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Géométrie brownienne

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ÉCOLE DOCTORALE de mathématiques Hadamard (EDMH)

MATHEMATIQUES ORSAY

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Quel est l'objectif de cette thèse ?

Cette thèse s'inscrit dans l'étude de la sphère brownienne en utilisant le mouvement brownien indexé par l'arbre brownien. Certaines variantes de la sphère brownienne sont apparues dans les dernières années, comme le plan brownien – qui est une version infinie de la sphère brownienne – et le disque brownien – qui apparaît comme limite d'échelle des quadrangulations avec frontière. Par analogie avec le mouvement brownien, nous parlerons de *géométrie brownienne*. L'objectif de cette thèse est de combiner différentes approches de cette théorie afin de développer une étude systématique de propriétés métriques de ces modèles telles que des propriétés de Markov spatiales, des formules explicites concernant des objets géométriques ou des propriétés isopérimétriques.

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Cette thèse est elle-même et ses circonstances. Chaque espace blanc doit donc être considéré comme un hommage non dit mais sans lequel ce travail n'aurait jamais pris forme et reposerait dans le vide. Je voudrais simplement finir par remercier ma famille et Lisa pour tout leur soutien et pour élargir toujours la vie elle-même ; j'espère que nous continuerons à rire et à danser porque juntos somos una canción simple.

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PARTIE I

INTRODUCTION

Cette introduction sera articulée en trois chapitres.

Dans le chapitre 1, nous présenterons le contexte des cartes aléatoires ainsi que la célèbre bijection de Cori-Vauquelin-Schaeffer afin de définir l'objet central de la géométrie brownienne: la sphère brownienne. Les chapitres 2 et 3 regrouperont nos contributions principales à cette théorie. Nous insisterons principalement sur les techniques et idées développées et renverrons le·a lecteur·rice intéressé·e aux parties ultérieures pour plus de détails. Mentionnons aussi qu'afin de preserver la cohérence de l'introduction, certaines notations utilisées dans cette partie diffèreront des notations des articles associés.
Le chapitre 2 sera centré sur nos travaux [77, 79, 93] ; ils représentent le cœur de cette thèse et rassemblent nos résultats liés à des propriétés de Markov spatiales en géométrie brownienne.
Finalement le chapitre 3 sera centré sur des formules explicites concernant des quantités géométriques telles que les longueurs de bords ou les volumes en géométrie brownienne. Dans cette partie, nous présenterons notre travail [78] ainsi qu'un chapitre correspondant à un projet en cours, contenant d'autres résultats pouvant être obtenus en utilisant les techniques développées dans nos précédents travaux.

Chapitre 1

Cartes planaires aléatoires et sphère brownienne

Nous introduirons ici les cartes planaires et présenterons la construction de la sphère brownienne, à l'aide du mouvement brownien indexé par l'arbre brownien en imitant la bijection CVS. Nous rappelerons également certaines propriétés géométriques de la sphère brownienne.

1.1 Cartes planaires

Commençons par rappeler qu'un graphe G est la donnée d'un ensemble de sommets $\mathcal{V}(G)$ et d'un ensemble d'arêtes $\mathcal{E}(G)$; chaque arête ayant comme extrémités des points de $\mathcal{V}(G)$. Un graphe est dit planaire s'il existe un plongement propre de celui-ci dans la sphère 2-dimensionelle $\mathbb{S}_2$. Ici par plongement propre d'un graphe G on comprend tout plongement de ses sommets $\mathcal{V}(G)$ et de ses arêtes $\mathcal{E}(G)$ dans $\mathbb{S}_2$ tel que les arêtes ne se croisent pas sauf éventuellement en leurs extrémités. En reprenant une formulation de Gilles Schaeffer, le plongement d'un graphe planaire se "contente d'exister" et, de manière générale, un graphe planaire a plusieurs plongements dans la sphère. Ce fait rend l'étude des graphes planaires compliquée lorsque l'on s'intéresse à leurs propriétés géométriques. Introduisons maintenant les cartes planaires:

Definition 1. *Une carte planaire est la donnée d'un graphe planaire connexe fini enraciné et d'un plongement propre de ce graphe dans la sphère* $\mathbb{S}_2$ *– ce dernier considéré à homéomorphisme de la sphère conservant l'orientation près.*

Le terme *enraciné* signifie simplement que nous avons distingué une arête orientée. Le sommet duquel part cette arête orientée est appelé *sommet racine* de la carte. Il peut être interprété comme un point depuis lequel on observe la carte en question. L'avantage d'utiliser les cartes planaires réside dans le fait que le plongement dans la sphère devient fixe, rendant ces objets plus robustes géométriquement que les graphes planaires. De manière générale nous utiliserons la notation m pour nous référer à une carte planaire. Nous noterons respectivement $\mathcal{V}(m)$ et $\mathcal{E}(m)$ l'ensemble des sommets et d'arêtes de m. Nous pouvons maintenant aussi définir $\mathcal{F}(m)$, l'ensemble des faces de m, comme étant l'ensemble des composantes connexes de $\mathbb{S}_2$ après avoir retiré les sommets et arêtes de m. Le degré d'une face est alors défini comme le nombre de demi-arêtes adjacentes à celle-ci. Il est également possible d'interpréter ce nombre comme le périmètre de la face en question. Nous distinguons la face à gauche de l'arête orientée; cette face est appelée la *face racine*. Un bon exemple de la robustesse des cartes planaires citée précédemment est la célèbre formule d'Euler qui permet de relier le cardinal des ensembles $\mathcal{V}(m)$, $\mathcal{E}(m)$ et $\mathcal{F}(m)$ par la relation:

$$\#\mathcal{V}(m) - \#\mathcal{E}(m) + \#\mathcal{F}(m) = 2. \tag{1.1}$$

De manière peut-être plus imagée, une carte aléatoire peut aussi être définie comme un ensemble fini de polygones recollés le long de leurs arêtes de manière à obtenir topologiquement une sphère (l'un des polygones ayant une arête orientée distinguée). Cette deuxième définition a l'avantage de mettre en évidence le fait que l'ensemble des cartes planaires à n faces est un ensemble fini. Grâce à cette remarque, on voit que les cartes planaires sont des objets adaptés aux probabilités discrètes.

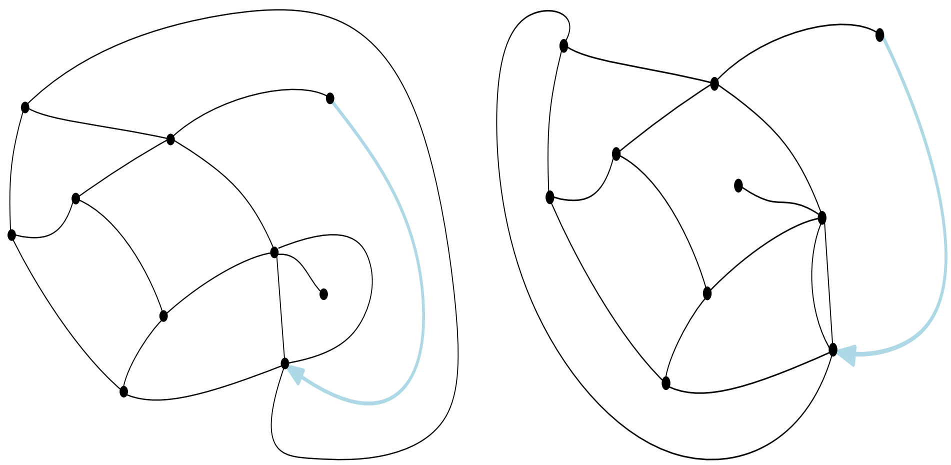


Figure 1.1 – Ces deux objets encodent le même graphe planaire mais sont différents en tant que cartes planaires. La figure de gauche est une quadrangulation.

Modèle de cartes planaires

Un modèle de cartes planaires est un sous-ensemble des cartes planaires vérifiant certaines contraintes combinatoires ou géométriques. On dira par exemple qu'une carte planaire est une p-angulation si toutes ses faces sont de degré p. Pour $p = 3$ et $p = 4$ on parlera respectivement de triangulations et de quadrangulations, voir figure 1.1 pour un exemple de quadrangulations. Remarquons que si m est une p-angulation alors on a $\#\mathcal{E}(m) = p/2 \cdot \#\mathcal{F}(m)$ puisque chaque arête est à l'interface de deux faces. Par conséquent, la formule d'Euler (1.1) nous permet aussi d'obtenir $\mathcal{V}(m) = (p/2 - 1)\#\mathcal{F}(m) + 2$. De manière plus conceptuelle et dans le cas des p-angulations, les trois paramètres donnant une notion naturelle de taille $\#\mathcal{V}(m)$, $\#\mathcal{E}(m)$ et $\#\mathcal{F}(m)$ n'ont qu'un seul degré de liberté. Ce petit fait est fondamental lorsque l'on parle d'une grande p-angulation. Les quadrangulations sont parmi les modèles les plus populaires et étudiés, du fait notamment de la bijection de Tutte, qui relie de manière bijective les cartes planaires aux quadrangulations comme suit:

Bijection de Tutte. Soit m une carte planaire. Dans chaque face f de m, on rajoute un sommet que l'on relie par une arête à chaque coin de f de telle manière que les arêtes ne se croisent pas. Puis en effaçant les arêtes de m on obtient une quadrangulation (on distingue l'arête orientée partant du sommet racine et pointant vers le sommet rajouté dans la face racine de m). Le nombre de faces de la quadrangulation obtenue est $\#\mathcal{E}(m)$.

De nombreuses propriétés des cartes générales peuvent alors être étudiées par l'intermédiaire des quadrangulations. Une autre bonne raison de considérer des quadrangulations est que ces objets peuvent être comptés très précisément. Notons Q_n l'ensemble des quadrangulations à n faces. On a:

$$\#Q_n = \frac{2}{n+2} 3^n \frac{1}{n+1} \binom{2n}{n}. \tag{1.2}$$

Nous renvoyons au census de Tutte [97] où l'on retrouve cette formule et bien d'autre résultats d'énumération de cartes.

Pourquoi étudier des cartes planaires aléatoires ?

Les cartes planaires comme modèles de géométrie aléatoire

Les cartes planaires sont des objets apparaissant dans une multiplicité de problématiques différentes. Une des motivations pour les étudier est, d'une part, que l'espace de toutes les cartes planaires est un espace dénombrable et donc sur lequel il est simple de définir des mesures de probabilité et, d'autre part, que les cartes planaires peuvent être interprétées comme des espaces métriques compacts. Expliquons ce deuxième point plus précisément. Soit m une carte, alors pour chaque couple de points $(u, v) \in \mathcal{V}(m)^2$, on peut définir sa distance de graphe $d_m(u, v)$ comme le nombre minimal d'arêtes à emprunter pour aller de u à v. L'espace $(\mathcal{V}(m), d_m)$ est un espace métrique compact et pour simplifier nous utiliserons parfois la notation (m, d_m) pour celui-ci. Les cartes planaires sont donc des modèles discrets de géométrie planaire nous permettant désormais de considérer des modèles de géométrie aléatoire. Une manière naturelle de faire cela est de fixer un modèle $\mathcal{M}$ de cartes puis, pour tout entier $n \geqslant 1$, de définir une carte uniforme m_n dans l'ensemble des cartes à n faces dans $\mathcal{M}$ (pourvu que cet ensemble soit fini et non vide). Les physiciens considèrent de tels modèles comme des modèles discrets de gravité quantique en dimension 2 et comme un moyen de donner sens à un analogue des intégrales de chemin de Feynman portant sur des surfaces (au lieu de l'intégrale de chemin plus "classique" portant sur des trajectoires).

La sphère brownienne

Un des aspects les plus remarquables de la théorie des cartes aléatoires est le fait que pour une grande variété de modèles, la suite m_n admet une limite d'échelle universelle appelée la sphère brownienne, un espace continu ayant la topologie de la sphère et qui peut être construit en utilisant le mouvement brownien indexé par l'arbre brownien. Nous expliquerons plus précisément ce que nous entendons par limite d'échelle (voir paragraphes 1.2 et 1.3). Mais de manière informelle le·a lecteur·rice pourra rapprocher naturellement ce résultat de la convergence des marches aléatoires sur $\mathbb{Z}$ (avec variance finie) vers le mouvement brownien. Parmi les modèles admettant la sphère brownienne comme limite d'échelle se trouvent les cartes uniformes, les triangulations et toutes p-angulations. Ce résultat constitue l'un des principaux accomplissements de la théorie et fut obtenu par Jean-François Le Gall [67] dans le cas des triangulations et des p-angulations pour p pair et de manière indépendante par Grégory Miermont [85] dans le cas des quadrangulations. Le cas des cartes uniformes sera traité postérieurement dans [21] et la convergence vers la carte brownienne des p-angulations pour p impair a été établi très récemment dans [6] par Louigi Addario-Berry et Marie Albenque. En fait, pourvu que le modèle considéré ne fasse pas intervenir de grandes faces, la sphère brownienne devrait apparaître comme limite d'échelle. Ceci peut être rendu rigoureux pour un grand nombre de modèles (voir [1, 4, 67, 82, 85] pour plus de détails). Par contre, dans le cas où le modèle présenterait de grandes faces nous sortirions du domaine d'universalité de la sphère brownienne. Nous renvoyons à [75] pour le cas "stable" qui reste encore très mystérieux. Les preuves de Jean-François Le Gall et de Grégory Miermont reposent toutes deux sur des bijections entre des cartes et des arbres planaires étiquetés. Cette étude permet de construire la sphère brownienne grâce au mouvement brownien indexé par l'arbre brownien.

Gravité quantique de Liouville

Il existe aussi une approche directement dans le continu pour aborder la sphère brownienne en utilisant le champ libre gaussien. Cette construction conçoit la sphère brownienne comme un objet muni d'une structure conforme, appartenant à la gravité quantique de Liouville. La gravité quantique de Liouville est une famille à un paramètre $\gamma \in (0, 2)$, et la sphère brownienne est apparentée avec le paramètre special $\gamma = \sqrt{8/3}$, ce dernier étant associé au modèle de gravité dit pure. Elle fut définie en toute rigueur par François David, Anti Kupiainen, Rémi Rhodes et Vincent Vargas [44]. Associer une métrique à cette structure conforme a constitué l'un des grands problèmes de la théorie et a été résolu récemment dans [47, 48, 54, 57] en utilisant une procédure de régularisation. Malgré cela, le fait que la gravité quantique de Liouville pour $\gamma = \sqrt{8/3}$ et la sphère brownienne soient bien le même objet est un résultat extrêmement délicat et profond. Jason Miller et Scott Sheffield ont fait le rapprochement de ces deux constructions dans une série d'articles [86, 87, 88, 89] mais ce lien demeure assez mal compris. L'approche de Jason Miller et de Scott Sheffield est de parvenir à encoder les boules métriques à l'aide d'une exploration markovienne du GFF: la *Quantum Loewner evolution* ou QLE. Le QLE commençant en un point z du plan complexe peut être approché par un processus SLE_6 commençant en z qu'on laisse évoluer une courte durée de temps δ (encodé par la $\sqrt{8/3}$–gravité quantique de Liouville) puis en réechantillonnant le point de départ du SLE_6 au hasard sur la zone découverte. Le QLE apparaît alors lorsque l'on considère la limite $\delta \to 0$. Le lien entre cette métrique et celle obtenu par des procedures de régularisation est rendu possible par une caractérisation axiomatique de ces deux distances établit dans [54].

Dans la suite de ce chapitre, nous décrirons le lien entre les quadrangulations et les arbres étiquetés. Nous définirons ensuite l'analogue continu des arbres étiquetés : le mouvement brownien indexé par l'arbre brownien. Nous donnerons aussi la construction de la sphère brownienne à partir de cet objet. Une fois ce contexte établi nous présenterons, dans les deux chapitres suivants, nos contributions à cette théorie.

Bijection de Cori-Vauquelin-Schaeffer

Les nombres de Catalan $\frac{1}{n+1}\binom{2n}{n}$ apparaissent de manière naturelle dans de nombreux problèmes de comptage; en particulier $\frac{1}{n+1}\binom{2n}{n}$ désigne le nombre d'arbres planaires à n arêtes. De ce fait, la formule (1.2) suggère qu'il existe une liaison dangereuse entre quadrangulations et arbres planaires. Trouver une telle relation a été le sujet d'une grande activité de recherche, en particulier puisque les arbres planaires sont beaucoup plus simples à étudier. Robert Cori et Bernard Vauquelin donneront la première interprétation dans ce sens en montrant que les quadrangulations peuvent être encodées par une classe d'arbres planaires étiquetés (voir [37]). Gilles Schaeffer popularisera et approfondira ces bijections en donnant surtout une interprétation métrique des étiquettes [94].

Le but de cette section est de présenter la bijection de Cori-Vauquelin-Schaeffer ou bijection CVS. Un arbre planaire est une carte planaire à une seule face ou, de manière équivalente, sans cycles. Nous enrichissons cette structure avec des étiquettes et appelons *coin d'un arbre* n'importe quel secteur angulaire entre deux arêtes adjacentes. Un étiquetage d'un arbre planaire $\mathcal{T}$ est une fonction $\ell : \mathcal{V}(\mathcal{T}) \mapsto \mathbb{Z}$. Nous dirons qu'un arbre planaire étiqueté $(\mathcal{T}, \ell)$ est *bien étiqueté* si la fonction ℓ vérifie les propriétés suivantes:

- L'étiquette de la racine est 0 ;
- Pour tout $u, v \in \mathcal{T}$ voisins, on a $\ell(u) - \ell(v) \in \{-1, 0, 1\}$.

Fixons $\boldsymbol{\mathcal{T}} = (\mathcal{T}, \ell)$, un arbre bien étiqueté et un entier $\varepsilon \in \{-1, 1\}$. Nous pouvons alors construire une carte planaire $\mathcal{S}(\boldsymbol{\mathcal{T}}, \varepsilon)$ de la manière suivante. Commençons par introduire un nouveau sommet v_* ne touchant pas $\mathcal{T}$. Ensuite, pour chaque coin c avec étiquette i, dessinons une arête reliant ce coin avec le premier coin d'étiquette $i - 1$ rencontré en suivant le contour de $\mathcal{T}$ (dans le sens des aiguilles d'une montre) à partir du coin c. S'il n'y a pas de tel coin, relions c avec v_*. Ces nouvelles arêtes peuvent être dessinées sans croisement. Nous distinguons alors l'arête tracée à partir du coin gauche de l'arête racine de $\mathcal{T}$: si $\varepsilon = 1$, nous l'orientons vers la racine de $\mathcal{T}$, et si $\varepsilon = -1$, nous l'orientons dans le sens inverse. En effaçant les arêtes de l'arbre $\mathcal{T}$ nous obtenons alors une quadrangulation à $\#\mathcal{V}(\boldsymbol{\mathcal{T}})+1$ sommets avec un point marqué v_*. De plus, les étiquettes sur les sommets encodent les distances vers v_* dans la quadrangulation $\mathcal{S}(\boldsymbol{\mathcal{T}}, \varepsilon)$ i.e. on a:

$$\mathrm{d}_{\mathrm{gr}}(v, v_*) = \ell(v) - \min \ell + 1, \tag{1.3}$$

pour tout sommet v de $\boldsymbol{\mathcal{T}}$.

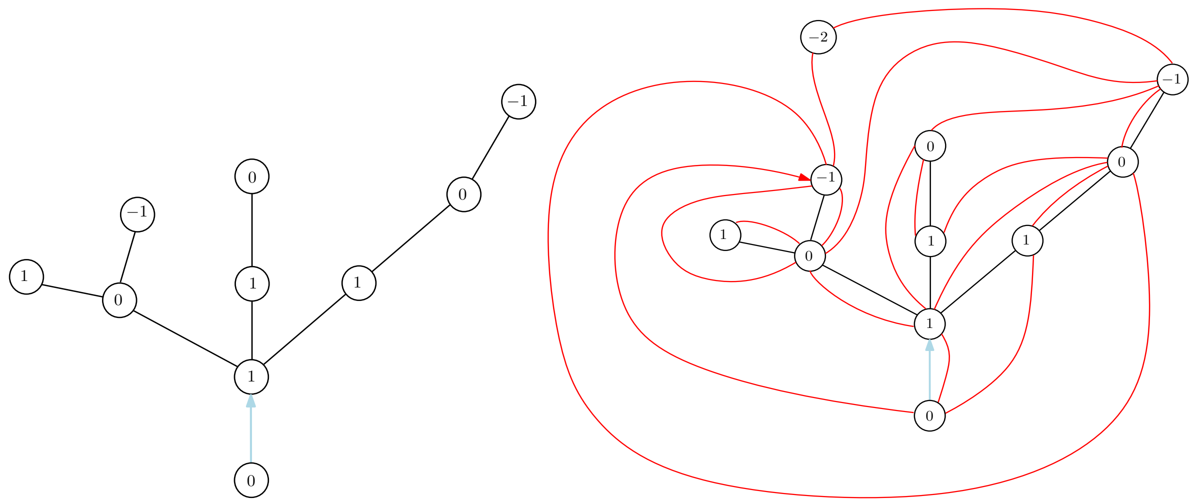


Figure 1.2 – Illustration de la bijection CVS. Ici on a $\varepsilon = 1$

L'application $\mathcal{S}$ définit une bijection entre les paires consistant en un arbre planaire bien étiqueté et un entier dans $\{-1, 1\}$, et les quadrangulations avec un point marqué. En particulier, nous pouvons obtenir une quadrangulation aléatoire uniforme avec $n+1$ sommets en considérant une variable uniforme ε dans $\{-1, 1\}$, un arbre planaire uniforme à n sommets, puis en tirant une variable uniforme dans l'ensemble $\{-1, 0, 1\}$ sur chaque arête de l'arbre. L'étiquette d'un sommet est alors la somme des étiquettes des arêtes sur la branche allant de la racine au sommet en question. Le·a lecteur·rice peut alors retrouver la formule 1.2.

Autres bijections

Depuis les travaux de Robert Cori, Bernard Vauquelin et Schaeffer, d'autres bijections ont été trouvées entre des modèles de cartes et des modèles d'arbres étiquetés. Citons quelques-unes des plus importantes. Jérémie Bouttier, Philippe Di Francesco et Emmanuel Guitter ont établi dans [26] une bijection de même nature pour toutes les cartes planaires sans restrictions. Cette bijection met en jeu des arbres avec quatre types de sommets mais elle prend une forme plus simple lorsque toutes les faces sont de degré pair. Grégory Miermont a établi une autre bijection entre les quadrangulations avec k points distingués et retards et les cartes planaires étiquetées à k faces [84]. Dans cette bijection les étiquettes permettent d'encoder des "cellules de Voronoï" par rapport aux k points distingués. Mentionnons finalement que Guillaume Chapuy, Michel Marcus et Gilles Schaeffer ont élargi la bijection CVS pour des topologies de genre supérieur [34].

1.2 Le mouvement brownien indexé par l'arbre brownien

Introduisons désormais l'analogue continu des arbres bien étiquetés. Dans ce travail nous considérerons des limites d'échelle d'espaces métriques compacts avec un point distingué. Pour ce faire il est nécessaire d'introduire une bonne notion de distance entre de tels espaces. Nous utiliserons ici la distance de Gromov-Hausdorff.

