46}$$

Theorem 4.3.3 in [43] states that for every $g : \partial D \to \mathbb{R}_+$ bounded measurable function, u_g^D solves the integral equation:

$$u_g^D(y) + \Pi_y\Big(\int_0^{\tau_D} \mathrm{d}t\, \psi(u_g^D(\xi_t))\Big) = \Pi_y\big(\mathbb{1}_{\{\tau_D<\infty\}} g(\xi_{\tau_D})\big). \tag{5.47}$$

By convention, we set $u_g^D(y) := g(y)$ for every $y \in \partial D$, and we stress that this convention is compatible with (5.47).

5.4 Construction of a measure supported on $\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$

From now on, we fix $x \in E$ and we consider the random set:

$$\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}, \quad \text{as well as its image on the tree } \mathcal{T}_H, \text{ viz.} \quad \{v \in \mathcal{T}_H : \xi_v = x\}. \tag{5.48}$$

In order to study the latter, we shall construct an additive functional $A := (A_t)_{t\in\mathbb{R}_+}$ of the Lévy snake supported on $\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$. The present section is devoted to the construction of A and to develop the machinery needed for our analysis. The study of $\{v \in \mathcal{T}_H : \xi_v = x\}$ is delayed to Section 5.5 and will heavily rely on the results of this section. Let us discuss now in detail the framework we will consider in the rest of this work.

Framework of Section 5.4 and 5.5: With the same notations as in previous sections, consider a strong Markov process ξ taking values in E with a.s. continuous sample paths and we make the following assumptions:

$$\boldsymbol{x} \textbf{ is regular, instantaneous and recurrent for } \xi, \tag{$\mathbf{H_2}$}$$

and

$$\int_{\mathbf{0}}^{\infty} \mathrm{d}\mathbf{t}\; \mathbb{1}_{\{\xi_\mathbf{t}=\boldsymbol{x}\}} = \mathbf{0}, \qquad \Pi_x - \text{a.s.} \tag{$\mathbf{H_3}$}$$

Under $(\mathbf{H_2})$ the local time of ξ at x is well defined up to a multiplicative constant (that we fix arbitrarily) and we denote it by $\mathcal{L}$. The recurrence hypothesis is assumed for convenience and we expect our results to hold with minor modifications without it. Set $E_* := E\backslash\{x\}$ and for $\mathrm{w} \in \mathcal{W}_E$, with the notation of Section 5.3 write

$$\tau_{E_*}(\mathrm{w}) = \inf\{h \in [0, \zeta_{\mathrm{w}}] : \mathrm{w}(h) = x\},$$

for the exit time of w from the open set E_*. Observe that since x is recurrent for ξ, we have

$$\Pi_y(\tau_{E_*} < \infty) = 1 \tag{5.49}$$

for every $y \in E_*$, and in particular $(\mathbf{H_1})$ holds. This will allow us to make use of the special Markov property established in the previous section. Assumption $(\mathbf{H_3})$ might seem a technicality but it plays a crucial role in our study: it will ensure, under $\mathbb{N}_y$ and $\mathbb{P}_{0,y}$, that the set of branching points of $\mathcal{T}_H$ and $\{\upsilon \in \mathcal{T}_H\backslash\{0\} : \xi_\upsilon = x\}$ are disjoint. We will explain properly this point after concluding the presentation of the section.

Let $\mathcal{N}$ be the excursion measure of ξ at x associated with $\mathcal{L}$ and, with a slight abuse of notation, still write σ_ξ for the lifetime of ξ under $\mathcal{N}$. The pair

$$\overline{\xi}_s = (\xi_s, \mathcal{L}_s), \quad s \geqslant 0,$$

is a strong Markov process taking values in the Polish space $\overline{E} := E \times \mathbb{R}_+$ equipped with the product metric $d_{\overline{E}}$. We set $\Pi_{y,r}$ for its law started from an arbitrary point $(y, r) \in \overline{E}$. Recall that we always work under the assumptions $(\mathbf{H}_0)$, which for $(\psi, \overline{\xi})$ takes the following form:

> **Hypothesis** $(\mathbf{H}'_0)$. There exists a constant $C_\Pi > 0$ and two positive numbers $p, q > 0$ such that,
> for every $y \in E$ and $t \geqslant 0$, we have:
>
> $$\Pi_{y,0}\Big(\sup_{0\leqslant u\leqslant t} d_{\overline{E}}\big((\xi_u, \mathcal{L}_u), (y, 0)\big)^p\Big) \leqslant C_\Pi \cdot t^q, \qquad \text{and} \qquad q\cdot(1-\Upsilon^{-1}) > 1, \tag{$\mathbf{H'_0}$}$$

where we recall the definition of Υ from (5.17). We will use respectively the notation $\overline{\Theta}$, $\overline{\mathcal{S}}$ for the sets defined as Θ, $\mathcal{S}$ in Section 5.2.3 but replacing the Polish space E by $\overline{E}$. It will be convenient to write the elements of $\mathcal{W}_{\overline{E}}$ as pairs $\overline{\mathrm{w}} = (\mathrm{w}, \ell)$, where $\mathrm{w} \in \mathcal{W}_E$ and $\ell : [0, \zeta_{\mathrm{w}}] \mapsto \mathbb{R}_+$ is a continuous function. Recall that under $(\mathbf{H'_0})$, the family of measures $(\mathbb{P}_{\mu,\overline{\mathrm{w}}} : (\mu, \overline{\mathrm{w}}) \in \overline{\Theta})$ are defined in the canonical space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$ and we denote the canonical process by (ρ, W, Λ), where $W_s : [0, \zeta_s(\overline{W}_s)] \mapsto E$ and $\Lambda_s : [0, \zeta_s(\overline{W}_s)] \mapsto \mathbb{R}_+$. Said otherwise, for each $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ the process

$$(\rho_s, W_s, \Lambda_s), \quad s \geqslant 0,$$

is the ψ-Lévy snake with spatial motion $\overline{\xi}$ started from $(\mu, \overline{\mathrm{w}})$ and we simply write $\overline{W}_s := (W_s, \Lambda_s)$. For every $(y, r_0) \in \overline{E}$, we denote the excursion measure of $(\rho, \overline{W})$ starting from $(0, y, r_0)$ by $\mathbb{N}_{y,r_0}$.

Recall that under $\mathbb{P}_{0,y,r_0}$ or $\mathbb{N}_{y,r_0}$, for each $s \geqslant 0$ and conditionally on ζ_s, the pair

$$(W_s, \Lambda_s) = \big((W_s(h), \Lambda_s(h)) : h \in [0, \zeta_s]\big)$$

has the distribution of $(\xi, \mathcal{L})$ under Π_{y,r_0} killed at ζ_s. In particular, the associated Lebesgue-Stieltjes measure of Λ_s is supported on the closure of $\{h \in [0, \zeta_s) : W_s(h) = x\}$, $\mathbb{P}_{0,y,r_0}$ and $\mathbb{N}_{y,r_0}$– a.e. We will restrict our analysis to the collection of initial conditions $(\mu, \overline{\mathrm{w}}) := (\mu, \mathrm{w}, \ell) \in \overline{\Theta}$ satisfying that:

(i) ℓ is a non-decreasing continuous function and the support of its Lebesgue-Stieltjes measure is
$$\overline{\{h \in [0, \zeta_{\mathrm{w}}) : \mathrm{w}(h) = x\}}.$$

(ii) The measure μ does not charge the set $\{h \in [0, \zeta_{\mathrm{w}}] : \mathrm{w}(h) = x\}$, viz.
$$\int_{[0,\zeta_{\mathrm{w}}]} \mu(\mathrm{d}h)\, \mathbb{1}_{\{\mathrm{w}(h)=x\}} = 0.$$

This subcollection of $\overline{\Theta}$ is denoted by $\overline{\Theta}_x$ and we will work with the process $\big((\rho, \overline{W}), (\mathbb{P}_{\mu,\overline{\mathrm{w}}} : (\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x)\big)$. Conditions (i) and (ii) are natural, since as a particular consequence of the next lemma, under $\mathbb{P}_{0,y,r_0}$ and $\mathbb{N}_{y,r_0}$ the Lévy snake $(\rho, \overline{W})$ takes values in $\overline{\Theta}_x$.

Lemma 5.13. *For every $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ and $(y, r_0) \in \overline{E}$, the process $(\rho, \overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r_0}$ takes values in $\overline{\Theta}_x$.*

Proof. First, we argue that $\mathbb{N}_{y,r_0}$–a.e. the pair $(\rho_t, \overline{W}_t)$ satisfies (i) and (ii) for each $t \in [0, \sigma]$. On the one hand, by formula (5.24), for every $(y, r_0) \in \overline{E}$ we have :
$$\mathbb{N}_{y,r_0}\left(\int_0^{\sigma_H} \mathrm{d}t\, \langle \rho_t, \{h \in [0, H_t] :\ W_t(h) = x\}\rangle\right) = \int_0^{\infty} \mathrm{d}a\ \exp(-\alpha a)\, E^0 \otimes \Pi_{y,r_0}\Big[\int_0^a J_a(\mathrm{d}h)\, \mathbb{1}_{\{\xi_h = x\}}\Big]$$
which vanishes. In the last claim we used that, by $(\mathbf{H_3})$ and the independence between ξ and J_∞, Campbell's formula yields $E^0 \otimes \Pi_{y,r_0}\big[\int_0^\infty J_\infty(\mathrm{d}h)\, \mathbb{1}_{\{\xi_h=x\}}\ \big] = 0$. On the other hand, by construction of the Lévy snake, for each fixed $t \geqslant 0$ the support of $\Lambda_t(\mathrm{d}h)$ is the closure of $\{h \in [0, H_t) : W_t(h) = x\}$ in $[0, H_t]$, $\mathbb{N}_{y,r_0}$–a.e.

Consequently, $\mathbb{N}_{y,r_0}$–a.e. , we can find a countable dense set $\mathcal{D} \subset [0, \sigma]$ such that we have
$$\langle \rho_t, \{h \in [0, H_t] : W_t(h) = x\}\rangle = 0 \text{ and supp } \Lambda_t(\mathrm{d}h) \text{ is the closure of } \{h \in [0, H_t) : W_t(h) = x\}$$
for every $t \in \mathcal{D}$. For instance, one can construct the set $\mathcal{D}$ by taking an infinite sequence of independent uniform points in $[0, \sigma]$. We now claim that ρ satisfies that $\mathbb{N}_{y,r_0}$–a.e., for every $s < t$, we have $\rho_s \mathbb{1}_{[0,m_H(s,t))} = \rho_t \mathbb{1}_{[0,m_H(s,t))}$, where we recall the notation $m_H(s,t) = \min_{[s,t]} H$. Indeed, remark that for fixed $s < t$, this holds by the Markov property and we can extend this property to all $0 \leqslant s < t \leqslant \sigma$ since ρ is right-continuous with respect to the total variation distance. Now, by the snake property we deduce that $\mathbb{N}_{y,r_0}$–a.e, for every $t \in [0, \sigma]$, we have
$$\langle \rho_t, \{h \in [0, H_t) :\ W_t(h) = x\}\rangle = 0 \quad \text{and} \quad \{h \in [0, H_t) : W_t(h) = x\} = \text{supp } \Lambda_t(\mathrm{d}h) \cap [0, H_t). \tag{5.50}$$
Taking the closure in the second equality we deduce that the closure of $\{h \in [0, H_t) : W_t(h) = x\}$ is exactly supp $\Lambda_t(\mathrm{d}h)$. However, to conclude that $\mathbb{N}_{y,r_0}$-a.e.
$$\langle \rho_t, \{h \in [0, H_t] :\ W_t(h) = x\}\rangle = 0, \qquad \text{for all } t \in [0, \sigma], \tag{5.51}$$
we still need an additional step. Arguing by contradiction, suppose that for some $t > 0$ the quantity (5.51) is non-null. Then, by (5.50) we must have $\rho_t(\{H_t\}) > 0$ and $W_t(H_t) = x$. By right-continuity of ρ with respect to the total variation metric, we get
$$\lim_{\varepsilon \to 0} |\rho_t(\{H_t\}) - \rho_{t+\varepsilon}(\{H_t\})| = 0,$$

and we deduce that for ε small enough, $\rho_u(\{H_t\}) > 0$ for all $u \in [t, t+\varepsilon)$; in particular $H(\rho_u) \geqslant H(\rho_t)$ for all $u \in [t, t+\varepsilon)$. Since $W_t(H_t) = x$, the snake property ensures that $W_u(H_t) = x$ for all $u \in [t, t+\varepsilon)$ and, since $\rho_u(\{H_t\}) > 0$ for every $u \in [t, t+\varepsilon)$, we obtain a contradiction with the fact that $\langle \rho_u, \{h \in [0, H_u] : W_u(h) = x\} \rangle = 0$ for every $u \in \mathcal{D}$.

Let us now deduce this result under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$. First, observe that the statement of the lemma follows directly under $\mathbb{P}_{0,y,r_0}$ by excursion theory. Next, fix $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ with $\overline{\mathrm{w}}(0) = (y, r_0)$, consider $(\rho, \overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and set $T_0^+ := \inf\{t \geqslant 0 : \langle \rho_t, 1 \rangle = 0\}$. The strong Markov property gives us that $((\rho_{T_0^+ + t}, \overline{W}_{T_0^+ + t}) : t \geqslant 0)$ is distributed according to $\mathbb{P}_{0,y,r_0}$ and consequently, $(\rho_{T_0^+ + t}, \overline{W}_{T_0^+ + t})_{t \geqslant 0}$ takes values in $\overline{\Theta}_x$. To conclude, it remains to prove the statement of the lemma under $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$. In this direction, under $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$, consider $\big((\alpha_i, \beta_i) : i \geqslant 0\big)$ the excursion intervals of $\langle \rho, 1 \rangle$ from its running infimum. Then, write $(\rho^i, \overline{W}^i)$ for the subtrajectories of $(\rho, \overline{W})$ associated with $[\alpha_i, \beta_i]$ and set $h_i := H_{\alpha_i}$. We recall from (5.22) that the measure:

$$\sum_{i \in \mathbb{N}} \delta_{(h_i, \rho^i, \overline{W}^i)},$$

is a Poisson point measure with intensity $\mu(\mathrm{d}h)\, \mathbb{N}_{\overline{\mathrm{w}}(h)}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. Since $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, it follows by the result under the excursion measures $(\mathbb{N}_{y,r_0} : (y, r_0) \in \overline{E})$ that $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$–a.s. the pair $(\rho_t, \overline{W}_t)$ belongs to $\overline{\Theta}_x$ for every $t \in [0, T_0^+]$, as wanted. $\square$

Finally, recall that the snake property ensures that the function $(\widehat{W}_s, \widehat{\Lambda}_s)_{s \geqslant 0}$ is well defined in the quotient space $\mathcal{T}_H$. Hence, we can think of $\overline{W}$ as a tree-indexed process, that we write with the usual abuse of notation as

$$(\xi_v, \mathcal{L}_v)_{v \in \mathcal{T}_H}.$$

Main results of Section 4: Now that we have introduced our framework, we can state the main results of this section. Much of our effort is devoted to the construction of a measure supported on the set $\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$ and satisfying suitable properties. In this direction, for every $r \geqslant 0$, we set $\tau_r(\overline{\mathrm{w}}) := \inf\{h \geqslant 0 : \overline{\mathrm{w}}(h) = (x, r)\}$ and remark that, for every $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, it holds that $\mu(\{\tau_r(\overline{\mathrm{w}})\}) = 0$, with the convention $\mu(\infty) = 0$. We can now state the main result of this section:

Theorem 5.14. *Fix $(y, r_0) \in \overline{E}$ and $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$. The convergence*

$$A_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathrm{d}u \int_{\mathbb{R}_+} \mathrm{d}r\, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}}, \tag{5.52}$$

holds uniformly in compact intervals in measure under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r_0}(\,\cdot \cap \{\sigma > z\})$ for every $z > 0$. Moreover, (5.52) *defines a continuous additive functional $A = (A_t)$ for the Lévy snake $(\rho, \overline{W})$ whose Lebesgue-Stieltjes measure* $\mathrm{d}A$ *is supported on $\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$.*

We will give another equivalent construction of the additive functional A in Proposition 5.22 but we are not yet in position to formulate the precise statement. Both constructions will be needed for our work. Next, the second main result of the section characterizes the support of the measure $\mathrm{d}A$ as follows:

Theorem 5.15. *Fix $(y, r_0) \in \overline{E}$, $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ and denote the support of the Stieltjets measure of A by* supp $\mathrm{d}A$. *Under $\mathbb{N}_{y,r_0}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, we have:*

$$\text{supp } \mathrm{d}A = \overline{\big\{t \in [0, \sigma] : \xi_{p_H(t)} = x \text{ and } p_H(t) \in \mathrm{Multi}_2(\mathcal{T}_H) \cup \{0\}\big\}}. \tag{5.53}$$

Observe that under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ with $\mathrm{w}(0) = x$, the root of $\mathcal{T}_H$ has infinite multiplicity and this is why we had to consider it separately in the previous display. This result is stated in a slightly different but equivalent form in Theorem 5.30. The identity (5.53) can be also formulated in terms of constancy intervals of $\widehat{\Lambda}$. More precisely, we will also establish in Theorem 5.30 that under $\mathbb{N}_{y,r_0}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, we have:

$$\text{supp}\ \mathrm{d}A = [0,\sigma]\backslash\Big\{t \in [0,\sigma] : \sup_{(t-\varepsilon,t+\varepsilon)} \widehat{\Lambda}_s = \inf_{(t-\varepsilon,t+\varepsilon)} \widehat{\Lambda}_s, \quad \text{for some } \varepsilon > 0\Big\}. \tag{5.54}$$

We conclude the presentation of our framework with a consequence of Lemma 5.13. Roughly speaking it states that, with the exception of the root under $\mathbb{P}_{0,x,0}$, the process $(\xi_v)_{v\in\mathcal{T}_H}$ can not take the value x at the branching points of $\mathcal{T}_H$. The precise statement is the following:

Proposition 5.16. *For every $(y, r_0) \in \overline{E}$ and $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, we have:*

$$\{t \in [0,\sigma] : \widehat{W}_t = x\} \cap \{t \in [0,\sigma] : p_H(t) \in \mathrm{Multi}_3(\mathcal{T}_H) \cup \mathrm{Multi}_\infty(\mathcal{T}_H),\ p_H(t) \neq 0\} = \varnothing,$$

under $\mathbb{N}_{y,r_0}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$.

Proof. We start by proving our result under $\mathbb{N}_{y,0}$. First, introduce the measure $\mathbb{N}^\bullet_{y,0}(\mathrm{d}\rho, \mathrm{d}\overline{W}, \mathrm{d}s)$ supported on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}}) \times \mathbb{R}_+$ defined by $\mathbb{N}^\bullet_{y,0} = \mathbb{N}_{y,0}(\mathrm{d}\rho, \mathrm{d}\overline{W})\,\mathrm{d}s \mathbb{1}_{\{s\leqslant\sigma\}}$ and write $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$ for the identity function $U(s) = s$. The law under $\mathbb{N}^\bullet_{y,0}$ of $(\rho, \overline{W}, U)$ is therefore given by

$$\mathbb{N}^\bullet_{y,0}\Big(\Phi(\rho, \overline{W}, U)\Big) = \mathbb{N}_{y,0}\Big(\int_0^\sigma \mathrm{d}s\, \Phi(\rho, \overline{W}, s)\Big).$$

The measure $\mathbb{N}^\bullet_{y,0}$ can be seen as a pointed version of $\mathbb{N}_{y,0}$. In particular, conditionally on $(\rho, \overline{W})$, the random variable U is a uniform point in $[0,\sigma]$. Under $\mathbb{N}^\bullet_{y,0}$ we still write $X_t := \langle \rho_t, 1\rangle$ and $H_t := H(\rho_t)$. Furthermore, we set $X^\bullet_t := X_{U+t} - X_U$ and $I^\bullet_t := \inf_{s\leqslant t} X^\bullet$, for every $t \geqslant 0$, and we denote the excursion intervals over the running infimum of $X^\bullet$ by $\big((\alpha_i, \beta_i) : i \in \mathbb{N}\big)$. The dependence on U is dropped to simplify notation. Finally, set

$$h^\bullet_i := H\big(\kappa_{-I^\bullet_{\alpha_i}} \rho_U\big),$$

and write $(\rho^{\bullet,i}, \overline{W}^{\bullet,i})$ for the corresponding subtrajectory associated with (α_i, β_i) occurring at height $h^\bullet_i$. Under $\mathbb{N}^\bullet_{y,0}$, the Markov property applied at time U and (5.22) gives that, conditionally on (ρ_U, W_U), the random measure

$$\mathcal{M}^\bullet := \sum_{i\in\mathbb{N}} \delta_{(h^\bullet_i, \rho^{\bullet,i}, \overline{W}^{\bullet,i})},$$

is a Poisson point measure with intensity $\rho_U(\mathrm{d}h)\, \mathbb{N}_{\overline{W}_U(h)}(\mathrm{d}\rho, \mathrm{d}W)$. In particular, the functional

$$F(\mathcal{M}^\bullet) = \#\Big\{(h^\bullet_i, \rho^{\bullet,i}, \overline{W}^{\bullet,i}) \in \mathcal{M}^\bullet : W^{\bullet,i}(0) = x\Big\},$$

conditionally on (ρ_U, W_U), is a Poisson random variable with parameter $\int \rho_U(dh) \mathbb{1}_{\{W_U(h)=x\}}$. However, by Lemma 5.13, we have $\int \rho_U(dh) \mathbb{1}_{\{W_U(h)=x\}} = 0$ and we derive that, $\mathbb{N}^\bullet_{y,0}$ –a.e., $F(\mathcal{M}^\bullet)$ is null. Heuristically, the previous argument shows that if we take – conditionally on σ – a point uniformly at random in $\mathcal{T}_H$, there is no branching point υ with label x on the right of the branch connecting the root to υ. Let us now show that this ensures that

$$\big\{t \in [0,\sigma] : \widehat{W}_t = x\big\} \cap \big\{t \in [0,\sigma] : p_H(t) \in \text{Multi}_3(\mathcal{T}_H) \cup \text{Multi}_\infty(\mathcal{T}_H)\big\} = \varnothing, \quad \mathbb{N}_{y,0}\text{–a.e.}$$

Since $\mathbb{N}^\bullet_{y,0}(\Phi(\rho,\overline{W})) = \mathbb{N}_{y,0}(\sigma \cdot \Phi(\rho,\overline{W}))$, it suffices to prove the previous display under $\mathbb{N}^\bullet_{y,0}$. Let $(\upsilon_i : i \in \mathbb{N})$ be an indexation of the branching points of $\mathcal{T}_H$ – an indexation measurable with respect to H. Pick a branching point $\upsilon_i \in \text{Multi}_3(\mathcal{T}_H) \cup \text{Multi}_\infty(\mathcal{T}_H)$ and let t_i be the smallest element of $p_H^{-1}(\upsilon_i)$. Arguing by contradiction, suppose that $\widehat{W}_{p_H(t_i)} = x$. Still under $\mathbb{N}^\bullet_{y,0}$, since υ_i is a branching point, we can find $0 \leqslant s_* < t_* \leqslant \sigma$ in $p_H^{-1}(\{\upsilon_i\})$ such that the event

$$\{\widehat{W}_{p_H(t_i)} = x\} \cap \{s_* < U < t_*\},$$

is included in $\{F(\mathcal{M}^U) > 0\}$. However $F(\mathcal{M}^U) = 0$, $\mathbb{N}^\bullet_{y,0}$–a.e. and we deduce that $\mathbb{N}^\bullet_{y,0}\big(\widehat{W}_{p_H(t_i)} = x, s_* < U < t_*\big) = 0$. Next, since conditionally on $(\rho,\overline{W})$, the variable U is uniformly distributed on $[0,\sigma]$, we conclude that $\mathbb{N}^\bullet_{y,0}\big(\widehat{W}_{p_H(t_i)} = x\big) = 0$. The desired result now follows, since the collection of branching points $(\upsilon_i : i \in \mathbb{N})$ is countable. Finally, we deduce the statement under $\mathbb{N}_{y,r_0}$ by the translation invariance of the local time and under $\mathbb{P}_{\mu,\overline{\text{w}}}$ by excursion theory – we omit the details since this is standard and one can apply the method described in Lemma 5.13. □

The section is organised as follows: In Section 5.4.1 we address several preliminary results needed to prove Theorems 5.14 and 5.15 and our results of Section 5.5. More precisely, Section 5.4.1 is essentially devoted to the study of a family of exit local times of $(\rho,\overline{W})$ that will be used as building block for our second construction of A. Then in Section 5.4.2 we shall prove Theorem 5.14 and establish our second construction of A in terms of the family of exit times studied in Section 5.4.1. The rest of the section is dedicated to the study of basic properties of A that we will frequently use in our computations. Finally, in Section 5.4.3 we turn our attention to the study of the support of the measure $\text{d}A$ and it will lead us to the proof of Theorem 5.15 and the characterisation (5.54).

5.4.1 Special Markov property of the local time

The first step towards constructing our additive functional A, with associated Lebesgue-Stieltjes measure $\text{d}A$ supported in $\{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$, consists in the study of a particular family of $[0,\infty)$-indexed exit local times of $(\rho,\overline{W})$. More precisely, for each $r \geqslant 0$, let $D_r \subset \overline{E} := E \times \mathbb{R}_+$ be the open domain

$$D_r := \overline{E} \backslash \{(x,r)\} \quad \text{and recall the notation} \quad \tau_r(\overline{\text{w}}) := \inf\{h \geqslant 0 : \ \overline{\text{w}}(h) = (x,r)\},$$

for every $\overline{\text{w}} \in \mathcal{W}_{\overline{E}}$. Notice that $\tau_r(\overline{\text{w}})$ is the exit time from D_r and we write τ_r instead of making use of the more cumbersome notation τ_{D_r}. We also recall that since $\tau_r(\overline{\text{w}}) \in \{h \in [0,\zeta_{\text{w}}] : \text{w}(h) = x\}$ as soon as $\tau_r(\overline{\text{w}}) < \infty$, for $(\mu,\overline{\text{w}}) \in \overline{\Theta}_x$ we have $\mu(\{\tau_r(\overline{\text{w}})\}) = 0$. Proposition 5.6 now yields that for any $(\mu,\overline{\text{w}}) \in \overline{\Theta}_x$ with $\overline{\text{w}}(0) \neq (x,r)$ we have

$$L^{D_r}_t(\rho,\overline{W}) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \text{d}s\, \mathbb{1}_{\{\tau_r(\overline{W}_s) < H_s < \tau_r(\overline{W}_s)+\varepsilon\}}, \tag{5.55}$$

where the convergence holds uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\overline{\text{w}}})$ and $L^1(\mathbb{N}_{\overline{\text{w}}(0)})$. Let us be more precise: recalling the notation $\overline{\text{w}} = (\text{w},\ell)$, first remark that if $\ell(0) < r$, for any $\text{w}(0) \in E$ we have $\Pi_{\mathrm{w}(0),\ell(0)}(\tau_r < \infty) = 1$ and in consequence ($\mathbf{H_1}$) holds. On the other hand, if $r < \ell(0)$, we simply have $L^{D_r} = 0$ since $\tau_r(\overline{W}_s) = \infty$ for every $s \geqslant 0$. Finally, if $\mathrm{w}(0) \neq x$ and $r \geqslant 0$, we have $\tau_{D_r}(\overline{W}_s) = \tau_{E_*}(W_s)$ for every $s \geqslant 0$, and recalling (5.49) it follows that $L^{D_r}(\rho, \overline{W}) = L^{E_*}(\rho, W)$. It will be usefully for our purposes to extend the definition to the remaining case $\overline{\mathrm{w}}(0) = (x, r)$, and that is precisely the content of the following lemma:

Lemma 5.17. *For $r \geqslant 0$, fix $(\mu, \overline{\mathrm{w}}) = (\mu, \mathrm{w}, \ell) \in \overline{\Theta}_x$ with $\overline{\mathrm{w}}(0) = (x, r)$. Then, the limit* (5.55) *exists for every $t \geqslant 0$, where the convergence holds uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\overline{\mathrm{w}}})$ and $L^1(\mathbb{N}_{x,r})$, and defines a continuous non-decreasing process that we still denote by L^{D_r}. Moreover, under $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{x,r}$, we have $L^{D_r}_\sigma = 0$.*

Proof. We work under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ since the result under $\mathbb{N}_{x,r}$ follows directly by excursion theory. For every $a \geqslant 0$ we set $T_a := \inf\{t \geqslant 0 : \ H_t = a\}$ and let $T_0^+ := \inf\{t \geqslant 0 : \ \langle \rho_t, 1\rangle = 0\}$. Since $\tau_r(\overline{W}_s) = 0$ for every $s \geqslant 0$, we have

$$\int_0^t \mathrm{d}s\, \mathbb{1}_{\{\tau_r(\overline{W}_s) < H_s < \tau_r(\overline{W}_s)+\varepsilon\}} = \int_0^t \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}} = \int_{T_\varepsilon \wedge t}^{T_0^+\wedge t} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}} + \int_{T_0^+\wedge t}^{T_0^+\wedge t} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}}.$$

Furthermore, by the strong Markov property and (5.9), we already know that $\varepsilon^{-1}\int_{T_0^+}^{T_0^++t} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}}$ converges as $\varepsilon \downarrow 0$ uniformly in compact intervals in $L^1(\mathbb{P}_{\mu,\mathrm{w}})$. To conclude, it suffices to show that:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \cdot \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\int_{T_\varepsilon}^{T_0^+} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}}\Big] = 0. \tag{5.56}$$

Write (α_i, β_i) for $i \in \mathbb{N}$ the excursion intervals of the killed process $(\langle \rho_t, 1\rangle : \ t \in [0, T_0^+])$ over its running infimum and let $(\rho^i, \overline{W}^i)$ be the subtrajectory associated with the excursion interval $[\alpha_i, \beta_i]$. To simplify notation, we also set $h_i = H(\alpha_i)$ and recall from (5.22) that the measure $\mathcal{M} := \sum_{i\in\mathbb{N}} \delta_{(h_i, \rho^i, \overline{W}^i)}$ is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\overline{\mathrm{w}}(h)}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. Next, notice that:

$$\int_{T_\varepsilon}^{T_0^+} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}} \leqslant \sum_{0\leqslant h_i \leqslant \varepsilon} \int_0^{\sigma(\overline{W}^i)} \mathrm{d}s\, \mathbb{1}_{\{0<H(\rho^i_s)<\varepsilon\}},$$

and we can now use that $\mathcal{M}$ is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\overline{\mathrm{w}}(h)}(\mathrm{d}\rho, \mathrm{d}\overline{W})$ to get that:

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\int_{T_\varepsilon}^{T_0^+} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}}\Big] \leqslant \mu([0,\varepsilon]) N\Big(\int_0^\sigma \mathrm{d}s \mathbb{1}_{\{0<H(\rho_s)<\varepsilon\}}\Big).$$

Finally, by the many-to-one formula (5.24), the previous display is $\varepsilon \cdot \mu([0,\varepsilon])$, which gives:

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} \cdot \mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\Big[\int_{T_\varepsilon}^{T_0^+} \mathrm{d}s\, \mathbb{1}_{\{0<H_s<\varepsilon\}}\Big] = \mu(\{0\}).$$

Now (5.56) follows since we have $\mu(\{0\}) = 0$, which holds since $\mathrm{w}(0) = x$ and $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$. □

Now, we give a regularity result for the double-indexed family $(L^{D_r}_s : (s, r) \in \mathbb{R}^2_+)$ that will be needed in the next section.

Lemma 5.18. *Let $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ with $\overline{\mathrm{w}} = (\mathrm{w}, \ell)$. There exists a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable function $(\mathscr{L}_t^r : (r,t) \in \mathbb{R}_+^2)$ satisfying the following properties:*

(i) *For every $r \geqslant 0$, the processes L^{D_r} and $\mathscr{L}^r$ are indistinguishables under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$.*

(ii) *$\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ almost surely, the mapping $t \mapsto \mathscr{L}_t^r$ is continuous for every $r \geqslant 0$.*

The result also holds under the measure $\mathbb{N}_{y,r_0}$, for every $(y, r_0) \in \overline{E}$, by the same type of arguments and we omit the details.

Proof. Fix an initial condition $(\mu, \overline{\mathrm{w}}) = (\mu, \mathrm{w}, \ell) \in \overline{\Theta}_x$. Since under $\mathbb{P}_{\mu,\mathrm{w},\ell}$, the distribution of $(\rho, W, \Lambda - \ell(0))$ is exactly $\mathbb{P}_{\mu,\mathrm{w},\ell-\ell(0)}$, without loss of generality we might assume that $\ell(0) = 0$. Next, by (5.55) and Lemma 5.17, for every $r \geqslant 0$ we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\sup_{s \leqslant t} |L_s^{D_r} - \varepsilon^{-1} \int_0^s \mathrm{d}u \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}}|\Big] = 0,$$

and hence for any subsequence (ε_n) converging to 0, the sequence of processes

$$Y_n(r,t) := \varepsilon_n^{-1} \int_0^t \mathrm{d}u \; \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon_n\}}, \qquad t \geqslant 0,$$

converges uniformly in compact intervals in probability towards L^{D_r}. Now, to simplify notation write $\boldsymbol{\omega} := (\mathsf{\rho}, \omega)$ for the elements of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$. Remark now that the mapping $(u, r, \boldsymbol{\omega}) \mapsto \tau_r(\overline{W}_u(\omega))$ is jointly measurable since for each $(u, \boldsymbol{\omega})$ it is rcll in r, while for each fixed r the mapping $(u, \boldsymbol{\omega}) \mapsto \tau_r(\overline{W}_u(\omega))$ is measurable. Consequently, by Fubini, for every fixed t the application

$$(r, \boldsymbol{\omega}) \mapsto \int_0^t \mathrm{d}u \; \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon_n\}}(\omega)$$

is measurable while for fixed $(r, \boldsymbol{\omega})$ it is continuous in t, and we deduce that Y_n is jointly measurable in $(r, t, \boldsymbol{\omega})$. It is now standard – see e.g. [80, Theorem 62] and its proof – to deduce that there exits a jointly measurable process $(r, t, \boldsymbol{\omega}) \mapsto Y(r, t, \boldsymbol{\omega})$ such that for every $(r, \boldsymbol{\omega}) \in \mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$, the mapping $t \mapsto Y(r, t, \boldsymbol{\omega})$ is continuous and for each fixed $r \geqslant 0$, $Y_n(r, \cdot) \mapsto Y(r, \cdot)$ as $n \uparrow \infty$ uniformly in compact intervals in probability. In particular for each $r \geqslant 0$, the process $(Y(r,t) : t \geqslant 0)$ is indistinguishable from $(L_t^{D_r} : \; t \geqslant 0)$ and we shall write $(\mathscr{L}_t^r : \; t \geqslant 0, \, r \geqslant 0)$ instead of $(Y(r,t) : \, t \geqslant 0, \, r \geqslant 0)$. □

We now turn our attention to the Markovian properties of $(\mathscr{L}_\sigma^r : r \geqslant 0)$ under the excursion measure $\mathbb{N}_{x,0}$. To simplify notation, for every $y \neq x$, we set:

$$u_\lambda(y) := \mathbb{N}_{y,0}\left(1 - \exp(-\lambda \mathscr{L}_\sigma^0)\right), \qquad \text{for } y \in E_*, \tag{5.57}$$

and remark that with the notation of (5.46) we have $u_\lambda(y) = u_\lambda^{E_*}(y)$. We shall use the usual convention $u_\lambda(x) = \lambda$.

Before stating our next result, we briefly recall from [64, Chapter II-1] that an $\mathbb{R}_+$–valued Markov process with semigroup $(P_t(y, \mathrm{d}z) : \; t, y \in \mathbb{R}_+)$ is called a branching process if its semigroup satisfies the branching property, viz. if for any $y, y' \in \mathbb{R}_+$, we have $P_t(y, \cdot) * P_t(y', \cdot) = P_t(y + y', \cdot)$. In order to fall in the framework of [64, Chapter II- Theorem 1] we also assume that $\int_{\mathbb{R}_+} P_t(y, \mathrm{d}z) z \leqslant y$ for every $t, y \in \mathbb{R}_+$. By the branching property it follows that for any

$t, y \in \mathbb{R}_+$ the distribution $P_t(y, \mathrm{d}z)$ is infinitely divisible and non-negative, and consequently, for every $t, y \in \mathbb{R}_+$, the Laplace transform of $P_t(y, \mathrm{d}z)$ takes the Lévy-Khintchine form:

$$\int_{\mathbb{R}_+} P_t(y, \mathrm{d}z) \exp(-\lambda z) = \exp\big(- y a_t(\lambda)\big), \qquad \text{for } \lambda \geqslant 0,$$

for some function $(a_t(\lambda) : t, \lambda \geqslant 0)$. By [64, Chapter II- Theorem 1], the function $(a_t(\lambda) : t, \lambda \geqslant 0)$ is the unique non-negative solution of the integral equation

$$a_t(\lambda) + \int_0^t \mathrm{d}u \, \Psi(a_u(\lambda)) = \lambda, \tag{5.58}$$

for a function $(\Psi(\lambda) : \lambda \geqslant 0)$ of the form,

$$\Psi(\lambda) = c_1\lambda + c_2\lambda^2 + \int_{\mathbb{R}_+} \nu(\mathrm{d}y) \,(\exp(-\lambda y) - 1 + \lambda y), \qquad \text{for } \lambda \geqslant 0,$$

where $c_1, c_2 \in \mathbb{R}_+$ and ν is a Lévy measure supported on $(0, \infty)$ satisfying $\int_{(0,\infty)} \nu(\mathrm{d}y)(y \wedge y^2) < \infty$. By (5.58), it follows that $a_t(\lambda)$ is the unique function that satisfies

$$\int_{a_t(\lambda)}^{\lambda} \frac{\mathrm{d}s}{\Psi(s)} = t, \qquad t, \lambda \geqslant 0.$$

The Markov process with semigroup (P_t) is then called a CSBP with branching mechanism Ψ, or in short a Ψ-CSBP. The exponent Ψ clearly fulfils (A1) and so does (A3) by [17, Corollary 2]. So to fall in our framework, it only need to satisfy (A4) – since as explained in the preliminaries (A4) implies (A2).

Proposition 5.19. *Under $\mathbb{N}_{x,0}$, the process $(\mathscr{L}_\sigma^r : r > 0)$ is a branching process with entrance measure $\nu_r(\mathrm{d}x) = \mathbb{N}_{x,0}(\mathscr{L}_\sigma^r \in \mathrm{d}x)$, for $r > 0$, and branching mechanism*

$$\widetilde{\psi}(\lambda) = \mathcal{N}\Big(\int_0^\sigma \mathrm{d}h \ \psi\big(u_\lambda(\xi_h)\big)\Big), \qquad \text{for } \lambda \geqslant 0. \tag{5.59}$$

Moreover, $\widetilde{\psi}$ satisfies the assumptions (A1) – (A4) *introduced in Section 5.2.1 and consequently we can associate to it a Lévy tree.*

Our result is a particular case, in the terminology of Lévy snakes, of Theorem 4 in [23] stated in the setting of superprocesses. Theorem 4 in [23] is more general and the family $(\mathscr{L}_\sigma^r)_{r>0}$ in our result correspond precisely to the total mass process of the superprocess considered in [23], for the same branching mechanism $\widetilde{\psi}$.

Proof. The proof is structured as follows: we start by introducing a family of probability kernels (P_t) and by showing that they form a semigroup of operators associated with a branching process. We then establish that $(\mathscr{L}_\sigma^r : r > 0)$ is a Markov process associated with the semigroup (P_t), with entrance measure $(\nu_r : r > 0)$. Finally, we conclude the proof by establishing that its branching mechanism is $\widetilde{\psi}$ and that it fulfils (A4).

We stress that we are only interested in the finite-dimensional distributions of $(\mathscr{L}_\sigma^r : r > 0)$. Recalling the notation (5.46), for any $r > 0$ and $\lambda \geqslant 0$, we write

$$u_\lambda^{D_r}(x, 0) = \mathbb{N}_{x,0}\big(1 - \exp(-\lambda \mathscr{L}_\sigma^r)\big) = \int \nu_r(\mathrm{d}y) \,(1 - \exp(-\lambda y)).$$

Moreover, since $\mathbb{N}_{x,0}(\mathscr{L}_\sigma^r > 0) \leqslant \mathbb{N}_{x,0}(\sup \widehat{\Lambda} \geqslant r) < \infty$, we have $\int_{(0,\infty)} \nu_r(\mathrm{d}y)(1 \wedge y) < \infty$, and we deduce that the function $\lambda \mapsto u_\lambda^{D_r}(x,0)$ is the Laplace exponent of an infinitely divisible random variable with Lévy measure $\nu_r(\cdot \cap (0,\infty))$. For each $t > 0$ and $y \in \mathbb{R}_+$ denote by $P_t(y, \mathrm{d}z)$ the probability measure with Laplace transform

$$\int P_t(y, \mathrm{d}z)\, \exp(-\lambda z) = \exp\big(-y \cdot u_\lambda^{D_t}(x,0)\big), \qquad \lambda \geqslant 0. \tag{5.60}$$

Remark now that the translation invariance of the local time of ξ implies that, under $\mathbb{P}_{0,x,r}$ (resp. $\mathbb{N}_{x,r}$) for $r \geqslant 0$, the distribution of $(W, \Lambda - r)$ is $\mathbb{P}_{0,x,0}$ (resp. $\mathbb{N}_{x,0}$). In particular, for every $s,t \geqslant 0$, we have

$$u_\lambda^{D_{t+s}}(x,s) = u_\lambda^{D_t}(x,0).$$

We deduce that the family $(P_t(y,\mathrm{d}z), t > 0, y \in \mathbb{R}_+)$ is a semigroup since, by the special Markov property applied at the domain D_s, it holds that

$$\begin{aligned}
\int P_{t+s}(y,\mathrm{d}z)\exp(-\lambda z) &= \exp\Big(-y\,\mathbb{N}_{x,0}\Big(1-\exp\big(-\lambda\mathscr{L}_\sigma^{t+s}\big)\Big)\Big)\\
&= \exp\Big(-y\,\mathbb{N}_{x,0}\Big(1-\exp\big(-\mathscr{L}_\sigma^{s}\cdot u_\lambda^{D_{t+s}}(x,s)\big)\Big)\Big)\\
&= \exp\Big(-y\,\mathbb{N}_{x,0}\Big(1-\exp\big(-\mathscr{L}_\sigma^{s}\cdot u_\lambda^{D_t}(x,0)\big)\Big)\Big)\\
&= \exp\Big(-y\cdot u^{D_s}_{u_\lambda^{D_t}(x,0)}(x,0)\Big),
\end{aligned}$$

which coincides with the Laplace transform of the measure $\int_{u\in\mathbb{R}_+} P_s(y,\mathrm{d}u)P_t(u,\mathrm{d}z)$. Since we have $\mathbb{N}_{x,0}(\mathscr{L}_\sigma^r) \leqslant 1$ by (5.28) and $1-\exp(-\lambda\mathscr{L}_\sigma^r) \leqslant \lambda\mathscr{L}_\sigma^r$, we deduce by dominated convergence and (5.60) that $\int_{\mathbb{R}_+} P_t(y,\mathrm{d}z)\, z = y\cdot\mathbb{N}_{x,0}(\mathscr{L}_\sigma^r) \leqslant y$. Since the semigroup clearly fulfils the branching property, it follows that there exits a CSBP associated with the semigroup (P_t).

Recall the notation $T_{D_\varepsilon} := \inf\{t \geqslant 0 : \tau_\varepsilon(\overline{W}_t) < \infty\}$ as well as the definition of the sigma field $\mathcal{F}^{D_\varepsilon}$ from (5.26). We will now show that for any $\varepsilon > 0$, the process $(\mathscr{L}_\sigma^{\varepsilon+r} : r \geqslant 0)$ under the probability measure $\mathbb{N}_{x,0}^{D_\varepsilon} := \mathbb{N}_{x,0}(\,\cdot\,|T_{D_\varepsilon} < \infty)$ has transition kernel (P_t). Fix $\varepsilon < a < b$; by considering the point process of excursions (5.45) outside D_a, we deduce by an application of the special Markov property that $\mathbb{N}_{x,0}^{D_\varepsilon}$–a.e.

$$\mathbb{N}_{x,0}^{D_\varepsilon}\Big(\exp\Big(-\lambda\mathscr{L}_\sigma^{b}\Big)\,|\mathcal{F}^{D_a}\Big) = \exp\Big(-\mathscr{L}_\sigma^{a}\mathbb{N}_{x,a}\Big(1-\exp(-\lambda\mathscr{L}_\sigma^{b})\Big)\Big) = \exp\Big(-\mathscr{L}_\sigma^{a}\cdot u_\lambda^{D_{b-a}}(x,0)\Big)$$

where in the last equality we used the translation invariance of the local time of ξ. We have obtained that, for every $\varepsilon > 0$, $(\mathscr{L}_\sigma^{r+\varepsilon} :\ r \geqslant 0)$ under $\mathbb{N}_{x,0}^{D_\varepsilon}$ is a CSBP with Laplace functional $(u_\lambda^{D_r}(x,0) :\ r > 0)$ and initial distribution $\mathbb{N}_{x,0}^{D_\varepsilon}(\mathscr{L}_\sigma^{\varepsilon} \in \mathrm{d}x)$ with respect to the filtration $(\mathcal{F}^{D_{\varepsilon+r}} : r \geqslant 0)$ (recall that $\mathscr{L}_\sigma^r$ is $\mathcal{F}^{D_r}$-measurable by Proposition 5.7 and Lemma 5.18). Now, we claim that for any $0 < r_1 < \cdots < r_k$ and any collection of non-negative measurable functions $f_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$,

$$\mathbb{N}_{x,0}\left(\prod_{i=1}^{k} f_i(\mathscr{L}_\sigma^{r_i})\right) = \int_{\mathbb{R}_+}\nu_{r_1}(\mathrm{d}z_1)f_1(z_1)\int_{\mathbb{R}_+} P_{r_2-r_1}(z_1,\mathrm{d}z_2)f_2(z_2)\ldots\int_{\mathbb{R}_+} P_{r_k-r_{k-1}}(z_{k-1},\mathrm{d}z_k)f_k(z_k). \tag{5.61}$$

This follows from the previous result, by observing that for any $\varepsilon < r_1$ we have

$$\mathbb{N}_{x,0}\left(\prod_{i=1}^{k} f_i(\mathscr{L}_\sigma^{r_i})\mathbb{1}_{\{T_{D_\varepsilon}<\infty\}}\right)$$
$$= \mathbb{N}_{x,0}\left(\mathbb{1}_{\{T_{D_\varepsilon}<\infty\}} f_1(\mathscr{L}_\sigma^{r_1}) \int_{\mathbb{R}_+} P_{r_2-r_1}(\mathscr{L}_\sigma^{r_1}, \mathrm{d}z_2) f_2(z_2) \dots \int_{\mathbb{R}_+} P_{r_k-r_{k-1}}(z_{k-1}, \mathrm{d}z_k) f_k(z_k)\right),$$

and we conclude taking the limit as $\varepsilon \downarrow 0$. The fact that the family $(\nu_t : t \geqslant 0)$ satisfies that $\nu_{t+s} = \nu_s P_t$ for $t, s \geqslant 0$ now follows from (5.61). Let us now identify $\widetilde{\psi}$. Recall from our discussion in (5.58) that the Laplace exponent $(u_\lambda^{D_r}(x,0) : r, \lambda \geqslant 0)$ is the unique solution to the equation

$$u_\lambda^{D_r}(x,0) + \int_0^r \mathrm{d}u\, \Psi\big(u_\lambda^{D_u}(x,0)\big) = \lambda, \tag{5.62}$$

where Ψ is the branching mechanism associated with (P_t), and that it is defined in a unique way by (5.62). In particular, Ψ characterizes completely the semigroup (P_t). To identify the branching mechanism we argue as follows: first, observe that the identity (5.47) applied at the domain D_r yields

$$u_\lambda^{D_r}(x,0) + \Pi_{x,0}\left(\int_0^{\tau_{D_r}} \mathrm{d}t\ \psi(u_\lambda^{D_r}(\xi_t, \mathcal{L}_t))\right) = \lambda, \tag{5.63}$$

for every $\lambda \geqslant 0$ and $r > 0$. Next, by excursion theory and $(\mathbf{H_3})$ we get:

$$\Pi_{x,0}\left(\int_0^{\tau_{D_r}} \mathrm{d}t\ \psi(u_\lambda^{D_r}(\xi_t, \mathcal{L}_t))\right) = \int_0^r \mathrm{d}u\ \mathcal{N}\left(\int_0^\sigma \mathrm{d}t\ \psi\left(u_\lambda^{D_r}(\xi_t, u)\right)\right)$$
$$= \int_0^r \mathrm{d}u\ \mathcal{N}\left(\int_0^\sigma \mathrm{d}t\ \psi\left(u_\lambda^{D_{r-u}}(\xi_t, 0)\right)\right),$$

where in the last equality we use the invariance by translation of the local time of ξ. Moreover, the special Markov property applied at the domain D_0 gives

$$u_\lambda^{D_r}(y,0) = u_{u_\lambda^{D_r}(x,0)}(y),$$

for every $y \in E\backslash\{x\}$ and $\lambda \geqslant 0$ – and the identity also holds for $y = x$. Putting everything together, by definition of $\widetilde{\psi}$, the identity (5.63) can be re-written as follows:

$$u_\lambda^{D_r}(x,0) + \int_0^r \mathrm{d}u\, \widetilde{\psi}(u_\lambda^{D_u}(x,0)) = \lambda. \tag{5.64}$$

Consequently, we deduce that the branching mechanism associated with the Laplace functional $u_\lambda^{D_r}(x,0)$ is $\widetilde{\psi}$. It remains to show that the conditions stated in Section 5.2.1 are satisfied by $\widetilde{\psi}$. As we already mentioned, it only remains to verify (A4). In this direction and recalling the notation $T_{D_r} = \inf\{t \geqslant 0 : \widehat{\Lambda}_t \geqslant r\}$, also by (5.64) we obtain that $f(\lambda, r) := u_\lambda^{D_r}(x,0)$ satisfies for every r,

$$\int_{f(\lambda,r)}^{\lambda} \frac{\mathrm{d}s}{\widetilde{\psi}(s)} = r, \tag{5.65}$$

where the limit $f(\infty, r) = \mathbb{N}_{x,0}(L_\sigma^{D_r} > 0)$ is finite, since $\{L_\sigma^{D_r} > 0\} \subset \{T_{D_r} < \infty\}$ and $\mathbb{N}_{x,0}(T_{D_r} < \infty) < \infty$ by the same argument used before Theorem 5.11. Hence, taking the limit as $\lambda \uparrow \infty$ in (5.65), we infer that the following conditions are fulfilled:

$$\widetilde{\psi}(\infty) = \infty \qquad \text{and} \qquad \int_\cdot^\infty \frac{\mathrm{d}s}{\widetilde{\psi}(s)} < \infty.$$

To derive the exact form of (A4), recall that $\widetilde{\psi}$ is convex and that we have $\widetilde{\psi}(0) = 0$ and $\widetilde{\psi}'(0+) \geqslant 0$. $\square$

Now that we have established that $\widetilde{\psi}$ is the Laplace exponent of a Lévy tree, let us briefly introduce some related notation and a few facts that will be used frequently in the upcoming sections. From now on, we set $\widetilde{X}$ a $\widetilde{\psi}$-Lévy process and we write $\widetilde{I}$ for the running infimum of $\widetilde{X}$. We also denote the excursion measure of the reflected process $\widetilde{X} - \widetilde{I}$ by $\widetilde{N}$ – where the associated local time is $-\widetilde{I}$. The usual notation introduced in Section 5.2.1 applied to $\widetilde{X}$ are indicated with a $\sim$. For instance, we denote the height process and the exploration process issued from $\widetilde{X}$ respectively by $\widetilde{H}$ and $\widetilde{\rho}$.

By convexity and the fact that $\widetilde{\psi}'(0+) \geqslant 0$, the only solution to $\widetilde{\psi}(\lambda) = 0$ is $\lambda = 0$. This implies that the mapping $\lambda \mapsto \widetilde{\psi}(\lambda)$ is invertible in $[0,\infty)$. By classical results in the theory of Lévy processes, $\widetilde{\psi}^{-1}$ is the Laplace exponent of the right-inverse of $-\widetilde{I}$ and, since $\widetilde{X} - \widetilde{I}$ does not spend time at 0, the former is a subordinator with no drift. So, recalling the relation between excursion lengths and jumps of the right-inverse of $-\widetilde{I}$, we derive that:

$$\widetilde{\psi}^{-1}(\lambda) = \widetilde{N}(1 - \exp(-\lambda\sigma)), \quad \lambda \geqslant 0. \tag{5.66}$$

For a more detailed discussion, we refer to Chapters IV and VII of [17].

We close this section with some useful identities in the same vein of (5.59), that will be used frequently in our computations. These identities allow to express some Laplace-like transforms concerning the process $\big(\psi(u_\lambda(\xi_t)) :\ t \geqslant 0\big)$, under the excursion measure $\mathcal{N}$, in terms of the $\widetilde{\psi}$. As an application of these computations, we will identify the drift and Brownian coefficients of $\widetilde{\psi}$. We summarise these identities in the following lemma.

Lemma 5.20. *For every $\lambda_1, \lambda_2 \in \mathbb{R}_+$ with $\lambda_1 \neq \lambda_2$, we have*

$$\mathcal{N}\left(1 - \exp\Big(-\int_0^\sigma \mathrm{d}s\ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)}\Big)\right) = \frac{\widetilde{\psi}(\lambda_1) - \widetilde{\psi}(\lambda_2)}{\lambda_1 - \lambda_2}. \tag{5.67}$$

Recalling the identities (5.23), remark that Lemma 8 allows to express the Laplace exponent of $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ in terms of $\mathcal{N}$ and ψ.

Proof. First note that the functions $\lambda \mapsto u_\lambda(y)$ and $\lambda \mapsto \psi(u_\lambda(y))$ are non-decreasing. So without loss of generality we can and will assume that $\lambda_1 > \lambda_2$. We set $T_x := \inf\{t \geqslant 0 : \xi_t = x\}$ and we write

$$\begin{aligned}
&\mathcal{N}\Big(1 - \exp\Big(-\int_0^\sigma \mathrm{d}s\ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)}\Big)\Big) \\
&\quad = \mathcal{N}\left(\int_0^\sigma \mathrm{d}s\ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)} \cdot \exp\Big(-\int_s^\sigma \mathrm{d}t\ \frac{\psi\big(u_{\lambda_1}(\xi_t)\big) - \psi\big(u_{\lambda_2}(\xi_t)\big)}{u_{\lambda_1}(\xi_t) - u_{\lambda_2}(\xi_t)}\Big)\right) \\
&\quad = \mathcal{N}\left(\int_0^\sigma \mathrm{d}s\ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)} \cdot \Pi_{\xi_s}\left(\exp\Big(-\int_0^{T_x} \mathrm{d}t\ \frac{\psi\big(u_{\lambda_1}(\xi_t)\big) - \psi\big(u_{\lambda_2}(\xi_t)\big)}{u_{\lambda_1}(\xi_t) - u_{\lambda_2}(\xi_t)}\Big)\right)\right)
\end{aligned}$$

where in the last equality we applied the Markov property. On the other hand, the definition of

$\widetilde{\psi}$ given in (5.59) yields

$$\frac{\widetilde{\psi}(\lambda_1) - \widetilde{\psi}(\lambda_2)}{\lambda_1 - \lambda_2} = \mathcal{N}\bigg(\int_0^\sigma \mathrm{d}s \ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{\lambda_1 - \lambda_2} \bigg)$$
$$= \mathcal{N}\bigg(\int_0^\sigma \mathrm{d}s \ \frac{\psi\big(u_{\lambda_1}(\xi_s)\big) - \psi\big(u_{\lambda_2}(\xi_s)\big)}{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)} \cdot \frac{u_{\lambda_1}(\xi_s) - u_{\lambda_2}(\xi_s)}{\lambda_1 - \lambda_2} \bigg).$$

Consequently, the lemma will follow as soon as we establish the identity:

$$\frac{u_{\lambda_1}(y) - u_{\lambda_2}(y)}{\lambda_1 - \lambda_2} = \Pi_y\Big(\exp\Big(- \int_0^{T_x} \mathrm{d}t \ \frac{\psi\big(u_{\lambda_1}(\xi_t)\big) - \psi\big(u_{\lambda_2}(\xi_t)\big)}{u_{\lambda_1}(\xi_t) - u_{\lambda_2}(\xi_t)} \Big)\Big).$$

In this direction, recall that under $\mathbb{N}_{y,0}$ with $y \neq x$ the processes $\mathscr{L}^0(\rho, \overline{W})$ and $L^{E_*}(\rho, W)$ are well defined and indistinguishables, and remark that

$$u_{\lambda_1}(y) - u_{\lambda_2}(y) = \mathbb{N}_{y,0}\Big(\exp\big(- \lambda_1 \int_0^\sigma \mathrm{d}\mathscr{L}_u^0 \big) - \exp\big(- \lambda_2 \int_0^\sigma \mathrm{d}\mathscr{L}_u^0 \big) \Big)$$
$$= (\lambda_1 - \lambda_2) \cdot \mathbb{N}_{y,0}\Big(\int_0^\sigma \mathrm{d}\mathscr{L}_s^0 \exp\big(- \lambda_1 \int_0^s \mathrm{d}\mathscr{L}_u^0 \big) \cdot \exp\big(- \lambda_2 \int_s^\sigma \mathrm{d}\mathscr{L}_u^0 \big) \Big).$$

Then, an application of the Markov property gives:

$$u_{\lambda_1}(y) - u_{\lambda_2}(y) = (\lambda_1 - \lambda_2) \cdot \mathbb{N}_{y,0}\Big(\int_0^\sigma \mathrm{d}\mathscr{L}_s^0 \exp\big(- \lambda_1 \mathscr{L}_s^0 \big) \cdot \mathbb{E}^{\dagger}_{\rho_s, \overline{W}_s}\big[\exp\big(- \lambda_2 \mathscr{L}_\sigma^0 \big) \big] \Big).$$

We can now apply the duality identity $\big((\rho_{(\sigma-t)-}, \eta_{(\sigma-t)-}, \overline{W}_{\sigma-t}) : \ t \in [0,\sigma] \big) \overset{(d)}{=} \big((\eta_t, \rho_t, \overline{W}_t) : \ t \in [0,\sigma] \big)$ under $\mathbb{N}_{y,0}$, to get that the previous display is equal to

$$(\lambda_1 - \lambda_2) \cdot \mathbb{N}_{y,0}\Big(\int_0^\sigma \mathrm{d}\mathscr{L}_s^0 \exp\big(- \lambda_1 \int_s^\sigma \mathrm{d}\mathscr{L}_t^0 \big) \cdot \mathbb{E}^{\dagger}_{\eta_s, \overline{W}_s}\big[\exp\big(- \lambda_2 \mathscr{L}_\sigma^0 \big) \big] \Big)$$
$$= (\lambda_1 - \lambda_2) \cdot \mathbb{N}_{y,0}\Big(\int_0^\sigma \mathrm{d}\mathscr{L}_s^0 \ \mathbb{E}^{\dagger}_{\rho_s, \overline{W}_s}\big[\exp\big(- \lambda_1 \mathscr{L}_\sigma^0 \big) \big] \cdot \mathbb{E}^{\dagger}_{\eta_s, \overline{W}_s}\big[\exp\big(- \lambda_2 \mathscr{L}_\sigma^0 \big) \big] \Big).$$

Remark that $(\eta, \overline{W})$ takes values in $\overline{\Theta}_x$ by duality and right-continuity of η with respect to the total variation distance. We are now in position to apply the many-to-one equation (5.24). In this direction, for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ with $\overline{\mathrm{w}}(0) = (y, 0)$ and $y \neq x$ we notice that

$$\mathbb{E}^{\dagger}_{\mu, \overline{\mathrm{w}}}\Big[\exp\big(- \lambda \mathscr{L}_\sigma^0 \big) \Big] = \exp\Big(- \int_0^{\tau_{D_0}(\overline{\mathrm{w}})} \mu(\mathrm{d}h) \ \mathbb{N}_{\overline{\mathrm{w}}(h)}\big(1 - \exp(-\lambda \mathscr{L}_\sigma^0) \big) \Big)$$
$$= \exp\Big(- \int_0^{\tau_{D_0}(\overline{\mathrm{w}})} \mu(\mathrm{d}h) \ u_\lambda(\mathrm{w}(h)) \Big),$$

for every $\lambda > 0$. Consequently, (5.24) gives:

$$\frac{u_{\lambda_1}(y) - u_{\lambda_2}(y)}{\lambda_1 - \lambda_2} = E^0 \otimes \Pi_y\Big(\exp\big(- \alpha T_x \big) \exp\Big(- \int_0^{T_x} J(\mathrm{d}s) \ u_{\lambda_1}(\xi_s) - \int_0^{T_x} \check{J}(\mathrm{d}s) \ u_{\lambda_2}(\xi_s) \Big) \Big).$$

Finally an application of (5.23) yields exactly the desired result (5.4.1). □

As an immediate consequence, we obtain two other useful identities taking $\lambda_2 = 0$ and letting $\lambda_2 \downarrow \lambda_1$ respectively. For every $\lambda > 0$, we have

$$\mathcal{N}\Big(1-\exp\Big(-\int_0^\sigma \mathrm{d}h\ \psi\big(u_\lambda(\xi_h)\big)/u_\lambda(\xi_h)\Big)\Big) = \widetilde{\psi}(\lambda)/\lambda,\ \mathcal{N}\Big(1-\exp\Big(-\int_0^\sigma \mathrm{d}h\ \psi'(u_\lambda(\xi_h))\Big)\Big) = \widetilde{\psi}'(\lambda), \tag{5.68}$$

where for the first one we used that $u_0(y) = 0$ since $\mathbb{N}_y(L_\sigma^{E_*} = \infty) = 0$. We also stress that (5.68) can be proved independently directly by the same arguments as the ones applied in the proof of (5.67).

Since by Proposition 5.19 the exponent $\widetilde{\psi}$ satisfies (A1) – (A4), it can be written in the following form

$$\widetilde{\psi}(\lambda) = \widetilde{\alpha}\lambda + \widetilde{\beta}\lambda^2 + \int_{\mathbb{R}_+} \widetilde{\pi}(\mathrm{d}x)\,(\exp(-\lambda x) - 1 + \lambda x),$$

where $\widetilde{\alpha}, \widetilde{\beta} \geqslant 0$ and $\widetilde{\pi}$ is a measure in $\mathbb{R}_+$ satisfying $\int \widetilde{\pi}(\mathrm{d}x)(x \wedge x^2) < \infty$. In the following corollary, we identify the coefficients $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Corollary 5.21. *We have* $\widetilde{\alpha} = \mathcal{N}\big(1 - \exp(-\alpha\sigma)\big)$ *and* $\widetilde{\beta} = 0$.

Proof. To simplify notation, for $\lambda \geqslant 0$ set $\psi^*(\lambda) := \psi(\lambda)/\lambda$, $\widetilde{\psi}^*(\lambda) := \widetilde{\psi}(\lambda)/\lambda$. Since $\widetilde{\psi}$ satisfies (A1)–(A4), by Fubini we derive that $\widetilde{\psi}^*$ is the Laplace exponent of a subordinator with exponent:

$$\widetilde{\alpha} + \widetilde{\beta}\lambda + \int_{\mathbb{R}_+} \mathrm{d}r\, \widetilde{\pi}([r,\infty))\big(1 - \exp(-\lambda r)\big). \tag{5.69}$$

Next, introduce the measure $\mathcal{N}^*(\mathrm{d}\xi) := \mathcal{N}(\exp(-\alpha\sigma)\mathrm{d}\xi)$ and observe that by (5.68), $\widetilde{\psi}^*(\lambda)$ can also be written in the form

$$\mathcal{N}\Big(1-\exp\Big(-\int_0^\sigma \mathrm{d}h\ \psi^*\big(u_\lambda(\xi_h)\big)\Big)\Big) = \mathcal{N}\big(1-\exp(-\alpha\sigma)\big)+\mathcal{N}^*\Big(1-\exp\Big(-\int_0^\sigma \mathrm{d}h\ \big(\psi^*\big(u_\lambda(\xi_h)\big)-\alpha\big)\Big)\Big). \tag{5.70}$$

Comparing with (5.69), our result will follow by showing that the second term on the right-hand side of (5.70) is the Laplace exponent of some pure-jump subordinator. In this direction, introduce under $E^0 \otimes \mathcal{N}^*$ and conditionally on (J_∞, ξ), a Poisson point measure

$$\mathcal{M}(\mathrm{d}h, \mathrm{d}\rho, \mathrm{d}\overline{W}) = \sum_{i\in\mathbb{N}} \delta_{(h_i, \rho^i, \overline{W}^i)},$$

with intensity $J_\sigma(\mathrm{d}h)\mathbb{N}_{\xi(h),0}\big(\mathrm{d}\rho, \mathrm{d}\overline{W}\big)$. This is always possible up to enlarging the measure space and for simplicity we still denote the underlying measure by $E^0 \otimes \mathcal{N}^*$. Next, define the functional $\sum_{i\in\mathbb{N}} \mathscr{L}_\sigma^0(\rho^i, \overline{W}^i)$ and denote its distribution by $\nu(\mathrm{d}x)$. By definition, we have:

$$\begin{aligned} E^0 \otimes \mathcal{N}^*\Big(1 - \exp\Big(- \lambda \sum_{i\in\mathbb{N}} \mathscr{L}_\sigma^0(\rho^i, W^i)\Big)\Big) &= E^0 \otimes \mathcal{N}^*\Big(1 - \exp\Big(- \int_0^\sigma J_\sigma(\mathrm{d}h) u_\lambda\big(\xi(h)\big)\Big)\Big) \\ &= \mathcal{N}^*\Big(1 - \exp\Big(- \int_0^\sigma \mathrm{d}h\ \big(\psi^*\big(u_\lambda(\xi_h)\big) - \alpha\big)\Big)\Big), \end{aligned}$$

where in the last equality we used that J_∞ is the Lebesgue-Stieltjes measure of a subordinator with exponent $\psi^*(\lambda) - \alpha$. Since the latter expression is finite, we deduce that ν is a Lévy measure satisfying $\int \nu(\mathrm{d}r)\,(1 \wedge r) < \infty$, and that the second term on the right-hand side of (5.70) is the Laplace exponent of a driftless subordinator with Lévy measure given by ν. □

5.4.2 Construction of the additive functional (A_t)

We are finally in position to introduce our additive function:

Proposition 5.22. *Fix $(y, r_0) \in \overline{E}$ and $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$. Under $\mathbb{N}_{y,r_0}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, the process defined as*

$$A_t = \int_{\mathbb{R}_+} \mathrm{d}r \mathscr{L}_t^r, \qquad \text{for } t \geqslant 0,$$

is a continuous additive functional of the Lévy snake taking finite values. Furthermore, we have

$$A_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathrm{d}u \int_{\mathbb{R}_+} \mathrm{d}r \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}}, \tag{5.71}$$

where the convergence holds uniformly in compact intervals in measure under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r_0}(\cdot \cap \{\sigma > z\})$ for every $z > 0$.

Proof. We start proving the proposition under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, where $(\mu, \overline{\mathrm{w}}) := (\mu, \mathrm{w}, \ell) \in \overline{\Theta}_x$. Remark that by the translation invariance of the local time we might assume that $\ell(0) = 0$ without loss of generality. For simplicity, we set $y := \mathrm{w}(0)$. Next, we write $\widehat{\Lambda}_t^* := \sup_{s \leqslant t} \widehat{\Lambda}_s$ and we note that it suffices to show that for any $t, K > 0$

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\sup_{s \leqslant t} \Big| \int_{\mathbb{R}_+} \mathrm{d}r \, \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \int_{\mathbb{R}_+} \mathrm{d}r \mathscr{L}_s^r \Big| \cdot \mathbb{1}_{\{\widehat{\Lambda}_t^* < K\}} \Big] \to 0,$$

as $\varepsilon \downarrow 0$. In this direction, we remark that the previous expression is bounded above by

$$\begin{aligned} &\int_{\mathbb{R}_+} \mathrm{d}r \, \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\sup_{s \leqslant t} \Big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \Big| \cdot \mathbb{1}_{\{\widehat{\Lambda}_t^* < K\}} \Big] \\ &\qquad \leqslant \int_{(0,K]} \mathrm{d}r \, \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\sup_{s \leqslant t} \Big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \Big| \Big], \end{aligned}$$

since on the event $\{\widehat{\Lambda}_t^* < K\}$ we have $\mathscr{L}^r = 0$ for every $r > K$. Now, by Lemma 5.18, it suffices to show that the expectation under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ in the previous display is uniformly bounded on $\varepsilon, r > 0$ since the desired result then follows by dominated convergence. To do so, we set $T_0^+ := \inf\big\{t \geqslant 0 : \langle \rho_t, 1 \rangle = 0\big\}$ and we notice that by the strong Markov property, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, the distribution of $(\rho_{T_0^+ + s}, \overline{W}_{T_0^+ + s} : \ s \geqslant 0)$ is $\mathbb{P}_{0,y,0}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. In particular we have the upper bound:

$$\begin{aligned} &\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big[\sup_{s \leqslant t} \Big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \Big| \Big] \\ &\leqslant \mathbb{E}^{\dagger}_{\mu,\overline{\mathrm{w}}}\Big[\frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}u \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} + \mathscr{L}_\sigma^r \Big] + \mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \Big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \Big| \Big]. \end{aligned}$$

So to conclude we need to prove both:

$$\text{(i)} \quad \sup_{\varepsilon > 0} \sup_{r > 0} \mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \Big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \Big| \Big] < \infty;$$

$$\text{(ii)} \quad \sup_{\varepsilon > 0} \sup_{r > 0} \mathbb{E}^{\dagger}_{\mu,\overline{\mathrm{w}}}\Big[\frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}u \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} + \mathscr{L}_\sigma^r \Big] < \infty.$$

Let us start showing (i). We are going to apply similar techniques to the ones used in the proof of Theorem 5.10. In this direction, we work under $\mathbb{P}_{0,y,0}$ and we fix $r, \varepsilon > 0$. Now, recall the definition of γ^{D_r}, σ^{D_r} and ρ^{D_r} introduced in Section 5.3.2 (keeping in mind the fact that here we work with $(\rho, \overline{W})$) and set

$$R_t^{D_r} := \int_0^t \mathrm{d}s \mathbb{1}_{\{\gamma_s^{D_r} > 0\}}, \qquad \text{for } t \geqslant 0,$$

which is the right inverse of σ^{D_r}. Next, for every $r > 0$, by definition we have $\tau_r(\rho_t, \overline{W}_t) = \tau_{D_r}(\rho_t, \overline{W}_t)$ and we derive that

$$\int_0^s \mathrm{d}u \; \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} = \int_0^{R_s^{D_r}} \mathrm{d}u \; \mathbb{1}_{\{0 < H(\rho_u^{D_r}) < \varepsilon\}},$$

since on $\{u \geqslant 0 : H(\rho_{\sigma_u^{D_r}}) > \tau_r(\overline{W}_{\sigma_u^{D_r}})\}$, we have $H(\rho_u^{D_r}) = H(\rho_{\sigma_u^{D_r}}) - \tau_r(\overline{W}_{\sigma_u^{D_r}})$. Recall from (5.43) that $\langle \rho^{D_r}, 1 \rangle$ is distributed as $\langle \rho, 1 \rangle$ under $\mathbb{P}_{0,y,0}$, which is a reflected ψ-Lévy process, and that we denote its local time at 0 by ℓ^{D_r}. In particular, the distribution of $(\langle \rho^{D_r}, 1 \rangle, \ell^{D_r})$ is the same as $\big((X_t - I_t, -I_t) : t \geqslant 0\big)$. Recalling from (5.44) that $\mathscr{L}_t^r = \ell^{D_r}(R_t^{D_r})$ and noticing that $R_s^{D_r} \leqslant s$, we derive the following inequality:

$$\begin{aligned}
&\mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \; \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}} - \mathscr{L}_s^r \big| \Big] \\
&= \mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \big| \frac{1}{\varepsilon} \int_0^{R_s^{D_r}} \mathrm{d}u \; \mathbb{1}_{\{0 < H(\rho_u^{D_r}) < \varepsilon\}} - \ell^{D_r}(R_s^{D_r}) \big| \Big] \\
&\leqslant \mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \; \mathbb{1}_{\{0 < H(\rho_u^{D_r}) < \varepsilon\}} - \ell^{D_r}(s) \big| \Big] \\
&= \mathbb{E}_{0,y,0}\Big[\sup_{s \leqslant t} \big| \frac{1}{\varepsilon} \int_0^s \mathrm{d}u \; \mathbb{1}_{\{0 < H(\rho_u) < \varepsilon\}} + I_s \big| \Big],
\end{aligned}$$

where in the first line we used that for each fixed $r > 0$, the processes $\mathscr{L}^r$ and L^{D_r} are indistinguishable. The latter quantity does not depend on r and by (5.9) it converges to 0 as $\varepsilon \downarrow 0$, giving (i).