Distance de Gromov-Hausdorff

Un espace métrique compact pointé (E, d, x) est la donnée d'un espace métrique compact (E, d) et d'un point x sur E. Le point distingué x est aussi interprété comme la racine de E. On appelle isométrie entre deux espaces métriques compacts pointés (E, d, x) et (E', d', x') toute isométrie $\phi : (E, d) \mapsto (E', d')$ vérifiant $\phi(x) = x'$, et on note $\mathbb{K}$ l'ensemble des espaces métriques compacts pointés à isométrie près. Afin de munir l'ensemble $\mathbb{K}$ d'une structure d'espace métrique nous introduisons maintenant la distance de Gromov-Hausdorff. Commençons par considérer (F, d), un espace métrique, et E, E' deux sous-ensembles compacts de F. La distance de Hausdorff entre E et E' est la quantité:

$$d_{\mathrm{H}}^F(E, E') := \inf\{\varepsilon > 0 : \; E \subset E'^{,\varepsilon} \text{ et } E' \subset E^{\varepsilon}\},$$

où pour un sous-ensemble A de F, nous notons A^{ε} le ε-voisinage de A dans F. Nous pouvons désormais définir la distance de Gromov-Hausdorff comme suit:

Definition 2. *Soient $\mathbf{E} := (E, d, x)$ et $\mathbf{E}' = (E', d', x')$ deux espaces métriques compacts pointés. La distance de Gromov-Hausdorff entre $\mathbf{E}$ et $\mathbf{E}'$ est la quantité:*

$$d_{GH}\big(\mathbf{E}, \mathbf{E}'\big) := \inf \Big(d_H^F(\phi(E), \phi'(E')) \vee \delta\big(\phi(x), \phi'(x')\big) \Big),$$

où l'infimum est pris sur l'ensemble des espaces métriques (F, δ) et des plongements isométriques $\phi : E \to F$ et $\phi' : E' \to F$.

Une des propriétés intéressantes de la distance de Gromov-Hausdorff est qu'on a $d_{\mathrm{GH}}\big(\mathbf{E}, \mathbf{E}'\big) = 0$ si et seulement si il existe une isométrie entre $\mathbf{E}$ et $\mathbf{E}'$. L'espace $(\mathbb{K}, d_{\mathrm{GH}})$ est donc un espace métrique et il peut être démontré qu'il est même polonais. C'est donc un bon espace sur lequel définir des variables aléatoires. Nous avons aussi besoin d'élargir la distance de Gromov-Hausdorff au cas non-compact et pour cela nous restreignons notre attention aux espaces géodésiques localement compacts et complets. Pour fixer les notations, posons $\mathbb{K}_\infty$ l'ensemble des espaces géodésiques localement compacts, complets et pointés à isométrie près. Pour tout $r > 0$ et $\mathbf{E} := (E, d, x)$ élément de $\mathbb{K}_\infty$, notons $B_r(\mathbf{E}) := \{y \in E : \; d(x, y) \leqslant r\}$. Munissons l'ensemble $B_r(\mathbf{E})$ de la restriction de la distance d et pointons-le en x. Comme E est localement compact et complet, l'ensemble $B_r(\mathbf{E})$ est un ensemble compact et $B_r(\mathbf{E})$ est donc un élément de $\mathbb{K}$. Nous pouvons maintenant équiper $\mathbb{K}_\infty$ de la distance:

$$d_{\mathrm{GH}}^{\mathrm{loc}}\big(\mathbf{E}, \mathbf{E}'\big) = \sum_{n=1}^{\infty} 2^{-n} \wedge d_{\mathrm{GH}}\big(B_n(\mathbf{E}), B_n(\mathbf{E}')\big).$$

Le mouvement brownien indexé par l'arbre brownien

Introduisons maintenant formellement l'analogue continu des arbres bien étiquetés: le mouvement brownien indexé par l'arbre brownien. Commençons par la structure d'arbre. Soit $\mathcal{T}_n$ une variable aléatoire uniforme dans l'ensemble des arbres planaires à n sommets. Notons respectivement ρ_n et $d_{\mathcal{T}_n}$ sa racine et sa distance de graphe. L'espace $(\mathcal{T}_n, d_{\mathcal{T}_n}, \rho_n)$ peut alors être conçu comme un élément de $\mathbb{K}$. La distance typique entre deux points de $\mathcal{T}_n$ est d'ordre $\sqrt{n}$ et de manière suprenante l'espace $(\mathcal{T}_n, n^{-\frac{1}{2}} \cdot d_{\mathcal{T}_n}, \rho_n)$ converge en distribution vers un objet continu nommé arbre brownien. L'arbre brownien est aussi connu sous le nom d'arbre d'Aldous, du nom de David Aldous qui l'introduisit et l'étudia [8, 9]. Donnons une construction de cet objet à l'aide de l'excursion brownienne. Soit $\mathbf{e}$ une excursion brownienne de durée de vie 1. Nous pouvons alors utiliser l'excursion brownienne pour introduire la pseudo-distance:

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \min_{[s \wedge t, s \vee t]} \mathbf{e}.$$

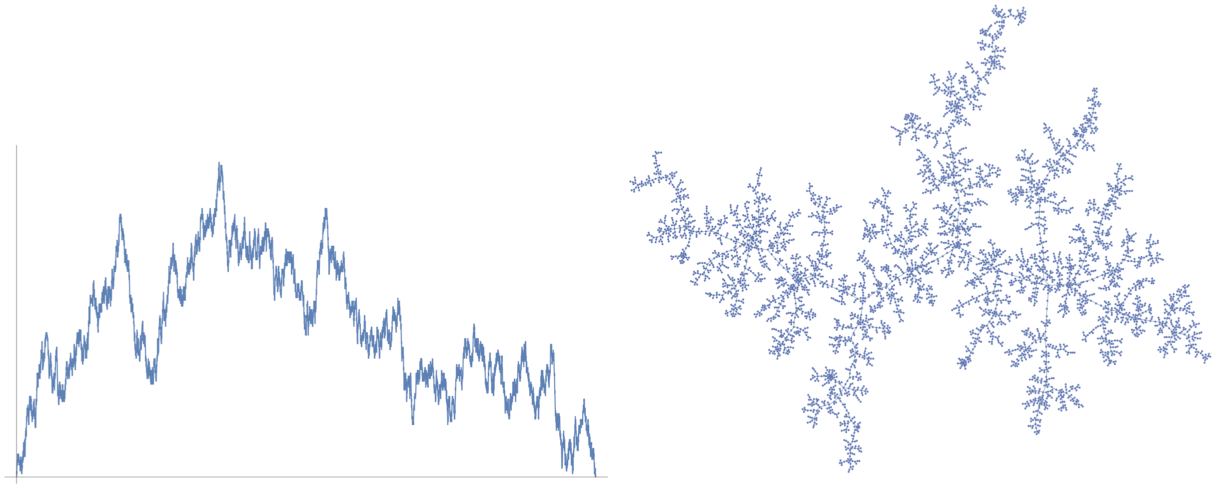


Figure 1.3 – La figure de gauche est une approximation d'une excursion brownienne obtenue en utilisant une marche aléatoire. La figure de droite représente l'approximation de l'arbre brownien en utilisant l'excursion de gauche.

En notant $s \sim_{d_{\mathbf{e}}} t$ lorsque $d_{\mathbf{e}}(s,t) = 0$, on obtient une relation d'équivalence sur $[0,1]$. De manière imagée, deux points s et t sont équivalents pour $\sim_{d_{\mathbf{e}}}$ si $e_s = e_t$ et si le segment connectant $(s, \mathbf{e}_s)$ et $(t, \mathbf{e}_t)$ est entièrement sous le graphe de $\mathbf{e}$ (sauf évidemment à ses extrémités). Pour simplifier les notations, nous notons 0 la classe d'équivalence de $0 \in \mathbb{R}$. L'espace $([0,1]/\sim_{d_{\mathbf{e}}}, d_{\mathbf{e}}, 0)$ est l'arbre brownien; il jouera le rôle d'analogue continu de la structure d'arbre apparaissant dans la bijection CVS. L'arbre brownien peut-être obtenu comme limite d'arbre uniforme, mais aussi comme limite d'arbre de Galton-Watson plus génériques. Ces convergences peuvent être étudier à l'aide de certaines propriétés de branchement Markovienne [58]. Intéressons-nous maintenant aux étiquettes. Dans le monde discret, les étiquettes le long d'une branche décrivent une marche aléatoire à incréments uniformes dans $\{-1, 0, 1\}$. De plus lorsqu'une branche fourche pour donner deux branches, la marche aléatoire se divise en deux marches indépendantes. Dans le monde continu on peut imiter cette dynamique en remplaçant les marches aléatoires par des mouvements browniens. Une manière de le faire est de remarquer que la fonction $(s,t) \mapsto \min_{[s\wedge t, s\vee t]} \mathbf{e}$ est symétrique et définie positive. On peut alors introduire $(\Lambda_s)_{s\in[0,1]}$ un processus gaussien vérifiant $\Lambda_0 = 0$ et avec fonction de covariance $(s,t) \mapsto \min_{[s\wedge t, s\vee t]} \mathbf{e}$. La construction même de ce processus entraîne que si $s \sim_{d_\mathbf{e}} t$ alors $\Lambda_s = \Lambda_t$, ce qui permet de factoriser le processus Λ par rapport à la relation d'équivalence $\sim_{d_\mathbf{e}}$. Nous obtenons ainsi un analogue continu des étiquettes. Cet objet est aussi connu sous le nom de mouvement brownien indexé par l'arbre brownien. Mentionnons qu'il est aussi possible de définir cet objet continu à l'aide d'un processus Markovien nommé *le serpent brownien*. Nous n'aurons pas besoin du serpent brownien dans cette introduction pour présenter nos résultats et nous ne l'introduisons pas ici pour ne pas alourdir la présentation. Mais il jouera un rôle fondamental dans les preuves de nos résultats. Au début de chaque article nous donnerons les préliminaires du serpent brownien nécessaires pour l'article en question. Pour le·a lecteur·rice intéressé·e nous renvoyons à [50, 65] pour plus d'informations concernant le serpent brownien.

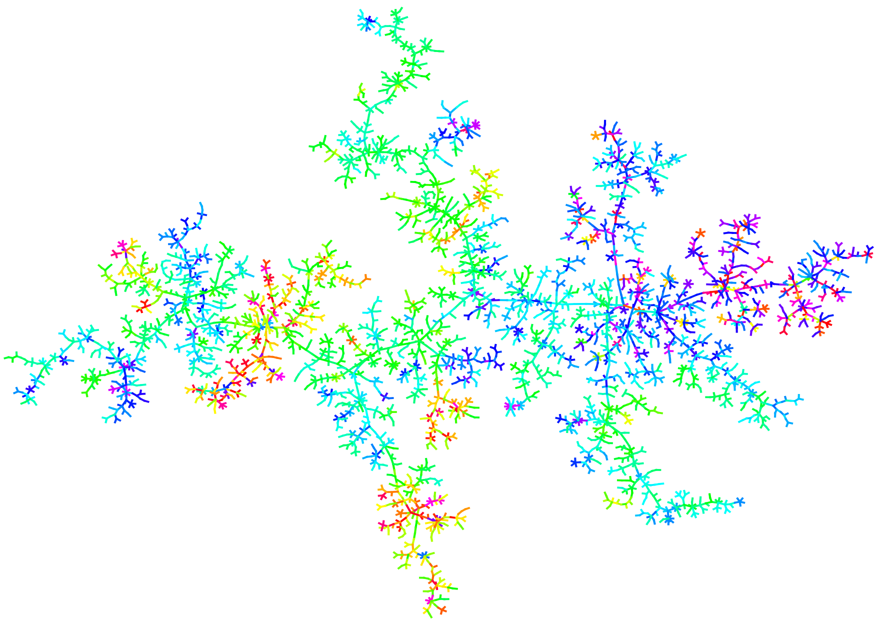


Figure 1.4 – Approximation du mouvement brownien indexé par l'arbre brownien (ici nous utilisons l'arbre de la figure 1.3). Les couleurs représentent les étiquettes: les couleurs jaune et verte correspondent à des étiquettes négatives alors que les couleurs bleue et violette correspondent à des étiquettes positives. La couleur rouge correspond aux valeurs extrêmes.

1.3 Construction de la sphère brownienne

Nous allons construire la sphère brownienne à l'aide du mouvement brownien indexé par l'arbre brownien. Pour cela nous utilisons les mêmes notations que dans la section précédente. Pour tout $s, t \in [0,1]$, introduisons la quantité:

$$D^\circ(s,t) := \Lambda_s + \Lambda_t - 2\max\big(\min_{[s,t]}\Lambda, \min_{[t,s]}\Lambda\big) \tag{1.4}$$

où l'on note $[u,v] = [0,v] \cup [u,1]$ si $v \leqslant u$. Définissons ensuite pour tout $s, t \in [0,1]$:

$$D^*(s,t) := \inf_{s=s_1,t_1\ldots,s_n,t_n=t} \sum_{i=1}^{n} D^\circ(s_i,t_i) \tag{1.5}$$

où l'infimum porte sur tous les entiers $n \geqslant 1$ et sur toutes les suites $s_1,\ldots,s_n$ telles que $(s_1,t_n) = (s,t)$ et $t_i \sim_{d_{\mathbf{e}}} s_{i+1}$ pour tout $1 \leqslant i \leqslant n-1$. La fonction D^* est une pseudo-distance sur $[0,1]$ et, comme dans le cas de l'arbre brownien, nous pouvons introduire la relation d'équivalence $\sim_{D^*}$ en notant $s \sim_{D^*} t$ si et seulement si $D^*(s,t) = 0$. En particulier, il est important de remarquer que si $s \sim_{d_{\mathbf{e}}} t$ alors nous avons aussi $s \sim_{D^*} t$. Il peut d'ailleurs être montré en utilisant (1.5) que D^* se factorise à travers la relation d'équivalence $\sim_{d_{\mathbf{e}}}$ et nous pouvons donc directement interpréter D^* comme une pseudo-distance sur l'arbre brownien. C'est cette interprétation sur l'arbre brownien qui est souvent présentée. Il peut être montré qu'il existe un unique t_*, tel que $\Lambda_{t_*} = \min_{[0,1]}\Lambda$, voir [76]. Nous notons ρ_* la classe d'équivalence pour $\sim_{D^*}$ de t_*. L'espace $([0,1]/\sim_{D^*}, D^*, \rho^*)$ est connu sous le nom de sphère brownienne. Il est aussi courant de pointer la sphère brownienne sur la classe d'équivalence de 0, mais ces deux constructions sont équivalentes. Nous verrons plus tard pourquoi nous préférons faire ce choix de racine. Comme expliqué précédemment, Jean-François Le Gall et Grégory Miermont ont montré indépendamment que la sphère brownienne apparaît comme limite d'échelle de modèles de cartes. Plus précisément si (m_n, d_{m_n}) est une quadrangulation uniforme à n faces, et ρ_{m_n} désigne la racine de m_n, alors la distance typique dans m_n est d'ordre $n^{\frac{1}{4}}$ et on a la convergence en loi suivante:

$$(m_n, n^{-\frac{1}{4}} d_{m_n}, \rho_{m_n}) \xrightarrow{(d)} ([0,1]/\sim_{D^*}, c\cdot D^*, \rho_*) \tag{1.6}$$

où c est simplement une constante positive. Le même résultat est vrai dans le cas des triangulations ou des p-angulations pourvu qu'on change la constante c.

Quelques propriétés géométriques

La sphère brownienne peut donc être définie comme le segment $[0,1]$ quotienté par une relation d'équivalence (fermée). Comprendre la topologie de la sphère brownienne revient alors, par des arguments de compacité, à caractériser les points d'annulation de D^*. Obtenir une telle caractérisation est le but de [66], où il est prouvé que:

$$D^*(s,t) = 0 \quad \Longleftrightarrow \quad \begin{cases} D^\circ(s,t) = 0 \\ \text{ou} \\ d_{\mathbf{e}}(s,t) = 0. \end{cases} \tag{1.7}$$

En utilisant (1.7), Jean-François Le Gall et Frédéric Paulin ont montré dans [76] que la sphère brownienne est presque sûrement homéomorphe à $\mathbb{S}_2$ (ils ont aussi établi que les deux conditions $D^\circ(s,t) = 0$ et $d_{\mathbf{e}}(s,t) = 0$ sont mutuellement exclusives dès lors que $s \neq t$). Mais la sphère brownienne est aussi un objet fractal et sa dimension de Hausdorff est presque sûrement égale à 4, voir [66]. Souvenons-nous maintenant que dans la bijection CVS les étiquettes encodent les distances au point d'étiquette minimale, voir (1.3). On montre alors facilement que pour tout $t \in [0,1]$ nous avons $D^*(t_*, t) = \Lambda_t - \Lambda_{t_*}$. Cette relation montre de plus que les étiquettes Λ peuvent aussi être factorisées par rapport à $\sim_{D^*}$ et donc définies sur la sphère brownienne. Les étiquettes représentent, à une translation près, les distances à ρ^*. C'est pour cette raison que nous avons choisi de pointer la sphère brownienne en ρ^*. Mentionnons aussi que la sphère brownienne est munie d'une mesure volume qui peut être définie comme la mesure image de la mesure de Lebesgue sur $[0,1]$ par la projection canonique associée à $\sim_{D^*}$. Cette mesure joue le rôle d'analogue continu de la mesure uniforme sur les sommets d'une carte aléatoire. Notons d'ailleurs que le volume total de la sphère brownienne est 1.

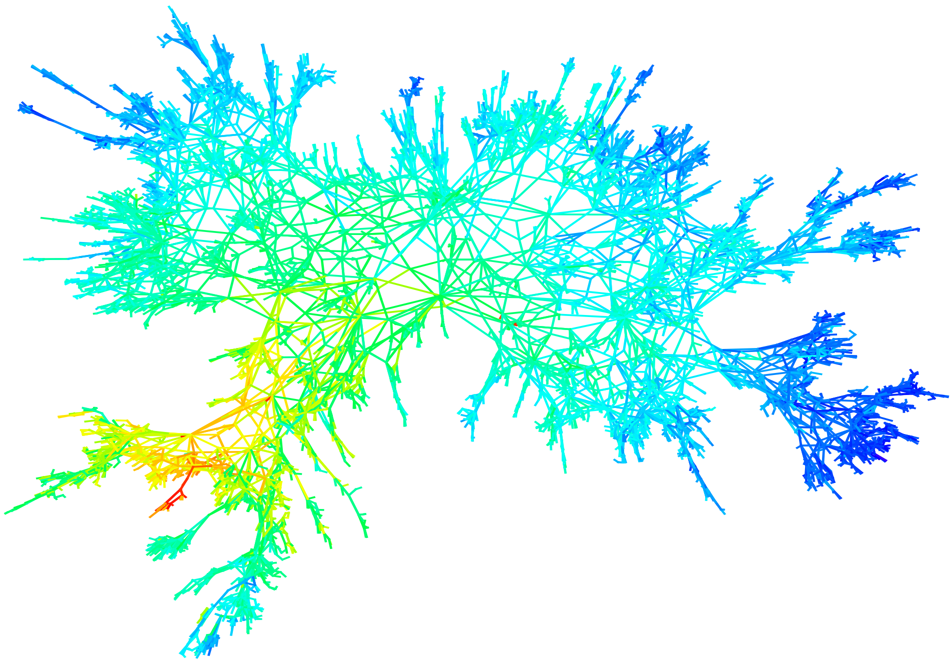


Figure 1.5 – Approximation de la sphère brownienne. Les couleurs représentent les distances au point racine, ce dernier est colorié en rouge.

Autres modèles de géométrie brownienne

Les dernières années ont vu apparaître plusieurs variantes de la sphère brownienne comme limite d'échelle de modèles discrets. Faisons un très rapide récapitulatif des modèles de géométrie brownienne qui apparaîtront dans cette thèse. Rappelons d'abord que le facteur de changement d'échelle $n^{-\frac{1}{4}}$ dans (1.6) correspond au fait que la distance typique dans une quadrangulation à n faces est d'ordre $n^{\frac{1}{4}}$. Si nous choisissons un facteur tendant vers 0 plus lentement que $n^{-\frac{1}{4}}$, nous sortons du domaine d'universalité de la sphère brownienne. Il y a encore une limite d'échelle mais qui est cette fois un espace géodésique aléatoire localement compact et complet appelé le plan brownien [39, 40]. D'autre part, nous pouvons aussi nous intéresser à des quadrangulations avec bord [7, 14, 20, 22, 55, 83]. Une quadrangulation avec bord est une carte planaire dont toutes les faces sont de degré 4 sauf éventuellement la face racine. Le bord est alors défini comme les arêtes adjacentes à cette face distinguée. Dans ce cas la limite d'échelle de ces objets dépend de la longueur du bord et du nombre total de faces. Plus précisément, si la longueur du bord est d'ordre n et le nombre de faces d'ordre n^2, nous pouvons diviser les distances dans la carte par le facteur d'échelle $n^{-\frac{1}{2}}$ pour trouver une nouvelle famille d'espaces compacts limite appelés disques browniens. Comme leurs noms le laissent deviner, ces objets ont la topologie du disque fermé du plan complexe. Le bord d'un disque brownien est alors défini comme l'ensemble de points sans voisinage homéomorphe au disque ouvert. De plus le disque brownien est équipé d'une mesure volume. La famille des disques browniens est une famille à deux paramètres (z, v), où z correspond à une notion de "périmètre" du bord et v correspond au volume total. Mentionnons aussi que les disques browniens peuvent être obtenus comme des sous-ensembles spéciaux de la sphère brownienne [71]. Par contre, si le nombre de faces est d'ordre supérieur à n^2, on obtient une famille d'espaces géodésiques localement compacts et complets appelées disques browniens de volume infini. Ces objets sont munis aussi d'une mesure volume mais le volume de chacun d'entre eux est infini. Leurs topologie est aussi déterminée, ils sont homéomorphes au complémentaire du disque unité ouvert dans le plan complexe, et en particulier nous pouvons définir leurs bords. Les disques browniens de volume infini sont une famille à un paramètre correspondant au périmètre du bord. Finalement, si le facteur de changement d'échelle tend vers 0 plus lentement que $n^{-\frac{1}{2}}$, nous obtenons le demi-plan brownien qui est un espace géodésique localement compact et complet ayant la topologie du demi-plan. Il faut noter que lorsque l'espace limite n'est pas compact les convergences ont lieu dans $\mathbb{K}_\infty$. Nous renvoyons enfin à [14] où Erich Baur, Grégory Miermont et Gourab Ray étudient toutes les limites d'échelle possibles de quadrangulations à bord.

Changement d'échelle

Soit (E, D, ρ_E) un élément de $\mathbb{K}$ ou $\mathbb{K}_\infty$ muni d'une mesure volume V_E. Pour tout $\lambda > 0$, nous notons $\lambda \cdot E$ l'espace $(E, \lambda D, \rho_E)$ muni de la mesure volume $\lambda^4 V_E$. Nous voyons cette opération comme une opération de changement d'échelle. Les modèles de géométrie brownienne sont stables par rapport à cette opération. Plus précisément, les lois du plan brownien et du demi-plan brownien sont invariantes par rapport à cette opération. Si E est un disque brownien de périmètre z et volume v, alors $\lambda \cdot E$ est un disque brownien de périmètre $\lambda^2 z$ et volume $\lambda^4 v$. Similairement, si E est un disque brownien de volume infini de périmètre z alors $\lambda \cdot E$ est un disque brownien de volume infini de périmètre $\lambda^2 z$. Enfin, si E est une sphère brownienne alors l'espace $\lambda \cdot E$ est appelé sphère brownienne de volume λ^4. Ce dernier espace peut être construit exactement comme la sphère brownienne en utilisant le mouvement brownien indexé par l'arbre brownien mais en prenant une excursion brownienne de durée de vie λ^4.

Modèles libres de géométrie brownienne

Pour des raisons techniques, lorsque l'on travaille avec des sphères browniennes ou des disques browniens, il est commode de considérer des versions de ces modèles où l'on ne fixe pas le volume. Ces modèles portent le nom de sphère brownienne et disque brownien libres et ils apparaissent aussi naturellement comme limite d'échelle de modèles de cartes aléatoires dont la taille n'est pas fixée (voir [22, 71] et section 2.1 ci-dessous pour un exemple). Concluons en donnant une définition de ces modèles. Une sphère brownienne libre est une sphère brownienne avec un volume distribué selon $(2\sqrt{2\pi})^{-1}v^{-3/2}\,\mathrm{d}v$, pour $v > 0$, et un disque brownien libre de périmètre z est un disque brownien de périmètre z dont le volume est distribué selon $z^3(2\pi v^5)^{-1/2}\exp(-z^2/2v)\,\mathrm{d}v$, pour $v > 0$.

La sphère brownienne libre peut aussi être obtenue de la manière suivante. On commence par introduire l'excursion brownienne $\mathbf{e}$ selon la mesure d'Itô $\mathbf{n}$ (avec normalisation $\mathbf{n}(\sup \mathbf{e} > 1) = 1/2$) et on note σ sa durée de vie. Nous pouvons alors utiliser cette excursion pour définir un arbre brownien (mais de volume σ) et lui associer des étiquettes Λ en adaptant la construction de la section 1.2. Notons $\mathbb{N}_0$ la loi du couple $(\mathbf{e}, \Lambda)$. Si on adapte la construction de la section 1.3 sous $\mathbb{N}_0$, nous obtenons la sphère brownienne libre. Nous concluons ce chapitre avec une remarque. Un disque brownien à périmètre et volume fixés est pointé sur un point de son intérieur.

Plusieurs nomenclatures sont possibles dans le cas du disque brownien libre. Ici nous suivons [22] et oublions le point distingué. Le disque brownien libre est donc un espace métrique compact non-pointé.

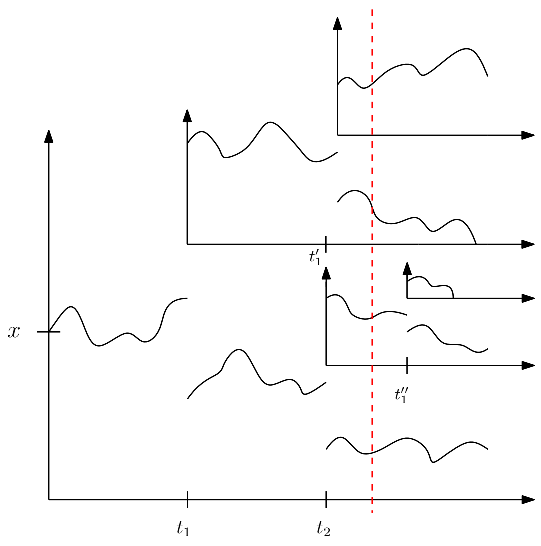
Chapitre 2

Propriétés de Markov spatiales

Nous présenterons ici nos résultats [77, 78, 93]. Nous commencerons par une introduction informelle aux processus de croissance-fragmentation. Nous donnerons aussi le lien entre ces processus et les triangulations à bords simples établis dans [19] de façon à motiver notre travail. Ceci permettra de donner une intuition dans le monde discret qui se rélèvera utile dans l'interprétation de certains résultats.

2.1 Processus de croissance-fragmentation et triangulations à bords simples

Soit X un processus de Markov prenant des valeurs positives. On suppose aussi que $X_t \to 0$ lorsque $t \to \infty$ et que tous les sauts de X sont négatifs. Le processus de croissance-fragmentation induit par X commençant en $x > 0$ peut être imagé comme suit :



Commençons avec une particule de masse initiale x et dont la masse évolue suivant la dynamique de X. Cette particule est appelée la particule Eve. Lorsque la masse de cette particule a un saut de taille $-\Delta$, une nouvelle particule, de masse initiale Δ, est produite. La masse de la nouvelle particule évolue alors de manière indépendante suivant la dynamique de X et commençant en Δ. Nous raisonnons alors par induction : chaque nouvelle particule a des enfants à ses instants de saut et nous réitérons la même construction (voir figure à gauche).

Pour tout $t \geqslant 0$, posons:

$$\mathbb{X}(t) := \text{Collection des masses à l'instant } t \text{ dans l'ordre décroissant.}$$

Le processus $\mathbb{X}$ est le processus de croissance-fragmentation induit par X (commençant en x). Il est important de souligner que la loi de $\mathbb{X}$ ne caractérise pas la dynamique du processus de Markov X. En d'autres mots, deux processus de Markov peuvent induire le même processus de croissance-fragmentation [95]. Nous mentionnons aussi qu'il est possible de définir des processus de croissance-fragmentation dans un cadre plus large, voir [17]. En particulier les cartes duales stables sont reliées à des processus de croissance-fragmentation induits par des processus de Markov avec sauts positifs et négatifs [18].

Nos objets d'étude ont des propriétés d'autosimilarité. C'est pour cette raison que nous sommes particulièrement intéressés par le cas où X est un processus de Markov autosimilaire. Dans ce cas il existe un nombre $\alpha \in \mathbb{R}$ appelé l'indice d'autosimilarité de X et un unique processus de Lévy ξ commençant en 0 tel que:

$$(X_t)_{t\geqslant 0} \stackrel{(d)}{=} \left(x\exp(\xi_{\gamma(tx^{-\alpha})})\right)_{t\geqslant 0}$$

où $\gamma(t) := \inf\{r \geqslant 0 : \int_0^r \exp(\alpha\xi_s)ds \geqslant t\}$. En particulier, un processus de Markov autosimilaire est caractérisé par son indice d'autosimilarité α et par l'exposant de Laplace de ξ que l'on note ψ (il est bien défini puisque ξ ne peut pas avoir de saut positifs). Pour simplifier les notations, nous dirons que X est un processus de Markov (α, ψ)-autosimilaire et de manière analogue que $\mathbb{X}$ est un processus de croissance-fragmentation (α, ψ)-autosimilaire. Expliquons maintenant, en suivant [19], le lien entre des processus de croissance-fragmentation et des modèles de cartes.

Triangulations du p-gone

Une triangulation du p-gone est une carte planaire telle que toutes ses faces sont des triangles à l'exception de la face racine qui peut avoir un périmètre arbitraire mais dont le bord est simple.

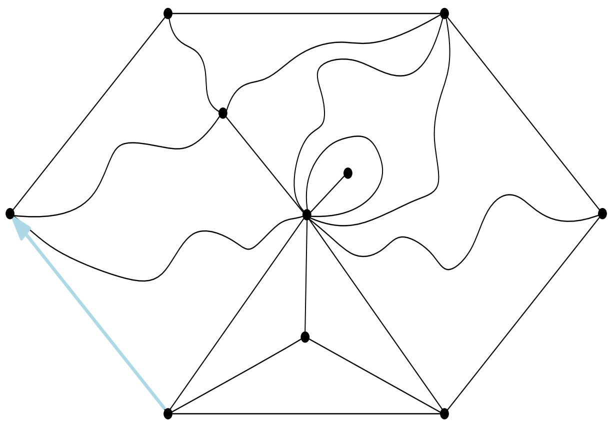


Figure 2.1 – Une triangulation du 6-gone.

Si on pose $\mathcal{T}_n^{(p)}$ l'ensemble des triangulations du p-gone à n sommets, on a $\#\mathcal{T}_n^{(p)} \underset{n\to\infty}{\sim} C_p \cdot n^{-\frac{5}{2}}(12\sqrt{3})^n$, où C_p est simplement une constante dépendant de p. Introduisons la mesure de probabilité sur l'espace des triangulations du p-gone définie par:

$$\forall m \in \bigcup_{n\geqslant 0} \mathcal{T}_n^{(p)},\ P_p(m) := \frac{(12\sqrt{3})^{-\#\mathcal{V}(m)}}{Z_p}$$

où Z_p est la constante de normalisation (elle est bien finie grâce au terme $n^{\frac{5}{2}}$). Cette mesure de probabilité est connue sous le nom de mesure de Boltzmann sur les triangulations du p-gone. Une triangulation aléatoire de loi P_p est naturellement appelée triangulation de Boltzmann (du p-gone). Nous nous intéressons ici à des explorations métriques. Pour cela, il sera très utile de concevoir une triangulation du p-gone à l'aide de sa représentation en cactus, c'est-à-dire en représentant chaque sommet par sa distance au bord (voir Figure 2.2).

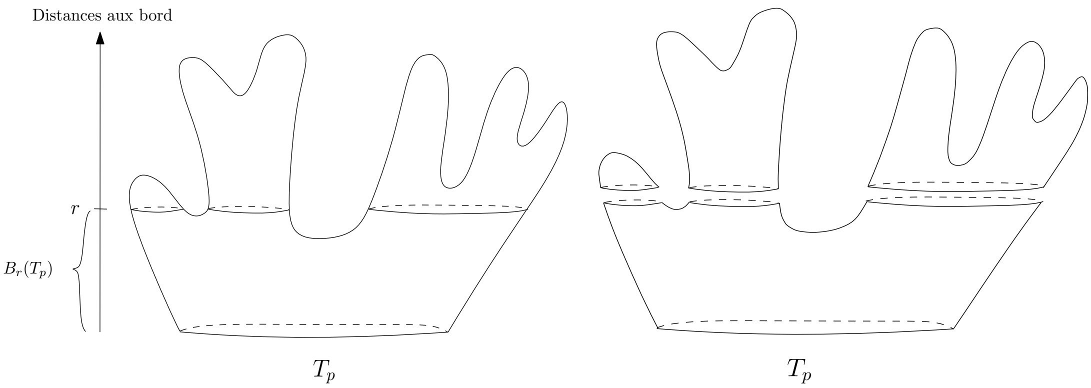


Figure 2.2 – représentation en Cactus.

Soit M_p une triangulation de Boltzmann de périmètre p et notons $B_r(M_p)$ l'ensemble des sommets à distance inférieure ou égale à r. En enlevant l'ensemble $B_r(M_p)$, on obtient une suite de composantes connexes $(C_i(r))_{i\geqslant 1}$. Evidemment il n'y a qu'un nombre fini de telles composantes, mais il est possible de compléter cette suite par un point cimetière $\varnothing$. Chacune de ces composantes connexes a un bord simple $\partial C_i(r)$, dont nous notons $|\partial C_i(r)|$ la longueur. Pour des raisons de cohérence pour chaque $C_i(r)$ nous distinguons une arête orientée sur $\partial C_i(r)$ par un algorithme déterministe et nous écrivons $\mathrm{Cl}(C_i(r))$ pour la carte reposant sur le cycle $\partial C_i(r)$. La carte $\mathrm{Cl}(C_i(r))$ peut être interprétée comme "l'adhérence" de $C_i(r)$ et remarquons que $\mathrm{Cl}(C_i(r))$ est en particulier une triangulation du $|\partial C_i(r)|$-gone. Par la définition même de la mesure de Boltzmann, et conditionnellement à la suite des périmètres $L^p(r) := (|\partial C_i(r)|)_{i\geqslant 1}$, les cartes $(\mathrm{Cl}(C_i(r)))_{i\geqslant 1}$ sont indépendantes et $\mathrm{Cl}(C_i(r))$ est une triangulation de Boltzmann de périmètre $|\partial C_i(r)|$. Les triangulations du p-gone vérifient alors une belle propriété markovienne lorsque l'on explore les triangulations en suivant les distances au bord.

Un processus de croissance-fragmentation comme limite d'échelle de L^p
Toute la difficulté est alors de comprendre l'évolution de $L^p(r)$ quand r augmente. C'est cette question qui est abordée en [19] où les auteurs montrent la convergence en loi suivante

$$\left(p^{-1}L^p(\sqrt{p}r)\right)_{r\geqslant 0} \xrightarrow[p\to\infty]{(d)} (\mathbb{X}(cr))_{r\geqslant 0} \tag{2.1}$$

où c est une constante positive et $\mathbb{X}$ un processus de croissance-fragmentation $(\frac{1}{2}, \psi)$-autosimilaire commençant en 1 avec

$$\psi(\lambda) := \sqrt{\frac{3}{2\pi}}\Big(-\frac{8}{3}\lambda + \int_{-\log 2}^{0} (e^{\lambda y} - 1 - \lambda(e^y - 1))\, e^{-3y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y\Big).$$

La constante c rend compte du modèle étudié; ce résultat devrait être vrai pour d'autres modèles discrets (modifiant la constante c). D'autre part, les triangulations de Boltzmann du p-gone convergent (lorsqu'on fait un changement d'échelle de $p^{-\frac{1}{2}}$ et qu'on fait tendre p vers l'infini) vers une version du disque brownien libre de périmètre 1 où l'on distingue le bord (voir [7]). Il est donc naturel de se demander s'il est possible de retrouver le processus de croissance-fragmentation $\mathbb{X}$ directement sur le disque brownien libre. C'est l'un des buts de notre travail [77]. Il serait envisageable d'essayer d'obtenir une convergence jointe des triangulations de Boltzmann et de la limite d'échelle (2.1), mais cela serait très délicat et nécessiterait des contrôles très fins sur les longueurs de tous les bords de manière simultanée tout en gardant trace de toute la géométrie de la triangulation de Boltzmann. C'est pour cette raison que dans [77] nous raisonnons exclusivement sur le modèle continu du disque brownien libre.

2.2 Exploration du disque brownien libre depuis son bord

Dans cette section nous présentons les résultats de [77]. Ce travail est le fruit d'une collaboration avec Jean-François Le Gall. Considérons un disque brownien libre de périmètre z que nous notons $(\mathbb{D}_z, \Delta, V)$. Ici Δ dénote la distance dans $\mathbb{D}_z$, et V la mesure volume sur $\mathbb{D}_z$. Rappelons que le disque brownien libre n'est pas pointé. Nous allons essayer de mimer ce qui se passe pour les triangulations du p-gone. Rappelons aussi que le bord du disque brownien libre est l'ensemble des points sans voisinage homéomorphe au disque ouvert que nous notons $\partial\mathbb{D}_z$. Nous pouvons alors considérer la représentation en cactus de $\mathbb{D}_z$ et nous écrivons $B_r(\mathbb{D}_z)$ pour l'ensemble des points à distance inférieure ou égale à r du bord. L'espace $\mathbb{D}_z\backslash B_r(\mathbb{D}_z)$ est composé de plusieurs composantes connexes $(\mathcal{C}_i(r))_{i\geqslant 1}$ et on note $\partial\mathcal{C}_i(r)$ la frontière de $\mathcal{C}_i(r)$. Dans le cas des triangulations, nous utilisions la notion de périmètre pour définir les longueurs des bords. Nous ne pouvons pas étendre directement cette définition au cas du disque brownien libre puisque les frontières $(\partial\mathcal{C}_i(r))_{i\geqslant 1}$ sont des objets fractals (de dimension de Hausdorff égale à 2) et en particulier leur longueur, par rapport à la distance Δ, est infinie. Il convient donc d'utiliser une autre notion pour généraliser le concept de périmètre. Pour cela nous considérons, pour toute composante connexe $\mathcal{C}_i(r)$, la couronne $\mathcal{C}_i(r) \cap B_{r+\varepsilon}(\mathbb{D}_z)$. Cette couronne est constituée des points de $\mathcal{C}_i(r)$ à distance inférieure ou égale à ε de sa frontière et son volume est d'ordre ε^2. Nous montrons le résultat suivant

Proposition 1

Presque sûrement, pour tout $i \geqslant 1$ et $r \geqslant 0$, la limite

$$|\partial\mathcal{C}_i(r)| := \lim_{\varepsilon\to 0} \varepsilon^{-2} V\big(\mathcal{C}_i(r) \cap B_{r+\varepsilon}(\mathbb{D}_z)\big)$$

existe.

Nous appelons la quantité $|\partial\mathcal{C}_i(r)|$ le périmètre (au sens généralisé) de $\partial\mathcal{C}_i(r)$. Des résultats semblables à ceux de la proposition 1 était bien connus pour un r fixe (voir [71]). Toute la difficulté était de montrer qu'on peut considérer ces limites simultanément pour tout $r \geqslant 0$. On peut d'ailleurs retrouver le périmètre de $\mathbb{D}_z$ grâce à la proposition 1 en prenant $i = 1$ et $r = 0$. Nous pouvons maintenant énoncer notre résultat principal dans [77].

Théorème 1

Le processus $\mathbf{L}(r) := \big(|\partial\mathcal{C}_i(r)|\big)_{i\geqslant 1}$ est un processus de croissance-fragmentation d'indice $(\frac{1}{2}, \psi)$ commençant en z. De plus, conditionnellement à $\mathbf{L}(r)$, pour tout $i \geqslant 1$, la distance intrinsèque sur $\mathcal{C}_i(r)$ se prolonge par continuité sur l'adhérence $\mathrm{Cl}(\mathcal{C}_i(r))$. On munit alors $\mathrm{Cl}(\mathcal{C}_i(r))$ de cette distance et de la restriction de V sur $\mathrm{Cl}(\mathcal{C}_i(r))$. Conditionnellement à $\mathbf{L}(r)$, les espaces métriques $\mathrm{Cl}(\mathcal{C}_i(r))$ sont indépendants et $\mathrm{Cl}(\mathcal{C}_i(r))$ est un disque brownien libre de périmètre $|\partial\mathcal{C}_i(r)|$.

Il est important de souligner que l'analogue discret de la deuxième partie du théorème était une conséquence directe de la définition de la mesure de Boltzmann. Cela n'est plus du tout le cas pour le disque brownien. Pour l'obtenir nous utilisons un résultat analogue connu pour la sphère brownienne [71] et un couplage entre la sphère brownienne et le disque brownien. Ce théorème a des conséquences géométriques, par exemple, si on note M la distance maximale d'un point au bord. On obtient pour tout $r > 1$:

$$c_1 r^{-6} \leqslant \mathbb{P}(M > r) \leqslant c_2 r^{-6}$$

où $c_1 < c_2$ sont deux constantes positives.

Illustration de la méthode utilisée pour démontrer le théorème 1

Nous donnons maintenant un aperçu de la méthode utilisée pour obtenir le théorème 1. Cette méthode est assez robuste et a été utilisée pour relier l'excursion brownienne bi-dimensionelle du demi-plan supérieur à un processus de croissance-fragmentation [11].

Soit $r_1 \geqslant 0$ et $\partial\mathcal{C}_{i_1}(r_1)$ un bord au niveau r_1. Nous disons qu'un bord $\partial\mathcal{C}_{i_2}(r_2)$, avec $r_2 \geqslant r_1$, est un descendant de $\partial\mathcal{C}_{i_1}(r_1)$ si tout chemin connectant $\partial\mathcal{C}_{i_2}(r_2)$ et $\partial\mathbb{D}_z$ touche $\partial\mathcal{C}_{i_1}(r_1)$, ou de manière équivalente si $\mathcal{C}_{i_1}(r_1)$ contient $\mathcal{C}_{i_2}(r_2)$. Comme le disque brownien libre a la topologie du disque fermé, cette notion de descendance est bien définie et donne une notion de généalogie pour les bords du disque brownien libre (voir figure 2.3 pour une illustration).

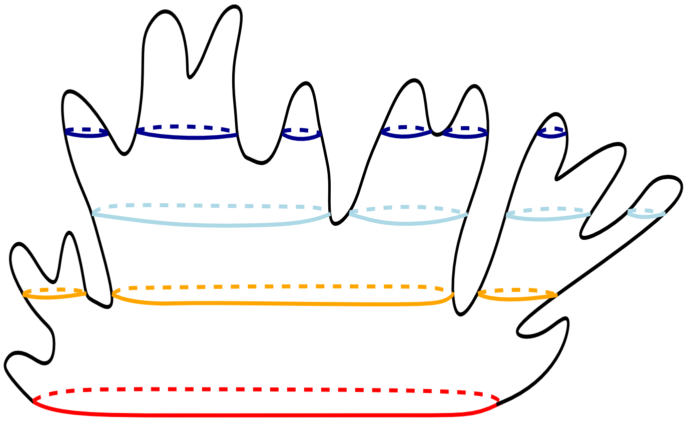


Figure 2.3 – Représentation en cactus du disque brownien libre.

L'idée est alors d'explorer depuis le bord $\partial\mathbb{D}_z$, qui va jouer le rôle de particule Eve du processus de croissance-fragmentation, en faisant augmenter les distances. Lorsque l'on croise un col notre bord se divise en deux, correspondant à un saut de la particule Eve, et on se retrouve avec deux bords à explorer. Il faut à ce moment-là un algorithme pour décider quel bord explorer en premier, et montrer une propriété d'indépendance entre les parties non découvertes. Dans le cas des triangulations on peut l'explorer triangle après triangle, mais dans le cas du disque brownien il n'y a pas de notion *a priori* naturelle d'exploration. Pour contourner cette difficulté, nous commençons par marquer le disque brownien libre en prenant un point uniforme $\rho^\bullet$ selon la mesure volume V. Il faut remarquer qu'en marquant un point nous biaisons la loi du disque brownien libre par la masse totale de V; cet objet est le disque brownien libre marqué. Il jouera un rôle important aussi dans notre travail [79] (voir section suivante). Pour le disque brownien libre marqué, il existe une manière naturelle de l'explorer; il suffit de suivre les bords menant vers $\rho^\bullet$, voir figure 2.4. Expliquons cela plus précisément. Notons $\mathbb{D}_z^\bullet$ le disque brownien libre marqué et $H^\bullet$ la distance de $\rho^\bullet$ au bord. Alors pour tout $r \leqslant H^\bullet$, on peut considérer $\partial\mathcal{C}^\bullet(r)$ l'unique bord au niveau r qui déconnecte $\rho^\bullet$ du bord du disque. Posons $X_r^\bullet := |\partial\mathcal{C}^\bullet(r)|$. Nous montrons qu'il est possible de coupler $\mathbb{D}_z^\bullet$ à la sphère brownienne, ce qui permet d'utiliser des résultats bien établis pour cette dernière [2, 71], et ainsi obtenir que:

- $X^\bullet$ est un processus de Markov $(\frac{1}{2}, \psi^\bullet)$-autosimilaire, avec

$$\psi^\bullet(\lambda) := \sqrt{\frac{3}{2\pi}} \int_{-\infty}^{0} \left(e^{\lambda y} - 1 - \lambda(e^y - 1)\right) e^{y/2} \left(1 - e^y\right)^{-5/2} \mathrm{d}y;$$

- les composantes connexes du complémentaire de $\bigcup_{r\leqslant H^\bullet} \partial\mathcal{C}^\bullet(r)$ sont en bijection avec les sauts de $X^\bullet$. De plus conditionnellement à $X^\bullet$, les adhérences de ces composantes connexes sont des disques browniens (libres) de périmètre le saut de $X^\bullet$ associé.