We now turn our attention to the proof of (ii). On the one hand, by Proposition 5.6 - (ii) and (5.28), for every $r > 0$ we have

$$\begin{aligned}
\mathbb{E}^{\dagger}_{\mu,\overline{\mathrm{w}}}\big[\mathscr{L}_\sigma^r\big] &= \int_{(0,\tau_r(\overline{\mathrm{w}}))} \mu(\mathrm{d}h) \; \mathbb{N}_{\overline{\mathrm{w}}(h)}\big(\mathscr{L}_\sigma^r\big) \\
&= \int_{(0,\tau_r(\overline{\mathrm{w}}))} \mu(\mathrm{d}h) \; E^0 \otimes \Pi_{\overline{\mathrm{w}}(h)}\big[\mathbb{1}_{\{\tau_r(\xi,\mathcal{L}) < \infty\}} \exp\big(-\alpha \tau_r(\xi, \mathcal{L})\big)\big] \leqslant \langle \mu, 1 \rangle.
\end{aligned}$$

On the other hand, the remaining term

$$\mathbb{E}^{\dagger}_{\mu,\overline{\mathrm{w}}}\Big[\frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}u \; \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u) + \varepsilon\}}\Big]$$

can be bounded similarly as we did in (5.33). More precisely, consider under $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$ the random measure $\sum_{i \in \mathbb{N}} \delta_{(h_i, \rho^i, \overline{W}^i)}$ defined in (5.22), set $T := \inf\{t > 0 : H_t = \tau_r(\overline{\mathrm{w}})\}$, with the convention $T = 0$ if $\tau_r(\overline{\mathrm{w}}) = \infty$, and remark that for every $s \in [0, T]$ we have $\tau_r(\overline{W}_s) = \tau_r(\overline{\mathrm{w}})$. Recalling

$\mu(\{\tau_r(\overline{\mathrm{w}})\}) = 0$, it follows by considering the excursion intervals of H over its running infimum and our previous remark, that the integral $\int_0^\sigma du \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u)+\varepsilon\}}$ can be written as

$$\sum_{h_i > \tau_r(\overline{\mathrm{w}})} \int_0^{\sigma(\overline{W}^i)} du \, \mathbb{1}_{\{\tau_r(\overline{\mathrm{w}}) < h_i + H(\rho_u^i) < \tau_r(\overline{\mathrm{w}})+\varepsilon\}} + \sum_{h_i < \tau_r(\overline{\mathrm{w}})} \int_0^{\sigma(\overline{W}^i)} du \, \mathbb{1}_{\{\tau_r(\overline{W}^i_u) < H(\rho_u^i) < \tau_r(\overline{W}^i_u)\}},$$

where the first term is now bounded above by $\sum_{h_i > \tau_r(\overline{\mathrm{w}})} \int_0^{\sigma(\overline{W}^i)} du \, \mathbb{1}_{\{0 < H(\rho_u^i) < \varepsilon\}}$. Consequently, by (5.24) we have

$$\begin{aligned}
&\mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\Big[\int_0^\sigma du \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u)+\varepsilon\}} \Big] \\
&\leqslant \mu((\tau_r(\overline{\mathrm{w}}), \infty)) N\Big(\int_0^\sigma ds \, \mathbb{1}_{\{0 < H(\rho_s) < \varepsilon\}} \Big) + \int_{(0,\, \tau_r(\overline{\mathrm{w}}))} \mu(dh) \mathbb{N}_{\overline{\mathrm{w}}(h)} \Big(\int_0^\sigma ds \, \mathbb{1}_{\{\tau_r(\overline{W}_s) < H_s < \tau_r(\overline{W}_s)+\varepsilon\}} \Big),
\end{aligned}$$

and again by the many-to-one formula (5.24), the previous display is bounded by $\varepsilon \cdot \langle \mu, 1 \rangle$. Putting everything together we deduce the upper bound

$$\mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\Big[\frac{1}{\varepsilon} \int_0^\sigma du \, \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u)+\varepsilon\}} + \mathscr{L}^r_\sigma \Big] \leqslant 2 \cdot \langle \mu, 1 \rangle,$$

which does not depend on the pair $r, \varepsilon > 0$ and concludes the proof of (ii).

Finally, we extend the result under the excursion measure $\mathbb{N}_{y,r_0}$. Working under $\mathbb{P}_{0,y,r_0}$ fix $z > 0$ and denote by $(\rho', \overline{W}') = (\rho_{(g+\cdot)\wedge d}, \overline{W}_{(g+\cdot)\wedge d})$ the first excursion with length $\sigma > z$. By the previous result, the quantity

$$\begin{aligned}
&\sup_{s \leqslant t} \Big| \, \varepsilon^{-1} \int_0^s du \int_{\mathbb{R}_+} dr \mathbb{1}_{\{\tau_r(\overline{W}'_u) < H(\rho'_u) < \tau_r(\overline{W}'_u)+\varepsilon\}} - \int_{\mathbb{R}_+} dr \mathscr{L}^r_s(\rho', \overline{W}') \, \Big| \\
&\qquad = \sup_{s \leqslant t \wedge (d-g)} \Big| \, \varepsilon^{-1} \int_g^{g+s} du \int_{\mathbb{R}_+} dr \mathbb{1}_{\{\tau_r(\overline{W}_u) < H_u < \tau_r(\overline{W}_u)+\varepsilon\}} - \int_{\mathbb{R}_+} dr (\mathscr{L}^r_{g+s} - \mathscr{L}^r_g) \, \Big|
\end{aligned}$$

converges in probability to 0, and it then follows that (5.71) holds in measure under $\mathbb{N}_{y,r_0}(\,\cdot \cap \{\sigma > z\})$.

□

As a straight consequence of the definition of A we deduce the following many-to-one formula:

Lemma 5.23. *For any non-negative measurable function Φ on $M_f(\mathbb{R}_+) \times M_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}}$ and $(y, r_0) \in \overline{E}$, we have*

$$\mathbb{N}_{y,r_0}\Big(\int_0^\sigma dA_s \, \Phi\big(\rho_s, \eta_s, \overline{W}_s\big) \Big) = \int_{r_0}^\infty dr \; E^0 \otimes \Pi_{y,r_0} \Big(\exp\big(-\alpha \tau_r\big) \cdot \Phi\big(J_{\tau_r}, \check{J}_{\tau_r}, (\xi_t, \mathcal{L}_t : t \leqslant \tau_r)\big) \Big). \tag{5.72}$$

Proof. By the translation invariance of the local time it is enough to prove the Lemma for $r_0 = 0$. Now recall that, under $\mathbb{N}_{y,0}$, for every fixed $r \geqslant 0$ the processes $\mathscr{L}^r$ and L^{D_r} are indistinguishable. Consequently, the left-hand side of (5.72) can be written in the form:

$$\int_0^\infty dr \, \mathbb{N}_{y,0} \Big(\int_0^\sigma dL_s^{D_r} \, \Phi\big(\rho_s, \eta_s, \overline{W}_s\big) \Big),$$

and hence we arrive at (5.72) applying (5.28). □

A first consequence of Lemma 5.23 is that for any $(y, r_0) \in \overline{E}$, we have

$$\operatorname{supp} \mathrm{d}A \subset \{t \in \mathbb{R}_+ : \widehat{W}_t = x\}, \quad \mathbb{N}_{y,r_0}\text{– a.e.} \tag{5.73}$$

Indeed, it suffices to observe that by (5.72), for any $\varepsilon > 0$, it holds that

$$\mathbb{N}_{y,r_0}\left(\int_0^\sigma \mathrm{d}A_s \mathbb{1}_{\{\mathrm{d}_E(\widehat{W}_s,x)>\varepsilon\}}\right) = 0,$$

where we recall that d_E stands for the metric of E. Let us comment on a few useful identities that will be used frequently in our computations:

Remark. Fix $(y, r_0) \in \overline{E}$ with $y \neq x$. Under $\mathbb{N}_{y,r_0}$ or $\mathbb{P}_{0,y,r_0}$, let (g, d) be an interval such that $H_s > H_g = H_d$, for every $s \in (g, d)$, and $\widehat{\Lambda}_g = r_0$ – remark that in particular we have $p_H(g) = p_H(d)$. We denote the corresponding subtrajectory, in the sense of Section 5.2.3, by $(\rho', \overline{W}')$ and its duration by $\sigma' = \sigma(W')$. Since for any $q \geqslant r$ and $s \geqslant 0$:

$$H_{(g+s)\wedge d} = H_g + H(\rho^i_{s\wedge\sigma_i}) \quad \text{ and } \quad \tau_q(\overline{W}_{(g+s)\wedge d}) = H_g + \tau_q(\overline{W}^i_{s\wedge\sigma_i}),$$

we deduce by the approximation (5.71) that the process $(A_{(g+t)\wedge d} - A_g : t \geqslant 0)$ only depends on $(\rho', \overline{W}')$ and it will be denoted by $(A_s(\rho', \overline{W}') : s \geqslant 0)$. Now we make the following observations:

(i) Working under $\mathbb{N}_{y,r_0}$, we denote the connected components of the open set $\{(H_s - \tau_{r_0}(\overline{W}_s))_+ > 0\}$ by $((\alpha_i, \beta_i) : i \in \mathcal{I})$ and we set $\sigma_i := \beta_i - \alpha_i$ its duration. We also write $(\rho^i, \overline{W}^i)$ for the excursions from D_{r_0} corresponding to the interval (α_i, β_i). By Proposition 5.22, the measure $\mathrm{d}A$ does not charge the set $\{s \geqslant 0 : H_s \leqslant \tau_{r_0}(\overline{W}_s)\}$ and we derive that:

$$A_\sigma = \sum_{i\in\mathcal{I}} \int_{(\alpha_i,\beta_i]} \mathrm{d}A_s = \sum_{i\in\mathcal{I}} A_{\sigma_i}(\rho^i, \overline{W}^i), \qquad \mathbb{N}_{y,r_0}\text{–a.e.} \tag{5.74}$$

(ii) We will now make similar remarks holding under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$, for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$. Under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$, denote the connected components of $\{s \geqslant 0 : \ H_s > \inf_{[0,s]} H\}$ by $((a_i, b_i) : i \in \mathbb{N})$ and write $(\rho^i, \overline{W}^i)$ for the subtrajectory associated with $[a_i, b_i]$. We also set $h_i = H_{a_i}$ and recall that the measure $\mathcal{M} = \sum_{i\in\mathcal{I}} \delta_{(h_i,\rho^i,\overline{W}^i)}$ is the Poisson point measure (5.22) associated with $(\rho, \overline{W})$. Moreover, we have:

$$\mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\big[|A_\sigma - \sum_{i\in\mathbb{N}} A_\sigma(\rho^i, \overline{W}^i)|\big] \leqslant \int_{\mathbb{R}_+} \mathrm{d}r \ \mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\big[|\mathscr{L}^r_\sigma - \sum_{i\in\mathbb{N}} \mathscr{L}^r_\sigma(\rho^i, \overline{W}^i)|\big].$$

Consequently, by Proposition 5.6 - (ii), the previous quantity is null and it follows that we still have

$$A_\sigma = \sum_{i\in\mathbb{N}} A_\sigma(\rho^i, \overline{W}^i), \qquad \mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}\text{– a.s.} \tag{5.75}$$

Recall now the definition (5.59) of $\widetilde{\psi}$ and the notation u_λ introduced in (5.57). The following proposition relates the Laplace transform of the total mass A_σ under $\mathbb{N}_{y,r_0}$ and the Laplace exponent $\widetilde{\psi}$. This identity will be needed to characterize the support of $\mathrm{d}A$ and will also play a central role in Section 5.5.

Proposition 5.24. *For every $r_0, \lambda \geqslant 0$ and $y \in E$, we have*

$$\mathbb{N}_{y,r_0}\Big(1 - \exp\big(-\lambda A_\infty\big)\Big) = u_{\widetilde{\psi}^{-1}(\lambda)}(y),$$

where we recall the convention $u_\lambda(x) = \lambda$, for every $\lambda \geqslant 0$. Moreover, for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, we have:

$$\mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\Big[\exp\big(-\lambda A_\infty\big)\Big] = \exp\Big(-\int \mu(\mathrm{d}h)\ u_{\widetilde{\psi}^{-1}(\lambda)}(\mathrm{w}(h))\Big).$$

The proposition has the following consequence: since $\widetilde{\psi}^{-1}(\lambda) = \widetilde{N}(1-\exp(-\lambda\sigma))$, the total mass A_∞ under $\mathbb{N}_{x,0}$ and σ under $\widetilde{N}$ have the same distribution. This connection is the tip of the iceberg of the results that will be established in the upcoming section, where we establish that the tree structure of the set $\{v \in \mathcal{T}_H : \ \xi_v = x\}$ is encoded by a $\widetilde{\psi}$–Lévy tree.

Proof. Under $\mathbb{N}_{y,r_0}$ with $y \neq x$ and $r_0 \geqslant 0$, set

$$T^* := \inf\{t \geqslant 0 : \tau_{r_0}(\overline{W}_t) < \infty\},$$

which is just the first hitting time of x by $(\widehat{W}_t)_{t\in[0,\sigma]}$. Notice that by (5.73), A_∞ vanishes on $\{T^* = \infty\}$ $\mathbb{N}_{y,r_0}$-a.e.. We set $G_\lambda := \mathbb{N}_{x,0}(1-\exp(-\lambda A_\infty))$, and remark that the identity (5.74) and the special Markov property applied to the domain D_{r_0} yields:

$$\mathbb{N}_{y,r_0}\Big(1 - \exp\big(-\lambda A_\infty\big)\Big) = \mathbb{N}_{y,r_0}\Big(1 - \exp\Big(-\mathscr{L}^{r_0}_\sigma \cdot \mathbb{N}_{x,r_0}\big(1-\exp\big(-\lambda A_\infty\big)\big)\Big)\Big).$$

Next, by the translation invariance of the local time $\mathcal{L}$, we derive that the previous display is equal to:

$$\mathbb{N}_{y,0}\Big(1 - \exp\Big(-\mathscr{L}^{0}_\sigma \cdot \mathbb{N}_{x,0}\big(1-\exp\big(-\lambda A_\infty\big)\big)\Big)\Big) = u_{G_\lambda}(y).$$

Moreover, for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ if we denote under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$ the Poisson process introduced in (5.22) by $\sum_{i\in\mathcal{I}} \delta_{(h_i,\rho^i,\overline{W}^i)}$, we get :

$$\begin{aligned}
\mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\Big[\exp\big(-\lambda A_\infty\big)\Big] &= \mathbb{E}^\dagger_{\mu,\overline{\mathrm{w}}}\Big[\exp\big(-\lambda \sum_{i\in\mathcal{I}} A_\infty(\rho^i, \overline{W}^i)\big)\Big] \\
&= \exp\Big(-\int \mu(\mathrm{d}h)\ \mathbb{N}_{\overline{\mathrm{w}}(h)}\big(1-\exp\big(-\lambda A_\infty\big)\big)\Big) \\
&= \exp\Big(-\int \mu(\mathrm{d}h)\ u_{G_\lambda}(\mathrm{w}(h))\Big),
\end{aligned}$$

where in the first equality we applied (5.75), and in the second we used that $\sum_{i\in\mathcal{I}} \delta_{(h_i,\rho^i,\overline{W}^i)}$ is a Poisson point measure with intensity $\mu(\mathrm{d}h)\mathbb{N}_{\overline{\mathrm{w}}(h)}(\mathrm{d}\rho, \mathrm{d}W)$. Consequently, the statement of the proposition will now follow if we establish that $G_\lambda = \widetilde{\psi}^{-1}(\lambda)$. In this direction, for $\lambda > 0$, notice that the Markov property implies that

$$G_\lambda = \lambda \cdot \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}A_s\ \exp\big(-\lambda\int_s^\sigma \mathrm{d}A_u\big)\Big) = \lambda \cdot \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}A_s\ \mathbb{E}^\dagger_{\rho_s,\overline{W}_s}\Big[\exp\big(-\lambda\int_0^\sigma \mathrm{d}A_u\big)\Big]\Big).$$

By the previous discussion under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$ and the many-to-one formula of A given in Lemma (5.23), we get:

$$\begin{aligned}
G_\lambda &= \lambda\int_0^\infty \mathrm{d}r\ E^0 \otimes \Pi_{x,0}\Big(\exp\big(-\alpha\tau_r\big)\exp\Big(-\int_0^{\tau_r} J_{\tau_r}(\mathrm{d}h)\ u_{G_\lambda}\big(\xi(h)\big)\Big)\Big) \\
&= \lambda\int_0^\infty \mathrm{d}r\ \Pi_{x,0}\Big(\exp\Big(-\int_0^{\tau_r}\mathrm{d}h\ \frac{\psi\big(u_{G_\lambda}(\xi(h))\big)}{u_{G_\lambda}\big(\xi(h)\big)}\Big)\Big),
\end{aligned}$$

where we recall that $\tau_r(\xi, \mathcal{L}) := \inf\{s \geqslant 0 : \mathcal{L}_s \geqslant r\}$ and in the second equality we used that $J_\infty(\mathrm{d}h)$ is the Lebesgue-Stieltjets measure of a subordinator with exponent $\psi(\lambda)/\lambda - \alpha$. Next, under $\Pi_{x,0}$, we consider $(s_i, t_i)_{i\geqslant 1}$ the connected components of $\{s \geqslant 0 : \xi_s \neq x\}$ and we remark that:

$$\int_0^{\tau_r} \mathrm{d}h \, \frac{\psi\big(u_{G_\lambda}(\xi(h))\big)}{u_{G_\lambda}\big(\xi(h)\big)} = \sum_{i\geqslant 1, \mathcal{L}_{s_i}<r} \int_{s_i}^{t_i} \mathrm{d}h \, \frac{\psi\big(u_{G_\lambda}(\xi(h))\big)}{u_{G_\lambda}\big(\xi(h)\big)},$$

since $\int_0^\infty \mathrm{d}h \mathbb{1}_{\{\xi_h = x\}} = 0$ by assumption $(\mathbf{H_3})$. Consequently, by excursion theory we get:

$$\Pi_{x,0}\Big(\exp\Big(- \int_0^{\tau_r} \mathrm{d}h \, \frac{\psi\big(u_{G_\lambda}(\xi(h))\big)}{u_{G_\lambda}\big(\xi(h)\big)}\Big)\Big) = \exp\Big(- r \cdot \mathcal{N}\Big(1 - \exp\big(- \int_0^\sigma \mathrm{d}h \, \frac{\psi\big(u_{G_\lambda}(\xi_h)\big)}{u_{G_\lambda}(\xi_h)}\big)\Big)\Big),$$

and hence

$$G_\lambda = \lambda \cdot \mathcal{N}\Big(1 - \exp\Big(- \int_0^\sigma \mathrm{d}h \, \frac{\psi\big(u_{G_\lambda}(\xi_h)\big)}{u_{G_\lambda}(\xi_h)}\Big)\Big)^{-1}.$$

However, by the first identity in (5.68), we have

$$\mathcal{N}\Big(1 - \exp\Big(- \int_0^\sigma \mathrm{d}h \frac{\psi(u_{G_\lambda}(\xi_h))}{u_{G_\lambda}(\xi_h)}\Big)\Big) = \frac{\widetilde{\psi}(G_\lambda)}{G_\lambda},$$

and we derive that $\widetilde{\psi}(G_\lambda) = \lambda$ for $\lambda > 0$ and equivalently $G_\lambda = \widetilde{\psi}^{-1}(\lambda)$. Finally, since $G_0 = 0$ the identity also holds for $\lambda = 0$. □

5.4.3 Characterization of the support of $\mathrm{d}A$

The rest of the section is devoted to the characterisation, under $\mathbb{N}_{y,r_0}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, of the support of the measure $\mathrm{d}A$. Our characterisation is given in terms of the constancy intervals of $\widehat{\Lambda}$, and of a family of special times for the Lévy snake that will be named *exit times from* x. Before giving a precise statement we will need several preliminary results under $\mathbb{N}_{x,0}$. First recall that under $\mathbb{N}_{x,0}$, for every $r > 0$ the processes $\mathscr{L}^r$ and L^{D_r} are indistinguishables – and in particular, by Proposition 5.7, $\mathscr{L}_\sigma^r$ is $\mathcal{F}^{D_r}$ measurable. Fix $r > 0$, recall the notation $\tau_r(\rho_t, \overline{W}_t) = \tau_{D_r}(\rho_t, \overline{W}_t)$ for $t \geqslant 0$, and denote the connected components of the open set $\{t \in [0,\sigma] : \tau_r(\overline{W}_t) < H_t\}$ by $\{(a_i^r, b_i^r) : i \in \mathcal{I}_r\}$. We write $\{(\rho^{i,r}, \overline{W}^{i,r}) : i \in \mathcal{I}_r\}$ for the corresponding subtrajectories, where as usual $\overline{W}^{i,r} = (W^{i,r}, \Lambda^{i,r})$. Next, recall the notation $\Gamma_s^D := \inf\big\{t \geqslant 0 : V_t^D > s\big\}$ for V^D defined by (5.25) and we set:

$$\theta_u^r := \inf\big\{s \geqslant 0 \, : \mathscr{L}^r_{\Gamma_s^{D_r}} > u\big\}, \qquad \text{for all } u \in [0, \mathscr{L}_\sigma^r).$$

Remark that $\mathrm{tr}_{D_r}(\widehat{W, \Lambda})_{\theta_u^r} = (x, r)$, for every $u \in [0, \mathscr{L}_\sigma^r)$. An application of the special Markov property applied at the domain D_r gives that, conditionally on $\mathcal{F}^{D_r}$, the point measure of the excursions from D_r

$$\mathcal{M}^{(r)} := \sum_{i \in \mathcal{I}_r} \delta_{(\mathscr{L}^r_{a_i^r}, \rho^{i,r}, \overline{W}^{i,r})}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\mathscr{L}_\sigma^r]}(u)\mathrm{d}u \, \mathbb{N}_{x,r}\big(\mathrm{d}\rho, \mathrm{d}\overline{W}\big)$.

Lemma 5.25. $\mathbb{N}_{x,0}$*–a.e., we have* $\{0, \sigma\} \in \operatorname{supp} \mathrm{d}A$.

Proof. We are going to show that for any $\varepsilon > 0$, we have $\mathbb{N}_{x,0}(A_{\varepsilon\wedge\sigma} = 0) = 0$ – the Lemma will follow since the symmetric statement $\mathbb{N}_x(A_{\sigma-\varepsilon} = 0) = 0$ will then hold by the duality identity (5.20). As previously we write

$$G_\lambda := \mathbb{N}_x\big(1 - \exp\big(-\lambda A_\infty\big)\big) = \widetilde{\psi}^{-1}(\lambda),$$

where the second equality holds by Proposition 5.19. For every positive rational numbers r and q, we introduce the stopping time $T_q^r := \inf\big\{s \geqslant 0 : \mathscr{L}_s^r > q\big\}$, with the convention $T_q^r = \infty$, if $\mathscr{L}_\sigma^r \leqslant q$. Let us prove that

$$\mathbb{N}_{x,0}\big(A_{T_q^r} = 0, \mathscr{L}_\sigma^r > 0\big) = 0. \tag{5.76}$$

In this direction, set $\mathbb{N}_{x,0}^r := \mathbb{N}_{x,0}^r(\mathrm{d}\rho, \mathrm{d}\overline{W}|\mathscr{L}_\sigma^r > 0)$ and using the fact that $\mathcal{M}^{(r)}$ is a Poisson point measure with intensity $\mathbb{1}_{[0,\mathscr{L}_\sigma^r]}(u)\mathrm{d}u\ \mathbb{N}_{x,r}\left(\mathrm{d}\rho, \mathrm{d}\overline{W}\right)$, remark that

$$\begin{aligned}
\mathbb{N}_{x,0}^r\Big(\exp\big(-\lambda A_{T_q^r}\big)\Big) &\leqslant \mathbb{N}_{x,0}^r\Big(\exp\big(-\lambda \sum_{i\in\mathcal{I}_r} A_\sigma(\rho^{i,r}, \overline{W}^{i,r})\mathbb{1}_{\{\mathscr{L}_{a_i}^r \leqslant q\}}\big)\Big) \\
&= \mathbb{N}_{x,0}^r\Big(\exp\Big(-(q \wedge \mathscr{L}_\sigma^r)\cdot \mathbb{N}_x\big(1 - \exp(-\lambda A_\infty)\big)\Big)\Big) \\
&= \mathbb{N}_{x,0}^r\Big(\exp\big(-(q \wedge \mathscr{L}_\sigma^r)\cdot G_\lambda\big)\Big),
\end{aligned}$$

and hence:

$$\mathbb{N}_{x,0}^r(A_{T_q^r} = 0) + \mathbb{N}_{x,0}^r\big(\exp\big(-\lambda A_{T_q^r}\big)\mathbb{1}_{\{A_{T_q^r} > 0\}}\big) \leqslant \mathbb{N}_{x,0}^r\left(\exp\big(-(q \wedge \mathscr{L}_\sigma^r)\cdot G_\lambda\big)\right).$$

Now (5.76) follows taking the limit as $\lambda \uparrow \infty$, since we are working under $\{\mathscr{L}_\sigma^r > 0\}$ and by Proposition 5.19 the function $\widetilde{\psi}$ satisfies (A4), which gives that G_λ goes to ∞ when $\lambda \uparrow \infty$. We stress that (5.76) holds for any positive rational numbers r and q. Now fix $\varepsilon > 0$, and notice that by the monotonicity of A, we have

$$\big\{A_{\varepsilon\wedge\sigma} = 0\ ;\ T_q^r < \varepsilon\big\} \subset \big\{A_{T_q^r} = 0\ ;\ T_q^r < \varepsilon\ ;\ \mathscr{L}_\sigma^r > 0\big\},$$

where the last set has null $\mathbb{N}_{x,0}$ measure by (5.76). The identity $\mathbb{N}_{x,0}(A_{\varepsilon\wedge\sigma} = 0) = 0$ now will follow as soon as we show that, $\mathbb{N}_{x,0}$-a.e. , there exists two positive rational numbers r and q satisfying that $T_q^r < \varepsilon$. Said otherwise, we need to establish that the origin is an accumulation point of $\{T_q^r :\ r, q \in \mathbb{Q}_+^*\}$. Arguing by contradiction, write

$$\Omega_0 = \bigcap_{r,q\in\mathbb{Q}_+^*} \big\{T_q^r \geqslant \varepsilon\big\} = \bigcap_{r\in\mathbb{Q}_+^*} \big\{T_q^r \geqslant \varepsilon\ :\ \forall q > 0\big\} = \bigcap_{r\in\mathbb{Q}_+^*} \big\{\mathscr{L}_\varepsilon^r = 0\big\}$$

where in the last equality we used (5.76), and suppose that $\mathbb{N}_{x,0}(\Omega_0) > 0$. To simplify notation, set $C(r) := \inf\{s \geqslant 0 : \widehat{\Lambda}_s > r\}$, and remark that the special Markov property, as stated in Theorem 5.11, applied to the domain D_r gives $\{\mathscr{L}_\varepsilon^r = 0\} = \{C(r) \geqslant \varepsilon\}$. We then derive that

$$0 < \mathbb{N}_{x,0}\Big(\bigcap_{r\in\mathbb{Q}_+^*} \{C(r) \geqslant \varepsilon\}\Big) = \mathbb{N}_{x,0}\Big(\widehat{\Lambda}_s = 0,\ \forall s \in [0, \varepsilon \wedge \sigma]\Big).$$

However, recalling the definition (5.19) of the excursion measure $\mathbb{N}_{x,0}$ this is in contradiction with the fact that for every $s \in (0, \sigma)$, $\mathbb{N}_{x,0}$ a.e., $\widehat{\Lambda}_s > 0$. Indeed, by definition of the Lévy snake under $\mathbb{N}_{x,0}$, for any fixed s the process $(W_s(t), \Lambda_s(t) : t \leqslant \zeta_s)$ has the distribution of a trajectory of the Markov process $(\xi_t, \mathcal{L}_t : t \geqslant 0)$ under $\Pi_{x,0}$ killed at ζ_s. We then have $\Lambda_s(t) > 0$, for every $t > 0$, since $\mathcal{L}_t > 0$, $\Pi_{x,0}$ a.s., and $\zeta_s = H(\rho_s)$ does not vanish on $(0, \sigma)$. $\square$

Define:

$$\mathcal{C}^* := \Big\{ t \in [0,\sigma] : \sup_{(t-\varepsilon,t+\varepsilon)\cap[0,\sigma]} \widehat{\Lambda} = \inf_{(t-\varepsilon,t+\varepsilon)\cap[0,\sigma]} \widehat{\Lambda} \ , \quad \text{for some } \varepsilon > 0 \Big\},$$

and remark that – the closure of the – connected components of $\mathcal{C}^*$ are exactly the constancy intervals of $\widehat{\Lambda}$. We will show that the support of dA is precisely the complement of $\mathcal{C}^*$. In this direction, our goal now is to give an equivalent definition of $\mathcal{C}^*$ in terms of H and W, and for this purpose we introduce the notion of exit times.

Definition 5.26. (Exit times from x) *A non negative number t is said to be an exit time from the point x for the process (ρ, W) if $\widehat{W}_t = x$ and there exists $s > 0$ such that*

$$H_t < H_{t+u}, \quad \textit{for all } u \in (0,s).$$

The collection of exit times from x is denoted by Exit(x).

Remark 5.27. Note that, for every $t \in \text{Exit}(x)$, the point $p_H(t)$ corresponds by definition to an interior point of the Lévy tree and in fact, recalling the result of Proposition 5.16, $p_H(t)$ is a point of multiplicity 2 in $\mathcal{T}_H$. In particular, for every $t \in \text{Exit}(x)$, there exists a unique $s > t$ such that $p_H(t) = p_H(s)$ and satisfying that:

$$\widehat{W}_s = x \quad \text{and} \quad H_{s-u} > H_t = H_s \quad \text{for all } u \in (0,v),$$

for some $v > 0$ – in this case, we can take $v := t - s$. By analogy, we write ${\downarrow}\text{Exit}(x)$ for the collection of times in $[0,\sigma]$ satisfying the previous display. Remark that the correspondence described above between Exit(x) and ${\downarrow}\text{Exit}(x)$ defines a bijection. We also stress that the inclusion $\text{Exit}(x) \cup {\downarrow}\text{Exit}(x) \subset \{t \in \mathbb{R}_+ : \widehat{W}_t = x\}$ is a priori strict since we are excluding in our definition potential times that will be mapped by p_H into leaves with label x.

Let us now prove the following technical lemma:

Lemma 5.28. *For every fixed $r > 0$, under $\mathbb{N}_{x,0}$, we have:*

$$\text{supp } \mathrm{d}\mathscr{L}^r = \overline{\{a_i^r, b_i^r : i \in \mathcal{I}_r\}} = \overline{\text{Exit}(x) \cap \big\{ s \in [0,\sigma] : \ \widehat{\Lambda}_s = r \big\}}, \tag{5.77}$$

and the same identity holds if we replace $\overline{\text{Exit}(x)}$ by $\overline{{\downarrow}\text{Exit}(x)}$. In particular, the measure dA *gives no mass to the complement of $\overline{\text{Exit}(x)}$ (or $\overline{{\downarrow}\text{Exit}(x)}$).*

Proof. First remark that if $\mathscr{L}^r_\sigma = 0$, by the special Markov property applied to the domain D_r, all the sets appearing in (5.77) are empty. Hence, it suffices to show (5.77) under $\mathbb{N}^r_x := \mathbb{N}_x(\cdot \,|\mathscr{L}^r_\sigma > 0)$. Moreover, notice that by definition we have:

$$\{a_i^r : i \in \mathcal{I}_r\} = \text{Exit}(x) \cap \big\{ s \in [0,\sigma] : \ \widehat{\Lambda}_s = r \big\}, \text{ and } \{b_i^r : i \in \mathcal{I}_r\} = {\downarrow}\text{Exit}(x) \cap \big\{ s \in [0,\sigma] : \ \widehat{\Lambda}_s = r \big\}.$$

To deduce (5.77), it is then enough to show that:

$$\text{supp } \mathrm{d}\mathscr{L}^r = \overline{\{a_i^r : i \in \mathcal{I}_r\}},$$

since the same equality will hold for $\{a_i^r : i \in \mathcal{I}_r\}$ replaced by $\{b_i^r : i \in \mathcal{I}_r\}$, using the duality identity (5.20) under $\mathbb{N}_{x,0}$.

So let us prove the previous display. We start showing the inclusion $\text{supp d}\mathscr{L}^r \subset \overline{\{a_i^r : i \in \mathcal{I}_r\}}$. In this direction, consider $s \in \text{supp d}\mathscr{L}^r$. By the special Markov property the set $\{\mathscr{L}^r_{a_i^r} : i \in \mathcal{I}_r\}$ is dense in $[0, \mathscr{L}^r_\sigma]$, which gives that for every ε there exists $i \in \mathcal{I}_r$ such that $\mathscr{L}^r_{(s-\varepsilon)+} < \mathscr{L}^r_{a_i^r} < \mathscr{L}^r_{s+\varepsilon}$. This ensures that $a_i^r \in (s-\varepsilon, s+\varepsilon)$ by monotonicity of $\mathscr{L}^r$. Consequently, the set supp $\text{d}\mathscr{L}^r$ is included in the closure of $\{a_i^r : i \in \mathcal{I}_r\}$. Let us now establish the reverse inclusion by showing that for every $j \in \mathcal{I}_r$, we have $a_j^r \in \text{supp d}\mathscr{L}^r$. In order to prove it, set $R_t := \sum_{\mathscr{L}^r_{a_i^r} \leqslant t} \sigma(\overline{W}^{i,r})$ for $t \geqslant 0$ and notice that it is a rcll process since $R_\infty \leqslant \sigma < \infty$. Now remark that by definition, for every $k \in \mathcal{I}_r$ with $a_k^r < a_j^r$, we have:

$$a_j^r - a_k^r \leqslant R_{\mathscr{L}^r_{a_j^r}-} - R_{\mathscr{L}^r_{a_k^r}-} + \theta^r_{\mathscr{L}^r_{a_j^r}} - \theta^r_{\mathscr{L}^r_{a_k^r}-}.$$

Since θ^r is monotone, it has a countable number of discontinuities and it follows by the special Markov property – using that θ^r is $\mathcal{F}^{D_r}$-measurable – that all the points $\{\mathscr{L}^r_{a_i^r} : i \in \mathcal{I}\}$ are continuity points of θ^r. Since R is rcll, this implies that for every $\varepsilon > 0$ there exists $k \in \mathcal{I}_r$ such that $a_j^r - \varepsilon < a_k^r < a_j^r$. All the values $\{\mathscr{L}^r_{a_i^r} : i \in \mathcal{I}_r\}$ being distinct we derive that $a_j^r \in \text{supp d}\mathscr{L}^r$, as wanted. As a consequence of (5.77), it follows that:

$$\mathbb{N}_{x,0}\left(\int_0^\sigma \text{d}A_s \mathbb{1}_{s \notin \overline{\text{Exit}(x)}}\right) = \int_0^\infty \text{d}r\, \mathbb{N}_{x,0}\left(\int_0^\sigma \text{d}\mathscr{L}^r_s \mathbb{1}_{s \notin \overline{\text{Exit}(x)}}\right) = 0,$$

and we deduce by duality that $\text{d}A$ gives no mass to the complement of $\overline{\text{Exit}(x)}$ – the same result holding for $\overline{{}^{\downarrow}\text{Exit}(x)}$. □

The next proposition establishes the connection between the constancy intervals of $\widehat{\Lambda}$, the exit times from x and the excursion intervals from D_r. This is the last result needed to characterise the support of $\text{d}A$.

Proposition 5.29. $\mathbb{N}_{x,0}$*–a.e., we have:*

$$\overline{\text{Exit}(x)} = \overline{{}^{\downarrow}\text{Exit}(x)} = \overline{\{a_i^r, b_i^r : r \in \mathbb{Q}_+^* \text{ and } i \in \mathcal{I}_r\}} = [0,\sigma] \backslash \mathcal{C}^*. \tag{5.78}$$

Proof. The first step consists in showing

$$\overline{\text{Exit}(x)} \subset \overline{\{a_i^r, b_i^r : r \in \mathbb{Q}_+^* \text{ and } i \in \mathcal{I}_r\}}. \tag{5.79}$$

Remark that by Lemma 5.28 the other inclusion is satisfied and still holds if we replace $\overline{\text{Exit}(x)}$ by $\overline{{}^{\downarrow}\text{Exit}(x)}$. In this direction, recall that by Lemma 5.13 the process $(\rho, \overline{W})$ takes values in $\overline{\Theta}_x$. In particular, we have

$$\mathbb{N}_{x,0}\text{–a.e., for all } q \in (0,\sigma),\ \overline{\{h < H_q : W_q(h) = x\}} = \text{supp } \Lambda_q(\text{d}h), \tag{$*$}$$

where we recall that supp $\Lambda_q(\text{d}h)$ is precisely the set

$$\Big\{t \in [0,\zeta_q] : \Lambda_q(t+h) > \Lambda_q(t) \text{ for any } 0 < h < (H_q - t) \text{ or } \Lambda_q(t) > \Lambda_q(t-h) \text{ for any } 0 < h < t\Big\}.$$

We let $\Omega_0 \subset \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$ be a measurable subset with $\mathbb{N}_{x,0}(\Omega_0^c) = 0$ at which property $(*)$ holds for every $(\mathsf{\rho}, \omega) \in \Omega_0$ and we argue for fixed $(\mathsf{\rho}, \omega) \in \Omega_0$. Fix $t \in \text{Exit}(x)$; by definition, for any $\varepsilon > 0$ we can find $t < q < t + \varepsilon$ such that $H_t < H_r$ for every $r \in (t, q]$. By our choice of Ω_0 and the snake property, it must hold either that:

(i) H_t is a time of right-increase for Λ_q (and in particular $\widehat{\Lambda}_q > \Lambda_q(H_t) = \widehat{\Lambda}_t$), or

(ii) H_t is *not* a time of right-increase for Λ_q, (and hence $\Lambda_q(H_t) > \Lambda_q(H_t - s)$, $\forall\, 0 < s < H_t$).

If (i) holds, set $s_k := \sup\{s \in [t,q] : \ \widehat{\Lambda}_s \leqslant 2^{-k}\lfloor 2^k \widehat{\Lambda}_t \rfloor + 2^{-k}\}$ and remark that we have $s_k \in \bigcup_{r\in\mathbb{Q}_+^*}\{a_i^r, b_i^r : i \in \mathcal{I}_r\}$, as soon as $\widehat{\Lambda}_{s_k} < \widehat{\Lambda}_q$. However, this is satisfied for k large enough. On the other hand, if (ii) holds we must have $\inf_{[t-\varepsilon,t]} H < H_t$ since t can not be a local infimum for H (otherwise, $p_H(t)$ would be a branching point with label $\widehat{W}_t = x$, in contradiction with Proposition 5.16). Now, the argument of case (i) holds by working with $s'_k := \sup\{s \in [0,t] : \ \widehat{\Lambda}_s \leqslant 2^{-k}\lfloor 2^k \widehat{\Lambda}_t \rfloor\}$. This implies that t belongs to the closure of $\bigcup_{r\in\mathbb{Q}_+^*}\{a_i^r, b_i^r \,:\, i \in \mathcal{I}_r\}$ giving (5.79). Moreover, by duality the contention (5.79) holds replacing $\text{Exit}(x)$ by ${}^{\downarrow}\text{Exit}(x)$, proving the first two equalities in (5.78). Consequently, to conclude it is enough to show that:

$$\overline{\{a_i^r, b_i^r : r \in \mathbb{Q}_+^* \text{ and } i \in \mathcal{I}_r\}} \subset [0,\sigma]\backslash \mathcal{C}^* \subset \overline{\text{Exit}(x) \cup {}^{\downarrow}\text{Exit}(x)}. \tag{5.80}$$

In this direction, notice that for every $r \in \mathbb{Q}_+^*$, under $\mathbb{N}_{x,r}$, we have $\widehat{\Lambda}_t > r$ for every $t \in (0,\sigma)$. Now, an application of the special Markov property applied to the domain D_r gives that:

$$\{a_i^r, b_i^r : \ i \in \mathcal{I}_r\} \subset [0,\sigma]\backslash \mathcal{C}^*, \quad \mathbb{N}_{x,0} - \text{a.e.},$$

for every $r \in \mathbb{Q}_+^*$, and the first inclusion $\subset$ in (5.80) follows. In order to obtain the remaining inclusion, let $t \in [0,\sigma]\backslash \mathcal{C}^*$. By definition, for every $\varepsilon > 0$ there exists $t - \varepsilon < t_1 < t_2 < t + \varepsilon$ such that $\widehat{\Lambda}_{t_1} < \widehat{\Lambda}_{t_2}$ or $\widehat{\Lambda}_{t_1} > \widehat{\Lambda}_{t_2}$. If the first holds, then $\sup\{s \in [t-\varepsilon, t_2] : \widehat{\Lambda}_s \leqslant \widehat{\Lambda}_{t_1}\}$ is an exit time and the other case follows by taking $\inf\{s \in [t_1,t_2] : \widehat{\Lambda}_s \leqslant \widehat{\Lambda}_{t_2}\}$. This ensures that t is in the closure of $\text{Exit}(x) \cup {}^{\downarrow}\text{Exit}(x)$ concluding our proof. □

Now, we are in position to state and prove the main result of the section:

Theorem 5.30. *Fix $(y, r_0) \in \overline{E}$ and $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$. Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r_0}$, we have*

$$\text{supp } \mathrm{d}A = \overline{\text{Exit}(x)} = \overline{{}^{\downarrow}\text{Exit}(x)} = [0,\sigma]\backslash \mathcal{C}^*,$$

where we recall the convention $[0,\infty] = [0,\infty)$.

Proof. First remark that by the special Markov property combined with (5.74) and (5.75), it is enough to prove the theorem under $\mathbb{N}_{x,0}$ and $\mathbb{P}_{0,x,0}$. We start by proving the theorem under $\mathbb{N}_{x,0}$ and remark that by Proposition 5.29 we only have to establish the first equality. Moreover, by Lemma 5.28 it only remains to show that under $\mathbb{N}_{x,0}$:

$$\text{supp } \mathrm{d}A \supset \overline{\text{Exit}(x)}. \tag{5.81}$$

However, by Lemma 5.25 we know that $\mathbb{N}_{x,0}(\{0,\sigma\} \cap \text{supp } \mathrm{d}A = \varnothing) = 0$, and then using that conditionally on $\mathcal{F}^{D_r}$ the measure $\mathcal{M}^{(r)}$ is a Poisson point measure with intensity measure given by $\mathbb{1}_{[0,\mathscr{L}_\sigma^r]}(\ell)\mathrm{d}\ell \; \mathbb{N}_{x,r}\left(\mathrm{d}\rho, \mathrm{d}\overline{W}\right)$, we derive that:

$$\mathbb{N}_{x,0} - \text{a.e., for all } r \in \mathbb{Q}_+^*, \ \{a_i^r, b_i^r : i \in \mathcal{I}_r\} \subset \text{supp } \mathrm{d}A.$$

Consequently, Proposition 5.29 implies (5.81). Finally, let us briefly explain how to obtain the result under $\mathbb{P}_{0,x,0}$. In this direction, under $\mathbb{P}_{0,x,0}$, denote the connected components of $\{s \in \mathbb{R}_+ :$

$X_s - I_s \neq 0\}$ by $\big((\alpha_i, \beta_i) : i \in \mathcal{I}\big)$. Excursion theory and our results under $\mathbb{N}_{x,0}$, give that, under $\mathbb{P}_{0,x,0}$, we have:

$$\text{supp } \mathrm{d}A \cap \cup_i(\alpha_i, \beta_i) = \overline{\text{Exit}(x)} \cap \cup_i(\alpha_i, \beta_i) = \overline{{\downarrow}\text{Exit}(x)} \ \cap \cup_i(\alpha_i, \beta_i) = \big([0, \sigma]\backslash\mathcal{C}^*\big) \cap \cup_i(\alpha_i, \beta_i),$$

$\alpha_i \in$ supp $\mathrm{d}A \cap \overline{\text{Exit}}(x) \cap \big([0, \infty)\backslash\mathcal{C}^*\big)$ and $\beta_i \in$ supp $\mathrm{d}A \cap {\downarrow}\text{Exit}(x) \cap \big([0, \infty)\backslash\mathcal{C}^*\big)$ for every $i \in \mathcal{I}$. The desired result now follows since the set $\{\alpha_i : \ i \in \mathcal{I}\}$ and $\{\beta_i : \ i \in \mathcal{I}\}$ are dense in $\{s \in \mathbb{R}_+ : X_s - I_s = 0\}$. □

5.5 The tree structure of $\{v \in \mathcal{T}_H : \ \xi_v = x\}$

In this section, we work under the framework introduced at the beginning of Section 5.4. Our goal now is to study the structure of the set $\{v \in \mathcal{T}_H : \xi_v = x\}$ and to do so, we encode it by *the subordinate tree of* $\mathcal{T}_H$ *with respect to the local time* $(\mathcal{L}_v : v \in \mathcal{T}_H)$. In this direction, we need to briefly recall the notion of subordination of trees defined in [66].

Subordination of trees by increasing functions. Let $(\mathcal{T}, d_\mathcal{T}, v_0)$ be an $\mathbb{R}$-tree and recall the standard notation $\preceq_\mathcal{T}$ and $\curlywedge_\mathcal{T}$ for the ancestor order and the first common ancestor. Next, consider $g : \mathcal{T} \to \mathbb{R}_+$ a non-negative continuous function. We say that g is non-decreasing if for every $u, v \in \mathcal{T}$:

$$u \preceq_\mathcal{T} v \text{ implies that } g(u) \leqslant g(v).$$

When the later holds, we can define a pseudo-distance on $\mathcal{T}$ by setting

$$d^g_\mathcal{T}(u, v) := g(u) + g(v) - 2 \cdot g(u \curlywedge_\mathcal{T} v), \qquad (u, v) \in \mathcal{T} \times \mathcal{T}. \tag{5.82}$$

The pseudo-distance $d^g_\mathcal{T}$ induces the following equivalence relation on $\mathcal{T}$: for $u, v \in \mathcal{T}$ we write

$$u \sim^g_\mathcal{T} v \iff d^g_\mathcal{T}(u, v) = 0,$$

and it was shown in [66] that $\mathcal{T}^g := (\mathcal{T}/ \sim^g_\mathcal{T}, d^g_\mathcal{T}, v_0)$ is a compact pointed $\mathbb{R}$-tree, where we still denoted the equivalency class of the root of $\mathcal{T}^g$ by v_0. The tree $\mathcal{T}^g$ is called the subordinate tree of $\mathcal{T}$ with respect to g and we write $p^\mathcal{T}_g : \mathcal{T} \to \mathcal{T}^g$ for the canonical projection which associates every $u \in \mathcal{T}$ with its $\sim^g_\mathcal{T}$–equivalency class. Observe that any two points $u, v \in \mathcal{T}$ are identified if and only if g stays constant on $[u, v]_\mathcal{T}$ and consequently the subordinate tree is obtained from $\mathcal{T}$ by identifying in a single point the components of $\mathcal{T}$ where g is constant.

Getting back to our setting, recall that under $\mathbb{N}_{x,0}$, $(\mathcal{L}_v : v \in \mathcal{T}_H)$ corresponds to $(\widehat{\Lambda}_t : t \geqslant 0)$ in the quotient space $\mathcal{T}_H = [0, \sigma]/ \sim_H$. This entails that the local time $(\mathcal{L}_v : v \in \mathcal{T}_H)$ is a non-decreasing function on $\mathcal{T}_H$ and we denote the induced subordinate tree by $\mathcal{T}^\mathcal{L}_H$. Recall that the exponent

$$\widetilde{\psi}(\lambda) = \mathcal{N}\left(\int_0^\sigma \mathrm{d}h \, \psi(u_\lambda(\xi_h))\right), \qquad \text{for } \lambda \geqslant 0,$$

is the exponent of a Lévy tree by Proposition 5.19. Hence, it satisfies (A1)—(A4) and by Corollary 5.21 it can be written in the following form:

$$\widetilde{\psi}(\lambda) := \tilde{\alpha}\lambda + \int_{(0,\infty)} \tilde{\pi}(\mathrm{d}x)(\exp(-\lambda x) - 1 + \lambda x),$$

where $\widetilde{\alpha} = \mathcal{N}\big(1-\exp(-\alpha\sigma)\big)$ and $\widetilde{\pi}$ is a sigma-finite measure on $\mathbb{R}_+\backslash\{0\}$ satisfying $\int_{(0,\infty)} \widetilde{\pi}(\mathrm{d}x)(x\wedge x^2) < \infty$. We will also use the notation $\widetilde{H}$ and $\widetilde{N}$ introduced prior to (5.66) for the height process and the excursion measure of a $\widetilde{\psi}$–Lévy tree. Finally, we recall that A stands for the additive functional introduced in Proposition 5.22 and we denote its right inverse by $A_t^{-1} := \inf\{s \geqslant 0 : A_s > t\}$, with the convention $A_t^{-1} = \sigma$ for every $t \geqslant A_\infty = A_\sigma$. Remark that the constancy intervals of A in $[0,\sigma]$ are the connected components of $[0,\sigma]\backslash\mathrm{supp}\,\mathrm{d}A$, which by Theorem 5.30 are precisely the connected components of $\mathcal{C}^*$ – the constancy intervals of the process $(\widehat{\Lambda}_t :\ t \in [0,\sigma])$. In particular, $(\widehat{\Lambda}_{A_t^{-1}} : t \geqslant 0)$ is a continuous non-negative process, with lifetime A_∞. We can now state the main result of this section:

Theorem 5.31. *The following properties hold:*

(i) *Under $\mathbb{N}_{x,0}$, the subordinate tree of $\mathcal{T}_H$ with respect to the local time $\mathcal{L}$, that we denote by $\mathcal{T}_H^{\mathcal{L}}$, is isometric to the tree coded by the continuous function $(\widehat{\Lambda}_{A_t^{-1}} :\ t \geqslant 0)$.*

(ii) *Moreover, we have the equality in distribution*

$$\Big((\widetilde{H}_t : t \geqslant 0),\ \textit{under } \widetilde{N}\Big) \overset{(d)}{=} \Big(\big(\widehat{\Lambda}_{A_t^{-1}} : t \geqslant 0\big),\ \textit{under } \mathbb{N}_{x,0}\Big). \tag{5.83}$$

In particular, $\mathcal{T}_H^{\mathcal{L}}$ is a Lévy tree with exponent $\widetilde{\psi}$.

Remark 5.32. Let us mention that when $\psi(\lambda) = \lambda^2/2$ and the underlying spatial motion ξ is a Brownian motion in $\mathbb{R}$, the previous theorem implies that under $\mathbb{N}_{0,0}$ the subordinate tree of $\mathcal{T}_H$ with respect to the local time $\mathcal{L}$ at 0 is a Lévy tree and – as a direct consequence of the scaling invariance of the Brownian motion – its exponent is of the form $\widetilde{\psi}(\lambda) = c\lambda^{3/2}$, for some constant $c > 0$. This result was already obtained by other methods in [66, Theorem 2].

We stress that the key result in (ii) is the identity in distribution (5.83): it entails that not only the function $(\widehat{\Lambda}_{A_t^{-1}} : t \geqslant 0)$ encodes the subordinate tree, but it is also the height process of a Lévy tree. The fact that $\mathcal{T}_H^{\mathcal{L}}$ is a $\widetilde{\psi}$-Lévy tree is then a direct consequence of (i) and (5.83). By a straightforward application of excursion theory one can deduce a version under $\mathbb{P}_{0,x,0}$ of Theorem 5.31, where now $\mathcal{T}_H^{\mathcal{L}}$ is a Lévy forest with exponent $\widetilde{\psi}$. The details are left to the reader.

The rest of the section is organised as follows: The section is devoted to the proof of Theorem 5.31. In Section 5.5.1 we start by showing (i) and we present the strategy that we follow to prove (ii). The proof of (ii) relies in all the machinery developed in previous sections combined with standard properties of Poisson point measures and is the content of Section 5.5.2.

5.5.1 The height process of the subordinate tree

In this short section we establish the first claim of Theorem 5.31 and settle the ground for the second part of the result. For every $u \in \mathcal{T}_H$, recall that $\mathcal{L}_u := \widehat{\Lambda}_s$ where s is any element of $p_H^{-1}(\{u\})$ (note that the definition is non ambiguous by the snake property) and that $\mathcal{L}$ is non-decreasing on $\mathcal{T}_H$. To simplify notation, we set:

$$H_t^A := \widehat{\Lambda}_{A_t^{-1}}, \quad t \geqslant 0,$$

which is a continuous process – as it was already mentioned in the discussion before Theorem 5.31. Let us start with the proof of Theorem 5.31-(i).

Proof of Theorem 5.31-(i). Our goal is to show that, under $\mathbb{N}_{x,0}$, the trees $\mathcal{T}_{H^A}$ and $\mathcal{T}_H^{\mathcal{L}}$ are isometric. In this direction, we start by introducing the pseudo-distance:

$$\widetilde{d}(s,t) := \widehat{\Lambda}_t + \widehat{\Lambda}_s - 2 \cdot \min_{s \wedge t, s \vee t} \widehat{\Lambda}\ , \quad s,t \in [0,\sigma],$$

and we write $s \approx t$ if and only if $\widetilde{d}(s,t) = 0$. By the snake property, we have $s \approx t$ for every $s \sim_H t$. Moreover, since $\mathcal{L}$ is increasing on $\mathcal{T}_H$, we get

$$\widetilde{d}(s,t) = \mathcal{L}_{p_H(t)} + \mathcal{L}_{p_H(s)} - 2 \cdot \mathcal{L}_{p_H(s) \wedge_{\mathcal{T}_H} p_H(t)},$$

for every $s,t \in [0,\sigma]$. The right-hand side of the previous display is exactly the definition of the pseudo-distance associated with the subordinate tree $\mathcal{T}_H^{\mathcal{L}}$ between $p_H(s)$ and $p_H(t)$ given in (5.82). We deduce that $([0,\sigma]/\approx, \widetilde{d}, 0)$ is isometric to $\mathcal{T}_H^{\mathcal{L}}$. It remains to show that $([0,\sigma]/\approx, \widetilde{d}, 0)$ is also isometric to $(\mathcal{T}_{H^A}, d_{H^A}, 0)$. In order to prove it, we notice that:

$$\widetilde{d}(A_{r_1}^{-1}, A_{r_2}^{-1}) = d_{H^A}(r_1, r_2),$$

for every $r_1, r_2 \in [0, A_\sigma]$. Furthermore, for every $t \in [0,\sigma]$ there exists $r \in [0, A_\sigma]$ such that $A_{r-}^{-1} \leqslant t \leqslant A_r^{-1}$ since by Lemma 5.25 the points 0 and σ are in the support of $\mathrm{d}A$. Moreover we have $\widetilde{d}(A_r^{-1}, t) = 0$, since by Theorem 5.30 the process $\widehat{\Lambda}$ stays constant on every interval of the form $[A_{r-}^{-1}, A_r^{-1}]$. This implies that $[0,\sigma]/\approx\ = \{A_r^{-1} : r \in [0, A_\infty]\}/\approx$ and we deduce by the previous display that $([0,\sigma]/\approx, \widetilde{d}, 0)$ and $(\mathcal{T}_{H^A}, d_{H^A}, 0)$ are isometric giving the desired result. □

The main difficulty to establish Theorem 5.31 (ii) comes from the fact that $\widetilde{H}$ is not a Markov process. To circumvent this, we are going to use the notion of marked trees embedded in a function.

Marked trees embedded in a function. A marked tree is a pair $\mathbf{T} := (\mathrm{T}, \{h_v : v \in \mathrm{T}\})$, where T is a finite rooted ordered tree and $h_v \geqslant 0$ for every $v \in \mathrm{T}$ – the number h_v is called the label of the individual v. For completeness let us give the formal definition of a rooted ordered tree. First, introduce Ulam's tree:

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \{1,2,...\}^n$$

where by convention $\{1,2,...\}^0 = \varnothing$. If $u = (u_1, ...u_m)$ and $v = (v_1, ..., v_n)$ belong to $\mathcal{U}$, we write uv for the concatenation of u and v, viz. $(u_1, ...u_m, v_1, ..., v_n)$. In particular, we have $u\varnothing = \varnothing u = u$. A (finite) rooted ordered tree T is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \mathrm{T}$;

(ii) If $v \in \mathrm{T}$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \{1,2,...\}$, then $u \in \mathrm{T}$;

(iii) For every $u \in \mathrm{T}$, there exists a number $k_u(\mathrm{T}) \geqslant 0$ such that $uj \in \mathrm{T}$ if and only if $1 \leqslant j \leqslant k_u(\mathrm{T})$.

If $u \in \mathrm{T}$ can be written as $u = vj$ for some $v \in \mathrm{T}$, $1 \leqslant j \leqslant k_v(\mathrm{T})$, we say that v is the parent of u. More generally, if $u = vy$ for some $v \in \mathrm{T}$ and $y \in \mathcal{U}$ with $y \neq \varnothing$, we say that v is an ancestor of u or equivalently that u is a descendant of v. On the other hand, if $u \in \mathrm{T}$ satisfies that $k_u(\mathrm{T}) = 0$,

u is called a leaf. The element $\varnothing$ is interpreted as the root of the tree and if v is a vertex of T, the branch connecting the root and v is the set of prefixes of v – considered with its corresponding family of labels.

Let us also introduce the concatenation of marked trees. If $\mathbf{T}_1, ..., \mathbf{T}_k$ are k marked trees and h is a non-negative real number, we write $[\mathbf{T}_1, ..., \mathbf{T}_k]_h$ for the marked tree defined as follows. The lifetime of $\varnothing$ is h, $k_\varnothing = k$, and for $1 \leqslant j \leqslant k$ the point ju belongs to the tree structure of $[\mathbf{T}_1, ..., \mathbf{T}_k]_h$ if and only if $u \in \mathrm{T}_j$ and its label is the label of u in $\mathbf{T}_j$. For convenience, we will identity a marked tree $\mathbf{T} := (\mathrm{T}, \{h_v : \ v \in \mathrm{T}\})$ with the set $\{(v, h_v) : \ v \in \mathrm{T}\}$.

We are now in position to define the embedded marked tree associated with a continuous function $(e(t))_{t\in[a,b]}$ and a given finite collection of times. We fix a finite sequence of times $a \leqslant t_1 \leqslant \dots t_n \leqslant b$ and we recall the notation $m_e(s,t) = \inf_{[s\wedge t, s\vee t]} e$. The embedded tree associated with the marks $t_1, \dots, t_n$ and the function e, $\theta(e, t_1, \dots, t_n)$, is defined inductively, according to the following steps:

- If $n = 1$, set $\theta(e, t_1) = (\varnothing, \{e(t_1)\})$.
- If $n \geqslant 2$, suppose that we know how to construct marked trees with less than n marks. Let $i_1, \dots, i_k$ be the distinct indices satisfying that $m_e(t_{i_q}, t_{i_q+1}) = m_e(t_1, t_n)$, and define the following restrictions for $1 \leqslant q \leqslant k-1$

  $$e^{(0)}(t) := (e(t) : t \in [t_1, t_{i_1}]),\ e^{(q)}(t) := (e(t) : t \in [t_{i_q+1}, t_{i_{q+1}}]),\ e^{(k)}(t) := (e(t) : t \in [t_{i_k+1}, t_n]).$$

  Next, consider the associated finite labelled trees,

  $$\theta(e^{(0)}, t_1, \dots, t_{i_1}),\ \theta(e^{(q)}, t_{i_q+1}, \dots, t_{i_{q+1}}),\ \theta(e^{(k)}, t_{i_k+1}, \dots, t_n), \qquad \text{for } 1 \leqslant q \leqslant k-1,$$

  and finally, concatenate them with a common ancestor with label $m_e(t_1, t_n)$, by setting

  $$\theta(e, t_1, \dots, t_n) := [\theta(e^{(0)}, t_1, \dots, t_{i_1}), \dots, \theta(e^{(k)}, t_{i_k+1}, \dots, t_n)]_{m_e(t_1,t_n)},$$

  and completing the recursion.

We say that the label h_v is the height of v in $\theta(e, t_1, \dots, t_n) = (\mathrm{T}, \{h_v : \ v \in \mathrm{T}\})$. Let us justify this terminology. First assume that $e(0) = 0$ and consider $\mathcal{T}_e$ the compact $\mathbb{R}$–tree induced by e. Then if $v_1, \dots, v_n$ are the leaves of $\mathbf{T}$ in lexicographic order, we have $(h_{v_1}, \dots, h_{v_n}) = (e(t_1), \dots, e(t_n))$. Moreover, if we write $v_i \curlywedge_{\mathrm{T}} v_j$ for the common ancestor of v_i and v_j in $\mathbf{T}$, it holds that $h_{v_j \curlywedge_{\mathrm{T}} v_i} = \inf_{[t_i\wedge t_j, t_i \vee t_j]} e$.[5]

Statements and main steps for the proof of Theorem 5.31 (ii). Our argument relies in identifying the distribution of the discrete embedded tree associated with $(\widehat{\Lambda}_{A_t^{-1}} : 0 \leqslant t \leqslant A_\infty)$ when the collection of marks are Poissonian. In this direction, we denote the law of a Poisson process $(\mathcal{P}_t : \ t \geqslant 0)$ with intensity λ by Q^λ and we work with the pair $(H_t^A, \mathcal{P}_t)_{t \leqslant A_\infty}$, under the product measure $\mathbb{N}_{x,0} \otimes Q^\lambda$. For convenience, we denote the law of $(\rho, \overline{W}, \mathcal{P}_{\cdot\wedge A_\infty})$ under $\mathbb{N}_{x,0} \otimes Q^\lambda$ by $\mathbb{N}^\lambda_{x,0}$ and we let $0 \leqslant \mathfrak{t}_1 < \dots < \mathfrak{t}_M \leqslant A_\infty$ be the jumping times of $(\mathcal{P}_t)$ falling in the excursion interval $[0, A_\infty]$, where $M := \mathcal{P}_{A_\infty}$. Finally, consider the associated embedded tree

$$\mathbf{T}^A := \theta\big(H^A, \mathfrak{t}_1, \dots, \mathfrak{t}_M\big), \qquad \text{under } \mathbb{N}^\lambda_{x,0}(\,\cdot\,| M \geqslant 1).$$

[5]The definition of $\theta(e, t_1, \dots, t_n)$ is directly connected with the classical notion of marginals trees – where the label of a point is the increment between its height and the height of its parent.

Remark that the probability measure $\mathbb{N}^\lambda_{x,0}(\,\cdot\,|M \geqslant 1)$ is well defined since by Proposition 5.24 we have

$$\mathbb{N}^\lambda_{x,0}\,(M \geqslant 1) = \mathbb{N}_{x,0}\,(1 - \exp(-\lambda A_\infty)) = \widetilde{\psi}^{-1}(\lambda).$$

Our goal is to show that $\mathbf{T}^A$ is distributed as the discrete embedded tree of a $\widetilde{\psi}$-Lévy tree associated with Poissonian marks with intensity λ. To state this formally, recall the notation $\widetilde{N}$ for the excursion measure of a $\widetilde{\psi}$-Lévy process, and that $\widetilde{H}$ stands for the associated height process. We write $\widetilde{N}^\lambda$ for the law of $(\widetilde{\rho}, \mathcal{P}_{\cdot \wedge \sigma_{\widetilde{H}}})$ under $\widetilde{N} \otimes Q^\lambda$ and remark that $\widetilde{M} := \mathcal{P}_{\sigma_{\widetilde{H}}}$ is the number of Poissonian marks in $[0, \sigma_{\widetilde{H}}]$. For simplicity, we denote the jumping times of $\mathcal{P}$ under $\widetilde{N}^\lambda$ by $\mathtt{t}_1, \dots, \mathtt{t}_{\widetilde{M}}$.

Proposition 5.33. *The discrete tree* $\mathbf{T}^A$ *under* $\mathbb{N}^\lambda_{x,0}(\,\cdot\,|M \geqslant 1)$ *has the same distribution as*

$$\widetilde{\mathbf{T}} := \theta\big(\widetilde{H}, \mathtt{t}_1, \dots, \mathtt{t}_{\widetilde{M}}\big) \qquad \text{under } \widetilde{N}^\lambda(\,\cdot\,|\widetilde{M} \geqslant 1).$$

The proof of Proposition 5.33 is rather technical and will be postponed to Section 5.5.2. The reason behind considering Poissonian marks to identify the distribution of H^A is to take advantage of the memoryless of Poissonian marks; this flexibility will allow us to make extensive use of the Markov property and excursion theory. Let us now explain how to deduce Theorem 5.31 (ii) from Proposition 5.33.

Proof of Theorem 5.31 (ii). First remark that the fact that $\mathcal{T}^{\mathcal{L}}_H$ is a $\widetilde{\psi}$-Lévy tree is a direct consequence of Theorem 5.31 (i) and (5.83). To conclude it remains to prove (5.83). In this direction, notice that the marked trees considered are ordered trees – the order of the vertices being the one induced by the marks. Recall that for every $1 \leqslant i \leqslant M$, the quantity $H^A_{\mathtt{t}_i}$ is the label of the i-th leaf in lexicographical order, and the same remark holds replacing $(H^A, M, \mathbf{T}^A)$ by $(\widetilde{H}, \widetilde{M}, \widetilde{\mathbf{T}})$. Consequently, the identity $\mathbf{T}^A \overset{(d)}{=} \widetilde{\mathbf{T}}$ of Proposition 5.33 yields the following equality in distribution

$$\Big(\big(\widetilde{M}, \widetilde{H}_{\mathtt{t}_1}, \dots, \widetilde{H}_{\mathtt{t}_{\widetilde{M}}}\big) :\ \widetilde{N}^\lambda(\,\cdot\,|\widetilde{M} \geqslant 1)\Big) \overset{(d)}{=} \Big(\big(M, H^A_{\mathtt{t}_1}, \dots, H^A_{\mathtt{t}_M}\big) :\ \mathbb{N}^\lambda_{x,0}(\,\cdot\,|M \geqslant 1)\Big).$$

Recall from Proposition 5.24 and the discussion after it, that A_∞ under $\mathbb{N}_{x,0}$ and $\sigma_{\widetilde{H}}$ under $\widetilde{N}$ have the same distribution. This ensures that, up to enlarging the measure space, we can define the height process $\widetilde{H}$ under the measure $\mathbb{N}^\lambda_{x,0}$ in such a way that its lifetime is precisely A_∞, viz. $\sigma_{\widetilde{H}} = A_\infty$, and then we might and will consider the same collection of Poisson marks $\mathtt{t}_1, \dots, \mathtt{t}_M$ to mark the processes H^A and $\widetilde{H}$. In the rest of the proof, we work with this coupling. In particular, under $\mathbb{N}^\lambda_{x,0}(\,\cdot\,|M \geqslant 1)$, our previous discussion entails

$$\Big(M, \widetilde{H}_{\mathtt{t}_1}, \dots, \widetilde{H}_{\mathtt{t}_M}\Big) \overset{(d)}{=} \Big(M, H^A_{\mathtt{t}_1}, \dots, H^A_{\mathtt{t}_M}\Big).$$

Let $(U_i :\ i \geqslant 1)$ be a collection of independent identically distributed uniform random variables in $[0, A_\infty]$ – and independent of all the rest. Remark that, conditionally on A_∞, $(\mathcal{P}_t : t \leqslant A_\infty)$ is independent of $\widetilde{H}$ and H^A, and the random variable M is Poisson with intensity (λA_∞). By conditioning on A_∞, we deduce that for any $m \geqslant 1$ and any measurable function $f : \mathbb{R}^m \mapsto \mathbb{R}_+$, we have

$$\mathbb{N}^{\lambda}_{x,0}\left(f(\widetilde{H}_{U^m_{(1)}},\dots,\widetilde{H}_{U^m_{(m)}})\frac{(\lambda A_\infty)^m}{m!}\exp\big(-\lambda A_\infty\big)\right)$$
$$= \mathbb{N}^{\lambda}_{x,0}\left(f(H^A_{U^m_{(1)}},\dots,H^A_{U^m_{(m)}})\frac{(\lambda A_\infty)^m}{m!}\exp\big(-\lambda A_\infty\big)\right), \tag{5.84}$$

where $(U^m_{(1)},\dots,U^m_{(m)})$ stands for the order statistics of $\{U_1,\dots,U_m\}$. Since the previous display holds for every $\lambda>0$ we get that

$$\left(A_\infty,\widetilde{H}_{U^m_{(1)}},\dots,\widetilde{H}_{U^m_{(m)}}\right)\overset{(d)}{=}\left(A_\infty,H^A_{U^m_{(1)}},\dots,H^A_{U^m_{(m)}}\right),$$

for every $m\geqslant 1$. Denote the unique continuous function vanishing at $\mathbb{R}_+\backslash(0,A_\infty)$ and linearly interpolating between the points $\{(A_\infty\cdot im^{-1},\widetilde{H}_{U_i^{(m)}}) : i\in\{1,\dots,m\}\}\cup\{(0,0),(A_\infty,0)\}$ by $(\widetilde{H}^m_t :\ t\geqslant 0)$. Similarly, let $H^{A,m}$ be the analogous function defined by replacing $\widetilde{H}$ by H^A. Next we remark that (5.84) ensures that, for every continuous bounded function $F:\mathbb{R}^k\mapsto\mathbb{R}_+$ and any fixed set of times $0\leqslant t_1\leqslant\dots\leqslant t_k$, we have

$$\mathbb{N}^{\lambda}_{x,0}\left(F(\widetilde{H}^m_{t_1},\dots,\widetilde{H}^m_{t_k})\cdot\big(1-\exp(-A_\infty)\big)\right)=\mathbb{N}^{\lambda}_{x,0}\left(F(H^{A,m}_{t_1},\dots,H^{A,m}_{t_k})\cdot\big(1-\exp(-A_\infty)\big)\right).$$

Using the fact that $U^m_{\lfloor tm\rfloor}\to A_\infty\cdot t$ a.s. for every $t\in[0,1]$, we derive that the pointwise convergences $\widetilde{H}^m\to\widetilde{H}$ and $H^{A,m}\to H^A$ as $m\uparrow\infty$. Finally since $\mathbb{N}^{\lambda}_{x,0}(1-\exp(-A_\infty))<\infty$, we deduce (5.83) by dominated convergence. □

5.5.2 Trees embedded in the subordinate tree

This section is devoted to the proof of Proposition 5.33. In short, the idea is to decompose inductively $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$ starting from their respective "left-most branches" – viz. the path connecting the root $\varnothing$ and the first leaf with the corresponding labels – and to show that they have the same law. Next, if we remove the left-most branch of $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$, we are left with two ordered collections of independent subtrees and we shall establish that they have respectively the same law as $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$. This will allow us to iterate this left-most branch decomposition in such a way that the branches discovered at step n in $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$ have the same law. Proposition 5.33 will follow since this procedure leads respectively to discover $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$. In order to state this formally let us introduce some notation.