Dans le deuxième point (quand nous parlons d'adhérence des composantes connexes), nous considérons chaque espace muni de la restriction de la mesure volume et de l'extension continue de la distance intrinsèque sur son intérieur (voir [71] pour l'existence de cet objet). Nous obtenons alors une décomposition similaire à celle du processus de croissance-fragmentation avec néanmoins pour défaut que le processus d'exploration dépend du point marqué $\rho^\bullet$. Pour surmonter cette difficulté, nous devons introduire un algorithme d'exploration déterministe ou du moins qui ne dépende pas des parties non découvertes. L'idée est de toujours explorer le bord le plus long lorsque l'on croise un col. Lorsque l'on explore le disque brownien libre de cette manière, par un argument de compacité, on tombe fatalement sur un point $\rho^\ell \in \mathbb{D}_z$ qui est défini de manière unique. Nous pouvons dès lors procéder comme pour le disque brownien libre marqué. Notons H la distance entre ρ^ℓ et le bord $\partial\mathbb{D}_z$ et, pour tout $r \geqslant 0$, nous écrivons $\partial\mathcal{C}^\ell(r)$ pour désigner l'unique bord au niveau r qui déconnecte ρ^ℓ du bord du disque. Enfin posons $X_r := |\partial\mathcal{C}^\ell(r)|$. De manière informelle, et par des raisonnements d'absolue continuité, X est distribué comme $X^\bullet$ sous le conditionnement $|\Delta X^\bullet_t| \leqslant X^\bullet_t$ pour tout $t \geqslant 0$. Ceci peut être rendu formel et nous parvenons de ce fait à montrer que:

- X est un processus de Markov $(\frac{1}{2}, \psi)$-autosimilaire, avec

$$\psi(\lambda) := \sqrt{\frac{3}{2\pi}}\Big(-\frac{8}{3}\lambda + \int_{-\log 2}^{0} (e^{\lambda y} - 1 - \lambda(e^y - 1))\, e^{-3y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y\Big);$$

- les composantes connexes du complémentaire de $\bigcup_{r \leqslant H} \partial\mathcal{C}^\ell(r)$ sont en bijection avec les sauts de X. De plus conditionnellement à X, les adhérences de ces composantes connexes sont des disques browniens (libres) dont le périmètre est le saut de X associé.

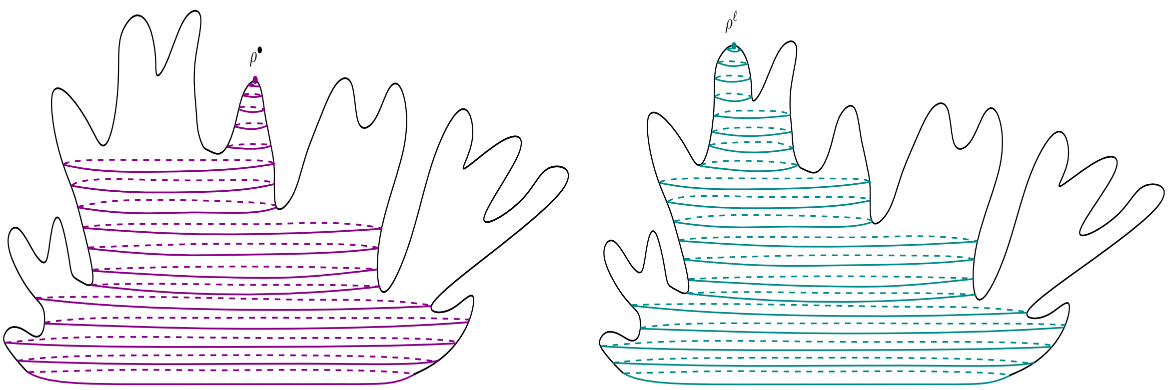


Figure 2.4 – Exploration du disque brownien libre en suivant les bords vers $\rho^\bullet$ à gauche et vers ρ^ℓ à droite.

Le théorème 1 peut alors être obtenu en appliquant le même argument à l'adhérence de chaque composante connexe du complémentaire de $\bigcup_{r \leqslant H} \partial\mathcal{C}^\ell(r)$.

Quelques conséquences

Grâce à nos résultats de couplage, nous pouvons déduire un résultat similaire pour la sphère brownienne. Pour cela nous considérons la sphère brownienne libre. Rappelons que ρ^* désigne la racine de la sphère brownienne libre. Remarquons aussi que la sphère brownienne contient un deuxième point distingué: le point correspondant à la racine de l'arbre brownien. Nous notons $\rho^\bullet$ ce deuxième point. Il peut être montré que conditionnellement au volume total de la sphère brownienne, le point $\rho^\bullet$ est un point uniforme pour la mesure V. Nous pouvons dès lors considérer $\rho^\bullet$ comme un point marqué. Dans le cas de la sphère brownienne nous explorons en suivant les distances par rapport à la racine ρ^*. Nous pouvons adapter simplement notre étude précédente concernant les composantes connexes du complémentaire des boules et les longueurs des bords de ces composantes connexes. Nous montrons que les longueurs des bords partant de ρ^* et allant vers $\rho^\bullet$ suivent un processus de Markov $(\frac{1}{2}, \psi^\bullet)$-autosimilaire partant de 0. Utilisons la notation $X^\bullet$ pour désigner ce processus. Les composantes connexes non découvertes sont alors en bijection avec les sauts de $X^\bullet$ et leurs adhérences sont, conditionnellement à $X^\bullet$, des disques browniens libres avec pour périmètre le saut associé. Il est alors possible d'encoder les longueurs des bords comme suit: on part du processus $X^\bullet$ et chacun de ses sauts produit un processus de croissance-fragmentation $(\frac{1}{2}, \psi)$-autosimilaire commençant en le saut en question. Nous mentionnons aussi que notre travail a des conséquences sur les temps locaux du mouvement brownien indexé par l'arbre brownien.

Une question concernant le plan brownien

Dans [77], nous montrons aussi, que dans le plan brownien, les bords des boules centrées en la racine sont encodés par un processus de croissance-fragmentation. Expliquons cela plus précisément. Introduisons dans un premier temps quelques notations qui seront aussi utiles dans les sections à venir. Soit $\mathcal{M}_\infty$ un plan brownien. Comme il va être souvent question du plan brownien dans cette introduction, nous utilisons la notation B_r pour désigner la boule fermée de rayon r centrée en la racine de $\mathcal{M}_\infty$. Comme le plan brownien a la topologie du plan complexe, le complémentaire de B_r possède une unique composante connexe non bornée, $\check{B}_r^\circ$. Nous écrivons $B_r^\bullet$ pour le complémentaire de $\check{B}_r^\circ$ – cet espace est connu sous le nom de *hull* de rayon r – et $\check{B}_r^\bullet$ pour l'adhérence de $\check{B}_r^\circ$. À nouveau nous pouvons donner un sens à la longueur des bords des composantes connexes de $\mathcal{M}_\infty \backslash B_r$. Ici l'infini joue le rôle de point marqué et les bords distingués sont les bords $(\partial \check{B}_r^\bullet)_{r\geqslant 0}$. Nous prouvons que le processus des longueurs des bords $(|\partial \check{B}_r^\bullet|)_{r\geqslant 0}$ est distribué comme le processus $X^\bullet$ de la sphère brownienne, conditionné à survivre. Ce processus est d'ailleurs étudié en profondeur dans [40]. Il est alors possible d'encoder les longueurs des bords comme suit: on part du processus $(|\partial \check{B}_r^\bullet|)_{r\geqslant 0}$ et chacun de ses sauts produit un processus de croissance-fragmentation $(\frac{1}{2}, \psi)$-autosimilaire commençant au saut en question. Plus précisément nous montrons que chaque saut de $\left(|\partial \check{B}_r^\bullet|\right)_{r\geqslant 0}$ produit un disque brownien libre dont le périmètre est le saut associé. Cependant une question subsiste dans [77]:

Quelle est la loi de $\check{B}_r^\bullet$ pour $r > 0$?

Répondre à cette question est l'une des motivations de notre travail [79], que nous présentons dans la section suivante, et joue un rôle central dans notre article [93] concernant les inégalités isopérimétriques du plan brownien.

2.3 Construction de modèles non-compacts

Dans [14], Erich Baur, Grégory Miermont et Gourab Ray caractérisent toutes les limites possibles de quadrangulations aléatoires avec bord. Ici nous nous intéresserons aux trois modèles principaux de géométrie brownienne non-compacts: le plan brownien, le disque brownien de volume infini et le demi-plan brownien. Nous présenterons notre travail [79] en collaboration avec Jean-François Le Gall et donnerons une approche unifiée pour construire ces objets en adaptant la construction de la sphère brownienne présentée dans la section 1.3. De plus, nos constructions permettent de contrôler les distances au bord dans le cas du disque brownien de volume infini et du demi-plan brownien et à la racine dans le cas du plan brownien. Par souci de simplification du vocabulaire employé, nous dirons que le bord du plan brownien est sa racine. Nos constructions diffèrent de celles apparaissant dans [14] où l'on contrôle les distances à l'infini. En particulier nos résultats nous permettront de répondre à la question posée à la fin de la section précédente, c'est-à-dire de donner la loi de $\check{B}^{\bullet}_r$ dans le plan brownien.

Une construction unifiée

Nos constructions reposent sur l'arbre brownien infini. Cet objet peut être construit à l'aide de deux mesures ponctuelles de Poisson

$$\mathcal{N} := \sum_{i\in I} \delta_{t_i,\mathbf{e}^i} \text{ et } \mathcal{N}' := \sum_{i\in J} \delta_{t_i,\mathbf{e}^i}$$

indépendantes d'intensité $2\mathbb{1}_{[0,\infty)}(t)dt\ \mathbf{n}(d\mathbf{e})$, où nous rappelons que $\mathbf{n}(d\mathbf{e})$ désigne la mesure d'Itô de l'excursion brownienne. Notons alors $\mathcal{T}_{e^i}$ l'arbre encodé par l'excursion $\mathbf{e}^i$ et introduisons l'union disjointe

$$[0,\infty) \cup \bigcup_{i\in I\cup J} \mathcal{T}_{e^i}\ .$$

Ici nous interprétons $[0,\infty)$ comme une ligne verticale infinie que nous appelons épine. Pour $i \in I$ (resp. $j \in J$) nous recollons l'arbre $\mathcal{T}_{e^i}$ à gauche (resp. à droite) de $[0,\infty)$ à hauteur t_i. L'espace obtenu est l'arbre brownien infini, que nous notons $\mathcal{T}_\infty$. Il peut aussi être obtenu comme un arbre brownien de durée de vie infinie. Nous assignons maintenant des étiquettes sur $\mathcal{T}_\infty$ de la manière suivante:

- Les étiquettes le long de l'épine $[0,\infty)$ suivent un processus de Bessel de dimension 3;
- Pour $i \in I \cup J$, les étiquettes le long de $\mathcal{T}_{e^i}$ suivent un mouvement brownien indexé par l'arbre $\mathcal{T}_{e^i}$ (commençant à l'étiquette du point de l'épine t_i).

Coupons alors, pour tout $i \in I \cup J$, les branches de $\mathcal{T}_{e^i}$ au premier point d'étiquette 0. Après cet élagage subsiste une structure d'arbre, mais les arbres recollés sur l'épine ne sont plus des arbres browniens. Notons $\widetilde{\mathcal{T}}_\infty$ l'arbre ainsi obtenu. Il y a un nombre infini de points d'étiquette 0 sur $\widetilde{\mathcal{T}}_\infty$ mais il est possible de définir une variable aléatoire Z rendant compte de la quantité de points d'étiquette 0. Dans [79] nous montrons qu'adapter la construction de la sphère brownienne à partir du mouvement brownien indexé par l'arbre brownien permet d'obtenir:

- Le plan brownien sous le conditionnement $Z = 0$;
- Le disque brownien de volume infini et de périmètre $z > 0$ sous le conditionnement $Z = z$;
- Le demi-plan brownien sous le conditionnement $Z = \infty$.

Figure 2.5 – Illustration de l'arbre $\widetilde{\mathcal{T}}_\infty$. La hauteur représente les distances à 0 des étiquettes. L'épine est coloriée en jaune et les points d'étiquette nulle sont coloriés en rouge.

De plus, dans ces constructions, les étiquettes correspondent aux distances au bord, celui-ci étant encodé à son tour par les points d'étiquette 0. Mentionnons qu'il faut adapter la construction de la sphère brownienne avec soin, puisqu'une adaptation directe recollerait tout le bord en un seul point (l'idée est de définir dans un premier temps la distance entre points d'étiquettes positives puis dans un deuxième temps de prolonger cette distance sur le bord par continuité). Les deux nouvelles contributions de ce travail sont celles du disque brownien de volume infini et du demi-plan brownien. Le cas du plan brownien correspond à la construction introduite dans [40]. Dans ce cas les étiquettes le long de l'épine évoluent comme un processus de Bessel de dimension 9, et les arbres recollés sur celle-ci sont des arbres browniens avec des étiquettes browniennes conditionnées à rester positives. Cependant, donner un sens au conditionnement du disque brownien de volume infini se révèle difficile puisqu'on a $Z = \infty$ presque sûrement. Cela rend le conditionnement pour le demi-plan brownien superflu et celui du disque brownien de volume infini dégénéré.

Le conditionnement du disque brownien de volume infini

Expliquons informellement comment nous parvenons à définir le conditionnement $Z = z > 0$. Pour ce faire, considérons un second arbre infini étiqueté en prenant un processus de Bessel de dimension 9 sur l'épine et en recollant sur celle-ci des arbres browniens avec des étiquettes conditionnées à rester strictement positives. Notons $\mathcal{T}'_\infty$ cet arbre étiqueté (qui est l'arbre associé au plan brownien, i.e. le conditionnement $Z = 0$). Dans ce cas la frontière est réduite à un seul point qui est la racine de $\mathcal{T}'_\infty$. Fixons maintenant $0 < r_1 < r_2$ et notons $\mathcal{T}'_\infty(r_1, r_2)$ le sous-arbre de $\mathcal{T}'_\infty$ correspondant à:

- Garder la partie de l'épine après le dernier temps de passage au niveau r_1 jusqu'au dernier temps de passage au niveau r_2 (du processus de Bessel de dimension 9);
- Garder les arbres collés sur cette partie de l'épine et couper les branches de ces arbres au premier point dont l'étiquette est égale à r_1.

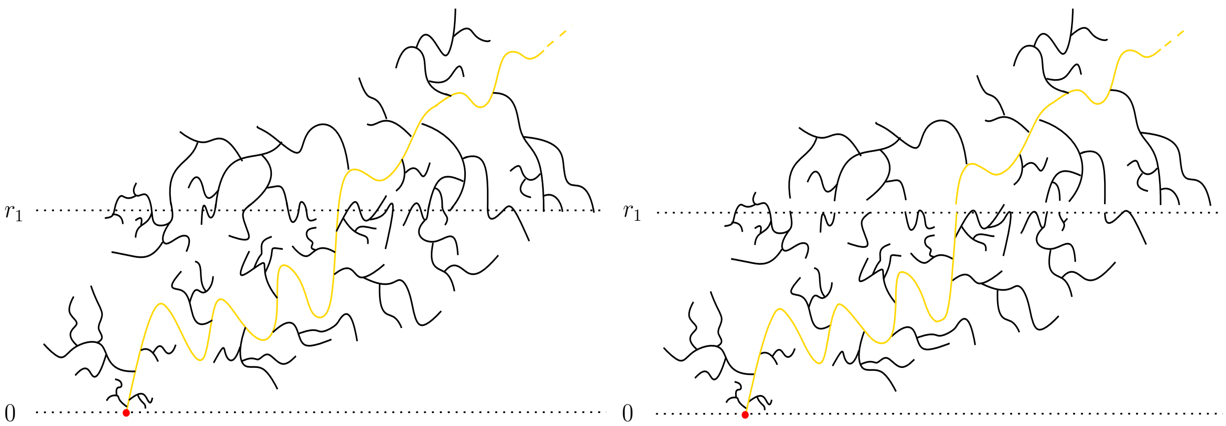


Figure 2.6 – Illustration de l'arbre $\mathcal{T}'_\infty$ avant et après l'avoir coupé au niveau r_1 avec $r_2 = \infty$. L'épine est coloriée en doré

Il est à nouveau possible de donner un sens à la quantité de points au niveau r_1 que nous notons Z'_{r_1,r_2}. Contrairement au cas avec épine étiquetée par un processus de Bessel de dimension 3, le support de la loi de Z'_{r_1,r_2} est $\mathbb{R}_+$ pour tout $r_2 > r_1$ (même pour $r_2 = \infty$). De plus, l'arbre $\mathcal{T}'_\infty(r_1, \infty)$ bénéficie d'une propriété de changement d'échelle provenant de celle du mouvement brownien et du processus de Bessel. Nous pouvons dès lors définir aisément la loi de $\mathcal{T}'_\infty(r_1, \infty)$ conditionnellement à $Z'_{r_1,\infty} = z > 0$. Nous montrons alors que, pour tout $r_2 > r_1$, il est possible de donner sens à la loi de $\mathcal{T}'_\infty(r_1, r_2)$, conditionnellement à $Z'_{r_1,r_2} = z > 0$, en coupant l'épine au dernier passage de ses étiquettes au niveau r_2. Il nous faut maintenant expliquer le lien entre cet objet et l'arbre initial $\widetilde{\mathcal{T}}_\infty$. Pour cela, pour tout $r > 0$, définissons $\widetilde{\mathcal{T}}_\infty(r)$ le sous-arbre de $\widetilde{\mathcal{T}}_\infty$ correspondant à :

- Garder la partie de l'épine avant le dernier temps de passage au niveau r ;

- Garder les arbres collés sur cette partie de l'épine sans modification.

La quantité totale de points d'étiquette 0 dans $\widetilde{\mathcal{T}}_\infty(r)$ est finie. Nous montrons, par des relations d'absolue continuité entre les processus de Bessel, que si l'on conditionne cette quantité à être égale à $z > 0$, alors $\widetilde{\mathcal{T}}_\infty(r)$ est distribué comme l'arbre $\mathcal{T}'_\infty(r_1, r_1 + r)$, conditionné à $Z'_{r_1,r_1+r} = z$, pourvu que l'on translate les étiquettes de $-r_1$. Ce résultat a un double intérêt: il permet d'une part de définir la loi de $\widetilde{\mathcal{T}}_\infty(r)$ conditionnellement à la quantité de points d'étiquette 0 et, d'autre part, il montre que la loi de $\mathcal{T}'_\infty(r_1, r_1 + r)$, conditionnellement à Z'_{r_1,r_1+r}, ne dépend pas de r_1 (pourvu évidemment que l'on translate les étiquettes par $-r_1$). Afin de définir la loi de $\widetilde{\mathcal{T}}_\infty$ conditionnellement à $Z = z > 0$, nous procédons à prendre la limite de la loi de $\widetilde{\mathcal{T}}_\infty(r)$ lorsque $r \to \infty$. Cela permet de définir la loi de $\widetilde{\mathcal{T}}_\infty$, conditionnellement à $Z = z > 0$, comme étant la loi de $\mathcal{T}'_\infty(r_1, \infty)$ sous le conditionnement $Z'_{r_1,\infty} = z$. Expliquons maintenant comment relier les différents conditionnements de $\widetilde{\mathcal{T}}_\infty$ aux différents modèles de géométrie brownienne. Pour cela, introduisons comme dans la section précédente le disque brownien libre marqué de périmètre z, que l'on note $\mathbb{D}^\bullet_z$. Rappelons aussi que $H^\bullet$ désigne la distance entre le point marqué et le bord dans $\mathbb{D}^\bullet_z$. Nous prouvons que la loi de $\widetilde{\mathcal{T}}_\infty(r)$ conditionnellement à avoir une quantité de points d'étiquette 0 égale à z permet d'encoder l'espace $\mathbb{D}^\bullet_z$ sous le conditionnement $H^\bullet = r$. Nous baptisons cet espace sous le nom de *disque brownien libre de périmètre z et marqué à hauteur r*. Ce dernier nous permet alors d'obtenir les liens entre $\widetilde{\mathcal{T}}_\infty$, le disque brownien de volume infini et le demi-plan brownien, par des passages à la limite s'appuyant sur [14]. En passant nous obtenons aussi le résultat suivant:

Proposition 2

Dans $\mathbb{D}^\bullet_1$, la distance entre le point marqué et le bord est une variable aléatoire de distribution:

$$p_1(r) := 9\, r^{-6}\Big(r + \frac{2}{3}\, r^3 - \Big(\frac{3}{2}\Big)^{1/2} \sqrt{\pi}\,(1 + r^2)\, \exp\Big(\frac{3}{2r^2}\Big)\, \mathrm{erfc}\Big(\sqrt{\frac{3}{2r^2}}\Big)\Big).$$

Concluons cette section en donnant quelques applications de nos constructions.

Etude des hulls dans le plan brownien

Rappelons la notation $\mathcal{M}_\infty$ pour le plan brownien et supposons qu'il est construit à partir de l'arbre $\mathcal{T}'_\infty$. Souvenons-nous aussi des notations B_r, $B^\bullet_r$, $\check{B}^\circ_r$ et $\check{B}^\bullet_r$ introduites à la fin de la section précédente. Nous montrons que l'arbre $\mathcal{T}'_\infty(r, \infty)$ correspond à $\check{B}^\bullet_r$, et que $\mathcal{T}'_\infty \backslash \mathcal{T}'_\infty(r, \infty)$ correspondent à l'intérieur de $B^\bullet_r$ que l'on note B°_r. En utilisant ces identifications nous montrons que les distances intrinsèques sur les ouverts B°_r et $\check{B}^\circ_r$ se prolongent par continuité respectivement sur $B^\bullet_r$ et $\check{B}^\bullet_r$. Nous munissons alors $B^\bullet_r$ et $\check{B}^\bullet_r$ de ces distances, et de la restriction de la mesure volume V. Nous pointons $B^\bullet_r$ à la racine de $\mathcal{M}_\infty$ et $\check{B}^\bullet_r$ au point de $\mathcal{M}_\infty$ correspondant à la racine de $\mathcal{T}'_\infty(r, \infty)$. Nous obtenons ainsi que $B^\bullet_r$ et $\check{B}^\bullet_r$ sont deux éléments de $\mathbb{K}_\infty$. Comme pour le disque brownien, nous pouvons définir la longueur de $\partial \check{B}^\bullet_r$ grâce à la formule:

$$|\partial \check{B}^\bullet_r| = \lim_{\varepsilon\to 0} \varepsilon^{-2} V\big(\check{B}^\bullet_r \cap B_{r+\varepsilon}\big).$$

Il se trouve que cette quantité est exactement $Z'_{r,\infty}$. Une des applications les plus importantes de nos constructions est le théorème suivant qui répond à la question posée à la fin de la section 2.2.

Théorème 2

Conditionnellement à $|\partial \check{B}^\bullet_r|$, le *hull* $B^\bullet_r$ et l'adhérence de son complementaire $\check{B}^\bullet_r$ sont indépendants et $\check{B}^\bullet_r$ est un disque brownien de volume infini de périmètre $|\partial \check{B}^\bullet_r|$.

Ce résultat est crucial pour notre étude sur les propriétés isopérimétriques [93].

Etude des horohulls dans le plan brownien

Notre construction du disque brownien avec un point marqué à une hauteur donnée s'avère aussi utile pour l'étude des *horohulls.* Rappelons rapidement la définition des *horohulls* du plan brownien $\mathcal{M}_\infty$. Notons Δ la distance de $\mathcal{M}_\infty$ et ρ_∞ la racine. Il peut être prouvé que, presque sûrement, pour tous $a, b \in \mathcal{M}_\infty$ la limite

$$\lim_{x\to\infty} (\Delta(a,x) - \Delta(b,x))$$

existe dans $\mathbb{R}$. Ici la limite $x \to \infty$ doit être prise dans le sens où x tend vers le point ∞ de la compactification d'Alexandroff du plan brownien. Nous pouvons alors écrire la limite précédente sous la forme $\mathcal{H}_a - \mathcal{H}_b$, où "l'horofonction" $a \mapsto \mathcal{H}_a$ est définie de manière unique en imposant $\mathcal{H}_{\rho_\infty} = 0$. L'horofonction $\mathcal{H}$ est interprétée comme donnant les distances relatives à l'infini. Pour tout $r > 0$, écrivons $\mathfrak{B}^\circ_r$ pour la composante connexe de l'ouvert $\{x \in \mathcal{M}_\infty : \mathcal{H}_x > -r\}$ contenant ρ_∞. Le *horohull* de rayon r est alors l'adhérence de $\mathfrak{B}^\circ_r$ que nous notons $\mathfrak{B}^\bullet_r$. Nous montrons que la limite:

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-2} V\big(\{x \in \mathfrak{B}^\bullet_r : \mathcal{H}_x < -r+\varepsilon\}\big)$$

existe presque sûrement. Ceci donne une bonne notion de longueur du bord de $\mathfrak{B}^\bullet_r$ que l'on note $|\partial\mathfrak{B}^\bullet_r|$. En utilisant la construction du disque brownien marqué à l'aide d'un arbre étiqueté nous montrons que la distance intrinsèque sur $\mathfrak{B}^\circ_r$ admet p.s. une extension continue sur $\mathfrak{B}^\bullet_r$. On munit le *horohull* $\mathfrak{B}^\bullet_r$ de cette distance, de la restriction de la mesure volume et nous pointons cet espace sur la racine de $\mathcal{M}_\infty$. Nous démontrons le résultat suivant

Théorème 3

Conditionnellement à $|\partial\mathfrak{B}^\bullet_r| = z$, le *horohull* $\mathfrak{B}^\bullet_r$ est un disque brownien libre de périmètre z et marqué à hauteur r.

Ceci nous permet en particulier de calculer la transformée de Laplace du couple $(|\partial\mathfrak{B}^\bullet_r|, V(\mathfrak{B}^\bullet_r))$. Plus précisément nous obtenons pour tous $r, \lambda, \mu \geqslant 0$:

$$\mathbb{E}\Big[\exp\big(-\lambda|\partial\mathfrak{B}^\bullet_r| - \mu V(\mathfrak{B}^\bullet_r)\big)\Big] = \frac{(\frac{2}{3}+\frac{\lambda}{3}\sqrt{2/\mu}\,)^{-1/2}\sinh((2\mu)^{1/4}r)+\cosh((2\mu)^{1/4}r)}{\Big((\frac{2}{3}+\frac{\lambda}{3}\sqrt{2/\mu}\,)^{1/2}\sinh((2\mu)^{1/4}r)+\cosh((2\mu)^{1/4}r)\Big)^3}.$$

Cette formule apparaît déjà dans [42], où les auteurs considérent des limites d'échelle de longueurs de bords et de volumes des *horohulls* discrets dans la quadrangulation infinie uniforme. Ceci n'est pas surprenant puisque la limite d'échelle de la quadrangulation infinie uniforme est le plan brownien. Il faut cependant mentionner qu'il ne serait pas facile de déduire notre résultat sur le plan brownien en utilisant le résultat analogue de la quadrangulation infinie uniforme. Nous parvenons aussi à déterminer la loi du processus $(|\partial\mathfrak{B}^{\bullet}_r|, V(\mathfrak{B}^{\bullet}_r))_{r\geqslant 0}$.

2.4 Inégalités isopérimétriques

Intéressons-nous maintenant à nos résultats de [93]. Le plan brownien, que nous notons $\mathcal{M}_\infty$, est presque sûrement homéomorphe au plan complexe $\mathbb{C}$. Il est, de plus, équipé d'une distance Δ ainsi que d'une mesure de volume V. Il est par conséquent possible de définir pour toute fonction continue $\gamma : [a,b] \to \mathcal{M}_\infty$ sa longueur par la formule:

$$\Delta(\gamma) := \sup_{a=a_1\leqslant a_2 \leqslant \ldots \leqslant a_n = b} \sum_{i=1}^{n} \Delta(\gamma(a_i), \gamma(a_{i+1})).$$

Il est alors naturel de se demander quelles sont les propriétés isopérimétriques de $\mathcal{M}_\infty$. Plus précisément, soit $\mathcal{K}$ pour l'ensemble des domaines (fermés) de Jordan de $\mathcal{M}_\infty$ dont l'intérieur contient la racine. Ici un domaine de Jordan est un ensemble fermé homéomorphe au disque unité fermé de $\mathbb{C}$. Pour tout $A \in \mathcal{K}$ son volume $V(A)$ est bien défini, puisque V est une mesure borélienne sur $\mathcal{M}_\infty$, et on peut définir la longueur de sa frontière $\Delta(\partial A)$ comme étant la longueur de n'importe quelle paramétrisation de ∂A. Une simple vérification montre que cette quantité ne dépend pas de la paramétrisation en question. Nous voulons comprendre jusqu'à quel point $\Delta(\partial A)$ peut être petit en fonction du volume $V(A)$. Pour cela, pour tout $r > 0$, rappelons que B_r désigne la boule de rayon r de $\mathcal{M}_\infty$ centrée en la racine, et introduisons la quantité:

$$L_r := \inf\big\{\Delta(\gamma) : \ \gamma \text{ cycle injectif séparant } B_r \text{ de l'infini}\big\}.$$

Le plan brownien $\mathcal{M}_\infty$ est invariant par changement d'échelle c'est-à-dire $\mathcal{M}_\infty \overset{(d)}{=} \lambda \cdot \mathcal{M}_\infty$, pour tout $\lambda > 0$. Ceci entraîne, pour tout $r > 0$, l'égalité en loi suivante:

$$L_r \overset{(d)}{=} rL_1. \tag{2.2}$$

En particulier, nous déduisons que pour $A \in \mathcal{K}$, les bonnes quantités à comparer sont $\Delta(\partial A)$ et $V(A)^{\frac{1}{4}}$. Dans [93] nous établissons le résultat suivant

Théorème 4

(i) On a

$$\limsup_{u\to\infty} \frac{\log\big(\mathbb{P}(L_1 > u)\big)}{u} \leqslant -\sup_{s>1} \frac{1}{2(s-1)} \log\big(\frac{s^2}{2s-1}\big).$$

Par conséquent, $\mathbb{P}(L_1 > u)$ décroît exponentiellement vite quand u tend vers l'infini.

(ii) Il existe deux constantes $0 < c_1 \leqslant c_2 < \infty$ telles que pour tout $\varepsilon > 0$:

$$c_1(\varepsilon^2 \wedge 1) \leqslant \mathbb{P}(L_1 < \varepsilon) \leqslant c_2\varepsilon^2.$$

Les résultats du théorème 4 s'étendent sans difficulté pour L_r avec $r > 0$ en utilisant (2.2). La preuve du théorème 4 repose sur le fait que l'écriture du plan brownien à l'aide d'un arbre étiqueté (voir section 2.3) permet de comprendre et de très bien contrôler les géodésiques vers la racine ainsi que les propriétés de confluence de ces géodésiques. Rappelons la notation $B_r^\bullet$ pour le *hull* de rayon r et $\check{B}_r^\bullet$ pour l'adhérence de son complémentaire. Une application du théorème de Jordan montre que tout cycle séparant B_r de l'infini doit aussi séparer $B_r^\bullet$ de l'infini. Rappelons que pour tout $r > 0$ nous avons:

$$|\partial \check{B}_r^\bullet| := \lim_{\varepsilon\to 0} \varepsilon^{-2} V\big(\check{B}_r^\bullet \cap B_{r+\varepsilon}\big), \text{ p.s. }, \tag{2.3}$$

mais il peut être montrer en utilisant nos résultats dans [78] que cette limite a lieu p.s. pour tout $r > 0$ simultanément. Afin d'étudier les propriétés isopérimétriques de $\mathcal{M}_\infty$, nous établissons une propriété de Markov "forte" pour $\mathcal{M}_\infty$. Plus précisément, nous notons $\mathcal{F}_r$ la tribu engendrée par $B_r^\bullet$ (vue comme un élément de $\mathbb{K}$ lorsqu'on le pointe à la racine et qu'on le munit de la restriction de la mesure V et de l'extension continue de la distance intrinsèque sur son intérieur, voir section précédente).

Théorème 5

Soit T un temps d'arrêt pour la filtration $(\mathcal{F}_{r+})_{r\geqslant 0}$ tel que $0 < T < \infty$ p.s. La distance intrinsèque sur l'intérieur de $\check{B}_T^\bullet$ a une unique extension continue sur $\check{B}_T^\bullet$. De plus, conditionnellement à $|\partial \check{B}_T^\bullet| = z$, l'espace $\check{B}_T^\bullet$ muni de cette extension continue et de la restriction de la mesure volume est un disque brownien de volume infini de périmètre z, indépendant de $B_T^\bullet$.

En toute rigueur il faudrait aussi préciser le point distingué de $\check{B}_T^\bullet$. Pour le faire il faut considérer l'écriture du plan brownien avec l'arbre $\mathcal{T}'_\infty$ et pointer $\check{B}_T^\bullet$ au point de $\mathcal{M}_\infty$ correspondant au dernier temps de passage du processus des étiquettes de l'épine au niveau T. En utilisant (2.3), nous obtenons que $T_z := \inf\{r \geqslant 0 : \ |\partial \check{B}_r^\bullet| = z\}$ est un temps d'arrêt pour la filtration $(\mathcal{F}_{r+})_{r\geqslant 0}$. De plus, $\big(|\partial \check{B}_r^\bullet|\big)_{r\geqslant 0}$ n'a pas de saut positif et tend vers l'infini lorsque $r \to \infty$, ce qui entraîne que $|\partial B_{T_z}^\bullet| = z$ p.s. Nous pouvons désormais appliquer le théorème 5, pour voir que $\check{B}_{T_z}^\bullet$ est un disque brownien de volume infini de périmètre z. Grâce à ce résultat, nous prouvons un analogue du théorème 4 pour le disque brownien infini, concernant les cycles injectifs séparant la frontière de l'infini. En combinant enfin notre étude des cycles séparants avec le théorème 5 nous établissons la propriété isopérimétrique suivante:

Théorème 6

Pour toute fonction croissante $f : \mathbb{R}_+ \mapsto (0, \infty)$:

(i) On a $\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{V(A)^{\frac{1}{4}}} f(|\log(V(A))|) = 0$, $\mathbb{P}$-p.s. , si $\sum_{m \in \mathbb{N}} f(m)^{-2} = \infty$.

(ii) On a $\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{V(A)^{\frac{1}{4}}} f(|\log(V(A))|) > 0$, $\mathbb{P}$-p.s. , si $\sum_{m \in \mathbb{N}} f(m)^{-2} < \infty$.

Ce résultat s'étend aisément pour le disque brownien de volume infini ainsi que pour la sphère brownienne. Mentionnons que dans [73], Jean-François Le Gall et Thomas Lehéricy obtiennent un analogue pour le modèle discret de la *quadrangulation infinie uniforme* du théorème 6 dans le cas particulier $f(x) := x^{4/3+\delta}$ pour tout $\delta > 0$.

Stratégie pour démontrer l'inégalité isopérimétrique

Donnons ci-après les grandes lignes de la preuve du théorème 6. Commençons par le point (i). Pour $z > 1$, et pour tout $n \in \mathbb{N}$, écrivons $B^{\bullet}_{T_{z^n}, T_{z^{n+1}}}$ pour l'adhérence de la couronne $B^{\bullet}_{T_{z^{n+1}}} \backslash B^{\bullet}_{T_{z^n}}$, et $L_{T_{z^n}, T_{z^{n+1}}}$ pour l'infimum des longueurs des cycles séparant la racine de l'infini tout en restant dans l'intérieur de $B^{\bullet}_{T_{z^n}, T_{z^{n+1}}}$. Par un argument de compacité, l'infimum $L_{T_{z^n}, T_{z^{n+1}}}$ est réalisé par un cycle injectif restant dans $B^{\bullet}_{T_{z^n}, T_{z^{n+1}}}$. Grâce au théorème de Jordan on sait alors que ce cycle délimite un élément de $\mathcal{K}$ contenant le *hull* $B^{\bullet}_{T_{z^n}}$ et contenu dans le *hull* $B^{\bullet}_{T_{z^{n+1}}}$. De ce fait, nous obtenons

$$\begin{aligned}\inf_{A \in \mathcal{K}} \frac{\Delta(\partial A)}{V(A)^{\frac{1}{4}}} f\big(|\log(V(A))|\big) &\leqslant \inf_{n \in \mathbb{N}} \frac{L_{T_{z^n}, T_{z^{n+1}}}}{V(B^{\bullet}_{T_{z^n}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{T_{z^{n+1}}}))|\big) \\ &\leqslant \liminf_{n \to \infty} \frac{L_{T_{z^{2n+1}}, T_{z^{2n+2}}}}{V(B^{\bullet}_{T_{z^{2n}}, T_{z^{2n+1}}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{T_{z^{2n+2}}}))|\big).\end{aligned}$$

Nous montrons alors que cette limite inférieure est égale à 0. Pour cela remarquons qu'une application du théorème 5 montre que les couronnes $\big(B^{\bullet}_{T_{z^n}, T_{z^{n+1}}}\big)_{n \geqslant 0}$ sont indépendantes et que l'invariance par changement d'échelle du plan brownien entraîne que $B^{\bullet}_{T_{z^n}, T_{z^{n+1}}}$ a la même loi que $z^{n/2} \cdot B^{\bullet}_{T_1, T_z}$. Nous obtenons donc que les variables $L_{T_{z^{2n+1}}, T_{z^{2n+2}}} \cdot V(B^{\bullet}_{T_{z^{2n}}, T_{z^{2n+1}}})^{-\frac{1}{4}}$, pour $n \geqslant 0$, sont indépendantes et identiquement distribuées. Nos méthodes pour étudier les variables L_r nous permettent aussi de montrer qu'il existe une constante c telle que:

$$\mathbb{P}\Big(\frac{L_{T_{z^{2n+1}}, T_{z^{2n+2}}}}{V(B^{\bullet}_{T_{z^{2n}}, T_{z^{2n+1}}})^{\frac{1}{4}}} \leqslant x \Big) \geqslant c \cdot x^2,$$

pour tout $x < 1$. Pour conclure la preuve de (i), nous prouvons qu'il existe $h > 0$, tel que $f(\log(V(B^{\bullet}_{T_{z^{2n+2}}})) < f(hn)$ pour tout n suffisamment grand. Une application du lemme de

Borel-Cantelli montre alors que si $\sum_{m\geqslant 1} f(m)^{-2} = \infty$ on a pour tout $\varepsilon > 0$:

$$\frac{L_{T_{z2n+1},T_{z2n+2}}}{V(B^{\bullet}_{T_{z2n},T_{z2n+1}})^{\frac{1}{4}}} f(hn) \leqslant \varepsilon$$

infiniment souvent. Ce qui permet d'obtenir (i). Expliquons maintenant comment démontrer (ii). Soit $A \in \mathcal{K}$ et m l'unique élément de $\mathbb{Z}$ tel que $V(B^{\bullet}_{2^m}) < V(A) \leqslant V(B^{\bullet}_{2^{m+1}})$, en particulier A ne peut pas être contenu dans $B^{\bullet}_{2^m}$. Il y a deux cas possibles, comme le montre la figure 2.7. Si ∂A touche le bord $\partial B^{\bullet}_{2^{m-1}}$, alors ∂A connecte $\partial B^{\bullet}_{2^{m-1}}$ et $\partial B^{\bullet}_{2^m}$ ce qui implique $\Delta(\partial A) \geqslant 2^{m-1}$. Par contre si ∂A n'intersecte pas $\partial B^{\bullet}_{2^{m-1}}$ alors ∂A doit séparer $B^{\bullet}_{2^{m-1}}$ de l'infini, entraînant $\Delta(\partial A) \geqslant L_{2^{m-1}}$. Donc on obtient pour tout $A \in \mathcal{K}$ que la quantité $\frac{\Delta(\partial A)}{V(A)^{\frac{1}{4}}} f(|\log(V(A))|)$ est bornée inférieurement par

$$\Big(\inf_{m\in\mathbb{Z}} \frac{2^{m-1}}{V(B^{\bullet}_{2^{m+1}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{2^m}))|\big)\Big) \wedge \Big(\inf_{m\in\mathbb{Z}} \frac{L_{2^{m-1}}}{V(B^{\bullet}_{2^{m+1}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{2^m}))|\big)\Big).$$

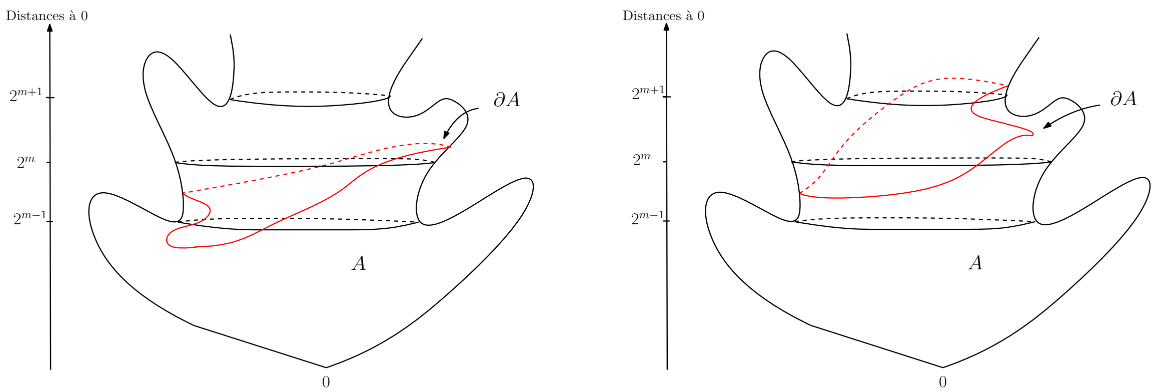


Figure 2.7 – Nous représentons en rouge le bord de A. À gauche nous illustrons le cas où ∂A intersecte $\partial B^{\bullet}_{2^{m-1}}$ et nous avons $\Delta(\partial A) \geqslant 2^{m-1}$. À droite nous illustrons le cas où ∂A ne touche pas $\partial B^{\bullet}_{2^{m-1}}$ et nous avons $\Delta(\partial A) \geqslant L_{2^{m-1}}$.

Pour prouver (ii) il suffit alors de montrer que p.s.:

$$\inf_{m\in\mathbb{Z}} \frac{2^{m-1}}{V(B^{\bullet}_{2^{m+1}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{2^m}))|\big) > 0 \text{ et } \inf_{m\in\mathbb{Z}} \frac{L_{2^{m-1}}}{V(B^{\bullet}_{2^{m+1}})^{\frac{1}{4}}} f\big(|\log(V(B^{\bullet}_{2^m}))|\big) > 0. \tag{2.4}$$

Nos méthodes d'étude des cycles séparants nous permettent de traiter avec souplesse ces deux quantités et de déduire (2.4) par un raisonnement utilisant à nouveau le lemme de Borel-Cantelli. Mentionnons que toutes les méthodes d'estimation et de contrôle nécessaires à la preuve du théorème 6 dépendent fortement d'une part des écritures du plan brownien et du disque brownien de volume infini à partir d'arbres étiquetés présentées dans la section précédente et d'autre part des résultats de [40], où Nicolas Curien et Jean-François Le Gall étudient le processus des *hulls* dans le plan brownien.

Chapitre 3

Calculs explicites concernant des longueurs de bords et des volumes

Dans ce chapitre nous chercherons à établir des formules explicites d'interprétation géométrique. Les résultats présentés reposeront encore une fois sur le mouvement brownien indexé par l'arbre brownien. Nous décrirons notre travail [78] ainsi que des résultats issus d'un article en préparation adaptant les techniques précédemment développées à l'étude du disque brownien.

3.1 Temps local et temps de vie

Dans cette section nous présenterons le travail [78] effectué en collaboration avec Jean-François Le Gall et nous travaillerons sous la mesure $\mathbb{N}_0$ (mesure encodant la sphère brownienne libre en utilisant la mesure d'Itô de l'excursion brownienne). Rappelons que le temps de vie σ de l'excursion $\mathbf{e}$ est une quantité aléatoire sous $\mathbb{N}_0$. Souvenons-nous aussi que $(\Lambda_s)_{s\in[0,\sigma]}$ désigne le processus des étiquettes et qu'il peut être factorisé par rapport à la relation d'équivalence $\sim_{d_{\mathbf{e}}}$ de l'arbre brownien ou par rapport à la relation d'équivalence $\sim_{D^*}$ de la sphère brownienne. De plus, dans l'écriture de la sphère brownienne, les étiquettes (après avoir été translatées par $-\min Z$) représentent les distances à la racine. Nous nous intéressons ici au temps local du mouvement brownien indexé par l'arbre brownien, $(\mathcal{L}^r)_{r\in\mathbb{R}}$, défini par la relation :

$$\int_0^\sigma ds\ F(\Lambda_s) = \int_{\mathbb{R}} dr\ F(r)\mathcal{L}^r,$$

voir [24, 25] pour l'existence de ce processus. Sous le conditionnement $\sigma = 1$, la mesure $\mathcal{L}^r\mathrm{d}r$ est souvent appelée ISE (pour Integrated Super-Brownian Excursion [10]). L'ISE apparaît dans de multiples théorèmes limite de modèles de probabilités discrètes mais aussi dans une variété de modèles de physique statistique [46, 80, 81]. Dans [24, 25] Mireille Bousquet-Mélou et Svante Janson montrent que conditionnellement au temps de vie σ on a :

$$\mathcal{L}^0 \overset{(d)}{=} \frac{2^{\frac{3}{4}}}{3} T^{-\frac{1}{2}} \cdot \sigma^3$$

où T est une variable positive stable d'indice (2/3) de transformée de Laplace $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{\frac{2}{3}})$. Leur preuve repose sur des passages à la limite en utilisant des modèles discrets d'arbres étiquetés. Une des motivations de notre travail [78] est de trouver un argument purement continu pour obtenir cette relation en loi.