If $\mathbf{T}:=(\mathrm{T},(h_v:\ v\in\mathrm{T}))$ is a discrete labelled tree and $n\geqslant 0$, we let $\mathbf{T}(n)$ be the set of all couples $(u,h_u)\in\mathbf{T}$ such that u has at most n entries in $\{2,3,...\}$. In particular $\mathbf{T}(0)$ is the branch connecting the root and the first leaf. Next, we introduce the collection

$$\mathbb{S}(\mathbf{T}):=\big((h_v,k_v(\mathrm{T})-1):\ v\text{ is a vertex of }\mathbf{T}(0)\big),$$

where the elements are listed in increasing order with respect to the height and we recall that $k_v(\mathrm{T})$ stands for the number of children of v. For simplicity, set $R:=\#\mathbf{T}(0)-1$, write $v_1,...,v_{R+1}$ for the vertices of $\mathbf{T}(0)$ in lexicographic order and observe that v_1 is the root while v_{R+1} is the first leaf – in particular $k_{v_{R+1}}(\mathrm{T})=0$. Heuristically, $\mathbb{S}(\mathbf{T})$ – or more precisely the measure

$\sum_i (k_{v_i}(\mathrm{T}) - 1)\delta_{h_{v_i}}$ – is a discrete version of the exploration process when visiting the first leaf of T and for this reason $\mathbb{S}(\mathbf{T})$ will be called the left-most spine of $\mathbf{T}$. Now, for every $1 \leqslant j \leqslant R$, set

$$K_j(\mathbf{T}) := \sum_{i=1}^{j} (k_{v_i}(\mathrm{T}) - 1),$$

with the convention $K_0(\mathbf{T}) = 0$ and remark that $K(\mathbf{T}) := K_R(\mathbf{T})$ stands for the number of subtrees attached "to the right" of $\mathbf{T}(0)$ in $\mathbf{T}$. To define these subtrees when $K(\mathbf{T}) \geqslant 1$, we need to introduce the following: for every $1 \leqslant i \leqslant K_R(\mathbf{T}) = K(\mathbf{T})$, let $a(i)$ be the unique index such that $K_{a(i)-1}(\mathbf{T}) < i \leqslant K_{a(i)}(\mathbf{T})$. Then, we introduce the marked tree

$$\mathbf{T}_i := \Big\{(u, h'_u) : \ \big(v_{a(i)}(K_{a(i)} + 2 - i)u, h_{v_{a(i)}} + h'_u\big) \in \mathbf{T}\Big\}. \tag{5.85}$$

Remark that the labels in each subtree $\mathbf{T}_i$ have been shifted by their relative height in $\mathbb{S}(\mathbf{T})$ and that the collection $(\mathbf{T}_i : 1 \leqslant i \leqslant K(\mathbf{T}))$ is listed in counterclockwise order.

We now apply this decomposition to $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$. For simplicity, we write $\widetilde{K} := K(\widetilde{\mathbf{T}})$ (resp. $K := K(\mathbf{T}^A)$) for the number of subtrees attached to the right of $\widetilde{\mathbf{T}}(0)$ (resp. $\mathbf{T}^A(0)$). When $\widetilde{K} \geqslant 1$ (resp. $K \geqslant 1$), we let $\widetilde{\mathbf{T}}_i$ (resp. $\mathbf{T}^A_i$) be the marked trees defined by (5.85) using $\widetilde{\mathbf{T}}$ (resp. $\mathbf{T}^A$). Proposition 5.33 can now be reduced to the following result:

Proposition 5.34. (i) *We have*

$$\Big(\mathbb{S}(\widetilde{\mathbf{T}}) \ : \ \widetilde{N}^\lambda(\cdot|\widetilde{M} \geqslant 1)\Big) \overset{(d)}{=} \Big(\mathbb{S}(\mathbf{T}^A) \ : \ \mathbb{N}^\lambda_{x,0}(\cdot|M \geqslant 1)\Big).$$

(ii) *Under* $\widetilde{N}^\lambda(\cdot\,|\widetilde{K}, \widetilde{M} \geqslant 1)$ *and conditionally on* $\mathbb{S}(\widetilde{\mathbf{T}})$*, the subtrees* $\widetilde{\mathbf{T}}_1, \dots \widetilde{\mathbf{T}}_{\widetilde{K}}$ *are distributed as* $\widetilde{K}$ *independent copies distributed as* $\widetilde{\mathbf{T}}$ *under* $\widetilde{N}^\lambda(\cdot\,|\widetilde{M} \geqslant 1)$*. Similarly, under* $\mathbb{N}^\lambda_{x,0}(\mathrm{d}\overline{W}, \mathrm{d}\mathcal{P}|K, M \geqslant 1)$ *and conditionally on* $\mathbb{S}(\mathbf{T}^A)$*, the subtrees* $\mathbf{T}^A_1, \dots, \mathbf{T}^A_K$ *are distributed as* K *independent copies distributed as* $\mathbf{T}^A$ *under* $\mathbb{N}^\lambda_{x,0}(\mathrm{d}\overline{W}, \mathrm{d}\mathcal{P}|M \geqslant 1)$.

Let us explain why Proposition 5.33 is a consequence of the previous result.

Proof of Proposition 5.33. We are going to show by induction that for every $n \geqslant 0$:

$$\widetilde{\mathbf{T}}(n) \text{ under } \widetilde{N}^\lambda(\cdot\,|\widetilde{M} \geqslant 1) \quad \text{ is distributed as } \quad \mathbf{T}^A(n) \text{ under } \mathbb{N}^\lambda_{x,0}(\cdot|M \geqslant 1). \tag{5.86}$$

First notice that Proposition 5.34 - (i) gives the previous identity in the case $n = 0$. Assume now that (5.86) holds for $n \geqslant 0$ and let us prove the identity for $n + 1$. First, remark that it is enough to argue with $\widetilde{\mathbf{T}}(n+1)$ under $\widetilde{N}^\lambda(\cdot\,|\widetilde{K}, \widetilde{M} \geqslant 1)$ and $\mathbf{T}^A(n+1)$ under $\mathbb{N}^\lambda_{x,0}(\cdot|K, M \geqslant 1)$ – since by Proposition 5.34, the variable $\widetilde{K}$ under $\widetilde{N}^\lambda(\cdot\,|\widetilde{M} \geqslant 1)$ is distributed as K under $\mathbb{N}^\lambda_{x,0}(\cdot|M \geqslant 1)$. Next, we see that $\widetilde{\mathbf{T}}(n+1)$ can be obtained by gluing the trees $\widetilde{\mathbf{T}}_i(n)$ to $\widetilde{\mathbf{T}}(0)$ at their respective positions after translating the labels by the associated heights. Moreover, these positions and heights are precisely the entries of $\mathbb{S}(\widetilde{\mathbf{T}})$. Since the same discussion holds when replacing $\widetilde{\mathbf{T}}$ by $\mathbf{T}^A$, the case $n + 1$ follows by Proposition 5.34 and the case n. Finally, since the trees $\widetilde{\mathbf{T}}$ and $\mathbf{T}^A$ are finite, (5.86) implies the desired result. □

Our goal now is to prove Proposition 5.34. In this direction, we will first encode the spines $\mathbb{S}(\widetilde{\mathbf{T}}), \mathbb{S}(\mathbf{T}^A)$ as well as the corresponding subtrees $\widetilde{\mathbf{T}}_i$, $\mathbf{T}_i^A$ in terms of $\widetilde{\rho}$, $(\rho, \overline{W})$ and $\mathcal{P}$. This will allow us to identify their law by making use of the machinery developed in previous sections. While $\mathbb{S}(\widetilde{\mathbf{T}})$ can be constructed directly in terms of $(\widetilde{\rho}_{\mathtt{t}_1+t} : t \geqslant 0)$ and the Poisson marks, the construction of $\mathbb{S}(\mathbf{T}^A)$ is more technical. Roughly speaking, the strategy consists in defining in terms of $(\rho, \overline{W})$ the exploration process for the subordinate tree at time $\mathtt{t}_1$, say $\rho^*_{\mathtt{t}_1}$, and then show – see Lemma 5.38 below – that $\widetilde{\rho}_{\mathtt{t}_1}$ and $\rho^*_{\mathtt{t}_1}$ have the same distribution. Needless to say that this statement is informal, since we have not yet shown that the subordinate tree is a Lévy tree. We will then deduce (i) by considering $\mathbb{S}(\widetilde{\mathbf{T}})$, $\mathbb{S}(\mathbf{T}^A)$ and conditioning respectively on $\widetilde{\rho}_{\mathtt{t}_1}$ and $\rho^*_{\mathtt{t}_1}$, Point (ii) will then follow easily by construction. For simplicity, from now on we write $\mathtt{t} := \mathtt{t}_1$.

We first start working under $\widetilde{N}^\lambda(\cdot \,|M \geqslant 1)$ and we introduce the following notation: let $\big((\widetilde{\alpha}_i, \widetilde{\beta}_i) : i \in \mathbb{N}\big)$ be the connected components of the open set

$$\big\{s \geqslant \mathtt{t} : \ \widetilde{H}_s > \inf_{[\mathtt{t},s]} \widetilde{H}\big\}.$$

As usual, we write $\widetilde{\rho}^i$ for the associated subtrajectory of the exploration process in the interval $[\widetilde{\alpha}_i, \widetilde{\beta}_i]$. We also consider $\widetilde{H}^i := (\widetilde{H}_{(\widetilde{\alpha}_i+s)\wedge\widetilde{\beta}_i} - \widetilde{H}_{\widetilde{\alpha}_i} : \ s \geqslant 0)$, $\widetilde{\mathcal{P}}^i := (\widetilde{\mathcal{P}}_{(\widetilde{\alpha}_i+t)\wedge\widetilde{\beta}_i} - \widetilde{\mathcal{P}}_{\widetilde{\alpha}_i} : \ t \geqslant 0)$ and note that in particular we have $H(\widetilde{\rho}^i) = \widetilde{H}^i$. Write $\widetilde{h}_i := \widetilde{H}(\widetilde{\alpha}_i)$, and consider the marked measure:

$$\widetilde{\mathcal{M}} := \sum_{i\in\mathbb{N}} \delta_{(\widetilde{h}_i, \widetilde{\rho}^i, \widetilde{\mathcal{P}}^i)}.$$

By the Markov property and (5.22), conditionally on $\mathcal{F}_\mathtt{t}$, the measure $\widetilde{\mathcal{M}}$ is a Poisson point measure with intensity $\widetilde{\rho}_\mathtt{t}(\mathrm{d}h)\widetilde{N}^\lambda(\mathrm{d}\rho, \mathrm{d}\mathcal{P})$. Now we can identify $\mathbb{S}(\widetilde{\mathbf{T}})$ in terms of functionals of $\widetilde{\mathcal{M}}$ and $\widetilde{H}_\mathtt{t}$. First, set $(\widetilde{h}_p^\circ : \ 1 \leqslant p \leqslant \widetilde{R})$ the collection of the different heights – in increasing order – among $(\widetilde{h}_i : \ i \in \mathbb{N})$ at which $\widetilde{\mathcal{P}}^i_{\sigma(\widetilde{\rho}^i)} \geqslant 1$. In particular, $\widetilde{R}$ gives the number of different heights $\widetilde{h}_j$ at which we can find at least one marked excursion above the running infimum of $(\widetilde{H}_{\mathtt{t}+t} : t \geqslant 0)$. Next, we write $\widetilde{M}_p^\circ$ for the number of atoms at level $\widetilde{h}_p^\circ$ in $\widetilde{\mathcal{M}}$ with at least one Poissonian mark. Now, remark that by construction we have:

$$\mathbb{S}(\widetilde{\mathbf{T}}) = \big((\widetilde{h}_1^\circ, \widetilde{M}_1^\circ), \ldots, (\widetilde{h}_{\widetilde{R}}^\circ, \widetilde{M}_{\widetilde{R}}^\circ), (\widetilde{H}_\mathtt{t}, -1)\big), \tag{5.87}$$

and in particular $\widetilde{K} = \sum_{i=1}^{\widetilde{R}} \widetilde{M}_i^\circ$. Finally, for later use denote the corresponding marked excursions arranged in counterclockwise order by $\widetilde{\mathscr{E}} := ((\widetilde{\rho}_\circ^q, \widetilde{H}_\circ^q, \widetilde{\mathcal{P}}_\circ^q) : 1 \leqslant q \leqslant \widetilde{K})$. Notice that the subtrees $(\widetilde{\mathbf{T}}_i : 1 \leqslant i \leqslant \widetilde{K})$ are precisely the respective embedded marked trees associated with $((\widetilde{H}_\circ^q, \widetilde{\mathcal{P}}_\circ^q) : 1 \leqslant q \leqslant \widetilde{K})$.

The main step remaining in our analysis under $\widetilde{N}^\lambda(\cdot \,|\widetilde{M} \geqslant 1)$ consists in characterizing the law of $(\widetilde{H}_\mathtt{t}, \widetilde{\rho}_\mathtt{t})$, and this is the content of the following lemma. Since $\widetilde{\mathcal{M}}$ conditionally on $\mathcal{F}_\mathtt{t}$ is a Poisson point measure with intensity $\widetilde{\rho}_\mathtt{t}(\mathrm{d}h)\widetilde{N}^\lambda(\mathrm{d}\rho, \mathrm{d}\mathcal{P})$, this will suffice to identify the distribution of $\mathbb{S}(\widetilde{\mathbf{T}})$. In this direction, Corollary 5.21 ensures that the measure $\widetilde{\rho}_\mathtt{t}$ is purely atomic and consequently by (5.7) it is of the form:

$$\widetilde{\rho}_\mathtt{t} := \sum_{i\in\mathbb{N}} \widetilde{\Delta}_i \cdot \delta_{\widetilde{h}^i} \,.$$

We stress that we have $\{\widetilde{h}^i : i \in \mathbb{N}\} = \{\widetilde{h}_i : i \in \mathbb{N}\}$ – even though the latter set has repeated elements.

Lemma 5.35. *Under $\widetilde{N}^\lambda(\cdot|\ \widetilde{M} \geqslant 1)$, the random variable $\widetilde{H}_{\mathfrak{t}}$ is exponentially distributed with intensity $\lambda/\widetilde{\psi}^{-1}(\lambda)$. Moreover, conditionally on $\widetilde{H}_{\mathfrak{t}}$, the measure $\sum \delta_{(\widetilde{h}^i, \widetilde{\Delta}_i)}$ is a Poisson point measure with intensity $\mathbb{1}_{[0,\widetilde{H}_{\mathfrak{t}}]}(\mathrm{d}h)\widetilde{\nu}(\mathrm{d}z)$, where $\widetilde{\nu}(\mathrm{d}z)$ is the measure supported on $\mathbb{R}_+$ characterised by:*

$$\int \widetilde{\nu}(\mathrm{d}z)\big(1 - \exp(-pz)\big) = \frac{\widetilde{\psi}(p) - \lambda}{p - \psi^{-1}(\lambda)} - \frac{\lambda}{\psi^{-1}(\lambda)}, \qquad p \geqslant 0. \tag{5.88}$$

Proof. Recall that by Proposition 5.24, we have $\widetilde{\psi}^{-1}(\lambda) = \widetilde{N}(1 - \exp(-\lambda\sigma)) = \widetilde{N}^\lambda(\widetilde{M} \geqslant 1)$. Consider two measurable functions $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $F : \mathcal{M}_f(\mathbb{R}_+) \mapsto \mathbb{R}_+$ and remark that

$$\widetilde{N}^\lambda\big(g(\widetilde{H}_{\mathfrak{t}})F(\widetilde{\rho}_{\mathfrak{t}})\mathbb{1}_{\{\widetilde{M}\geqslant 1\}}\big) = \lambda \cdot \widetilde{N}\Big(\int_0^\sigma \mathrm{d}s \exp(-\lambda s) g(\widetilde{H}_s)F(\widetilde{\rho}_s)\Big).$$

By duality (5.20) and the Markov property, the previous expression can be written in the form:

$$\begin{aligned}\lambda \cdot \widetilde{N}\Big(\int_0^\sigma \mathrm{d}s\ g(\widetilde{H}_s)F(\widetilde{\eta}_s)\exp(-\lambda(\sigma - s))\Big) &= \lambda \cdot \widetilde{N}\Big(\int_0^\sigma \mathrm{d}s\ g(\widetilde{H}_s)F(\widetilde{\eta}_s)\widetilde{E}_{\widetilde{\rho}_s}[\exp(-\lambda\sigma)]\Big) \\ &= \lambda \cdot \widetilde{N}\Big(\int_0^\sigma \mathrm{d}s\ g(\widetilde{H}_s)F(\widetilde{\eta}_s)\exp(-\widetilde{\psi}^{-1}(\lambda)\langle\widetilde{\rho}_s, 1\rangle)\Big),\end{aligned}$$

where in the last line we use the identity $\widetilde{\psi}^{-1}(\lambda) = \widetilde{N}(1 - \exp(-\lambda\sigma))$. Consider under P^0 the pair of subordinators $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ with Laplace exponent (5.23), defined replacing ψ by $\widetilde{\psi}$, and denote its Lévy measure by $\widetilde{\gamma}(\mathrm{d}u_1, \mathrm{d}u_2)$. We stress that since $\widetilde{\psi}$ does not have Brownian part, the subordinators $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ does not have drift. The many-to-one formula (5.24) applied to $\widetilde{\psi}$ gives:

$$\widetilde{N}^\lambda\big(g(\widetilde{H}_{\mathfrak{t}})F(\widetilde{\rho}_{\mathfrak{t}})\big|\ \widetilde{M} \geqslant 1\big) = \frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}a\ \exp(-\widetilde{\alpha}a)g(a)E^0\big[F(\mathbb{1}_{[0,a]}\mathrm{d}\widetilde{U}^{(1)})\exp(-\widetilde{\psi}^{-1}(\lambda)\widetilde{U}_a^{(2)})\big]. \tag{5.89}$$

We shall now deduce from the later identity that the pair $\big(\widetilde{H}_{\mathfrak{t}}, \sum \delta_{(\widetilde{h}^i, \widetilde{\Delta}_i)}\big)$ has the desired distribution. In this direction, let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a measurable function satisfying $f(h, 0) = 0$, for every $h \geqslant 0$. By (5.89), we derive that

$$\begin{aligned}&\widetilde{N}^\lambda\Big(g(\widetilde{H}_{\mathfrak{t}})\exp\Big(-\sum_{i\in\mathbb{N}} f(\widetilde{h}_i, \widetilde{\Delta}_i)\Big)\Big|\widetilde{M} \geqslant 1\Big) \\ &\quad= \frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}a\, g(a)\exp(-\widetilde{\alpha}a)E^0\Big[\exp\Big(-\sum_{h\leqslant a}\big(f(h, \Delta\widetilde{U}_h^{(1)}) + \widetilde{\psi}^{-1}(\lambda)\Delta\widetilde{U}_h^{(2)}\big)\Big)\Big].\end{aligned} \tag{5.90}$$

Moreover, by the exponential formula it follows that the expectation under E^0 in the previous display equals

$$\exp\Big(-\int_0^a \mathrm{d}h \int \widetilde{\gamma}(\mathrm{d}u_1, \mathrm{d}u_2)\big(1 - \exp(-f(h, u_1) - \widetilde{\psi}^{-1}(\lambda)u_2)\big)\Big),$$

and notice that we can write:

$$\begin{aligned}&\int \widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)\big(1-\exp(-f(h,u_1)-\widetilde{\psi}^{-1}(\lambda)u_2)\big)\\&=\int \widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)\exp(-\widetilde{\psi}^{-1}(\lambda)u_2)\big(1-\exp(-f(h,u_1))\big)+\int \widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)\big(1-\exp(-\widetilde{\psi}^{-1}(\lambda)u_2)\big).\end{aligned}$$

To simplify this expression, introduce the measure $\widetilde{\gamma}'(\mathrm{d}u_1):=\int_{u_2\in\mathbb{R}}\widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)\exp(-\widetilde{\psi}^{-1}(\lambda)u_2)$ and observe that (5.23) entails $\int\widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)(1-\exp(-\widetilde{\psi}^{-1}(\lambda)u_2))=\lambda/\widetilde{\psi}^{-1}(\lambda)-\widetilde{\alpha}$. We deduce that (5.90) can be written in the following form:

$$\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}a\, g(a)\exp(-\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}a)\exp\Big(-\int_0^a \mathrm{d}h\int\widetilde{\gamma}'(\mathrm{d}u_1)\Big(1-\exp\big(-f(h,u_1)\big)\Big)\Big),$$

and to conclude it suffices to remark that $\widetilde{\gamma}'=\widetilde{\nu}$, since by (5.23) we have

$$\begin{aligned}\int\widetilde{\gamma}'(\mathrm{d}u_1)\big(1-\exp(-pu_1)\big)&=\int\widetilde{\gamma}(\mathrm{d}u_1,\mathrm{d}u_2)\big(\exp(-\widetilde{\psi}^{-1}(\lambda)u_2)-\exp(-pu_1-\widetilde{\psi}^{-1}(\lambda)u_2)\big)\\&=\frac{\widetilde{\psi}(p)-\lambda}{p-\widetilde{\psi}^{-1}(\lambda)}-\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)},\end{aligned}$$

for every $p\geqslant 0$. □

We now turn our attention to the other side of the picture, and we now work under $\mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1)$. The objective is to obtain analogue results for the spine $\mathbb{S}(\mathbf{T}^A)$. In this direction, recall the notation $G_\lambda:=\widetilde{\psi}^{-1}(\lambda)$ and we start with the following technical lemma characterizing the law of $(\rho,\overline{W})$ at time $A^{-1}_{\mathfrak{t}}$.

Lemma 5.36. *For any non-negative measurable function f in $M_f(\mathbb{R}_+)\times\mathcal{W}_{\overline{E}}$, we have:*

$$\begin{aligned}&\mathbb{N}^\lambda_{x,0}\Big(f(\rho_{A^{-1}_{\mathfrak{t}}},\overline{W}_{A^{-1}_{\mathfrak{t}}})\mathbb{1}_{\{M\geqslant 1\}}\Big)\\&\qquad=\lambda\int_0^\infty\mathrm{d}a\,E^0\otimes\Pi_x\Big(\exp(-\alpha\tau_a)f(J_{\tau_a},(\xi_t,\mathcal{L}_t)_{t\leqslant\tau_a})\exp\big(-\int_0^{\tau_a}\check{J}_{\tau_a}(\mathrm{d}h)\,u_{G_\lambda}(\xi_h)\big)\Big).\end{aligned}$$

Proof. Since $\{M\geqslant 1\}=\{\mathfrak{t}\leqslant A_\infty\}$, we have:

$$\begin{aligned}\mathbb{N}^\lambda_{x,0}\Big(f(\rho_{A^{-1}_{\mathfrak{t}}},\overline{W}_{A^{-1}_{\mathfrak{t}}})\mathbb{1}_{\{M\geqslant 1\}}\Big)&=\lambda\cdot\mathbb{N}_{x,0}\Big(\int_0^{A_\infty}\mathrm{d}s\,f(\rho_{A^{-1}_s},\overline{W}_{A^{-1}_s})\exp(-\lambda s)\Big)\\&=\lambda\cdot\mathbb{N}_{x,0}\Big(\int_0^\sigma\mathrm{d}A_s\,f(\rho_s,\overline{W}_s)\exp(-\lambda A_s)\Big)\\&=-\lambda\cdot\mathbb{N}_{x,0}\Big(\int_0^\sigma\mathrm{d}A_{\sigma-s}\,f(\rho_{\sigma-s},\overline{W}_{\sigma-s})\exp(-\lambda A_{\sigma-s})\Big).\end{aligned}$$

Moreover, by time reversal (5.20), we know that:

$$(\rho_{(\sigma-s)-},\overline{W}_{\sigma-s},A_{\sigma-s}:0\leqslant s\leqslant\sigma)\overset{(d)}{=}(\eta_s,\overline{W}_s,A_\sigma-A_s:0\leqslant s\leqslant\sigma),$$

and we remark that $\{s\in[0,\sigma]:\rho_s\neq\rho_{s-}\}\subset\{s\in[0,\sigma]:\rho_s(\{H_s\})>0\}$ which has null $\mathrm{d}A$ measure $\mathbb{N}_{x,0}$– a.e by the many-to-one formula of Lemma 5.23. This implies:

$$-\mathbb{N}_{x,0}\Big(\int_0^\sigma\mathrm{d}A_{\sigma-s}\,f(\rho_{\sigma-s},\overline{W}_{\sigma-s})\exp(-\lambda A_{\sigma-s})\Big)=\mathbb{N}_{x,0}\Big(\int_0^\sigma\mathrm{d}A_s\,f(\eta_s,\overline{W}_s)\exp(-\lambda\int_s^\sigma\mathrm{d}A_s)\Big).$$

Next, by making use of the strong Markov property, we derive that

$$\begin{aligned}\mathbb{N}^{\lambda}_{x,0}\left(f(\rho_{A_{\mathrm{t}}^{-1}},\overline{W}_{A_{\mathrm{t}}^{-1}})\mathbb{1}_{\{M\geqslant 1\}}\right) &= \lambda\cdot\mathbb{N}_{x,0}\left(\int_0^\sigma \mathrm{d}A_s\, f(\eta_s,\overline{W}_s)\exp\big(-\lambda\int_s^\sigma \mathrm{d}A_s\big)\right)\\ &= \lambda\cdot\mathbb{N}_{x,0}\left(\int_0^\sigma \mathrm{d}A_s\, f(\eta_s,\overline{W}_s)\mathbb{E}^{\dagger}_{\rho_s,\overline{W}_s}\Big[\exp\big(-\lambda\int_0^\sigma \mathrm{d}A_s\big)\Big]\right)\\ &= \lambda\cdot\mathbb{N}_{x,0}\left(\int_0^\sigma \mathrm{d}A_s\, f(\eta_s,\overline{W}_s)\exp\Big(-\int \rho_s(\mathrm{d}h)\, u_{G_\lambda}(W_s(h))\Big)\right),\end{aligned}$$

where in the last line we used Proposition 5.24. The statement of the lemma now follows applying (5.72) and recalling that $(J_\infty,\check{J}_\infty)\overset{(d)}{=}(\check{J}_\infty,J_\infty)$, under P^0. □

For simplicity, in the rest of the section we write:

$$(\rho^A_{\mathrm{t}},\overline{W}^A_{\mathrm{t}}) := (\rho_{A_{\mathrm{t}}^{-1}},\overline{W}_{A_{\mathrm{t}}^{-1}}),$$

and $\overline{W}^A_{\mathrm{t}} := (W^A_{\mathrm{t}},\Lambda^A_{\mathrm{t}})$ – remark that in particular we have $H^A_{\mathrm{t}} = \widehat{\Lambda}^A_{\mathrm{t}}$. Let us now decompose $\overline{W}^A_{\mathrm{t}}$ in terms of its excursion intervals away from x. To be more precise, we need to introduce some notation. For every $r > 0$ and $\overline{\mathrm{w}} := (\mathrm{w},\ell) \in \mathcal{W}_{\overline{E}}$, we set:

$$\tau^+_r(\overline{\mathrm{w}}) := \inf\big\{h \geqslant 0 : \ell(h) > r\big\}.$$

Remark that since ℓ is continuous, $r \mapsto \tau^+_r(\overline{\mathrm{w}})$ is right-continuous. Moreover, $\tau(\overline{\mathrm{w}})$ and $\tau^+(\overline{\mathrm{w}})$ are related by the relation $\tau_r(\overline{\mathrm{w}}) = \tau^+_{r-}(\overline{\mathrm{w}})$. Similarly, under $\Pi_{y,0}$ for $y \in E$, we will write $\tau^+_r(\xi) := \inf\{t \geqslant 0 : \mathcal{L}_t > r\}$. The advantage of working with $\tau^+(\xi)$ instead of $\tau(\xi)$ is that, under $\Pi_{x,0}$, the process $\tau^+(\xi)$ is a subordinator. Moreover, by excursion theory, it is well known that its Lévy-Itô decomposition is given by

$$\tau^+_r(\xi) = \sum_{s\leqslant r}\Delta\tau^+_s(\xi), \qquad r \geqslant 0,$$

since $(\mathbf{H_3})$ ensures that the process $\tau^+(\xi)$ does not have drift part – equivalently $\tau^+(\xi)$ is purely discontinuous. For simplicity, when there is no risk of confusion the dependency on ξ is dropped.

Getting back to our discussion, under $\mathbb{N}^\lambda_{x,0}(\cdot\,|M \geqslant 1)$, let $(r_j : j \in \mathcal{J})$ be an enumeration of the jumping times of the right-continuous process $(\tau^+_r(\overline{W}^A_{\mathrm{t}}) : 0 \leqslant r < H^A_{\mathrm{t}})$ – for technical reasons the indexing is assumed to be measurable with respect to $\overline{W}^A_{\mathrm{t}}$. For each $j \in \mathcal{J}$, set

$$\overline{W}^{A,j}_{\mathrm{t}} := \Big(\big(W^A_{\mathrm{t}}\big(h+\tau_{r_j}(\overline{W}^A_{\mathrm{t}})\big),\Lambda^A_{\mathrm{t}}\big(h+\tau_{r_j}(\overline{W}^A_{\mathrm{t}})\big)-\Lambda^A_{\mathrm{t}}(\tau_{r_j}(\overline{W}^A_{\mathrm{t}}))\big) : h \in [0,\tau^+_{r_j}(\overline{W}^A_{\mathrm{t}})-\tau_{r_j}(\overline{W}^A_{\mathrm{t}})]\Big),$$

and

$$\langle\rho^{A,j}_{\mathrm{t}},f\rangle := \int \rho^A_{\mathrm{t}}(\mathrm{d}h)f(h-\tau_{r_j}(\overline{W}^A_{\mathrm{t}}))\mathbb{1}_{\{\tau_{r_j}(\overline{W}^A_{\mathrm{t}})<h<\tau^+_{r_j}(\overline{W}^A_{\mathrm{t}})\}}.$$

The first coordinates of the family $(\overline{W}^{A,j}_{\mathrm{t}} : j \in \mathcal{J})$ correspond to the excursion of W^A_{t} away from x while the second coordinate is identically zero. We also stress that since $(x,0) \in \overline{\Theta}_x$, by Lemma 5.13 the support of ρ^A_{t} is included in $\bigcup_{j\in\mathcal{J}}(\tau_{r_j}(\overline{W}^A_{\mathrm{t}}),\tau^+_{r_j}(\overline{W}^A_{\mathrm{t}}))$. Our goal now is to identify the law of $\sum_{j\in\mathcal{J}}\delta_{(r_j,\rho^{A,j}_{\mathrm{t}},W^{A,j}_{\mathrm{t}})}$. As we shall see, the restriction to the first two coordinates of this measure is, roughly speaking, a biased version of the excursion point measure of ξ under $\Pi_{x,0}$. More precisely, let $(E^0\otimes\mathcal{N})_*(\mathrm{d}J,\mathrm{d}\xi)$ be the measure in $\mathcal{M}_f(\mathbb{R}_+)\otimes\mathbb{D}(\mathbb{R}_+,E)$ defined by

$$(E^0\otimes\mathcal{N})_*\big[F(J,\xi)\big] := E^0\otimes\mathcal{N}\Big[\exp\big(-\int\check{J}_\sigma(\mathrm{d}h)u_{G_\lambda}(\xi_h)-\alpha\sigma\big)F(J_\sigma,\xi)\Big].$$

Lemma 5.37. *Under $\mathbb{N}^\lambda_{x,0}(\cdot|M \geqslant 1)$, the random variable $H^A_{\mathfrak{t}}$ is exponentially distributed with parameter $\lambda/\psi^{-1}(\lambda)$. Moreover, conditionally on $H^A_{\mathfrak{t}}$, the measure:*

$$\sum_{j\in\mathcal{J}} \delta_{(r_j,\rho^{A,j}_{\mathfrak{t}},W^{A,j}_{\mathfrak{t}})},$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,H^A_{\mathfrak{t}}]}(r)\mathrm{d}r(E^0\otimes\mathcal{N})_*(\mathrm{d}J,\mathrm{d}\xi)$.

Proof. First, we fix two measurable functions $g:\mathbb{R}_+\mapsto\mathbb{R}_+$ and $f:\mathbb{R}_+\times\mathcal{M}_f(\mathbb{R}_+)\times\mathbb{D}(\mathbb{R}_+,E)\mapsto\mathbb{R}_+$. The statement of the lemma will follow by establishing that:

$$\begin{aligned}&\mathbb{N}^\lambda_{x,0}\big(g(H^A_{\mathfrak{t}})\exp(-\sum_{j\in\mathcal{J}} f(r_j,\rho^{A,j}_{\mathfrak{t}},W^{A,j}_{\mathfrak{t}}))\mid M\geqslant 1\big)\\ &\quad=\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}r\,\exp\big(-r\cdot\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\big)g(r)\exp\Big(-\int_0^r \mathrm{d}s\,(E^0\otimes\mathcal{N})_*\big[1-\exp\big(-f(s,J,\xi)\big)\big]\Big).\end{aligned}\tag{5.91}$$

To simplify notation, for every $\mu\in\mathcal{M}(\mathbb{R}_+)$ and $a,b\geqslant 0$, we write $\phi(\mu,a,b)$ for the measure ν defined by:

$$\int\nu(\mathrm{d}h)F(h)=\int_{(a,b)}\mu(\mathrm{d}h)F(h-a).$$

Next, under $\Pi_{x,0}$, denote the excursion point measure of ξ by $\sum_j\delta_{(r_j,\xi^j)}$. Now an application of Lemma 5.36 gives

$$\begin{aligned}&\mathbb{N}^\lambda_{x,0}\big(g(H^A_{\mathfrak{t}})\exp(-\sum_{j\in\mathcal{J}} f(r_j,\rho^{A,j}_{\mathfrak{t}},W^{A,j}_{\mathfrak{t}}))\mid M\geqslant 1\big)\\ &=\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}r\,g(r)E^0\otimes\Pi_{x,0}\Big(\exp(-\alpha\tau_r)\exp\big(-\sum_{r_j\leqslant r} f(r_j,\phi(J_\infty,\tau_{r_j},\tau^+_{r_j}),\xi^j)\big)\\ &\qquad\qquad\cdot\exp\big(-\int\check{J}_{\tau_r}(\mathrm{d}h)u_{G_\lambda}(\xi_h)\big)\Big)\\ &=\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}r\,g(r)E^0\otimes\Pi_{x,0}\Big(\exp\Big(-\sum_{r_j\leqslant r}\Big\{f\big(r_j,\phi(J_\infty,\tau_{r_j},\tau^+_{r_j}),\xi^j\big)\\ &\qquad\qquad+\int_{\tau_{r_j}}^{\tau^+_{r_j}}\check{J}_\infty(\mathrm{d}h)u_{G_\lambda}(\xi_h)+\alpha\sigma(\xi^j)\Big\}\Big)\Big),\end{aligned}$$

where in the last equality we used the fact that τ^+ is purely discontinuous and that thanks to $(\mathbf{H_3})$, under $P^0\otimes\Pi_{x,0}$, we can write $\check{J}_\infty(\mathrm{d}h)=\sum_{r_j}\check{J}_\infty(\mathrm{d}h)\mathbb{1}_{[\tau_{r_j},\tau^+_{r_j}]}(h)$. We are going to conclude using standard techniques of excursion theory. First remark that if we introduce an i.i.d. collection of measures $(J^j_\infty,\check{J}^j_\infty)_{j\in\mathbb{N}}$ distributed as $(J_\infty,\check{J}_\infty)$ under P^0, the previous display can be written in the form:

$$\frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)}\int_0^\infty \mathrm{d}r\,g(r)E^0\otimes\Pi_{x,0}\Big(\exp\Big(-\sum_{r_j\leqslant r}\Big\{f(r_j,J^j_{\sigma(\xi^j)},\xi^j)+\int\check{J}^j_{\sigma(\xi^j)}(\mathrm{d}h)u_{G_\lambda}(\xi^j_h)+\alpha\sigma(\xi^j)\Big\}\Big)\Big).\tag{5.92}$$

Since by excursion theory $\sum_{r_j\leqslant r}\delta_{(r_j,J^j_\infty,\check{J}^j_\infty,\xi^j)}$ is a Poisson point measure with intensity measure given by $\mathbb{1}_{[0,r]}(\mathrm{d}s)E^0\otimes\mathcal{N}(\mathrm{d}J_\infty,\mathrm{d}\check{J}_\infty,\mathrm{d}\xi)$, we deduce that the expectation under $E^0\otimes\Pi_{x,0}$ in

(5.92) writes:

$$\exp\Big(- \int_0^r \mathrm{d}s\, E^0 \otimes \mathcal{N}\Big[1 - \exp\big(- f(s, J_\sigma, \xi) - \int \check{J}_\sigma(\mathrm{d}h) u_{G_\lambda}(\xi_h) - \alpha\sigma\big)\Big]\Big).$$

Next, we remark that the previous display equals:

$$\exp\Big(- \int_0^r \mathrm{d}s\, (E^0 \otimes \mathcal{N})_* \Big[1 - \exp\big(- f(s, J, \xi)\big)\Big]\Big)$$
$$\cdot \exp\Big(- r \cdot E^0 \otimes \mathcal{N}\Big[1 - \exp\big(- \int \check{J}_\sigma(\mathrm{d}h) u_{G_\lambda}(\xi_h) - \alpha\sigma\big)\Big]\Big).$$

Moreover, by (5.23) the measure $\check{J}_\infty$ is the Lebesgue-Stieltjes measure of a subordinator with Laplace exponent $p \mapsto \psi(p)/p - \alpha$, which yields

$$E^0 \otimes \mathcal{N}\big[1 - \exp\big(- \int \check{J}_\sigma(\mathrm{d}h) u_{G_\lambda}(\xi_h) - \alpha\sigma\big)\big] = \mathcal{N}\Big(1 - \exp\Big(- \int_0^\sigma \mathrm{d}h \frac{\psi(u_{G_\lambda}(\xi_h))}{u_{G_\lambda}(\xi_h)}\Big)\Big) = \frac{\lambda}{\widetilde{\psi}^{-1}(\lambda)},$$

where in the first equality we applied (5.23) and in last one we used (5.68). Putting everything together we obtain the desired identity (5.91). □

To identify the law of $\mathbb{S}(\mathbf{T}^A)$, we now define the natural candidate of the exploration process of the subordinate tree at time $\mathfrak{t}$ – as we already mentioned, this statement is purely heuristic. Let us start by introducing some notations. Still under $\mathbb{N}^\lambda_{x,0}(\cdot | M \geqslant 1)$ denote the connected components of the open set

$$\big\{ s \geqslant A^{-1}_{\mathfrak{t}} : \ H_s > \inf_{[A^{-1}_{\mathfrak{t}}, s]} H \big\}$$

by $((\alpha_i, \beta_i) : i \in \mathbb{N})$, and as usual write $(\rho^i, \overline{W}^i) := (\rho^i, W^i, \Lambda^i)$ for the subtrajectory associated with the excursion interval $[\alpha_i, \beta_i]$. Further, set $h_i := H_{\alpha_i}$ and consider the measure:

$$\sum_{i \in \mathbb{N}} \delta_{(h_i, \rho^i, \overline{W}^i)}\,. \tag{5.93}$$

By the strong Markov property and (5.22), conditionally on $(\rho^A_{\mathfrak{t}}, \overline{W}^A_{\mathfrak{t}})$, the measure (5.93) is a Poisson point measure with intensity $\rho^A_{\mathfrak{t}}(\mathrm{d}h)\mathbb{N}_{\overline{W}^A_{\mathfrak{t}}(h)}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. Next, for every $j \in \mathcal{J}$ we set:

$$L^j := \sum_{\tau_{r_j}(\overline{W}^A_{\mathfrak{t}}) < h_i < \tau^+_{r_j}(\overline{W}^A_{\mathfrak{t}})} \mathscr{L}^{r_j}_\sigma(\rho^i, \overline{W}^i), \tag{5.94}$$

which is the total amount of exit local time from the domain D_{r_j} generated by the excursions glued on the right-spine of $\overline{W}^A_{\mathfrak{t}}$ at the interval $\big(\tau_{r_j}(\overline{W}^A_{\mathfrak{t}}), \tau^+_{r_j}(\overline{W}^A_{\mathfrak{t}})\big)$. Finally, we introduce the measure $\rho^*_{\mathfrak{t}} := \sum_{j \in \mathcal{J}} L^j \cdot \delta_{r_j}$.

Lemma 5.38. *We have the following identity in distribution:*

$$\big((\widetilde{H}_{\mathfrak{t}}, \widetilde{\rho}_{\mathfrak{t}}) : \widetilde{N}^\lambda(\cdot | \widetilde{M} \geqslant 1)\big) \overset{(d)}{=} \big((H^A_{\mathfrak{t}}, \rho^*_{\mathfrak{t}}) : \ \mathbb{N}^\lambda_{x,0}(\cdot | M \geqslant 1)\big).$$

In particular, Lemma 5.38 implies that $H(\rho^*_{\mathfrak{t}}) = H^A_{\mathfrak{t}}$.

Proof. We start noticing that, by Lemmas 5.35 and 5.37, we already have:

$$\big(\widetilde{H}_{\mathfrak{t}} : \widetilde{N}^{\lambda}(\cdot|\widetilde{M} \geqslant 1)\big) \overset{(d)}{=} \big(H^A_{\mathfrak{t}} : \mathbb{N}^{\lambda}_{x,0}(\cdot|M \geqslant 1)\big).$$

Consequently, again by Lemma 5.35 the desired result will follow by showing that, under $\mathbb{N}^{\lambda}_{x,0}(\cdot|M \geqslant 1)$ and conditionally on $H^A_{\mathfrak{t}}$, the measure

$$\sum_{j\in\mathcal{J}} \delta_{(r_j, L^j)}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,H^A_{\mathfrak{t}}]}(\mathrm{d}h)\widetilde{\nu}(\mathrm{d}z)$, where the measure $\widetilde{\nu}$ is characterised by (5.88). In this direction, we work in the rest of the proof under $\mathbb{N}^{\lambda}_{x,0}(\cdot|M \geqslant 1)$ and recall that, conditionally on $(\rho^A_{\mathfrak{t}}, \overline{W}^A_{\mathfrak{t}})$, the measure (5.93) is a Poisson point measure with intensity $\rho^A_{\mathfrak{t}}(\mathrm{d}h)\mathbb{N}_{\overline{W}^A_{\mathfrak{t}}(h)}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. In particular, (5.94) entails that conditionally on $(\rho^A_{\mathfrak{t}}, \overline{W}^A_{\mathfrak{t}})$, the random variables $(L^j : j \in \mathcal{J})$ are independent. Moreover, since by definition $u_p(y) = \mathbb{N}_{y,0}(1 - \exp(-p\mathscr{L}^0_{\sigma}))$, the translation invariance of the local time $\mathcal{L}$ gives

$$\begin{aligned}\mathbb{N}^{\lambda}_{x,0}\big(\exp(-pL^j) \mid \rho^A_{\mathfrak{t}}, \overline{W}^A_{\mathfrak{t}}\big) &= \exp\Big(-\int_{\tau_{r_j}(\overline{W}^A_{\mathfrak{t}})}^{\tau^+_{r_j}(\overline{W}^A_{\mathfrak{t}})} \rho^A_{\mathfrak{t}}(\mathrm{d}h) u_p\big(W^A_{\mathfrak{t}}(h)\big)\Big)\\ &= \exp\Big(-\int \rho^{A,j}_{\mathfrak{t}}(\mathrm{d}h) u_p\big(W^{A,j}_{\mathfrak{t}}(h)\big)\Big),\end{aligned}$$

for every $j \in \mathcal{J}$. It will be then convenient to introduce, for $(\mu, \mathrm{w}) \in \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, the probability measure $\mathrm{m}_{\mu,\mathrm{w}}$ in $\mathbb{R}_+$ defined through its Laplace transform:

$$\int \mathrm{m}_{\mu,\mathrm{w}}(\mathrm{d}z)\exp(-pz) = \exp\Big(-\int \mu(\mathrm{d}h)u_p\big(\mathrm{w}(h)\big)\Big),$$

if $H(\mu) = \zeta(\mathrm{w})$ and $\mathrm{m}_{\mu,\mathrm{w}} = 0$ otherwise. The map $(\mu, \mathrm{w}) \mapsto \mathrm{m}_{\mu,\mathrm{w}}$ takes values in $\mathcal{M}_f(\mathbb{R}_+)$ and it is straightforward to see that it is measurable. Next, remark that by our previous discussion we have:

$$\begin{aligned}&\mathbb{N}^{\lambda}_{x,0}\Big(G(H^A_{\mathfrak{t}})\exp(-\sum_{j\in\mathcal{I}} f(r_j, L^j)) \mid M \geqslant 1\Big)\\ &\quad= \mathbb{N}^{\lambda}_{x,0}\Big(G(H^A_{\mathfrak{t}})\prod_{j\in\mathcal{I}}\int \mathrm{m}_{\rho^{A,j}_{\mathfrak{t}}, W^{A,j}_{\mathfrak{t}}}(\mathrm{d}z)\exp(-f(r_j, z)) \Big| M \geqslant 1\Big)\\ &\quad= \mathbb{N}^{\lambda}_{x,0}\Big(G(H^A_{\mathfrak{t}})\exp(-\sum_{j\in\mathcal{J}} f^*(r_j, \rho^{A,j}_{\mathfrak{t}}, W^{A,j}_{\mathfrak{t}})) \Big| M \geqslant 1\Big),\end{aligned}$$

where $f^*(r, \mu, \mathrm{w}) := -\log\big(\int \mathrm{m}_{\mu,\mathrm{w}}(\mathrm{d}z)\exp(-f(r,z))\big)$. Now, we can apply Lemma 5.37 to get:

$$\begin{aligned}&\mathbb{N}^{\lambda}_{x,0}\Big(G(H^A_{\mathfrak{t}})\exp(-\sum_{j\in\mathcal{I}} f(r_j, L^j)) \Big| M \geqslant 1\Big)\\ &\quad= \mathbb{N}^{\lambda}_{x,0}\Bigg(G(H^A_{\mathfrak{t}})\exp\Big(-\int_0^{H^A_{\mathfrak{t}}} \mathrm{d}r\,(E^0 \otimes \mathcal{N})_*\Big[\int \mathrm{m}_{J,\xi}(\mathrm{d}z)\big(1 - \exp -f(r,z)\big)\Big]\Big)\Bigg),\end{aligned}$$

and it follows that conditionally on $H^A_{\mathfrak{t}}$ the measure $\sum \delta_{(r_j, L^j)}$ is a Poisson point measure with intensity:

$$\mathbb{1}_{[0,H^A_{\mathfrak{t}}]}(r)\mathrm{d}r\,(E^0 \otimes \mathcal{N})_*\big[\mathrm{m}_{J,\xi}(\mathrm{d}z)\big].$$

To conclude, we need to show that the measure $(E^0 \otimes \mathcal{N})_*\big[\mathrm{m}_{J,\xi}(\mathrm{d}z)\big]$ is precisely $\widetilde{\nu}(\mathrm{d}z)$. In this direction, remark that:

$$\begin{aligned}
(E^0 \otimes \mathcal{N})_*\Big[\int \mathrm{m}_{J,\xi}(\mathrm{d}z)(1-\exp(-pz))\Big] &= (E^0 \otimes \mathcal{N})_*\Big[1-\exp(-\int J(\mathrm{d}h)u_p(\xi(h))\Big] \\
= E^0 \otimes \mathcal{N}\Big(1-\exp\big(-\int J_\sigma(\mathrm{d}h)\; u_p(\xi(h)) - \int \check{J}_\sigma(\mathrm{d}h)u_{G_\lambda}(\xi(h)) - \alpha\sigma\big)\Big) \\
- E^0 \otimes \mathcal{N}\Big(1-\exp\big(-\int \check{J}_\sigma(\mathrm{d}h)u_{G_\lambda}(\xi(h)) - \alpha\sigma\big)\Big).
\end{aligned}$$

Then, (5.23) entails that the previous display is equal to

$$\mathcal{N}\Big(1-\exp\big(-\int_0^\sigma \mathrm{d}h\frac{\psi(u_p(\xi(h)))-\psi(u_{G_\lambda}(\xi(h)))}{u_p(\xi(h))-u_{G_\lambda}(\xi(h))}\big)\Big) - \mathcal{N}\Big(1-\exp\big(-\int_0^\sigma \mathrm{d}h\frac{\psi(u_{G_\lambda}(\xi(h)))}{u_{G_\lambda}(\xi(h))}\big)\Big).$$

However, by Lemma 5.20 the previous display is precisely (5.88). □

We can now identify $\mathbb{S}(\mathbf{T}^A)$ in terms of our functionals. In this direction, for every $i \in \mathbb{N}$, we introduce $(\rho^{i,k}, W^{i,k}, \Lambda^{i,k})_{k\in\mathcal{K}_i}$ the excursions from $D_0 = \overline{E}\setminus\{(x,0)\}$ of $(\rho^i, W^i, \Lambda^i - \Lambda^i_0)$. In particular, the family $(\rho^{i,k}, \overline{W}^{i,k})_{k\in\mathcal{K}_i}$ is in one-to-one correspondence with the connected components $[a_{i,k}, b_{i,k}]$, $k \in \mathcal{K}_i$, of the open set $\{s \in [0, \sigma(\overline{W}^i)] : \tau_{\Lambda^i_0}(\overline{W}^i_s) < \zeta_s(\overline{W}^i_s)\}$, in such a way that $(\rho^{i,k}, W^{i,k}, \Lambda^{i,k} + \Lambda^i_0)$ is the subtrajectory of $(\rho^i, \overline{W}^i)$ associated with the interval $[a_{i,k}, b_{i,k}]$. In the time scale of $((\rho_s, \overline{W}_s) : s \geqslant 0)$, the excursion $(\rho^{i,k}, W^{i,k}, \Lambda^{i,k} + \Lambda^i_0)$ corresponds to the subtrajectory associated with $[\alpha_{i,k}, \beta_{i,k}]$, where $\alpha_{i,k} := \alpha_i + a_{i,k}$ and $\beta_{i,k} := \alpha_i + b_{i,k}$. Next, for each $k \in \mathcal{K}_i$, we introduce the point process $\mathcal{P}^{i,k}_t := \mathcal{P}_{(A_{\alpha_{i,k}}+t)\wedge A_{\beta_{i,k}}} - \mathcal{P}_{A_{\alpha_{i,k}}}$ and we set:

$$\mathcal{M} := \sum_{i\in\mathbb{N}}\sum_{k\in\mathcal{K}_i} \delta_{(\Lambda^i_0(0), \rho^{i,k}, \overline{W}^{i,k}, \mathcal{P}^{i,k})}.$$

An application of the Markov property at time $A^{-1}_{\mathfrak{t}}$ and the special Markov property applied to the domain D_0 shows that, conditionally on $\rho^*_{\mathfrak{t}}$, the measure $\mathcal{M}$ is a Poisson point measure with intensity $\rho^*_{\mathfrak{t}}(\mathrm{d}r)\mathbb{N}^\lambda_{x,0}(\mathrm{d}\rho, \mathrm{d}\overline{W}, \mathrm{d}\mathcal{P})$. For every $j \in \mathcal{J}$, consider

$$M_j := \#\Big\{\big(\Lambda^i_0(0), \rho^{i,k}, \overline{W}^{i,k}, \mathcal{P}^{i,k}\big) \in \mathcal{M} : \; \Lambda^i_0(0) = r_j \text{ and } \mathcal{P}^{i,k}_{A_\sigma(\overline{W}^{i,k})} \geqslant 1\Big\},$$

and denote the elements of $\{(r_j, M_j), j \in \mathcal{J} : \; M_j \geqslant 1\}$ arranged in increasing order with respect to r_j by $\big((r^\circ_1, M^\circ_1), \dots, (r^\circ_R, M^\circ_R)\big)$. We now remark that by construction we have:

$$\mathbb{S}(\mathbf{T}^A) = \big((r^\circ_1, M^\circ_1), \dots, (r^\circ_R, M^\circ_R), (H^A_{\mathfrak{t}}, -1)\big), \tag{5.95}$$

and, in particular, $K = \sum_{p=1}^R M^\circ_p$ which is the number of atoms $(\Lambda^i_0(0), \rho^{i,k}, \overline{W}^{i,k}, \mathcal{P}^{i,k}) \in \mathcal{M}$ with at least one Poissonian mark. Finally, we write $\mathscr{E} := ((\rho^q_\circ, \overline{W}^q_\circ, \mathcal{P}^q_\circ) : \; 1 \leqslant q \leqslant K)$ for the collection of these marked excursions enumerated in counterclockwise order. Remark that, for every $1 \leqslant q \leqslant K$, $\mathbf{T}^A_i$ is the embedded tree associated with $\widehat{\Lambda}^q_\circ$ – time changed by $A(\rho^q_\circ, \overline{W}^q_\circ)$ – and marked by $\mathcal{P}^q_\circ$. We are now in position to prove Proposition 5.34.

Proof of Proposition 5.34. For every $h \geqslant 0$ with $\widetilde{\rho}_{\mathfrak{t}}(\{h\}) > 0$, we write $\widetilde{\mathcal{M}}^{(h)} := \widetilde{\mathcal{M}}\mathbb{1}_{\{\widetilde{h}^i = h\}}$. Similarly, for every $r \geqslant 0$ satisfying $\rho^*_{\mathfrak{t}}(\{r\}) > 0$, we set $\mathcal{M}^{(r)} := \mathcal{M}\mathbb{1}_{\{\Lambda^i_0(0)=r\}}$. Next, we introduce the following families respectively under $\widetilde{N}^\lambda(\cdot|\widetilde{M} \geqslant 1)$ and $\mathbb{N}^\lambda_{x,0}(\cdot|M \geqslant 1)$:

$$\Big\{\big(h\mathbf{1}_{\{\widetilde{\mathcal{M}}^{(h)}(\widetilde{M}\geqslant 1)\geqslant 1\}}, \widetilde{\mathcal{M}}^{(h)}(\widetilde{M} \geqslant 1)\big) : h \geqslant 0, \, \widetilde{\rho}_{\mathfrak{t}}(\{h\}) > 0\Big\} \cup \{(\widetilde{H}_{\mathfrak{t}}, -1)\}, \tag{5.96}$$

and

$$\big\{\big(r\mathbf{1}_{\{\mathcal{M}^{(r)}(M\geqslant 1)\geqslant 1\}},\, \mathcal{M}^{(r)}(M\geqslant 1)\big) : r\geqslant 0,\ \rho^*_{\mathtt{t}}(\{r\})>0\big\}\cup\{(H^A_{\mathtt{t}},-1)\}, \tag{5.97}$$

where by Lemma 5.38, we have respectively that $H(\rho^A_{\mathtt{t}}) = H^A_{\mathtt{t}}$, $H(\widetilde{\rho}_{\mathtt{t}}) = \widetilde{H}_{\mathtt{t}}$. Recall that, under $\widetilde{N}^\lambda(\cdot|\widetilde{M}\geqslant 1,\widetilde{\rho}_{\mathtt{t}})$, the measure $\widetilde{\mathcal{M}}$ is a Poisson point measure with intensity $\widetilde{\rho}_{\mathtt{t}}(\mathrm{d}h)\widetilde{N}^\lambda(\mathrm{d}\rho,\mathrm{d}\mathcal{P})$ and similarly, under $\mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1,\rho^*_{\mathtt{t}})$, the measure $\mathcal{M}$ is a Poisson point measure with intensity $\rho^*_{\mathtt{t}}(\mathrm{d}r)\mathbb{N}^\lambda_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W},\mathrm{d}\mathcal{P})$. Consequently, by restriction properties of Poisson measures, under $\widetilde{N}^\lambda(\cdot|\widetilde{M}\geqslant 1,\widetilde{\rho}_{\mathtt{t}})$, the variables $\big(\widetilde{\mathcal{M}}^{(h)}(\widetilde{M}\geqslant 1) : \widetilde{\rho}_{\mathtt{t}}(\{h\})>0\big)$ are independent Poisson random variables with intensity $\widetilde{\rho}_{\mathtt{t}}(\{h\})\widetilde{N}^\lambda(M\geqslant 1)$ and, under $\mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1,\rho^*_{\mathtt{t}})$, the variables $\big(\mathcal{M}^{(r)}(M\geqslant 1) : \rho^*_{\mathtt{t}}(\{r\})>0\big)$ are also independent Poisson random variables, this time with intensity $\rho^*_{\mathtt{t}}(\{r\})\mathbb{N}^\lambda_{x,0}(M\geqslant 1)$. Now, recall from Lemma 5.38 the identity

$$(\widetilde{\rho}_{\mathtt{t}} : \widetilde{N}^\lambda(\cdot|\widetilde{M}\geqslant 1)) \overset{(d)}{=} (\rho^*_{\mathtt{t}} :\ \mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1)).$$

Since $\widetilde{N}^\lambda(\widetilde{M}\geqslant 1) = \mathbb{N}^\lambda_{x,0}(M\geqslant 1)$, this ensures that the families (5.96) and (5.97) have the same distribution. Moreover, the measures $\widetilde{\rho}_{\mathtt{t}}$ and $\rho^*_{\mathtt{t}}$ being atomic, the families (5.87), (5.95) correspond respectively to the subset of elements of (5.96) and (5.97) with non-null entries. This gives the first statement of the proposition.

To establish (ii), it suffices to show that conditionally on $\mathbb{S}(\widetilde{\mathbf{T}})$, the marked excursions $\widetilde{\mathscr{E}}$ are distributed as $\widetilde{K}$ independent copies with law $\widetilde{N}^\lambda(\mathrm{d}H,\mathrm{d}\mathcal{P}|\widetilde{M}\geqslant 1)$ and that, conditionally on $\mathbb{S}(\mathbf{T}^A)$, the excursions $\mathscr{E}$ are distributed as K independent copies with law $\mathbb{N}^\lambda_{x,0}(\mathrm{d}\overline{W},\mathrm{d}\mathcal{P}|M\geqslant 1)$. Remark that our previous reasoning already implies that $\widetilde{\mathscr{E}}$ and $\mathscr{E}$ satisfy the desired property if we do not take into account the ordering. However, this is not enough and to keep track of the ordering we proceed as follows:

We start studying $\widetilde{\mathscr{E}}$ under $\widetilde{N}^\lambda(\cdot|\widetilde{M}\geqslant 1)$ and we introduce $(\widetilde{\mathscr{I}_s} : s\geqslant \mathtt{t})$, the running infimum of $(\langle\widetilde{\rho}_s,1\rangle - \langle\widetilde{\rho}_{\mathtt{t}},1\rangle : s\geqslant\mathtt{t})$. Next, we consider the measure

$$\sum_{i\in\mathbb{N}} \delta_{(-\widetilde{\mathscr{I}}_{\widetilde{\alpha}_i},\widetilde{\rho}^i,\widetilde{\mathcal{P}}^i)}, \tag{5.98}$$

and we stress that, by the strong Markov property and the discussion below (5.21), conditionally on $\mathcal{F}_{\mathtt{t}}$ this measure is a Poisson point measure with intensity $\mathbb{1}_{[0,\langle\widetilde{\rho}_{\mathtt{t}},1\rangle]}(u)\mathrm{d}u\ \widetilde{N}^\lambda(\mathrm{d}\rho,\mathrm{d}\mathcal{P})$. Moreover, its image by the transformation $s\mapsto H(\kappa_s\widetilde{\rho}_{\mathtt{t}})$ on its first coordinate gives precisely $\widetilde{\mathcal{M}}$. In particular, the collection $\big((\widetilde{h}^\circ_1,\widetilde{M}^\circ_1),\dots,(\widetilde{h}^\circ_{\widetilde{R}},\widetilde{M}^\circ_{\widetilde{R}}),(\widetilde{H}_{\mathtt{t}},-1)\big)$ only depends on $\widetilde{\rho}_{\mathtt{t}}$ and $\big(\widetilde{\mathscr{I}}_{\widetilde{\alpha}_i} :\ i\geqslant 0 \text{ with } \widetilde{\mathcal{P}}^i_{\sigma(\widetilde{\rho}^i)}\geqslant 1\big)$. Remark that the ordered marked excursions $\widetilde{\mathscr{E}}$ correspond precisely to the atoms $H(\widetilde{\rho}^i)$ of (5.98) with $\widetilde{\mathcal{P}}^i_{\sigma(\widetilde{\rho}^i)}\geqslant 1$, when considered in decreasing order with respect to $-\widetilde{\mathscr{I}}_{\widetilde{\alpha}_i}$. Since $H(\widetilde{\rho}_{\mathtt{t}}) = \widetilde{H}_{\mathtt{t}}$, we deduce by restriction properties of Poisson measures that, conditionally on $(\widetilde{\rho}_{\mathtt{t}},\widetilde{K})$, the collection $\widetilde{\mathscr{E}}$ is independent of $\mathbb{S}(\widetilde{\mathbf{T}})$ and formed by $\widetilde{K}$ i.i.d. variables with distribution $\widetilde{N}^\lambda(\mathrm{d}\rho,\mathrm{d}\mathcal{P}|\widetilde{M}\geqslant 1)$, as wanted.

Let us now turn our attention to the distribution of $\mathscr{E}$ under $\mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1)$. Similarly, under $\mathbb{N}^\lambda_{x,0}(\cdot|M\geqslant 1)$ we consider $(\mathscr{I}_s :\ s\geqslant A^{-1}_{\mathtt{t}})$, the running infimum of $(\langle\rho_s,1\rangle - \langle\rho_{A^{-1}_{\mathtt{t}}},1\rangle : s\geqslant A^{-1}_{\mathtt{t}})$ as well as the measure

$$\sum_{i\in\mathbb{N}} \delta_{(-\mathscr{I}_{\alpha_i},\rho^i,W^i)}. \tag{5.99}$$

Once again, by the strong Markov property and (5.21), conditionally on $\mathcal{F}_{A_{\mathrm{t}}^{-1}}$, the measure (5.99) is a Poisson point measure with intensity $\mathbb{1}_{[0,\langle \rho_{\mathrm{t}}^A,1\rangle]}(u)\mathrm{d}u\ \mathbb{N}^{\lambda}_{\overline{W}^A_{\mathrm{t}}(H(\kappa_u \rho_{\mathrm{t}}^A))}(\mathrm{d}\rho,\mathrm{d}\overline{W})$. We now introduce the process:

$$V_t := \sum_{i\in\mathbb{N}} \mathscr{L}^{\Lambda_0^i}_{t\wedge\beta_i - t\wedge\alpha_i}(\rho^i,\overline{W}^i), \quad t \geqslant 0,$$

where $V_\infty = \langle \rho_{\mathrm{t}}^*, 1\rangle < \infty$ by Lemma 5.38. Recall that $(\rho^{i,k},\overline{W}^{i,k})_{k\in\mathcal{K}_i}$ stands for the excursions of $(\rho^i, W^i, \Lambda^i - \Lambda_0^i)$ outside D_0 and we stress that in the time scale of $((\rho_s,\overline{W}_s) : s \geqslant 0)$, the excursion $(\rho^{i,k}, W^{i,k}, \Lambda^{i,k} + \Lambda_0^i)$ corresponds to the subtrajectory associated with $[\alpha_{i,k},\beta_{i,k}]$, where $\alpha_{i,k} := \alpha_i + a_{i,k}$ and $\beta_{i,k} := \alpha_i + b_{i,k}$. To simplify notation set $\mathrm{Tr}(\rho^i,\overline{W}^i)$ for the truncation of $(\rho^i,\overline{W}^i)$ to the domain $D_{\Lambda_0^i}$. An application of the strong Markov property combined with the special Markov property in the form given in Theorem 5.11 implies that, conditionally on $\sum_i \delta_{(-\mathscr{I}_{\alpha_i},\mathrm{Tr}(\rho^i,\overline{W}^i))}$, the measure:

$$\sum_{i\in\mathbb{N},k\in\mathcal{K}_i} \delta_{(V_{\alpha_{i,k}},\rho^{i,k},\overline{W}^{i,k},\mathcal{P}^{i,k})} \tag{5.100}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,\langle\rho_{\mathrm{t}}^*,1\rangle]}(p)\mathrm{d}p\ \mathbb{N}^\lambda_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W},\mathrm{d}\mathcal{P})$. The conclusion is now similar to the previous discussion on $\widetilde{\mathscr{E}}$. First, remark that the collection of variables $((r_1^\circ, M_1^\circ),\dots,(r_R^\circ,M_R^\circ),(H_{\mathrm{t}}^A,-1))$ can be recovered from

$$\sum_{i\in\mathbb{N}} \delta_{(-\mathscr{I}_{\alpha_i},\mathrm{Tr}(\rho^i,\overline{W}^i))} \quad \text{and} \quad \Big(V_{\alpha_{i,k}} : i\in\mathbb{N}, k\in\mathcal{K}_i \text{ with } \mathcal{P}^{i,k}_{A_\sigma(\overline{W}^{i,k})} \geqslant 1\Big)$$

by making use of the mapping $r \mapsto \sum_{(-\mathscr{I}_{\alpha_i})\leqslant r} \mathscr{L}^{\Lambda_0^i}_\sigma(\rho^i,\overline{W}^i)$ and the fact that $\Lambda_0^i(0)$ can be read from $\mathrm{Tr}(\rho^i,\overline{W}^i)$. In our last claim we used that $\mathscr{L}^{\Lambda_0^i}_\sigma(\rho^i,\overline{W}^i)$ is measurable with respect to $\mathrm{Tr}(\rho^i,\overline{W}^i)$ – by Proposition 5.7 – as well as the equality $H_{\mathrm{t}}^A = \sup_{i\in\mathbb{N}}\Lambda_0^i(0)$ – which holds since $\mathcal{M}$ conditionally on ρ_{t}^* is Poisson $\rho_{\mathrm{t}}^*(\mathrm{d}r)\mathbb{N}^\lambda_{x,0}$ and $H(\rho_{\mathrm{t}}^*) = H_{\mathrm{t}}^A$ by Lemma 5.38. Furthermore, the ordered marked excursions $\mathscr{E}$ correspond precisely to the atoms of (5.100) with $\mathcal{P}^{i,k}_{A_\sigma(\overline{W}^{i,k})} \geqslant 1$ in decreasing order with respect to the process V – since V is non-decreasing and all the values $\{V_{\alpha_{i,k}} : i\in\mathbb{N}, k\in\mathcal{K}_i\}$ are distinct. Putting everything together, we deduce by restriction properties of Poisson measures that, conditionally on $\sum_i \delta_{(-\mathscr{I}_{\alpha_i},\mathrm{Tr}(\rho^i,\overline{W}^i))}$ and K, the collection $\mathscr{E}$ is independent of $\mathbb{S}(\mathbf{T}^A)$ and composed by K i.i.d. variables with distribution $\mathbb{N}^\lambda_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W},\mathrm{d}\mathcal{P}|M \geqslant 1)$. This completes the proof of Proposition 5.34. □

Chapter 6

The excursion theory

The content of this chapter is taken from the work in progress [16], written in collaboration with Armand Riera. This version constitutes an early preliminary version and the content is subject to modifications. Occasionally, only sketches of proof are provided and some measurably questions have not been properly addressed yet.

Abstract. We develop an excursion theory for Markov processes indexed by Lévy trees, away from a regular instantaneous point x. Excursion components are defined as the connected components of the complement of the set of points in the tree with label x. The excursion corresponding to an excursion component is the restriction to the motion to such component. The family of excursion are indexed by means of the additive functional introduced in [82, Section 4]. We prove that, as in classical excursion theory, the excursion process is a Poisson point process with intensity $\mathrm{d}t \otimes \mathbb{N}^*_x$. We refer to the measure $\mathbb{N}^*_x$ as the excursion measure away from x, and we provide a precise description of the latter. Finally, we address the reconstruction of the tree-indexed process in terms of its excursion process.

6.1 Introduction

The content of this chapter is work in progress, and this short introduction serves the sole purpose of giving a brief outline of the results obtained in [83].

Excursion theory for time-indexed Markov processes has been an active topic of research for decades, and has shown to be a powerful tool in settings of very different natures. In short, it is classic that if x is a regular, instantaneous point for a time-indexed Markov process $(\xi_t)_{t\in\mathbb{R}}$ taking values in a Polish space E with rcll paths, one can decompose the path in a family of excursions away from x. More precisely, set $\mathscr{Z}^\circ$ for the set of times at which the Markov process visits x, and write $(a_i, b_i)_{i\in\mathbb{N}}$ for the connected components of the open set $\mathbb{R}_+\backslash\mathscr{Z}^\circ$. We shall call the piece of path $\xi^i := (\xi_{(a_i+t)\wedge b_i} : t \geqslant 0)$ the excursion associated with the excursion interval (a_i, b_i). The family of excursions $(\xi^i)_{i\in\mathbb{N}}$ can be then studied by means of a remarkable additive functional $\mathcal{L} = (\mathcal{L}_t)_{t\in\mathbb{R}_+}$ called the local time at x of the Markov process ξ. Namely, the point process of excursions indexed by their respective local time $\sum_{i\in\mathbb{N}} \delta_{(\mathcal{L}_{a_i}, \xi^i)}$, often referred to as the excursion process, is a Poisson point process in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, E)$ with intensity $\mathrm{d}t \otimes \mathcal{N}$. The (infinite) measure $\mathcal{N}$ is the so-called excursion measure away from x of the Markov process. Moreover, the path $(\xi_t)_{t\in\mathbb{R}_+}$ can be recovered from the excursion process, making use of the fact that the order induced by $\mathcal{L}$ is precisely the temporal order.

In this work, and in contrast with the time-indexed setting, the Markov processes we consider are indexed by a random set. Namely, the indexing set is now a so-called Lévy tree. The purpose of this work is to develop an excursion theory for continuous Markov processes indexed by a Lévy tree away from a regular and instantaneous point x for the Markov process. When the tree is the Brownian tree, the Markov process a Brownian motion and $x = 0$, an excursion theory was developed in [1] by different methods. The results we obtain both complement and extend the results from the work [1] to arbitrary Lévy trees and more general continuous spatial motions. Let us give a brief overview of the object we shall be working with.

Lévy trees are a family of rooted, compact $\mathbb{R}$-trees introduced in [43]. These can be constructed canonically by considering the tree coded by the height function H under the excursion measure N away from 0 of a spectraly positive Lévy process, with Laplace exponent ψ, reflected at its infimum. Namely, if we write σ for the duration of an excursion under N and for every $s, t \in \mathbb{R}_+$ we set $s \sim_H t$ if $H_s = H_t = \min_{[s,t]} H$, the ψ-Lévy tree $\mathcal{T}_H$ is defined as the quotient space $\mathcal{T}_H := [0, \sigma]/ \sim_H$, and we write p_H for the projection mapping every element $t \in [0, \sigma]$ to its equivalence class. One can think of $(p_H(t) : t \geqslant 0)$ as a clockwise exploration of $\mathcal{T}_H$, starting at the root $\varnothing := p_H(0)$. If we denote the law of the ψ-Lévy process by P_0, this construction can still be performed under P_0 but the resulting tree $\mathcal{T}_H$ is no longer compact, and is referred instead as a ψ-Lévy forest. The height function H is in general not a Markov process, and its study often relies in a measure valued strong Markov process $\rho = (\rho_t : t \geqslant 0)$ called the exploration process taking values in the space of finite measure in $\mathbb{R}_+$, that we denote by $\mathcal{M}_f(\mathbb{R}_+)$. Roughly speaking, the exploration process ρ encodes the branching structure of $\mathcal{T}_H$; moreover, if for an arbitrary $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ we write $H(\mu) := \sup \operatorname{supp} \mu$ for the topological support of μ, the exploration process and H are linked by the relation $\rho_t = H(\rho_t)$ for $t \geqslant 0$. We refer to Section 6.2.1 for a detailed discussion. Now, informally, the Markov process $(\xi, \mathcal{L})$ indexed by the Lévy tree $\mathcal{T}_H$ can be understood as follows. We start the Markov process ξ, paired with its local time at x, at some point $(y, r) \in E \times \mathbb{R}_+$ at the root $\varnothing$ of $\mathcal{T}_H$. Then, the pair travels along the branches of $\mathcal{T}_H$ away from the root, and at each branching point it splits in independent copies with same law. We shall henceforth denote this process by $(\xi_a, \mathcal{L}_a)_{a \in \mathcal{T}_H}$. We stress that this description is informal, and to define this process formally we rely in the formalism of Lévy snakes in the sense of [43]. In this direction, we write $\mathcal{W}_{E \times \mathbb{R}_+}$ the space of finite $E \times \mathbb{R}_+$-valued paths; every element $(\mathrm{w}, \ell) \in \mathcal{W}_E$ consists in a pair of continuous functions $\mathrm{w} : [0, \zeta_{\mathrm{w}}] \to E$, $\ell : [0, \zeta_{\mathrm{w}}] \to \mathbb{R}_+$ with finite lifetime $0 \leqslant \zeta_{\mathrm{w}} < \infty$. We write $(\widehat{\mathrm{w}}, \widehat{\ell})$ for the tip of the path (w, ℓ), viz. $(\mathrm{w}(\zeta_{\mathrm{w}}), \ell(\zeta_{\mathrm{w}}))$. In short, the ψ-Lévy snake with spatial motion $(\xi, \mathcal{L})$ is a time indexed strong Markov process (ρ, W, Λ), taking values in $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{E \times \mathbb{R}_+}$ and such that for every $t \geqslant 0$, (W_t, Λ_t) encodes the labels of $(\xi_a, \mathcal{L}_a)_{a \in \mathcal{T}_H}$ along the geodesic path $[\![\varnothing, p_H(t)]\!] \subset \mathcal{T}_H$, connecting the root to $p_H(t)$. If with a slight abuse of notation, for $(y, r) \in E \times \mathbb{R}_+$ we still write (y, r) for the path with null lifetime started at (y, r), we denote by $\mathbb{P}_{0,y,r}$ the law of the Lévy snake started from $(0, y, r)$. For an overview of the Lévy snake, we refer to Section 6.2.4.