Présentation de nos résultats

Comme expliqué précédemment, nous raisonnons sous la mesure $\mathbb{N}_0$ et posons $\sigma_+ := \int_0^\sigma dr \mathbb{1}_{\Lambda_r>0}$ ainsi que $\sigma_- := \int_0^\sigma dr \mathbb{1}_{\Lambda_r<0}$. Nous étudierons ici le triplet $(\mathcal{L}^0, \sigma_+, \sigma_-)$. En quelques mots, σ_+ represente les points d'étiquettes positives et σ_- ceux avec étiquettes négatives; nous avons en particulier $\sigma = \sigma_+ + \sigma_-$ puisque la masse de points d'étiquette nulle est négligeable. Nous renvoyons à la Figure 3.1 pour une représentation visuelle de ces quantités. L'un des résultats principaux de notre travail [78] est de caractériser la transformée de Laplace du triplet $(\mathcal{L}^0, \sigma_+, \sigma_-)$ i.e.

$$\mathbb{N}_0(1 - \exp(-\lambda\mathcal{L}^0 - \mu_1\sigma_+ - \mu_2\sigma_-)) , \quad \lambda, \mu_1, \mu_2 > 0.$$

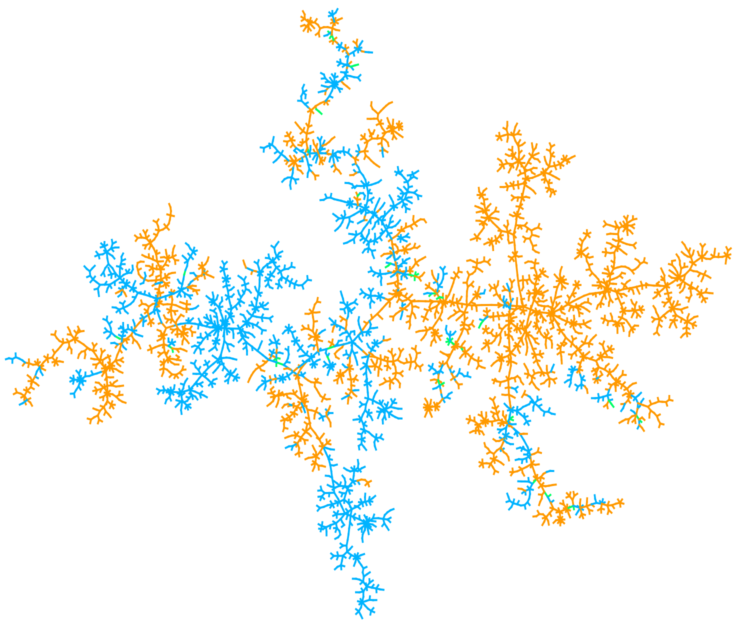


Figure 3.1 – Nous réutilisons l'approximation du mouvement brownien indexé par l'arbre brownien de la figure 1.4 et nous colorions les points en fonction du signe des étiquettes. En bleu sont représentés les points d'étiquette positive et en orange les points d'étiquette négative. En vert sont représentés les points d'étiquette (presque) nulle.

Plus précisément nous montrons le résultat suivant:

Proposition 3

Pour tous $\lambda, \mu_1, \mu_2 > 0$, la quantité $\mathbb{N}_0(1 - \exp(-\lambda\mathcal{L}^0 - \mu_1\sigma_+ - \mu_2\sigma_-))$ est l'unique solution positive de l'équation $h_{\mu_1,\mu_2}(v) = \sqrt{6}\,\lambda$, où, pour tout $v \geqslant 0$,

$$h_{\mu_1,\mu_2}(v) = \sqrt{\sqrt{2\mu_1} + v}\,\Big(2v - \sqrt{2\mu_1}\Big) + \sqrt{\sqrt{2\mu_2} + v}\,\Big(2v - \sqrt{2\mu_2}\Big).$$

Lorsque $\mu_1 = \mu_2$, cette équation peut être résolue de manière explicite. On obtient la formule suivante:

$$\mathbb{N}_0\Big(1 - \exp(-\lambda\mathcal{L}^0 - \mu\sigma)\Big) = \begin{cases} \sqrt{2\mu}\, \cos\Big(\frac{2}{3}\, \arccos\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{si } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \leqslant 1, \\ \sqrt{2\mu}\, \cosh\Big(\frac{2}{3}\, \operatorname{arcosh}\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{si } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \geqslant 1. \end{cases} \tag{3.1}$$

Mentionnons que nous pouvons retrouver la loi conditionnelle de $\mathcal{L}^0$ sachant $\sigma = \sigma_+ + \sigma_-$ en utilisant (3.1). Il est cependant plus simple d'utiliser la proposition 3 pour retrouver le résultat de Mireille Bousquet-Mélou et Svante Janson à l'aide d'une inversion de Lagrange. Un autre cas qui peut être étudié avec nos méthodes est celui où $\mu_2 = 0$. Dans ce cas, en faisant appel à des méthodes combinatoires telles que l'inversion de Lagrange et la recherche de bonne paramétrisation rationnelle, nous montrons que:

Théorème 7

Conditionnellement à σ_+, le temps local $\mathcal{L}^0$ est distribué comme $(2^{9/4}/3)\, D\, T^{-1/2} \cdot \sigma_+^3$ où D est une variable aléatoire de densité $2x\,\mathbf{1}_{[0,1]}(x)$ indépendante de T.

Interprétation en géométrie brownienne

Expliquons brièvement l'interprétation de ces résultats pour la sphère brownienne (libre). Comme rappelé plus haut, les étiquettes représentent les distances à la racine de la sphère brownienne. Souvenons-nous aussi que la sphère brownienne a un point marqué, correspondant à la racine de l'arbre brownien. Ce point est uniforme dans la sphère brownienne et a pour étiquette 0. La quantité $\mathcal{L}^0$ peut alors être interprétée comme une bonne notion de la taille des points d'étiquettes 0, i.e. de la sphère centrée en la racine et de rayon $-\min\Lambda$, qui correspond à la distance entre les deux points distingués de la sphère brownienne. D'autre part, σ_- est exactement le volume de la boule centrée en la racine et de rayon $-\min\Lambda$, et σ_- est le volume du complémentaire de cette boule. Le résultat de Mireille Bousquet-Mélou et Svante Janson, que nous retrouvons, donne donc la loi de la taille de la sphère de rayon $-\min\Lambda$ conditionnellement au volume total. Similairement, le théorème 7 donne la loi de la taille de la sphère de rayon $-\min\Lambda$ conditionnellement au volume de la boule qu'elle délimite.

Illustration de la méthode pour obtenir la proposition 3

Concluons en donnant l'idée de la démonstration de la proposition 3. Prenons l'arbre brownien et enlevons tous les points d'étiquette 0. Nous obtenons ainsi une famille dénombrable $(\mathcal{C}_i)_{i\in\mathbb{N}}$ de composantes connexes. Sur la figure 3.1, ces composantes connexes sont les zones monochromatiques bleues ou oranges. D'après les résultats de [2], nous pouvons définir une notion de longueur pour chaque bord de $\mathcal{C}_i$ que l'on note $Z(\mathcal{C}_i)$. Nous montrons alors qu'il existe un processus $(Y_r)_{r\geqslant 0}$, pouvant être interprété comme l'excursion d'un processus de branchement avec mécanisme $\frac{8}{3}\lambda^{\frac{3}{2}}$, tel que, $\mathbb{N}_0$ – presque partout, les sauts de Y coïncident exactement avec les quantités $\big(Z(\mathcal{C}_i)\big)_{i\in\mathbb{N}}$ et:

$$\mathcal{L}^0 = \int_0^\infty dr\ Y_r.$$

Rappelons que le long des branches de l'arbre brownien, les étiquettes décrivent un mouvement brownien. Intuitivement, Y_r correspond au nombre de points v d'étiquette 0 sur l'arbre tels que le mouvement brownien partant de la racine jusqu'à v a accumulé un temps local en 0 égal à r. Pour faire le lien avec la section 2.2, Y est distribué exactement comme le processus $X^\bullet$ de la sphère brownienne. Toujours d'après [2], conditionnellement à Y nous avons les propriétés suivantes

- Les variables $(\mathcal{C}_i)_{i\in\mathbb{N}}$ sont indépendantes;
- Pour tout $i \in \mathbb{N}$, les étiquettes sur $\mathcal{C}_i$ sont positives avec probabilité $\frac{1}{2}$ et négatives sinon;
- Pour tout $i \in \mathbb{N}$, le volume de $\mathcal{C}_i$ est distribué comme $Z(\mathcal{C}_i)^2 \cdot U$ où U est une variable indépendante de tout le reste et de transformée de Laplace:

$$\forall \mu > 0,\ \mathbb{E}[\exp(-\mu U)] = (1+\sqrt{2\mu})\exp(-\sqrt{2\mu}).$$

Nous obtenons ainsi, pour tout $\lambda, \mu_1, \mu_2 > 0$, l'équation:

$$\mathbb{N}_0\big(1-\exp(-\lambda\mathcal{L}^0-\mu_1\sigma_+-\mu_2\sigma_-)\big) = \mathbb{N}_0\Big(1-\exp\Big(-\lambda\int_0^\infty \mathrm{d}r\, Y_r\Big)\prod_{i=0}^\infty F(\mu_1,\mu_2,(\Delta Y_{r_i})^2)\Big),$$

où $(r_i)_{i\geqslant 0}$ est la collection des instants de saut de Y et F est la fonction définie pour tout $\mu_1, \mu_2, x > 0$ par:

$$F(\mu_1,\mu_2,x) := \frac{1}{2}\Big((1+\sqrt{2\mu_1 x})\ \exp(-\sqrt{2\mu_1 x}) + (1+\sqrt{2\mu_2 x})\ \exp(-\sqrt{2\mu_2 x})\Big).$$

Le processus Y est bien connu et le reste de la preuve repose sur des procédures classiques de théorie des processus.

3.2 Calculs concernant le disque brownien

Présentons finalement un projet en cours – que nous avons décidé d'intégrer à ce manuscrit dans une forme très préliminaire – afin de rendre compte de la flexibilité des méthodes présentées dans les sections précédentes. Cette section peut être conçue comme une présentation de prolongements à venir.

Commençons par rappeler que $\mathbb{D}_z$ désigne le disque brownien libre de périmètre z. Cet espace n'est pas pointé, a un volume aléatoire et son périmètre est fixe. Dans la section 2.2, nous nous sommes intéressés à l'exploration du disque brownien libre en suivant les distances par rapport au bord. Il nous a semblé naturel de nous demander s'il était aussi possible d'étudier une exploration métrique depuis un point du bord. L'une des motivations du chapitre 8 sera de donner quelques réponses dans cette direction. Pour ce faire nous introduirons une version à périmètre aléatoire du disque brownien libre avec deux points distingués sur le bord.
Expliquons cela plus en détail. Le disque brownien libre est muni d'une mesure volume mais aussi d'une mesure uniforme sur son bord, voir [71, 72]. Notons $\mathbb{D}_z^{b,\bullet\bullet}$ la loi du disque brownien libre à périmètre z après avoir pris deux points aléatoires – l'un après l'autre – selon la mesure uniforme du bord du disque brownien. Dans cette introduction, le symbole b fait référence au fait que les points distingués appartiennent au bord et non pas à l'intérieur du disque brownien comme dans les sections 2.2 et 2.3. Nous notons alors $\mathbb{D}^{b,\bullet\bullet}$ pour un disque brownien libre bi-pointé (sur son bord) mais dont le périmètre est distribué selon la mesure $\sqrt{3/(2\pi)}z^{-3/2}\mathrm{d}z$, pour $z > 0$. Par analogie avec la sphère brownienne, nous appelons le premier point *la racine* (noté ρ^*) et le second point *le point marqué* (noté $\rho^\bullet$). La raison pour laquelle nous considérons cette version bi-pointée est qu'il est désormais possible de définir une notion de *hull* par rapport à la racine. Afin d'éviter certaines complications dues à des conditionnements, nous avons de plus décidé de considérer un périmètre aléatoire. De manière un peu étonnante, nos méthodes nous permettent aussi d'étudier des cellules de Voronoï dans $\mathbb{D}^{b,\bullet\bullet}$. Mentionnons que l'étude des cellules de Voronoï dans des modèles aléatoires tels que les arbres de Lévy ou les modèles de géométrie brownienne reste un sujet mal compris. Dans [33], Guillaume Chapuy conjecture qu'en prenant n points uniformes sur la sphère brownienne, les volumes de ces cellules de Voronoï suivent une loi de Dirichlet$(1,\dots,1)$. Il conjecture également ce résultat pour d'autres modèles, et l'unique modèle à ce jour résolu est celui de l'arbre brownien [5]. Le cas $n = 2$ pour la sphère brownienne a été établi par Emmanuel Guitter [52] en considérant de grandes quadrangulations et en passant à la limite à l'aide de la bijection avec retard de Grégory Miermont [84]. Nos méthodes permettent d'obtenir certains résultats concernant les cellules de Voronoï sur le disque brownien et pourraient amener à retrouver les résultats d'Emmanuel Guitter. Donnons ci-après un aperçu de nos résultats préliminaires.

Constructions d'espaces métriques en utilisant une épine finie

Nos résultats reposent sur des constructions de modèles de géométrie brownienne à partir d'arbres étiquetés. Le cadre dans lequel nous travaillons est particulièrement inspiré de notre travail sur les modèles non-compacts ainsi que de l'article [14]. À toute fonction continue $g : [0,\zeta_g] \mapsto \mathbb{R}_+$ restant strictement positive sur $]0,\zeta_g[$ et vérifiant $g(0) = g(\zeta_g) = 0$, nous associons un arbre étiqueté comme suit. Commençons par introduire une mesure ponctuelle de Poisson:

$$\mathcal{P}_g = \sum_{i\in I} \delta_{t_i,\mathbf{e}^i} \ ,$$

d'intensité $2\mathbb{1}_{[0,\zeta_g]}(t)\mathrm{d}t\,\mathbf{n}(\mathrm{d}\mathbf{e})$. Nous considérons alors l'union disjointe:

$$[0,\zeta_g] \cup \bigcup_{i\in I} \mathcal{T}_{e^i} \ ,$$

et nous interprèterons ici le segment $[0,\zeta_g]$ comme une ligne horizontale que nous nommerons *épine*. Pour $i \in I$, "nous recollons l'arbre $\mathcal{T}_{e^i}$ à gauche de $[0,\zeta_g]$ sur le point t_i" et notons $\mathfrak{H}_g$ l'arbre ainsi obtenu. Nous rajoutons des étiquettes sur $\mathfrak{H}_g$ de la manière suivante:

- Les étiquettes le long de l'épine $[0,\zeta_g]$ suivent la fonction g;
- Pour $i \in I$, les étiquettes le long de $\mathcal{T}_{e^i}$ suivent un mouvement brownien indexé par l'arbre $\mathcal{T}_{e^i}$ (et commençant à l'étiquette du point de l'épine t_i i.e. $g(t_i)$).

Comme dans la section 2.3, nous coupons, pour tout $i \in I$, les branches de $\mathcal{T}_{e^i}$ au premier point d'étiquette 0. Nous écrivons $\widetilde{\mathfrak{H}}_g$ pour la structure d'arbre étiqueté ainsi obtenue.

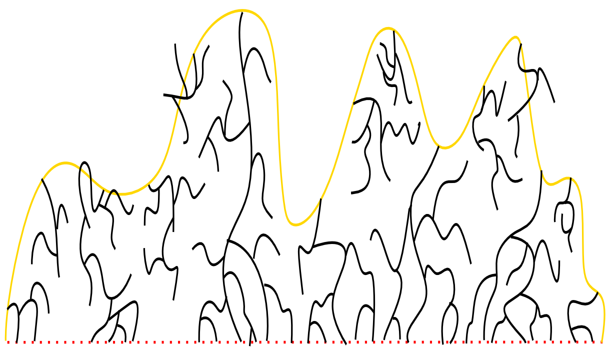


Figure 3.2 – Illustration de l'arbre étiqueté $\widetilde{\mathfrak{H}}_g$. Nous représentons en rouge le niveau 0 et en doré l'épine. Par soucis de clarté de l'image nous avons représenté les arbres presque exclusivement sous l'épine. En réalité les arbres croissent l'épine infiniment souvent.

Si la fonction g vérifie de bonnes conditions, nous pouvons adapter la construction de la sphère brownienne à l'aide du mouvement brownien indexé par l'arbre brownien. Cela nous permet d'associer à $\widetilde{\mathfrak{H}}_g$ un élément (aléatoire) de $\mathbb{K}$ noté $\Theta(\widetilde{\mathfrak{H}}_g)$. La racine correspond au point 0 de l'épine $[0,\zeta_g]$. Exactement comme dans la section 2.3 nous devons adapter la construction de la sphère brownienne avec soin pour éviter de recoller ensemble tous les points d'étiquette 0. Il est à nouveau possible de définir une variable aléatoire $Z(\mathcal{P}_g)$ rendant compte de la quantité de points d'étiquette 0. Pour fixer les notations, nous désignerons par $\partial_1\widetilde{\mathfrak{H}}_g$ l'ensemble des points d'étiquette 0 et par $\partial_2\widetilde{\mathfrak{H}}_g$ l'épine; nous verrons $\partial_1\widetilde{\mathfrak{H}}_g \cup \partial_2\widetilde{\mathfrak{H}}_g$ comme le bord de $\widetilde{\mathfrak{H}}_g$. Nous interpréterons alors la variable ζ_g comme la longueur de $\partial_2\widetilde{\mathfrak{H}}_g$ et $Z(\widetilde{\mathfrak{H}}_g)$ comme la longueur de $\partial_1\widetilde{\mathfrak{H}}_g$. Finalement nous introduirons une variable $\mathcal{Y}(\widetilde{\mathfrak{H}}_g)$ donnant sens au volume total de points de l'arbre $\widetilde{\mathfrak{H}}_g$. Les quantités $(\zeta_g, Z(\widetilde{\mathfrak{H}}_g), \mathcal{Y}(\widetilde{\mathfrak{H}}_g))$ auront des interprétations géométriques naturelles quand nous relierons l'espace $\Theta(\widetilde{\mathfrak{H}}_g)$ au disque brownien $\mathbb{D}^{b,\bullet\bullet}$. De plus dans l'espace $\Theta(\widetilde{\mathfrak{H}}_g)$, les étiquettes correspondent aux distances à la partie encodée par $\partial_1\widetilde{\mathfrak{H}}_g$. Avant de présenter les applications de cette construction à l'étude des *hulls* et des cellules de Voronoï de $\mathbb{D}^{b,\bullet\bullet}$, nous rappelons que $\mathbf{n}$ désigne la mesure d'Itô de l'excursion Brownienne (avec normalisation $\mathbf{n}(\sup \mathbf{e} > 1) = 1/2$). Nous introduisons aussi la mesure σ-finie $\mathbf{n}^\bullet$ définie par:

$$\mathbf{n}^\bullet(F(t^\bullet, \mathbf{e})) := \mathbf{n}\Big(\int_0^\sigma \mathrm{d}t\, F(t, \mathbf{e})\Big) \ ,$$

qui est une version biaisée par le temps de vie σ de la mesure $\mathbf{n}$. Sous les mesures $\mathbf{n}$ et $\mathbf{n}^\bullet$, nous considérons aussi, conditionnellement à $\mathbf{e}$, les deux arbres étiquetés $\mathfrak{H}_{\mathbf{e}}$ et $\mathfrak{H}_{\sqrt{3}\mathbf{e}}$.

Études des hulls

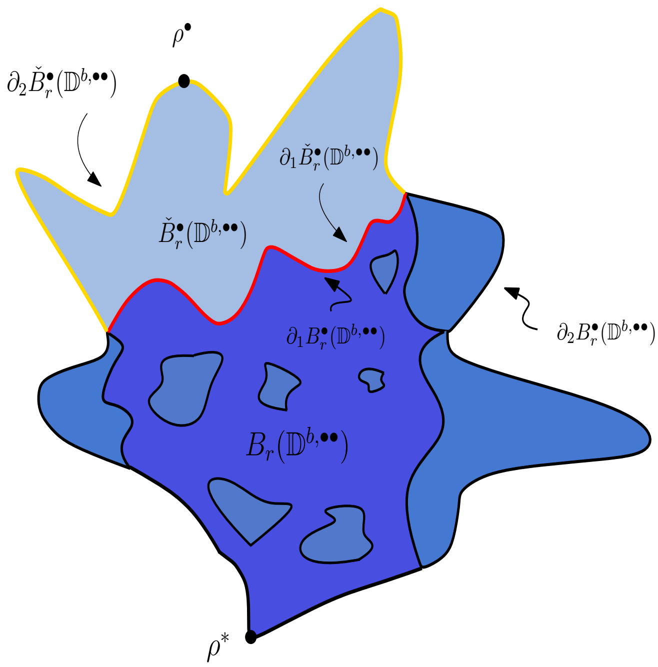
Nous allons maintenant montrer comment les espaces $\widetilde{\mathfrak{H}}_g$, pour des choix convenables de fonctions g aléatoires, interviennent dans la description de certains sous-ensembles remarquables de $\mathbb{D}^{b,\bullet\bullet}$. Commençons par fixer quelques notations. Pour tout $r \geqslant 0$, nous écrivons $B_r(\mathbb{D}^{b,\bullet\bullet})$ pour la boule fermée de rayon r centrée en ρ^*. Rappelons que $\mathbb{D}^{b,\bullet\bullet}$ est muni d'une distance et d'une mesure volume que l'on note respectivement Δ et V pour simplifier les notations. Fixons $r > 0$ et raisonnons, dans le reste de ce paragraphe, sous l'événement $\{\Delta(\rho^*, \rho^\bullet) > r\}$. Le *hull* de rayon r (noté $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$) est l'ensemble des points v de $\mathbb{D}^{b,\bullet\bullet}$ tel que tout chemin connectant v et $\rho^\bullet$ touche la boule $B_r(\mathbb{D}^{b,\bullet\bullet})$. Nous écrivons aussi $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ pour l'adhérence du complémentaire de $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$. Il peut être montré, en utilisant le théorème de Jordan, que les deux espaces $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ ont la topologie du disque unité fermé du plan complexe. Comme pour le disque brownien, nous notons $\partial B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ (resp. $\partial \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$) l'ensemble des points de $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ (resp. $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$) n'ayant pas de voisinage homéomorphe au disque ouvert. Nous voyons alors $\partial B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\partial \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ comme les bords respectifs de $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$, et posons:

$$\partial_1 B_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) = B_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) \cap \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}),$$

$$\partial_2 B_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) = B_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) \cap \partial\mathbb{D}^{b,\bullet\bullet}$$

$$\partial_1 \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) = B_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) \cap \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}),$$

$$\partial_2 \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) = \check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) \cap \partial\mathbb{D}^{b,\bullet\bullet}.$$



Nous pouvons définir une bonne notion de longueur pour tous ces bords de manière analogue aux sections précédentes et nous utiliserons la notation $|\cdot|$ pour nous référer à ces longueurs. Notre objectif est de comprendre la loi des espaces $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$, et de savoir comment ils dépendent l'un de l'autre. Pour donner un énoncé formel, nous devons voir $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ comme des éléments de $\mathbb{K}$. Des méthodes semblables à celles employées pour étudier les *hulls* dans le plan brownien permettent de montrer qu'on peut munir $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ (resp. $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$) de l'extension continue de la distance intrinsèque sur son intérieur. Nous équipons aussi $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ de la restriction de la mesure volume. Nous enracinons $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ au point racine ρ^*. Le choix du point distingué pour $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ est moins évident, et nous décidons de l'enraciner dans un des points de $\partial_1\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet}) \cap \partial_2\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ (par planarité, il peut être montré que cet espace est réduit à deux points). Nous montrons alors la propriété d'indépendance suivante:

Théorème 8

Conditionnellement à $|\partial_1\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})| = z$, le *hull* $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ et l'adhérence de son complémentaire $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ sont indépendants et $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ est distribué comme $\Theta(\widetilde{\mathfrak{H}}_{\sqrt{3}\mathbf{e}})$ sous la mesure $\mathbf{n}^\bullet\big(\cdot \mid Z(\widetilde{\mathfrak{H}}_{\sqrt{3}\mathbf{e}}) = z\big)$.

Nous observons notamment le même phénomène que pour le plan brownien: conditionnellement à la longueur de $|\partial_1\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})|$, la loi de $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ ne dépend pas de r. Posons $c_1 = 4/\sqrt{3}$. Nous conjecturons in fine le résultat suivant:

Conjecture 1

Sous $c_1\mathbf{n}$, l'espace $\Theta(\widetilde{\mathfrak{H}}_{\sqrt{3}\mathbf{e}})$, avec deux points distingués correspondant aux deux extrémités de l'épine, est distribué comme un disque brownien libre à périmètre libre bi-pointé.

En particulier, sous $c_1\mathbf{n}$ et conditionnellement à $\sigma + Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}) = z$, l'espace $\Theta(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ devrait être un disque brownien libre de périmètre z (après oubli de ses points distingués). Nous donnons quelques résultats dans la direction de cette conjecture. Nous montrons par exemple que sous le conditionnement $\sigma + Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}) = z$, le volume total de $\Theta(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ est distribué selon $z^3(2\pi v^5)^{-1/2}\exp(-z^2/2v)\,\mathrm{d}v$ (comme pour le disque brownien libre) et que les deux points distingués sont bien distribués de manière uniforme sur le bord. Observons que c'est la mesure $\mathbf{n}^\bullet$ et non pas $\mathbf{n}$ qui intervient dans le théorème 8, ce qui s'explique par le fait que l'espace $\check{B}_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ vient avec un point distingué supplémentaire, le point $\rho^\bullet$, qui appartient à la partie de la frontière correspondant à l'épine de $\widetilde{\mathfrak{H}}_{\sqrt{3}\mathbf{e}}$. Ce point distingué supplémentaire correspond au temps $t^\bullet$ apparaissant dans la définition de $\mathbf{n}^\bullet$.

Dans le chapitre 8 nous montrons aussi que le *hull* $B_r^\bullet(\mathbb{D}^{b,\bullet\bullet})$ peut être encodé par un arbre étiqueté. Ces codages permettent d'obtenir toute une variété de formules explicites, nous donnons ici l'une d'entre elles:

Proposition 4

Pour tous $\gamma, \mu \geqslant 0$ et $z > 0$, on a:

$$\mathbf{n}^{\bullet}\Big(\exp\big(-\gamma|\partial_2 B_r^{\bullet}(\mathbb{D}^{b,\bullet\bullet})| - \mu V(B_r^{\bullet}(\mathbb{D}^{b,\bullet\bullet}))\big)\,\Big|\, |\partial_1 B_r^{\bullet}(\mathbb{D}^{b,\bullet\bullet})| = z\Big)$$
$$= \frac{2}{3}r^2 \cdot \Big(\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{\frac{3}{2}}\cdot(2\mu)^{\frac14}\coth\Big((2\mu)^{\frac14}r\Big)\Big)^2$$
$$\cdot \exp\Big(-\sqrt{\frac{8}{3}}\cdot\sqrt{\gamma+\sqrt{2\mu}}\cdot r\Big)$$
$$\cdot \exp\Big(-z\Big(\sqrt{\frac{\mu}{2}}\Big(3\coth^2\big((2\mu)^{\frac14}r\big)-2\Big)-\frac{3}{2r^2}\Big)\Big).$$

La preuve de la conjecture 1 est un travail en cours.

Cellules de Voronoï

Présentons maintenant nos résultats concernant les cellules de Voronoï. Si nous enlevons les deux points distingués ρ^* et $\rho^{\bullet}$, le bord $\partial\mathbb{D}^{b,\bullet\bullet}$ est divisé en deux composantes connexes $\mathcal{L}_1^{\circ}$ et $\mathcal{L}_2^{\circ}$, toutes deux homéomorphes à l'intervalle ouvert $(0,1)$. Écrivons $\mathcal{L}_1$ et $\mathcal{L}_2$ pour désigner l'adhérence de ces ensembles. Nous pouvons utiliser la mesure uniforme sur le bord (ou des méthodes semblables à celles introduites dans les sections précédentes) pour définir les longueurs de $\mathcal{L}_1$ et de $\mathcal{L}_2$, notées respectivement $|\mathcal{L}_1|$ et $|\mathcal{L}_2|$. Rappelons que $\mathbb{D}^{b,\bullet\bullet}$ est muni d'une distance Δ. Introduisons maintenant $\mathcal{V}_1$ (resp. $\mathcal{V}_2$), l'ensemble des points $x \in \mathbb{D}^{b,\bullet\bullet}$ tels que $\Delta(x,\mathcal{L}_1) = \Delta(x,\partial\mathbb{D}^{b,\bullet\bullet})$ (resp. $\Delta(x,\mathcal{L}_2) = \Delta(x,\partial\mathbb{D}^{b,\bullet\bullet})$). En d'autres termes, $\mathcal{V}_1$ est la cellule de Voronoï de $\mathcal{L}_1$ par rapport à $\mathcal{L}_2$ et inversement. Nous notons $\mathcal{V}_1^{\circ}$ l'intérieur de $\mathcal{V}_1\backslash\partial\mathbb{D}^{b,\bullet\bullet}$. Nous montrons alors que la distance intrinsèque sur $\mathcal{V}_1^{\circ}$ se prolonge par continuité en une distance sur $\mathcal{V}_1$. Nous pouvons donc munir $\mathcal{V}_1$ de cette extension ainsi que de la restriction de la mesure volume, et le pointer sur la racine ρ^*. Le même raisonnement peut être appliqué à $\mathcal{V}_2$, ce qui nous permet de voir $\mathcal{V}_1$ et $\mathcal{V}_2$ comme deux éléments aléatoires de $\mathbb{K}$.

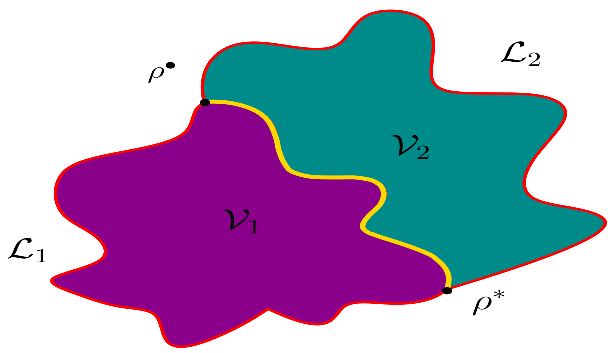


Figure 3.3 – Représentation des cellules de Voronoï $\mathcal{V}_1$ et $\mathcal{V}_2$. Nous colorons en doré l'interface entre les deux cellules.

Le couple $(\mathcal{V}_1, \mathcal{V}_2)$ ainsi obtenu peut être encodé par deux arbres étiquetés comme suit. Prenons l'excursion brownienne $\mathbf{e}$ et, conditionnellement à $\mathbf{e}$, considérons l'arbre étiqueté $\mathfrak{H}_{\mathbf{e}}$ ainsi qu'une copie indépendante de $\mathfrak{H}_{\mathbf{e}}$ notée $\mathfrak{H}'_{\mathbf{e}}$. Nous démontrons le résultat suivant:

Théorème 9

Le couple $(\mathcal{V}_1, \mathcal{V}_2)$ est distribué comme $\big(\Theta(\widetilde{\mathfrak{H}}_{\mathbf{e}}), \Theta(\widetilde{\mathfrak{H}}'_{\mathbf{e}})\big)$ sous $2\mathbf{n}$.

Nous pouvons de plus retrouver la loi de $\mathbb{D}^{b,\bullet\bullet}$ en recollant de manière métrique les cellules $\Theta(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ et $\Theta(\widetilde{\mathfrak{H}}'_{\mathbf{e}})$ le long des points correspondants à l'épine de telle sorte à ce que $\Theta(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ devienne la première cellule de Voronoï et $\Theta(\widetilde{\mathfrak{H}}'_{\mathbf{e}})$ la seconde cellule de Voronoï. L'épine joue ici le rôle d'interface entre les deux cellules de Voronoï. En effectuant cette identification, la variable aléatoire σ donne une bonne notion de longueur de l'interface entre ces deux cellules. Nous obtenons aussi que $Z(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ (resp. $Z(\widetilde{\mathfrak{H}}'_{\mathbf{e}})$) est égale à $|\mathcal{L}_1|$ (resp. $|\mathcal{L}_2|$) et que $\mathcal{Y}(\widetilde{\mathfrak{H}}_{\mathbf{e}})$ (resp. $\mathcal{Y}(\widetilde{\mathfrak{H}}'_{\mathbf{e}})$) correspond au volume total de la première (resp. seconde) cellule de Voronoï. La transformée de Laplace de $(\sigma, Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}), \mathcal{Y}(\widetilde{\mathfrak{H}}_{\mathbf{e}}))$ peut être caractérisée de la manière suivante:

Proposition 5

Pour touq $\lambda, \gamma, \mu \in \mathbb{R}_+$, on a :

$$\mathbf{n}\Big(1 - \exp\big(-(\gamma - \sqrt{\mu})\sigma - (\lambda - \sqrt{\mu})Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}) - \mu/2\mathcal{Y}(\widetilde{\mathfrak{H}}_{\mathbf{e}})\big)\Big) = \sqrt{\frac{2}{3}} \cdot \frac{2\lambda^{\frac{3}{2}} + 2\sqrt{3}(\lambda - \sqrt{\mu})\sqrt{\gamma} + 3(\gamma - \sqrt{\mu})\sqrt{\lambda} + \sqrt{3}\gamma^{\frac{3}{2}}}{2\lambda + 2\gamma - \sqrt{\mu} + 2\sqrt{3}\sqrt{\gamma\lambda}}.$$

Nous donnerons dans le chapitre 8 quelques conséquences directes de la proposition 5. Nous parviendrons aussi à déterminer une formule explicite pour la transformée de Laplace du quadruplet $(Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}), \mathcal{Y}(\widetilde{\mathfrak{H}}_{\mathbf{e}}), Z(\widetilde{\mathfrak{H}}'_{\mathbf{e}}), \mathcal{Y}(\widetilde{\mathfrak{H}}'_{\mathbf{e}}))$. Soulignons enfin que ces transformées de Laplace sont malheureusement difficiles à interpréter bien qu'il semblerait possible de retrouver les résultats d'Emmanuel Guitter concernant les cellules de Voronoï de la sphère brownienne directement dans le continu grâce à notre formule pour la transformée de Laplace de $(Z(\widetilde{\mathfrak{H}}_{\mathbf{e}}), \mathcal{Y}(\widetilde{\mathfrak{H}}_{\mathbf{e}}), Z(\widetilde{\mathfrak{H}}'_{\mathbf{e}}), \mathcal{Y}(\widetilde{\mathfrak{H}}'_{\mathbf{e}}))$. Pour rendre ce rapprochement rigoureux, il faudrait démontrer la conjecture 1.

PARTIE II

PROPRIÉTÉS DE MARKOV SPATIALES EN GÉOMÉTRIE BROWNIENNE ET APPLICATIONS

Chapitre 4

Growth-fragmentation processes in Brownian motion indexed by the Brownian tree

Les resultats de ce chapitre sont issus de l'article [77], écrit en collaboration avec Jean-Francois Le Gall et publié dans *Annals of Probability*.

We consider the model of Brownian motion indexed by the Brownian tree. For every $r \geqslant 0$ and every connected component of the set of points where Brownian motion is greater than r, we define the boundary size of this component, and we then show that the collection of these boundary sizes evolves when r varies like a well-identified growth-fragmentation process. We then prove that the same growth-fragmentation process appears when slicing a Brownian disk at height r and considering the perimeters of the resulting connected components.

Contents

4.1 Introduction

The main goal of the present work is to prove that the collection of boundary sizes of excursions of Brownian motion indexed by the Brownian tree above a fixed level evolves according to a well-identified growth-fragmentation process when the level increases. Because of the close connections between Brownian motion indexed by the Brownian tree and the Brownian map or the Brownian disk, this result also implies that the collection of boundary sizes of the connected components of the set of points of a Brownian disk whose distance from the boundary is greater than r evolves according to the same growth-fragmentation process. The latter fact may be viewed as a continuous analog of a recent result of Bertoin, Curien and Kortchemski [19] identifying the growth-fragmentation process arising as the scaling limit for the collection of lengths of cycles obtained by slicing random Boltzmann triangulations with a boundary at a given height, when the size of the boundary grows to infinity. In fact, the growth-fragmentation process of [19] is the same as in our main results, and this strongly suggests that the results of [19] could be extended to more general planar maps with a boundary (see also [18] for related results).

In order to give a more precise description of our main results, we first need to recall the notion of Brownian motion indexed by the Brownian tree. The Brownian tree of interest here is a variant of Aldous' continuum random tree, which is also called the CRT. This tree is conveniently defined as the tree $\mathcal{T}_\zeta$ coded by a Brownian excursion $(\zeta_s)_{0\leqslant s\leqslant\sigma}$ under the σ-finite Itô measure of positive excursions (see e.g. [74], or Section 5.2.1 below for the definition of this coding). We write ρ for the root of $\mathcal{T}_\zeta$, and we note that $\mathcal{T}_\zeta$ is canonically equipped with a volume measure vol$(\cdot)$. We then consider Brownian motion indexed by $\mathcal{T}_\zeta$, which we denote by $(V_a)_{a\in\mathcal{T}_\zeta}$ – we sometimes also say that V_a is a Brownian label assigned to a. Informally, conditionally on $\mathcal{T}_\zeta$, $(V_a)_{a\in\mathcal{T}_\zeta}$ is just the centered Gaussian process such that $V_\rho = 0$, and $\mathbb{E}[(V_a - V_b)^2] = d_\zeta(a,b)$ for every $a,b\in\mathcal{T}_\zeta$, where d_ζ is the distance on $\mathcal{T}_\zeta$. A formal definition leads to certain technical difficulties because the indexing set is random, but these difficulties can be overcome easily using the formalism of snake trajectories as recalled in Section 5.2.1. Within this formalism, the Brownian tree $\mathcal{T}_\zeta$, and the Brownian motion $(V_a)_{a\in\mathcal{T}_\zeta}$ are defined under a σ-finite measure $\mathbb{N}_0$ – see Section 5.2.2 for more details. We note that both the CRT and Brownian motion indexed by the Brownian tree are important probabilistic objects that appear as scaling limits for several combinatorial models, interacting particle systems and statistical physics models (see the introduction of [2] for a few related references). Furthermore, Brownian motion indexed by the Brownian tree is very closely related to the measure-valued process called super-Brownian motion (see in particular [65]).

Let us now discuss growth-fragmentation processes, referring to [17] and [18] for additional details. The basic ingredient in the construction of a (self-similar) growth-fragmentation process is a self-similar Markov process $(X_t)_{t\geqslant 0}$ with values in $[0,\infty)$ and only negative jumps, which is stopped upon hitting 0. Suppose that $X_0 = z > 0$, and view $(X_t)_{t\geqslant 0}$ as the evolution in time of the mass of an initial particle called the Eve particle. At each time t where the process X has a jump, we consider that a new particle with mass $-\Delta X_t$ (a child of the Eve particle) is born, and the mass of this new particle evolves (from time t) again according to the law of the process X, independently of the evolution of the mass of the Eve particle. Then each child of the Eve particle has children at discontinuity times of its mass process, and so on. We consider that a particle dies when its mass vanishes. Under suitable assumptions (see [17]), we can make sense of the process $(\mathbf{X}(t))_{t\geqslant 0}$ giving for every time t the sequence (in nonincreasing order) of masses of all particles alive at that time (if there are only finitely many such particles, the sequence is completed by adding terms all equal to 0). The process $\mathbf{X}$ is Markovian and is called the growth-fragmentation process with Eve particle process X. In the preceding description, the process starts from $(z, 0, 0, \ldots)$, but we can get a more general initial value by considering several or countably many Eve particles that evolve independently – some assumption is needed on the initial values of these Eve particles so that at every time t the masses of the particles alive can be ranked in a nonincreasing sequence.

Theorem 4.1. *Almost everywhere under the measure $\mathbb{N}_0$, for every $r \geqslant 0$ and for every connected component $\mathcal{C}$ of the open set $\{a \in \mathcal{T}_\zeta : V_a > r\}$, the limit*

$$|\partial\mathcal{C}| := \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{vol}(\{a \in \mathcal{C} : V_a < r+\varepsilon\})$$

exists in $(0,\infty)$ and is called the boundary size of $\mathcal{C}$. For every $r \geqslant 0$, let $\mathbf{X}(r)$ denote the sequence of boundary sizes of all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ ranked in nonincreasing order. Then, under $\mathbb{N}_0$, the process $(\mathbf{X}(r))_{r\geqslant 0}$ is a growth-fragmentation process whose Eve particle process $(X_t)_{t\geqslant 0}$ can be described as follows. The process $(X_t)_{t\geqslant 0}$ is the self-similar Markov process with index $\frac{1}{2}$ which in the case $X_0 = 1$ can be represented as

$$X_t = \exp(\xi(\chi(t))),$$

where $(\xi(s))_{s\geqslant 0}$ is the Lévy process with only negative jumps and Laplace exponent

$$\psi(\lambda) = \sqrt{\frac{3}{2\pi}}\Big(-\frac{8}{3}\lambda + \int_{-\log 2}^{0} (e^{\lambda y} - 1 - \lambda(e^y - 1))\, e^{-3y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y\Big), \tag{4.1}$$

and $(\chi(t))_{t\geqslant 0}$ is the time change

$$\chi(t) = \inf\Big\{s \geqslant 0 : \int_0^s e^{\xi(v)/2}\,\mathrm{d}v > t\Big\}. \tag{4.2}$$

In the setting of Theorem 4.1, we consider the infinite measure $\mathbb{N}_0$, but the statement still makes sense by conditioning on the initial value $\mathbf{X}(0)$. The representation of the self-similar Markov process X in terms of the Lévy process ξ is the classical Lamperti representation of self-similar Markov processes [62]. We note that the process ξ drifs to $-\infty$ and $\chi(t) = \infty$ for $t \geqslant H_0 := \int_0^\infty e^{\xi(v)/2}\,\mathrm{d}v$, which simply means that X_t is absorbed at 0 at time H_0.

It is interesting to relate the growth-fragmentation process of Theorem 4.1 to the local times of the process $(V_a)_{a\in\mathcal{T}_\zeta}$. It is known [25] (see also [96] for closely related results concerning super-Brownian motion) that there exists, $\mathbb{N}_0(\mathrm{d}\omega)$ a.e., a continuous function $(\mathcal{L}_x, x \in \mathbb{R})$ such that, for every nonnegative measurable function f on $\mathbb{R}$,

$$\int_{\mathcal{T}_\zeta} \mathrm{vol}(\mathrm{d}a)\, f(V_a) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\,\mathcal{L}_x,$$

and we call $\mathcal{L}_x$ the local time at level x. Then, for every $r > 0$, if $\mathcal{N}_\varepsilon^r$ denotes the number of connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ with boundary size greater than ε, Proposition 4.11 below gives

$$\varepsilon^{3/2}\,\mathcal{N}_\varepsilon^r \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{\sqrt{6\pi}}\,\mathcal{L}_r\ , \quad \mathbb{N}_0 \text{ a.e.}$$

In other words the suitably rescaled number of fragments of $\mathbf{X}(r)$ with size greater than ε converges to $\mathcal{L}_r$. As a side remark, one might expect the process $(\mathcal{L}_r)_{r\geqslant 0}$ to be Markovian under $\mathbb{N}_0$, by analogy with the classical Ray-Knight theorems for local times of linear Brownian motion. This is not the case, but the previous display shows that $\mathcal{L}_r$ is a function of the Markov process $\mathbf{X}(r)$ which obviously contains more information than the local time.

Thanks to the excursion theory developed in [2], we can in fact deduce Theorem 4.1 from a simpler statement valid under the "positive Brownian snake excursion measure" $\mathbb{N}_0^*$ introduced and studied in [2]. We refer to Section 5.2.2 for more details, but note that we can still make sense of the "genealogical tree" $\mathcal{T}_\zeta$ and the "labels" V_a, $a \in \mathcal{T}_\zeta$ under $\mathbb{N}_0^*$. However, we now have $V_a \geqslant 0$ for every $a \in \mathcal{T}_\zeta$, and more precisely the labels V_b are positive along the ancestral line of a, except at the root and possibly at a. Informally the measure $\mathbb{N}_0^*$ describes the subtree and the labels corresponding under $\mathbb{N}_0$ to a connected component of the set of points with positive labels. One can make sense under $\mathbb{N}_0^*$ of the boundary size

$$\mathcal{Z}_0^* := \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{vol}(\{a \in \mathcal{T}_\zeta : V_a < \varepsilon\})\ , \qquad \mathbb{N}_0^* \text{ a.e.}$$

and define the conditional probability measures $\mathbb{N}_0^{*,z}(\cdot) = \mathbb{N}_0^*(\cdot \mid \mathcal{Z}_0^* = z)$ for every $z > 0$.

Theorem 4.2. *Let $z > 0$. Almost surely under the measure $\mathbb{N}_0^{*,z}$, for every $r \geqslant 0$ and for every connected component $\mathcal{C}$ of the open set $\{a \in \mathcal{T}_\zeta : V_a > r\}$, the limit*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{vol}(\{a \in \mathcal{C} : V_a < r + \varepsilon\})$$

exists in $(0, \infty)$ and is called the boundary size of $\mathcal{C}$. For every $r \geqslant 0$, let $\mathbf{Y}(r)$ denote the sequence of boundary sizes of all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ ranked in nonincreasing order. Then, under $\mathbb{N}_0^{*,z}$, the process $(\mathbf{Y}(r))_{r\geqslant 0}$ is distributed as the growth-fragmentation process of Theorem 4.1 with initial value $\mathbf{Y}(0) = (z, 0, 0, \ldots)$.