The notions of excursion components and excursions away from x of $(\xi_a)_{a \in \mathcal{T}_H}$ should be heuristically clear. Namely, if we set $\mathscr{Z}$ for the set of points with label x in $\mathcal{T}_H$, the excursion components $(C_u^0)_{u \in D}$ consist in the connected components of the open set $\mathcal{T}_H \backslash \mathscr{Z}$, and the excursion ξ^u associated to the excursion component C_u^0 is the restriction of the motion to the closure of C_u^0, say C_u. The point in C_u closest to the root $\varnothing$ shall be denoted by u, and we refer to it as the debut point of the excursion ξ^u. For every ξ^u, one can construct a $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$ valued process (ρ^u, W^u) in terms of deterministic operations on (ρ, W) encoding both C_u and ξ^u. We refer to Definition 6.8 for a precise definition. The study of the collection of excursions $(\xi^u)_{u \in D}$ is preformed in the sequel through the family of processes $(\rho^u, W^u)_{u\in D}$, and with a slight abuse of notation we shall refer to them still as the excursions away from x.

From our previous discussion, there are two main objects of interest: on the one hand, the random subset set $\mathscr{Z}$, and on the other hand the family of excursions away from x. The study of the set $\mathscr{Z}$ was conducted in [82], and it strongly relies in an additive functional $A = (A_t)_{t\in\mathbb{R}_+}$ of (ρ, W) introduced in [82, Proposition 6], with Stieltjes measure dA supported in a (well identified) subset of $\{t \geqslant 0 : \widehat{W}_t = x\}$ - we refer to [82, Theorem 4.3] for a precise statement. The set $\mathscr{Z}$ possesses a natural genealogical structure as it is a subset of $\mathcal{T}_H$. It has been shown in [82] that this genealogy can be encoded in the random tree obtained from subordinating - in the sense of [66] - the tree $\mathcal{T}_H$ by the continuous non-decreasing function [1] $(\mathcal{L}_a)_{a\in\mathcal{T}_H}$. The resulting tree, that we denote by $\mathcal{T}_H^{\mathcal{L}}$, is a Lévy tree with explicit exponent $\widetilde{\psi}$, see [82, Theorem 4.1]. Since the tree $\mathcal{T}_H^{\mathcal{L}}$ has been explicitly constructed in terms of (ρ, W, Λ), it is natural to look for explicit constructions of the functionals related to $\mathcal{T}_H^{\mathcal{L}}$ in terms of (ρ, W). A first result in this direction was obtained in [82]. Namely, Theorem [82, Theorem 4.1] yields that the tree coded by the continuous function

$$\widehat{\Lambda}_{A_t^{-1}}, \quad t \geqslant 0,$$

is isometric to $\mathcal{T}_H^{\mathcal{L}}$, and has the law of the height process of a $\widetilde{\psi}$-Lévy tree. We shall henceforth denote the process in the last display by $\widetilde{H}$, and write $\widetilde{X}$ for the Lévy process associated with such height process - we refer to Proposition 6.1 for a precise statement.

This work is devoted to the study of the family of excursions $(\rho^u, W^u)_{u\in D}$; let us give an overview of our main results. First, in Section 6.4 we introduce an infinite measure $\mathbb{N}_x^*$ in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ that corresponds, roughly speaking, to the law of the ψ-Lévy snake with spatial motion and excursion under $\mathcal{N}$, trimmed at its first return time to x. This description is informal since the excursion measure $\mathcal{N}$ is an infinite measure. This explicit description allows to construct, making use of the theory of exit local times - see e.g. Section 6.6.1 for a brief overview - a notion of fractal measure for the set of points $\{t \geqslant 0 : \widehat{W}_t = x\}$ under $\mathbb{N}_x^*$. More precisely, under $\mathbb{N}_x^*$ we introduce a continuous non-decreasing process $(L_t^*)_{t\in\mathbb{R}_+}$ that at each time $t \geqslant 0$, measures the fractal size of the set $\{s \in [0,t] : \widehat{W}_s = x\}$.

We now turn our attention to the intricate relationship between the measure $\mathbb{N}_x^*$ and the family of excursions away from x. In analogy with the time-indexed setting, to index this family we shall rely in the additive functional A. Write $\mathbb{D}(\mathbb{R}_+, E \times \mathbb{R}_+)$ for the space of $E \times \mathbb{R}_+$-valued rcll paths equipped with the Skorokhod topology. We can now introduce the main result of this work. Theorem 6.28 states that if we set $g(u) := \inf\{t \geqslant 0 : p_H(t) = u\}$ for the first time the exploration p_H visits the debut point u, for any $(y, r) \in E \times \mathbb{R}_+$ and under $\mathbb{P}_{0,y,r}$, the point measure

$$\mathcal{E} = \sum_{u\in D} \delta_{(A_{g(u)}, \rho^u, W^u)}$$

is a Poisson point process in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, E \times \mathbb{R}_+)$ with intensity $\mathrm{d}t \otimes \mathbb{N}_x^*$. For this reason, we baptise $\mathbb{N}_x^*$ the excursion measure away from x of $(\xi_a)_{a\in\mathcal{T}}$ - we refer to Theorem 6.28 for a more precise and general statement. As a byproduct of our results, we deduce that when the tree is the Brownian tree, the motion a Brownian motion and $x = 0$, the measure $\mathbb{N}_0^*$ coincides with the excursion measure introduced, by different methods, by C. Abraham and J.-F. Le Gall in [1]. The proof of Theorem 6.28 is achieved in two steps.

[1] with respect to the genealogical order

Step 1: We show in Proposition 6.30 and Corollary 6.31 that the measure $\mathcal{E}$ is a Poisson measure with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}_x^*$, where $\widehat{\mathbb{N}}_x^*$ is a sigma-finite measure in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$.

Step 2: Proving that $\widehat{\mathbb{N}}_x^*$ is precisely the measure $\mathbb{N}_x^*$ introduced in Section 6.4. This identification is made in Proposition 6.33 by making use of spinal decompositions of the Lévy snake under $\mathbb{N}_{x,0}$ and $\mathbb{N}_x^*$.

Theorem 6.28 shares striking similarities with the results we stated in the setting of time-indexed Markov processes.

Finally, Section 6.8 is devoted to reconstruction related questions, as we obtain explicit descriptions for the different functionals of $\mathcal{T}_{\widetilde{H}}$ and (ρ, W) in terms of the excursion process $\mathcal{E}$. More precisely, we prove the following results.

(R2) The jump measure of $\widetilde{X}$ is given by $\sum_{u \in D_+} \delta_{(A_{g(u)}, L^*_\infty(\rho^u, W^u))}$.

(R3) The local times at the branching points of $\mathcal{T}_{\widetilde{H}}$ (in the sense of Lemma 6.2) and the family of process $(L^*(\rho^u, W^u) : u \in D)$ differ by an explicit time-change

(R4) The Lévy snake (ρ, W), an therefore the tree-indexed process $(\xi_a)_{a \in \mathcal{T}_H}$, can be recovered from the excursion process $\mathcal{E}$.

As a corollary of (R2), we identify the law of $(\rho^u, W^u)_{u \in D}$ conditionally on $\widetilde{H}$. We stress that the content of this last section is at an early stage and is subject to change.

6.2 Preliminaries

6.2.1 The height process and the exploration process

In this section, we introduce the framework that we will be working with for the rest of this study. We begin with standard considerations on Lévy processes, deferring the construction of the corresponding Lévy trees to the next section. In this direction, we write X for the canonical process on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, the space of right-continuous paths with left limits endowed with the Skorokhod topology, and we denote the law of an arbitrary Lévy process started from 0 by P. We write $(\mathcal{F}_t : \ t \in [0, \infty])$ for the canonical filtration, completed as usual by the class of all P-negligible sets of $\mathcal{F}_\infty$. In what follows, we shall always assume that under P, the following assumptions hold:

(A1) X does not have negative jumps;

(A2) The paths of X are of infinite variation;

(A3) X does not drift to $+\infty$.

We shall write $\psi : \mathbb{R}_+ \mapsto \mathbb{R}$ for the Laplace exponent of X, viz. the function defined by the relation

$$E[\exp(-\lambda X_1)] = \exp(\psi(\lambda)), \qquad \text{for every } \lambda \geqslant 0.$$

We recall that ψ is well defined since X is spectrally positive. If we denote the Lévy measure of X by π, condition (A1) ensures that π is supported on $(0, \infty)$. Further, it is straightforward to see that (A2) holds if and only if ψ has Gaussian component or

$$\int_{(0,1)} \pi(\mathrm{d}x)\ x = \infty,$$

while (A3) yields that the Lévy measure satisfies the integrability condition $\int_{(0,\infty)} \pi(\mathrm{d}x)(x \wedge x^2) < \infty$. Now, it is not difficult to check that under (A1)-(A3), the Laplace exponent ψ can be written in the following form:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}x)(\exp(-\lambda x) - 1 + \lambda x), \tag{6.1}$$

for some $\alpha, \beta \in \mathbb{R}_+$. As discussed in [43], for every such function ψ, it is possible to construct a random tree out of a fundamental functional of X called the height process associated with X. To this end, we shall as well impose the following additional condition on the Laplace exponent:

(A4)

$$\int_1^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)} < \infty.$$

As we will explain below, (A4) is a necessary and sufficient condition for the continuity of the height process. This in turn will ensure the compactness of the corresponding tree. For the rest of this work, we shall work under assumptions (A1)-(A4). Examples of processes that verify these assumptions include Brownian motion and spectrally positive α-stable processes with $\alpha \in (1,2)$.

Let us now give a brief introduction to the height process; our presentation follows [43, Section 1.2]. Inspired by its discrete analogue for Galton-Watson trees (see Section 0.2 in [9]), the height process $H := (H_t : t \geqslant 0)$ is a functional of X defined in such a way that, at each fixed t, the variable H_t measures the size of the set:

$$\big\{s \in [0,t] :\ X_{s-} \leqslant \inf_{s \leqslant r \leqslant t} X_r\big\}. \tag{6.2}$$

To make our description precise, we shall make use of local times and a time-reversal argument. Let us start by introducing some notation. For each $t \geqslant 0$, we consider the time-reversed process

$$\widehat{X}^{(t)}_s := X_t - X_{(t-s)-} \quad \text{and} \quad \widehat{S}^{(t)}_s := \sup_{[0,s]} \widehat{X}^{(t)}, \quad \text{for } 0 \leqslant s \leqslant t, \tag{6.3}$$

with the convention $\widehat{X}^{(t)}_t = X_t$. Then, it is well known that $(X_s : 0 \leqslant s \leqslant t)$ has the same distribution as the time-reversed process $(\widehat{X}^{(t)}_s : 0 \leqslant s \leqslant t)$. Further, the point 0 is instantaneous and regular for the strong Markov process $S - X = (\sup_{[0,s]} X - X_s : s \geqslant 0)$. Now, for every $s \geqslant 0$, let us consider the functional $\Gamma_s : \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}_+$ defined for every $e \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ by the relation:

$$\Gamma_s(e) := \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^s \mathrm{d}r\, 1_{\{\sup_{[0,r]} e - e(r) < \varepsilon_k\}},$$

for some arbitrary fixed decreasing sequence $(\varepsilon_k)_{k\geqslant 0}$ of positive numbers converging to 0. Under P one can find a sub-sequence, that we still write $(\varepsilon_k)_{k\geqslant 0}$, such that for each fixed t the process $\Gamma(\widehat{X}^{(t)}) := (\Gamma_s(\widehat{X}^{(t)}) : 0 \leqslant s \leqslant t)$ exists a.s. The process $\Gamma(\widehat{X}^{(t)})$ is a local time at 0 for $\widehat{S}^{(t)} - \widehat{X}^{(t)}$, and note that the set

$$\{s \in [0,t] : \widehat{S}^{(t)}_s - \widehat{X}^{(t)}_s = 0\}$$

is precisely (6.2) under the mapping $s \mapsto t - s$. Now, for every $t \geqslant 0$ we set:

$$H_t := \Gamma_t(\widehat{X}^{(t)}). \tag{6.4}$$

We shall refer to $H = (H_t : t \geqslant 0)$ as the height process of X. Condition (A4) ensures that H possesses a continuous modification [43, Theorem 1.4.3], that we consider from now on and that we still denote by H. Let us stress that despite the fact that H can be defined when condition (A4) fails, in that case the behaviour of H is highly irregular: in any interval $[r, s] \subset \mathbb{R}_+$, the image of H contains a half-line $[a, \infty)$ for some $a \geqslant 0$.

One of the main difficulties arising in the study of the height process is that it is not Markovian as soon as $\pi \neq 0$. To circumvent this difficulty, we will need to introduce a measure-valued Markov process – called the exploration process – which roughly speaking carries the information needed to make H markovian. Let us start by introducing some notation. For every $0 \leqslant s \leqslant t$, we set

$$I_{s,t} := \inf_{s \leqslant u \leqslant t} X_u$$

and write $\mathrm{d}_s I_{s,t}$ for the Lebesgue-Stieltjets measure associated with the non-decreasing, continuous mapping $u \mapsto I_{u,t}$ for $u \in [0, t]$. We denote the space of finite measures in $\mathbb{R}_+$ equipped with the topology of weak convergence by $\mathcal{M}_f(\mathbb{R}_+)$ and we still write 0 for the identically nul measure. The exploration process is the $\mathcal{M}_f(\mathbb{R}_+)$ – valued process $\rho = (\rho_t : t \geqslant 0)$ defined, for every nonnegative measurable function f, by the relation

$$\langle \rho_t, f \rangle := \int_{[0,t]} \mathrm{d}_s I_{s,t} \; f(H_s), \quad t \geqslant 0.$$

Note that in particular, the total mass $\langle \rho_t, 1 \rangle$ of ρ_t is $X_t - I_t$. Despite its rather technical definition, the exploration process possesses crucial properties making its study viable. For instance, by [43, Proposition 1.2.3], $(\rho_t : t \geqslant 0)$ is an $\mathcal{M}_f(\mathbb{R}_+)$–valued càdlàg strong Markov process and the decomposition of the measure ρ_t on its continuous and purely discontinuous parts is given by:

$$\rho_t(\mathrm{d}r) = \beta \mathbb{1}_{[0,H_t]}(r)\mathrm{d}r + \sum_{\substack{0 < s \leqslant t \\ X_{s-} < I_{s,t}}} (I_{s,t} - X_{s-}) \, \delta_{H_s}(\mathrm{d}r), \quad t \geqslant 0. \tag{6.5}$$

It was later established in [2] that the exploration process is a Feller process. Let us now briefly explain the connection between ρ and H. To this end, we use the notation μ for an arbitrary element of $\mathcal{M}_f(\mathbb{R}_+)$ and we denote the supremum of its topological support by $H(\mu)$, viz. $H(\mu) := \sup(\mathrm{supp}\mu)$, with the convention $H(0) = 0$. The following properties hold P - a.s.

(i) We have the equality between the processes $(H_t : t \geqslant 0) = (H(\rho_t) : t \geqslant 0)$.

(ii) The process $t \mapsto \rho_t$ is càdlàg with respect to the total variation distance.

(iii) Almost surely, the following sets are equal:

$$\{t \geqslant 0 : \rho_t = 0\} = \{t \geqslant 0 : X_t - I_t = 0\} = \{t \geqslant 0 : H_t = 0\}. \tag{6.6}$$

From our previous discussion, under P the exploration process starts from $\rho_0 = 0$. We shall now explain introduce the law of ρ started from an arbitrary $\mu \in \mathcal{M}_f(\mathbb{R}_+)$. To this end, we shall make use of two deterministic operations on the elements of $\mathcal{M}_f(\mathbb{R}_+)$.

Pruning. For $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and $a \geqslant 0$, we write $\kappa_a\mu$ for the element of $\mathcal{M}_f(\mathbb{R}_+)$ characterised by the relation:

$$\kappa_a\mu([0,r]) = \mu([0,r]) \wedge (\langle \mu, 1\rangle - a), \tag{6.7}$$

with the convention $\kappa_a\mu = 0$ if $a \geqslant \langle \mu, 1\rangle$. Roughly speaking, the operation $\mu \mapsto \kappa_a\mu$ prunes the measure μ from the tip of its topological support so that the remaining mass is $\langle \mu, 1\rangle - a$. Remark that despite the fact that $H(\mu)$ might be infinite, the measure $\kappa_a\mu$ has compact support for every $a > 0$. Further, if μ has compact support and supp $\mu = [0, H(\mu)]$, the mapping $\mathbb{R}_+ \ni a \mapsto H(\kappa_a\mu)$ is continuous.

Concatenation. Consider μ, ν two element of $\mathcal{M}_f(\mathbb{R}_+)$ and assume $H(\mu) < \infty$. We write $[\mu, \nu]$ for the element of $\mathcal{M}_f(\mathbb{R}_+)$ defined as follows:

$$\langle [\mu, \nu], f\rangle := \int_{[0,H(\mu)]} \mu(\mathrm{d}r)\, f(r) + \int_{\mathbb{R}_+} \nu(\mathrm{d}r) f(H(\mu) + r),$$

where f is an arbitrary, non-negative measurable function on $\mathbb{R}_+$.

We are now in position to define the law of the exploration process started from an arbitrary measure $\mu \in \mathcal{M}_f(\mathbb{R}_+)$. Under P, we write $\rho^\mu = (\rho_t^\mu : t \geqslant 0)$ for the $\mathcal{M}_f(\mathbb{R}_+)$ – valued process defined at time $t = 0$ as $\rho_0^\mu := \mu$ and for $t > 0$ by

$$\rho_t^\mu := [\kappa_{-I_t}\mu, \rho_t]. \tag{6.8}$$

Remark that the right-hand side is well defined since $\kappa_a\mu$ has compact support for every $a > 0$. We will use the notation $\mathbf{P}_\mu$ to denote the law of ρ^μ in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$, the space of right-continuous $\mathcal{M}_f(\mathbb{R}_+)$-valued paths. If for $r \geqslant 0$ we set $T_r := \inf\{t \geqslant 0 : -I_t > r\}$, it follows that

$$\langle \rho_t^\mu, 1\rangle = X_t + \langle \mu, 1\rangle, \text{ for } 0 \leqslant t \leqslant T_{\langle \mu, 1\rangle}, \quad \text{and} \quad \langle \rho_t^\mu, 1\rangle = X_t - I_t, \text{ for } t \geqslant T_{\langle \mu,1\rangle}.$$

Said otherwise, the process $(\langle \rho_t^\mu, 1\rangle : 0 \leqslant t \leqslant T_{\langle \mu,1\rangle})$ is distributed as the Lévy process X started from $\langle \mu, 1\rangle$ and stopped when reaching 0, and $(\langle \rho^\mu_{t+T_{\langle \mu,1\rangle}}, 1\rangle : t \geqslant 0)$ has the same law as $(\langle \rho_t, 1\rangle : t \geqslant 0)$ under P. We shall write $\mathbf{P}_\mu^\dagger$ for the law of $(\rho^\mu_{t\wedge T_{\langle \mu,1\rangle}} : t \geqslant 0)$ under P.

Let us now introduce a closely related process to ρ that will be used frequently in this work. Under P, we write $\eta := (\eta_t : t \geqslant 0)$ for the measure-valued process defined as

$$\eta_t(\mathrm{d}r) := \beta \mathbb{1}_{[0,H_t]}(r)\mathrm{d}r + \sum_{\substack{0<s\leqslant t \\ X_{s-}<I_{s,t}}} (X_s - I_{s,t})\, \delta_{H_s}(\mathrm{d}r), \quad t \geqslant 0. \tag{6.9}$$

The process η is càdlàg with respect to the total variation distance of measures [43, Corollary 3.1.6] and takes values in $\mathcal{M}_f(\mathbb{R}_+)$ [43, Lemma 3.1.1]. Further, we have $H(\eta_t) = H(\rho_t)$ for every $t \geqslant 0$ and the set $\{t \geqslant 0 : \eta_t = 0\}$ coincides with (6.6). The process η is often referred to as the dual of ρ – the terminology is justified by the identity in distribution (6.11) below – and the pair (ρ, η), is a strong Markov process [43, Proposition 3.1.2]. For a complete account on $(\eta_t : t \geqslant 0)$ we refer to [43, Section 3.1].

For our purposes, it will be crucial to define the height process H and *a fortiori*, the pair (ρ, η), under the excursion measure of the reflected process $X - I = (X_t - \inf_{[0,t]} X_s : t \geqslant 0)$. Let us be more precise: under our current hypothesis, 0 is regular and instantaneous for the Markov process $X - I$, and it is well known that P –a.s. the Lebesgue measure of the set $\{t \in \mathbb{R}_+ : X_t = I_t\}$ is null. The process $-I$ is a local time of $X - I$ and we denote the associated excursion measure from 0 by N. For $e \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$, we write σ_e for the lifetime of e, viz. $\sigma_e := \sup\{t \geqslant 0 : e(t) \neq 0\}$, with the usual convention $\inf\{\varnothing\} = \infty$. Now, denote the excursion intervals of $X - I$ away from 0 by $(a_i, b_i)_{i\in\mathbb{N}}$ – we recall that these are defined as the connected components of the open set $\{t \geqslant 0 : X_t - I_t > 0\}$. For every $i \in \mathbb{N}$, we set $e_i = (X_{(a_i+t)\wedge b_i} - I_{a_i} : t \geqslant 0)$ for the corresponding excursion. Remark that since $-I_t \to \infty$ as $t \uparrow \infty$ by (A3), every excursion interval has finite length. Moreover by (iii), the intervals $(a_i, b_i)_{i\in\mathbb{N}}$ are precisely the excursion intervals away from 0 of ρ and H. The key observation is that H, and therefore (ρ, η), restricted to an arbitrary excursion interval $[a_i, b_i]$ can be written in terms of a functional that only depends on the corresponding excursion e_i. Informally, this should not come as a surprise: note that the definition of H in (6.4) only depends on the excursion straddling t, we refer to the discussion preceding Lemma 1.2.4 in [43] for a more detailed account. This implies that the same holds for the measures defined in (6.5) and (6.9), and with some abuse of notation, we write $(\rho(e), \eta(e))$ for the corresponding functional on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. In particular, it holds that $(\rho_{(a_i+t)\wedge b_i}, \eta_{(a_i+t)\wedge b_i}) = (\rho_t(e_i), \eta_t(e_i))$ for every $t \geqslant 0$. Now, by considering the first excursion $e_j = (X_{(g+t)\wedge d} - I_g : t \geqslant 0)$ with duration $\sigma_{e_j} > \varepsilon$, we can define the law of (ρ, η) under $N(\mathrm{d}e|\sigma_e > \varepsilon)$ as the law of $(\rho_t(e_j), \eta_t(e_j) : t \geqslant 0)$ under P. By repeating this procedure for every $\varepsilon > 0$, defines the law of (ρ, η) under N. Similarly, the functional $e \mapsto H(\rho(e))$ on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ extends the construction of H under N and we often write $H(e)$ for $H(\rho(e))$. Note that in particular, P-a.s. for every $i \in \mathbb{N}$, we have $H_{(a_i+t)\wedge b_i} = H_t(e_i)$ for $t \geqslant 0$. As a straightforward consequence of our previous discussion and excursion theory for the reflected Lévy process $X - I$, we deduce that the random measure in $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2$ defined as

$$\sum_{i\in\mathbb{N}} \delta_{(-I_{a_i}, \rho_{(a_i+\cdot)\wedge b_i}, \eta_{(a_i+\cdot)\wedge b_i})} \tag{6.10}$$

is a Poisson point measure with intensity $\mathbb{1}_{\mathbb{R}_+}(u)\mathrm{d}u\, N(\mathrm{d}\rho, \mathrm{d}\eta)$. Note that (iii) and our discussion on the process η immediately yields that under P, the measure $(0, 0)$ is regular and instantaneous for the Markov process (ρ, η), and that $-I$ is a local time. Therefore, by (6.10) the excursion measure of (ρ, η), associated with $-I$, is precisely $N(\mathrm{d}\rho, \mathrm{d}\eta)$. We also recall for later use the identity in distribution:

$$\big((\rho_t, \eta_t) : t \geqslant 0\big) \overset{(d)}{=} \big((\eta_{(\sigma-t)-}, \rho_{(\sigma-t)-}) : t \geqslant 0\big), \quad \text{under } N, \tag{6.11}$$

and we refer to [43, Corollary 3.1.6] for a proof. Clearly $-I$ is as well a local time for ρ and the corresponding excursion measure is given by $N(\mathrm{d}\rho)$. The strong Markov property of the exploration process under N takes the following form. Let T be a $(\mathcal{F}_t)$ stopping time and Φ a bounded $\mathcal{F}_T$-measurable function. For every bounded functional F on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$, we have

$$N\big(\mathbb{1}_{\{0<T<\infty\}}\Phi \cdot F(\rho_{T+s} : s \geqslant 0)\big) = N\big(\mathbb{1}_{\{0<T<\infty\}}\Phi \cdot \mathbf{E}^{\dagger}_{\rho_T}[F]\big). \tag{6.12}$$

We stress that under N, we still have $H(\rho_t) = H_t$ and that $\langle \rho_t, 1\rangle = X_t$, for every $t \geqslant 0$.

Let us close our discussion with some reconstruction related questions concerning H, ρ and X. Namely, we shall be interested in addressing when does one can reconstruct, given one of these tree processes, the remaining two. We argue under P but analogous arguments hold under N. First, by (i) the height process is a functional of ρ and since we have that $\langle \rho, 1\rangle = X - I$ where $-I$ is a local time for ρ, it readily follows that one can recover the Lévy process X out of ρ. Further, by construction both H and ρ are functionals of X, so it remains to check if one can recover X (and therefore ρ) from the height process. To this end, let us recall that by [43, Lemma 1.3.2] and the monotonicity of $t \mapsto I_t$ we have:

$$\lim_{\varepsilon \to 0} E\Big[\sup_{s\in[0,t]} \Big|\frac{1}{\varepsilon}\int_0^s \mathrm{d}u \, \mathbb{1}_{\{H_u < \varepsilon\}} + I_s\Big|\Big] = 0, \qquad \text{for every } t \geqslant 0. \tag{6.13}$$

The convergence in the previous display yields that $-I$ can also be thought of as the local time at 0 of H. This leads us to the following important result:

Proposition 6.1. *Under P and N, the processes ρ and X are measurable functions of H.*

Proof. Let us start arguing under P. Observe from our previous discussion and (6.13) that it plenty suffices to show that we can construct the reflected Lévy process $X - I$ from H. Therefore, by right continuity, this will follow as soon as we show that for every rational t, we have the convergence in probability

$$X_t - I_t = \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_t^{t+T_0\circ\theta_t} \mathrm{d}s \, \mathbf{1}_{\{H_s \leqslant \min_{[t,s]} H + \varepsilon\}},$$

where in the last display we write $T_0 \circ \theta_t := \inf\{s \geqslant 0 : X_{t+s} - I_{t+s} = 0\}$. By conditioning on $\mathcal{F}_t$ and applying the Markov property it suffices to establish that for every $\mu \in \mathcal{M}_f^0$, under $\mathbf{P}_\mu^\dagger$ the following convergence holds in probability

$$\langle \mu, 1\rangle = \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^\sigma \mathrm{d}s \, \mathbf{1}_{\{H_s \leqslant \min_{[0,s]} H + \varepsilon\}}$$

where $\sigma = \inf\{t \geqslant 0 : \rho_s = 0 \text{ for every } s \geqslant t\}$. Since under P, by (6.8) we can write $H(\rho_s^\mu) = H(\kappa_{-I_s}\mu) + H(\rho_s)$ and $\min_{[0,s]} H(\rho^\mu) = H(\kappa_{-I_s}\mu)$, this is equivalent to showing that the convergence in the last display holds in probability under P, replacing σ by $\inf\{t \geqslant 0 : -I_t = \langle \mu, 1\rangle\}$. However, this follows from (6.13). Now, the result under N can be obtained by similar arguments by applying the Markov property under the excursion measure (6.12). □

6.2.2 Trees coded by excursions and Lévy trees

In this section, we introduce the notion of a Lévy tree with branching mechanism ψ (or, in short, a ψ-Lévy tree), in the sense of [43]. We start by briefly recalling standard notation and basic properties of $\mathbb{R}$-trees. An $\mathbb{R}$-tree $(\mathcal{T}, d)$ is a uniquely arcwise connected metric space, in which each arc is isometric to a compact interval of $\mathbb{R}$. In this work we shall exclusively work with rooted $\mathbb{R}$-trees, which further imposes that $\mathcal{T}$ possesses a distinguished point $\rho \in \mathcal{T}$, called the root. In this work, trees are considered modulo isometries preserving the root, and with a slight abuse of notation we shall still denote them by $(\mathcal{T}, d)$.

For every $u, v \in \mathcal{T}$, we write $[\![u, v]\!]$ for the unique injective path connecting u and v, and we denote their common ancestor by $u \curlywedge v$, viz. the unique element of $\mathcal{T}$ verifying the relation $[\![\rho, u\curlywedge v]\!] = [\![\rho, u]\!] \cap [\![\rho, v]\!]$. It is therefore natural to define a partial order $\preceq$ encoding the genealogy of $\mathcal{T}$. Namely, we shall write $u \preceq v$ if $u \in [\![\rho, v]\!]$, and when the latter holds we say that u is an ancestor of v. The multiplicity of u is defined as the (possibly infinite) number of connected components of $\mathcal{T}\backslash\{u\}$. For every $k \in \mathbb{N}^* \cup \infty$, we write $\text{Multi}_k(\mathcal{T})$ for the family of points in $\mathcal{T}$ with multiplicity k. This allows us to extend classic notions from discrete trees to the framework of $\mathbb{R}$-trees. For instance, we shall make use of the following standard nomenclature:

- The elements of $\text{Multi}_1(\mathcal{T})$ are the *leaves* of $\mathcal{T}$;
- The elements with multiplicity at least 2 are the *skeleton* of $\mathcal{T}$;
- The elements with multiplicity at least 3 are the *branching points* of $\mathcal{T}$.

Let us now present a canonical way to construct $\mathbb{R}$-trees using continuous, non-negative functions. This method is standard and we refer to [43] for a thorough study on coding of $\mathbb{R}$-trees. Fix a continuous, non-negative function $e : \mathbb{R}_+ \mapsto \mathbb{R}_+$, recall the notation $\sigma_e = \sup\{t \geqslant 0 : \; e(t) \neq 0\}$ and that by convention, we write $[0, \infty)$ for $[0, \sigma_e]$ if $\sigma_e = \infty$. For every $s, t \in [0, \sigma_e]$ with $s \leqslant t$, let

$$m_e(s,t) := \inf_{s \leqslant u \leqslant t} e_u,$$

and define the pseudometric d_e in $[0, \sigma_e]$ by setting

$$d_e(s,t) := e_s + e_t - 2 \cdot m_e(s \wedge t, s \vee t), \quad \text{for } s,t \in [0, \sigma_e].$$

For every $s, t \in [0, \sigma_e]$ such that $d_e(s,t) = 0$, we write $s \sim_e t$ and note that this is equivalent to the condition $m_e(s \wedge t, s \vee t) = 0$. Now, we set $\mathcal{T}_e := [0, \sigma_e]/ \sim_e$ for the corresponding quotient space and let $p_e : [0, \sigma_e] \mapsto \mathcal{T}_e$ be the canonical projection. The metric space $(\mathcal{T}_e, d_e)$ is an $\mathbb{R}$-tree, and the function e is referred to as the coding function of $\mathcal{T}_e$. If we further assume that $\sigma_e < \infty$, the resulting tree $\mathcal{T}_e$ is compact. By convention, $\mathcal{T}_e$ is rooted at $p_e(0)$ and with a slight abuse of notation we still denote its root by 0. We stress that *a priori*, the coding function e can not be recovered in general from $\mathcal{T}_e$. Roughly speaking, e encodes the tree $\mathcal{T}_e$ as well as a canonical orientation of it.

We are now in position to introduce Lévy trees. These are precisely the (random) trees obtained from using as coding function the height process H of a Lévy process under the excursion measure N. Let us be more precise. Let X be a Lévy process with Laplace exponent ψ and, under the excursion measure N of $X - I$, consider the height process H. The random tree $(\mathcal{T}_H, d_H)$ coded by H under N is called the Lévy tree with branching mechanism ψ (or in short, ψ-Lévy tree). We mention that when $\psi(\lambda) = \lambda^2/2$ for $\lambda \geqslant 0$, the corresponding tree $\mathcal{T}_H$ is the so-called Brownian tree, and $\mathcal{T}_H$ under $N(\cdot\,|\sigma_H = 1)$ is the celebrated Brownian continuum random tree. If we work instead under P, the height process H is still a continuous, non-negative function on $\mathbb{R}_+$ and thus $(\mathcal{T}_H, d_H)$ is still well defined under P. Recall the equality between the sets (6.6), write $(a_i, b_i)_{i \in \mathbb{N}}$ for the excursion intervals of H away from 0 and for every $i \in \mathbb{N}$, set $H^i := H(e_i)$. The tree $\mathcal{T}_H$ under P can be interpreted as the concatenation at the root, with respect to the order induced by the local time $-I$, of the collection of trees $(\mathcal{T}_{H^i} : i \in \mathbb{N})$. For this reason, under P we refer to $\mathcal{T}_H$ as a forest of ψ - Lévy trees.

Let us now briefly discuss some geometric properties of $\mathcal{T}_H$ under the excursion measure N. By [44, Theorem 4.6.], for every non-negative integer $k \notin \{1, 2, 3, \infty\}$, the sets $\text{Multi}_k(\mathcal{T}_H)$ are empty - in particular, branching points in a Lévy tree are either of multiplicity 3 or infinite. Moreover, $\text{Multi}_\infty(\mathcal{T}_H)$ is nonempty if and only if the Lévy measure π is non-null, and every element of

$\{t \geqslant 0 : \Delta X_t > 0\}$ is in bijection by the projection p_H with an point of $\text{Multi}_\infty(\mathcal{T}_H)$. If $\Delta X_t > 0$, it can be argued that ΔX_t is the fractal mass of the corresponding branching point $p_H(t)$ and that it encodes the number of trees rooted at $p_H(t)$. In this work we shall need a more precise notion of mass, measuring at any time t the number of sub-trees in $\mathcal{T}_H$ rooted at $p_H(t)$ attached to the left and to the right of the geodesic path $[\![0, p_H(t)]\!]$. To this end, we introduce a pair of continuous processes that we shall refer to as the local times at the branching point $p_H(t)$. Let us be more precise.

The local time at a branching point. First, under N or P and for every $\upsilon \in \{t \geqslant 0 : \Delta X_t > 0\}$, we set $z(\upsilon) := \inf\{t \geqslant \upsilon : H_t < H_\upsilon\}$. Note that by the strong Markov property of the exploration process and the fact that 0 is regular for $(-\infty, 0)$ under P for the Lévy process X, the latter coincides with $\inf\{t \geqslant \upsilon : X_t \leqslant X_{\upsilon-}\}$. Now, we define a continuous non-decreasing process $\lambda^{\upsilon,\ell} = (\lambda_t^{\upsilon,\ell} : t \geqslant 0)$ by the relation:

$$\lambda_t^{\upsilon,\ell} := X_\upsilon - I_{\upsilon,t}, \qquad \text{for } t \in [\upsilon, z(\upsilon)],$$

with $\lambda_t^{\upsilon,\ell} = 0$ if $0 \leqslant t < \upsilon$ and $\lambda_t^{\upsilon,\ell} = \Delta X_\upsilon$ if $t > z(\upsilon)$. We refer to $\lambda_\infty^{\upsilon,\ell}$, or equivalently ΔX_υ, as the mass at the branching point $p_H(\upsilon)$. Next, for $t \geqslant 0$ we let $\lambda_t^{\upsilon,r} := \Delta X_\upsilon - \lambda_t^{\upsilon,\ell}$. In particular, this gives that

$$\lambda_t^{\upsilon,r} = I_{\upsilon,t} - X_{\upsilon-}, \qquad \text{for } t \in [\upsilon, z(\upsilon)]$$

where now $\lambda_t^{\upsilon,r} = \Delta X_t$ for $t \leqslant \upsilon$ and $\lambda_t^{\upsilon,r} = 0$ if $t \geqslant z(\upsilon)$. We shall now justify our terminology. Consider the connected components of $\{\upsilon \leqslant t \leqslant z(\upsilon) : H_t > \min_{[\upsilon,t]} H\}$ as well as the corresponding excursions of $(H_t : \upsilon \leqslant t \leqslant z(\upsilon))$ over its running infimum. If $(a, b) \subset \mathbb{R}_+$ is an arbitrary interval, we set $n_\upsilon((a, b), \varepsilon)$ for the number of these excursions starting in (a, b) and reaching a height greater than ε. Observe that every excursion interval is mapped by p_H in a sub-tree of $\mathcal{T}_H$ rooted at $p_H(t)$.

Lemma 6.2. *For every $\varepsilon > 0$, we set $V(\varepsilon) := N(\sup H > \varepsilon)$. Under P and N, a.e. for every $\upsilon \in \{s \geqslant 0 : \Delta X_s > 0\}$ and $t \geqslant 0$, we have the following point-wise convergences:*

$$\lambda_t^{\upsilon,\ell} = \lim_{\varepsilon \to 0} \frac{n_\upsilon\big((0, t), \varepsilon\big)}{V(\varepsilon)}, \qquad \lambda_t^{\upsilon,r} = \lim_{\varepsilon \to 0} \frac{n_\upsilon\big((t, z(\upsilon)), \varepsilon\big)}{V(\varepsilon)}. \tag{6.14}$$

Moreover, the family $(\lambda^{\upsilon,\ell}, \lambda^{\upsilon,r} : \upsilon \in [0, \sigma]$ with $\Delta X_\upsilon > 0)$ is H-measurable, and will be referred to as the local times at the branching points of $\mathcal{T}_H$.

Proof. We shall only argue under the excursion measure N. Let T be an arbitrary stopping time satisfying that on the event $\{T < \sigma\}$, we have $\Delta X_T > 0$. On $\{T < \sigma\}$, we introduce the notations $X^{(T)} := (X_{T+t} - X_T : t \geqslant 0)$, $I^{(T)} := \inf_{(0,t]} X^{(T)}$ for the running infimum and note that by standard properties of Lévy processes we can write $z(T) = T + \inf\{t \geqslant 0 : -I_t^{(T)} = \Delta X_T\}$. Consider $(a_i, b_i)_{i \in \mathbb{N}}$ the connected components of $\{t \geqslant 0 : X_t^{(T)} > I_t^{(T)}\}$, write $(e_i)_{i \in \mathbb{N}}$ the corresponding excursions of X^T and set $H^i := H(e_i)$. By the strong Markov property and excursion theory for the reflected Lévy process, on the event $\{T < \sigma\}$ and conditionally on $\mathcal{F}_T$, the measure $\mu_T := \sum_i \delta_{(-I_{a_i}^{(T)}, H^i)} 1_{\{-I_{a_i}^{(T)} < \Delta X_T\}}$ is a Poisson measure with intensity $1_{[0, \Delta X_T]} \mathrm{d}u \, N(\mathrm{d}H)$. Remark that $(H^i)_{i \in \mathbb{N}}$ are precisely the excursions of $(H_t : \upsilon \leqslant t \leqslant z(\upsilon))$ over its running infimum. Still on the event $\{T < \sigma\}$, we set $\upsilon := T$ and to simplify notation, for $q, \varepsilon > 0$ we write

$N(q,\varepsilon) := \#\{(-I^{(T)}_{a_i}, H^i) \in \mu_T : I^{(T)}_{a_i} < q, \sup_u H^i_u > \varepsilon\}$. Now, the first convergence of the lemma will follows as soon as we prove that a.e. for every $0 \leqslant t \leqslant \inf\{u \geqslant 0 : -I^{(T)}_u = \Delta X_T\}$ we have the pointwise convergence:

$$-I^{(T)}_t = \lim_{\varepsilon \to 0} \frac{N(-I^{(T)}_t, \varepsilon)}{V(\varepsilon)}. \tag{6.15}$$

To this end, we work on $\{T < \sigma\}$ and conditionally on ΔX_T. We recall from Corollary 1.4.2 of [43] that the function $V = (V(a) : a > 0)$ is determined by the relation $\int_{V(a)}^{\infty} \mathrm{d}\lambda \psi(\lambda)^{-1} = a$. Since ψ is strictly convex with $\psi(0) = 0$ and $\psi'(0) \geqslant 0$, it follows that V is continuous and $\lim_{\varepsilon \to 0} V(\varepsilon) = \infty$. Now, this yields that for every fixed $r > 0$, the process $P_t := N(q, e^{-t})$ for $t \geqslant 0$ is a counting process with independent increments and continuous predicable compensator $\nu_t := qV(e^{-t})$ for $t \geqslant 0$. By Corollary 25.26 of [57], the time-changed process $(P_{\nu_t^{-1}} : t \geqslant 0)$ is a standard Poisson process and by the law of large numbers of Poisson processes we get that $N(q,\varepsilon)/V(\varepsilon) \to q$ a.e. as $\varepsilon \downarrow 0$. Furthermore, by monotonicity the convergence holds a.e. for every $0 \leqslant q \leqslant \Delta X_T$. Taking $q = I^{(T)}_t$ for $0 \leqslant t \leqslant \inf\{u \geqslant 0 : -I^{(T)}_u = \Delta X_T\}$ yields (6.15). Now, since the set $\{s \geqslant 0 : \Delta X_s > 0\}$ can be exhausted by a collection of stopping times, each one of them satisfying that $1_{\{T<\infty\}} \Delta X_T > 0$, we infer that the first desired convergence of the lemma holds. The second convergence follows by similar arguments, we skip the details.

Finally, observe that the family of jump-times $\{t \geqslant 0 : \Delta X_t > 0\}$ is H measurable by Lemma 6.50, and that the same holds for $n_v((a,b),\varepsilon)$ for every $(a,b) \subset \mathbb{R}_+$ and $\varepsilon > 0$ by definition of n_v. Now, the approximation (6.14) show that $(\lambda^{v,\ell}, \lambda^{v,r} : v \in [0,\sigma]$ with $\Delta X_v > 0)$ is H-measurable. $\square$

Observe that the exploration process ρ as defined in (6.5) can be written in terms of the family $(\lambda^{v,r} : v \in [0,\sigma]$ with $\Delta X_v > 0)$. Namely, under P and N, we have

$$\rho_t(\mathrm{d}h) := \beta \mathbb{1}_{[0,H_t]}(h)\mathrm{d}h + \sum_{\substack{0<s\leqslant t \\ \Delta X_s > 0}} \lambda^{s,r}_t \delta_{H_s}(\mathrm{d}h), \quad t \geqslant 0. \tag{6.16}$$

We are now in position to introduce the notion of a *Markov process indexed by a Lévy tree.* To do so we shall rely in the formalism of Lévy snakes.

6.2.3 Snake driven by a function

This section provides an overview of the theory of Lévy snakes, which are a class of time-indexed Markov processes first introduced in [43] and further developed in [82]. After a brief introduction to this remarkable family of time-indexed processes, we fix the framework that we shall be working with for the rest of this work. Our presentation draws upon [43, 82].

Let us start by fixing a Polish space (E, d_E). We write $\mathcal{W}_E$ for the collection of E-valued finite paths in E. More precisely, every element w of $\mathcal{W}_E$ is a continuous mapping $\mathrm{w} : [0, \zeta_{\mathrm{w}}] \mapsto E$, where ζ_{w} is a finite non-negative number called the lifetime of w. The endpoint or tip of the path of w is denoted by $\widehat{\mathrm{w}} := \mathrm{w}(\zeta_{\mathrm{w}})$. For every $y \in E$, we write $\mathcal{W}_{E,y} \subseteq \mathcal{W}_E$ for the collection of continuous finite paths starting from y. With a slight abuse of notation, we still denote by y the unique element of $\mathcal{W}_{E,y}$ with lifetime $\zeta_{\mathrm{w}} = 0$. If for every $\mathrm{w}, \mathrm{w}' \in \mathcal{W}_E$ we set

$$d_{\mathcal{W}_E}(\mathrm{w}, \mathrm{w}') := |\zeta_{\mathrm{w}} - \zeta_{\mathrm{w}'}| + \sup_{r \geqslant 0} d_E\big(\mathrm{w}(r \wedge \zeta_{\mathrm{w}}), \mathrm{w}'(r \wedge \zeta_{\mathrm{w}'})\big),$$

the mapping $d_{\mathcal{W}_E} : \mathcal{W}_E \times \mathcal{W}_E \mapsto \mathbb{R}$ is a metric on $\mathcal{W}_E$ and $(\mathcal{W}_E, d_{\mathcal{W}_E})$ is a Polish space.

The first step towards defining the Lévy snake consists in introducing the notion of a snake driven by a continuous (deterministic) function, with spatial motion a continuous Markov process with values E. In this direction, we denote the space of all continuous E-valued functions endowed with the uniform metric by $\mathbb{C}(\mathbb{R}_+, E)$, and we write $\xi = (\xi_t : t \geqslant 0)$ for the canonical process on $\mathbb{C}(\mathbb{R}_+, E)$. We next fix an E-valued strong Markov process with continuous sample paths and for every $y \in E$, we let Π_y be its law in $\mathbb{C}(\mathbb{R}_+, E)$ started from y. Finally, we set $\Pi := (\Pi_y)_{y \in E}$.

Now, fix an arbitrary finite path $\mathrm{w} \in \mathcal{W}_E$. For every $a, b \in \mathbb{R}_+$ with $a \leqslant \zeta_{\mathrm{w}}$ and $a \leqslant b$, we let $R_{a,b}(\mathrm{w}, \mathrm{dw}')$ be the probability measure on $\mathcal{W}_E$ characterised by the following properties:

- $R_{a,b}(\mathrm{w}, \mathrm{dw}')$-a.s., $\mathrm{w}'(s) = \mathrm{w}(s)$ for every $s \in [0, a]$.
- $R_{a,b}(\mathrm{w}, \mathrm{dw}')$-a.s., $\zeta_{\mathrm{w}'} = b$.
- Under $R_{a,b}(\mathrm{w}, \mathrm{dw}')$, $(\mathrm{w}'(s + a))_{s \in [0, b-a]}$ is distributed as $(\xi_s)_{s \in [0, b-a]}$ under $\Pi_{\mathrm{w}(a)}$.

Namely, w' under $R_{a,b}(\mathrm{w}, \mathrm{dw}')$ coincides with w up to time a, and then it is distributed as the Markov process $(\xi_t : 0 \leqslant t \leqslant b - a)$ under $\Pi_{\mathrm{w}(a)}$.

We will now endow $\mathcal{W}_E^{\mathbb{R}_+}$ with a probability measure. In this direction, write $W = (W_t : t \geqslant 0)$ for the canonical process on $\mathcal{W}_E^{\mathbb{R}_+}$. We next fix a continuous, non-negative function h such that $\zeta_{\mathrm{w}} = h(0)$, and for $s, t \geqslant 0$ recall the notation $m_h(s, t)$ for the infimum of h on $[s \wedge t, s \vee t]$. We let $Q^h_{\mathrm{w}}(\mathrm{d}W)$ be the probability measure on $\mathcal{W}_E^{\mathbb{R}_+}$ characterised, for every $n \geqslant 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n$, by the relation:

$$Q^h_{\mathrm{w}}\Big(W_{t_0} \in A_0, \dots, W_{t_n} \in A_n\Big) = \mathbb{1}_{A_0}(\mathrm{w}) \int_{A_1 \times \cdots \times A_n} R_{m_h(0,t_1), h(t_1)}(\mathrm{w}, \mathrm{dw}_1) \dots R_{m_h(t_{n-1}, t_n), h(t_n)}(\mathrm{w}_{n-1}, \mathrm{dw}_n).$$

The canonical process W under Q^h_{w} is a time-inhomogenous $\mathcal{W}_E$-valued Markov process, referred to as *the snake driven by h, with spatial motion* Π *started from* w. The function h is called the driving function since for every $t \geqslant 0$, Q^h_{w}-a.s. it holds that $\zeta_{W_t} = h(t)$. Furthermore, the term *snake* stems from the following key property: for every fixed $0 \leqslant s < t$, Q^h_{w}-a.s. we have

$$W_s(r) = W_t(r), \qquad \text{for every } 0 \leqslant r \leqslant m_h(s, t). \tag{6.17}$$

We stress that this property only holds Q^h_{w}-a.s. for fixed $s, t \in \mathbb{R}_+$. In the sequel, we will refer to it as the (weak) snake property. Turning now our attention to the path regularity of W, let us recall from [82] sufficient conditions on the pair (h, Π) ensuring that W under Q^h_{w} has a continuous modification under the metric $d_{\mathcal{W}_E}$. To this end, we recall the convention $[a, \infty] := [a, \infty)$ for $a < \infty$ and we proceed to introduce some relevant terminology. Consider a family of disjoint intervals $([a_i, b_i], i \in \mathcal{J})$ indexed by an arbitrary subset $\mathcal{J} \subseteq \mathbb{N}$, with $a_i < b_i$, for $a_i, b_i \in \mathbb{R}_+ \cup \{\infty\}$. A continuous, non-negative function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be locally r-Hölder-continuous in $([a_i, b_i] : i \in \mathcal{J})$ for some $r \in (0, 1]$ if, for every $n > 0$, there exists a constant $C_n > 0$ we have:

$$|h(s) - h(t)| \leqslant C_n |s - t|^r, \quad \text{ for every } s, t \in [a_i \wedge n, b_i \wedge n],\ i \in \mathcal{J}.$$

Now, let us consider the following assumptions on the pair (h, Π).

(i) There exists a constant $C_\Pi > 0$ and two positive numbers $p, q > 0$ such that, for every $y \in E$ and $t \geqslant 0$, we have:

$$\Pi_y\big(\sup_{0 \leqslant u \leqslant t} d_E(\xi_u, y)^p\big) \leqslant C_\Pi \cdot t^q.$$

(ii) If we let $((a_i, b_i) : \ i \in \mathcal{J})$ be the excursion intervals of h above its running infimum, the function h is locally r-Hölder continuous in $([a_i, b_i] : i \in \mathcal{J})$, with $qr > 1$.

Proposition 1 in [82] states that under conditions (i) and (ii) on (Π, h), for every $\mathrm{w} \in \mathcal{W}_E$ with $\zeta_{\mathrm{w}} = h(0)$, the process W under Q^h_{w} possesses a continuous modification. Therefore, the measure Q^h_{w} can be defined in the Skorokhod space of $\mathcal{W}_E$-valued right-continuous paths $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_E)$ and, with a slight abuse of notation, we still denote it by Q^h_{w}. From now on, when (i) and (ii) hold, the measure Q^h_{w} will always be considered as a measure in $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_E)$. One crucial consequence is that, under (i) - (ii), the identity (6.17) now holds Q^h_{w}-a.s simultaneously for every $0 \leqslant s < t < \infty$, by continuity of W. This key property will be used frequently and we refer to it as the (strong) snake property. Crucially, it allows to define $\widehat{W} = (\widehat{W}_t : t \geqslant 0)$ as a process indexed by $\mathcal{T}_h$, the tree coded by h. Let us be more precise. Since h is a continuous, non-negative function in $\mathbb{R}_+$, we can consider the corresponding tree $\mathcal{T}_h = [0, \sigma_h]/ \sim_h$. The snake property entails that the continuous, $\mathbb{R}_+ \times E$-valued process $(h, \widehat{W}) := ((h_s, \widehat{W}_s) : s \geqslant 0)$ satisfies that,

$$\text{for every } 0 \leqslant s \leqslant t \text{ such that, } s \sim_h t \text{ we have } \widehat{W}_s = \widehat{W}_t. \tag{6.18}$$

Said otherwise, the function $\widehat{W}$ is compatible with the equivalence relation $\sim_h$ and therefore, is well defined in the quotient space $\mathcal{T}_h$. With a slight abuse of notation, we write $(\widehat{W}_a : a \in \mathcal{T}_H)$ for the E-valued function in $\mathcal{T}_H$ defined, for every $a \in \mathcal{T}_h$, by the relation

$$\widehat{W}_a := \widehat{W}_s, \quad \text{where } s \text{ is an arbitrary element of } p_H^{-1}(a).$$

Remark: Let us briefly comment on our definitions. In the terminology of [1], a continuous $\mathcal{W}_E$-valued mapping $\omega = (\omega_s, s \geqslant 0)$ fulfilling the (strong) snake property and with finite lifetime $\sigma(\omega) = \sup\{t \geqslant 0 : \zeta_{\omega_t} \neq 0\}$, is called a *snake trajectory.* A continuous $\mathbb{R}_+ \times E$-valued pair $(h, \widehat{\omega})$ with finite lifetime satisfying (6.18), is a so-called *tree-like path*. These two families are in bijection, see e.g. Section 2.2 in [1] for a more detailed discussion - keeping in mind that the paths ω considered in [1] start at $\omega_0 = y$, for some $y \in E$.

In this work, the driving function of the snakes we consider is random, and more precisely consists in the height process $H(\rho)$ of a Lévy process. However, the corresponding snake is not in general a Markov process. This can be solved by working with the pair, conformed by the exploration process ρ and the respective snake driven by $H(\rho)$. In order to give a precise description of this process, we shall now introduce the notion of a *snake path.* First, write

$$\mathcal{M}_f^0 := \big\{\mu \in \mathcal{M}_f(\mathbb{R}_+) : \ H(\mu) < \infty \ \text{ and supp } \mu = [0, H(\mu)]\big\} \cup \{0\},$$

and let Θ be the collection of pairs $(\mu, \mathrm{w}) \in \mathcal{M}_f^0 \times \mathcal{W}_E$ satisfying that $H(\mu) = \zeta_{\mathrm{w}}$.

Definition 6.3. *A pair $(\rho, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ is called a snake path started from $(\mu, \mathrm{w}) \in \Theta$ if the mapping $s \mapsto \omega_s$ is continuous in $(\mathcal{W}_E, d_{\mathcal{W}_E})$ and the following properties hold:*

(S1) $(\rho_0, \omega_0) = (\mu, \mathrm{w})$.

(S2) *For every $s \geqslant 0$, we have $(\rho_s, \omega_s) \in \Theta$ - in particular $H(\rho) = \zeta(\omega)$.*

(S3) *ω satisfies the snake property: for every $0 \leqslant s \leqslant t$,*

$$\omega_s(r) = \omega_t(r), \quad \text{for every } \ 0 \leqslant r \leqslant \inf_{[s,t]} \zeta(\omega).$$

The family of snake paths started from some fixed $(\mu, \mathrm{w}) \in \Theta$ is denoted by $\mathcal{S}_{\mu,\mathrm{w}}$, and when (μ, w) is of the form $(0, y)$ for some $y \in E$, we simply write $\mathcal{S}_y$. Note that the snake property (S3) yields that for every $(\rho, \omega) \in \mathcal{S}_y$, the mapping $\widehat{\omega}$ is well defined in $\mathcal{T}_{H(\rho)}$. Now, we set

$$\mathcal{S} := \bigcup_{(\mu,\mathrm{w})\in\Theta} \mathcal{S}_{\mu,\mathrm{w}}.$$

The elements of $\mathcal{S}$ are simply referred to as snake paths and the duration of an arbitrary $(\rho, \omega) \in \mathcal{S}$ is indifferently denoted by

$$\sigma_{H(\rho)} = \sigma(\omega) = \sup\{t \geqslant 0 : \zeta_{\omega_t} \neq 0\}.$$

We will frequently consider restrictions of snake paths $(\rho, \omega) \in \mathcal{S}$ to intervals $[a, b] \subset \mathbb{R}_+$ at which $H(\rho_a) = H(\rho_b)$ with $H(\rho_a) < H(\rho_t)$ for every $t \in (a, b)$ – note that this condition ensures that $p_{H(\rho)}(a) = p_{H(\rho)}(b)$. To this end, we introduce the notion of subtrajectories.

Subtrajectories. Let us start by introducing some notation. First, for $(\mu, \mathrm{w}) \in \Theta$ and fixed $0 < z \leqslant H(\mu)$, we let $\theta_z(\mu, \mathrm{w}) = (\theta_z\mu, \theta_z\mathrm{w})$ be the element of Θ defined by the relations:

$$\langle \theta_z\mu, f\rangle := \int \mu(\mathrm{d}r) f(r - z)1_{\{r>z\}}, \qquad \theta_z\mathrm{w} := \mathrm{w}(z + r),\ r \in [0, \zeta_{\mathrm{w}} - z], \tag{6.19}$$

where $f : \mathbb{R}_+ \mapsto \mathbb{R}$ is an arbitrary measurable bounded function. Now, consider an arbitrary $(\rho, \omega) \in \mathcal{S}$ and fix $0 \leqslant a < b$ such that $H(\rho_a) = H(\rho_b)$ with $H(\rho_a) < H(\rho_t)$ for every $t \in (a, b)$. The subtrajectory of (ρ, ω) in $[a, b]$ is the element of $\mathcal{S}_{\widehat{\omega}_a}$, defined by

$$\mathscr{T}_a(\rho, \omega) := \theta_{H(\rho_a)}(\rho_{(a+t)\wedge b}, \omega_{(a+t)\wedge b}), \qquad \text{for } t \geqslant 0.$$

More precisely, if we let $(\rho', \omega') = \mathscr{T}_a(\rho, \omega)$, for every $t \in [0, b - a]$ we have

$$\langle \rho'_t, f\rangle := \int \rho_{a+t}(\mathrm{d}r) f(r - H(\rho_a))\mathbb{1}_{\{r>H(\rho_a)\}} \qquad \text{and} \qquad \omega'_t := \omega_{a+t}(H(\rho_a) + \cdot\,),$$

with $(\rho'_t, \omega'_t) = (0, \hat{\omega}_a)$ for every $t \geqslant b - a$. In particular, the lifetime of (ρ', ω') writes

$$\zeta(\omega'_t) = H(\rho_{a+t}) - H(\rho_a) = H(\rho'_t), \qquad \text{for every } t \in [0, b - a].$$

Notice that the subtrajectory $\mathscr{T}_a(\rho, \omega)$ is an element of $\mathcal{S}_{\widehat{\omega}_a}$, and that it encodes the labels $(\hat{\omega}_\upsilon : \upsilon \in p_{H(\rho)}([s, t]))$ in the sub-tree $p_{H(\rho)}([a, b]) \subset \mathcal{T}_{H(\rho)}$.

6.2.4 The Lévy snake with spatial motion $(\xi, \mathcal{L})$

In this work, we will be interested mostly in spatial motions consisting of pairs formed by an E-valued continuous strong Markov process coupled with its local time at some fixed point $x \in E$. Let us be more precise: for the rest of this work, we consider a continuous strong Markov process ξ taking values in E, and satisfying the following assumptions:

$$\boldsymbol{x} \textbf{ is regular, instantaneous and recurrent for } \xi, \tag{$\mathbf{H_1}$}$$

and

$$\int_{\mathbf{0}}^{\infty} \mathrm{d}\mathbf{t}\ \mathbb{1}_{\{\xi_\mathbf{t}=\boldsymbol{x}\}} = \mathbf{0}, \qquad \Pi_x - \text{a.s.} \tag{$\mathbf{H_2}$}$$

Let us briefly comment on these assumptions. Hypothesis ($\mathbf{H_1}$) ensures the existence of a local time for ξ at x, that we denote by $\mathcal{L} = (\mathcal{L}_t : t \geqslant 0)$. Since x is recurrent we have $\mathcal{L}_\infty = \infty$. Recall that $\mathcal{L}$ is unique up to a multiplicative constant, that we fix arbitrarily, and we write $\mathcal{N}$ for the corresponding (infinite) excursion measure. The pair

$$\overline{\xi}_s := (\xi_s, \mathcal{L}_s), \quad s \geqslant 0,$$

is a strong Markov process taking values in the Polish space $\overline{E} := E \times \mathbb{R}_+$ equipped with the product metric $d_{\overline{E}}$. We set $\Pi_{y,r}$ for its law in $\mathbb{C}(\mathbb{R}_+, \overline{E})$ started from an arbitrary point $(y, r) \in \overline{E}$ and it will be convenient to assume that $\overline{\xi}$ is the canonical process in $\mathbb{C}(\mathbb{R}_+, \overline{E})$. With a slight abuse of notation, Under $\Pi_{y,r}$ and $\mathcal{N}$, we write σ_ξ for the lifetime of ξ defined as $\sigma_\xi := \sup\{t \geqslant 0 : \xi_t \neq x\}$. The main implications of ($\mathbf{H_2}$) are postponed. Let us just note for latter use that, if we write $(a_i, b_i)_{i\in\mathbb{N}}$ for the excursion intervals away from x of ξ and we let $(\xi^i)_{i\in\mathbb{N}}$ be the corresponding excursions with respective (finite) durations $\sigma_i := \sigma_{\xi^i}$, condition ($\mathbf{H_2}$) ensures that for any non-negative function $f : E \mapsto \mathbb{R}_+$ we have $\int_0^\infty \mathrm{d}s f(\xi_s) = \sum_{i\in\mathbb{N}} \int_0^{\sigma_i} \mathrm{d}s\, f(\xi^i_s)$, $\Pi_{x,0}$–a.s. This fact will be used frequently in our computations.

Getting back to the setting of snakes driven by continuous functions, the role of the Polish space E is taken-over in our framework by the product space $\overline{E}$. Therefore, every element of $\mathcal{W}_{\overline{E}}$ writes $\overline{\mathrm{w}} = (\mathrm{w}, \ell)$ for some $\mathrm{w} \in \mathcal{W}_E$ and $\ell \in \mathcal{W}_{\mathbb{R}_+}$ with identical lifetimes. Letting $\overline{\Pi} = (\Pi_{y,r})_{(y,r)\in\overline{E}}$, we stress that as soon as conditions (i) and (ii) from Section 6.2.3 are fulfilled by $(\overline{\Pi}, h)$, for every $\overline{\mathrm{w}} \in \mathcal{W}_{\overline{E}}$ with $\zeta_{\overline{\mathrm{w}}} = h(0)$ the measure $Q^h_{\overline{\mathrm{w}}}$ is well defined in $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_{\overline{E}})$.

The Lévy snake. We are now in position to introduce the ψ-Lévy Snake with spacial motion $\overline{\Pi}$ (abbreviated as the $(\psi, \overline{\Pi})$-Lévy snake). In short, this process is defined by considering as (random) driving function for the snake with spacial motion $\overline{\Pi}$, the height process of a ψ-Lévy process. Fix a Laplace exponent ψ satisfying (A1) – (A4) and for $\mu \in \mathcal{M}^0_f$, recall the notation $\mathbf{P}_\mu$ for the law in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$ of the exploration process started from μ. With a slight abuse of notation, we still write ρ for the canonical process in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+))$. Recall that under $\mathbf{P}_0$, the exploration process ρ takes values in $\mathcal{M}^0_f$. From the definition of $\mathbf{P}_\mu$ given in (6.9), we get that for every $\mu \in \mathcal{M}^0_f$, the exploration process under $\mathbf{P}_\mu$ takes values as well in $\mathcal{M}^0_f$, and that $H(\rho)$ is continuous $\mathbf{P}_\mu$-a.s. To ensure the continuity of the spacial positions of the $(\psi, \overline{\Pi})$-Lévy snake, we still need to impose one last condition on the pair $(\psi, \overline{\Pi})$. In this direction, we set

$$\Upsilon := \sup\left\{r \geqslant 0 : \lim_{\lambda\to\infty} \lambda^{-r}\psi(\lambda) = \infty\right\}$$

and remark that $\Upsilon \geqslant 1$ by convexity of ψ. For the rest of this work, we impose the following assumption on the pair $(\psi, \overline{\Pi})$:

> **Hypothesis** ($\mathbf{H}_3$). There exists a constant $C_{\overline{\Pi}} > 0$ and two positive numbers $p, q > 0$ such that,
> for every $y \in E$ and $t \geqslant 0$, we have:
>
> $$\Pi_{y,0}\Big(\sup_{0\leqslant u\leqslant t} d_{\overline{E}}\big((\xi_u, \mathcal{L}_u), (y, 0)\big)^p \Big) \leqslant C_{\overline{\Pi}} \cdot t^q, \qquad \text{and} \qquad q \cdot (1 - \Upsilon^{-1}) > 1, \qquad (\mathbf{H_3})$$

This last assumption has the following implication. Write $\overline{\Theta}$ for the collection of pairs $(\mu, \overline{\mathrm{w}}) \in \mathcal{M}^0_f \times \mathcal{W}_{\overline{E}}$ satisfying the condition $H(\mu) = \zeta_{\overline{\mathrm{w}}}$. If for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$ and under $\mathbf{P}_\mu$ we write $(a_i, b_i)_{i\in\mathbb{N}}$

for the excursion intervals of $H(\rho)$ over its running infimum, the second condition in $(\mathbf{H_3})$ combined with [43, Theorem 1.4.4] ensures that for some $r \in (0,1]$ with $qr > 1$, the height process $H(\rho)$ is a.s. locally r-Holder continuous in $(a_i, b_i)_{i\in\mathbb{N}}$. Therefore, for every such r, $\mathbf{P}_\mu$–a.s. condition (ii) from Section 6.2.3 is satisfied; for a detailed discussion we refer to Section 2.3 in [82]. We infer that for every $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$, $\mathbf{P}_\mu$–a.s. the measure $Q^{H(\rho)}_{\overline{\mathrm{w}}}$ is well defined in $\mathbb{D}(\mathbb{R}_+, \mathcal{W}_{\overline{E}})$.

Now, consider the canonical process (ρ, W, Λ) in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_{\overline{E}})$, the space of $\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_{\overline{E}}$-valued, right continuous paths, where $W_s : [0, \zeta_{W_s}] \mapsto E$ and $\Lambda_s : [0, \zeta_{W_s}] \mapsto \mathbb{R}_+$. With a slight abuse of notation, we denote its canonical filtration by $(\mathcal{F}_t : t \geqslant 0)$ and to simplify notation, we write $\overline{W} := (W, \Lambda)$. For every, $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$, we let $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ be the probability measure in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$ defined by the relation

$$\mathbb{P}_{\mu,\overline{\mathrm{w}}}(\mathrm{d}\rho,\, \mathrm{d}\overline{W}) := \mathbf{P}_\mu(\mathrm{d}\rho)\, Q^{H(\rho)}_{\overline{\mathrm{w}}}(\mathrm{d}\overline{W}).$$

In other terms, the law of $(\rho, \overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ is characterised by the following conditions:

- Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, the law of ρ is $\mathbf{P}_\mu$.
- Conditionally on ρ, the distribution of $\overline{W}$ is $Q^{H(\rho)}_{\overline{\mathrm{w}}}$.

The process

$$\Big((\rho, \overline{W}), \big(\mathbb{P}_{\mu,\overline{\mathrm{w}}} : (\mu, \overline{\mathrm{w}}) \in \overline{\Theta}\ \big)\Big)$$

is a strong Markov process with respect to the filtration $(\mathcal{F}_{t+})$, known as the ψ-Lévy snake with spatial motion $\overline{\Pi}$. Note that under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, the process $(H, \overline{W})$ has continuous paths. This entails that the strong snake property holds in the following sense: $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$-a.s., for every $s \leqslant t$, we have

$$W_s(r) = W_t(r), \qquad \text{for } 0 \leqslant r \leqslant m_H(s,t).$$

Moreover, $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$-a.s.

$$\zeta_{W_s} = H(\rho_s), \quad \text{for every } s \geqslant 0.$$

To simplify notation we often write ζ_s for ζ_{W_s}. Note from the definition of $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ that for every $(y, r) \in \overline{E}$ and under $\mathbb{P}_{0,y,r}$, for each fixed $s \geqslant 0$ and conditionally on ζ_s, the pair $(W_s, \Lambda_s) = \big((W_s(h), \Lambda_s(h)) : h \in [0, \zeta_s]\big)$ has the distribution of $(\xi, \mathcal{L})$ under $\Pi_{y,r}$ killed at ζ_s. In particular, the associated Lebesgue-Stieltjes measure of Λ_s is supported on the closure of $\{h \in [0, \zeta_s) : W_s(h) = x\}$, $\mathbb{P}_{0,y,r}$–a.e. Note however that the later property might fail if we work under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ for an arbitrary $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$. Therefore, it will be convenient for our purposes to impose further restrictions on the initial conditions $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}$ that we shall work with. In this direction, we let $\overline{\Theta}_x$ be the subset of $\overline{\Theta}$ conformed by pairs $(\mu, \overline{\mathrm{w}})$, with $\overline{\mathrm{w}} = (\mathrm{w}, \ell)$, satisfying the following conditions:

(i') ℓ is a non-decreasing continuous function and the support of its Lebesgue-Stieltjes measure is

$$\overline{\big\{h \in [0, \zeta_{\mathrm{w}}) : \mathrm{w}(h) = x\big\}}.$$

(ii') The measure μ does not charge the set $\{h \in [0, \zeta_{\mathrm{w}}] : \mathrm{w}(h) = x\}$, viz.

$$\int_{[0,\zeta_{\mathrm{w}}]} \mu(\mathrm{d}h)\, \mathbb{1}_{\{\mathrm{w}(h)=x\}} = 0.$$

The following lemma taken from [82] justifies our definition for the subset $\overline{\Theta}_x$.

Lemma 6.4. [82, Lemma 5] *For every $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$, the process $(\rho,\overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ takes values in the subset $\overline{\Theta}_x$.*

Since this hold for instance for initial conditions of the form $(0,y,r)$ for $(y,r)\in\overline{E}$, the space $\overline{\Theta}_x$ is the natural subset of initial condition to work with. From now on, we will work with $(\rho,\overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ for $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$.

It now follows from our definitions and our previous discussion on the regularity of the Lévy snake that for every $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$, the process $(\rho,\overline{W})$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ takes vales in $\mathcal{S}_{\mu,\overline{\mathrm{w}}}$. In particular, if we write $\mathcal{T}_H$ for the Lévy tree coded by $H(\rho)$, the snake property yields that $(\widehat{W}_s,\widehat{\Lambda}_s : s\in[0,\sigma_H])$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ is well defined in the quotient space $\mathcal{T}_H$. With a slight abuse of notation, we write

$$((\xi_a,\mathcal{L}_a) : a\in\mathcal{T}_H)$$

for the $\mathcal{T}_H$-valued function defined, for every $a\in\mathcal{T}_H$, as $(\xi_a,\mathcal{L}_a):=(\widehat{W}_s,\widehat{\Lambda}_s)$ for any $s\in p_H^{-1}(a)$. When working under $\mathbb{P}_{0,y,r}$ for some $(y,r)\in\overline{E}$, we refer to the process in the previous display as the Markov process $(\xi,\mathcal{L})$ indexed by the ψ-Lévy tree $\mathcal{T}_H$, started from (y,r). At this point, let us mention the crucial role played by assumption $(\mathbf{H_2})$. By [82, Proposition 4], it ensures that the set of branching points of $\mathcal{T}_H$ and $\{v\in\mathcal{T}_H\backslash\{0\} : \xi_v=x\}$ are disjoint. More precisely, for every $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$, $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$–a.e. we have:

$$\Big\{t\in[0,\sigma] : \widehat{W}_t=x\Big\}\cap\Big\{t\in[0,\sigma] : p_H(t)\in\mathrm{Multi}_3(\mathcal{T}_H)\cup\mathrm{Multi}_\infty(\mathcal{T}_H),\ p_H(t)\neq 0\Big\}=\varnothing. \quad (6.20)$$

Excursion measures of the Lévy snake. The identity (6.6) combined with the snake property yields that for every $(y,r)\in\overline{E}$, the point $(0,(y,r))\in\mathcal{M}_f^0\times\mathcal{W}_{\overline{E}}$ is instantaneous and regular for $(\rho,\overline{W})$ and that $-I$ is a local time for $(\rho,\overline{W})$ at $(0,(y,r))$. We write $\mathbb{N}_{y,r}$ for the corresponding excursion measure in $\mathbb{D}(\mathbb{R}_+,\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_{\overline{E}})$. Let $(\alpha_i,\beta_i)_{i\in\mathbb{N}}$ be the connected components of the complement of $\{t\geqslant 0 : \rho_t=0\}$ and for every $i\in\mathbb{N}$, write $(\rho^i,\overline{W}^i)$ for the subtrajectory of (ρ,W) corresponding to the excursion interval (α_i,β_i). By excursion theory, under $\mathbb{P}_{0,y,r}$ the measure

$$\mathcal{M}=\sum_{i\in\mathcal{I}}\delta_{(-I_{\alpha_i},\rho^i,\overline{W}^i)} \quad (6.21)$$

is a Poisson point measure with intensity $\mathbb{1}_{\mathbb{R}_+}(u)\mathrm{d}u\,\mathbb{N}_{y,r}(\mathrm{d}\rho,\mathrm{d}\overline{W})$.

Since the excursion measure of (ρ,η) under $\mathbb{P}_{0,y,r}$ is $N(\mathrm{d}\rho,\mathrm{d}\eta)$, it readily follows from the form of the conditional law of W given (ρ,η) that the measure $\mathbb{N}_{y,r}$ writes:

$$\mathbb{N}_{y,r}(\mathrm{d}\rho,\ \mathrm{d}\eta,\ \mathrm{d}\overline{W})=N(\mathrm{d}\rho,\ \mathrm{d}\eta)\,Q_{y,r}^{H(\rho)}(\mathrm{d}\overline{W}).$$

In other words, (ρ,η) under $\mathbb{N}_{y,r}$ is distributed as (ρ,η) under the excursion measure N and, conditionally on the pair, $\overline{W}$ has the law of a snake driven by $H(\rho)$ with spatial motion $\overline{\xi}$. Under the excursion measure $\mathbb{N}_{y,r}$, we have the following identity in distribution:

$$\big((\rho_t,\eta_t,\overline{W}_t) : t\in[0,\sigma]\big)\overset{(d)}{=}\big((\eta_{(\sigma-t)-},\rho_{(\sigma-t)-},\overline{W}_{\sigma-t}) : t\in[0,\sigma]\big). \quad (6.22)$$

We refer to the identity in the previous display as the duality property of the Lévy snake.

The strong Markov property under the excursion measure $\mathbb{N}_{y,r}$ takes the following form. Let $T > 0$ be an arbitrary $\mathcal{F}_{t+}$-stopping time and Φ be a bounded $\mathcal{F}_{T+}$-measurable function. For every nonegative measurable functional F on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f^0 \times \mathcal{W}_{\overline{E}})$, we have

$$\mathbb{N}_{y,r}\big(1_{\{T<\infty\}}\Phi \cdot F(\rho_{T+s}, \overline{W}_{T+s} : s \geqslant 0)\big) = \mathbb{N}_{y,r}\big(1_{\{T<\infty\}}\Phi \cdot \mathbb{E}^{\dagger}_{\rho,\overline{W}}[F]\big),$$

where we denoted by $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$ the law of $((\rho_{s\wedge\sigma}, \overline{W}_{s\wedge\sigma}) : s \geqslant 0)$ under $\mathbb{P}_{\mu,\mathrm{w}}$. We will be henceforth frequently interested in computing quantities of the form $\mathbb{E}^{\dagger}_{\mu,\overline{\mathrm{w}}}[F]$ for different functionals of interest F of the Lévy snake. To this end, the following observation will be of use. Recall that $\langle \rho, 1\rangle$ under $\mathbb{P}^{\dagger}_{\mu,\mathrm{w}}$ is distributed as the Lévy process X under P started from $\langle \mu, 1\rangle$ and killed at its first passage time at 0. If under $\mathbb{P}^{\dagger}_{\mu,\mathrm{w}}$ and with a slight abuse of notation, we write $I_t = \inf_{[0,t]}\langle \rho, 1\rangle - \langle \mu, 1\rangle$ for $t \geqslant 0$, the measure $\mathcal{M}$ under $\mathbb{P}^{\dagger}_{\mu,\mathrm{w}}$ is a Poisson point measure with intensity

$$1_{[0,\langle \mu,1\rangle]}(u)\mathrm{d}u\, \mathbb{N}_{\overline{\mathrm{w}}((H(\kappa_u\mu))}(\mathrm{d}\rho, \mathrm{d}\overline{W}). \tag{6.23}$$

We refer to [43, Lemma 4.2.4] and its proof for details on these statements. The strong Markov property will be often combined with this fact.

We conclude the section with a reconstruction lemma, which states that the Lévy snake can be recovered solely from its lifetime process and the tip of the snake path.

Corollary 6.5. *For every $(y, r) \in \overline{E}$, under $\mathbb{P}_{0,y,r}$ and $\mathbb{N}_{y,r}$, the process (ρ, W) can be recovered from $(H, \widehat{W})$.*

This corollary is an immediate consequence of Corollary 6.1 and the snake property, we leave the details to the reader.

6.3 Debut points, debut times and excursions

In this section, we shall introduce the notion of an excursion away from x for the tree-indexed process $(\xi_a)_{a\in\mathcal{T}_H}$. In contrast with the classic setting of time-indexed Markov processes, the family of excursions away from x of $(\xi_a)_{a\in\mathcal{T}_H}$ possesses a significantly richer geometric structure. For latter use, we shall now address some of its basic geometric properties. More precise versions of the results that we present had already been established for the Brownian motion indexed by the Brownian tree in [1], and we shall rely on similar arguments. Recall that a point of $\mathcal{T}_H$ with multiplicity at least 3 is called a branching point, and that the collection of branching points of $\mathcal{T}_H$ and the set $\{a \in \mathcal{T}_H : \xi_a = x\}$ are disjoint by (6.20). We recall as well that for every $a, b \in \mathcal{T}_H$, we write $[\![a, b]\!]$ for the unique geodesic path connecting the points a, b.

Definition 6.6. *Under $\mathbb{P}_{0,y,r}$ or $\mathbb{N}_{y,r}$ for $(y, r) \in \overline{E}$, a point $u \in \mathcal{T}_H$ is called an excursion debut for $(\xi_a)_{a\in\mathcal{T}_H}$ if the following properties hold:*

(i) *We have $\xi_u = x$.*

(ii) *We can find $v \succ u$ such that $\xi_a \neq x$ for every a in $]\!]u, v]\!]$.*

We denote the collection of excursion debuts by D. For every $u \in D$, we write C_u for the subset of points $v \in \mathcal{T}_H$ fulfilling that $v \succ u$ with $\xi_a \neq x$ for every $a \in]\!]u, v[\![$. In particular, if u is a debut point, then it belongs to C_u.

For the rest of the section, we work under $\mathbb{N}_{y,r}$ and $\mathbb{P}_{0,y,r}$ for $(y,r) \in \overline{E}$ but our results are often only established under the excursion measure $\mathbb{N}_{y,r}$. We start our discussion with some elementary geometric properties of the set $\{a \in \mathcal{T}_H : \xi_a = x\}$.

Lemma 6.7. *For every $u \in D$, set $C_u^0 := C_u \cap \{a \in \mathcal{T}_H : \xi_a \neq x\}$. Then, the family $(C_u^0)_{u\in D}$ are the connected components of the open set $\{a \in \mathcal{T}_H : \xi_a \neq x\}$.*

In particular, this yields that the collection D is countable, since for every $u \in D$, the non-empty subset $p_H^{-1}(C_u)$ of $(0,\sigma)$ is open and consequently has non-null Lebesgue measure. Remark that *a priori*, and in contrast with the Brownian case treated in [1], we do not have $\mathrm{Int}(C_u) = C_u^0$. Indeed, remark that for instance, we can not rule out the existence of an isolated point w of the set $C_u \cap \{a \in \mathcal{T}_H : \xi_a = x\}$ satisfying $w \in \mathrm{Multi}_1(\mathcal{T}_H)$. For such w, we have both $w \in \mathrm{Int}(C_u)$ and $\xi_w = x$. This scenario can not occur for the Brownian motion indexed by the Brownian tree by Lemma 16 of [1].

Proof. Since $(\widehat{W}_t)_{t\in[0,\sigma]}$ is continuous and compatible with the equivalence relation $\sim_H$, we infer that $(\xi_a)_{a\in\mathcal{T}_H}$ is also continuous. In particular, the set $\{a \in \mathcal{T}_H : \xi_a \neq x\}$ is open in $\mathcal{T}_H$ and for every $u \in D$, the connected component C_u^0 is open. Next, we claim that $\cup_{u\in D} C_u^0 = \{x \in \mathcal{T}_H : \xi_a \neq x\}$. Note that the inclusion $\cup_{u\in D} C_u^0 \subset \{x \in \mathcal{T}_H : \xi_a \neq x\}$ follows by definition. To obtain the reverse inclusion, consider $w \in \mathcal{T}_H$ such that $\xi_w \neq x$ and remark that the set $[\![0,w]\!] \cap \{a \in \mathcal{T}_H : \xi_a \neq x\}$ is nonempty. Then by continuity of $(\xi_a)_{a\in\mathcal{T}_H}$, we can find a unique $u \in [\![0,w]\!]$ satisfying $\xi_u = x$ and such that for every $a \in]\!]u,w]\!]$, we have $\xi_a \neq x$. It now follows from our definitions that u is an excursion debut and that $w \in C_u^0$, which proves the reverse inclusion. It remains to prove that the sets C_u^0, $u \in D$, are disjoint and connected. The latter follows directly from the fact that for any $w_1, w_2 \in C_u^0$, we have $[\![w_1,w_2]\!] \subset C_u^0$ since u is not a branching point, and thus $u \notin [\![w_1,w_2]\!]$. Finally, let us check that if u, u' are distinct debut points, then C_u and $C_{u'}$ are disjoint. Arguing by contradiction, if we had $v \in C_u \cap C_{u'}$, then it must hold that $u \prec u' \prec v$ or $u' \prec u \prec v$. In any case, we get respectively that $u' \in]\!]u,v[\![$ with $\xi_{u'} = x$ and $u \in]\!]u',v[\![$ with $\xi_u = x$, in contradiction with the fact that v belongs to $C_u \cap C_{u'}$. □

For $u \in D$, we set

$$g(u) := \inf\big\{t \geqslant 0 : p_H(t) = u\big\}, \quad \text{and} \quad d(u) := \sup\big\{t \geqslant g(u) : p_H(t) = u\big\}. \tag{6.24}$$

Note that $p_H^{-1}(u) = \{g(u), d(u)\}$ since u is not a branching point. Every $u \in D$ can be identified with a snake path (ρ^u, W^u) started from $(0,x)$, that we shall refer to as an excursion away from x of $(\rho, \overline{W})$. In short, it consists in the sub-trajectory associated to $[g(u), d(u)]$ truncated at its first return time to x. To give a precise definition we first need to introduce some deterministic operations. For $\overline{\mathrm{w}} = (\mathrm{w}, \ell) \in \mathcal{W}_{\overline{E}}$, we write $\tau_x^*(\overline{\mathrm{w}}) := \inf\{t > 0 : \mathrm{w}(t) = x\}$ for the hitting time of x by $\overline{\mathrm{w}}$ and for $(\mu, \mathrm{w}) \in \Theta_x$, $0 < z \leqslant \zeta_{\mathrm{w}}$, recall from Section 6.2.3 the notation $\theta_z(\mu, \mathrm{w}) = (\theta_z\mu, \theta_z\mathrm{w})$ for the translation of (μ, w) to time z. Fix $(\mathsf{\rho}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}) \times \mathcal{W}_E)$ and for $t \geqslant 0$, we set

$$V_t^*(\mathsf{\rho}, \omega) := \int_0^t \mathrm{d}s \, 1_{\{H(\mathsf{\rho}_s) \leqslant \tau_x^*(\omega_s)\}}.$$

We denote the right-inverse of $(V_t^*(\mathsf{\rho}, \omega) : 0 \leqslant t \leqslant \sigma)$ by $(\Gamma_t^*(\mathsf{\rho}, \omega) : 0 \leqslant t < V_\sigma^*(\mathsf{\rho}, \omega))$, viz.

$$\Gamma_t^*(\mathsf{\rho}, \omega) = \inf\{s \geqslant 0 : V_s^*(\mathsf{\rho}, \omega) > t\} \tag{6.25}$$

with the convention $\Gamma_t^*(\rho,\omega) = \sigma(\omega)$ if $t \geqslant V_\sigma^*(\rho,\omega)$. We shall write $\mathrm{tr}_*(\rho,\omega)$ for the element of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}) \times \mathcal{W}_E)$ defined by the relation:

$$\mathrm{tr}_*(\rho,\omega)_t := (\rho_{\Gamma_t^*(\rho,\omega)}, \omega_{\Gamma_t^*(\rho,\omega)}), \qquad \text{for } t \geqslant 0. \tag{6.26}$$

Informally, $\mathrm{tr}_*(\omega)$ removes the trajectories ω_s from ω hitting x and glues the remaining endpoints. Note that if (ρ,ω) is an element of $\mathcal{S}_x$, then we have $\mathrm{tr}_*(\rho,\omega) \in \mathcal{S}_x$. We can now introduce the notion of an excursion away from x. Under $\mathbb{N}_{x,r}$ recall that for every $u \in D$, there exits exactly two times $g(u) < d(u)$ such that $p_H(g(u)) = p_H(d(u)) = u$, and note that for $s \in [g(u), d(u)]$ we have $H_s > H_{g(u)}$. Finally, for every $u \in D$, we set:

$$(\rho^u, W^u) := \mathrm{tr}_*\big(\mathscr{T}_{g(u)}(\rho, W)\big).$$

Definition 6.8. *The family $((\rho^u, W^u) : u \in D)$ is referred to as the family of excursion away from x of $(\rho, \overline{W})$.*

The excursions of $(\xi_a)_{a\in\mathcal{T}_H}$ away from x, that we denote by $(\xi^u)_{u\in D}$, can now be defined for every $u \in D$ as $(\widehat{W}_a^u : a \in \mathcal{T}_{H(\rho^u)})$. The following lemma shows that the family of excursions away from x can be indexed by the value of the local time at the respective connected component.

Lemma 6.9. *For every $u \in D$, the process $(\mathcal{L}_a)_{a\in\mathcal{T}_H}$ is constant on C_u and we denote its value by ℓ_u. Moreover, if u' is another arbitrary element of D with $u \neq u'$, we have $\ell_u \neq \ell'_u$.*

Proof. For every debut $u \in D$, we set $\ell_u = \mathcal{L}_u$. We recall from Lemma 5 in [82] that $\mathbb{N}_{x,r}$ a.e., the Lévy snake $(\rho, \overline{W})$ takes values in $\overline{\Theta}_x$. Therefore, we can consider a measurable subset Ω_0 of full $\mathbb{N}_{x,0}$-measure such that for every $(\rho,\omega) \in \Omega_0$, the process $(\rho_t(\rho), \overline{W}_t(\omega))_{t\geqslant 0}$ stays in $\overline{\Theta}_x$. Without loss of generality, we work under Ω_0. Let us now prove that, on every $\mathcal{C}_u$, the local time $\mathcal{L}$ is constant and identically equal to ℓ_u. To this end, let $u \in D$ and consider $a \in C_u \backslash \{u\}$. Recall that by definition, it holds that $\xi_v \neq x$ for every $v \in]\!]u, a[\![$. Next, consider $t := \inf\{s \geqslant 0 : p_H(s) = a\}$ and note that we have $g(u) < t$ with $H_t > \inf_{[g(u),t]} H = H_{g(u)}$. Since the image under p_H of $I := \big\{r \in (g(u), t) : \inf_{[r,t]} H = H_r\big\}$ is $]\!]u, a[\![$, it must hold that $\widehat{W}_s \neq x$ for every $s \in I$. By the snake property, we get that $W_t(h) \neq x$ for every $h \in (H_{g(u)}, H_t)$ and since $(\rho_t, \overline{W}_t) \in \overline{\Theta}_x$, we infer that $\widehat{\Lambda}_t = \Lambda_t(H_{g(u)}) = \widehat{\Lambda}_{g(u)}$. This shows that $\mathcal{L}_u = \mathcal{L}_a$ and since a is arbitrary, we infer that $\mathcal{L}$ is identically equal to ℓ_u on C_u.

Let us now show that if $u \neq u'$, then $\ell_u \neq \ell_{u'}$. For $t, t' \in \mathbb{Q}$, we write $t \curlywedge t'$ for the smallest element of $p_H^{-1}(p_H(t) \curlywedge p_H(t'))$. Note from the definition of $\mathbb{N}_{x,0}$ that conditionally on $(W_t(h) : 0 \leqslant h \leqslant H_{t\curlywedge t'})$, the processes

$$L := \big(\Lambda_t(H_{t\curlywedge t'} + h) - \Lambda_t(H_{t\curlywedge t'}) : h \in [0, H_t - H_{t\curlywedge t'}]\big)$$
$$L' := \big(\Lambda_{t'}(H_{t\curlywedge t'} + h) - \Lambda_{t'}(H_{t\curlywedge t'}) : h \in [0, H_{t'} - H_{t\curlywedge t'}]\big)$$

are independent, and distributed as the local time $\mathcal{L}$ under $\Pi_{W_t(H_{t\curlywedge t'}),0}$ stopped respectively at $H_t - H_{t\curlywedge t'}$ and $H_{t'} - H_{t\curlywedge t'}$. Further, set $\mathcal{Z}^\circ := \{t \geqslant 0 : \widehat{W}_t = x\}$ and for $t \in (\mathcal{Z}^\circ)^c$, write $u(t)$ for the unique element $u \in D$ such that $p_H(t) \in C_u$ - in particular remark that $\ell_{u(t)} = \widehat{\Lambda}_t = \mathcal{L}_{p_H(t)}$, and note that the unicity is guaranteed by Lemma 6.7. Consider another arbitrary $t' \in \mathcal{Z}^\circ$. The lemma will shortly follow as soon as we prove that we have $\ell_{u(t)} \neq \ell_{u(t')}$ if L' or L is not identically null; in this direction, we shall make use of the following remark.

Under $\Pi_{x,0}$, if for $r \geqslant 0$ we let $\tau_r^+(\mathcal{L}) = \inf\{t \geqslant 0 : \mathcal{L}_t > r\}$, recall that $(\tau_r^+(\mathcal{L}) : r \geqslant 0)$ is a subordinator. Under $\Pi_{y,0}$ for $y \in E$, consider an independent copy $(\xi', \mathcal{L}')$ of $(\xi, \mathcal{L})$ and note that under $\Pi_{x,0}$, the family of jump-times of the independent subordinators $\tau^+(\mathcal{L})$, $\tau^+(\mathcal{L}')$ are disjoint. If we write $(a_i, b_i)_{i\in I}$ and $(a'_i, b'_i)_{i\in I'}$ for the excursion intervals away from x of ξ and ξ' under $\Pi_{y,0}$ and we fix $t_0 > 0$, it follows that the collections $\{\mathcal{L}_{a_i} : i \in I, 0 < a_i < t_0\}$, $\{\mathcal{L}'_{a'_i} : i \in I', 0 < a'_i < t_0\}$ are disjoint on the event $\{\tau_x(\xi) < t_0 \text{ or } \tau_x(\xi') < t_0\}$.

Let us then conclude the statement of the lemma. If we consider two arbitrary $u, u' \in D$ with $u \neq u'$, we can find $t, t' \in \mathbb{Q} \cap (\mathcal{Z}^\circ)^c$ such that $p_H(t) \in \mathcal{C}_u$, $p_H(t') \in \mathcal{C}_{u'}$; note that by definition $u(t) = u$ and $u(t') = u'$. If $u(t) \prec u(t')$, since $\xi_{u(t')} = x$ we get from the fact that $\mathcal{L}_{p_H(t)} = \mathcal{L}_{u(t)}$ the inequality $\mathcal{L}_{p_H(t)} < \mathcal{L}_{p_H(t')}$ - we stress that in the last assertion we used that under $\mathbb{N}_{x,0}$, the Lévy snake $(\rho, \overline{W})$ takes values in $\overline{\Theta}_x$. The case $u(t') \prec u(t)$ follows by analogous reasoning, and it remains to address when the common ancestor $u(t') \curlywedge u(t)$ is neither $u(t')$ nor $u(t)$. When this holds, we must have $u(t) \in \rrbracket p_H(t') \curlywedge p_H(t), p_H(t) \llbracket$ since $u(t)$ is not a branching point. Since $\xi_{u(t)} = x$, with the same notation as before, we get that the process L is not identically equal to 0 and from our previous discussion we infer that $\ell_u \neq \ell_{u'}$.

□

6.4 The excursion measure $\mathbb{N}^*_x$ away from x

Now that we have defined the notion of an excursion away from x for $(\rho, \overline{W})$, the next step consists in constructing a candidate for the corresponding (infinite) excursion measure, that we shall denote by $\mathbb{N}^*_x$. We shall use the notation $\Pi^\dagger_y$ for the law of ξ started from y and stopped at its first passage time to x. For every driving function h and $\mathrm{w} \in \mathcal{W}_E$ with $\zeta_{\mathrm{w}} = h(0)$, we denote by Q^h_{w} the law of the snake driven by h with spatial motion $\Pi^\dagger := (\Pi^\dagger_y)_{y\in E}$. For every $\mathrm{w} \in \mathcal{W}_E$, we consider the family of measures $R_{a,b}(\mathrm{w}, \mathrm{dw}')$ for $0 \leqslant a \leqslant \zeta_{\mathrm{w}}$ and $b \geqslant a$ introduced in Section 6.2.3 associated to the spatial motion $\Pi^\dagger$. For $0 < t \leqslant \sigma_h$ we write $\nu^h_t(\mathrm{dw})$ for the law of $(\xi_s : 0 \leqslant s \leqslant h(t))$ under the excursion measure $\mathcal{N}$ - note that ν^h_t is a measure on $\mathcal{W}_E$. The interest on the family $(\nu^h_t : t > 0)$ stems from the following property.

Proposition 6.10. *Fix a driving function h with $\sigma_h < \infty$ and $h_0 = h_{\sigma_h} = 0$. There exists $Q^h_{\mathcal{N}}(\mathrm{d}W)$ a unique probability measure on $\mathcal{W}_E^{\mathbb{R}_+}$ such that, for every $n \geqslant 1$ and $0 = t_0 < t_1 < t_2 \cdots < t_n$,*

$$Q^h_{\mathcal{N}}\Big(W_{t_0} \in A_0, W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\Big) = \mathbb{1}_{A_0}(x) \int_{A_1\times\cdots\times A_n} \nu^h_{t_1}(\mathrm{dw}_1) R_{m_h(t_1,t_2),h(t_2)}(\mathrm{w}_1, \mathrm{dw}_2) \dots R_{m_h(t_{n-1},t_n),h(t_n)}(\mathrm{w}_{n-1}, \mathrm{dw}_n).$$

Proof. First remark that for every $0 < s < t$ and f measurable function f on $\mathcal{W}_E$, we have

$$\int_{\mathcal{W}_E} \nu^h_s(\mathrm{dw}) \int_{\mathcal{W}_E} R_{m_h(s,t),h(t)}(\mathrm{w}, \mathrm{dw}') f(\mathrm{w}') = \mathcal{N}\big(f(\xi_r : 0 \leqslant r \leqslant h(t))\big) = \nu^h_t(f),$$

where in the second equality we used the Markov property at time $m_h(s,t)$. Therefore, we have:

$$\nu^h_t(\mathrm{dw}') = \int_{\mathcal{W}_E} \nu^h_s(\mathrm{dw}) R_{m_h(s,t)}(\mathrm{w}, \mathrm{dw}').$$

This entails that the family of measures $Q^h_{\mathcal{N}}\big(W_{t_0} \in \mathrm{dw}_0, \dots, W_{t_n} \in \mathrm{dw}_n\big)$, for $n \geqslant 1$ and $0 = t_0 < t_1 < \cdots < t_n$ satisfy Kolmogorov's consistency criterion. The proposition now follows by Kolmogorov's theorem. □

Informally, the canonical process W under $Q^h_{\mathcal{N}}$ can be interpreted as the snake driven by h with spatial motion an excursion under $\mathcal{N}$. We now turn our attention to some basic properties of $Q^h_{\mathcal{N}}$.

Lemma 6.11. *Fix a driving function h with $\sigma_h < \infty$, $h_0 = h_{\sigma_h} = 0$ and set $h_{\sigma-\cdot} := (h_{\sigma_h - t} : 0 \leqslant t \leqslant \sigma_h)$. The following properties hold:*

(i) *The distribution of $(W_{(\sigma-t)} : 0 \leqslant t \leqslant \sigma_h)$ under $Q^h_{\mathcal{N}}$ coincides with the law of $(W_t : 0 \leqslant t \leqslant \sigma_h)$ under $Q^{h_{\sigma-\cdot}}_{\mathcal{N}}$.*

(ii) *Let $q > 0$ be as in $(\mathbf{H_3})$. If we suppose that h is r-Hölder-continuous with $qr > 1$, then the canonical process W under $Q^h_{\mathcal{N}}$ possesses a continuous modification.*

Proof. For $0 \leqslant s < t \leqslant \sigma_h$, we shall write $P^h_{s,t}$ and $P^{h_{\sigma-\cdot}}_{s,t}$ for the transition semigroup from time s to time t of the time-inhomogenous Markov process W under $Q^h_{\mathcal{N}}$ and $Q^{h_{\sigma-\cdot}}_{\mathcal{N}}$ respectively. Turning our attention to (i), first note that the result will follow as soon as we establish that

$$\nu^h_{\sigma-t} f_2 P^h_{\sigma-t,\sigma-s} f_1 = \nu^{h_{\sigma-\cdot}}_s f_1 P^{h_{\sigma-\cdot}}_{s,t} f_2, \tag{6.27}$$

for every $0 < s < t < \sigma_h$ and bounded measurable functions f_1, f_2 on $\mathcal{W}_E$. Indeed, if the previous identity holds, then by inductively applying (6.27) and noting that $\nu^{h_{\sigma-\cdot}}_t = \nu^h_{\sigma-t}$ we infer

$$\nu^h_{\sigma-t_k} f_k P^h_{\sigma-t_k,\sigma-t_{k-1}} f_{k-1} \dots P^h_{\sigma-t_2,\sigma-t_1} f_1 = \nu^{h_{\sigma-\cdot}}_{t_1} f_1 P^{h_{\sigma-\cdot}}_{t_1,t_2} f_2 \dots P^{h_{\sigma-\cdot}}_{t_{k-1},t_k} f_k,$$

for every $0 < t_1 < \cdots < t_k < \sigma_h$ and bounded measurable functions $f_1, \dots, f_k$ on $\mathcal{W}_E$. The latter equality yields that $(W_{\sigma-t} : 0 \leqslant t \leqslant \sigma_h)$ under $Q^h_{\mathcal{N}}$ and $(W_t : 0 \leqslant t \leqslant \sigma_h)$ under $Q^{h_{\sigma-\cdot}}_{\mathcal{N}}$ have the same finite-dimensional distributions, proving (i). Now, note that the left-hand side of (6.27) is given by

$$\nu^h_{\sigma-t} f_2 P^h_{\sigma-t,\sigma-s} f_1 = \int_{\mathcal{W}_E} \nu^h_{\sigma-t}(\mathrm{dw}) f_2(\mathrm{w}) \int_{\mathcal{W}_E} R_{m_h(\sigma-t,\sigma-s),h(\sigma-s)}(\mathrm{w}, \mathrm{dw}') f_1(\mathrm{w}').$$

Said otherwise, the law of $(W_{\sigma-t}, W_{\sigma-s})$ under $Q^h_{\mathcal{N}}$ is characterised by the following: up to time $m_h(\sigma-s,\sigma-t)$, the paths $(W_{\sigma-t}, W_{\sigma-s})$ coincide and are distributed as an excursion under $\mathcal{N}$ restricted to the time-interval $[0, m_h(\sigma-t,\sigma-s)]$. Moreover, by the Markov property under $\mathcal{N}$ and the definition of the measures $(R_{a,b}(\mathrm{w}, \mathrm{dw}') : \mathrm{w} \in \mathcal{W}_E, 0 \leqslant a \leqslant \zeta_{\mathrm{w}}$ and $b \geqslant a)$, conditionally on $(W_{\sigma-t}(u) : 0 \leqslant r \leqslant m_h(\sigma-t,\sigma-s))$, the restrictions

$$W_{\sigma-t}\big(m_h(\sigma-t,\sigma-s) + r\big), \quad r \in \big[0, h(\sigma-t) - m_h(\sigma-t,\sigma-s)\big]$$

and

$$W_{\sigma-s}\big(m_h(\sigma-t,\sigma-s) + r\big), \quad r \in \big[0, h(\sigma-s) - m_h(\sigma-t,\sigma-s)\big]$$

are independent with distributions ξ under $\Pi^\dagger_{W(m_h(\sigma-t,\sigma-s))}$ stopped at time $h(\sigma-t) - m_h(\sigma-t,\sigma-s)$ and $h(\sigma-s) - m_h(\sigma-t,\sigma-s)$ respectively. A similar inspection of the right-hand side of (6.27) gives that this is precisely the law of (W_s, W_t) under $Q_{\mathcal{N}}^{h_{\sigma-\cdot}}$, and concludes the proof of (i).

Now let us prove (ii). Recall that we are working under ($\mathbf{H_3}$) and observe that, since $((\Pi_y^{\dagger})_{y\in E}, h)$ satisfies conditions (i) and (ii) of Section 6.2.3, under $Q_{\mathcal{N}}^h$ the process W possesses a continuous modification on $(\sigma_h/3, \sigma_h)$. Furthermore, by (ii) we have the equality in distribution,

$$(W_s : \varepsilon \leqslant s \leqslant \sigma_h) \text{ under } Q_{\mathcal{N}}^{h_{\sigma-\cdot}} \overset{(d)}{=} (W_{\sigma-s} : \varepsilon \leqslant s \leqslant \sigma_h) \text{ under } Q_{\mathcal{N}}^h.$$

This implies that W under $Q_{\mathcal{N}}^h$ possesses as well a continuous modification in $[0, 2\sigma_h/3)$ and therefore in the interval $[0, \sigma_h]$. □

Now, we randomise the driving function h by setting:

$$\mathbf{N}_x^*(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}W) := N(\mathrm{d}\rho, \mathrm{d}\eta) Q_{\mathcal{N}}^{H(\rho)}(\mathrm{d}W).$$

Note that $\mathbf{N}_x^*$ is a sigma-finite measure - consider for example the events $\{\sup\{r \geqslant 0 : W_\varepsilon(r) \neq x\} > \varepsilon\}$, for $\varepsilon > 0$ with the convention that $\sup\{\varnothing\} = 0$. In order to study the properties of $\mathbf{N}_x^*$ we consider as well the following family of closely related measures. In this direction, we shall denote the collection of pairs $(\mu, \mathrm{w}) \in \mathcal{M}_f^0 \times \mathcal{W}_E$ with $H(\mu) = \zeta_{\mathrm{w}}$ verifying condition (ii') from Section 6.2.4 by Θ_x.

- For $y \in E$, we let $\mathbf{N}_y$ be the excursion measure of the $(\psi, \Pi^{\dagger})$-Lévy snake at $(0, y)$.
- For $(\mu, \mathrm{w}) \in \Theta_x$, we let $\mathbf{P}_{\mu,\mathrm{w}}^{\dagger}$ be the law of the $(\psi, \Pi^{\dagger})$ - Lévy snake started at (μ, w) and stopped at time $\inf\{s \geqslant 0 : \rho_s = 0\}$.

We stress that $\mathbf{N}_x$ and $\mathbf{N}_x^*$ are of drastically different nature; for instance, under $\mathbf{N}_x$ we have $\widehat{W}_t = x$ for every $t \geqslant 0$. We shall start investigating the properties of $\mathbf{N}_x^*$ and then address its relation with the family of measures we just introduced. Since $H(\rho)$ under $N(\mathrm{d}\rho)$ is a.e. r-Hölder continuous with $qr > 1$, by Lemma 6.11-(ii), under $\mathbf{N}_x^*$ the process W possesses a continuous modification. Therefore, the measure $\mathbf{N}_x^*$ is well defined in the canonical space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, and from now on it will be implicitly assumed that $\mathbf{N}_x^*$ is a measure on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$. Heuristically, the canonical process (ρ, W) under $\mathbf{N}_x^*$ can be interpreted as the ψ - Lévy snake with spatial motion distributed according to $\mathcal{N}$. This description is informal, since $\mathcal{N}$ is an infinite measure.

The definition of $\mathbf{N}_x^*$, combined with Lemma 6.11-(i) and (6.22), allows us to recover the so-called duality property of the Lévy snake under $\mathbf{N}_x^*$.

Corollary 6.12. *Under $\mathbf{N}_x^*$, the processes $(\rho_s, W_s : 0 \leqslant s \leqslant \sigma)$ and $(\eta_{(\sigma-s)-}, W_{\sigma-s} : 0 \leqslant s \leqslant \sigma)$ have the same distribution.*

Let us now address the Markovian character of $\mathbf{N}_x^*$.

Proposition 6.13. *For every $(\mathcal{F}_{t+})$-stopping time $T > 0$, every non-negative $\mathcal{F}_{T+}$-measurable function Φ and non-negative measurable function F on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, we have*

$$\mathbf{N}_x^*\big(\mathbb{1}_{\{T<\infty\}} \Phi \cdot F\big((\rho_{T+s}, W_{T+s} : s \geqslant 0)\big)\big) = \mathbf{N}_x^*\big(\mathbb{1}_{\{T<\infty\}} \Phi \cdot \mathbf{E}^{\dagger}_{\rho_T, W_T}[F]\big). \tag{6.28}$$

Proof. Our arguments and notation follow closely Section 4.1.3 of [43]. In this direction, it suffices to prove the result for an arbitrary bounded stopping time $T > 0$, that we fix from now on. For $t \geqslant 0$ we let $\lceil t \rceil$ be the smallest integer satisfying $t < \lceil t \rceil$ and for every $n \geqslant 1$, we set $T^{(n)} := \lceil Tn \rceil / n$. We write d_{TV} for the total variation metric on $\mathcal{M}_f(\mathbb{R}_+)$ and recall that the Prokhorov metric d_P on $\mathcal{M}_f(\mathbb{R}_+)$ is bounded above by d_{TV}, viz. $d_P \leqslant d_{\text{TV}}$. Consider $f : \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E \mapsto \mathbb{R}_+$ a bounded Lipschitz-continuous function with respect to the product metric $d_{\text{TV}} \wedge 1 + d_{\mathcal{W}_E} \wedge 1$ as well as a bounded $\mathcal{F}_{T+}$-measurable random variable Φ, non-null on a set with finite $\mathbf{N}_x^*$ measure. If we let $(\mathbf{Q}_t^\dagger : t > 0)$ be the transition semi-group of (ρ, W) under $\mathbf{N}_x^*$, the statement of the proposition will follow by showing that

$$\mathbf{N}_x^*\big(\Phi \cdot f(\rho_{T+t}, W_{T+t})\big) = \mathbf{N}_x^*\big(\Phi \cdot \mathbf{Q}_t^\dagger f(\rho_T, W_T)\big).$$

In this direction, for every $n \geqslant 1$, by the simple Markov property we have

$$\begin{aligned}\mathbf{N}_x^*(\Phi \cdot f(\rho_{T^{(n)}+t}, W_{T^{(n)}+t})) &= \sum_{k=1}^{\infty} \mathbf{N}_x^*(\Phi \cdot \mathbb{1}_{\{T^{(n)}=\frac{k}{n}\}} f(\rho_{T^{(n)}+t}, W_{T^{(n)}+t})) \\ &= \sum_{k=1}^{\infty} \mathbf{N}_x^*\big(\Phi \cdot \mathbb{1}_{\{T^{(n)}=\frac{k}{n}\}} \mathbf{Q}_t^\dagger f(\rho_{\frac{k}{n}}, W_{\frac{k}{n}})\big) \\ &= \mathbf{N}_x^*\big(\Phi \cdot \mathbf{Q}_t^\dagger f(\rho_{T^{(n)}}, W_{T^{(n)}})\big) \qquad (6.29)\end{aligned}$$

By right-continuity of (ρ, W) under $\mathbf{N}_x^*$, we have that $\lim_{n\to\infty} f(\rho_{T^{(n)}+t}, W_{T^{(n)}+t}) = f(\rho_{T+t}, W_{T+t})$ and to conclude, it suffices to prove that $\limsup_{n\to\infty} |\mathbf{Q}_t^\dagger f(\rho_T, W_T) - \mathbf{Q}_t^\dagger f(\rho_{T^{(n)}}, W_{T^{(n)}})| = 0$ a.e. To this end, for every $\varepsilon > 0$ and $(\mu, \mathrm{w}) \in \Theta$, set

$$\begin{aligned}\mathcal{V}_\varepsilon(\mu, \mathrm{w}) := \Big\{ &(\mu', \mathrm{w}') \in \Theta : \exists\, \varepsilon_0, \varepsilon_1 \in [0, \varepsilon) \text{ such that } \kappa_{\varepsilon_0}\mu = \kappa_{\varepsilon_1}\mu', \\ &\text{and } (\mathrm{w}(h) : 0 \leqslant h \leqslant H(\kappa_{\varepsilon_0}\mu)) = (\mathrm{w}'(h) : 0 \leqslant h \leqslant H(\kappa_{\varepsilon_1}\mu')) \Big\}, \qquad (6.30)\end{aligned}$$

where we recall that κ is the cutting operation defined in (6.7). We introduce this set since it is plain that, $\mathbf{N}_x^*$-a.e., for every $s > 0$ small enough, the pair (ρ_{T+s}, W_{T+s}) belongs to $\mathcal{V}_\varepsilon(\rho_T, W_T)$. Therefore, $\mathbf{N}_x^*$ a.e.,

$$\limsup_{n\to\infty} |\mathbf{Q}_t^\dagger f(\rho_T, W_T) - \mathbf{Q}_t^\dagger f(\rho_{T^{(n)}}, W_{T^{(n)}})| \leqslant \sup_{(\mu', \mathrm{w}') \in \mathcal{V}_\varepsilon(\rho_T, W_T)} |\mathbf{Q}_t^\dagger f(\rho_T, W_T) - \mathbf{Q}_t^\dagger(\mu', \mathrm{w}')|.$$

Now, it was established in the proof of [43, Lemma 4.1.3] by means of a coupling argument that that for every $(\mu, \mathrm{w}) \in \Theta$, we have

$$\lim_{\varepsilon\to 0} \sup_{(\mu', \mathrm{w}') \in \mathcal{V}_\varepsilon(\mu, \mathrm{w})} |\mathbf{Q}_t^\dagger f(\mu, \mathrm{w}) - \mathbf{Q}_t^\dagger(\mu', \mathrm{w}')| = 0. \qquad (6.31)$$

This completes the proof of the proposition.

□

For $(\mathsf{p}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}) \times \mathcal{W}_E)$, recall the definition of $\Gamma^*(\mathsf{p}, \omega)$ and $\mathrm{tr}_*(\mathsf{p}, \omega)$ given respectively in (6.25) and (6.26).

Definition 6.14. *We denote by* $\mathbb{N}_x^*$ *the law of* $\mathrm{Tr}_*(\rho, W)$ *under* $\mathbf{N}_x^*$*. The measure* $\mathbb{N}_x^*$ *is referred to as the excursion measure of* $(\xi_a)_{a\in\mathcal{T}_H}$ *away from* x*.*

The terminology might be slightly misleading and for instance, the measure $\mathbb{N}_x^*$ should not be confused with $\mathbb{N}_x$, the excursion measure of (ρ, W) away from the measure 0 and the path x. As before, we consider the following family of measures closely related to $\mathbb{N}_x^*$: we set

- $\mathbb{N}_y^*$ with $y \in E\backslash\{x\}$ the law of $\mathrm{Tr}_*(\rho, W)$ under $\mathbf{N}_y$.
- $\mathbb{P}_{\mu,\mathrm{w}}^*$ with Θ_x the law of $\mathrm{Tr}_*(\rho, W)$ under $\mathbf{P}_{\mu,\mathrm{w}}^\dagger$.

Before concluding the section, let us briefly address some properties of $\mathbb{N}_x^*$ that will be used frequently in this work. The time-changed process $\mathrm{Tr}_*(\rho, W)$ is adapted to the filtration $\mathcal{G}_t := \mathcal{F}_{\Gamma_t^*+}$ for $t \geqslant 0$ and $(\mathcal{G}_t)$ is right-continuous. If we consider a $(\mathcal{G}_t)$ - stopping time $T > 0$, it readily follows that Γ_T^* is a $(\mathcal{F}_{t+})$-stopping time and that $\mathcal{G}_T = \mathcal{F}_{\Gamma_T^*+}$. We refer to [27, Exercise 2.11] for a more detailed discussion. Therefore, with the same notation as in (6.28), we infer from the strong Markov property of (ρ, W) under $\mathbf{N}_x^*$ that for every non-negative $\mathcal{G}_T$-measurable function Φ, we have

$$\mathbb{N}_x^*\big(1_{\{T<\infty\}}\Phi \cdot F\big((\rho_{T+s}, W_{T+s} : s \geqslant 0)\big)\big) = \mathbb{N}_x^*\big(1_{\{T<\infty\}}\Phi \cdot \mathbb{E}^*_{\rho_T, W_T}[F]\big). \tag{6.32}$$

Now, recall the identity in distribution under $\mathbf{N}_x^*$ of Corollary 6.12. Let us infer that the same identity holds under $\mathbb{N}_x^*$.

Corollary 6.15. *Under $\mathbb{N}_x^*$, the processes $(\rho_s, W_s : 0 \leqslant s \leqslant \sigma)$ and $(\eta_{(\sigma-s)-}, W_{\sigma-s} : 0 \leqslant s \leqslant \sigma)$ have the same distribution.*

Proof. Under $\mathbf{N}_x^*$, to simplify notation, we write $\rho_t^* := \rho_{\Gamma_t^*}$, $\eta_t^* := \eta_{\Gamma_t^*}$, $W_t^* := W_{\Gamma_t^*}$, $\sigma_* := V_\sigma^*$ and recall that the lifetime of W^* is σ^*. To prove the corollary it suffices to show that, under $\mathbf{N}_x^*$, we have:

$$(\eta^*_{(\sigma^*-s)-}, W^*_{\sigma^*-s} : 0 \leqslant s \leqslant \sigma^*) \overset{(d)}{=} (\rho_s^*, W_s^* : 0 \leqslant s \leqslant \sigma^*). \tag{6.33}$$

In this direction, note that the processes ρ^* and η^* are càdlàg. Moreover, it is straightforward to check that for every $s \geqslant 0$, we have the identity

$$\Gamma^*_{(\sigma^*-s)-} = \sigma - \Gamma_s^*(W_{\sigma-\cdot}).$$

Now, fix $s \geqslant 0$ and let $(s_n)_{n\geqslant 0}$ be an arbitrary decreasing sequence with $s_n \downarrow s$. Then, since $\Gamma^*_{\sigma^*-s_n} \uparrow \Gamma^*_{(\sigma^*-s)-}$ as $n \to \infty$, we infer that

$$\eta^*_{(\sigma^*-s)-} = \lim_{n\to\infty} \eta^*_{\sigma^*-s_n} = \eta_{(\Gamma^*_{(\sigma^*-s)-})-} = \eta_{(\sigma-\Gamma_s^*(W_{\sigma-\cdot}))-}$$

as well as the equality $W^*_{\sigma^*-s} = W_{\sigma-\Gamma_s^*(W_{\sigma-\cdot})}$. Finally, the duality of $\mathbf{N}_x^*$ established in Corollary 6.12 yields the identity in distribution

$$\big(\eta_{(\sigma-\Gamma_s^*(W_{\sigma-\cdot}))-}, W_{\sigma-\Gamma_s^*(W_{\sigma-\cdot})} : 0 \leqslant s \leqslant \sigma^*\big) \overset{(d)}{=} \big(\rho_s^*, W_s^* : 0 \leqslant s \leqslant \sigma^*\big).$$

This concludes the proof of (6.33). □

Let us conclude the section with a lemma that will be used frequently in our computations, in conjunction with the strong Markov property under $\mathbb{N}_x^*$.

Lemma 6.16. *Fix an arbitrary* $(\mu, \mathrm{w}) \in \Theta_x$ *satisfying that* $\mathrm{w}(0) = x$, *and set* $u_0 := \inf\{u \geqslant 0 : \tau_*(\mathrm{w}(H(\kappa_u\mu))) = \infty\}$. *Under* $\mathbb{P}^*_{\mu,\mathrm{w}}$, *write* $(\alpha_i, \beta_i)_{i\in\mathcal{I}}$ *for the excursion intervals of* $\langle\rho, 1\rangle - \langle\rho_0, 1\rangle$ *over its running infimum* $I_t = \inf_{[0,t]}\langle\rho_t, 1\rangle - \langle\rho_0, 1\rangle$ *and for every* $i \in \mathcal{I}$ *let* (ρ^i, W^i) *be the subtrajectory associated to the interval* (α_i, β_i). *Then, the measure*

$$\mathcal{M} = \sum_{i\in\mathcal{I}} \delta_{(-I_{\alpha_i}, \rho^i, W^i)}.$$

is a Poisson point measure with intensity $\mathbb{1}_{(0,\langle\mu,1\rangle-u_0]}(u)\mathrm{d}u\, \mathbb{N}^*_{\mathrm{w}(\kappa_u\mu)}(\mathrm{d}\rho, \mathrm{d}W)$

Proof. Recall that the law of $\mathrm{Tr}_*(\rho, W)$ under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ is precisely $\mathbb{P}^*_{\mu,\mathrm{w}}$. For the rest of the proof we argue under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ and we shall work with an arbitrary fixed initial condition (μ, w) satisfying that $\tau_*(\mathrm{w}) < \infty$; when $\tau_*(\mathrm{w}) = \infty$, we have $u_0 = 0$ and the arguments are simpler. We maintain the notation I for the running infimum of $\langle\rho_{\Gamma^*_t}, 1\rangle - \langle\rho_{\Gamma^*_0}, 1\rangle$ and $(\alpha_i, \beta_i)_{i\in\mathcal{I}}$ for the corresponding excursion intervals. Next, we write $(a'_j, b'_j)_{j\in\mathcal{J}}$ for the excursion intervals of $\langle\rho, 1\rangle - \langle\mu, 1\rangle$ over its running infimum I', set $(\rho^{j,\prime}, W^{j,\prime})_{j\in\mathcal{J}}$ for the corresponding subtrajectories and recall from Section 6.2.4 that the measure $\sum_{j\in\mathcal{J}} \delta_{(-I'_{a_j}\rho^{j,\prime}, W^{j,\prime})}$ is a Poisson measure with intensity $\mathbb{1}_{[0,\langle\mu,1\rangle]}\mathrm{d}u\, \mathbf{N}_{\mathrm{w}(\kappa_u\mu)}(\mathrm{d}\rho, \mathrm{d}W)$. Let us start verifying that

$$\Gamma^*_0(\rho, W) = \inf\{s \geqslant 0 : -I'_s = u_0\}. \tag{6.34}$$

To see this, recall the definition of the exploration process started from μ from (6.8) as well as the definition of W under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$, and write T_{u_0} for the stopping time in the right-hand side. Now, on the one hand, note that for every $t < T_{u_0}$ it holds that $H_t \geqslant \tau_*(\mathrm{w}) = \tau_*(W_t)$ where the equality $H_t = \tau_*(W_t)$ can only hold if $\langle\rho_t, 1\rangle = I'_t$. Since the set of such time has null Lebesgue measure, we get that $\Gamma^*_0(\rho, W) \geqslant T_{u_0}$. To prove the converse inequality it suffices to show that for every $t > T_{u_0}$, we have $H(\kappa_{-I'_t}\mu) < \tau_*(\mathrm{w})$. To see this, we first note that $H(\kappa_{-I'_{T_{u_0}}}\mu) = \tau_*(\mathrm{w})$. Since $(-\infty, 0)$ is regular for X under P, by the strong Markov property we get that for every $t > T_{u_0}$, we have $I'_t < I'_{T_{u_0}}$ and by definition of T_{u_0} it follows that $H(\kappa_{-I'_t}\mu) < H(\kappa_{-I'_{T_u}}\mu) = \tau_*(\mathrm{w})$.

Now, noting that for every $t > T_{u_0}$ it holds that $\tau_*(\mathrm{w}(\kappa_{-I_t}\mu)) = \infty$, we deduce that for every (α'_j, β'_j) with $\alpha'_j > T_{u_0}$ we have that $\tau_*(\mathrm{w}(\kappa_{-I_{\alpha'_j}}\mu)) = \infty$. It now follows from our previous observations that the sets $\{(\alpha'_j, \beta'_j) : j \in \mathcal{J} \text{ and } \alpha'_j > T_{u_0}\}$ and $\{(\Gamma^*_{\alpha_i}, \Gamma^*_{\beta_i}) : i \in \mathcal{I}\}$ are identical. Moreover, the subtrajectory of $\mathrm{Tr}_*(\rho, W)$ corresponding to the interval (α_i, β_i) is precisely $\mathrm{Tr}_*(\rho^{j,\prime}, W^{j,\prime})$, where $j \in \mathcal{J}$ is the unique index satisfying $(\alpha'_j, \beta'_j) = (\Gamma^*_{\alpha_i}, \Gamma^*_{\beta_i})$. Since by (6.34) we can write $\inf_{(0,t]}\langle\rho_{\Gamma^*_t}, 1\rangle - \langle\rho_{\Gamma^*_0}, 1\rangle = I'_{\Gamma^*_t} - u_0$, we conclude that the measure $\mathcal{M}$ under $\mathbb{P}^*_{\mu,\mathrm{w}}$ has the same distribution as the following measure under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$

$$\sum_{j\in\mathcal{J}} \mathbb{1}_{\{-I'_{\alpha'_j} > u_0\}} \delta_{(-I'_{\alpha'_j} + u_0, \mathrm{Tr}_*(\rho^{j,\prime}, W^{j,\prime}))},$$

the latter being a Poisson measure with intensity $\mathbb{1}_{(0,\langle\mu,1\rangle-u_0]}(u)\mathrm{d}u\mathbf{N}_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{Tr}_*(\rho, W) \in \cdot)$. Finally, since the push-forward measure of the Lebesgue measure in $[0, \langle\mu, 1\rangle]$ by the mapping $u \mapsto H(\kappa_u\mu)$ is precisely μ, the condition $\mu(\{0 \leqslant h \leqslant \zeta_\mathrm{w} : \mathrm{w}(h) = x\}) = 0$ on the pair (μ, w) ensures that the Lebesgue measure of the set $\{u \in (0, \langle\mu, 1\rangle - u_0] : \mathrm{w}(H(\kappa_u\mu)) = x\}$ is null, so we can replace $\mathbf{N}_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{Tr}_*(\rho, W) \in \cdot)$ by $\mathbb{N}^*_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{d}\rho, \mathrm{d}W)$. □

6.5 Spinal decompositions

6.5.1 Spinal decomposition of the Lévy snake

On the forthcoming sections, the study of spatial properties of the Lévy snake and its excursions away from x rely in a precise description of the law of $(\rho, \eta, \overline{W})$ at a typical time viz. sampled uniformly in $[0, \sigma]$. To this end, for an arbitrary $(y, r) \in \overline{E}$ we introduce the pointed measure $\mathbb{N}^\bullet_{y,r} := \mathbb{N}_{y,r}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W})\mathrm{d}s\, 1_{\{s \leqslant \sigma\}}$ on $\mathbb{D}(\mathbb{R}_+, \big(\mathcal{M}_f(\mathbb{R}_+)\big)^2 \times \mathcal{W}_{\overline{E}}) \times \mathbb{R}_+$ and write $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$ for the identity function $U(s) = s$. Then, the law of the triplet $(\rho, \eta, \overline{W}, U)$ under $\mathbb{N}^\bullet_{y,r}$ is characterised by the relation

$$\mathbb{N}^\bullet_{y,r}(\Phi(\rho, \eta, \overline{W}, U)) = \mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\, \Phi(\rho, \eta, \overline{W}, s)\Big).$$

Roughly speaking U is a point taken, conditionally on $(\rho, \overline{W})$, uniformly at random with respect to the Lebesgue measure on $[0, \sigma]$. In particular, $(\rho_U, \eta_U, \overline{W}_U)$ under $\mathbb{N}^\bullet_{y,r}$ should be interpreted as the law of the Lévy snake at a typical time, taken with respect to the Lebesgue measure, under the biased measure $\mathbb{N}^\bullet_{y,r}$. It will be crucial for our purposes to not only characterise the law of $(\rho_U, \eta_U, \overline{W}_U)$ under $\mathbb{N}^\bullet_{y,r}$, but to include as well the subtrajectories in the left and right spine of $(\rho_U, \eta_U, \overline{W}_U)$; the functional encoding such information shall be denoted by $\mathtt{Sp}(\rho, \overline{W})_U$. The definition of the latter requires us to introduce some notation.

We argue under $\mathbb{N}^\bullet_{y,r}$ for some arbitrary $(y, r) \in \overline{E}$. We fix some $s \in (0, \sigma)$ and we shall start by defining a point measure that encodes the right spine of (ρ_s, W_s). Denote by $(\alpha_i(s), \beta_i(s))_{i \in \mathcal{I}_s}$ the connected components of $\{t \geqslant s : \langle \rho, 1 \rangle_t > \inf_{[s,t]} \langle \rho, 1 \rangle_t\}$. For each $i \in \mathcal{I}_s$, we let $(\rho^i, \eta^i, \overline{W}^i)$ be the subtrajectory associated to the interval $(\alpha_i(s), \beta_i(s))$. To simplify notation, when there is no risk of confusion we write α_i, β_i instead of $\alpha_i(s), \beta_i(s)$. Finally, for $t \geqslant s$ we let $I^{(r)}_t := \inf_{[s,t]} \langle \rho, 1 \rangle - \langle \rho_u, 1 \rangle$ and we set

$$\mathcal{P}^{(r)}_s(\rho, \overline{W}) := \sum_{i \in \mathcal{I}_s} \delta_{(-I^{(r)}_{\alpha_i}, \rho^i, \eta^i, \overline{W}^i)}. \tag{6.35}$$

We can now perform an analogous construction to encode the left spine. Namely, we consider $(\alpha_j(s), \beta_j(s))_{j \in \mathcal{J}_s}$ the connected components of $\{0 \leqslant t \leqslant s : \langle \eta_t, 1 \rangle > \inf_{[t,s]} \langle \eta, 1 \rangle\}$ and write $(\rho^i, \eta^i, \overline{W}^i)$ be the corresponding sub-trajectories. Finally, we set $I^{(\ell)}_t := \inf_{[t,s]} \langle \eta, 1 \rangle - \langle \eta_s, 1 \rangle$ for $0 \leqslant t \leqslant s$. With the same convention as before, we can define an analogous measure

$$\mathcal{P}^{(\ell)}_s(\rho, \overline{W}) := \sum_{j \in \mathcal{J}_s} \delta_{(-I^{(\ell)}_{\alpha_j}, \rho^j, \eta^j, \overline{W}^j)} \tag{6.36}$$

encoding now the left spine of $(\rho_s, \eta_s, \overline{W}_s)$. For convenience, the sets $\mathcal{J}_s$, $\mathcal{I}_s$ are assumed to be disjoint. In the sequel, to simplify notation we simply write $t_j := -I^{(\ell)}_{\alpha_j}$, $t_i := -I^{(r)}_{\alpha_i}$ for $j \in \mathcal{J}_s$, $i \in \mathcal{I}_s$.

The triplet

$$\mathtt{Sp}(\rho, \overline{W})_s := \big((\rho_s, \eta_s, \overline{W}_s), \mathcal{P}^{(\ell)}_s(\rho, \overline{W}), \mathcal{P}^{(r)}_s(\rho, \overline{W})\big), \tag{6.37}$$

is referred to as the spine decomposition of $(\rho, \overline{W})$ at time s, $(\rho_s, \eta_s, \overline{W}_s)$ being the spine. Our goal now is to characterise the distribution of $\mathtt{Sp}(\rho, W)_U$ under $\mathbb{N}^\bullet_{y,r}$. In this direction, in an auxiliary probability space $(\Omega_0, \mathcal{F}^0, P^0)$, consider a 2-dimensional subordinator $(U^{(1)}, U^{(2)})$ with

Laplace exponent given by:

$$-\log E^0\Big[\exp\big(-\lambda_1 U_1^{(1)} - \lambda_2 U_1^{(2)}\big)\Big] := \begin{cases} \frac{\psi(\lambda_1)-\psi(\lambda_2)}{\lambda_1-\lambda_2} - \alpha & \text{if } \lambda_1 \neq \lambda_2, \\ \psi'(\lambda_1) - \alpha & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

In particular, we have the identity in law $(U^{(1)}, U^{(2)}) \overset{(d)}{=} (U^{(2)}, U^{(1)})$ and note that both subordinators $U^{(1)}$, $U^{(2)}$ have Laplace exponent $\psi(\lambda)/\lambda - \alpha$, for $\lambda \geqslant 0$. Finally, still under P^0 and for $a \in (0,\infty]$, we write $(J_a, \check{J}_a)$ for the measure in $\mathbb{R}_+^2$ defined by the relation,

$$(J_a, \check{J}_a) := \big(\mathbb{1}_{[0,a]}(t)\ \mathrm{d}U_t^{(1)}, \mathbb{1}_{[0,a]}(t)\ \mathrm{d}U_t^{(2)}\big)$$

with the usual convention $[0,\infty] := [0,\infty)$. We shall write $M_p := M_p(\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}})$ for the collection of point measures in $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}}$. We can now state:

Proposition 6.17. *For fixed $(y,r) \in \overline{E}$ and under $\Pi_{y,r} \otimes E^0$, for every $a \geqslant 0$ we let $(\mathcal{M}_a^{(\ell)}, \mathcal{M}_a^{(r)})$ be a pair of point measure on $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}}$ such that conditionally on $(\check{J}_a, J_a, (\overline{\xi}_s : s \leqslant a))$, they are independent Poisson measures with respective intensities*

$$1_{[0,\langle \check{J}_a, 1\rangle]} \mathrm{d}u\, \mathbb{N}_{\overline{\xi}(H(\kappa_u \check{J}_a))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W}), \qquad \textit{and} \qquad 1_{[0,\langle J_a, 1\rangle]} \mathrm{d}u\, \mathbb{N}_{\overline{\xi}(H(\kappa_u J_a))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W}).$$

For every non-negative measurable functional Φ in $\mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}} \times M_p^2$, we have:

$$\mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\, \Phi\big(\mathtt{Sp}(\rho, \overline{W})_s\big)\Big) = \int_0^\infty \mathrm{d}a\, \exp\big(-\alpha a\big) \cdot E^0 \otimes \Pi_{y,r}\Big(\Phi(J_a, \check{J}_a, (\overline{\xi}_s : s \leqslant a), \mathcal{M}_a^{(\ell)}, \mathcal{M}_a^{(r)})\Big). \tag{6.38}$$

Proof. Our proof follows by similar arguments to the ones used in Proposition 2 of [72]. First, note that if instead of Φ we consider a non-negative measurable function Φ_1 on $\mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}}$, identity (6.38) follows by [82, Lemma 1]. Namely, we have

$$\mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\, \Phi_1\big(\rho_s, \eta_s, \overline{W}_s\big)\Big) = \int_0^\infty \mathrm{d}a\, \exp\big(-\alpha a\big) \cdot E^0 \otimes \Pi_{y,r}\Big(\Phi(J_a, \check{J}_a, (\overline{\xi}_s : s \leqslant a))\Big).$$

Returning to the general setting, remark that it suffices to prove the result for an arbitrary functional Φ of the form $\Phi_1\Phi_2\Phi_3$, for Φ_1 as before and Φ_2, Φ_3 non-negative measurable functionals on M_p. For $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, we shall write $\mathbb{Q}_{\mu,\mathrm{w}}(\mathrm{d}\mathcal{P})$ for the law of a Poisson measure with intensity measure $1_{[0,\langle \mu, 1\rangle]} \mathrm{d}u \mathbb{N}_{\overline{\mathrm{w}}(H(k_u \mu))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W})$ and recall from Section 6.2.4 that the law of the measure $\mathcal{M}$ defined in (6.21) under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$ is precisely $\mathbb{Q}_{\mu,\overline{\mathrm{w}}}$. By the Markov property at time s under $1_{\{s<\sigma\}}\mathbb{N}_{y,r}$ and a change of variable, we get

$$\begin{aligned} &\mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\, \Phi_1\big(\rho_s, \eta_s, \overline{W}_s\big) \Phi_2(\mathcal{P}_s^{(\ell)}) \Phi_3(\mathcal{P}_s^{(r)})\Big) \\ &= \mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\, \Phi_1\big(\rho_s, \eta_s, \overline{W}_s\big) \Phi_2(\mathcal{P}_s^{(\ell)}) \mathbb{Q}_{\rho_s, \overline{W}_s}(\Phi_3)\Big) \\ &= \mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}u\, \Phi_1\big(\rho_{\sigma-u}, \eta_{\sigma-u}, \overline{W}_{\sigma-u}\big) \Phi_2(\mathcal{P}_{\sigma-u}^{(\ell)}) \mathbb{Q}_{\rho_{\sigma-u}, \overline{W}_{\sigma-u}}(\Phi_3)\Big). \end{aligned}$$

The proof will now follow by carefully applying the duality property of the Lévy snake (6.22). In this direction, write $\texttt{Rev}$ the time reversal operator, defined by the relation $\texttt{Rev}(\rho,\eta,\overline{W}) = (\rho_{(\sigma-\cdot)-},\eta_{(\sigma-\cdot)-},\overline{W}_{\sigma-\cdot})$. Recall that by definition, we have:

$$\mathcal{P}^{(\ell)}_{\sigma-u}(\rho,\overline{W}) = \sum_{j\in\mathcal{J}_{\sigma-u}} \delta_{(-I^{(\ell)}_{\alpha_j},\rho^j,\eta^j,\overline{W}^j)}$$

where $(\alpha_j,\beta_j)_{j\in\mathcal{J}_{\sigma-u}}$ are the connected components of $\{t\in[0,\sigma-u] : \langle \eta_t,1\rangle > \inf_{[t,\sigma-u]}\langle\eta,1\rangle\}$, $(\rho^j,\eta^j,\overline{W}^j)_{j\in\mathcal{J}_{\sigma-u}}$ are the corresponding subtrajectories and $I^{(\ell)}_t = \inf_{[t,\sigma-u]}\langle\eta,1\rangle - \langle\eta_{\sigma-u},1\rangle$ for $t\in[0,\sigma-u]$. Let us express this family of functionals of $(\rho,\eta,\overline{W})$ in terms of the time-reversed process $(\rho_{(\sigma-\cdot)-},\eta_{(\sigma-\cdot)-},\overline{W}_{\sigma-\cdot})$. In what follows, we shall make repeated use of the duality property (6.22) without explicit mention. With a slight abuse of notation, we let $(\alpha_{j'},\beta_{j'})_{j'\in\mathcal{J}'_u}$ be the connected components of $\{t\in[u,\sigma] : \langle\eta_{(\sigma-t)-},1\rangle > \inf_{[u,t]}\langle\eta_{(\sigma-\cdot)-},1\rangle\}$, and write $(\rho^{j'},\eta^{j'},\overline{W}^{j'})_{j\in\mathcal{J}'_u}$ the corresponding subtrajectories of $(\rho_{(\sigma-\cdot)-},\eta_{(\sigma-\cdot)-},\overline{W}_{\sigma-\cdot})$. Further, we set $I^{(\ell')}_t := \inf_{[u,t]}\langle\eta_{\sigma-\cdot-},1\rangle - \langle\eta_{(\sigma-u)-},1\rangle$, for $t\in[u,\sigma]$. Observe that the set of jump-times of η is countable and under $\mathbb{N}_{y,r}$, for every fixed $u>0$ the sets $\{t\geqslant u : (\rho_t,\eta_t)\neq(\rho_{t-},\eta_{t-})\}$, $\{t\geqslant u : X_t = \inf_{[u,t]}X\}$ are disjoint. It follows that for $\mathrm{d}u$ almost every $u\in(0,\sigma)$, under $\mathbb{N}_{y,r}$ the processes $I^{(\ell')}$ and $(\inf_{[u,t]}\langle\eta_{\sigma-\cdot},1\rangle - \langle\eta_{\sigma-u},1\rangle : t\in[u,\sigma])$ are indistinguishable. We infer that for $\mathrm{d}u$ almost every $u\in(0,\sigma)$, there exists a bijection $j'\leftrightarrow j$ between $\mathcal{J}'_u$ and $\mathcal{J}_{\sigma-u}$ such that we have

$$(-I^{(\ell)}_{\alpha_j},\rho^i,\eta^i,\overline{W}^i) = (-I^{(\ell')}_{\alpha_{j'}},\texttt{Rev}(\rho^{j'},\eta^{j'},\overline{W}^{j'})).$$

In the last identity we used that (ρ,η) are continuous at the extremities of excursion intervals $\{\alpha_j,\beta_j : j\in\mathcal{J}_u\}$. Therefore, for $\mathrm{d}u$ almost every $u\in(0,\sigma)$ we can write:

$$\mathcal{P}^{(\ell)}_{\sigma-u}(\rho,\overline{W}) = \sum_{j'\in\mathcal{J}'_u}\delta_{(-I^{(\ell')}_{\alpha_{j'}},\texttt{Rev}(\rho^{j'},\eta^{j'},\overline{W}^{j'}))}$$

where by the Markov property and duality (6.22), conditionally on $\sigma((\eta_{(\sigma-s)-},\rho_{(\sigma-s)-},\overline{W}_{\sigma-s}) : 0\leqslant s\leqslant u)$ the measure in the right-hand side is a Poisson measure with intensity $\mathbb{Q}_{\eta_{(\sigma-u)-},\overline{W}_{\sigma-u}}$. We stress that in the last claim we used that under $\mathbb{N}_{y,r}$, the distribution of $(\eta_{(\sigma-\cdot)-},\rho_{(\sigma-\cdot)-},\overline{W}_{\sigma-\cdot})$ is $(\rho,\eta,\overline{W})$. Putting everything together, we deduce

$$\mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\,\Phi_1\big(\rho_s,\eta_s,\overline{W}_s\big)\Phi_2(\mathcal{P}^{(\ell)}_s)\Phi_3(\mathcal{P}^{(r)}_s)\Big) = \mathbb{N}_{y,r}\Big(\int_0^\sigma \mathrm{d}s\,\Phi_1\big(\eta_s,\rho_s,\overline{W}_s\big)\mathbb{Q}_{\rho_s,\overline{W}_s}(\Phi_2)\mathbb{Q}_{\eta_s,\overline{W}_s}(\Phi_3)\Big).$$

Now, the proof follows from the case we covered initially for $\Phi=\Phi_1$, recalling that $(\check{J}_a,J_a)$ has the same distribution as $(J_a,\check{J}_a)$. □

We shall now apply the same treatment to the Lévy snake under a pointed version of the excursion measure $\mathbb{N}^*_x$. This will allow us in the next section to obtain a second connection between the measures $\mathbb{N}_x$ and $\mathbb{N}^*_x$ through a spinal decomposition in excursions under $\mathbb{N}_x$. Note that the functional (6.37) is well defined under both $\mathbf{N}^*_x$ and $\mathbb{N}^*_x$ as soon as we remove the process Λ as well as the subtrajectoiries Λ^i,Λ^j from our definitions. The same notations are maintained, replacing $\overline{W}$ by W, and $\overline{W}_i$ by W_i for $i\in\mathcal{I}_s\cup\mathcal{J}_s$, for every $s\geqslant 0$. In the same vain as before, we introduce the pointed measure $\mathbb{N}^{*,\bullet}_x := \mathbb{N}^*_x(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W})\mathrm{d}s\,\mathbb{1}_{\{s\leqslant\sigma\}}$ and we characterise the law of $\texttt{Sp}(\rho,W)_U$ under $\mathbb{N}^{*,\bullet}_x$. We shall write $M^*_p := M_p(\mathbb{R}_+\times\mathcal{M}_f(\mathbb{R}_+)^2\times\mathcal{W}_E)$ for the space of point measures in $\mathbb{R}_+\times\mathcal{M}_f(\mathbb{R}_+)^2\times\mathcal{W}_E$.

Proposition 6.18. *Under $E^0 \otimes \mathcal{N}$, for $a \geqslant 0$ let $(\mathcal{M}_a^{\ell,*}, \mathcal{M}_a^{r,*})$ be a pair of point measure on $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E$ such that on the event $\{\sigma \geqslant a\}$ and conditionally to $(\check{J}_a, J_a, (\xi_s : s \leqslant a))$, they are independent Poisson measures with respective intensities*

$$1_{[0,\langle \check{J}_a,1\rangle]}\mathrm{d}u\, \mathbb{N}^*_{\xi(H(\kappa_u \check{J}_a))}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}W), \qquad \textit{and} \qquad 1_{[0,\langle J_a,1\rangle]}\mathrm{d}u\, \mathbb{N}^*_{\xi(H(\kappa_u J_a))}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}W).$$

For every non-negative measurable functional Φ in $(\mathcal{M}_f(\mathbb{R}_+))^2 \times \mathcal{W}_E \times (M_p^*)^2$, we have:

$$\mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, \Phi\big(\mathtt{Sp}\,(\rho,W)_s\big) = E^0 \otimes \mathcal{N}\Big(\int_0^\sigma \mathrm{d}a\, \exp\big(-\alpha a\big)\cdot \Phi(J_a, \check{J}_a, (\xi_s : s \leqslant a), \mathcal{M}_a^{\ell,*}, \mathcal{M}_a^{r,*})\Big). \tag{6.39}$$

Proof. The proof follows by the same arguments as in Proposition 6.17 after a few considerations. In this direction, suppose first that Φ_1 is a bounded measurable function on $(\mathcal{M}_f(\mathbb{R}_+))^2 \times \mathcal{W}_E$. Since for every fixed $s \geqslant 0$, conditionally on (ρ,η) the variable W_s is distributed as an excursion under $\mathcal{N}$ stopped at time H_s, we deduce from [43, Proposition 3.1.3] that for every non-negative measurable functional Φ taking values in $\mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E$, we have:

$$\mathbf{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, \Phi_1\big(\rho_s,\eta_s,W_s\big)\Big) = \int_0^\infty \mathrm{d}a\, \exp\big(-\alpha a\big)\cdot E^0 \otimes \mathcal{N}\left[\Phi\big(J_a, \check{J}_a, (\xi_t : t \leqslant a)\big)\right] \tag{6.40}$$

Now, by definition of $\mathbb{N}^*_x$ and a change of variable we have

$$\begin{aligned}\mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, \Phi_1\big(\rho_s,\eta_s,W_s\big)\Big) &= \mathbf{N}^*_x\Big(\int_0^{V^*_\sigma} \mathrm{d}s\, \Phi_1\big(\rho_{\Gamma^*_s},\eta_{\Gamma^*_s},W_{\Gamma^*_s}\big)\Big)\\ &= \mathbf{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, 1_{\{H_s \leqslant \tau^*_x(W_s)\}}\, \Phi_1\big(\rho_s,\eta_s,W_s\big)\Big).\end{aligned}$$

We deduce from (6.40) that the term in the right-hand side writes

$$\int_0^\infty \mathrm{d}a\, \exp\big(-\alpha a\big)\cdot E^0 \otimes \mathcal{N}\left[1_{\{a\leqslant\sigma\}}\Phi\big(J_a, \check{J}_a, (\xi_t : t \leqslant a)\big)\right]$$

proving the result for our choice of Φ. In the last reasoning we used that under $\mathcal{N}$, we have $\tau^*_x(\xi) = \sigma(\xi)$. To prove the result for an arbitrary functional Φ as in the statement of the proposition, we need one last remark. Let Φ_2 be a non-negative measurable functional on M_p^* and write $\mathbb{Q}^*_{\mu,\mathrm{w}}$ for the law of a Poisson point measure with intensity $1_{[0,\langle\mu,1\rangle]}\mathrm{d}u\, \mathbb{N}^*_{\mathrm{w}(H(\kappa_u\mu))}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}W)$. Making use of (6.40), by the same arguments as in [82, Lemma 5] we infer that under $\mathbf{N}^*_x$, the Lévy snake takes values in the set $\Theta^e := \{(\mu,\mathrm{w}) \in \Theta : \mathrm{w}(0) = x, \tau^*_x(\mathrm{w}) > 0 \text{ and } \mu(\{0,\tau^*_x(\mathrm{w})\}) = 0)\}$, with the convention that $\mu(\infty) = 0$. We deduce from this that under $\mathbb{N}^*_x$, the process (ρ, W) takes values in $\Theta^e \cap \{(\mu,\mathrm{w}) \in \Theta : \tau^*_x(\mathrm{w}) \in \{\zeta_{\mathrm{w}},\infty\}\}$. Hence, by Lemma 6.16, for every fixed $s \geqslant 0$ we get that conditionally on $\mathcal{F}_s$ and on $\{s < \sigma\}$, the law of $\mathcal{P}^{(r)}_s$ is precisely $\mathbb{Q}^*_{\rho_s,W_s}$. The general case now follows exactly as in the proof of Proposition 6.17, by making use of the Markov property (6.32) under $\mathbb{N}^*_x$, and the duality property of Lemma 6.15 under $\mathbb{N}^*_x$. □

We maintain the notation

$$\Theta^e := \{(\mu,\mathrm{w}) \in \Theta : \mathrm{w}(0) = x, \tau^*_x(\mathrm{w}) > 0 \text{ and } \mu(\{0,\tau^*_x(\mathrm{w})\}) = 0)\}.$$

For latter use, we gather from the previous proof the following result.

Corollary 6.19. *Under $\mathbf{N}^*_x$ resp. $\mathbb{N}^*_x$, the process (ρ, W) takes values in Θ^e resp. $\Theta^e \cap \{(\mu,\mathrm{w}) \in \Theta : \tau^*_x(\mathrm{w}) \in \{\zeta_{\mathrm{w}},\infty\}\}$.*

6.5.2 Spinal relation between $\mathbb{N}_x$ and $\mathbb{N}_x^*$

In this section, we shall relate the measures $\mathbb{N}_x$, $\mathbb{N}_x^*$ through a spinal decomposition in excursion away from x under $\mathbb{N}_x$. Let us start by introducing some notation: for every $r \geqslant 0$ and $\overline{\mathrm{w}} := (\mathrm{w}, \ell) \in \mathcal{W}_{\overline{E}}$, we set:

$$\tau_r^+(\overline{\mathrm{w}}) := \inf\big\{h \geqslant 0 : \ell(h) > r\big\}.$$

Under $\Pi_{y,r}$ for $(y,r) \in \overline{E}$, we still write $\tau_r^+(\xi) := \inf\{t \geqslant 0 :\ \mathcal{L}_t > r\}$ and as usual when there is no risk of confusion the dependence on ξ is dropped. Until further notice we argue under $\mathbb{N}_{x,0}$. In what follows it will be convenient to index each excursion W^u with the value taken by the local time $\widehat{\Lambda}$ in $p_H^{-1}(C_u)$. In other terms, we will work with the family of pairs $(\ell^u, W^u)_{u \in D}$, where $\ell^u := \widehat{\Lambda}_{g(u)}$. Recall from Lemma 6.9 that if u, u' are two excursion debuts with $u \neq u'$, we have $\ell^u \neq \ell^{u'}$ $\mathbb{N}_{x,0}$ –a.e. For fixed $t > 0$ we set,

$$\mathcal{J}(\Lambda_t) := \{r \in [0, \widehat{\Lambda}_t) : \tau_r(\Lambda_t) < \tau_r^+(\Lambda_t)\}.$$

For $u \in D$, we say that W^u is *present in the spine at time* t if $\ell^u \in \mathcal{J}(\Lambda_t) \cup \{\widehat{\Lambda}_t\}$. Let us justify our terminology: for fixed $t > 0$ and $\mathbb{N}_{x,0}$–a.e., when ℓ^u is present in the spine at time t, the path $(W_t(\tau_{\ell^u} + s) : s \in [0, \tau^+_{\ell^u} \wedge H_t - \tau_{\ell^u}])$ is non-trivial and coincides with $(\xi_a : a \in [\![0, p_H(t)]\!] \cap C_u)$. Moreover and still for fixed $t > 0$, it follows from the definition of debut points that $\mathbb{N}_{x,0}$–a.e. for every $r \in \mathcal{J}(\Lambda_t) \cup \{\widehat{\Lambda}_t\}$, there exists a unique pair (ℓ^u, W^u) with $\ell^u = r$. In the two last assertions we used that $\mathbb{N}_{x,0}$–a.e. it holds that $\widehat{W}_t \neq x$ by Lemma 6.17. Note that $\mathbb{N}^\bullet_{x,0}$ – a.e. for every $u \in D$ we have

$$\{U \in [g(u), d(u)]\} = \{\ell^u \in \mathcal{J}(\Lambda_U) \cup \{\widehat{\Lambda}_U\}\}$$

since the set $\{\ell_u = \widehat{\Lambda}_U, U \notin [g(u), d(u)]\}$ is $\mathbb{N}^\bullet_{x,0}$ null - the latter being a consequence of Proposition 6.17 and Lemma 6.9. To study the family of excursions present in the spine at time U, we decompose (ρ_U, η_U, W_U) in terms of the excursion intervals away from x of W_U. Namely, consider an enumeration $(r_j : j \in \{1, 2, \dots\})$ of the elements of $\mathcal{J}(\Lambda_U)$ and for each $r_j \in \mathcal{J}(\Lambda_U)$, we set

$$W_U^j := \Big(\big(W_U\big(h + \tau_{r_j}(\overline{W}_U)\big) :\ h \in [0, \tau^+_{r_j}(\overline{W}_U) - \tau_{r_j}(\overline{W}_U)]\Big),$$

$$\langle \rho_U^j, f\rangle := \int \rho_U(\mathrm{d}h) f(h - \tau_{r_j}(\overline{W}_U)) 1_{\{\tau_{r_j}(\overline{W}_U) < h < \tau^+_{r_j}(\overline{W}_U)\}},$$

and

$$\langle \eta_U^j, f\rangle := \int \eta_U(\mathrm{d}h) f(h - \tau_{r_j}(\overline{W}_U)) 1_{\{\tau_{r_j}(\overline{W}_U) < h < \tau^+_{r_j}(\overline{W}_U)\}}.$$

The family $(W_U^j : r_j \in \mathcal{J}(\Lambda_U))$ are the excursions of W_U away from x, excluding the excursion straddling H_U. It will be crucial for our purposes to also keep track of the latter. To this end, for $\mathrm{w} \in \mathcal{W}_{E,x}$, we recall the notation $\ell_x(\mathrm{w})$ for the last passage time to x of w, viz.

$$\ell_x(\mathrm{w}) := \sup\{h < \zeta_{\mathrm{w}} : \mathrm{w}(h) = x\}.$$

In particular, the excursion straddling H_U writes

$$(\rho_U^0, \eta_U^0, W_U^0) := \theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U).$$

We set $r_0 := \widehat{\Lambda}_U$ for the corresponding value of the local time at this last excursion. We stress that since $(x, 0) \in \overline{\Theta}_x$, by [82, Lemma 5] the support of ρ_U is included in the union of the excursion intervals $(\tau_{r_j}(\overline{W}_U), \tau^+_{r_j}(\overline{W}_U))$ for $j \geqslant 1$ and $(\tau_{r_0}, H_U]$.

As in the previous section, to encode the left and right spine of each $(\rho_U^j, \eta_U^j, W_U^j)$ for $j \geqslant 0$, we introduce a family of measures $(\mathcal{P}_U^{\ell,j}, \mathcal{P}_U^{r,j})_{j\geqslant 0}$. Let us start by introducing the necessary notation to define $(\mathcal{P}_U^{r,j})_{j\geqslant 0}$. First, for every $j \geqslant 0$, we set $T_{r_j} = \inf\{t \geqslant U : \widehat{\Lambda}_t = r_j\}$ and $T_{r_j}^+ = \inf\{t \geqslant U : \widehat{\Lambda}_t < r_j\}$. Let $(\alpha_k(r_j), \beta_k(r_j))_{k\in\mathcal{K}_j}$ be the excursion intervals of $(\langle \rho_{T_{r_j}+t}, 1\rangle - \langle \rho_{T_{r_j}}, 1\rangle : t \in [0, T_{r_j}^+ - T_{r_j}])$ over its running infimum, a process that we denote by $I^{(r_j)}$. If we write $(\rho^{j,k}, \eta^{j,k}, W^{j,k}, \Lambda^{j,k})_{k\in\mathcal{K}_j}$ for the corresponding subtrajectories, for $k \in \mathcal{K}_j$ we set $\overline{W}^{j,k} := (W^{j,k}, \Lambda^{j,k} - r_j)$ and consider the point measure:

$$\mathcal{P}_U^{r,j} := \sum_{k\in\mathcal{K}_{r_j}} \delta_{(-I^{(r_j)}_{\alpha_k(r_j)}, \rho^{j,k}, \eta^{j,k}, \overline{W}^{j,k})}.$$

Analogously, for every $j \geqslant 0$, we let $S_{r_j} = \inf\{t \geqslant 0 : \widehat{\Lambda}_{U-t} = r_j\}$ and $S_{r_j}^+ = \inf\{t \geqslant 0 : \widehat{\Lambda}_{U-t} < r_j\}$. Write $(\alpha'_k(r_j), \beta'_k(r_j))_{k\in\mathcal{K}'_j}$ for the excursion intervals of $(\langle \eta_{U-S_{r_j}-t}, 1\rangle - \langle \eta_{U-S_{r_j}}, 1\rangle : t \in [0, S_{r_j}^+ - S_{r_j}])$ over its running infimum $I'^{(r_j)}$ and denote the corresponding sub-trajectories by $(\rho^{j,k}, W^{j,k}, \Lambda^{j,k})_{k\in\mathcal{K}'_j}$, where the indexing sets $\mathcal{K}_j$ and $\mathcal{K}'_j$ are supposed disjoint. For $k \in \mathcal{K}'_j$ we set $\overline{W}^{j,k} = (W^{j,k}, \Lambda^{j,k} - r_j)$. Note that these subtrajectories are parameterised counterclockwise. Finally, set:

$$\mathcal{P}_U^{\ell,j} := \sum_{k\in\mathcal{K}'_{r_j}} \delta_{(-I'^{(r_j)}_{\alpha'_k(r_j)}, \mathtt{Rev}(\rho^{j,k}, \eta^{j,k}, \overline{W}^{j,k}))}.$$

Our goal now consists in identifying the law of the pair :

$$\sum_{r_j\in\mathcal{J}(\Lambda_U)} \delta_{(r_j, \rho_U^j, \eta_U^j, W_U^j, \mathcal{P}_U^{\ell,j}, \mathcal{P}_U^{r,j})} \quad \text{and} \quad \big(\theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U), \mathcal{P}_U^{\ell,0}, \mathcal{P}_U^{r,0}\big).$$

In this direction, we set $\mathcal{N}_*(\mathrm{d}\xi) := \mathcal{N}(\mathrm{d}\xi)e^{-\alpha\sigma}$ for the biased excursion measure and under $E^0 \otimes \mathcal{N}_*$ and $E^0 \otimes \mathcal{N}$ we consider $(\mathcal{M}^\ell, \mathcal{M}^r)$ a pair of point measures on $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_{\overline{E}}$ such that, conditionally on $(J_\sigma, \check{J}_\sigma, \xi)$, they are independent Poisson measures with respective intensities:

$$1_{[0,\langle \check{J}_\sigma, 1\rangle]}(u)\mathrm{d}u\, \mathbb{N}_{\xi(H(\kappa_u \check{J}_\sigma)),0}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W}), \qquad \text{and} \qquad 1_{[0,\langle J_\sigma, 1\rangle]}(u)\mathrm{d}u\, \mathbb{N}_{\xi(H(\kappa_u J_\sigma)),0}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}\overline{W}). \tag{6.41}$$

Finally, for every $0 < a < \sigma$, we write $(\mathcal{M}_a^\ell, \mathcal{M}_a^r)$ for the restrictions $(\mathcal{M}^\ell 1_{[0,\langle \check{J}_a, 1\rangle]}, \mathcal{M}^r 1_{[0,\langle J_a, 1\rangle]})$ and recall the notation $\widetilde{\alpha} := \mathcal{N}(1 - \exp(-\alpha\sigma))$. Now we can state:

Proposition 6.20. *Under $\mathbb{N}^\bullet_{x,0}$, the random variable $\widehat{\Lambda}_U$ has density $1_{\{x>0\}} \exp(-x\widetilde{\alpha})$ with respect to the Lebesgue measure in $\mathbb{R}$. Further, the pair*

$$\Big(\mathcal{O}_U := \sum_{r_j\in\mathcal{J}(\Lambda_U)} \delta_{(r_j, \rho_U^j, \eta_U^j, W_U^j, \mathcal{P}_U^{\ell,j}, \mathcal{P}_U^{r,j})},\ \big(\theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U), \mathcal{P}_U^{\ell,0}, \mathcal{P}_U^{r,0}\big)\Big)$$

is independent and the joint law is characterised by the following properties:

- (i) *The measure $\mathcal{O}_U$ is a Poisson measure with intensity $1_{[0,\widehat{\Lambda}_U]}(y)\mathrm{d}y E^0 \otimes \mathcal{N}_*((J_\sigma, \check{J}_\sigma, \xi, \mathcal{M}^\ell, \mathcal{M}^r) \in \mathrm{d}z)$.*
- (ii) *The triplet $\big(\theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U), \mathcal{P}_U^{\ell,0}, \mathcal{P}_U^{r,0}\big)$ is independent from $\widehat{\Lambda}_U$ and its law is characterised by the relation:*

$$\mathbb{N}^\bullet_{x,0}\Big(F\big(\theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U), \mathcal{P}_U^{\ell,0}, \mathcal{P}_U^{r,0}\big)\Big) = E^0 \otimes \mathcal{N}\Big(\int_0^\sigma \mathrm{d}a\, e^{-a\alpha} F(J_a, \check{J}_a, \xi^a, \mathcal{M}_a^\ell, \mathcal{M}_a^r)\Big) \tag{6.42}$$

where to simplify notation we write $\xi^a := (\xi_t : 0 \leqslant t \leqslant a)$.

Observe that the right-hand side of (6.42) is essentially (6.39), the only difference being that the atoms of $\mathcal{M}^\ell$, $\mathcal{M}^r$ are not truncated at the exit time from $E\backslash\{x\}$. We mention that we shall as well identify the characteristic measure of $\mathcal{O}_U$ in Proposition 6.37 below.

Proof. Let us start by identifying the law of $\widehat{\Lambda}_U$ under $\mathbb{N}^\bullet_{x,0}$ and proving that conditionally on $\widehat{\Lambda}_U$, the pair

$$\Big(\sum_{r_j\in\mathcal{J}(\Lambda_U)} \delta_{(r_j,\rho^j_U,\eta^j_U,W^j_U)},\, \theta_{\ell_x(W_U)}(\rho_U,\eta_U,W_U)\Big)$$

is independent, the measure being a Poisson point measure with intensity $1_{[0,\widehat{\Lambda}_U]}(r)\mathrm{d}r(E^0\otimes \mathcal{N}_*)(\mathrm{d}J_\sigma,\mathrm{d}\check{J}_\sigma,\mathrm{d}\xi)$. In this direction, we fix measurable functions $g:\mathbb{R}_+\mapsto\mathbb{R}_+$, $f:\mathbb{R}_+\times\mathcal{M}_f(\mathbb{R}_+)^2\times\mathcal{W}_E\mapsto\mathbb{R}_+$ and $F:\mathcal{M}_f(\mathbb{R}_+)^2\times\mathcal{W}_E\mapsto\mathbb{R}_+$. This first statement will follow by establishing that:

$$\begin{aligned}\mathbb{N}^\bullet_{x,0}\Big(g(\widehat{\Lambda}_U)\exp\big(-\sum_{r_j\in\mathcal{J}(\Lambda_U)} f(r_j,\rho^j_U,\eta^j_U,W^j_U)\big)F\big(\theta_{\ell_x(W_U)}(\rho_U,\eta_U,W_U)\big)\Big)\\ =\int_0^\infty \mathrm{d}y\,e^{-y\tilde{\alpha}}g(y)\exp\Big(-\int_0^y \mathrm{d}s\,(E^0\otimes\mathcal{N}_*)\big[1-\exp\big(-f(s,J_\sigma,\check{J}_\sigma,\xi)\big)\big]\Big)\\ \cdot E^0\otimes\mathcal{N}\Big(\int_0^\sigma \mathrm{d}a\,e^{-a\alpha}F(J_a,\check{J}_a,\xi^a)\Big).\end{aligned} \tag{6.43}$$

To simplify notation, we shall henceforth suppose that the functions f and F only depend on (r_j,ρ^j_U,W^j_U), and (ρ^0_U,W^0_U) respectively, the general case follows by the same type of arguments we now describe. Under $\Pi_{x,0}$, denote the excursion point measure of ξ by $\sum_{j\in\mathcal{I}}\delta_{(r_j,\xi^j)}$. With our convention, the first moment formula of Lemma 6.17 gives that the left-hand side in the previous display writes:

$$\int_0^\infty \mathrm{d}a\; E^0\otimes\Pi_{x,0}\Big(\exp(-\alpha a)g(\mathcal{L}_a)\cdot\exp\big(-\sum_{r_j<\mathcal{L}_a} f(r_j,\phi(J_\infty,\tau_{r_j},\tau^+_{r_j}),\xi^j)\big)F\big(\theta_{\ell_x(\xi^a)}(J_a,\xi^a)\big)\Big), \tag{6.44}$$

where we denoted by $\phi(J_\infty,\tau_{r_j},\tau^+_{r_j})$ the measure under $E^0\otimes\Pi_{x,0}$ defined by the relation

$$\phi(J_\infty,\tau_{r_j},\tau^+_{r_j}):=\int J_\infty(\mathrm{d}h)f(h-\tau_{r_j})1_{\{\tau_{r_j}<h<\tau^+_{r_j}\}}.$$

Now, since $\mathcal{L}$ is constant on the excursion intervals $(\tau_{r_j},\tau^+_{r_j})$, we can write (6.44) as:

$$\begin{aligned}E^0\otimes\Pi_{x,0}\Big(\sum_{i\in\mathcal{I}} g(r_i)\cdot\exp\big(-\sum_{r_j<r_i} f(r_j,\phi(J_\infty,\tau_{r_j},\tau^+_{r_j}),\xi^j)\big)\exp\big(-\alpha\sum_{r_j<r_i}\sigma(\xi^j)\big)\\ \cdot\int_0^{\sigma(\xi^i)}\mathrm{d}a\,\exp(-\alpha a)F(\phi(J_\infty,\tau_{r_i},\tau_{r_i}+a),(\xi^i_t:0\leqslant t\leqslant a))\Big).\end{aligned}$$

Hence, if we consider under P^0 an i.i.d. collection $(J^i)_{i\in\mathcal{I}}$ with same law as J_∞, by an application of the compensation formula we get that the previous display writes:

$$\int_0^\infty \mathrm{d}y\,g(y)E^0\otimes\Pi_{x,0}\Big(\exp\big(-\sum_{r_j\leqslant y}\alpha\sigma(\xi^j)+f(r_j,J^j_{\sigma^j},\xi^j)\big)\Big)E^0\otimes\mathcal{N}\Big(\int_0^{\sigma(\xi)}\mathrm{d}a\,\exp(-\alpha a)F(J_a,\xi^a)\Big).$$

To conclude, by the exponential formula we have

$$E^0\otimes\Pi_{x,0}\Big(\exp\big(-\sum_{r_j\leqslant y}\alpha\sigma(\xi^j)+f(r_j,J^j_{\sigma^j},\xi^j)\big)\Big)=\exp\Big(-\int_0^y \mathrm{d}s\,(E^0\otimes\mathcal{N})\big[1-\exp\big(-\alpha\sigma-f(s,J_\sigma,\xi)\big)\big]\Big),$$

and since $\widetilde{\alpha}=\mathcal{N}(1-\exp(-\alpha\sigma))$, we deduce the desired identity (6.43).

To conclude the proof of the proposition we still need one argument. Recall that we write $(\rho_U^0,\eta_U^0,W_U^0):=\theta_{\ell_x(W_U)}(\rho_U,\eta_U,W_U)$. It remains to show that conditionally on the pair of variables $(\widehat{\Lambda}_U,\sum_{r_j\in\mathcal{J}(\Lambda_U)}\delta_{(r_j,\rho_U^j,\eta_U^j,W^j)},(\rho_U^0,\eta_U^0,W_U^0))$, the measures $(\mathcal{P}_U^{\ell,j},\mathcal{P}_U^{r,j})_{j\geqslant 0}$ are independent Poisson measure with respective intensities:

$$1_{[0,\langle\eta_U^j,1\rangle]}(u)\mathrm{d}u\mathbb{N}_{W_U^j(H(\kappa_u\eta_U^j)),0}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W})\quad\text{ and }\quad 1_{[0,\langle\rho_U^j,1\rangle]}(u)\mathrm{d}u\mathbb{N}_{W_U^j(H(\kappa_u\rho_U^j)),0}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W}).\tag{6.45}$$

With the notations of Proposition 6.17, recall that by the Markov property and (6.23), conditionally on $(\rho_U,\overline{W}_U)$, the measure $\mathcal{P}_U^{(r)}$ is a Poisson measure with intensity measure given by $1_{[0,\langle\rho_U,1\rangle]}\mathrm{d}u\,\mathbb{N}_{\overline{W}_U(H(k_u\rho_U))}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W})$. For every $j\in\mathbb{N}$ we set

$$m_j=\int\rho_U(\mathrm{d}h)1_{(\tau^+_{r_j}(\overline{W}_U),\infty)},\quad m_j^+=\int\rho_U(\mathrm{d}h)1_{(\tau_{r_j}(\overline{W}_U),\infty)}.$$

It follows that the family of restricted measures $\mathcal{P}_U^{(r)}1_{(m_j,m_j^+)}$ for $j\in\mathbb{N}$ are, conditionally on $(\rho_U,\overline{W}_U)$, independent Poisson measures with respective intensities

$$1_{(m_j,m_j^+)}(u)\mathrm{d}u\,\mathbb{N}_{\overline{W}_U(H(\kappa_u\rho_U))}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W}).$$

Finally, consider the mapping G_j defined by the relation $G_j(u,\rho,\eta,W,\Lambda)=(u-m_j,\rho,\eta,W,\Lambda-\Lambda_0)$ and remark that $\mathcal{P}_U^{r,j}$ is precisely the image of $\mathcal{P}_U^{(r)}1_{(m_j,m_j^+)}$ under G_j. Noting that $\langle\rho_U^j,1\rangle=m_j^+-m_j$, we get from classic properties of Poisson measures and straightforward computations that the intensity of $\mathcal{P}_U^{r,j}$ is given by the second measure in (6.45). The conditional law of $(\mathcal{P}_U^{\ell,j})_{j\in\mathbb{N}}$ follows by duality by similar arguments, we skip the details. □

6.6 The excursion point process

In this section we turn our attention to the study of the family $(\rho^u,W^u)_{u\in D}$ of excursions away from x. As in classic excursion theory of time-indexed Markov processes, to index this family, we shall make use of an additive functional of the Lévy snake introduced in [82, Section 4]. To this end, in this first section we shall recall its definition as well as some of its basic properties. We state as well the so-called special Markov property, a spacial version of the classic Markov property crucial for the study of the Lévy snake.

6.6.1 Additive functionals of the Lévy snake and the special Markov property

Let us start by introducing some notations that will be used from now on. Fix $(y,r)\in\overline{E}$ and an arbitrary open subset $D\subset\overline{E}$ containing (y,r). For $\overline{\mathrm{w}}\in\mathcal{W}_{\overline{E}}$ with $\overline{\mathrm{w}}(0)=(y,r)$, set

$$\tau_D(\overline{\mathrm{w}}):=\inf\big\{t\in[0,\zeta_{\overline{\mathrm{w}}}]:\ \overline{\mathrm{w}}(t)\notin D\big\},$$

with the usual convention $\inf\{\varnothing\} = \infty$. With a slight abuse of notation, under $\Pi_{y,r}$ we write τ_D for $\tau_D(\overline{\xi}) := \inf\{t \geqslant 0 : \overline{\xi}_t \notin D\}$. Unless stated otherwise, we will always assume that D fulfils:

$$\Pi_{y,r}(\tau_D < \infty) > 0. \tag{6.46}$$

Exit local time. The notions that we are about to present rely on some deterministic operations on snake paths, that we shall now briefly introduce. For $(\mathsf{\rho}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \overline{E})$, we define the functional

$$V_t^D(\mathsf{\rho}, \omega) := \int_0^t \mathrm{d}s\, \mathbb{1}_{\{\zeta_{\omega_s} \leqslant \tau_D(\omega_s)\}}, \qquad t \geqslant 0. \tag{6.47}$$

Roughly speaking, the variable V_t^D measures the amount of time the snake trajectory ω spent in D up to time t. We write $\Gamma^D(\mathsf{\rho}, \omega)$ for its right-inverse, viz. for the right-continuous process defined for every $s \in \big[0, V_{\sigma(\omega)}^D(\mathsf{\rho}, \omega)\big)$ as

$$\Gamma_s^D(\mathsf{\rho}, \omega) := \inf\big\{t \geqslant 0 : V_t^D(\mathsf{\rho}, \omega) > s\big\},$$

with the convention $\Gamma_s^D(\mathsf{\rho}, \omega) := \sigma(\omega)$, if $s \geqslant V_{\sigma(\omega)}^D(\mathsf{\rho}, \omega)$. The truncation of $(\mathsf{\rho}, \omega)$ to D is the element of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \overline{E})$ defined by the relation:

$$\mathrm{tr}_D\big(\mathsf{\rho}, \omega\big) := \big(\mathsf{\rho}_{\Gamma_s^D(\mathsf{\rho},\omega)}, \omega_{\Gamma_s^D(\mathsf{\rho},\omega)}\big)_{s \in \mathbb{R}_+}.$$

Furthermore, if $(\mathsf{\rho}, \omega) \in \mathcal{S}_{y,x}$, then $\mathrm{tr}_D\big(\mathsf{\rho}, \omega\big)$ is still in $\mathcal{S}_{y,x}$ - we refer to Section 3.1 of [82] for a more detailed discussion. Roughly speaking, $\mathrm{tr}_D(\mathsf{\rho}, \omega)$ encodes the trajectories of $(\mathsf{\rho}, \omega)$ that stay in D during their entire lifetime.

Recall that $(\rho, \overline{W})$ stands for the canonical process in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$, and that it takes values in $\mathcal{S}_{y,r}$ under $\mathbb{N}_{y,r}$ and $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ with $\overline{\mathrm{w}}(0) = (y, r)$. To encode the information gathered by the trajectories that stay in D, we introduce the following sigma field in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_{\overline{E}})$

$$\mathcal{F}^D := \sigma\big(\mathrm{tr}_D(\rho, \overline{W})_s : s \geqslant 0\big). \tag{6.48}$$

When working under $\mathbb{P}_{0,y,r}$ and $\mathbb{N}_{y,r}$, it will be implicitly assumed that the sigma field $\mathcal{F}^D$ has been completed with the respective negligible subsets – to simplify notation, we still denote it by $\mathcal{F}^D$.

Now, consider $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ satisfying that $\overline{\mathrm{w}}(0) \in D$, and further assume that $\mu(\{\tau_D(\overline{\mathrm{w}})\}) = 0$ if $\tau_D(\overline{\mathrm{w}}) < \infty$. Then, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, there exists a continuous, non-decreasing process L^D with associated Lebesgue-Stieltjes measure $\mathrm{d}L^D$ supported on $\{t \in \mathbb{R}_+ : \widehat{W}_t \in \partial D\}$ defined, for every $t \geqslant 0$, by the limit

$$L_t^D = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathrm{d}s\, \mathbb{1}_{\{\tau_D(W_s) < H_s < \tau_D(W_s) + \varepsilon\}} \tag{6.49}$$

where the convergence holds uniformly in compact intervals in $L^1(\mathbb{N}_{y,r})$ and $L^1(\mathbb{P}_{\mu,\overline{\mathrm{w}}})$. We refer to Propositions 4.3.1 and 4.3.2 in [43] as well as Proposition 3 in [82] for a proof of this statement. The process L^D is called the exit local time from D. Heuristically, L_t^D measures the number of connected components not containing the root of $p_H([0,t]) \backslash \{p_H(s) : \tau_D(W_s) = \zeta_s,\, s \leqslant t\}\}$. In particular, if $\inf\{t \geqslant 0 : \tau_D(\overline{W}_t) < \infty\} = \infty$, we have $L_\sigma^D = 0$. It was established in Proposition 2 of [82] that under $\mathbb{N}_{y,r}$ and $\mathbb{P}_{0,y,r}$, the time-changed process

$$\widetilde{L}^D := \big(L_{\Gamma_s^D}^D\big)_{s \in \mathbb{R}_+}$$

is $\mathcal{F}^D$-measurable – note that in particular, this yields that the total mass L^D_σ is $\mathcal{F}_D$-measurable. Finally, for latter use, we recall from [43, Proposition 4.3.2] the first moment formula:

$$\mathbb{N}_{y,r}\left(\int_0^\sigma \mathrm{d}L^D_s\ \Phi(\rho_s,\eta_s,\overline{W}_s)\right) = E^0 \otimes \Pi_{y,r}\Big(\mathbb{1}_{\{\tau_D<\infty\}}\exp(-\alpha\tau_D)\Phi\big(J_{\tau_D},\check{J}_{\tau_D},(\overline{\xi}_t : t \leqslant \tau_D)\big)\Big).$$

The special Markov property. One of the key properties of the exit local time is that it can be used to index the subtrajectories of $(\rho,\overline{W})$ that exit the domain D. Let us be more precise: first, denote the connected components of the open set

$$\big\{t \geqslant 0 :\ \tau_D(\overline{W}_t) < \zeta_t\big\}$$

by $\big((a_i,b_i) : i \in \mathcal{I}\big)$, where $\mathcal{I}$ is an indexing set that might be empty. Condition (6.46) and the first moment formula of Lemma 6.17 ensure that under $\mathbb{N}_{y,r}$ and $\mathbb{P}_{0,y,r}$ the set in the previous display is non-empty with non-null measure. Recall that the processes ζ and H are indistinguishable and note that for every $i \in \mathcal{I}$, we have $\tau_D(W_{a_i}) = H_{a_i} = H_{b_i}$ with $H_s > H_{a_i}$ for every $s \in (a_i,b_i)$. In fact, Lemma 2 of [82] states that the multiplicity of $p_H(a_i)$ is exactly 2. For every $i \in \mathcal{I}$, let $(\rho^i,\overline{W}^i)$ be the subtrajectory of $(\rho,\overline{W})$ associated with the interval $[a_i,b_i]$ (in the sense of Section 6.2.3) and with lifetime process given by $\zeta^i = (\zeta_{(a_i+s)\wedge b_i} - \tau_D(\overline{W}_{a_i}))_{s\geqslant 0}$. By the snake property, we have $\overline{W}^i_0(0) = \overline{W}^i_s(0)$ for every $s \in [0,b_i-a_i]$ and note that $\overline{W}^i_0(0) \in \partial D$. We refer to the collection $((\rho^i,\overline{W}^i) : i \in \mathcal{I})$ as the family of subtrajectories of $(\rho,\overline{W})$ that exit the domain D. Remark that each subtrajectory $(\rho^i,\overline{W}^i)$ encodes the labelled sub-tree of $\mathcal{T}_H$ rooted at $p_H(a_i)$ and starting at the boundary point $\overline{W}^i_0(0) \in \delta D$. Now, set $T_D = \inf\{t \geqslant 0 : \tau_D(\overline{W}_t) < \infty\}$ and write

$$\theta_r := \inf\big\{s \geqslant 0\ : L^D_{\Gamma^D_s} > r\big\}, \qquad \text{for } r \in [0, L^D_\sigma)$$

for the right-inverse of $\widetilde{L}^D$. Now we can state the special Markov property.

Theorem 6.21. [82, Theorem 3.1, Corolary 1] *Under $\mathbb{P}_{0,y,r}$ and $\mathbb{N}_{y,r}(\cdot\,|T_D<\infty)$, conditionally on $\mathcal{F}^D$, the point measure*

$$\sum_{i\in\mathcal{I}} \delta_{(L^D_{a_i},\rho^i,\overline{W}^i)}(\mathrm{d}\ell,\ \mathrm{d}\rho,\ \mathrm{d}\omega)$$

is a Poisson point measure with intensity

$$\mathbb{1}_{[0,L^D_\sigma]}(u)\,\mathrm{d}u\,\mathbb{N}_{\mathrm{tr}_D(\widehat{W},\widehat{\Lambda})_{\theta_u}}(\mathrm{d}\rho,\ \mathrm{d}\omega),$$

where under $\mathbb{P}_{0,y,r}$, we have $L^D_\sigma = \infty$ a.s.

Note that in the previous statement we are relying crucially in the fact that $\widetilde{L}^D$ is $\mathcal{F}_D$ measurable.

The local time at x of $(\widehat{W}_t : t \geqslant 0)$. Recall conditions (i'), (ii') in the definition of $\overline{\Theta}_x$ from Section 6.2.3 and fix an arbitrary $(\mu,\overline{\mathrm{w}}) \in \overline{\Theta}_x$, $(y,r) \in \overline{E}$. For every $r \geqslant 0$, consider the open domain $D_r := \overline{E}\backslash\{(x,r)\}$ and to simplify notation, write

$$\tau_r(\overline{\mathrm{w}}) := \inf\{h \geqslant 0 :\ \overline{\mathrm{w}}(h) = (x,r)\}$$

for the exit time from D_r of $\overline{\mathrm{w}}$. We stress that in contrast with our previous discussion, $\overline{\mathrm{w}}$ and (y,r) are arbitrary, and for instance we are no longer assuming that $\overline{\mathrm{w}}(0), (y,r) \in D_r$. Under $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$

and $\mathbb{N}_{y,r}$, there exists a continuous, non-decreasing process $A = (A_t)_{t\in\mathbb{R}_+}$ null at 0 and defined by the relation

$$A_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathrm{d}u \int_{\mathbb{R}_+} \mathrm{d}r\, \mathbb{1}_{\{\tau_r(\overline{W}_u)<H_u<\tau_r(\overline{W}_u)+\varepsilon\}}$$

where the convergence holds uniformly in compact intervals in measure under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}(\,\cdot\,\cap \{\sigma > z\})$ for every $z > 0$. Moreover, the support of the Stieltjes measure $\mathrm{d}A$, that we denote by $\mathrm{supp}\;\mathrm{d}A$, satisfies $\mathrm{supp}\;\mathrm{d}A \subseteq \{t \in [0,\sigma] : \widehat{W}_t = x\}$; we refer to [82, Section 4.2] for a proof of this statements. For this reason, the process A is called the local time at x of $\widehat{W}$. A useful consequence of the fact that $\mathrm{d}A$ is supported on the subset $\{t \in [0,\sigma] : \widehat{W}_t = x\}$ is that under $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$, we can decompose A_σ in the contributions made by each sub-trajectory attached to the spine $(\mu,\overline{\mathrm{w}})$. More precisely, let $I_t = \inf_{[0,t]}\langle\rho,1\rangle - \langle\mu,1\rangle$ for $t \geqslant 0$, and write $(\alpha_i,\beta_i)_{i\in\mathbb{N}}$ the excursion intervals of $\langle\rho,1\rangle$ over its running infimum. We denote the subtrajectory associated with the interval (α_i,β_i) by (ρ^i,W^i). If we set $\mathcal{M} = \sum_{i\in\mathbb{N}}\delta_{(-I_{\alpha_i},\rho^i,\overline{W}^i)}$, we can write

$$A_\sigma = \sum_{i\in\mathbb{N}} A_\sigma(\rho^i,\overline{W}^i), \qquad \mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}-\text{ a.s.} \tag{6.50}$$

We refer to the remark following [82, Lemma 9] for a proof of this identity. This identity will be used in the following setting. For $s \in [0,\sigma]\backslash\{t \in [0,\sigma] : \widehat{W}_t = x\}$, we shall write $u(s)$ for the unique debut $u \in D$ satisfying that $p_H(s) \in C_u$ - note that the unicity is a consequence of Lemma 6.7 - and for $\mathrm{w} \in \mathcal{W}_{E,x}$, we recall the notation $\ell_x(\mathrm{w})$ for the last passage time to x of w. As a consequence of identity (6.50) we infer the following lemma, that we state for latter use, and whose proof might be skipped in a first lecture:

Lemma 6.22. *Under $\mathbb{N}_{x,0}$ and for $s \in (0,\sigma)$, recall the notation $\mathcal{P}_s^{(\ell)},\mathcal{P}_s^{(r)}$ for the left and right spines at time s in the sense of Section 6.5.1. For every $j \in \mathcal{J}_s$, $i \in \mathcal{I}_s$ we write $h_j := H(\kappa_{t_j}\eta_s)$, $h_i := H(\kappa_{t_i}\rho_s)$ and note that in particular we have $h_j = H_{\alpha_j}, h_i = H_{\alpha_i}$. For any non-negative measurable functions g_1, g_2 on $\mathbb{R}$ and F on $\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_E$, we have*

$$\mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g_1(A_{g(u(s))})F(\rho_s,W_s)g_2(A_\sigma - A_{d(u(s))})\Big)$$
$$= \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g_1\big(\sum_{h_j<\ell_x(W_s)} A_\sigma(\rho^j,W^j)\big)F(\rho_s,W_s)g_2\big(\sum_{h_i<\ell_x(W_s)} A_\sigma(\rho^i,W^i)\big)\Big).$$

Note that despite the fact that $u(s)$ is not defined for s in the set $\{t \in [0,\sigma] : \widehat{W}_t = x\}$, the identity in last display is well defined since we have $\mathbb{N}_x(\int_0^\sigma \mathrm{d}s\mathbb{1}_{\{\widehat{W}_s=x\}}) = 0$ by Lemma 6.17 and $(\mathbf{H_2})$.

Proof. First, note that for every fixed $s \in (0,\sigma)$, on the event $\{\widehat{W}_s \neq x\}$ we can write $d(u(s)) = \inf\{t > s : H_t = \ell_x(W_s)\}$. Using the fact that $\rho_s(\{\ell_x(W_s)\}) = 0$ as well as the strong Markov property, the latter coincides with $\inf\{t > s : H_t < \ell_x(W_s)\}$. By an application of the strong

Markov property and the identity (6.50) we infer that

$$\begin{aligned}
&\mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g_1(A_{g(u(s))})F(\rho_s,W_s)g_2(A_\sigma - A_{d(u(s))})\Big)\\
&\quad= \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g_1(A_{g(u(s))})F(\rho_s,W_s)g_2\big(\sum_{h_i<\ell_x(W_s)} A_\sigma(\rho^i,W^i)\big)\Big)\\
&\quad= \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g_1(A_{g(u(\sigma-s))})F(\rho_{\sigma-s},W_{\sigma-s})g_2\big(\sum_{h_i<\ell_x(W_{\sigma-s})} A_\sigma(\rho^i,W^i)\big)\Big).
\end{aligned}$$

Now, recall the identity in distribution (6.22) and note that we can write

$$A_{g(u(\sigma-s))} = A_\sigma(\eta_{(\sigma-\boldsymbol{\cdot})-},\overline{W}_{\sigma-\boldsymbol{\cdot}}) - A_{d(u(s))}(\eta_{(\sigma-\boldsymbol{\cdot})-},\overline{W}_{\sigma-\boldsymbol{\cdot}}).$$

where the of debut in the previous display and the functionals g,d should be considered with respect to the time reversed process $(\eta_{\sigma-\boldsymbol{\cdot}-},W_{\sigma-\boldsymbol{\cdot}})$. The proof of the lemma now follows by making use of the same reasoning as before for the time-reversed process $(\eta_{(\sigma-s)-},W_{\sigma-s}: 0\leqslant s\leqslant \sigma)$. □

We now turn our attention to the support of $\mathrm{d}A$ and its connection with the family of debuts. In this direction, we recall from [82, Proposition 8] that the support of the measure $\mathrm{d}A$ can be fully characterised both in terms of the constancy intervals of $\widehat{\Lambda}=(\widehat{\Lambda}_t: t\geqslant 0)$, and in terms of a family of random times of the Lévy snake called *exit times from* x. More precisely, a time $t\in[0,\sigma]$ is called an exit time from x for $(\rho,\overline{W})$ if $\widehat{W}_t = x$ and there exists some $s>t$ such that

$$H_t < H_r, \quad \text{for every } r\in(t,s].$$

We denote by $\mathrm{Exit}(x)$ the collection of exit times from x. On the other hand, we write $\mathcal{C}^*$ for the subset of $\mathbb{R}_+$ defined by the relation: $t\in\mathcal{C}^*$ if and only if $\widehat{\Lambda}$ is constant on some open neighbourhood of t. Then, Proposition 8 in [82] states that for every $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$, $(y,r)\in\overline{E}$, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, a.e. we have

$$\operatorname{supp}\mathrm{d}A = [0,\sigma]\backslash\mathcal{C}^* = \overline{\mathrm{Exit}(x)}. \tag{6.51}$$

We refer to [82, Proposition 8] for a proof of this statements as well as for equivalent formulations. In fact, a closer look to the identity in the last display yields the following result:

Corollary 6.23. *Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, a time $t\geqslant 0$ is a point of left (resp. right) increase for A if and only if for every $\varepsilon>0$, we can find $s\in(t-\varepsilon,t]$ (resp. $s\in[t,t+\varepsilon)$) such that $\widehat{\Lambda}_s\neq\widehat{\Lambda}_t$.*

In the sequel, we will make use of A to index the family of excursions; namely, we will work with the family of pairs $(A_{g(u)},(\rho^u,W^u))$. It will be crucial for our purposes to prove that the set $\{g(u): u\in D\}$ belongs to $\operatorname{supp}\mathrm{d}A$ and to have a description of this set in terms of the pair (H,Λ). To this end, we introduce the notion of a debut time for the Lévy snake:

Definition 6.24. *A time $t\in[0,\sigma]$ is called a debut time for $(\rho,\overline{W})$ if it satisfies the following properties:*

(i) *There exists $s>t$ such that $H_t<H_r$ for every $r\in(t,s]$, and $\widehat{\Lambda}_s=\widehat{\Lambda}_t$.*

(ii) *For every $\delta>0$, we have $\inf_{((t-\delta)_+,t]}\widehat{\Lambda}<\widehat{\Lambda}_t$.*

The family of debut times is denoted by D°.

We stress that under $\mathbb{N}_{y,r}$, condition (ii) does not hold for neither 0 nor σ and therefore $\{0,\sigma\}$ are not a debut times. Analogously, under $\mathbb{P}_{0,y,r}$ the point 0 is not a debut time. The following lemma justifies our terminology.

Lemma 6.25. *Under $\mathbb{P}_{0,y,r}$ and $\mathbb{N}_{y,r}$, the mapping $g : D \mapsto (0,\sigma)$ is a bijection between excursion debuts and debut times, and its inverse is given by p_H.*

Proof. We shall only prove the statement under $\mathbb{N}_{x,0}$. Recall that under $\mathbb{N}_{x,0}$, a.e. the process $(\rho,\overline{W})$ takes values in $\overline{\Theta}_x$. Consider a measurable subset Ω_0 of $\mathbb{D}(\mathbb{R}_+,\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_{\overline{E}})$ at which this property holds for every $(\rho,\omega)\in\Omega_0$ and work for fixed $(\rho,\omega)\in\Omega_0$. We shall start by proving that $g(D)\subseteq D^\circ$, and then proceed to show that the mapping $g : D\mapsto D^\circ$ is bijective with inverse $p_H : D^\circ\to D$.

First, consider an arbitrary $u\in D$ and let us show that $g(u)\in D^\circ$. Condition (ii) in Definition 6.6 yields that we can find $s\in(g(u),d(u))$ such that we have $W_s(h)\neq x$ for $h\in(H_{g(u)},H_s]$. In particular, the path Λ_s is constant on $[H_{g(u)},H_s]$ and therefore $\widehat{\Lambda}_{g(u)}=\widehat{\Lambda}_s$. Since u is not a branching point, it must hold that $H_r>H_t$ for every $r\in(t,s]$ which shows that Definition 6.24-(i) is fulfilled by $g(u)$. Still for s as before, by condition (i) in Definition 6.6, we have $x=\widehat{W}_{g(u)}=W_s(H_{g(u)})$ and since $W_s(h)\neq x$ for $h\in(H_{g(u)},H_s]$ it must hold that $H_{g(u)}$ is a time of left increase for the path Λ_s. This gives that $\Lambda_s((H_{g(u)}-\delta)_+)<\Lambda_s(H_{g(u)})$ for every $\delta>0$. Now, since u is not a branching point, $g(u)$ can not be a local infimum for H and we get that $\inf_{[0,(g(u)-\delta)_+]}H<H_{g(u)}$ for every $\delta>0$. Now, by the snake property we deduce that (ii) holds.

Let us now prove that the mapping $g : D\to D^\circ$ is surjective. In this direction, consider an arbitrary $t\in D^\circ$ and let us show that $u:=p_H(t)$ is an excursion debut. This proves that the mapping g is surjective since it is plain that $g\circ p_H(t)=t$; indeed, since t fulfils condition (i) of Definition 6.24, for $d : D\to(0,\sigma)$ as in (6.24) we necessarily have $d\circ p_H(t)>t$. First, by considering s as in Definition 6.24 - (i) and arguing as before, the snake property yields that $\widehat{W}_t=x$ and that $W_s(h)\neq x$ for $h\in(H_t,H_s)$. If we set $w:=p_H(s)$, we infer that $w\succ u$ with $\mathcal{L}_w=\mathcal{L}_u$ and $\xi_a\neq x$ for every $a\in]\!]u,w[\![$. It readily follows from the fact that $H_s>H_t$ and the support properties of local times that we can further find w' with $w\succ w'\succ u$ such that $\xi_a\neq x$ for every $a\in]\!]u,w']\!]$, proving that condition (ii) of Definition 6.6 holds. The fact that g is injective is clear since $p_H\circ g(u)=u$ by definition of $g(u)$; this concludes the proof of the lemma. □

Note however that in the previous lemma we worked with a restricted subset of initial conditions, which leads us to the following remark.

Remark 6.26. Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ for an arbitrary starting condition $(\mu,\overline{\mathrm{w}})\in\overline{\Theta}_x$ with $\zeta_{\mathrm{w}}>0$, the definition of excursion debuts D given in Definition 6.6 still make sense, but *a priori* we no longer have a bijection between D and D°. Indeed, for every excursion away from x of the starting condition w we have an excursion debut u in the sense of Definition 6.6, but now the variables $g(u),d(u)$ coincide and $g(u)$ is no longer an element of D°. Therefore, it will be convenient to extend the definition of D under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, $\mathbb{P}^\dagger_{\mu,\overline{\mathrm{w}}}$ or $\mathbb{N}_{y,r}$ by the relation

$$D:=p_H(D^\circ).$$

In particular, the set D considers only debuts that will be visited twice by the exploration $t\mapsto p_H(t)$ - which was always the case under $\mathbb{P}_{0,y,r},\mathbb{N}_{y,r}$. Note that if we take $(\mu,\overline{\mathrm{w}})=(0,y,r)$ or (y,r), by Lemma 6.25 this definition is consistent with Definition 6.6. Moreover, with this definition the statement of Lemma 6.25 now holds as well under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$. Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and for every $u \in D$, we shall still write $g(u), d(u)$ for respectively $\inf\{t \geqslant 0 : p_H(t) = u\}$ and $\sup\{t \geqslant 0 : p_H(t) = u\}$, and we set $(\rho^u, W^u) := \mathrm{tr}_*\big(\mathscr{T}_{g(u)}(\rho, W)\big)$ for the respective excursion.

We now turn our attention to a technical lemma summing up two important properties of debut times.

Lemma 6.27. *Under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, for every $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$ and $(y, r) \in \overline{E}$, every debut time $t \in D^\circ$ is an element of* Exit(x) *and a point of left increase for A.*

Proof. One easily gets from the arguments employed in the proof Lemma 6.25 (or as a straight consequence of the latter) that under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$ for $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, $(y, r) \in \overline{E}$, if t is a debut time, it must hold that $\widehat{W}_t = x$. Therefore every debut time belongs to Exit(x). Moreover, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, by (ii) and Corollary 6.23 any debut time is a point of left-increase for A. □

The genealogy of the excursions $(\rho^u, W^u)_{u\in D}$ can be encoded in a random tree that was studied in [82]. To this end, we shall now briefly recall its definition as well as some of the main results obtained in this work. Much of the effort in [82] was directed towards studying the structure of the following random subset of $\mathcal{T}_H$

$$\mathcal{Z} = \{\upsilon \in \mathcal{T}_H : \xi_\upsilon = x\}.$$

Since the image under p_H of $\{t \in [0, \sigma] : \widehat{W}_t = x\}$ is precisely $\mathcal{Z}$, the study of the latter is closely related with the additive functional A that we just introduced. First, we shall introduce a random tree that encodes the genealogical structure of $\mathcal{Z}$. In this direction, since the mapping $a \mapsto \mathcal{L}_a$ in non-decreasing in $\mathcal{T}_H$, this can be achieved by making use of the notion of subordination of trees by non decreasing functions introduced in [66]. Namely, we define a pseudodistance in $\mathcal{T}_H$ by the relation

$$d_{\mathcal{L}}(a, b) := \mathcal{L}_a + \mathcal{L}_b - 2\mathcal{L}_{a\curlywedge b}, \quad a, b \in \mathcal{T}_H$$

and we for any $a, b \in \mathcal{T}_H$, we shall write $a \sim_{\mathcal{L}} b$ if and only if $d_{\mathcal{L}}(a, b) = 0$. It readily follows that $\sim_{\mathcal{L}}$ is an equivalence relation on $\mathcal{T}_H$. Now, by [66, Proposition 4] the metric space $\mathcal{T}_H^{\mathcal{L}} := (\mathcal{T}_H / \sim_{\mathcal{L}}, d_{\mathcal{L}})$ is still a tree. Heuristically, this tree is obtained by contracting the excursion components $(C_u)_{u\in D}$ in $\mathcal{T}_H$ in a single point. If we write $p_{\mathcal{L}}$ for the canonical projection mapping any element of $\mathcal{T}_H$ to its equivalence class in $\mathcal{T}_H^{\mathcal{L}}$, by convention $\mathcal{T}_H^{\mathcal{L}}$ is rooted at $p_{\mathcal{L}}(0)$. In the terminology of [66], we refer to $\mathcal{T}_H^{\mathcal{L}}$ as the tree obtained by subordinating $\mathcal{T}_H$ by the local time $(\mathcal{L}_a)_{a\in\mathcal{T}_H}$. For a more detailed account, we refer to [66, 82]. By construction, the tree $\mathcal{T}_H^{\mathcal{L}}$ encodes the genealogy of the elements in the set $\mathcal{Z}$, and in the sequel we shall argue that it can be used as well to (partially) encode the genealogy of the excursions away from x. Recalling our discussion on trees coded by continuous functions, since $\mathcal{T}_H^{\mathcal{L}}$ was constructed explicitly in terms of $(\rho, \overline{W})$, it is then natural to ask for an explicit coding function for the tree $\mathcal{T}_H^{\mathcal{L}}$ given in terms of the Lévy snake. This leads us to the following result:

Theorem 5.1-(i) in [82]. *Under $\mathbb{N}_{x,0}$, the subordinate tree of $\mathcal{T}_H$ with respect to the local time $\mathcal{L}$, that we denote by $\mathcal{T}_H^{\mathcal{L}}$, is isometric to the tree coded by the continuous function $(\widehat{\Lambda}_{A_t^{-1}} :\ t \geqslant 0)$.*

Let us now turn our attention to the Markovian character of $\mathcal{T}_H^{\mathcal{L}}$. Set $\overline{E}_* := E\backslash\{x\} \times \mathbb{R}_+$, for $\lambda \geqslant 0$ write $u_\lambda(y) := \mathbb{N}_{y,0}(1-\exp(-\lambda L_\sigma^{\overline{E}_*}))$ and we define a function $\widetilde{\psi} : \mathbb{R}_+ \mapsto \mathbb{R}$ by the relation:

$$\widetilde{\psi}(\lambda) = \mathcal{N}\Big(\int_0^\sigma \mathrm{d}h\ \psi\big(u_\lambda(\xi_h)\big)\Big), \qquad \text{for } \lambda \geqslant 0. \tag{6.52}$$

By [82, Proposition 5], the function $\widetilde{\psi}$ is the characteristic exponent of a Lévy tree - in the sense that it is the Laplace exponent of a Lévy process satisfying conditions (A1) - (A4). Therefore, $\widetilde{\psi}$ writes as (6.1) for some coefficient $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}_+$ and a Lévy measure $\widetilde{\pi}$. Further, by [82, Corollary2] the Gaussian component $\widetilde{\beta}$ is null, and the drift is given by $\widetilde{\alpha} = \mathcal{N}(1-\exp(-\alpha\sigma))$.

Now, consider in another probability space a Lévy process $\widetilde{X}$ with exponent $\widetilde{\psi}$. We write $\widetilde{N}$ for its excursion measure and $\widetilde{H}$ for the corresponding height process. In particular, the tree $\mathcal{T}_{\widetilde{H}}$ coded by $\widetilde{H}$ under $\widetilde{N}$ is a $\widetilde{\psi}$-Lévy tree. Finally, we can state the main result of Section 5 in [82].

Theorem 5.1-(ii) in [82]. *With the notations introduced above, we have the equality in distribution*

$$\Big((\widetilde{H}_t : t \geqslant 0),\ \textit{under } \widetilde{N}\Big) \overset{(d)}{=} \Big((\widehat{\Lambda}_{A_t^{-1}} : t \geqslant 0),\ \textit{under } \mathbb{N}_{x,0}\Big). \tag{6.53}$$

In particular, $\mathcal{T}_H^{\mathcal{L}}$ is a Lévy tree with exponent $\widetilde{\psi}$.

6.6.2 The Poisson point process of excursions

Recall the (extended) definition for the set of debuts D under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and $\mathbb{N}_{y,r}$, for $(\mu,\overline{\mathrm{w}}) \in \overline{\Theta}_x$ and $(y,r) \in \overline{E}$ from Remark 6.26 as well as the notation $g(u) := \inf\{t \in [0,\sigma] : p_H(t) = u\}$ for $u \in D$, and write (ρ^u, W^u) for the corresponding excursion. Finally, recall the definition of the measure $\mathbb{N}_x^*$ from Section 6.4. We are now in position to state the main result of this work.

Theorem 6.28. *For every $(\mu,\overline{\mathrm{w}}) \in \overline{\Theta}_x$, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ the measure*

$$\mathcal{E} := \sum_{u\in D} \delta_{(A_{g(u)},\rho^u,W^u)}$$

is a Poisson measure on $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ with intensity $\mathrm{d}t \otimes \mathbb{N}_x^*$.

As a byproduct of our reasoning, we shall deduce that when $\psi(\lambda) = \lambda^2/2$, the spatial motion is a real Brownian motion and $x = 0$, the measure $\mathbb{N}_0^*$ coincides with the excursion measure introduced in [1]. Theorem 6.28 shares striking similarities with the celebrated result by Itô in the time-indexed setting, where now the role of the local time is taken over by $(A_t)_{t\in\mathbb{R}_+}$. The measure $\mathcal{E}$ is referred to as the point process of excursions away from x, or in short, the excursion process of $(\xi_a)_{a\in\mathcal{T}_H}$. The next two sections are devoted to the proof of this result and its proof is broken down in two main steps:

- *Step 1:* Showing in Proposition 6.30 and Corollary 6.31 that under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$, the measure $\mathcal{E}$ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}_x^*$, where $\widehat{\mathbb{N}}_x^*$ is a sigma-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ that does not depend on $(\mu,\overline{\mathrm{w}})$.
- *Step 2:* Proving that $\widehat{\mathbb{N}}_x^*$ is precisely the measure $\mathbb{N}_x^*$ introduced in Section 6.4. This identification is done in Proposition 6.33.

In this section, we shall prove the first step and we postpone the proof of the second step to the next section. In fact, we are going to prove a more precise version of Theorem 6.28: in Proposition 6.30 and Corollary 6.31, we establish that $\mathcal{E}$ is a Poisson measure with respect to a filtration $(\mathcal{G}_r)$ that we now introduce.

The filtration of the excursion process. We shall work under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ for an arbitrary initial condition $(\mu,\overline{\mathrm{w}}) \in \overline{\Theta}_x$. To simplify notation, for every $r \geqslant 0$ we shall write $(\rho_r^A, \overline{W}_r^A)$ for $(\rho, \overline{W})$ at time A_r^{-1}. Consider $(\mathscr{I}_s^{(r)} : \ s \geqslant A_r^{-1})$, the running infimum of $(\langle \rho_s, 1\rangle - \langle \rho_r^A, 1\rangle : s \geqslant A_r^{-1})$ and set $T := \inf\{t \geqslant A_r^{-1} : \langle \rho_s, 1 \rangle = 0\}$. Denote the excursion intervals of $(\langle \rho_s, 1\rangle - \langle \rho_r^A, 1\rangle : A_r^{-1} \leqslant s \leqslant T)$ over its running infimum by $(\alpha_i, \beta_i)_{i\in\mathbb{N}}$, write $(\rho^i, \overline{W}^i)_{i\in\mathbb{N}}$ for the the corresponding sub-trajectories, and consider the measure

$$\sum_{i\in\mathcal{I}_r^A} \delta_{(-\mathscr{I}_{\alpha_i}^{(r)}, \rho^i, \overline{W}^i)}. \tag{6.54}$$

By the strong Markov property, and more precisely the discussion preceding (6.23), conditionally on $\mathcal{F}_{A_r^{-1}}$, the measure in the previous display is a Poisson measure with intensity given by

$$\mathbb{1}_{[0,\langle \rho_r^A, 1\rangle]}(s)\mathrm{d}s \ \mathbb{N}_{\overline{W}_r^A(H(\kappa_s \rho_r^A))}(\mathrm{d}\rho, \mathrm{d}\overline{W}).$$

Recall the notation $\mathrm{tr}_*(\rho^i, \overline{W}^i)$ for the truncation of $(\rho^i, \overline{W}^i)$ at its first return time to x and set:

$$\mathcal{G}_r := \mathcal{F}_{A_r^{-1}} \vee \sigma\Big(\sum_{i\in\mathcal{I}_r^A} \delta_{(-\mathscr{I}_{\alpha_i}^{(r)}, \mathrm{tr}_*(\rho^i, \overline{W}^i))}\Big).$$

In the sequel, for each $(\mu, \overline{\mathrm{w}})$ as before, when working under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ it will be implicitly assumed that we work with the filtration $(\mathcal{G}_r)$ completed with the set of $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ negligible subsets, and we still denote it the same. Now, let us write $\mathcal{E}1_{[0,r]}$ resp. $\mathcal{E}1_{[0,r)}$ for the restriction of $\mathcal{E}$ to the atoms $(A_{g(u)}, \rho^u, W^u)$ satisfying that $A_{g(u)} \leqslant r$ resp. $A_{g(u)} < r$.

Lemma 6.29. *For every $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$, under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and for every fixed $r \geqslant 0$, we have that $\mathcal{E}1_{[0,r]} = \mathcal{E}1_{[0,r)}$ a.s. Moreover, the measure $\mathcal{E}1_{[0,r]}$ is $\mathcal{G}_r$ - measurable.*

Proof. For the rest of the proof, we work for some fixed $r \geqslant 0$. Writing $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, let us start by showing that $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ a.s. we have $\mathcal{E}(\{r\} \times \mathbb{D}) = 0$. First, remark that neither A_r^{-1} nor A_{r-}^{-1} are debut times: by the strong Markov property at time A_r^{-1}, we have that $\inf_{[A_r^{-1}, t]} H < H_{A_r^{-1}}$ for any $t > A_r^{-1}$, which gives that A_r^{-1} is not an element of $\mathrm{Exit}(x)$. In our reasoning we used that by Lemma 6.4, we have $\rho_r^A(\{H_r^A\}) = 0$ since $\widehat{W}_r^A = x$; remark that the same argument holds if we consider instead A_{r-}^{-1}. Further, since by Lemma 6.25 the point $g(u)$ is a debut time, Corollary 6.23 yields that $g(u) \in \mathrm{supp}\, \mathrm{d}A$ and therefore, no debut time $g(u)$ can fall in the interval (A_{r-}^{-1}, A_r^{-1}). Now, since A_r^{-1} is a point of right-increase for A, for every $g(u) > A_r^{-1}$ we have that $A_{g(u)} > A_{A_r^{-1}} = r$. Analogously, A_{r-}^{-1} is a point of left-increase for A and by a similar reasoning we get that for every $g(u) < A_{r-}^{-1}$, we have $A_{g(u)} < A_{A_{r-}^{-1}} = r$. This proves that for every fixed $r \geqslant 0$, a.s. we have $\mathcal{E}(\{r\} \times \mathbb{D}) = 0$.

It remains to show that $\mathcal{E}1_{[0,r)}$ is $\mathcal{G}_r$-measurable - modulo considering another enumeration of its atoms. We fix one element $(A_{g(u)}, \rho^u, W^u) \in \mathcal{E}1_{[0,r)}$ and observe it must hold that $g(u) < A_{r-}^{-1}$. From the definition of excursion debut-times and crucially point 2 of Lemma 6.27 it follows that the family $\{(g(u), A_{g(u)}) : g(u) < A_{r-}^{-1}\}$ is $\mathcal{G}_r$-measurable. The same holds for the family of excursions (ρ^u, W^u) satisfying that $d(u) < A_r^{-1}$ and it remains to prove that the same holds for those satisfying $g(u) < A_r^{-1} \leqslant d(u) < \inf\{t \geqslant A_r^{-r} : \rho_t = 0\}$. However, this follows from the fact that in that case, the corresponding excursion can be recovered in terms of $(\rho_t, W_t : t \leqslant A_r^{-1})$ and the (truncated) right spine $\sum_{i\in\mathcal{I}_r^A} \delta_{(-\mathscr{I}_{\alpha_i}, \mathrm{tr}_*(\rho^i, \overline{W}^i))}$, by making use of the snake property and standard arguments from excursion theory. We leave the details to the reader. □

We introduce the shift operator θ_r on $\mathcal{E}$, defined for every fixed $r > 0$ by the relation

$$\theta_r\mathcal{E} := \sum_{u\in D, A_{g(u)}>r} \delta_{(A_{g(u)}-r, \rho^u, W^u)}.$$

Note that $\theta_r\mathcal{E}$ is $(\rho_{A_r^{-1}+t}, \overline{W}_{A_r^{-1}+t} : t \geqslant 0)$ - measurable, and that by Lemma 6.29, in the last display one can change the strict inequality $A_{g(u)} > r$ by $A_{g(u)} \geqslant r$.

Proposition 6.30. *Under $\mathbb{P}_{0,x,0}$ the measure $\mathcal{E}$ is a $(\mathcal{G}_r)$ - Poisson measure on $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}_x^*$, where $\widehat{\mathbb{N}}_x^*$ is a sigma-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$.*

Proof. For every arbitrary measurable subset $\mathcal{U} \subset \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, we consider the $[0, \infty]$-valued process

$$N_{\mathcal{U}}(r) := \mathcal{E}\big([0, r] \times \mathcal{U}\big) = \#\big\{(A_{g(u)}, \rho^u, W^u) \in \mathcal{E} : A_{g(u)} \leqslant r \text{ and } (\rho^u, W^u) \in \mathcal{U}\big\}, \quad r \geqslant 0$$

and we set $\widehat{\mathbb{N}}_x^*(\mathcal{U}) := \mathbb{E}_{0,x,0}(N_{\mathcal{U}}(1))$. The proof of the proposition mainly consists in establishing the two following properties:

(i) For every measurable $\mathcal{U} \subset \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ such that $N_{\mathcal{U}}(t) < \infty$ for $t \geqslant 0$, $N_{\mathcal{U}}$ is a $(\mathcal{G}_r)$ - Poisson process with rate $\mathbb{N}_x^*(\mathcal{U})$.

(ii) For every disjoint measurable subsets $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ such that $\mathbb{N}_x^*(\mathcal{U}_i) < \infty$, the processes $N_{\mathcal{U}_1}, \ldots, N_{\mathcal{U}_n}$ are independent.

It will then follow from classic arguments that $\widehat{\mathbb{N}}_x^*$ is a sigma-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ and that $\mathcal{E}$ is a Poisson measure in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}_x^*$.

Let us start by addressing (i). It suffices to show that $N_{\mathcal{U}}$ is a counting process and that it is a $(\mathcal{G}_r)$ - Lévy process; viz. $N_{\mathcal{U}}$ is $(\mathcal{G}_r)$-adapted and for every fixed $r \geqslant 0$, the process $(N_{\mathcal{U}}(t + r) - N_{\mathcal{U}}(r) : t \geqslant 0)$ is independent of $\mathcal{G}_r$ and with same distribution as $N_{\mathcal{U}}$ under $\mathbb{P}_{0,x,0}$. Starting with the former, from our definitions it is clear that $N_{\mathcal{U}}$ is non-decreasing and that it takes values in the non-negative integers. Therefore, it remains to show that it only has jumps of unitary size, which boils down to proving that if $u_1, u_2 \in D$ are distinct, we have $A_{g(u_1)} \neq A_{g(u_2)}$. But now, this fact is an immediate consequence of Lemma 6.27. Let us now show that $N_{\mathcal{U}}$ is a $(\mathcal{G}_r)$-Lévy process. In this direction, by Lemma 6.29 the process $N_{\mathcal{U}}$ is $(\mathcal{G}_r)$-adapted and notice that for every fixed $r \geqslant 0$, we can write

$$N_{\mathcal{U}}(r + h) - N_{\mathcal{U}}(r) = \mathcal{E}\big((r, r + h] \times \mathcal{U}\big) = \theta_r\mathcal{E}\big([0, h] \times \mathcal{U}\big), \quad \text{for } h \geqslant 0.$$

Now, remark that the first point (i) will follow as soon as we prove that the measure $\theta_r\mathcal{E}$ is independent from $\mathcal{G}_r$ and distributed as $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$, since this gives that for every bounded $\mathcal{G}_r$-measurable function Φ we have

$$\mathbb{E}_{0,x,0}\Big(\Phi \cdot F\big(\theta_r\mathcal{E}\big([0, h] \times \mathcal{U}\big), h \geqslant 0\big)\Big) = \mathbb{E}_{0,x,0}(\Phi) \cdot \mathbb{E}_{0,x,0}\Big(F\big(\mathcal{E}\big([0, h] \times \mathcal{U}\big), h \geqslant 0\big)\Big), \quad (6.55)$$

where F is an arbitrary non-negative measurable function on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. To achieve this, we shall encode $\theta_r \mathcal{E}$ in terms of a point measure independent from $\mathcal{G}_r$. Still under $\mathbb{P}_{0,x,0}$, consider the measure (6.54) and we introduce the process:

$$V_t := \sum_{i \in \mathcal{I}_r^A} L^{\overline{E}_*}_{t\wedge\beta_i - t\wedge\alpha_i}(\rho^i, \overline{W}^i), \quad t \geqslant 0. \tag{6.56}$$

Note that V_∞ is $\mathcal{G}_r$-measurable since for every $i \in \mathcal{I}_r^A$, by [82, Proposition 3] $L^{\overline{E}_*}_\sigma(\rho^i, W^i)$ is $\text{tr}_*(\rho^i, \overline{W}^i)$-measurable, and that $V_\infty < \infty$ by. For every $i \in \mathcal{I}_r^A$, let $(\rho^{i,k}, \overline{W}^{i,k})_{k\in\mathcal{K}_i}$ be the sub-trajectories of $(\rho^i, W^i, \Lambda^i - \Lambda_0^i)$ escaping the domain $\overline{E}_*$ and write $(a_{i,k}, b_{i,k})_{k\in\mathcal{K}_i}$ for the respective excursion intervals. Note that in the time scale of $((\rho_s, \overline{W}_s) : s \geqslant 0)$, the excursion $(\rho^{i,k}, W^{i,k}, \Lambda^{i,k} + \Lambda_0^i)$ is the subtrajectory associated to the interval $[\alpha_{i,k}, \beta_{i,k}]$, where $\alpha_{i,k} := \alpha_i + a_{i,k}$ and $\beta_{i,k} := \alpha_i + b_{i,k}$. An application of the strong Markov property combined with the special Markov property [Theorem 6.21] yields that conditionally on $\mathcal{G}_r$, the measure:

$$\sum_{i\in\mathbb{N}, k\in\mathcal{K}_i} \delta_{(V_{\alpha_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,V_\infty]}(p)\mathrm{d}p \ \mathbb{N}_{x,0}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. Next, set $T := \inf\{t \geqslant A_r^{-1} : \langle \rho_t, 1\rangle = 0\}$ and remark that by the strong Markov property, the process $(\rho_t, W_t : t \geqslant T)$ is distributed $\mathbb{P}_{0,x,0}$ and is independent from $\mathcal{F}_T$ - and in particular, independent from the measure in the previous display. If we write $(c_i, d_i)_{i\in\mathbb{N}}$ for the sub-collection of excursion intervals away from 0 of $\langle \rho, 1\rangle$ occurring in $[T, \infty)$, it follows from our previous observation, excursion theory and by conditioning with respect to $\mathcal{G}_r$, that the measure

$$\mathcal{M}' = \sum_{i\in\mathbb{N}, k\in\mathcal{K}_i} \delta_{(V_{\alpha_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})} + \sum_{i\in\mathbb{N}} \delta_{(V_\infty - I_{c_i}, \rho^i, \overline{W}^i)}$$

is a Poisson measure independent from $\mathcal{G}_r$, with intensity $\mathrm{d}t \otimes \mathbb{N}_{x,0}$. In particular, it has the same distribution as the measure $\mathcal{M}$ defined in (6.21) under $\mathbb{P}_{0,x,0}$. Let us now infer from these observations that $\theta_r\mathcal{E}$ is $\mathcal{M}'$-measurable and distributed as $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$. To do so, it suffices to argue that both measures $\theta_r\mathcal{E}$ and $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$ can be written in terms of a functional of $\mathcal{M}'$ and $\mathcal{M}$ respectively, this functional being the same for both. First, by classical arguments from excursion theory we can recover the Lévy Snake (ρ, W) from $\mathcal{M}$ and, since $\mathcal{M}'$ has the same distribution as $\mathcal{M}$, by the same procedure we can construct from $\mathcal{M}'$ a process $(\rho', \overline{W}')$ with same distribution as $(\rho, \overline{W})$. Let D' be the excursion debuts of $(\rho', \overline{W}')$, write $(\rho'^{u'}, W'^{u'})_{u'\in D'}$ for the corresponding family of excursions, and set

$$\mathcal{E}(\rho', \overline{W}') = \sum_{u'\in D'} \delta_{(A_{g(u')}(\rho', W'), \rho'^{u'}, W'^{u'})}$$

for the associated excursion process. Note from our construction that there exists a bijection between $\{u \in D : g(u) \geqslant A_r^{-1}\}$ and D' satisfying that, for every debut point u in the set $\{u \in D : g(u) \geqslant A_r^{-1}\}$, we can find $u' \in D'$ with $W^u = W'^{u'}$. Moreover, for such a pair u, u' it holds that

$$A_{g(u)}(\rho, \overline{W}) - r = A_{g(u')}(\rho', \overline{W}').$$

Let us stress however that the processes $(A_{A_r^{-1}+t} - r : t \geqslant 0)$ and $A(\rho', \overline{W}')$ differ. To obtain the identity in the previous display we used (6.50) combined with the fact that $(\rho_{A_r^{-1}}, W_{A_r^{-1}})$ belongs to Θ_x. This shows that $\theta_r\mathcal{E}$ is precisely $\mathcal{E}(\rho', W')$, and therefore it is distributed as $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$. Since $\mathcal{M}'$ is independent of $\mathcal{G}_r$, equality (6.55) holds, concluding the proof of (i).

To establish the second point (ii), note that a slight variation of our previous argument gives that $(N_{\mathcal{U}_1}, \dots, N_{\mathcal{U}_n})$ has independent and stationary increments. Hence, to show that the Poisson processes $N_{\mathcal{U}_1}, \dots, N_{\mathcal{U}_n}$ are independent, it suffices to show that they jump at different times. But once again, this property follows immediately from the fact that if $u_1, u_2 \in D$ are distinct, we have $A_{g(u_1)} \neq A_{g(u_2)}$.

To conclude that $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$ is a Poisson measure with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}^*_x$ we still need one argument. If for arbitrary $\delta > 0$ we set $\mathcal{U}_\delta := \{(\mathsf{\rho}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E) : \sigma(\omega) > \delta\}$, we have $N_{\mathcal{U}_\delta}(t) < \infty$ for every $t \geqslant 0$ and therefore $\mathbb{E}_{0,x,0}(N_{\mathcal{U}_\delta}(1)) < \infty$. Indeed, this can be deduced from observing that

$$\sum_{u \in D} \sigma(W^u) 1_{\{A_{g(u)} \leqslant r\}} \leqslant \inf\{t > A_r^{-1} : \langle \rho_t, 1 \rangle = 0\} < \infty.$$

Since $\mathcal{U} = \cup_{\delta > 0} \mathcal{U}_\delta$, we deduce from standard arguments that $\widehat{\mathbb{N}}^*_x$ is a sigma-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ and that under $\mathbb{P}_{0,x,0}$, the measure $\mathcal{E}$ is a Poisson measure with characteristic measure $\mathrm{d}t \otimes \widehat{\mathbb{N}}^*_x$. □

Let us now extend the previous result under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$.

Corollary 6.31. *For every* $(\mu, \overline{\mathrm{w}}) \in \overline{\Theta}_x$*, under* $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ *the measure* $\mathcal{E}$ *is a* $(\mathcal{G}_r)$*-Poisson measure on* $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$ *with intensity* $\mathrm{d}t \otimes \widehat{\mathbb{N}}^*_x$ *and therefore, with same distribution as* $\mathcal{E}$ *under* $\mathbb{P}_{0,x,0}$.

We stress that the characteristic measure of $\mathcal{E}$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ does not depend on the choice of $(\mu, \overline{\mathrm{w}})$.

Proof. We work for a fixed arbitrary initial condition $(\mu, \mathrm{w}) \in \overline{\Theta}_x$. The proof follows from very similar arguments to the ones employed in Proposition 6.30, so we will only give a brief sketch. Consider two disjoint indexing sets $\mathcal{I}$, $\mathcal{J}$ and denote the excursion intervals of $\langle \rho, 1\rangle - \langle \mu, 1\rangle$ over its running infimum $I = (I_t : t \geqslant 0)$ occurring before time $T = \inf\{t \geqslant 0 : \langle \rho, 1 \rangle = -\langle \mu, 1\rangle\}$ by $(\alpha_i, \beta_i)_{i \in \mathcal{I}}$, and by $(c_i, d_i)_{i \in \mathcal{J}}$ those falling in $[T, \infty)$. We write $(\rho^i, \overline{W}^i)_{i \in \mathcal{I} \cup \mathcal{J}}$ for the corresponding sub-trajectories. For every $i \in \mathcal{I}$, we denote the excursions of $(\rho^i, W^i, \Lambda^i - \Lambda^i_0)$ outside $\overline{E}_*$ by $(\rho^{i,k}, \overline{W}^{i,k})_{k \in \mathcal{K}_i}$. If we let (V_t) be the process defined as in (6.56) in terms of $(\rho^i, \overline{W}^i)_{i \in \mathcal{I}}$, the measure

$$\sum_{i \in \mathbb{N}, k \in \mathcal{K}_i} \delta_{(V_{\alpha_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})}$$

is a Poisson point measure with intensity $\mathbb{1}_{[0,V_\infty]}(p)\mathrm{d}p \; \mathbb{N}_{x,0}(\mathrm{d}\rho, \mathrm{d}\overline{W})$.

Next, let us first assume that $\mathrm{w}(0) \neq x$ and recall the notation $\overline{\mathrm{w}} = (\mathrm{w}, \ell)$. Then, by the strong Markov property, the process $((\rho_{T+t}, W_{T+t}) : t \geqslant 0)$ is independent of $\mathcal{F}_T$ and has law $\mathbb{P}_{0,y,0}$. Every $(\rho^i, \overline{W}^i)$ with $i \in \mathcal{J}$, is then an excursions away from $(0, y, \ell(0))$. For every such excursion, we denote the subtrajectories of $(\rho^i, W^i, \Lambda^i - \ell(0))$ outside $\overline{E}_*$ by $(\rho^{i,k}, \overline{W}^{i,k})_{k \in \mathcal{K}_i}$ and we write the corresponding excursion intervals by $(c_{i,k}, d_{i,k})_{k \in \mathcal{K}_i}$. If we let (V^*_t) be the process defined as in (6.56) in terms of $(\rho^i, W^i, \Lambda^i - \ell(0))_{i \in \mathcal{J}}$, the measure

$$\mathcal{M}' = \sum_{i \in \mathcal{I}, k \in \mathcal{K}_i} \delta_{(V_{\alpha_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})} + \sum_{i \in \mathcal{J}, k \in \mathcal{K}_i} \delta_{(V_\infty + V^*_{c_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})} \tag{6.57}$$

is a Poisson measure with intensity $\mathrm{d}t \otimes \mathbb{N}_{x,0}$. Therefore, if we write $(\rho', \overline{W}')$ for the Lévy snake constructed in terms of $\mathcal{M}'$, the law of $(\rho', \overline{W}')$ under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ is $\mathbb{P}_{0,x,0}$. In particular, if we denote the excursion process of $(\rho', \overline{W}')$ by $\mathcal{E}(\rho', \overline{W}')$, we infer that the distribution of $\mathcal{E}(\rho', \overline{W}')$ is the one of $\mathcal{E}$ under $\mathbb{P}_{0,x,0}$. Moreover, arguing as in the proof of Proposition 6.30 we deduce that under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ we have the a.s. equality $\mathcal{E} = \mathcal{E}(\rho', \overline{W}')$.

Let us now assume that $\mathrm{w}(0) = x$. Then, the shifted process $(\rho_{T+t}, \overline{W}_{T+t} : t \geqslant 0)$ is independent of $\mathcal{F}_T$, its law is $\mathbb{P}_{0,x,\ell(0)}$ and the process $(-I_{T+t} + I_T : t \geqslant 0)$ is a local time. Now, to prove that $\mathcal{E}$ is a Poisson point process with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}^*_x$ we can proceed as before by considering instead of (6.57) the following measure

$$\mathcal{M}'' = \sum_{i \in \mathbb{N}, k \in \mathcal{K}_i} \delta_{(V_{\alpha_{i,k}}, \rho^{i,k}, \overline{W}^{i,k})} + \sum_{i \in \mathbb{N}} \delta_{(V_\infty - I_{c_i} + I_T, \rho^i, \overline{W}^i - \ell(0))}, \tag{6.58}$$

which is a Poisson measure with intensity $\mathrm{d}t \otimes \mathbb{N}_{x,0}$.

The fact that $\mathcal{E}$ is a $(\mathcal{G}_r)$-Poisson point process follows from very similar arguments as in Proposition 6.30, we skip the details. $\square$

6.6.3 Identification of the intensity measure

The objective of this section is to show that the measure $\widehat{\mathbb{N}}^*_x$ is precisely $\mathbb{N}^*_x$. Recall from Lemma 6.9 that for every fixed $u \in D$, the process $(\mathcal{L}_a)_{a \in \mathcal{T}_H}$ is constant on C_u and that we write ℓ_u for its value. Recall as well the notation $\widehat{\Lambda}^A_r$ for $\widehat{\Lambda}$ at time A_r^{-1}. Under $\mathbb{N}_{y,r}$, the measure $\mathcal{E}$ is no longer a Poisson measure but we still have the following averaging formula:

Proposition 6.32. *For every fixed $(y, r) \in \overline{E}$ and every non-negative measurable functions $\Phi : \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}) \times \mathcal{W}_E) \mapsto \mathbb{R}_+$ and $g : \mathbb{R}^2 \mapsto \mathbb{R}_+$, we have*

$$\mathbb{N}_{y,r}\Big(\sum_{u \in D} g(A_{g(u)}, \ell_u) \Phi(\rho^u, W^u) \Big) = \mathbb{N}_{y,r}\Big(\mathbb{1}_{\{A_\sigma > 0\}} \int_0^{A_\sigma} \mathrm{d}r \, g(r, \widehat{\Lambda}^A_r) \Big) \widehat{\mathbb{N}}^*_x(\Phi). \tag{6.59}$$

Let us start by commenting on some important consequences of this result. In this direction, recall the notation $\widetilde{N}$ for the excursion measure of a reflected Lévy process with characteristic exponent $\widetilde{\psi}$ and write $\widetilde{H}$ for the corresponding height process. By the identity in distribution (6.53) the process $(\widehat{\Lambda}^A_r : 0 \leqslant r \leqslant A_\sigma)$ under $\mathbb{N}_{x,0}$ has the same distribution as $(\widetilde{H}_s : 0 \leqslant s \leqslant \sigma)$ under $\widetilde{N}$. Now, it immediately follows from this fact, Corollary 6.32 and 6.17 that for every measurable non-negative function g on $\mathbb{R}_+$ we have

$$\mathbb{N}_{x,0}\Big(\sum_{u \in D} g(\ell_u) \Phi(\rho^u, W^u) \Big) = \int_0^\infty \mathrm{d}a \, \exp(-\widetilde{\alpha} a) g(a) \widehat{\mathbb{N}}^*_x(\Phi). \tag{6.60}$$

where we recall that $\widetilde{\alpha}$ stands for the drift coefficient of $\widetilde{\psi}$. When the spatial motion ξ is a Brownian motion, $\mathcal{T}_H$ is the Brownian tree and $x = 0$, identity (6.60) was already obtained in [1, Theorem 23] by different methods for an excursion measure introduced in [1, Theorem 23], and (6.60) proves that the latter coincides with $\widehat{\mathbb{N}}^*_0$. Another important consequence of (6.60) is that the measure $\widehat{\mathbb{N}}^*_x$ is invariant under the time-reversal operation $\mathtt{Trev}$ on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E)$, defined by the relation $\mathtt{Trev} : (\rho, \eta, \omega) \mapsto (\eta_{(\sigma - \cdot)-}, \rho_{(\sigma-\cdot)-}, \omega_{\sigma - \cdot})$. More precisely, by duality (6.22) under $\mathbb{N}_{x,0}$ we have :

$$\mathbb{N}_{x,0}\big(\sum_{u \in D} g(\ell_u) \Phi(W^u, \rho^u, \eta^u) \big) = \mathbb{N}_{x,0}\big(\sum_{u \in D} g(\ell_u) \Phi(\mathtt{Trev}(W^u, \rho^u, \eta^u)) \big),$$

and an application of (6.60) yields the equality $\widehat{\mathbb{N}}^*_{x,0}(F(\rho,\eta,W)) = \widehat{\mathbb{N}}^*_{x,0}(F(\mathtt{Trev}(\rho,\eta,W)))$.

Proof. First, fix an arbitrary initial condition $(\mu,\overline{\mathrm{w}}) \in \overline{\Theta}_x$ with $(\mathrm{w}(0),\ell(0)) = (y,r)$ and $H(\mu) > 0$. The first step consists in showing that (6.59) holds if we replace $\mathbb{N}_{x,0}$ by $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$, and σ by $T_0 := \inf\{t \geqslant 0 : \langle \rho, 1\rangle_t = 0\}$. In this direction, we shall argue under $\mathbb{P}_{\mu,\overline{\mathrm{w}}}$ and further assume that the function g is bounded and compactly supported. Since the process $(\widehat{\Lambda}^A_r)_{r\in\mathbb{R}_+}$ is continuous and adapted to $(\mathcal{G}_r)$, it is $(\mathcal{G}_r)$-predictable and for every $u \in D$ we can write $\ell_u = \widehat{\Lambda}_{g(u)} = \widehat{\Lambda}^A_{A_{g(u)}}$. Now, by Corollary 6.31 the measure $\mathcal{E}$ is a $(\mathcal{G}_r)$-Poisson point process with intensity $\mathrm{d}t \otimes \widehat{\mathbb{N}}^*_x$ and hence

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D} g(A_{g(u)},\ell_u)\Phi(\rho^u,W^u)\Big) = \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D} g(A_{g(u)},\widehat{\Lambda}^A_{A_{g(u)}})\Phi(\rho^u,W^u)\Big)$$
$$= \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_0^\infty \mathrm{d}r\, g(r,\widehat{\Lambda}^A_r)\Big)\widehat{\mathbb{N}}^*_x(\Phi). \tag{6.61}$$

The goal now consists in computing

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D} g(A_{g(u)},\ell_u)\Phi(\rho^u,W^u)1_{\{A_{g(u)}>A_{T_0}\}}\Big).$$

To make our arguments precise it will be convenient to introduce the shift operator θ_r for $r > 0$ on $\mathbb{D}(\mathbb{R}_+,\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_E)$, defined by the relation $\theta_r(\rho,\omega) = (\rho_{r+t},\omega_{r+t} : t \geqslant 0)$. In particular, the additive property of A writes for every $s,t > 0$ as $A_{s+t} = A_s + A_t\circ\theta_s$. With our notations, we can write

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D} g(A_{g(u)},\ell_u)\Phi(\rho^u,W^u)1_{\{A_{g(u)}>A_{T_0}\}}\Big)$$
$$= \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D\circ\theta_{T_0}} g(A_{T_0} + A_{g(u)}\circ\theta_{T_0},\, \widehat{\Lambda}^A_{A_{g(u)}}\circ\theta_{T_0})\Phi\big((\rho^u,W^u)\circ\theta_{T_0}\big)\Big).$$

Now, note that by the strong Markov property, A_{T_0} is independent from $(\rho,\overline{W})\circ\theta_{T_0}$ and the distribution of $(\rho,\overline{W})\circ\theta_{T_0}$ is $\mathbb{P}_{0,x,0}$. In particular, the measure $\mathcal{E}\circ\theta_{T_0}$ is a Poisson measure with intensity $\mathrm{d}t\otimes\widehat{\mathbb{N}}^*_x$. Therefore, arguing as before but with the shifted process $(\rho,\overline{W})\circ\theta_{T_0}$ we get

$$\mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\sum_{u\in D} g(A_{g(u)},\ell_u)\Phi(\rho^u,W^u)1_{\{A_{g(u)}>A_{T_0}\}}\Big) = \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_0^\infty \mathrm{d}r\, g(A_{T_0}+r,\widehat{\Lambda}^A_r\circ\theta_{T_0})\Big)\widehat{\mathbb{N}}^*_x(\Phi)$$
$$= \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_0^\infty \mathrm{d}r\, g(A_{T_0}+r,\widehat{\Lambda}^A_{A_{T_0}+r})\Big)\widehat{\mathbb{N}}^*_x(\Phi)$$
$$= \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_{A_{T_0}}^\infty \mathrm{d}r\, g(r,\widehat{\Lambda}^A_r)\Big)\widehat{\mathbb{N}}^*_x(\Phi), \tag{6.62}$$

In the second equality we used that, by the identity $A^{-1}_{A_{t_0}+r} = T_0 + A^{-1}_r\circ\theta_{T_0}$, we can write

$$\widehat{\Lambda}^A_r\circ\theta_{T_0} = \widehat{\Lambda}_{T_0+A^{-1}_r\circ\theta_{T_0}} = \widehat{\Lambda}^A_{A_{T_0}+r}.$$

We now infer from (6.61) and (6.62) that we have:

$$\mathbb{E}^{\dagger}_{\mu,\mathrm{w}}\Big(\sum_{u\in D} g(A_{g(u)})\Phi(\rho^u,W^u)\Big) = \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_0^\infty \mathrm{d}r\, g(r,\widehat{\Lambda}^A_r)\Big)\widehat{\mathbb{N}}^*_x(\Phi) - \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_{A_{T_0}}^\infty g(r,\widehat{\Lambda}^A_r)\Big)\widehat{\mathbb{N}}^*_x(\Phi)$$
$$= \mathbb{E}_{\mu,\overline{\mathrm{w}}}\Big(\int_0^{A_{T_0}} \mathrm{d}r\, g(r,\widehat{\Lambda}^A_r)\Big)\widehat{\mathbb{N}}^*_x(\Phi).$$

This proves the identity (6.59) if we replace $\mathbb{N}_{x,0}$ by $\mathbb{P}^{\dagger}_{\mu,\overline{\mathrm{w}}}$. The result of the corollary will now follow by applying the Markov property under $\mathbb{N}_{x,0}$. More precisely, for every $\varepsilon > 0$, note that on the event $\{A_\varepsilon^{-1} < \infty\}$ we can write $A_{A_\varepsilon^{-1}+t} = A_t \circ \theta_{A_\varepsilon^{-1}} + \varepsilon$ for $t \geqslant 0$. Hence, from a very similar reasoning as before, the strong Markov property and the identity in the previous display yield

$$\begin{aligned}
&\mathbb{N}_{y,r}\Big(1_{\{A_\varepsilon^{-1}<\infty\}} \sum_{u\in D} g\big(A_{g(u)}, \widehat{\Lambda}^A_{A_{g(u)}}\big)\Phi(\rho^u, W^u) 1_{\{A_{g(u)}>\varepsilon\}}\Big) \\
&\qquad = \mathbb{N}_{y,r}\Big(1_{\{A_\varepsilon^{-1}<\infty\}} \mathbb{E}^{\dagger}_{\rho_{A_\varepsilon^{-1}} \overline{W}_{A_\varepsilon^{-1}}}\Big(\int_0^{A_{T_0}} \mathrm{d}r\, g(\varepsilon + r, \widehat{\Lambda}^A_r)\Big)\Big) \widehat{\mathbb{N}}^*_x(\Phi) \\
&\qquad = \mathbb{N}_{y,r}\Big(1_{\{A_\varepsilon^{-1}<\infty\}} \int_0^{A_\sigma \circ \theta_{A_\varepsilon^{-1}}} \mathrm{d}r\, g(\varepsilon + r, \widehat{\Lambda}^A_r \circ \theta_{A_\varepsilon^{-1}})\Big) \widehat{\mathbb{N}}^*_x(\Phi) \\
&\qquad = \mathbb{N}_{y,r}\Big(1_{\{A_\varepsilon^{-1}<\infty\}} \int_\varepsilon^{A_\sigma} \mathrm{d}r\, g(r, \widehat{\Lambda}^A_r)\Big) \widehat{\mathbb{N}}^*_x(\Phi)
\end{aligned}$$

where in the last equality we used that $\widehat{\Lambda}^A_r \circ \theta_{A_\varepsilon^{-1}} = \widehat{\Lambda}^A_{\varepsilon+r}$ as well as the additive property of A. By right-continuity we have $A_\varepsilon^{-1} \downarrow A_0^{-1} = 0$ as $\varepsilon \downarrow 0$ and note that $\{A_0^{-1} < \infty\} = \{A_\sigma > 0\}$. In particular, since by [82, Lemma 10] under $\mathbb{N}_{x,r}$ the point 0 belongs to the support of the measure $\mathrm{d}A$, we have $A_0^{-1} = 0$. Identity (6.59) now follows by monotone convergence taking the limit as $\varepsilon \downarrow 0$ noting that $\mathbb{N}_{y,r}$-a.e., $\mathcal{E}(\{0\} \times \mathbb{D}) = 0$. Indeed, this fact follows if $y = x$ since 0 belongs to the support of $\mathrm{d}A$ and is not a debut time, while for $y \neq x$ it can be deduced from the fact that no debut time can occur in $[0, A_0^{-1})$ by the first point of Lemma 6.27 and A_0^{-1} is not a debut time - by the same reasoning employed in the proof of Lemma 6.29. □

Finally, we are in position to conclude the proof of Theorem 6.28, by proving that the intensity measure $\widehat{\mathbb{N}}^*_x$ is precisely the measure $\mathbb{N}^*_x$ introduced in Section 6.4

Proposition 6.33. *The intensity measure* $\widehat{\mathbb{N}}^*_x$ *is precisely the excursion measure* $\mathbb{N}^*_x$.

Proof. Recall the spine decomposition under the measure $\mathbb{N}^*_x$ obtained in Proposition 6.18. The result of the proposition will be obtained by showing that the same identity holds if we replace $\widehat{\mathbb{N}}^*_x$ by $\mathbb{N}^*_x$.

Let us start by supposing that this result holds and let us explain how to deduce from it that the measures $\mathbb{N}^*_x$ and $\widehat{\mathbb{N}}^*_x$ are identical. Recall from the discussion of Section 6.5.1 that, under $\mathbb{N}^*_x$ or $\widehat{\mathbb{N}}^*_x$, for every $s \in (0, \sigma)$ we can reconstruct by means of some functional F in $(\mathcal{M}_f(\mathbb{R}_+))^2 \times \mathcal{W}_E \times (M_p^*)^2$ the snake path (ρ, W) from its spine $\mathtt{Sp}(\rho_s, W_s)$. We stress that F does not depend on the election of the time $s \in (0, \sigma)$. By making use of identity (6.39) and the analogue version under $\widehat{\mathbb{N}}^*_x$, we get for that choice of functional F that for every non-negative function f on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E)$, we have

$$\mathbb{N}^*_x(f(\rho, W)) = \mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, f \circ F\big(\mathtt{Sp}\,(\rho, W)_s\big)\Big) = \widehat{\mathbb{N}}^*_x\Big(\int_0^\sigma \mathrm{d}s\, f \circ F(\mathtt{Sp}\,(\rho, W)_s)\Big) = \widehat{\mathbb{N}}^*_x(f(\rho, W)). \tag{6.63}$$

Since both $\mathbb{N}^*_x$ and $\widehat{\mathbb{N}}^*_x$ are measures in $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$, this proves the equality $\mathbb{N}^*_x = \widehat{\mathbb{N}}^*_x$,

Let us then prove that identity (6.39) holds if we replace $\mathbb{N}^*_x$ by $\widehat{\mathbb{N}}^*_x$, for an arbitrary non-negative function F on $(\mathcal{M}_f(\mathbb{R}_+))^2 \times \mathcal{W}_E \times (M_p^*)^2$. Fix an arbitrary non-negative function g on $\mathbb{R}$ and consider the functional $\Phi : \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E) \mapsto \mathbb{R}_+$ defined by the relation

$\Phi(\rho, W) = \int_0^\sigma \mathrm{d}s\, F(\mathtt{Sp}(\rho_s, W_s))$. On the one hand, Proposition 6.32 immediately gives:

$$\mathbb{N}_{x,0}\Big(\sum_{u \in D} g(A_{g(u)}) \Phi(\rho^u, W^u) \Big) = \mathbb{N}_{x,0}\Big(\int_0^{A_\sigma} \mathrm{d}x\, g(x) \Big) \widehat{\mathbb{N}}^*_x \Big(\int_0^\sigma \mathrm{d}s\, F(\mathtt{Sp}(\rho_s, W_s)) \Big). \tag{6.64}$$

We shall now compute the left-hand side of the previous display by making use of Proposition 6.20 and we shall make extensive use of the notations introduced in the latter. For $(\mu, \mathrm{w}) \in \Theta_x$, recall the notation $\theta_r(\mu, \mathrm{w}) = (\theta_r \mu, \theta_r \mathrm{w})$ for the translation of (μ, w) to time r defined in (6.19) and that for $s \in [0, \sigma] \backslash \{t \geqslant 0 : \widehat{W}_t = x\}$, we write $u(s)$ for the unique debut $u \in D$ satisfying that $p_H(s) \in C_u^0$. Recall from Lemma 6.7 that the family $(C_u^0)_{u \in D}$ are the connected components of $\mathcal{T}_H \backslash \mathcal{Z}$ and therefore, $p_H^{-1}(C_u \backslash C_u^0) \subset \{t \geqslant 0 : \widehat{W}_t = x\}$. Note that by Proposition 6.17 and $(\mathbf{H_2})$ yield that $\mathbb{N}_x(\int_0^\sigma \mathrm{d}s 1_{\{\widehat{W}_s = x\}}) = 0$. In particular, under $\mathbb{N}_{x,0}$, the mapping $s \mapsto u(s)$ is a.e. well defined for almost every $s \in (0, \sigma)$. Now, by definition of the functional Φ and our previous observations, for every $u \in D$ we can write

$$g(A_{g(u)}) \Phi(\rho^u, W^u) = \int_{p_H^{-1}(C_u^0)} \mathrm{d}s\, g(A_{g(u(s))}) F\big(\theta_{\ell_x(W_s)}(\rho_s, W_s), \mathcal{P}_s^{\ell,0}, \mathcal{P}_s^{r,0}\big), \quad \mathbb{N}_{x,0} \text{ a.e.}$$

where in the last display we used that $A_{g(u(s))}$ is identically equal to $A_{g(u)}$ for $s \in p_H^{-1}(C_u)$. It now follows that the left hand side of (6.64) can be written as

$$\begin{aligned} \mathbb{N}_{x,0}\Big(\sum_{u \in D} g(A_{g(u)}) \Phi(\rho^u, W^u) \Big) &= \mathbb{N}_{x,0}\Big(\sum_{u \in D} \int_{p_H^{-1}(C_u^0)} \mathrm{d}s\, g(A_{g(u(s))}) F\big(\theta_{\ell_x(W_s)}(\rho_s, W_s), \mathcal{P}_s^{\ell,0}, \mathcal{P}_s^{r,0}\big) \Big) \\ &= \mathbb{N}_{x,0}\Big(\int \mathrm{d}s\, 1_{p_H^{-1}(\cup_u C_u^0)}\, g(A_{g(u(s))}) F\big(\theta_{\ell_x(W_s)}(\rho_s, W_s), \mathcal{P}_s^{\ell,0}, \mathcal{P}_s^{r,0}\big) \Big) \\ &= \mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g(A_{g(u(s))}) F\big(\theta_{\ell_x(W_s)}(\rho_s, W_s)\big), \mathcal{P}_s^{\ell,0}, \mathcal{P}_s^{r,0} \Big), \end{aligned}$$

where in the last equality we used again that $\mathbb{N}_{x,0}(\int_0^\sigma \mathrm{d}s 1_{\{\widehat{W}_s = x\}}) = 0$. To compute the expression in the last display we shall use the spinal decomposition of $(\rho, \overline{W})$ in excursions under $\mathbb{N}_{x,0}$ obtained in Section 6.5.2. More precisely, recall from Section 6.5.2 the definition of the measure $\mathcal{O}_U$ as well as the family of measures $\mathcal{P}_U^{\ell,j} = \sum_{j \in \mathcal{K}_j'} \delta_{(t_j, \rho^{j,k}, \eta^{j,k}, \overline{W}^{j,k})}$ for $r_j \in \mathcal{J}(\Lambda_U)$. If for an obvious choice of functional G, we set

$$\sum_{r_i \in \mathcal{J}(\Lambda_U)} \sum_{k \in \mathcal{K}_j} A_\sigma(\rho^{i,k}, W^{i,k}) =: G(\mathcal{O}_U)$$

we deduce by Lemma 6.22, followed by an application of Proposition 6.20 that

$$\begin{aligned} &\mathbb{N}_{x,0}\Big(\int_0^\sigma \mathrm{d}s\, g(A_{g(u(s))}) F\big(\theta_{\ell_x(W_s)}(\rho_s, W_s), \mathcal{P}_s^{\ell,0}, \mathcal{P}_s^{r,0}\big) \Big) \\ &\qquad = \mathbb{N}_{x,0}^\bullet\Big(g \circ G(\mathcal{O}_U) F\big(\theta_{\ell_x(W_U)}(\rho_U, W_U), \mathcal{P}_U^{\ell,0}, \mathcal{P}_U^{r,0}\big) \Big) \\ &\qquad = \mathbb{N}_{x,0}^\bullet\big(g \circ G(\mathcal{O}_U) \big) \cdot \mathcal{N} \otimes E^0 \Big(\int_0^\sigma \mathrm{d}a\, e^{-\alpha a} F(J_a, \check{J}_a, \xi^a, \mathcal{M}_a^\ell, \mathcal{M}_a^r) \Big). \end{aligned} \tag{6.65}$$

Putting everything together, we have established that (6.65) and (6.64) coincide. Therefore, comparing with (6.39), it now remains to show that for some function $g : \mathbb{R} \to \mathbb{R}_+$, we have

$$\mathbb{N}_{x,0}\Big(\int_0^{A_\sigma} \mathrm{d}x\, g(x) \Big) = \mathbb{N}_{x,0}^\bullet\big(g \circ G(\mathcal{O}_U) \big) < \infty. \tag{6.66}$$

In this direction, consider the function $g(x) = e^{-\lambda x}$, for $x \in \mathbb{R}$, $\lambda \geqslant 0$. First, by Proposition 7 in [82] the left hand side in the previous display is given by $\lambda^{-1}\mathbb{N}_{x,0}(1 - e^{-\lambda A_\sigma}) = \widetilde{\psi}^{-1}(\lambda)/\lambda$. On the other hand, if for $y \neq x$ we set $v_\lambda(y) := \mathbb{N}_y(1 - e^{\lambda A_\sigma})$, by the description of the law of $\mathcal{O}_U$ given in Proposition 6.20 we obtain that the right-hand side of (6.66) writes

$$\int_0^\infty \mathrm{d}r\, e^{-\widetilde{\alpha} r} \exp\Big(r \cdot E^0 \otimes \mathcal{N}_*\Big(1 - \exp\big(- \int_0^\sigma J(\mathrm{d}h) v_\lambda(\xi(h))\big)\Big)\Big).$$

Recalling that $J_\sigma(\mathrm{d}h)$ under $E^0 \otimes \mathcal{N}_*$ is the Lebesgue-Stieltiets measure of a subordinator with Laplace exponent $\psi(\lambda)/\lambda - \alpha$ for $\lambda \geqslant 0$ stopped at time σ, as well as the definition of the biased measure $\mathcal{N}_*(\mathrm{d}\xi) = \mathcal{N}(\mathrm{d}\xi)e^{-\alpha\sigma}$ and $\widetilde{\alpha} = \mathcal{N}(1 - \exp(-\alpha\sigma))$, we infer that

$$\begin{aligned}\mathbb{N}^\bullet_{x,0}\big(g \circ G(\mathcal{O}_U)\big) &= \int_0^\infty \mathrm{d}r e^{-\widetilde{\alpha} r} \exp\Big(r \cdot E^0 \otimes \mathcal{N}_*\Big(1 - \exp\big(- \sigma\alpha - \int_0^\sigma \mathrm{d}h\, \psi\big(v_\lambda(\xi(h))\big)/v_\lambda(\xi(h))\big)\Big)\Big) \\ &= \int_0^\infty \mathrm{d}r\, \exp\Big(- r\mathcal{N}\Big(1 - \exp\Big(- \int_0^\sigma \mathrm{d}h\, \psi\big(v_\lambda(\xi_h)\big)/v_\lambda(\xi_h)\Big)\Big)\Big)\end{aligned}$$

Now, by Proposition 7 of [66] we can write $v_\lambda(y) = u_{\widetilde{\psi}^{-1}(\lambda)}(y)$. This fact combined with identity (4.21) from [66] yields that the last display is precisely $\widetilde{\psi}^{-1}(\lambda)/\lambda$. This concludes the proof of identity (6.66) and of the proposition. □

This concludes the proof of Theorem 6.28.

6.7 Construction of L^*

This section is devoted to introducing an additive functional $L^* := (L^*_t : t \geqslant 0)$ of the Lévy snake under the excursion measure $\mathbb{N}^*_x$ that will be crucial for the development of Section 6.8. Roughly speaking, at each time t, the variable L^*_t measures the number of subtrajectories of W that have returned to x up to time t and in particular, the total mass L^*_σ can be interpreted as measuring the (fractal) size of the set $\{t \in (0, \sigma) : \widehat{W}_t = x\}$ under $\mathbb{N}^*_x$. We stress however that this description is imprecise. The definition of L^* under $\mathbb{N}^*_x$ is given in Proposition 6.35 and relies on preliminary constructions under both $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ and $\mathbf{N}^*_x$ that we shall now introduce.

In this direction, recall the notation Θ^e for the subset of Θ consisting in pairs (μ, w) with $\mathrm{w}(0) = x$, $\tau^*_x(\mathrm{w}) > 0$ and satisfying $\mu(\{0, \tau^*_x(\mathrm{w})\}) = 0$. Recall from Lemma 6.19 that (ρ, W) under $\mathbf{N}^*_x$ takes values in Θ^e. For any $(\mu, \mathrm{w}) \in \Theta^e$ and under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$, denote by $(\alpha_i, \beta_i)_{i\in\mathbb{N}}$ the excursion intervals of $\langle\rho, 1\rangle - \langle\mu, 1\rangle$ over its running infimum $(I_t : t \geqslant 0)$. If we write (ρ^i, η^i, W^i) for the subtrajectory associated to the interval (α_i, β_i), recall that the measure

$$\mathcal{M} := \sum_{i\in\mathcal{I}} \delta_{(-I_{\alpha_i}, \rho^i, \eta^i, W^i)} \tag{6.67}$$

is a Poisson measure with intensity $1_{[0,\langle\mu,1\rangle]}\mathrm{d}u\, \mathbf{N}_{\mathrm{w}(H(k_u\mu))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}W)$. If for every $i \in \mathcal{I}$ we set

$$h_i := H(\kappa_{-I_{\alpha_i}}\mu) = H_{\alpha_i},$$

it is straightforward to check from standard properties of Poisson measures that $\sum_{i\in\mathcal{I}} \delta_{(h_i,\rho^i,\eta^i,W^i)}$ is a Poisson measure with intensity $\mu(\mathrm{d}h)\mathbf{N}_{\mathrm{w}(h)}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}W)$. In the second equality of the last display, we used the definition (6.8) of ρ under $\mathbf{P}_{\mu,\mathrm{w}}$ as well as the identity (6.6). We stress that the definition of Θ^e ensures that μ does not charge the set $\{0\} \cup \{\tau_*(\mathrm{w})\}$. This yields that $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$–a.s. there are no excursions W^i with starting point $W^i_0 = x$ at height $h_i \in [0, \tau_*(\mathrm{w})]$. Finally, recall the notation $(\Gamma^*_t : t \geqslant 0)$ for the right-inverse of $V^*_t = \int_0^t \mathrm{d}s\, 1_{\{H_s \leqslant \tau^*_x(W_s)\}}$ for $t \geqslant 0$.

Lemma 6.34. *For every fixed $(\mu, \mathrm{w}) \in \Theta^e$ and under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$, there exits a continuous, non-decreasing process $(L^*_t : t \geqslant 0)$, with $L^*_\sigma < \infty$ a.e. defined by:*

$$\lim_{\varepsilon \to 0} \sup_{s \leqslant t \wedge \sigma} \Big|\frac{1}{\varepsilon}\int_0^s \mathrm{d}u 1_{\{\tau^*_x(W_u) < H_u < \tau^*_x(W_u) + \varepsilon\}} - L^*_s\Big| \tag{6.68}$$

the convergence holding for every $t > 0$ in $L_1(\mathbf{P}^\dagger_{\mu,\mathrm{w}})$. Moreover, still under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ the following properties hold:

(i) *Under $\mathbf{N}_y$ with $y \neq x$, we consider L^{E_*} the exit local time from E_*. Then, $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ – a.e., we have*

$$L^*_t = \sum_{\tau_*(\mathrm{w}) > h_i,\, i \in \mathcal{I}} L^{E_*}_{t \wedge \beta_i - t \wedge \alpha_i}(\rho^i, W^i), \qquad \textit{for } t \geqslant 0. \tag{6.69}$$

(ii) *The process $(L^*_{\Gamma^*_t} : 0 \leqslant t \leqslant V^*_\sigma)$ is $\mathrm{Tr}_*(\rho, W)$ measurable.*

Proof. We work under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ for a fixed arbitrary initial condition $(\mu, \mathrm{w}) \in \Theta^e$. We shall first establish the existence of a continuous, non-decreasing process L^* defined by the relation (6.68). This will be achieved by showing that for any $t > 0$, the sequence

$$\Big(\frac{1}{\varepsilon}\int_0^s \mathrm{d}u 1_{\{\tau^*_x(W_u) < H_u < \tau^*_x(W_u) + \varepsilon\}} : 0 \leqslant s \leqslant t\Big)_{\varepsilon > 0} \tag{6.70}$$

is Cauchy in $L_1(\mathbf{P}^\dagger_{\mu,\mathrm{w}})$ with respect to the uniform norm as $\varepsilon \downarrow 0$. We will make use of similar arguments to the ones employed in Proposition 2 of [82]. We shall then infer (i) and (ii) from our definitions.

We start with some preliminary remarks and constructions. First, this lemma shares obvious similarities with the theory of exit local times recalled in Section 6.6.1, but we stress that we can not directly make use of (6.49) in the current setting. Indeed, since for $\mathrm{w} \in \Theta^e$ we have $\mathrm{w}(0) = x$, this gives that $\tau^*_x(\mathrm{w}) = \inf\{h > 0 : \mathrm{w}(h) = x\}$ is not an exit time. Let us now explain how this minor difficulty can be circumvented. For every $0 < r < H(\mu) \wedge \sigma(\mathrm{w})$ and $(\uprho, \omega) \in \mathcal{S}$ with $(\uprho_0, \omega_0) = (\mu, \mathrm{w})$, we let $(\uprho^{(r)}, \omega^{(r)})$ be the trajectory defined by the relation:

$$(\uprho^{(r)}_t, \omega^{(r)}_t) := \begin{cases} \theta_r(\uprho_t, \omega_t) & \text{for } 0 \leqslant t \leqslant \inf\{t > 0 : \langle \theta_r \uprho_t, 1\rangle = 0\}, \\ (0, \mathrm{w}(r)) & \text{otherwise.} \end{cases}$$

In particular, we have $\sigma(\omega^{(r)}) = \inf\{t > 0 : \langle \theta_r \uprho_t, 1\rangle = 0\}$ and $(\uprho^{(r)}_0, \omega^{(r)}_0) = \theta_r(\mu, \mathrm{w})$. Now, fix $r \in (0, H(\mu) \wedge \sigma(\mathrm{w}))$ and note that $(\rho^{(r)}, W^{(r)})$ under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ is distributed as (ρ, W) under $\mathbf{P}^\dagger_{\theta_r(\mu,\mathrm{w})}$, where now, the initial condition fulfils $\theta_r \mathrm{w}(0) \neq x$. Therefore, for every $t \geqslant 0$, $\tau^*_x(W^{(r)}_t)$ coincides with $\tau_{E_*}(W^{(r)}_t)$, the exit time from the open set E_* by $W^{(r)}_t$. It now follows from (6.49) that there exists a continuous, non-decreasing process $L^{*,r}$ such that for every $t > 0$,

$$\lim_{\varepsilon \to 0} \mathbf{E}^\dagger_{\mu,\mathrm{w}}\Big(\sup_{s \leqslant t \wedge \sigma^{(r)}} \Big|\frac{1}{\varepsilon}\int_0^s \mathrm{d}u\, 1_{\{\tau^*_x(W^{(r)}_u) < H(\rho^{(r)}_u) < \tau^*_x(W^{(r)}_u) + \varepsilon\}} - L^{*,r}_s\Big|\Big) = 0.$$

We stress that $L^{*,r}$ is just the exit local time $L^{E_*}(\rho^{(r)}, W^{(r)})$. Now, let $T_r := \sigma(\rho^{(r)})$ and remark that $T_r = \inf\{t > 0 : H_t = r\}$. The key now is that we have

$$\frac{1}{\varepsilon}\int_0^{t\wedge T_r} \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} = \frac{1}{\varepsilon}\int_0^{t} \mathrm{d}u\, 1_{\{\tau_x^*(W_u^{(r)})<H(\rho_u^{(r)})<\tau_x^*(W_u^{(r)})+\varepsilon\}}, \quad \text{for } t \geqslant 0, \quad (6.71)$$

which gives that the approximation (6.68) holds if we replace σ by T_r setting $(L_t^* : 0 \leqslant t \leqslant T_r) := (L_t^{*,r} : 0 \leqslant t \leqslant T_r)$. Note that this holds for any $0 < r < H(\mu) \wedge \sigma(\mathrm{w})$. Since $T_r \uparrow \sigma$, by (6.71) the process $(L_t^* : 0 \leqslant t < \sigma)$ is well defined, and is continuous and non-decreasing.

Let us extend the construction to the closed interval $[0, \sigma]$ by showing that the sequence (6.70) is Cauchy in $L_1(\mathbf{P}^{\dagger}_{\mu,\mathrm{w}})$ – in particular this gives that $L_\sigma^* < \infty$, $\mathbf{P}^{\dagger}_{\mu,\mathrm{w}}$ – a.e. By our previous discussion, this will follow as soon as we show that

$$\lim_{r\downarrow 0}\sup_{\varepsilon>0} \mathbf{E}^{\dagger}_{\mu,\mathrm{w}}\Big(\frac{1}{\varepsilon}\int_{T_r}^{\sigma} \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}}\Big) = 0. \quad (6.72)$$

Under $\mathbf{P}^{\dagger}_{\mu,\mathrm{w}}$, recall that the measure $\sum_{i\in\mathcal{I}}\delta_{(h_i,\rho^i,W^i)}$ is a Poisson measure with intensity given by $\mu(\mathrm{d}h)\mathbf{N}_{\mathrm{w}(h)}(\mathrm{d}\rho, \mathrm{d}W)$. Since $\mathbf{P}^{\dagger}_{\mu,\mathrm{w}}$ – a.e. the Lebesgue measure of the set $\{s \geqslant 0 : \langle \rho_s, 1\rangle = \inf_{u\leqslant s}\langle \rho_u, 1\rangle\}$ is null, we infer from basic properties of Poisson measures and Lemma 6.17 that

$$\begin{aligned}
\mathbf{E}^{\dagger}_{\mu,\mathrm{w}}\Big(\frac{1}{\varepsilon}\int_{T_r}^{\sigma} \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}}\Big) &= \int_{[0,r]}\mu(\mathrm{d}h)\mathbf{N}_{\mathrm{w}(h)}\Big(\frac{1}{\varepsilon}\int_0^{\sigma} \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}}\Big)\\
&= \int_{[0,r]}\mu(\mathrm{d}h)\,\Pi^{\dagger}_{\mathrm{w}(h)}\Big(\frac{1}{\varepsilon}\int_0^{\infty} \mathrm{d}a\, \exp(-\alpha a)1_{\{\tau_x^*(\xi)<a<\tau_x^*(\xi)+\varepsilon\}}\Big)\\
&\leqslant \mu[0, r].
\end{aligned}$$

Now, since $(\mu, \mathrm{w}) \in \Theta^e$, we have $\mu(\{0\}) = 0$ and we deduce that the limit as $\varepsilon \downarrow 0$ in the previous display is 0. This proves (6.72) and concludes the proof of the existence of L^*.

Let us now establish (i) and (ii). First, note that the excursion intervals (α_i, β_i) with $\beta_i \leqslant T_r$ away from 0 of $(\langle \rho_t, 1\rangle - \inf_{s\leqslant t}\langle \rho_s, 1\rangle : t \geqslant 0)$ are precisely the ones of $(\langle \rho_t^{(r)}, 1\rangle - \inf_{s\leqslant t}\langle \rho_s^{(r)}, 1\rangle : t \geqslant 0)$. Moreover, the subtrajectory of (ρ, W) and $(\rho^{(r)}, W^{(r)})$ associated to the interval (α_i, β_i) coincide. By Proposition 2 - (ii) of [82] applied to $(\rho^{(r)}, W^{(r)})$ at the exit time τ_{E_*}, we have that

$$L_{T_r}^* = L_{\sigma^{(r)}}^{*,r}(\rho^{(r)}, W^{(r)}) = \sum_{\tau_x^*(\mathrm{w})>h_i\geqslant r} L_\sigma^{E_*}(\rho^i, W^i).$$

Now, the identity (6.69) follows by monotonicity taking the limit as $r \downarrow 0$ since $\mu(\{0\}) = 0$. Finally, in order to prove (ii), we shall decompose $L^* \circ \Gamma^*$ in terms of $\mathrm{Tr}_*(\rho, W)$-measurable functionals. In this direction, write $\mathcal{I}'$ for the subset of $\mathcal{I}$ defined by the relation: $i \in \mathcal{I}'$ if and only if $h_i < \tau_*(\mathrm{w})$. For every $i \in \mathcal{I}'$ we set $(\alpha_i^*, \beta_i^*) := (V_{\alpha_i}^*, V_{\beta_i}^*)$. Remark that this is the unique pair satisfying the relation $(\Gamma^*_{\alpha_i^*}, \Gamma^*_{\beta_i^*}) = (\alpha_i, \beta_i)$. Indeed, for every $i \in \mathcal{I}'$, we have $\tau(W_{\alpha_i}) = \tau(W_{\beta_i}) = \infty$, which gives that Γ^* is piece-wise linear in a neighbourhood of $V_{\alpha_i}^*$ and $V_{\beta_i}^*$. The family $((\alpha_i^*, \beta_i^*) : i \in \mathcal{I}')$ are precisely the excursion intervals away from 0 of the time-changed process $(\langle \rho_{\Gamma_t^*}, 1\rangle - \inf_{s\leqslant t}\langle \rho_{\Gamma_s^*}, 1\rangle : t \geqslant 0)$ and therefore are $\mathrm{Tr}_*(\rho, W)$ measurable. Further, for every $r \in \{u \geqslant 0 : \langle \rho_{\Gamma_u^*}, 1\rangle = \inf_{s\leqslant u}\langle \rho_{\Gamma_s^*}, 1\rangle\}$ we have $\tau_*(W_{\Gamma_r^*}) = \infty$, which gives that $\mathrm{d}\Gamma^*$ does not charge the set $\{u \geqslant 0 : \langle \rho_{\Gamma_u^*}, 1\rangle = \inf_{s\leqslant u}\langle \rho_{\Gamma_s^*}, 1\rangle\}$ since the Lebesgue measure of

$\{t \geqslant 0 : \langle \rho_t, 1\rangle = \inf_{s \leqslant t}\langle \rho_s, 1\rangle\}$ is null. If for every $i \in \mathcal{I}'$ we let $\Gamma^*_i(t) := \Gamma^*_t(\rho^i, W^i)$, this gives that

$$\Gamma^*_t = \sum_{i\in\mathcal{I}'} \Gamma^*_{t\wedge\alpha^*_i} - \Gamma^*_{t\wedge\beta^*_i} = \sum_{i\in\mathcal{I}'} \Gamma^*_i(t\wedge\beta^*_i - t\wedge\alpha^*_i).$$

We infer from the identity in the last display that, still for $i \in \mathcal{I}'$, we have

$$\begin{aligned} L^{E_*}_{\Gamma^*_t\wedge\beta_i - \Gamma^*_t\wedge\alpha_i}(\rho^i, W^i) &= L^{E_*}_{\Gamma^*_t\wedge\Gamma^*_{\beta^*_i} - \Gamma^*_t\wedge\Gamma^*_{\alpha^*_i}}(\rho^i, W^i) \\ &= L^{E_*}_{\Gamma^*_{t\wedge\beta^*_i} - \Gamma^*_{t\wedge\alpha^*_i}}(\rho^i, W^i) \\ &= L^{E_*}_{\Gamma^*_i(t\wedge\beta^*_i - t\wedge\alpha^*_i)}(\rho^i, W^i), \end{aligned}$$

and now (6.69) yields that we can write

$$L^*_{\Gamma^*_t} = \sum_{h_i<\tau^*_x(\mathrm{w}),\, i\in\mathcal{I}} L^{E_*}_{\Gamma^*_i(t\wedge\beta^*_i - t\wedge\alpha^*_i)}(\rho^i, W^i) \quad \text{for } t \geqslant 0.$$

The second point (ii) follows from recalling that under $\mathbf{N}_y$, the process $(L^{E_*} \circ \Gamma^{E_*}_t : 0 \leqslant t \leqslant V^{E_*}_\sigma)$ is $\mathrm{Tr}_{E_*}(\rho, W)$ – measurable by Proposition 3 of [82]. □

We shall now use Lemma 6.34 to extend our construction under the excursion measure $\mathbf{N}^*_x$. In this direction, we argue under $\mathbf{N}^*_x$ and we start introducing some notation. For fixed $0 < t < \sigma$ recall from Section 6.5.2 that we write $\mathtt{Sp}(\rho, W)_t = (\rho_t, \eta_t, W_t, \mathcal{P}^{(\ell)}_t, \mathcal{P}^{(r)}_t)$ the spinal decomposition at time t of (ρ, W), where $\mathcal{P}^{(\ell)}_t = \sum_{j\in\mathcal{J}_t} \delta_{(-I^{(\ell)}_{\alpha_j}, \rho^j, \eta^j, W^j)}$ and $\mathcal{P}^{(r)}_t = \sum_{i\in\mathcal{I}_t} \delta_{(-I^{(r)}_{\alpha_i}, \rho^i, \eta^i, W^i)}$. Set $T^{(t,r)}_* = \inf\{s \geqslant t : \tau_*(W_s) = \infty\}$, $T^{(t,\ell)}_* = t - \inf\{s \geqslant 0 : \tau_*(W_{t-s}) = \infty\}$ and denote by $\mathcal{I}'_t$ (resp. $\mathcal{J}'_t$) the subset of $\mathcal{I}_t$ (resp. $\mathcal{J}_t$) defined by the relation: $i \in \mathcal{I}'_t$ (resp. $j \in \mathcal{J}'_t$) if and only if we have $\alpha^{(r)}_i(t) \geqslant T^{(t,r)}_*$ (resp. $\alpha^{(\ell)}_j(t) \leqslant T^{(t,\ell)}_*$).

Proposition 6.35. *Under $\mathbf{N}^*_x$, there exits a continuous, non-decreasing process (L^*_t) defined by the relation:*

$$\lim_{\varepsilon\to 0} 1_V \sup_{s\leqslant t} \Big|\frac{1}{\varepsilon}\int_0^s \mathrm{d}u\, 1_{\{\tau^*_x(W_u) < H_u < \tau^*_x(W_u)+\varepsilon\}} - L^*_s\Big| \tag{6.73}$$

the convergence holding uniformly in compact intervals in measure for any $V \in \mathcal{F}$ with $\mathbf{N}^*_x(V) < \infty$. Moreover, the following holds:

(i) $(L^*_{\Gamma^*_t} : 0 \leqslant t \leqslant V^*_\sigma)$ *is* $\mathrm{Tr}_*(\rho, W)$ *measurable and under* $\mathbb{N}^*_x$ *it is simply denoted by* (L^*_t).

(ii) *Under $\mathbf{N}^*_x$, a.s. for every $t \in (0, \sigma)$ we have the identities:*

$$L^*_t = \sum_{j\in\mathcal{J}'_t} L^{E_*}_\sigma(\rho^j, W^j) \qquad \text{and} \qquad L^*_\sigma - L^*_t = \sum_{i\in\mathcal{I}'_t} L^{E_*}_\sigma(\rho^i, W^i). \tag{6.74}$$

Note that in particular, we get from (ii) that

$$L^*_\sigma = \sum_{j\in\mathcal{J}'_t} L^{E_*}_\sigma(\rho^j, W^j) + \sum_{i\in\mathcal{I}'_t} L^{E_*}_\sigma(\rho^i, W^i).$$

Proof. Let us start proving the existence of (L_t^*). In this direction, for every $\varepsilon > 0$, $\delta \geqslant 0$ we set

$$I_t(\delta,\varepsilon) := \frac{1}{\varepsilon}\int_{\delta\wedge t}^{t} \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}}, \quad t \geqslant 0.$$

It suffices to show that for any $\eta, t > 0$ and $V \in \mathcal{F}$ with $\mathbf{N}_x^*(V) < \infty$ – that we fix from now on – we have

$$\lim_{\varepsilon,\varepsilon'\to 0} \mathbf{N}_x^*\big(\sup_{s\leqslant t\wedge\sigma} |I_s(0,\varepsilon) - I_s(0,\varepsilon')| \geqslant \eta, V\big) = 0. \tag{6.75}$$

For every $\delta > 0$, we set $V_\delta := \{\sigma > \delta, \sigma(W_\delta) > \delta\}$. Remark that V_δ is a $\mathcal{F}_\delta$-measruable subset with finite $\mathbf{N}_x^*$-measure, and $V_\delta \uparrow \Omega$ as $\delta \downarrow 0$. In particular for every $\ell > 0$, we can find some $\delta > 0$ small enough such that $\mathbf{N}_x^*(V\backslash V_\delta) < \ell$. Let us start by showing that for every $\delta > 0$,

$$\lim_{\varepsilon,\varepsilon'\to 0} \mathbf{N}_x^*\Big(\sup_{s\leqslant t} |I_s(\delta,\varepsilon) - I_s(\delta,\varepsilon')| > \eta, V\Big) = 0. \tag{6.76}$$

Remark that it suffices to show that the convergence in the last display holds if we replace V by V_δ, for every arbitrary small $\delta > 0$. Now, by the Markov property, we have:

$$\lim_{\varepsilon,\varepsilon'\to 0} \mathbf{N}_x^*\Big(\sup_{s\leqslant t} |I_s(\delta,\varepsilon)-I_s(\delta,\varepsilon')| > \eta, V_\delta\Big) \leqslant \lim_{\varepsilon\to 0} \mathbf{N}_x^*\Big(1_{V_\delta}\mathbf{P}^\dagger_{\rho_\delta,W_\delta}\big(\sup_{s\leqslant t}|I_s(\delta,\varepsilon)-I_s(\delta,\varepsilon')| > \eta\big)\Big) = 0,$$

where in the last equality we used Lemma 6.34 and the dominated convergence theorem. We stress that in the last display we used that (ρ_δ, W_δ) belongs to Θ^e, $\mathbf{N}_x^*$ – a.e.

Let us now infer from our previous reasoning the convergence (6.75). Note that for every $\delta > 0$, we can bound

$$\sup_{s\leqslant t} |I_s(0,\varepsilon) - I_s(0,\varepsilon)| \leqslant I_\delta(0,\varepsilon) + I_\delta(0,\varepsilon') + \sup_{s\leqslant t} |I_s(0,\varepsilon) - I_s(0,\varepsilon)|.$$

Now, (6.75) will follow as soon as we show that

$$\lim_{\delta\to 0}\sup_{\varepsilon>0} \mathbf{N}_x^*\Big(\varepsilon^{-1}\int_0^\delta 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} \geqslant \eta, V\Big) = 0. \tag{6.77}$$

Remark that by duality, we can write

$$\mathbf{N}_x^*\Big(\varepsilon^{-1}\int_0^\delta 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} \geqslant \eta, V\Big) = \mathbf{N}_x^*\Big(\varepsilon^{-1}\int_{\sigma-\delta}^\sigma 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} \geqslant \eta, \widetilde{V}\Big)$$

where $\widetilde{V}$ is the image of V by the mapping $(\rho,\omega) \mapsto \big(\rho_{\sigma-t}, \omega_{\sigma-t} : 0 \leqslant t \leqslant \sigma(\rho)\big)$. Now, to prove (6.77) it suffices to check that for every $\delta > 0$, it holds that

$$\lim_{\varepsilon\to 0} \mathbf{N}_x^*\Big(\frac{1}{\varepsilon}\int_\delta^\sigma \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} \geqslant \eta, V_\delta\Big) = 0. \tag{6.78}$$

First, by an application of the Markov property we have

$$\mathbf{N}_x^*\Big(\frac{1}{\varepsilon}\int_\delta^\sigma \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}} \geqslant \eta, V_\delta\Big) \leqslant \mathbf{N}_x^*\Big(\mathbf{P}^\dagger_{\rho_\delta,W_\delta}\big(I_\sigma(0,\varepsilon) \geqslant \eta\big)1_{V_\delta}\Big).$$

Further, for any $(\mu, \mathrm{w}) \in \Theta^e$, Markov's inequality and the first-moment formula (6.38) give that

$$\mathbf{P}^\dagger_{\mu,\mathrm{w}}\big(I_\sigma(0,\varepsilon) \geqslant \eta\big) \leqslant \eta^{-1}\mathbf{E}^\dagger_{\mu,\mathrm{w}}\Big(\varepsilon^{-1}\int_0^\sigma \mathrm{d}u\, 1_{\{\tau_x^*(W_u)<H_u<\tau_x^*(W_u)+\varepsilon\}}\Big) \leqslant \eta^{-1}\langle\mu, 1\rangle.$$

This yields that we have the pointwise convergence:

$$\lim_{\delta\to 0} \mathbf{P}^{\dagger}_{\rho_\delta, W_\delta}\big(I_\sigma(0,\varepsilon) \geqslant \eta\big) \leqslant \lim_{\delta\to 0} \eta^{-1}\langle \rho_\delta, 1\rangle = 0 \quad \mathbf{N}_x^* - \text{a.e.}$$

and it follows by dominated convergence that (6.78) holds, concluding the proof of (6.75).

Property (i) follows immediately from Lemma 6.34 and we turn our attention to (ii). Let us start by constructing a version of the family $(\sum_{i\in\mathcal{I}_s'} L_\sigma^{E_*}(\rho^i, W^i) : 0 < s < \sigma)$ defined outside of a $\mathbf{N}_x^*$– negligible set. The uniform convergence (6.73) gives that for every $V \in \mathcal{F}$ measurable with $\mathbf{N}_x(V) < \infty$, there exists some sub-sequence $(\varepsilon_n)_{n\geqslant 0}$ such that $\mathbf{N}_x$ a.s. for every $0 < a < b$, we have the pointwise convergence,

$$\lim_{\varepsilon\to 0} 1_V \Big|\frac{1}{\varepsilon}\int_a^b \mathrm{d}u\, 1_{\{\tau_x^*(W_u) < H_u < \tau_x^*(W_u)+\varepsilon\}} - (L_b^* - L_a^*)\Big|.$$

Now, it easily follows from our definition that for every fixed t and $\mathbf{N}_x$ a.e. for every $i \in \mathcal{I}_t$ we have $L_\sigma^{E_*}(\rho^i, W^i) = L^*_{\beta_i^r(t)} - L^*_{\alpha_i^r(t)}$. From now on, for every $0 < t < \sigma$ and $i \in \mathcal{I}_t$, we set $L_\sigma^{E_*}(\rho^i, W^i) := L^*_{\beta_i^r(t)} - L^*_{\alpha_i^r(t)}$. In particular, the family $(\sum_{i\in\mathcal{I}_s'} L_\sigma^{E_*}(\rho^i, W^i) : s \in (0,\sigma))$ is defined outside of a $\mathbf{N}_x^*$-null set. An analogous reasoning can be applied to $(\sum_{i\in\mathcal{J}_s'} L_\sigma^{E_*}(\rho^i, W^i) : s \in (0,\sigma))$ and in therefore, statement (ii) makes sense. By Lemma 6.34-(i), for fixed $s > 0$ and on the event $\{s < \sigma\}$, the Markov property at time s gives that $\mathbf{N}_x^*$–a.s., we have

$$(L_\sigma^* - L_s^*) = \sum_{i\in\mathcal{I}_s'} L_\sigma^{E_*}(\rho^i, W^i). \tag{6.79}$$

Consider a measurable subset $\Omega_0 \subset \Omega$ with full $\mathbf{N}_x^*$ - measure such that for every $(\uprho, \omega) \in \Omega_0$, the equality in the last display holds for every rational s. Let us now prove that the equality in the previous display holds in Ω_0 for every $s \in (0,\sigma)$. We pick an arbitrary $q > 0$ and fix arbitrary rationals s, t such that $s < q < t$. By continuity of L^*, our claim will follow if we prove that we have

$$L_\sigma^* - L_t^* \leqslant \sum_{i\in\mathcal{I}_q'} L_\sigma^{E_*}(\rho^i, W^i) \leqslant L_\sigma^* - L_s^*. \tag{6.80}$$

The second inequality immediately follows from our definitions and we turn our attention to the first equality. By considering a smaller rational t we can assume that t belongs to an interval of the form $(q + \alpha_j^{(r)}(q), q + \beta_j^{(r)}(q))$ for some $j \in \mathcal{I}_q$. Now, by comparison of the summands we get that $\sum_{i\in\mathcal{I}_t'} L_\sigma^{E_*}(\rho^i, W^i) \leqslant \sum_{i\in\mathcal{I}_q'} L_\sigma^{E_*}(\rho^i, W^i)$ and by our choice of Ω_0 we infer that (6.80) holds. Finally, by duality, we get that $\mathbf{N}_x^*$–a.e. for every $s \in (0,\sigma)$ we have

$$L_{\sigma-s}^* = L_\sigma^*(\rho_{\sigma-\cdot}, W_{\sigma-\cdot}) - L_s^*(\rho_{\sigma-\cdot}, W_{\sigma-\cdot}) = \sum_{i\in\mathcal{J}_{\sigma-s}'} L_\sigma^{E_*}(\rho^i, W^i)$$

concluding the proof of the lemma. □

The identities (6.69) and (6.74) yield analogue results under $\mathbb{P}^*_{\mu,\mathrm{w}}$ and $\mathbb{N}_x^*$. Let us briefly comment on this. First, recall that under $\mathbf{N}_y$ for $y \neq x$, the time-changed exit local time $(L^{E_*}_{\Gamma_t^*} : 0 \leqslant t \leqslant V_\sigma)$ is $\mathcal{F}_{E_*}$ – measurable. With some abuse of notation, under $\mathbf{N}_y$ we shall write

$$L_t^{E_*}(\mathrm{Tr}(\rho, W)) := L^{E_*}_{\Gamma_t^*}, \qquad \text{for } 0 \leqslant t \leqslant V_\sigma^* \tag{6.81}$$

and in particular, we have $L^{E_*}_{V^*_\sigma}(\mathrm{Tr}(\rho, W)) = L^{E_*}_\sigma$. Hence, L^{E_*} is defined under $\mathbb{N}^*_y$ and as usual the dependence on (ρ, W) is omitted. Further, when working under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ for $(\mu, \mathrm{w}) \in \Theta^e$ with $\tau^*_x(\mathrm{w}) \in \{\zeta_\mathrm{w}, \infty\}$ and with the same notation as in Lemma 6.34-(i), it follows from our definitions that we can write

$$L^{E_*}_\sigma = \sum_{h_i<\tau^*_x(\mathrm{w}),\, i\in\mathcal{I}} L^{E_*}_\sigma(\rho^i, W^i) = \sum_{h_i<\tau^*_x(\mathrm{w}),\, i\in\mathcal{I}} L^{E_*}_{V^*_\sigma}(\mathrm{Tr}(\rho^i, W^i))$$

and we set $L^{E_*}_\sigma(\mathrm{Tr}(\rho, W)) := \sum_{h_i<\tau^*_x(\mathrm{w})} L^{E_*}_\sigma(\mathrm{Tr}(\rho^i, W^i))$. Remark that the truncated measure $\mathcal{M}^{\mathrm{tr}} := \sum_{h_i<\tau_{E_*}(\mathrm{w})} \delta_{(-I_{\alpha_i}, \mathrm{Tr}(\rho^i, W^i))}$ is $\mathrm{Tr}_*(\rho, W)$ – measurable and that since the initial condition (μ, w) belongs to Θ^e with $\tau^*_x(\mathrm{w}) \in \{\zeta_\mathrm{w}, \infty\}$, for every $i \in \mathcal{I}$ we have $h_i < \tau_*(\mathrm{w})$. Finally, under $\mathbb{P}^*_{\mu,\mathrm{w}}$ recall the definition of the measure $\mathcal{M}$ introduced in Lemma 6.16 and that $(\mathcal{M}^{\mathrm{tr}}, \mathrm{Tr}_*(\rho, W))$ under $\mathbf{P}^\dagger_{\mu,\mathrm{w}}$ has the same distribution as $(\mathcal{M}, (\rho, W))$ under $\mathbb{P}^*_{\mu,\mathrm{w}}$. Now, we can state:

Corollary 6.36. *The following identities hold:*

(i) *Let* $\mathcal{M} = \sum_{i\in\mathcal{I}} \delta_{(-I_{\alpha_i}, \rho^i, W^i)}$ *be the measure introduced in Lemma 6.16. For* $(\mu, \mathrm{w}) \in \Theta^e$ *with* $\tau^*_x(\mathrm{w}) \in \{\zeta_\mathrm{w}, \infty\}$ *and under* $\mathbb{P}^*_{\mu,\mathrm{w}}$*, we have:*

$$L^{E_*}_\sigma = \sum_{i\in\mathcal{I}} L^{E_*}_\sigma(\rho^i, W^i), \quad \mathbb{P}^*_{\mu,\mathrm{w}} - a.e. \tag{6.82}$$

(ii) *Under* $\mathbb{N}^*_x$ *and for every* $t \in (0, \sigma)$*, consider* $\mathtt{Sp}(\rho, W)_t$ *the spinal decomposition of* (ρ, W) *at time* t*. Then, a.e. for every* $t \in (0, \sigma)$*,*

$$L^*_t = \sum_{i\in\mathcal{J}_t} L^{E_*}_\sigma(\rho^i, W^i) \quad \text{and} \quad (L^*_\sigma - L^*_t) = \sum_{i\in\mathcal{I}_t} L^{E_*}_\sigma(\rho^i, W^i). \tag{6.83}$$

The first point (i) follows from our previous observations while (ii) is a consequence of identities (6.74); we skip the details. We now turn our attention to a description for the law of (ρ, W) observed at a typical time taken with respect to the measure $\mathrm{d}L^*$.

Proposition 6.37. *Under* $E^0 \otimes \mathcal{N}$*, let* $(\mathcal{M}^{\ell,*}, \mathcal{M}^{r,*})$ *be a pair of point measures on* $\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E$ *such that conditionally on* $(\check{J}_\sigma, J_\sigma, (\xi_s : s \leqslant \sigma))$*, they are independent Poisson measures with respective intensities*

$$\mathbb{1}_{[0,\langle \check{J}_\sigma, 1\rangle]} \mathrm{d}u\, \mathbb{N}^*_{\xi(H(\kappa_u \check{J}_\sigma))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}W), \qquad \text{and} \qquad \mathbb{1}_{[0,\langle J_\sigma, 1\rangle]} \mathrm{d}u\, \mathbb{N}^*_{\xi(H(\kappa_u J_\sigma))}(\mathrm{d}\rho, \mathrm{d}\eta, \mathrm{d}W).$$

For every non-negative measurable function F *on* $\mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E \times M_p(\mathbb{R}_+ \times \mathcal{M}_f(\mathbb{R}_+)^2 \times \mathcal{W}_E)^2$*, we have*

$$\mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}L^*_s\, F(\mathtt{Sp}(\rho, W)_s)\Big) = E^0 \otimes \mathcal{N}\big(\exp(-\alpha\sigma) F(J_\sigma, \check{J}_\sigma, (\xi_s : s \leqslant \sigma), \mathcal{M}^{\ell,*}, \mathcal{M}^{r,*})\big).$$

Observe that the measure in the right-hand side of the previous display coincides with the characteristic measure of $\mathcal{O}_U$ obtained in Proposition 6.20, modulo a truncation of the atoms of $(\mathcal{M}^\ell, \mathcal{M}^r)$ at their exit from $E\backslash\{x\}$.

Proof. First, we shall consider a continuous bounded function F on $\mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E$ vanishing on

$$\big\{(\mu, \mathrm{w}) \in \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E : \zeta_{\mathrm{w}} < \ell \text{ or } \mathrm{w}(\ell) = x\big\},$$

for some fixed arbitrary $\ell > 0$. Let us then define

$$V := \{(\mathsf{\rho}, \omega) \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{W}_E) : \text{ for some } s \in (0, \sigma(\omega)) \text{ it holds that } \zeta_{\omega_s} \geqslant \ell, \omega_s(\ell) \neq x\}.$$

Note that $F(\rho_s, W_s)$ vanishes for every $s \geqslant 0$ on the complement of V and that $\mathbf{N}^*_x(V) < \infty$. By Proposition 6.35, we can find a sub-sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that the convergence (6.73) holds a.e. for every $0 \leqslant t \leqslant \sigma$ along $(\varepsilon_n)_{n\in\mathbb{N}}$. Therefore, along this sub-sequence, the sequence of measures $\mathrm{d}s\, 1_{\{\tau_*(W_s) < H_s < \tau_*(W_s)+\varepsilon_n\}}$ for $n \geqslant 0$ converge weakly towards $\mathrm{d}L^*_s$. Since the mappings $s \mapsto F(\rho_s, W_s)$ and $s \mapsto F(\rho_{s-}, W_s)$ are respectively upper-semicontinuous and lower semicontinuous, we have the $\mathbf{N}^*_x$-a.e. convergences:

$$\limsup_{n\to\infty} \frac{1}{\varepsilon_n} \int_0^\sigma \mathrm{d}s\, 1_{\{\tau^*_x(W_s) < H_s < \tau^*_x(W_s)+\varepsilon_n\}} F(\rho_s, W_s) \leqslant \int_0^\sigma \mathrm{d}L^*_s F(\rho_s, W_s),$$

and

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_0^\sigma \mathrm{d}s\, 1_{\{\tau^*_x(W_s) < H_s < \tau^*_x(W_s)+\varepsilon_n\}} F(\rho_{s-}, W_s) \geqslant \int_0^\sigma \mathrm{d}L^*_s F(\rho_{s-}, W_s).$$

To conclude, we first need a couple of observations. First, remark that the approximation (6.35) yields that the measure $\mathrm{d}L^*_s$ is $\mathbf{N}^*_x$–a.e. supported on the set $\{s \in \mathbb{R}_+ : H_s = \tau^*_x(W_s)\}$. On the other hand, since by Lemma 6.19 the process (ρ, W) takes vales in Θ^e, it holds that $\mathbf{N}^*_x$–a.e. for every s we have $\rho_s(\{\tau^*_x(W_s)\}) = 0$. Putting this two facts together yields that ρ is continuous at $\mathrm{d}L^*_s$ almost every s. Since the set of discontinuities of ρ is countable, we infer that we can replace ρ_{s-} by ρ_s in both terms of the last display. It now follows from this observation and Proposition 6.35 that we have

$$\begin{aligned}\mathbf{N}^*_x\Big(\int_0^\sigma \mathrm{d}L^*_s\, F(\rho_s, W_s)\Big) &= \mathbf{N}^*_x\Big(1_V \int_0^\sigma \mathrm{d}L^*_s\, F(\rho_s, W_s)\Big)\\ &= \lim_{\varepsilon_n\to 0} \mathbf{N}^*_x\Big(1_V \frac{1}{\varepsilon_n} \int_0^\sigma \mathrm{d}s\, 1_{\{\tau^*_x(W_s) < H_s < \tau^*_x(W_s)+\varepsilon_n\}} F(\rho_s, W_s)\Big).\end{aligned}$$

Now, by (6.38) we can write:

$$\begin{aligned}&\mathbf{N}^*_x\Big(\int_0^\sigma \mathrm{d}s\, 1_{\{\tau^*_x(W_s) < H_s < \tau^*_x(W_s)+\varepsilon\}} F(\rho_s, W_s)\Big)\\ &\qquad = \mathcal{N} \otimes E^0\Big(\int_0^\infty \mathrm{d}a\, e^{-\alpha a} 1_{\{\sigma(\xi) < a < \sigma(\xi)+\varepsilon\}} F\big(J_a, (\xi_t : 0 \leqslant t \leqslant a)\big)\Big)\end{aligned}$$

where in the last equality we used that $\tau^*_x(\xi) = \sigma(\xi)$ under $\mathcal{N}$. By an application of the dominated convergence theorem, we infer from the continuity of F and right-continuity of J_a at time $a = \sigma$ that

$$\mathbf{N}^*_x\Big(\int_0^\sigma \mathrm{d}L^*_s F(\rho_s, W_s)\Big) = \mathcal{N} \otimes E^0\big(\exp(-\alpha\sigma) F(J_\sigma, (\xi_s : 0 \leqslant s \leqslant \sigma))\big).$$

We stress that in our last argument we used that the set $\{\sigma \geqslant \ell\}$ has finite $\mathcal{N}$-measure as well as our standing hypothesis on the function F. By the usual approximation methods the previous equality holds for any non-negative measurable F on $\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{W}_E$. Since $\mathbf{N}_x^*$ – a.e. the measure $\mathrm{d}L_s^*$ is supported on $\{s\in\mathbb{R}_+ : H_s = \tau_x^*(W_s)\}$, we get by time-change (see e.g. [81, Proposition V-1.4]) that

$$\int_0^\sigma \mathrm{d}L_s^* F(\rho_s, W_s) = \int_0^{V_\sigma^*} \mathrm{d}L_{\Gamma_s^*}^* F(\rho_{\Gamma_s^*}, W_{\Gamma_s^*}), \qquad \mathbf{N}_x^* \text{ a.e.}$$

Finally, noting that $(V_\sigma^*, (L_{\Gamma_s^*}^*), \mathrm{Tr}_*(\rho, W))$ under $\mathbf{N}_x^*$ is distributed as $(\sigma, (L_s^*), (\rho, W))$ under $\mathbb{N}_x^*$, we deduce the desired result for our choice of F. The proof when considering an arbitrary F as in the statement of the proposition follows by the same type of arguments as in Proposition 6.17, we skip the details. □

Let us conclude the section with an application of Proposition 6.37.

Corollary 6.38. *Recall the definition of the Laplace exponent* $\widetilde{\psi}$ *from* (6.52). *If we write* $\tilde{\pi}$ *for its correspodning Lévy measure on* $(0,\infty)$, *we have* $\tilde{\pi}(\mathrm{d}z) = \mathbb{N}_x^*(L_\sigma^* \in \mathrm{d}z)$.

Since by [82, Corollary 2] the drift coefficient of $\widetilde{\psi}$ is given by $\widetilde{\alpha} = \mathcal{N}(1-\exp(-\alpha\sigma))$ and the Gaussian component $\widetilde{\beta}$ is null, this completely characterises the Lévy-Kintchine triplet of $\widetilde{\psi}$.

Proof. Since the Laplace exponent $\widetilde{\psi}$ has no Brownian component [82, Corollary 2], this is equivalent to proving that for $\lambda \geqslant 0$, we have the identity

$$\mathbb{N}_x^*\big(\exp(-\lambda L_\sigma^*) - 1 + \lambda L_\sigma^*\big) = \widetilde{\psi}(\lambda) - \widetilde{\alpha}\lambda.$$

We shall follow similar arguments to the ones employed in the proof of Lemma 8 in [66]. In this direction, note that by the Markov property under $\mathbb{N}_x^*$, we have

$$\begin{aligned}\mathbb{N}_x^*\big(\exp(-\lambda L_\sigma^*) - 1 + \lambda L_\sigma^*\big) &= \lambda\cdot\mathbb{N}_x^*\left(\int_0^\sigma \mathrm{d}L_s^*\Big(1-\exp(-\lambda\int_s^\sigma \mathrm{d}L_u^*)\Big)\right)\\ &= \lambda\cdot\mathbb{N}_x^*\left(\int_0^\sigma \mathrm{d}L_s^*\,\mathbb{E}_{\rho_s,W_s}^*\Big(1-\exp(-\lambda\int_0^\sigma \mathrm{d}L_u^*)\Big)\right). \qquad (6.84)\end{aligned}$$

In order to compute the last expression, fix $(\mu, \mathrm{w}) \in \Theta^e$ and recall the identity (6.82). The formula for the Laplace transform for integrals with respect to Poisson random measures yields

$$\mathbb{E}_{\mu,\mathrm{w}}^*\big[\exp(-\lambda L_\sigma^*)\big] = \exp\Big(-\int_0^{\tau_x^*(\mathrm{w})}\mu(\mathrm{d}h)\,\mathbb{N}_{\mathrm{w}(h)}^*\big(1-\exp(-\lambda L_\sigma^*)\big)\Big) = \exp\Big(-\int_0^{\tau_*(\mathrm{w})}\mu(\mathrm{d}h)\,u_\lambda\big(\mathrm{w}(h)\big)\Big),$$

where in the last equality we used that by definition, the distribution of $L_\sigma^{E_*}$ under $\mathbb{N}_y^*$ for $y\neq x$ is precisely the one of $L_\sigma^{E_*}$ under $\mathbb{N}_y$. Getting back to (6.84), we infer from the identity in the last display and Proposition 6.37 that:

$$\begin{aligned}\mathbb{N}_x^*\big(\exp(-\lambda L_\sigma^*) - 1 + \lambda L_\sigma^*\big) &= \lambda\cdot\mathbb{N}_x^*\Big(\int_0^\sigma \mathrm{d}L_s^*\,1-\exp\big(-\int\rho_s(\mathrm{d}h)\,u_\lambda\big(W_s(h)\big)\big)\Big)\\ &= \lambda\cdot\mathcal{N}\otimes E^0\Big(\exp(-\alpha\sigma)\Big(1-\exp\big(-\int J_\sigma(\mathrm{d}h)\,u_\lambda(\xi_h)\big)\Big)\Big)\\ &= \lambda\cdot\mathcal{N}\otimes E^0\Big(\exp(-\sigma\alpha)-\exp\big(-\int_0^\sigma \mathrm{d}h\,\psi(u_\lambda(\xi_h))/u_\lambda(\xi_h)\big)\Big),\end{aligned}$$

where in the last equality we used that J_∞ under P^0 is the jump measure of a subordinator with Laplace exponent $\psi(\lambda)/\lambda - \alpha$. Finally, we observe that the last display writes:

$$\lambda \cdot \mathcal{N} \otimes E^0\Big(1 - \exp\big(-\int_0^\sigma \mathrm{d}h\, \psi(u_\lambda(\xi_h))/u_\lambda(\xi_h)\big)\Big) - \lambda \cdot \mathcal{N} \otimes E^0\Big(1 - \exp(-\sigma\alpha)\Big).$$

The result of the corollary now immediately follows from the identity (4.21) in [66]. □

We conclude the section with some technical results that will be needed in Section 6.8. With the notations of Proposition 6.20 under $E^0 \otimes \mathcal{N}_*$, let us write $\mathcal{M}^\ell = \sum_{k\in\mathcal{K}^{(\ell)}} \delta_{(t_k^{(\ell)},\rho^k,\eta^k,\overline{W}^k)}$, $\mathcal{M}^r = \sum_{k\in\mathcal{K}^{(r)}} \delta_{(t_k^{(r)},\rho^k,\eta^k,\overline{W}^k)}$ where $\mathcal{K}^{(\ell)}$, $\mathcal{K}^{(r)}$ are two disjoint indexing sets. Next, for $k \in \mathcal{K}^{(\ell)}$ set $\alpha_k := \sum_{t_i^{(\ell)} < t_k^{(\ell)}} \sigma(W^i)$, $\beta_k := \sum_{t_i^{(\ell)} \leqslant t_k^{(\ell)}} \sigma(W^i)$ and introduce the process

$$V_t(\mathcal{M}^\ell) := \sum_{k\in\mathcal{K}^{(\ell)}} L^{E_*}_{t\wedge\beta_k - t\wedge\alpha_k}(\rho^k, \overline{W}^k), \quad t \geqslant 0.$$

To simplify notation and when there is no risk of confusion, we denote the process in the previous display by $V^{(\ell)}$. Note that this construction is a deterministic function of $\mathcal{M}^\ell$ and for instance can be applied to $\mathcal{M}^r$. Analogously, we write $V^{(r)}$ for $V(\mathcal{M}^r)$.

Lemma 6.39. *Under $E^0 \otimes \mathcal{N}_*$, for every $\varepsilon > 0$ we have $V_\varepsilon(\mathcal{M}^\ell), V_\varepsilon(\mathcal{M}^r) > 0$ a.e.*

Proof. Recall from (6.41) the intensity measure of $\mathcal{M}^r$ conditionally on (ξ, J). Since $(\check{J}_\sigma, \xi, \mathcal{M}^r)$ and $(J_\sigma, \xi, \mathcal{M}^\ell)$ have the same distribution, it suffices to show that for any $\varepsilon > 0$, we have $E^0 \otimes \mathcal{N}_*(V_\varepsilon^{(r)} = 0) = 0$.

Consider a deterministic increasing sequence $(a_n : n \geqslant 0)$ with $a_n \uparrow 1$. It suffices to show that $E^0 \otimes \mathcal{N}_*(L^*_{\sigma_i}(\rho^i, W^i) = 0, \forall\, i \in \mathcal{K}^{(r)}$ such that $H(\kappa_{t_i} J_\sigma) > \sigma a_n) = 0$. If for $y \in E_*$ we write $v(y) = \mathbb{N}_y(L_\sigma^{E_*} \neq 0)$, this is equivalent to proving that $E^0 \otimes \mathcal{N}_*$ a.e.,

$$\exp\big(-\int_{a_n\sigma}^\sigma J_\sigma(\mathrm{d}h) v(\xi(h))\big) = 0. \tag{6.85}$$

Under $\mathbb{P}^*_{\mu,\mathrm{w}}$ and $\mathbb{N}^*_x$ and for every $n \geqslant 0$ we set $T_n = \inf\{t \geqslant 0 : H_t < a_n H_0\}$. Since by Corollary 6.36-(i) for $(\mu, \mathrm{w}) \in \Theta^e$ with $\tau^*_x(\mathrm{w}) = \zeta_\mathrm{w}$ we have $\mathbb{P}^*_{\mu,\mathrm{w}}(L^{E_*}_{T_n} = 0) = \exp\big(-\int_{a_n\zeta_\mathrm{w}}^{\zeta_\mathrm{w}} \mu(\mathrm{d}h) v(\mathrm{w}(h))\big)$, by the first moment formula of Proposition 6.37 we get

$$\begin{aligned} E^0 \otimes \mathcal{N}\Big(\exp(-\alpha\sigma)\exp\big(-\int_{a_n\sigma}^\sigma J_\sigma(\mathrm{d}h) v(\xi(h))\big)\Big) &= \mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}L^*_s\, \mathbb{P}^*_{\rho_s,W_s}(L^{E_*}_{T_n} = 0)\Big) \\ &= \mathbb{N}^*_x\Big(\int_0^\sigma \mathrm{d}L^*_s\, \mathbb{1}_{\{L^*_{s+T_n\circ\theta_s} - L^*_s = 0\}}\Big) \end{aligned} \tag{6.86}$$

where in the last equality we applied the Markov property as well as the identities of Lemma 6.36. Now, note that the interval $(s, s+T_n\circ\theta_s)$ is open and non-empty and therefore the cardinality of $C(n) = \{s \in \operatorname{supp} \mathrm{d}L^* : \mathrm{d}L^*(s, s+T_n\circ\theta_s) = 0\}$ is countable. Indeed, any element of this set is the right end of a connected component of the open set $\mathbb{R}_+ \backslash \operatorname{supp} \mathrm{d}L^*$. Therefore, by continuity of L^* it holds that $\mathrm{d}L^*(C(n)) = 0$ and it follows that the formula in the last display is null. We infer that (6.85) holds, concluding the proof of the lemma. Since J_∞ under E_0 is the jump measure of a subordinator with Laplace exponent $\psi(\lambda)/\lambda - \alpha$, remark that in particular we deduce from the fact that (6.86) vanishes that $\mathcal{N}-$ a.e., we have $\int_{\sigma a_n}^\sigma \mathrm{d}h\, \psi(v(\xi(h)))/v(\xi(h)) = \infty$, a fact that *a priory* is not simple to establish directly. □

Let us conclude the section with an important consequence of Lemma 6.39. We shall still make use of the notations of Proposition 6.20. Under $E_0 \otimes \mathcal{N}_*$ consider the measure $\mathcal{M}^\ell = \sum_{k\in\mathcal{K}^{(\ell)}} \delta_{(t_k^{(\ell)},\rho^k,\eta^k,\overline{W}^k)}$ and for every $k \in \mathcal{K}^{(\ell)}$, write $(\rho^{k,j},\overline{W}^{k,j})_{j\in\mathcal{K}_k^{(\ell)}}$ for the subtrajectories of $(\rho^k,\overline{W}^k)$ outside $E_* \times \mathbb{R}_+$. Denote the corresponding excursion intervals by $(a_{k,j},b_{k,j})_{j\in\mathcal{K}_k^{(\ell)}}$ and for $j \in \mathcal{K}_k^{(\ell)}$ set $\alpha_{k,j} := \alpha_k + a_{k,j}$. After performing the analogous construction in terms of the measure $\mathcal{M}^r = \sum_{k\in\mathcal{K}^{(r)}} \delta_{(t_k^{(r)},\rho^k,\eta^k,\overline{W}^k)}$, with an obvious choice of notation we introduce the following pair of measures:

$$\big(\Xi^*(\mathcal{M}^\ell),\Xi^*(\mathcal{M}^r)\big) := \Big(\sum_{k\in\mathcal{K}^{(\ell)},j\in\mathcal{K}_k^{(\ell)}} \delta_{(V^{(\ell)}_{\alpha_{k,j}},\rho^{k,j},\eta^{k,j},\overline{W}^{k,j})}, \sum_{k\in\mathcal{K}^{(r)},j\in\mathcal{K}_k^{(r)}} \delta_{(V^{(r)}_{\alpha_{k,j}},\rho^{k,j},\eta^{k,j},\overline{W}^{k,j})}\Big). \quad (6.87)$$

We infer from the special Markov property that, conditionally on $(J_\sigma,\check{J}_\sigma,\xi)$ and

$$\Big(\sum_{k\in\mathcal{K}^{(\ell)}} \delta_{(t_k^{(\ell)},\mathrm{Tr}_*(\rho^k,\overline{W}^k))}, \sum_{k\in\mathcal{K}^{(r)}} \delta_{(t_k^{(r)},\mathrm{Tr}_*(\rho^k,\overline{W}^k))}\Big), \quad (6.88)$$

the pair $\big(\Xi^*(\mathcal{M}^\ell),\Xi^*(\mathcal{M}^r)\big)$ are independent Poisson measures with respective intensities

$$\mathbb{1}_{[0,V_\infty^{(\ell)}]}(u)\mathrm{d}u\, \mathbb{N}_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W}) \quad \text{and} \quad \mathbb{1}_{[0,V_\infty^{(r)}]}(u)\mathrm{d}u\, \mathbb{N}_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W}). \quad (6.89)$$

Lemma 6.39 ensures that both measures are non trivial $E^0 \otimes \mathcal{N}^*$–a.e. Let us now briefly perform an analogous construction to the tip of the spine, viz. for $((\rho_U^0,\eta_U^0,W_U^0),\, (\mathcal{P}_U^{\ell,0},\mathcal{P}_U^{r,0}))$ under $\mathbb{N}^\bullet_{x,0}$. Still from Proposition 6.20 recall that under $\mathbb{N}^\bullet_{x,0}$, conditionally on $(\rho_U^0,\eta_U^0,W_U^0)$, the pair of measures $(\mathcal{P}_U^{\ell,0},\mathcal{P}_U^{r,0})$ are independent Poisson random measures with respective intensities

$$\mathbb{1}_{[0,\langle\eta_U^0,1\rangle]}(u)\mathrm{d}u\, \mathbb{N}_{W_U^0(H(\kappa_u\eta_U^0)),0}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W}) \quad \text{and} \quad \mathbb{1}_{[0,\langle\rho_U^0,1\rangle]}(u)\mathrm{d}u\, \mathbb{N}_{W_U^0(H(\kappa_u\rho_U^0)),0}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W}).$$

Then, if we now consider $(V_t(\mathcal{P}^{\ell,0}),V_t(\mathcal{P}_U^{r,0}))_{t\geqslant 0}$ as well as the pair of measures $\big(\Xi^*(\mathcal{P}_U^{\ell,0}),\Xi^*(\mathcal{P}_U^{r,0})\big)$ we still have that conditionally on $(\rho_U^0,\eta_U^0,W_U^0)$ and $(V_\infty(\mathcal{P}_U^{\ell,0}),V_\infty(\mathcal{P}_U^{r,0}))$ the pair of variables $\big(\Xi^*(\mathcal{P}_U^{\ell,0}),\Xi^*(\mathcal{P}_U^{r,0})\big)$ are independent Poisson measures with respective intensities (6.89). Note however that in contrast with our previous case, we might have $V_\infty(\mathcal{P}_U^{\ell,0}) = 0$ or $V_\infty(\mathcal{P}_U^{r,0}) = 0$.

6.8 Reconstructions

The content of this section is at an early stage.

It is well know from classic excursion theory of time indexed Markov processes that one can recover the initial path of the Markov process from its excursion measure. This is essentially achieved by concatenating the excursions, using the fact that the ordering induced by the local time is precisely the temporal order. In the setting of Markov processes indexed by Lévy trees, the inherent complexity of these objects gives rise to a several natural analogous questions that we shall now address. We shall start with an overview of the main results of the section.

In this direction, recall that we write $\widetilde{H}$ for the height process $\widehat{\Lambda}^A$ of the subordinate tree and we let $\widetilde{X}$ be the Lévy process associated to this height process. The first part of this section is devoted to proving that one can recover the Lévy process $\widetilde{X}$ - or equivalently, the height process

$\widetilde{H}$ - from the excursion process $\mathcal{E}$. To be more precise, let us introduce some notation. Under $\mathbb{N}_{x,0}$ and $\mathbb{P}_{0,x,0}$, recall that we write $\mathcal{E} = \sum_{u\in D} \delta_{(A_{g(u)},\rho^u,W^u)}$ for the excursion process of (ρ, W) and consider the following subset of D

$$D_+ := \{u \in D : L_\sigma^*(W^u) > 0\}.$$

We shall write $\sum_{u\in D_+} \delta_{(A_{g(u)},L_\sigma^*(\rho^u,W^u))}$ for the image of $\mathcal{E}1_{\{L_\sigma^*(\rho,W)>0\}}$ under the mapping given by $(A_{g(u)}, \rho^u, W^u) \mapsto (A_{g(u)}, L_\sigma^*(\rho^u, W^u))$. The first main result of the section is the following theorem:

Theorem 6.40. *Under $\mathbb{P}_{0,x,0}$, the measure $\sum_{u\in D_+} \delta_{(A_{g(u)},L_\sigma^*(\rho^u,W^u))}$ is the jump measure of $\widetilde{X}$.*

The proof of this result is achieved in two steps:

Step 1: We first prove in Proposition 6.45 that the set $\{A_{g(u)} : u \in D_+\}$ is precisely the set of jump-times of $\widetilde{X}$. This is the main content of Section 6.8.1.

Step 2: We then show in Section 6.8.2 that for every $u \in D_+$ we have $\Delta\widetilde{X}_{A_{g(u)}} = L_\sigma^*(\rho^u, W^u)$. The proof of the later will be a straight consequence of Proposition 6.47, which links the local times at the branching points of $\mathcal{T}_{\widetilde{H}}$ with the family of processes $(L^*(\rho^u, W^u) : u \in D_+)$ via an explicit time-change. Section 6.8.2 is devoted to the proof of the latter.

As a consequence of Theorem 6.40 we deduce both, the reconstruction of $\widetilde{X}$ in terms of $\mathcal{E}$, and identify the law of the family of excursions $(\rho^u, W^u)_{u\in D}$ given $\widetilde{H}$. This result is closely related to [1, Theorem 40].

Corollary 6.41. *The Lévy process $\widetilde{X}$ is $\mathcal{E}$-measurable. In particular, if for every $u \in D$ we let $z_u = L_\sigma^*(\rho^u, W^u)$, we have*

$$\mathbb{N}_{x,0}\left(g(\widetilde{H})\exp\Big(\sum_{u\in D} f(A_{g(u)},\rho^u,W^u)\Big)\right) = \mathbb{N}_{x,0}\left(g(\widetilde{H})\prod_{u\in D}\mathbb{N}_x^*\Big(\exp\big(f(A_{g(u)},\rho,W)\big)\Big|L_\sigma^*=z_u\Big)\right).$$

In other words, conditionally on $\widetilde{H}$, the family $(\rho^u, W^u)_{u\in D}$ are independent with respective law given by $\mathbb{N}_x^*(\mathrm{d}\rho, \mathrm{d}W|L_\sigma^* = z_u)$.

We now turn our attention to the reconstruction of the Lévy snake in terms of the excursion process $\mathcal{E}$. Unsurprisingly, this procedure turns out to be significantly more delicate than in the classic setting of time-indexed Markov processes, but we still have the following analogue theorem:

Theorem 6.42. *Recall the notation $\mathcal{E}$ for the excursion process of (ρ, W). The Lévy snake (ρ, W) can be recovered from $\mathcal{E}$.*

Remark that by Corollary 6.5, to obtain this result it suffices to show that the pair $(H, \widehat{W})$ can be recovered from $\mathcal{E}$. This is precisely the content of Proposition 6.49. Section 6.8.3 is devoted to the proof of this result, and relies strongly again in Proposition 6.47.

In what follows, we shall make extensive use of the spinal decomposition in excursions of Proposition 6.20 as well as the closing remarks of Section 6.5.1. The same notations are maintained through this section. Roughly speaking, our arguments often rely on proving desired properties for the excursions on a typical Snake path - viz. by working under $\mathbb{N}_{x,0}^\bullet$ and using Proposition 6.20 - and then transferring such properties to every excursion (ρ^u, W^u) of (ρ, W). This last step relies on the following lemma:

Lemma 6.43. *Let $A \in \mathcal{F}$, $B \in \mathcal{F} \otimes \mathbb{R}_+$ and suppose that for some (random) $I \subset \mathcal{B}(\mathbb{R}_+)$ with $\mathbb{N}_{x,0}$ -a.e. positive Lebesgue measure it holds that $B^c \cap \{U \in I\} \supseteq A^c \times \{U \in I\}$. If B has full $\mathbb{N}^{\bullet}_{x,0}$ measure, then A has full $\mathbb{N}_{x,0}$-measure.*

Proof. Since B has full $\mathbb{N}^{\bullet}_{x,0}$ measure, then $0 = \mathbb{N}^{\bullet}_{x,0}(B^c) \geqslant \mathbb{N}^{\bullet}_{x,0}(B^c \cap \{U \in I\}) \geqslant \mathbb{N}_{x,0}(\mathcal{L}(I); A^c)$ where we denoted by $\mathcal{L}(I)$ the Lebesgue measure of I. Since $\mathcal{L}(I) > 0$ $\mathbb{N}_{x,0}$ a.e. this gives that A has full $\mathbb{N}_{x,0}$ measure. □

The rest of the section is organised as follows: in Section 6.8.1 we study the family of jump-times of $\widetilde{X}$ and prove that the sets $\{t \geqslant 0 : \Delta \widetilde{X}_t > 0\}$ and $\{A_{g(u)} : u \in D_+\}$ coincide [Proposition 6.45]. In Section 6.8.2 we establish that the local times at the branching points of $\mathcal{T}_{\widetilde{H}}$ and $(L^*(\rho^u, W^u) : u \in D_+)$ differ by a time change [Proposition 6.92]. In particular, we shall deduce from this result the identity $\Delta \widetilde{X}_{A_{g(u)}} = L^*_\sigma(\rho^u, W^u)$ for $u \in D_+$ which concludes the proof of Theorem 6.40. Finally, Section 6.8.3 is devoted to the proof of Theorem 6.8. Namely, we shall prove in Proposition 6.49 that the pair $(H, \widehat{W})$ can be reconstructed from $\mathcal{E}$.

6.8.1 The jump-times of $\widetilde{X}$

This short section is devoted to proving that the two following sets $\{t \geqslant 0 : \Delta \widetilde{X}_t > 0\}$ and $\{A_{g(u)} : u \in D_+\}$ coincide. This is the first step towards proving Theorem 6.40 and we shall start by covering some preliminary results. Recall that we write D° for the set of debut times [Definition 6.24], and that by Lemma 6.25 the relation $u \mapsto g(u)$ is a bijection between D and D°. In particular, the subset $D_+ \subset D$ is in bijection with a subset of D° that we now characterise:

Lemma 6.44. *For $t \in [0, \sigma]$, we introduce the condition*

(i') There exists $s > t$ such that $H_s > H_t$ with $\inf_{[t,s]} H = H_t$ and $\widehat{\Lambda}_s > \widehat{\Lambda}_t$.

The subset of debut times that satisfy (i') *is denoted by D°_+. Then, under $\mathbb{N}_{x,0}$ and $\mathbb{P}_{0,x,0}$ a debut $u \in D$ belongs to D_+ if an only if $g(u)$ belongs to D°_+.*

Proof. Recall that under $\mathbb{N}_{x,0}$, for every fixed $t \in (0, \sigma)$ such that $\widehat{W}_t \neq x$, we let $u(t) \in D$ be the unique debut such that $\ell_{u(t)} = \widehat{\Lambda}_t$. Since $\mathbb{N}^{\bullet}_{x,0}$-a.e. we have $\widehat{W}_U \neq x$, we can write $u(U)$ for the unique debut point satisfying $\ell_{u(U)} = \widehat{\Lambda}_U$. Let us start arguing under $\mathbb{N}^{\bullet}_{x,0}$. By the discussion following Lemma 6.39, condition (i') is satisfied by $g(u(U))$ if and only if $V_\infty(\mathcal{P}^{\ell,0}_U) \neq 0$ or $V_\infty(\mathcal{P}^{r,0}_U) \neq 0$. Moreover, since by (6.83) we can write $L^*_\infty(\rho^{u(U)}, W^{u(U)}) = V_\infty(\mathcal{P}^{\ell,0}_U) + V_\infty(\mathcal{P}^{r,0}_U)$, it follows that the union of the (disjoint) sets $\{g(u(U)) \in D^\circ_+, L^*_\sigma(\rho^{u(U)}, W^{u(U)}) > 0\}$, $\{g(u(U)) \notin D^\circ_+, L^*_\sigma(\rho^{u(U)}, W^{u(U)}) = 0\}$ has full $\mathbb{N}^{\bullet}_{x,0}$ measure. The statement of the lemma now immediately follows by an application of Lemma 6.43. Since this type of reasoning will be often used, we shall provide the details for reader's convenience, but in the sequel they will be systematically omitted. Fix an arbitrary $u \in D$. There exists a non-empty interval $(g'(u), d'(u)) \subset (g(u), d(u))$ such that $\{U \in (g'(u), d'(u))\} \subseteq \{\widehat{\Lambda}_U = \ell_u\}$. Therefore, from our previous reasoning we get

$$\begin{aligned} 0 &= \mathbb{N}^{\bullet}_{x,0}\big(g(u(U)) \in D^\circ_+, L^*_\sigma(\rho^{u(U)}, W^{u(U)}) = 0, U \in (g'(u), d'(u))\big) \\ &= \mathbb{N}_{x,0}\big((d'(u) - g'(u)); g(u) \in D^\circ_+, L^*_\sigma(\rho^u, W^u) = 0\big) \end{aligned}$$

and analogously we obtain as well that $\mathbb{N}_x\big((d'(u) - g'(u)); g(u) \notin D^\circ_+, L^*_\sigma(\rho^u, W^u) > 0\big) = 0$. Since $\mathbb{N}_{x,0}$-a.e. it holds that $d'(u) - g'(u) > 0$, we infer that $\mathbb{N}_{x,0}$-a.e. the sets $\{g(u) \in D^\circ_+, L^*_\sigma(\rho^u, W^u) >$

$0\}$, $\{g(u) \notin D_+^\circ, L_\sigma^*(\rho^u, W^u) = 0\}$ are a partition of Ω. It follows that $\mathbb{N}_{x,0}$-a.e. we have that $g(u) \in D_+$ if and only if $g(u)$ belongs to D_+°. □

We can now prove the remarkable connection between D_+ and the set of jump-times of $\widetilde{X}$. Recall the notation $\widetilde{H}$ for the height process $\widehat{\Lambda}^A$ of the subordinate three $\widetilde{\mathcal{T}}_{\widetilde{H}}$.

Proposition 6.45. *Under $\mathbb{P}_{0,x,0}$ and $\mathbb{N}_{x,0}$, the mapping $\tilde{g} : D_+ \mapsto \mathbb{R}_+$ defined by the relation $\tilde{g}(u) := A_{g(u)}$ is a bijection between the sets D_+ and $\{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}$.*

Proof. It suffices to prove the result under $\mathbb{N}_{x,0}$ and for clarity, we divide the proof in two main steps. In this direction, fix an arbitrary $u \in D_+$ and let us start proving that:

- $\widetilde{g}(u)$ *is a jump time for* $\widetilde{X}$. To achieve this, by Lemma 6.50 it suffices to prove that $\mathbb{N}_{x,0}$-a.e.,

1. For every $\delta > 0$, we have $\inf_{[\widetilde{g}(u)-\delta,\, \widetilde{g}(u)]} \widetilde{H} < \widetilde{H}_{\widetilde{g}(u)}$.

2. For every $\varepsilon > 0$ such that $\inf_{[\widetilde{g}(u),\, \widetilde{g}(u)+\varepsilon]} \widetilde{H} = \widetilde{H}_{\widetilde{g}(u)}$, there exists $r \in (\widetilde{g}(u), \widetilde{g}(u)+\varepsilon)$ satisfying $\widetilde{H}_{\widetilde{g}(u)} = \widetilde{H}_r$.

To establish this two properties we shall make use of the results of Section 6.5.2. Let us start with some preliminary remarks. Under $E_0 \otimes \mathcal{N}_*$ and to simplify notation, write $\big(\Xi_\ell^*, \Xi_r^*\big)$ for the pair of measures (6.87) and introduce the processes:

$$A_t(\Xi_\ell^*) := \sum_{k \in \mathcal{K}^{(\ell)}, j \in \mathcal{K}_k^{(r)}} A_{t\wedge\beta_{k,j} - t\wedge\alpha_{k,j}}(\rho^{k,j}, \overline{W}^{k,j}), \quad \widehat{\Lambda}_t(\Xi_\ell^*) := \sum_{k \in \mathcal{K}^{(\ell)}, j \in \mathcal{K}_k^{(r)}} \widehat{\Lambda}^{k,j}_{t\wedge\beta_{k,j} - t\wedge\alpha_{k,j}} 1_{t \in (\alpha_{k,j}, \beta_{k,j})}.$$

Recall that by Lemma 6.39 we have $V_\infty^{(\ell)} > 0$ and that conditionally on (J_σ, ξ) and $V_\infty^{(\ell)}$, the measure Ξ_ℓ^* is a Poisson measure with intensity $1_{[0,V_\infty^{(\ell)}]}(u)\mathrm{d}u\, \mathbb{N}_{x,0}(\mathrm{d}\rho, \mathrm{d}\overline{W})$. We infer that conditionally on (J_σ, ξ) and (6.88), the pair $\big(A_t(\Xi_\ell^*), \widehat{\Lambda}_t(\Xi_\ell^*)\big)_{t\geqslant 0}$ is distributed as $(A_t, \widehat{\Lambda}_t)_{t\geqslant 0}$ under $\mathbb{P}_{0,x,0}$ stopped at time $\inf\{s \geqslant 0 : -I_s = V_\infty^{(\ell)}\}$ and in particular, $A_\infty(\Xi_\ell^*) > 0$ by Lemma 10 of [82]. In our last argument we used that under $\mathbb{P}_{0,x,0}$, the measure $\mathrm{d}A$ does not charge the set $\{t \geqslant 0 : I_t = X_t\}$. This follows noting that $\mathbb{E}_{0,x,0}\big(\mathrm{d}A(\{s \geqslant 0 : H_s = 0\})\big) = 0$, since for every $r > 0$ and writing $\mathscr{L}^r$ for the exit local time from the domain $E \times [0, r)$, we have $\mathrm{d}\mathscr{L}_s^r(\{s \geqslant 0 : H_s = 0\}) = 0$ were the last assertion follows form the integral representation of A given in [82, Proposition 6]. Now, we deduce that the origin is regular and instantaneous for the time changed process $\widehat{\Lambda}^A(\Xi_\ell^*)$, from identity (6.6) combined with the fact that $\widehat{\Lambda}_A$ under $\mathbb{P}_{0,x,0}$ is the height process of a $\widetilde{\psi}$ - Lévy tree. Note that the same holds for the time reversed process $\mathtt{Trev}\widehat{\Lambda}^A(\Xi_\ell^*)$.

Now we work under $\mathbb{N}_{x,0}^\bullet$ and recall that we write $\widetilde{H}$ for $\widehat{\Lambda}^A$. For a fixed arbitrary $\ell_u \in \mathcal{J}(\Lambda_U)$, let us prove that the corresponding $\widetilde{g}(u) = A_{g(u)}$ satisfies conditions 1 and 2. Starting with the latter, recall that by Proposition 6.20, conditionally on $\widehat{\Lambda}_U$ the measure $\mathcal{O}_U$ is a Poisson measure with intensity $1_{[0,\widehat{\Lambda}_U]}\mathrm{d}s E^0 \otimes \mathcal{N}_*((J_\sigma, \check{J}_\sigma, \xi, \mathcal{M}^\ell, \mathcal{M}^r) \in \mathrm{d}z)$ and let $(r_i, \rho_U^i, \eta_U^i, W_U^i, \mathcal{P}_U^{\ell,i}, \mathcal{P}_U^{r,i})$ be the unique atom of $\mathcal{O}_U$ such that $r_i = \ell_u$. To simplify notation, we write $\Xi_\ell^{*,i}$ for $\Xi^*(\mathcal{P}_U^{\ell,i})$. By our previous considerations under $E^0 \otimes \mathcal{N}_*$ and (4.27) in [82], we have $A_\infty(\mathcal{P}_U^{\ell,i}) = A_\infty(\Xi_\ell^{*,i}) > 0$ and note that we can write

$$\Big(\widehat{\Lambda}^A_{A_{g(u)}+t} : 0 \leqslant t \leqslant A_\infty(\mathcal{P}_U^{\ell,i})\Big) = \mathtt{Trev}\Big(\widehat{\Lambda}^A_t(\Xi_\ell^{*,i}) + r_i : 0 \leqslant t < A_\infty(\Xi_\ell^{*,i})\Big).$$

Further, from our reasoning under $E^0 \otimes \mathcal{N}_*$ the point 0 is regular and instantaneous for the process in the right-hand side, and we infer that condition 2 is fulfilled by $A_{g(u)}$. Condition 1 will follow if we can find an increasing sequence of times $(t_n)_{n\geqslant 0}$ with $t_n \uparrow g(u)$ and such that $\widetilde{H}_{A_{t_n}} < \widetilde{H}_{A_{g(u)}}$ for every $n \geqslant 0$. Recall that $g(u)$ is a debut time by Lemma 6.25 and in particular, by Definition 6.24-(ii) we have $\inf_{(g(u)-\varepsilon, g(u)]} \widehat{\Lambda} < \widehat{\Lambda}_{g(u)}$ for every $\varepsilon > 0$. Now, it readily follows that we can find a sequence $(t_n)_{n\geqslant 0}$ with $t_n \uparrow t$ such that $\widetilde{H}_{A_{t_n}} = \widehat{\Lambda}_{t_n} < \widehat{\Lambda}_{g(u)} = \widetilde{H}_{A_{g(u)}}$.

Our previous reasoning shows that under $\mathbb{N}^{\bullet}_{x,0}$, the set $B := \{\forall \ell_u \in \mathcal{J}(\Lambda_U), A_{g(u)} \in \{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}\}$ has full measure. Let us deduce from this that under $\mathbb{N}_{x,0}$ and for every fixed $u' \in D_+$, the point $A_{g(u')}$ is a jump-time for $\widetilde{X}$. Using the fact that $L^*_\sigma(\rho^{u'}, W^{u'}) > 0$, it is not difficult to check that there exists a nonempty interval $(g'(u'), d'(u')) \subset (g(u'), d(u'))$ such that $\{U \in (g'(u'), d'(u'))\} \subset \{\ell^{u'} \in \mathcal{J}(\Lambda_U)\}$. Now, our claim readily follows by an application of Lemma 6.43 to the sets B, $A := \{A_{g(u')} \in \{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}\}$ and $I := (g'(u'), d'(u'))$.

• *The mapping* $\widetilde{g} : D_+ \mapsto \{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}$ *is bijective.* Let us start proving that $\widetilde{g}$ is injective. Consider $u, u' \in D$ with $u \neq u'$ and without loss of generality suppose that $g(u') < g(u)$. Since $g(u)$ is a debut time, we infer from condition (i) in Definition 6.24 and Lemma 6.27 that $g(u)$ is a point of left increase for A. This gives that $A_{g(u')} < A_{g(u)}$ and proves that the mapping $\widetilde{g}$ is injective. Let us now check the surjectiviy of $\widetilde{g}$. Fix an arbitrary time t in the set $\{t \geqslant 0 : \Delta\widetilde{X}_t > 0\}$, so that conditions (i) and (ii) of Lemma 6.40 are satisfied by $\widetilde{H}$ at time t. We start checking that A^{-1}_{t-} is an element of D°_+ viz. that $(H, \widehat{\Lambda})$ at time A^{-1}_{t-} fulfils conditions (i),(ii) of Definition 6.24 as well as (i') of Lemma 6.44. We consider two different cases. Suppose first that $A^{-1}_{t-} < A^{-1}_t$. By condition (i) of Lemma 6.40, we can find an increasing sequence of times $s_n \uparrow t$ such that $\widetilde{H}_{s_n} < \widetilde{H}_t$ for every $n \geqslant 0$. By left continuity, we have $A^{-1}_{s_n} \uparrow A^{-1}_{t-}$ which yields that A^{-1}_{t-} satisfies Definition 6.6-(ii). Further, noting that $\widehat{\Lambda}$ is constant on $[A^{-1}_{t-}, A^{-1}_t]$, by the snake property it must hold that $H_r \geqslant H^A_{t-}$ for every $r \in [A^{-1}_{t-} < A^{-1}_t]$. Since H has no constancy intervals, we can find some time s on this interval at which $H_s > H^A_{t-}$. This proves that A^{-1}_{t-} satisfies as well condition (i) and therefore is a debut time. Let us now assume that $A^{-1}_{t-} = A^{-1}_t$. Arguing as before, condition Definition 6.24-(ii) is fulfilled by A^{-1}_{t-} so let us prove that Definition 6.24-(i) also holds. Suppose by contradiction that we can not find $s > A^{-1}_t$ such that $H_s > H^A_t$ with both $\min_{[A^{-1}_t, s]} H = H^A_t$ and $\widehat{\Lambda}^A_t = \widehat{\Lambda}_s$. Since t is a jump time of $\widetilde{H}$, we can find a decreasing sequence $s_n \downarrow t$ satisfying that $s_n \in \{r \geqslant t : \widetilde{H}_r = \widetilde{H}_t = \min_{[t,r]} \widetilde{H}\}$. By right-continuity, $A^{-1}_{s_n} \downarrow A^{-1}_t$ and for every n, it must hold that $H^A_{s_n} = H^A_t$. It follows that conditions (i) and (ii) of Lemma 6.40 are satisfied by H at time A^{-1}_t, which gives that A^{-1}_t is a jump time for X. Further, since $\mathrm{d}A$ is supported on $\{t \geqslant 0 : \widehat{W}_t = x\}$, we have $\widehat{W}^A_t = x$ which contradicts (6.20). This concludes the proof that A^{-1}_{t-} is a debut-time for $(\rho, \overline{W})$. Moreover, clearly condition Lemma 6.44-(i') is fulfilled as well by H at time A^{-1}_{t-} proving that A^{-1}_t belongs to D°_+. Now, recall from Lemma 6.25 that the mapping $g : u \mapsto g(u)$ for $u \in D$ is a bijection between D and D°. If we write $u := g^{-1}(A^{-1}_{t-})$ for the corresponding excursion debut associated to A^{-1}_{t-} by the bijection g, by Lemma 6.44 we have that u belongs to D_+. Finally, since $\widetilde{g}(u) = A_{g(u)} = t$ this proves that the mapping $\widetilde{g}$ is surjective, and concludes the proof of the proposition. □

By Proposition 6.45, the first entries of the atoms on the following pair of measures

$$\sum_{u \in D_+} \delta_{(A_{g(u)}, L^*_\sigma(W^u))}, \qquad \text{and} \qquad \sum_{t \in \mathbb{R}_+} 1_{\{\tilde{X}_{t-} \neq \tilde{X}_t\}} \delta_{(t, \Delta\tilde{X}_t)}$$

coincide $\mathbb{P}_{0,x,0}$ and $\mathbb{N}_{x,0}$ – a.e. To establish Theorem 6.40 it remains to show that for every $u \in D_+$, we have $L^*_\sigma(\rho^u, W^u) = \Delta \widetilde{X}_{A_{g(u)}}$. This fact will be obtained, as we mentioned previously, as a straight consequence of the crucial relationship linking the local times at the branching points of $\mathcal{T}_{\widetilde{H}}$, and the family of processes $(L^*(\rho^u, W^u) : u \in D_+)$.

6.8.2 The local time at the branching points of $\mathcal{T}_{\widetilde{H}}$

For every $u \in D$, write $(\alpha_i(u), \beta_i(u))_{i \in \mathcal{Q}_u}$ for the connected components of the complement in $(0, \sigma(W^u))$ of supp $\mathrm{d}L^*(\rho^u, W^u)$, with the convention that if $L^*_\sigma(\rho^u, W^u) = 0$ we let $\mathcal{Q}_u := \{0\}$ and therefore $(\alpha_0(u), \beta_0(u)) := (0, \sigma(W^u))$. We set $\mathcal{X} := \{(\alpha_i(u), \beta_i(u)) : u \in D,\, i \in \mathcal{Q}_u\}$. Next, for every $u \in D$ and $t \geqslant 0$, we introduce the time change

$$\sigma_u(t) := \int_0^t \mathrm{d}s \mathbb{1}_{\{\widehat{\Lambda}_s = \ell_u\}} \quad \text{as well as} \quad \sigma_u^{-1}(t) := \inf\{s \geqslant 0 : \sigma_u(s) > t\}.$$

We write $\sigma_u^{-1}(t-)$ for the left limit of σ_u^{-1} at t. Roughly speaking, $\sigma_u(t)$ measures the amount of time spent by $(p_H(s) : s \geqslant 0)$ in C_u up to time t. Next, consider the mapping q on $\mathcal{X}$ defined for every $(\alpha_i(u), \beta_i(u)) \in \mathcal{X}$ by the relation:

$$\mathrm{q}(\alpha_i(u), \beta_i(u)) := (\sigma_u^{-1}(\alpha_i(u)), \sigma_u^{-1}(\beta_i(u)-)).$$

To simplify notation the interval on the right-hand side is denoted by $(g(u,i), d(u,i))$. Finally, denote the family of connected components of $(\text{supp } \mathrm{d}A)^c$ by $\mathcal{X}'$.

Lemma 6.46. *Under $\mathbb{N}_{x,0}$ and $\mathbb{P}_{0,x,0}$, the mapping $\mathrm{q} : \mathcal{X} \to \mathcal{X}'$ is a bijection between the sets $\mathcal{X}, \mathcal{X}'$. The bijection q is characterised by the following property: for every fixed $u \in D, i \in \mathcal{Q}_u$, we have*

$$\big(\theta_{\ell_x(W_{g(u,i)})}(\rho, W, \widehat{\Lambda})_{(t+g(u,i))\wedge d(u,i)} : t \geqslant 0\big) = \big((\rho^u, W^u, \ell_u)_{(t+\alpha_i(u))\wedge \beta_i(u)} : t \geqslant 0\big). \tag{6.90}$$

Note that (6.90) in particular ensures both that $d(u,i) - g(u,i) = \alpha_i(u) - \beta_i(u)$, and that for $t \in (g(u,i), d(u,i))$, the mapping σ_u is linearly increasing while every other $\sigma_{u'}$ with $u' \neq u$ it remains constant.

Proof. Recall from (6.51) that the support of $\mathrm{d}A$ is precisely the complement of the constancy intervals of $\widehat{\Lambda}$. We start arguing under $\mathbb{N}^{\bullet}_{x,0}$. First, since $\widehat{W}_U \neq x$ a.e., we can find a unique $(g_*, d_*) \in \mathcal{X}'$ such that $U \in (g_*, d_*)$. Namely, $g_* := \sup\{s \leqslant U : \widehat{\Lambda}_s \neq \widehat{\Lambda}_U\}$, $d_* := \inf\{s > U : \widehat{\Lambda}_s \neq \widehat{\Lambda}_U\}$. On the other hand, let $u_0 := u(U)$ be the unique excursion such that $p_H(U) \in C_{u_0}$. With the notations of Proposition 6.20, note that $\sigma_{u_0}(U) = \sum_{k \in \mathcal{K}'_{r_0}} \sigma(\mathrm{Tr}_*(\rho^k, W^k))$ as well as the identity

$$(\theta_{\ell_x(W_U)}(\rho_U, \eta_U, W_U), \mathcal{P}^{\ell,0}_U \circ \mathrm{Tr}_*^{-1}, \mathcal{P}^{r,0}_U \circ \mathrm{Tr}_*^{-1}) = \mathtt{Sp}(\rho^{u_0}, W^{u_0})_{\sigma_{u_0}(U)}. \tag{6.91}$$

Further, there exist a unique connected component in the complement of supp $\mathrm{d}L^*(\rho^{u_0}, W^{u_0})$, that we denote by $(\alpha_*(u_0), \beta_*(u_0)) \in \mathcal{X}$, such that we have $\sigma_{u_0}(U) \in (\alpha_*(u_0), \beta_*(u_0))$. Namely, $\alpha_*(u_0) := \sup\{s \leqslant \sigma_{u_0}(U) : s \in \text{supp } \mathrm{d}L^*\}$, $\beta_*(u_0) := \inf\{t \geqslant \sigma_{u_0}(U) : L^*_t(\rho^{u_0}, W^{u_0}) > L^*_{\sigma_{u_0}(U)}(\rho^{u_0}, W^{u_0})\}$. Let us now prove that $\mathrm{q}(\alpha_*(u_0), \beta_*(u_0)) = (g_*, d_*)$. In this direction, with the notations introduced in Section 6.5.2 and at the end of Section 6.7, we consider the process

$$V_t(\mathcal{P}^{r,0}_U) := \sum_{k \in \mathcal{K}_{r_0}} L^{E_*}_{t\wedge\beta_k - t\wedge\alpha_k}(\rho^k, \overline{W}^k), \quad t \geqslant 0, \quad \text{and} \quad \Xi^*(\mathcal{P}^{r,0}_U) = \sum_{k \in \mathcal{K}_{r_0}, j \in \mathcal{K}_{r_0,k}} \delta_{(V^{(r)}_{\alpha_{k,j}}, \rho^{k,j}, \eta^{k,j}, \overline{W}^{k,j})}.$$

Recall from Proposition 6.20 that conditionally on $(\widehat{\Lambda}_U, V_\infty(\mathcal{P}_U^{r,0}))$, the measure $\Xi^*(\mathcal{P}_U^{r,0})$ is a Poisson measure with intensity $1_{[0,V_\infty(\mathcal{P}_U^{r,0})]}(r)\mathrm{d}r\mathbb{N}_{x,0}(\mathrm{d}\rho,\mathrm{d}\eta,\mathrm{d}\overline{W})$. By the relation (6.83), for $t \geqslant 0$ we can write $V_t(\mathcal{P}_U^{r,0}) = L^*_{\sigma_{u_0}(U+t)}(\rho^{u_0}, W^{u_0}) - L^*_{\sigma_{u_0}(U)}(\rho^{u_0}, W^{u_0})$ and we distinguish two cases. Suppose first that $V_\infty(\mathcal{P}_U^{\ell,0}) > 0$, and for $t \geqslant 0$ we set

$$\widehat{\Lambda}_t(\,\Xi^*_{r,0}) := \sum_{k\in\mathcal{K}_{r_0}, j\in\mathcal{K}_{r_0,k}} \widehat{\Lambda}^{k,j}_{t\wedge\beta_{k,j}-t\wedge\alpha_{k,j}} 1_{t\in(\alpha_{k,j},\beta_{k,j})}.$$

As a straight consequence of the special Markov property and the fact that under $\mathbb{N}_{x,0}$, $\widehat{\Lambda}_t > 0$ for every $t \in (0,\sigma)$, we get that $\inf\{t \geqslant 0 : \widehat{\Lambda}_t(\Xi^*_{r,0}) > 0\}$ and $\inf\{t \geqslant 0 : V_t(\mathcal{P}_U^{r,0}) > 0\}$ coincide. This implies that d_* and $\inf\{t \geqslant 0 : L^*_{\sigma_{u_0}(t)}(\rho^{u_0}, W^{u_0}) > L^*_{\sigma_{u_0}(U)}(\rho^{u_0}, W^{u_0})\}$ coincide, and we deduce that

$$\sigma_{u_0}^{-1}(\beta_*(u_0)-) = \inf\{t \geqslant 0 : \sigma_{u_0}(t) = \beta_*(u_0)\} = \inf\{t \geqslant 0 : L^*_{\sigma_{u_0}(t)} > L^*_{\sigma_{u_0}(U)}(\rho^{u_0}, W^{u_0})\} = d_*.$$

We stress that to derive the second equality we used the special Markov property. Suppose now that $V_\infty(\mathcal{P}_U^{r,0}) = 0$. On the one hand, this yields that $\beta_*(u_0) = \sigma(W^{u_0})$, while on the other hand arguing as before we deduce that $\widehat{\Lambda}$ is constant on $(U, d(u_0))$. Noting that $\inf_{[d(u_0),d(u_0)+\varepsilon]} \widehat{\Lambda} < \widehat{\Lambda}_{d(u_0)}$ for every $\varepsilon > 0$, this proves both that $\sigma_{u_0}^{-1}(\beta_*(u_0)-) = d(u_0)$ and that the later coincides with d_*.

From our previous reasoning and the relationship between the respective spines (6.91) we infer as well that $\big(\theta_{\ell_x(W_U)}(\rho, W, \widehat{\Lambda})_{(t+U)\wedge d_*} : t \geqslant 0\big) = \big((\rho^{u_0}, W^{u_0}, \ell_{u_0})_{(t+\sigma_{u_0}(U))\wedge\beta_*(u_0)} : t \geqslant 0\big)$. An analogous inspection of the left spine gives that $\sigma_{u_0}^{-1}(\alpha_*(u_0)) = g_*$ as well as the identity

$$(\theta_{\ell_x(W_U)}(\rho, W, \widehat{\Lambda})_{(t+g_*)\wedge d_*} : t \geqslant 0) = ((\rho^{u_0}, W^{u_0}, \ell_{u_0})_{(t+\alpha_*(u_0))\wedge\beta_*(u_0)} : t \geqslant 0).$$

By making use of Lemma 6.43 it plenty follows from our reasoning that q is a bijection between the sets $\mathcal{X}$, $\mathcal{X}'$ and that it fulfils property (6.90). $\square$

We are now in position to prove the crucial relation between the family of local times at the branching points of the subordinate tree $\mathcal{T}_{\widetilde{H}}$ - in the sense of Lemma 6.2 - and the family $(L^*(\rho^u, W^u) : u \in D)$. This will conclude the proof of Theorem 6.40 and will be crucial as well for establishing Theorem 6.42.

Proposition 6.47. *For every $u \in D_+$ and with a slight abuse of notation, we write $\widetilde{\lambda}^{\ell,u}, \widetilde{\lambda}^{r,u}$ for the local times at the branching point $p_{\widetilde{H}} \circ \widetilde{g}(u)$ in $\mathcal{T}_{\widetilde{H}}$. Then, $\mathbb{N}_{x,0}$-a.e. for every $u \in D_+$ we have*

$$\widetilde{\lambda}^{\ell,u}_{A_t} = L^*_{\sigma_u(t)}(\rho^u, W^u), \quad \widetilde{\lambda}^{r,u}_{A_t} = L^*_{\sigma}(\rho^u, W^u) - L^*_{\sigma_u(t)}(\rho^u, W^u) \tag{6.92}$$

and in particular $\Delta\widetilde{X}_{A_{g(u)}} = L^*_\sigma(\rho^u, W^u)$.

Proof. We shall make use of the notations and results of Proposition 6.20 and in this direction we start arguing under $E_0 \otimes \mathcal{N}^*$. Recall that conditionally on the triplet $(J_\sigma, \check{J}_\sigma, \xi)$, the measures $\Xi^*(\mathcal{M}^\ell), \Xi^*(\mathcal{M}^r)$ are independent Poisson measures with respective intensities given by $1_{[0,V_\infty(\mathcal{M}^\ell)]}(r)\mathrm{d}r\mathbb{N}_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W})$ and $1_{[0,V_\infty(\mathcal{M}^r)]}(r)\mathrm{d}r\mathbb{N}_{x,0}(\mathrm{d}\rho,\mathrm{d}\overline{W})$. By the exact same argument employed in the proof of Lemma 6.2, using the fact that $\mathbb{N}_{x,0}(\sup_t \widehat{\Lambda}_t > \varepsilon) = \widetilde{N}(\sup_t \widetilde{H}_t > \varepsilon)$ by

(6.53), we have

$$V_\infty(\mathcal{M}^\ell) = \lim_{\varepsilon\downarrow 0} \frac{\#\{\Lambda^{k,j} \in \Xi^*(\mathcal{M}^\ell) : \sup_t \widehat{\Lambda}^{k,j}_t > \varepsilon\}}{\mathbb{N}_{x,0}(\sup_t \widehat{\Lambda}_t > \varepsilon)}, \tag{6.93}$$
$$V_\infty(\mathcal{M}^r) = \lim_{\varepsilon\downarrow 0} \frac{\#\{\Lambda^{k,j} \in \Xi^*(\mathcal{M}^r) : \sup_t \widehat{\Lambda}^{k,j}_t > \varepsilon\}}{\mathbb{N}_{x,0}(\sup_t \widehat{\Lambda}_t > \varepsilon)}$$

the convergences holding point-wise $E_0 \otimes \mathcal{N}^*$–a.e. On the other hand, by Lemma 6.2 and with the notations of Lemma 6.2, we have the following a.e. convergence under $\mathbb{N}^\bullet_{x,0}$

$$\widetilde{\lambda}^{\ell,u}_{A_U} = \lim_{\varepsilon\to 0} \frac{\#\{\widetilde{H}^i : a'_i < A_U,\, \sup_t \widetilde{H}^i_t > \varepsilon\}}{\widetilde{N}(\sup_t \widetilde{H}_t > \varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{\#\{\Lambda^{i,k} \in \Xi^*(\mathcal{P}^{\ell,i}_U) : \sup_t \widehat{\Lambda}^{i,k}_t > \varepsilon\}}{\mathbb{N}_{x,0}(\sup_t \widehat{\Lambda}_t > \varepsilon)} \tag{6.94}$$

with an analogous result holding for $\widetilde{\lambda}^{r,u}_{A_U}$. Since the measure $\mathcal{O}_U$ is a Poisson measure with intensity $1_{[0,\widehat{\Lambda}_U]}(r)\mathrm{d}rE^0 \otimes \mathcal{N}_*((J_\sigma, \check{J}_\sigma, \xi, \mathcal{M}^\ell, \mathcal{M}^r) \in \mathrm{d}z)$, this proves that $\mathbb{N}^\bullet_{x,0}$ a.e., for every $\ell_u \in \mathcal{J}(\Lambda_U)$ and if we let $(r_i, \rho^i_U, \eta^i_U, W^i_U, \mathcal{P}^{\ell,i}_U, \mathcal{P}^{r,i}_U)$ be the unique atom of $\mathcal{O}_U$ such that $r_i = \ell_u$, we have the identities $\widetilde{\lambda}^{\ell,u}_{A_U} = V_\infty(\mathcal{P}^{\ell,i}_U)$, $\widetilde{\lambda}^{r,u}_{A_U} = V_\infty(\mathcal{P}^{r,i}_U)$. By making use of a similar reasoning for the tip $(\rho^0_U, \eta^0_U, W^0_U, \mathcal{P}^{\ell,0}_U, \mathcal{P}^{r,0}_U)$ these identities hold as well for $i = 0$, $\ell_u = \widehat{\Lambda}_U$ as soon as $u \in D_+$. Since $V_\infty(\mathcal{P}^{\ell,i}_U) = L^*_{\sigma_{u_0}(U)}(\rho^u, W^u)$, $V_\infty(\mathcal{P}^{r,i}_U) = L^*_\sigma(\rho^u, W^u) - L^*_{\sigma_u(U)}(\rho^u, W^u)$ it follows that $\mathbb{N}_{x,0}$ a.e. for a dense set of times t in $(0,\sigma)$ identity (6.92) holds for every $u \in D_+$. The statement of the lemma now follows by continuity and recalling that for every $t \geqslant 0$, we have $\widetilde{\lambda}^{\ell,u}_t + \widetilde{\lambda}^{r,u}_t = \Delta\widetilde{X}_{A_{g(u)}}$. □

The previous proposition concludes the proof of Theorem 6.40.

6.8.3 Reconstruction of the tree-like path $(H, \widehat{W})$

This last section is devoted to proving Theorem 6.42. Recall that to achieve this it suffices to show that $(H, \widehat{W})$ can be recovered from the excursion process $\mathcal{E}$. In this direction, with the notations of Lemma 6.46, for every $u \in D, i \in \mathcal{Q}_u$ we let $H^{u,i} := (H^u_{(\alpha_i(u)+t)\wedge\beta_i(u)} : t \geqslant 0)$, $\widehat{W}^{u,i} := (\widehat{W}_{(\alpha_i(u)+t)\wedge\beta_i(u)} : t \geqslant 0)$. Recall that $\{(g(u,i), d(u,i)) : u \in D, i \in \mathcal{Q}_u\}$ are the connected components of $(\mathrm{supp}\ \mathrm{d}A)^c$ as well as the relation (6.90). Now, we consider the measure

$$\mathcal{E}' := \sum_{u\in D, i\in\mathcal{Q}_u} \delta_{(A_{g(u,i)}, H^{u,i}, \widehat{W}^{u,i})}.$$

In Proposition 6.49 below we shall provide a reconstruction of the pair $(H, \widehat{W})$ written in terms of $\mathcal{E}'$ and $\widetilde{H}$. Therefore, the following lemma is the last steeping stone towards establishing Theorem 6.42.

Lemma 6.48. *The measure $\mathcal{E}'$ can be constructed from $\mathcal{E}$.*

Proof. Since the family $(A_{g(u)}, H^{u,i}, \widehat{W}^{u,i})_{u\in D, i\in\mathcal{Q}_u}$ is $\mathcal{E}$-measurable, it remains to show that the same still holds if we replace $A_{g(u)}$ by $A_{g(u,i)}$. Now, this will follow as soon as we show that, for every $u \in D$ and $i \in \mathcal{Q}_u$ it holds that

$$A_{g(u,i)} = \begin{cases} \inf\{a \geqslant 0 : \widetilde{\lambda}^{\ell,u}_a = L^*_{\alpha_i(u)}(\rho^u, W^u)\} & \text{if } L^*_{\alpha_i(u)}(\rho^u, W^u) > 0, \\ A_{g(u)} & \text{if } L^*_{\alpha_i(u)}(\rho^u, W^u) = 0. \end{cases} \tag{6.95}$$

Let us be more precise: the family $(A_{g(u)}, \widetilde{\lambda}^{\ell,u})_{u\in D_+}$ is a function of $\widetilde{H}$, which on its turn is a functional of $\mathcal{E}$ by Theorem 6.40. Since the family $(A_{g(u)}, L^*_{\alpha_i(u)}(\rho^u, W^u))_{u\in D_+, i\in \mathcal{Q}_u}$ can also be obtained from $\mathcal{E}$, this would prove that $A_{g(u,i)}$ is $\mathcal{E}$-measurable. Fix $u \in D$, $i \in \mathcal{Q}_u$ and let us assume first that $L^*_{\alpha_i(u)}(\rho^u, W^u) = 0$. Then, it must hold that $\alpha_i(u) = 0$ and by definition of q we get $g(u,i) = \mathrm{q}(\alpha_i(u)) = g(u)$. This proves the second equality in (6.95). On the other hand, if $L^*_{\alpha_i(u)}(\rho^u, W^u) > 0$ the variable $A_{g(u,i)}$ is a time of left increase for $\widetilde{\lambda}^{\ell,u}$ - see argument below - and therefore, we have that

$$A_{g(u,i)} = \inf\{a \geqslant 0 : \widetilde{\lambda}^{\ell,u}_a = \widetilde{\lambda}^{\ell,u}_{A_{g(u,i)}}\}.$$

Now, by Proposition 6.92 and definition of the bijection q we have the equalities:

$$\widetilde{\lambda}^{\ell,u}_{A_{g(u,i)}} = L^*_{\sigma_u(g(u,i))}(\rho^u, W^u) = L^*_{\alpha_i(u)}(\rho^u, W^u)$$

proving the first equality in (6.95). This concludes the proof of the corollary and it remains to show that $A_{g(u,i)}$ is a point of left increase for $\widetilde{\lambda}^{\ell,u}$ when $L^*_{\alpha_i(u)}(\rho^u, W^u) > 0$. Arguing by contradiction, suppose that the latter does not hold. Then, since $g(u,i)$ is a point of left increase for A, for some $\varepsilon > 0$ small enough it must hold that

$$L^*_{\sigma_u(g(u,i)-\varepsilon)}(\rho^u, W^u) = \widetilde{\lambda}^{\ell,u}_{A_{g(u,i)-\varepsilon}} = \widetilde{\lambda}^{\ell,u}_{A_{g(u,i)}} = L^*_{\alpha_i(u)}(\rho^u, W^u).$$

Since $\alpha_i(u)$ is a point of left increase for $L^*(\rho^u, W^u)$, the process σ_u must be constant in the interval $(g(u,i)-\varepsilon, g(u,i))$. The hypothesis $L^*_{\sigma_u(g(u,i))}(\rho^u, W^u) > 0$ ensures that $\widehat{\Lambda}$ in such neighbourhood must be strictly larger than ℓ_u, and therefore there must exists an excursion of $\widehat{\Lambda}$ away from ℓ_u ending at $d(u,i)$. In particular, on the event $U \in (g(u,i), d(u,i))$ we can find an event of positive measure at which the measure $\Xi(\mathcal{P}^{\ell,0}_U)$ has an atom with first coordinate equal to $L^*_{\alpha_i(u)}(\rho^u, W^u)$. However, this is impossible since the family $(L^*_{\alpha_i(u)}(\rho^u, W^u) : i \in \mathcal{Q}_u)$ is measurable with respect to (ρ^u, W^u). □

The following proposition concludes the proof of Theorem 6.42.

Lemma 6.49. *The process $(H_t, \widehat{W}_t : t \geqslant 0)$ can be recovered from the excursion process $\mathcal{E}$.*

Proof. By Lemma 6.48 and Corollary 6.41, it would suffice to show that we can express $(H, \widehat{W})$ in terms of the measure $\mathcal{E}'$ and $\widetilde{H}$. While the reconstruction of $\widehat{W}$ from $\mathcal{E}'$ does not present additional difficulties, the one of H still requires of some additional work that we shall now address. In this direction, we argue under $\mathbb{N}^\bullet_{x,0}$. For every $u \in D_+$ we set $T_u := \inf\{t \geqslant 0 : L^*_t(\rho^u, W^u) > \widetilde{\lambda}^{u,\ell}_{A_U}\}1_{\{\widetilde{\lambda}^{u,\ell}_{A_U} > 0\}}$, while if $u \in D \backslash D_+$ we simply let $T_u = 0$. Note that U belongs to some element of $\mathcal{X}'$, say $(g(u_0, i_0), d(u_0, i_0)) \in \mathcal{X}'$, so we can replace A_U by $A_{g(u_0,i_0)}$ in our definition of T_u. The bulk of the proof consists in proving that H_U can be decomposed as follows:

$$H_U = \sum_{u\in D, u\neq u_0} H^u_{T_u} + H^{u_0}_{\sigma_{u_0}(U)} \tag{6.96}$$

with our usual notation $u_0 := u(U)$. First, recalling the law of $(H_U, \overline{W}_U)$ under $\mathbb{N}^\bullet_{x,0}$ from Proposition 6.17, we can write

$$\begin{aligned} H_U &= \inf\{h \geqslant 0 : \Lambda_U(h) = \widehat{\Lambda}_U\} + \theta_{\ell_x(W_U)}(H_U) \\ &= \sum_{r_i < \widehat{\Lambda}_U} (\tau^+_{r_i}(\overline{W}_U) - \tau_{r_i}(\overline{W}_U)) + H^{u_0}_{\sigma_{u_0}(U)} \end{aligned} \tag{6.97}$$

where in the second equality we used $(\mathbf{H_2})$. We next fix an arbitrary $\ell_u \in \mathcal{J}(\Lambda_U)$ and let $(r_i, \rho^i_U, \eta^i_U, W^i_U, \mathcal{P}^{\ell,i}_U, \mathcal{P}^{r,i}_U)$ be the unique atom of $\mathcal{O}_U$ such that $r_i = \ell_u$. Since the characteristic measure of $\mathcal{O}_U$ is precisely $E^0 \otimes \mathcal{N}^*$, Lemma 6.39 ensures that the corresponding debut u belongs to D_+. Further, let us check that the following identities hold:

$$\tau^+_{r_i}(\overline{W}_U) - \tau_{r_i}(\overline{W}_U) = H^u_{\sigma_u(U)} = H^u_{T_u} \tag{6.98}$$

The first equality follows from observing that $\sigma_u(U) = \sum_{k \in \mathcal{K}'_{r_i}} \sigma(\mathrm{Tr}_*(\rho^k, W^k))$ - where the atoms in the right hand side belong to $\mathcal{P}^{\ell,i}_U$ - while the second one is essentially a consequence of Proposition 6.47 and Lemma 6.39. Let us be more precise on this last point: first, by Proposition 6.47 we have $L^*_{\sigma_u(U)}(\rho^u, W^u) = \widetilde{\lambda}^{\ell,i}_{A_U}$ which gives that $T_u \geqslant \sigma_u(U)$. On the other hand, Lemma 6.39 and an application of the special Markov property yields that

$$L^*_{\sigma_u(U)}(\rho^u, W^u) < L^*_{\sigma_u(U)+\varepsilon}(\rho^u, W^u)$$

for every $\varepsilon > 0$. This shows that $T_u = \sigma_u(U)$ and (6.98) follows. Observe that if $\ell_u \notin \mathcal{J}(\Lambda_U) \cup \{\widehat{\Lambda}_U\}$, then $\sigma_u(U)$ is either null or equal to $\sigma(W^u)$. If we further assume that $u \in D_+$, by Proposition 6.47 we infer that $\widetilde{\lambda}^{\ell,u}_{A_U}$ is either null or equal to the total mass $L^*_{\sigma(W^u)}(\rho^u, W^u)$, and therefore $T_u \in \{0, \infty\}$. Noting that if $u \in D \backslash D_+$ by definition we have $T_u = 0$, we infer that in any case it holds that $H^u_{T_u} = 0$. Identity (6.96) now follows from combining this fact with (6.98) and equality (6.97). Our argument under $\mathbb{N}^\bullet_{x,0}$ yields that for a countable dense set of times in $(0, \sigma)$, we have the identity

$$H_t = \sum_{u \in D, u \neq u(t)} H^u_{T_u} + H^{u(t)}_{\sigma_{u(t)}(t)} \tag{6.99}$$

where the process on the right-hand side is continuous on $(g(u,i), d(u,i))$ for every $u \in D, i \in \mathcal{Q}_u$.

Let us now finally describe the reconstruction of the pair $(H, \widehat{W})$ in terms of $(\mathcal{E}', \widetilde{H})$. First, consider the following càglàd process

$$Z(a) := \sum_{A_{g(u,i)} \in \mathcal{E}'} \sigma(W^{u,i}) 1_{\{A_{g(u,i)} < a\}}, \qquad \text{for } a \geqslant 0.$$

We write $Z(a+)$ for its right limit at time a and observe that $\sigma(W^{u,i}) = d(u,i) - g(u,i)$. If we fix an arbitrary $(g(u',i'), d(u',i')) \in \mathcal{X}'$, the fact that $\mathbb{N}_{x,0}$ a.e. the integral $\int_0^\infty \mathrm{d}s 1_{\{\widehat{W}_s = x\}}$ is null ensures that $(Z(A_{g(u',i')}), Z(A_{g(u',i')}+)) = (g(u',i'), d(u',i'))$. Let us then construct $(H_t, \widehat{W}_t)$ for for every $t \in (g(u',i'), d(u',i'))$ in terms of $(\mathcal{E}', \widetilde{H})$. In this direction, we let $s := t - g(u',i')$ and we proceed as follows:

1. Consider the family of local times $(\widetilde{\lambda}^{\ell,u})_{u \in D_+}$ at the branching points of $\widetilde{H}$.
2. For every $u \in D_+$, we set $T_u := \inf\{t \geqslant 0 : L^*_t(\rho^u, W^u) = \widetilde{\lambda}^{\ell,u}_{A_{g(u',i')}}\} 1_{\{\widetilde{\lambda}^{\ell,u}_{A_{g(u',i')}} > 0\}}$ and let $T_u = 0$ if $u \in D \backslash D_+$.
3. Making use of identity (6.99), we can write

$$H_t := \sum_{u \in D, u \neq u'} H^u_{T_u} + H^{u',i'}_s \quad \text{and} \quad \widehat{W}_t = \widehat{W}^{u',i'}_s.$$

This concludes the proof of the proposition. □

6.9 Appendix

In this appendix we include a technical lemma that was excluded from the main discussion for readability purposes.

Lemma 6.50. *For $t \in \mathbb{R}_+$, let $z(t) := \inf\{s \geqslant t : H_s < H_t\}$. Under P and N, a time $t \in \mathbb{R}_+$ is a jump time of X if and only if*

(i) *For every $\delta > 0$, $\inf_{[t-\delta,t]} H < H_t$*

(ii) *For every $\varepsilon > 0$, there exists $u \in (t, t+\varepsilon)$ such that $H_t = H_u$ with $\inf_{[t,t+\varepsilon]} H = H_t$.*

When these properties hold, $z(t)$ coincides with $\inf\{s \geqslant t : X_s = X_{t-}\}$

Note that since H has no constancy intervals, (ii) implies that for every $\varepsilon > 0$, we can also find $u \in (t, t+\varepsilon)$ such that $H_t < H_u$.

Proof. We shall only prove the result under P. The collection of jump-times of X is countable and can be written as a disjoint union of stopping times. We write T for an arbitrary stopping time such that $\Delta X_T > 0$. We shall also write $\mathcal{Y}$ for the collection of times that fulfil both (i) and (ii).

• Let us start by proving that every fixed stopping time T at which $\Delta X_T > 0$ fulfils properties (i) and (ii). Let us start by establishing the former and in this direction we shall make use of the notations introduced in (6.3). First, note that since for every fixed $t > 0$, the process $(X_s : 0 \leqslant s \leqslant t)$ has the same distribution as $(X_t - X_{(t-s)-} : 0 \leqslant s \leqslant t)$, the strong Markov property combined with the fact that 0 is regular for $(0, \infty)$ for X yields T can not be a local infimum. Now, fix an arbitrary rational $t > T$ such that $X_{T-} \leqslant I_{T,t}$. Note that by the definition of H_t as the local time at 0 of the time reversed process $\hat{S}^{(t)} - \hat{X}^{(t)}$ in $[0,t]$, we have that for any $u_1 < u_2 < u_3$ belonging to the set $\{s \in [0,t] : X_{s-} \leqslant I_{s,t}\}$, it holds that $H_{u_1} < H_{u_3}$. Moreover, since T is not a local infimum, for any $\varepsilon > 0$ we can find some $u \in (T-\varepsilon, T]$ belonging to the set $\{s \in [0,T] : X_{s-} \leqslant I_{s,T}\}$. It now follows that both $u, T \in \{s \in [0,t] : X_{s-} \leqslant I_{s,t}\}$ and we deduce that $H_u < H_T$. We now turn our attention to (ii) and in this direction let us start by introducing some notation. For $t \geqslant 0$ we write $X_t^{(T)} := X_{T+t} - X_T$, $I_t^{(T)} = \inf_{[0,t]} X^{(T)}$ and $\rho_t^{(T)}$ for the corresponding exploration process. Further, for $r \in \mathbb{R}_+$ we set $T_r(I) := \inf\{t \geqslant 0 : -I_t \geqslant r\}$. Since $\rho_T(\{H_T\}) = \Delta X_T$, it follows by the strong Markov property of the exploration process (see (1.13) in [43]) that for any $0 \leqslant t \leqslant T_{\Delta X_T}(I^{(T)})$, we have

$$H(\rho_{T+t}) = H(\kappa_{-I_t^{(T)}}\rho_T) + H(\rho_t^{(T)}) = H_T + H(\rho_t^{(T)}).$$

Since $X^{(T)}$ has the same distribution as X, the point 0 is regular and instantaneous for $X^{(T)} - I^{(T)}$, and since a.e. we have $\{t \geqslant 0 : X_t^{(T)} - I_t^{(T)} = 0\} = \{t \geqslant 0 : \rho_t^{(T)} = 0\}$ we infer that (ii) holds. Now, the fact that $z(T)$ coincides with $T + \inf\{r \geqslant 0 : -I_r^{(T)} = \Delta X_T\}$ readily follows again by the strong Markov property.

• Let us now show that every $T \in \mathcal{Y}$ is a jump time for X. To achieve this, it suffices to show that such a time t must be a discontinuity time for ρ – with respect to the total variation distance of measures. Indeed, by (1.12) in [43] we know that the discontinuity times of ρ are precisely of the form $\rho_u = \rho_{u-} + \delta_{H_u}\Delta X_u$, for $u \in \{s \geqslant 0 : \Delta X_s > 0\}$. Now, to show that ρ is discontinuous at time t, it plenty suffices to show that $\rho_{t-}(\{H_t\}) = 0$ while $\rho_t(\{H_t\}) > 0$. Starting with the former, note that by (i), there exists a decreasing sequence of positive numbers $(\delta_n)_{n\geqslant 0}$ with $\delta_n \downarrow 0$ such that $H(\rho_{t-\delta_n}) < H(\rho_t)$. Hence, for every $n \geqslant 0$ it holds that $\rho_{t-\delta_n}(\{H_t\}) = 0$, and it follows by left-continuity that $\rho_{t-}(\{H_t\}) = 0$. Let us now show that $\rho_t(\{H_t\}) > 0$. It readily follow from our definitions that we can construct a non-increasing sequence of stopping times $\tau_n \downarrow t$ satisfying for every $n \geqslant 0$, that $H_t = H_{\tau_n} = \min_{[\tau,\tau+q]} H$ for some $q > 0$ which depends on n. Remark that the condition $\inf_{[\tau_n,\tau_n+q]} H = H_{\tau_n}$ ensures, by the strong Markov property of the exploration process, that $\rho_{\tau_n}(\{H_{\tau_n}\}) > 0$ and consequently $\rho_{\tau_n}(\{H_t\}) > 0$. Moreover, since if s, s' are distinct jump-times it must hold that $H_s \neq H_{s'}$, the sequence $(\rho_{\tau_n}(\{H_t\}) : n \geqslant 0)$ must be non-decreasing and by right-continuity of ρ we get $\rho_t(\{H_t\}) > 0$. This proves that t is a jump-time for ρ and concludes the proof of the lemma. □

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