Theorem 4.1 will be derived as a straightforward consequence of Theorem 4.2 and the excursion theory of [2]. Both Theorems 4.1 and 4.2 have direct applications to the models of random geometry known as the Brownian map and the Brownian disk. Recall that the Brownian map is a random compact metric space homeomorphic to the sphere $\mathbb{S}^2$, which is the scaling limit of various classes of random planar maps equipped with the graph distance (see in particular [67, 85]). Similarly, the Brownian disk is a random compact metric space homeomorphic to the closed unit disk of the plane, which appears as the scaling limit of rescaled Boltzmann quadrangulations with a boundary, when the size of the boundary grows to infinity (see [20, 22, 55]). We note that the papers [20, 22] consider Brownian disks with fixed boundary size and volume, but in the present work we will be interested in the free Brownian disk [22, Section 1.5] which has a fixed boundary size but a random volume. Let us write $\mathbb{D}_z$ for the free Brownian disk with boundary size $z > 0$. The space $\mathbb{D}_z$ is equipped with a volume measure denoted by $\mathbf{V}(\mathrm{d}x)$. The boundary $\partial\mathbb{D}_z$ may be defined as the set of all points of $\mathbb{D}_z$ that have no neighborhood homeomorphic to the open unit disk, and for every $x \in \mathbb{D}_z$, we write $H(x)$ for the "height" of x, meaning the distance from x to the boundary $\partial\mathbb{D}_z$.

Theorem 4.3. *Almost surely, for every $r \geqslant 0$, for every connected component $\mathcal{C}$ of $\{x \in \mathbb{D}_z : H(x) > r\}$, the limit*

$$|\partial\mathcal{C}| := \lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathbf{V}(\{x \in \mathcal{C} : H(x) < r + \varepsilon\})$$

exists and is called the boundary size of $\mathcal{C}$. For every $r \geqslant 0$, let $\mathbf{Z}(r)$ denote the sequence of boundary sizes of all connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$ ranked in nonincreasing order. Then, the process $(\mathbf{Z}(r))_{r\geqslant 0}$ is distributed as the growth-fragmentation process of Theorem 4.1 with initial value $\mathbf{Z}(0) = (z, 0, 0, \ldots)$.

As the similarity between the two statements suggests, Theorem 4.3 is closely related to Theorem 4.2, and in fact can be derived from the latter result thanks to the construction of the free Brownian disk (with boundary size z) from a snake trajectory distributed according to $\mathbb{N}_0^{*,z}$, which is developed in [71]. Similarly, we could use Theorem 4.1 to derive a result analogous to Theorem 4.3 for the free Brownian map, thanks to the construction of the latter metric space from a snake trajectory distributed according to $\mathbb{N}_0$ (see e.g. [71, Section 3]). Rather than writing down this statement about the free Brownian map, we give in Section 4.11 an analog of Theorem 4.3 for the Brownian plane, which is an infinite-volume version of the Brownian map that has been shown [29, 39] to be the universal scaling limit of infinite random lattices such as the UIPT or the UIPQ. Theorem 4.7 below shows that the collection of boundary sizes of the connected components of the complement of the ball of radius r centered at the root of the Brownian plane evolves like the same growth-fragmentation process with indefinite growth starting from 0 (see [18, Section 4.2] for a thorough discussion of this process).

We next state another result for the Brownian disk, which is closely related to Theorem 4.3.

Theorem 4.4. *Let $r > 0$. On the event $\{\sup\{H(x) : x \in \mathbb{D}_z\} > r\}$, let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ be the connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$ ranked in nonincreasing order of their boundary* *sizes, and for every $j = 1, 2, \ldots$, let d_j denote the intrinsic metric induced by the Brownian disk metric on the open set $\mathcal{C}_j$. Then, a.s. on the event $\{\sup\{H(x) : x \in \mathbb{D}_z\} > r\}$, for every $j = 1, 2, \ldots$ the metric d_j has a continuous extension to the closure $\overline{\mathcal{C}}_j$ of $\mathcal{C}_j$ in $\mathbb{D}_z$, and this extension is a metric on $\overline{\mathcal{C}}_j$. Furthermore, conditionally on the sequence of boundary sizes $(|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \ldots)$, the metric spaces $(\overline{\mathcal{C}}_1, d_1), (\overline{\mathcal{C}}_2, d_2), \ldots$ are independent free Brownian disks with respective boundary sizes $|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \ldots$.*

In the same way as Theorem 4.3 follows from Theorem 4.2, Theorem 4.4 is a consequence of a statement (Theorem 4.6 below) that describes the conditional distribution of the snake trajectories corresponding to the "excursions above level h" under $\mathbb{N}_0^{*,z}$, conditionally on the boundary sizes of these excursions. One might expect that Theorem 4.6, which essentially corresponds to the branching property of growth-fragmentation processes, would be a basic tool for the proof of Theorem 4.2, but in fact our proof of Theorem 4.2 does not use this branching property. We also note that Theorem 4.4 is a Brownian disk analog of a result of [71] showing that the connected components of the complement of a ball in the Brownian map are independent Brownian disks conditionally on their volumes and boundary sizes.

Let us finally mention an interesting corollary to our results.

Corollary 4.1. *There exist positive constants $\mathbf{c}_1$ and $\mathbf{c}_2$ such that, for every $r \geqslant 1$,*

$$\mathbf{c}_1\, r^{-6} \leqslant \mathbb{N}_0^{*,1}\Big(\sup_{a\in\mathcal{T}_\zeta} V_a > r\Big) = \mathbb{P}\Big(\sup_{x\in\mathbb{D}_1} H(x) > r\Big) \leqslant \mathbf{c}_2\, r^{-6}.$$

Corollary 4.1 immediately follows from Theorem 4.2 and Theorem 4.3, by using the asymptotics for the extinction time of growth-fragmentation processes found in [18, Corollary 4.5].

The proof of Theorem 4.2 occupies much of the remaining part of the paper. Let us briefly outline the main steps of this proof. For every $a \in \mathcal{T}_\zeta$ such that $V_a > 0$, one can define a function $(Z_r^{(a)})_{0\leqslant r<V_a}$ such that, for every $r \in [0, V_a)$, $Z_r^{(a)}$ is the boundary size of the connected component of $\{b \in \mathcal{T}_\zeta : V_b > r\}$ that contains a (see Proposition 4.5 below). The function $r \mapsto Z_r^{(a)}$ is càdlàg (right-continuous with left limits) with only negative jumps, and every discontinuity time r_0 of this function corresponds to a "splitting" of the connected component containing a into two components, namely the one containing a, which has boundary size $Z_{r_0}^{(a)}$, and another one with boundary size $|\Delta Z_{r_0}^{(a)}|$. It is not a priori obvious that a splitting cannot yield more than two components, but this follows from the fact that local minima of the process V are distinct, see Section 4.3.1 below. It turns out (Proposition 4.6) that there exists a unique $a^\bullet \in \mathcal{T}_\zeta$, called the terminal point of the locally largest evolution, such that, for every discontinuity time r_0 of $r \mapsto Z_r^{(a^\bullet)}$, we have $Z_{r_0}^{(a^\bullet)} > |\Delta Z_{r_0}^{(a^\bullet)}|$ (meaning that $a^\bullet$ "stays" in the component with the larger boundary size) and $V_{a^\bullet}$ is maximal among the labels of points satisfying the latter property. Furthermore, the distribution of $(Z_r^{(a^\bullet)})_{0\leqslant r<V_{a^\bullet}}$ is the law of the process X of Theorem 4.1 up to its hitting time of 0 (Proposition 4.7). The process $(Z_r^{(a^\bullet)})_{0\leqslant r<V_{a^\bullet}}$ thus plays the role of the evolution of the mass of the Eve particle. Furthermore, one verifies that, conditionally on $(Z_r^{(a^\bullet)})_{0\leqslant r<V_{a^\bullet}}$, for every discontinuity time r_0, the connected component that splits off the one containing $a^\bullet$ at

time r_0 is represented by a snake trajectory distributed according to $\mathbb{N}_0^{*,|\Delta Z^{(a)}_{r_0}|}$ (Proposition 4.8). This provides the recursive structure needed to identify the process $\mathbf{Y}(r)$ of Theorem 4.2 as a growth-fragmentation process.

We finally mention a few recent papers that are related to the present work. We refer to [17, 18, 95] for the theory of growth-fragmentation processes. As we already mentioned, Theorem 4.3 can be viewed as a continuous version of the main result of [19]. In addition to [20, 22, 55], free Brownian disks are discussed in the paper [86], which develops an axiomatic characterization of the Brownian map as part of a program aiming to equip the Brownian map with a canonical conformal structure. Brownian disks also play an important role in the recent papers [53, 56] of Gwynne and Miller motivated by the study of statistical physics models on random planar maps. Finally we observe that there is an interesting analogy between Theorem 4.1 and the fragmentation process occurring when cutting the CRT at a fixed height. According to [16], the sequence of volumes of the connected components of the complement of the ball of radius r centered at the root in the CRT is a self-similar fragmentation process whose dislocation measure has the form $(2\pi)^{-1/2}(x(1-x))^{-3/2}\,\mathrm{d}x$. Notice that the Lévy measure of the process ξ of Theorem 4.1 is the push forward of the measure $\mathbf{1}_{[1/2,1]}(x)\,\sqrt{3/2\pi}\,(x(1-x))^{-5/2}\,\mathrm{d}x$ under the mapping $x \mapsto \log x$.

The paper is organized as follows. Section 5.2 gives a number of preliminaries. In particular, we recall the formalism of snake trajectories, which provides a convenient set-up for the study of Brownian motion indexed by the Brownian tree. We also give a "re-rooting" representation of the measure $\mathbb{N}_0^{*,z}$, which is a key tool in several subsequent proofs. Section 4.3 discusses the connected components of the tree $\mathcal{T}_\zeta$ above a fixed level and also the components "above the minimum": the independence and distributional properties of the latter have been studied already in the paper [2] and play a basic role in the proof of Theorem 4.2. Section 4.4 is devoted to the existence and properties of the boundary size processes $(Z_r^{(a)})_{0\leqslant r<V_a}$. In this section, we rely on the theory of exit measures for the Brownian snake [65]. Section 4.5 introduces the locally largest evolution, and Section 4.6 identifies the law of the associated boundary size process (Proposition 4.7). A key tool for this identification is Proposition 4.4, which gives the distribution under $\mathbb{N}_0$ of the exit measure process time-reversed at its last visit to $z>0$. Section 4.7 studies the excursions from the locally largest evolution. Roughly speaking, this study provides the recursive structure that shows that the "children" of the Eve particle evolve according to the same Markov process. Theorems 4.1 and 4.2 are then proved in Section 4.8, and Theorem 4.3 is derived from Theorem 4.2 in Section 4.9. Section 4.10 gives the proof of Theorem 4.4. Finally, Section 4.11 contains some complements. In particular, we provide a direct derivation of the cumulant function associated with our growth-fragmentation processes, which is independent of the proof of the main results. We also discuss the analog of Theorem 4.3 for the Brownian plane, and we investigate the relations between local times of $(V_a)_{a\in\mathcal{T}_\zeta}$ and the growth-fragmentation process of Theorem 4.1. The Appendix gives the proof of two technical results.

4.2 Preliminaries

4.2.1 Snake trajectories

Most of this work is devoted to the study of random processes indexed by continuous random trees. The formalism of snake trajectories, which has been introduced in [2], provides a convenient framework for this study, and we recall the main definitions that will be needed below.

A (one-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})}$ is called the lifetime of w. We let $\mathcal{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geqslant 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. We set $\mathcal{W}_0 = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = 0\}$. The trivial element of $\mathcal{W}_0$ with zero lifetime is identified with the point 0 of $\mathbb{R}$. Occasionally we will use the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leqslant t \leqslant \zeta_{(\mathrm{w})}\}$.

Definition 4.1. *A snake trajectory (with initial point* 0*) is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_0$ which satisfies the following two properties:*

(i) *We have $\omega_0 = 0$ and the number $\sigma(\omega) := \sup\{s \geqslant 0 : \omega_s \neq 0\}$, called the duration of the snake trajectory ω, is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = 0$ for every $s \geqslant 0$).*

(ii) *For every $0 \leqslant s \leqslant s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s \leqslant r \leqslant s'} \zeta_{(\omega_r)}]$.*

We will write $\mathcal{S}_0$ for the set of all snake trajectories. If $\omega \in \mathcal{S}_0$, we often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geqslant 0$. The set $\mathcal{S}_0$ is equipped with the distance

$$d_{\mathcal{S}_0}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geqslant 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

A snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$: See [2, Proposition 8].

Let $\omega \in \mathcal{S}_0$ be a snake trajectory and $\sigma = \sigma(\omega)$. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact $\mathbb{R}$-tree, which will be denoted by $\mathcal{T}_\zeta$ and called the *genealogical tree* of the snake trajectory. This $\mathbb{R}$-tree is the quotient space $\mathcal{T}_\zeta := [0, \sigma]/\sim$ of the interval $[0, \sigma]$ for the equivalence relation

$$s \sim s' \text{ if and only if } \ \zeta_s = \zeta_{s'} = \min_{s \wedge s' \leqslant r \leqslant s \vee s'} \zeta_r,$$

and $\mathcal{T}_\zeta$ is equipped with the distance induced by

$$d_\zeta(s, s') = \zeta_s + \zeta_{s'} - 2 \min_{s \wedge s' \leqslant r \leqslant s \vee s'} \zeta_r.$$

(notice that $d_\zeta(s, s') = 0$ if and only if $s \sim s'$, and see e.g. [74, Section 3] for more information about the coding of $\mathbb{R}$-trees by continuous functions). Let $p_\zeta : [0, \sigma] \longrightarrow \mathcal{T}_\zeta$ stand for the canonical projection. By convention, $\mathcal{T}_\zeta$ is rooted at the point $\rho := p_\zeta(0) = p_\zeta(\sigma)$, and the volume measure $\mathrm{vol}(\cdot)$ on $\mathcal{T}_\zeta$ is defined as the push forward of Lebesgue measure on $[0,\sigma]$ under p_ζ. For every $a, b \in \mathcal{T}_\zeta$, $[\![a,b]\!]$ denotes the line segment from a to b, and the ancestral line of a is the segment $[\![\rho,a]\!]$ (a point b of $[\![\rho,a]\!]$ is called an ancestor of a, and we also say that a is a descendant of b). We use the notation $]\!]a,b[\![$ or $]\!]a,b]\!]$ with an obvious meaning. Branching points of $\mathcal{T}_\zeta$ are points c such that $\mathcal{T}_\zeta\backslash\{c\}$ has at least 3 connected components.

Let us now make a crucial observation: By property (ii) in the definition of a snake trajectory, the condition $p_\zeta(s) = p_\zeta(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ can be viewed as defined on the quotient space $\mathcal{T}_\zeta$ (this is indeed the main motivation for introducing snake trajectories: replacing mappings defined on trees, which later will be random trees, by mappings defined on intervals of the real line). For $a \in \mathcal{T}_\zeta$, we set $V_a(\omega) := \widehat{W}_s(\omega)$ whenever $s \in [0,\sigma]$ is such that $a = p_\zeta(s)$ – by the previous observation this does not depend on the choice of s. We interpret V_a as a "label" assigned to the "vertex" a of $\mathcal{T}_\zeta$. Notice that the mapping $a \mapsto V_a$ is continuous on $\mathcal{T}_\zeta$.

We will use the notation

$$\begin{aligned} W_* &:= \min\{W_s(t) : s \geqslant 0, t \in [0,\zeta_s]\} = \min\{V_a : a \in \mathcal{T}_\zeta\}, \\ W^* &:= \max\{W_s(t) : s \geqslant 0, t \in [0,\zeta_s]\} = \max\{V_a : a \in \mathcal{T}_\zeta\}. \end{aligned}$$

Finally, we will use the notion of a subtrajectory. Let $\omega \in \mathcal{S}_0$ and assume that the mapping $s \mapsto \zeta_s(\omega)$ is not constant on any nontrivial subinterval of $[0,\sigma]$ (this will always hold in our applications). Let $a \in \mathcal{T}_\zeta\backslash\{\rho\}$ such that a has strict descendants and a is not a branching point. Then there exist two times $s_1 < s_2$ in $(0,\sigma)$ such that $p_\zeta(s_1) = p_\zeta(s_2) = a$, and the set $p_\zeta([s_1,s_2])$ consists of all descendants of a in $\mathcal{T}_\zeta$. We define a new snake trajectory ω' with duration $s_2 - s_1$ by setting, for every $s \geqslant 0$,

$$\omega'_s(t) := \omega_{(s_1+s)\wedge s_2}(\zeta_{s_1} + t) - \widehat{\omega}_{s_1}\,, \quad \text{for } 0 \leqslant t \leqslant \zeta'_s := \zeta_{(s_1+s)\wedge s_2} - \zeta_{s_1}.$$

We call ω' the subtrajectory of ω rooted at a. Informally, ω' represents the subtree of descendants of a and the associated labels.

4.2.2 Re-rooting and truncation of snake trajectories

We now introduce two important operations on snake trajectories in $\mathcal{S}_0$. The first one is the re-rooting operation on $\mathcal{S}_0$ (see [2, Section 2.2]). Let $\omega \in \mathcal{S}_0$ and $r \in [0,\sigma(\omega)]$. Then $\omega^{[r]}$ is the snake trajectory in $\mathcal{S}_0$ such that $\sigma(\omega^{[r]}) = \sigma(\omega)$ and for every $s \in [0,\sigma(\omega)]$,

$$\begin{aligned} \zeta_s(\omega^{[r]}) &= d_\zeta(r, r \oplus s), \\ \widehat{W}_s(\omega^{[r]}) &= \widehat{W}_{r\oplus s} - \widehat{W}_r, \end{aligned}$$

where we use the notation $r \oplus s = r + s$ if $r + s \leqslant \sigma$, and $r \oplus s = r + s - \sigma$ otherwise. By a remark following the definition of snake trajectories, these prescriptions completely determine $\omega^{[r]}$.

We will write $\zeta_s^{[r]}(\omega) = \zeta_s(\omega^{[r]})$ and $W_s^{[r]}(\omega) = W_s(\omega^{[r]})$. The tree $\mathcal{T}_{\zeta^{[r]}}$ is then interpreted as the tree $\mathcal{T}_\zeta$ re-rooted at the vertex $p_\zeta(r)$: More precisely, the mapping $s \mapsto r \oplus s$ induces an isometry from $\mathcal{T}_{\zeta^{[r]}}$ onto $\mathcal{T}_\zeta$, which maps the root of $\mathcal{T}_{\zeta^{[r]}}$ to $p_\zeta(r)$. Furthermore, the vertices of $\mathcal{T}_{\zeta^{[r]}}$ receive the "same" labels as in $\mathcal{T}_\zeta$, shifted so that the label of the root is still 0.

The second operation is the truncation of snake trajectories. For any $\mathrm{w} \in \mathcal{W}_0$ and $y \in \mathbb{R}$, we set

$$\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}\,,\ \tau_y^*(\mathrm{w}) := \inf\{t \in (0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$$

with the usual convention $\inf \varnothing = \infty$ (this convention will be in force throughout this work unless otherwise indicated). Notice that $\tau_y(\mathrm{w}) = \tau_y^*(\mathrm{w})$ except possibly if $y = 0$.

Let $\omega \in \mathcal{S}_0$ and $y \in \mathbb{R}$. We set, for every $s \geqslant 0$,

$$\eta_s(\omega) = \inf\left\{t \geqslant 0 : \int_0^t \mathrm{d}u\, \mathbf{1}_{\{\zeta_{(\omega_u)} \leqslant \tau_y^*(\omega_u)\}} > s\right\}$$

(note that the condition $\zeta_{(\omega_u)} \leqslant \tau_y^*(\omega_u)$ holds if and only if $\tau_y^*(\omega_u) = \infty$ or $\tau_y^*(\omega_u) = \zeta_{(\omega_u)}$). Then, setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geqslant 0$ defines an element ω' of $\mathcal{S}_0$, which will be denoted by $\mathrm{tr}_y(\omega)$ and called the truncation of ω at y (see [2, Proposition 10]). The effect of the time change $\eta_s(\omega)$ is to "eliminate" those paths ω_s that hit y (at a positive time when $y = 0$) and then survive for a positive amount of time. The genealogical tree of $\mathrm{tr}_y(\omega)$ is canonically and isometrically identified with the closed subset of $\mathcal{T}_\zeta$ consisting of all a such that $V_b(\omega) \neq y$ for every strict ancestor b of a (excluding the root when $y = 0$). By abuse of notation, we often write $\mathrm{tr}_y(W)$ instead of $\mathrm{tr}_y(\omega)$.

4.2.3 Measures on snake trajectories

We will be interested in two important measures on $\mathcal{S}_0$. First the Brownian snake excursion measure $\mathbb{N}_0$ is the σ-finite measure on $\mathcal{S}_0$ that satisfies the following two properties: Under $\mathbb{N}_0$,

(i) the distribution of the lifetime function $(\zeta_s)_{s \geqslant 0}$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_0\Big(\sup_{s \geqslant 0} \zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon};$$

(ii) conditionally on $(\zeta_s)_{s \geqslant 0}$, the tip function $(\widehat{W}_s)_{s \geqslant 0}$ is a centered Gaussian process with covariance function

$$K(s, s') = \min_{s \wedge s' \leqslant r \leqslant s \vee s'} \zeta_r.$$

Informally, the lifetime process $(\zeta_s)_{s \geqslant 0}$ evolves under $\mathbb{N}_0$ like a Brownian excursion, and conditionally on $(\zeta_s)_{s \geqslant 0}$, each path W_s is a linear Brownian path started from 0, which is "erased" from its tip when ζ_s decreases and is "extended" when ζ_s increases. The measure $\mathbb{N}_0$ can be interpreted as the excursion measure away from 0 for the Markov process in $\mathcal{W}_0$ called the Brownian snake. We refer to [65] for a detailed study of the Brownian snake. For every $r > 0$, we have

$$\mathbb{N}_0(W^* > r) = \mathbb{N}_0(W_* < -r) = \frac{3}{2r^2}$$

(see e.g. [65, Section VI.1]).

The following scaling property is often useful. For $\lambda > 0$, for every $\omega \in \mathcal{S}_0$, we define $\theta_\lambda(\omega) \in \mathcal{S}_0$ by $\theta_\lambda(\omega) = \omega'$, with

$$\omega'_s(t) := \sqrt{\lambda}\,\omega_{s/\lambda^2}(t/\lambda)\ , \quad \text{for } s \geqslant 0,\ 0 \leqslant t \leqslant \zeta'_s := \lambda\zeta_{s/\lambda^2}.$$

Then $\theta_\lambda(\mathbb{N}_0) = \lambda\,\mathbb{N}_0$.

Under $\mathbb{N}_0$, the paths W_s, $0 < s < \sigma$, take both positive and negative values, simply because they behave like one-dimensional Brownian paths started from 0. We will now introduce another important measure on $\mathcal{S}_0$, which is supported on snake trajectories taking only nonnegative values. For $\delta \geqslant 0$, let $\mathcal{S}_0^{(\delta)}$ be the set of all $\omega \in \mathcal{S}_0$ such that $\sup_{s\geqslant 0}(\sup_{t\in[0,\zeta_s(\omega)]} |\omega_s(t)|) > \delta$. Also set

$$\mathcal{S}_0^+ = \{\omega \in \mathcal{S}_0 : \omega_s(t) \geqslant 0 \text{ for every } s \geqslant 0, t \in [0, \zeta_s(\omega)]\} \cap \mathcal{S}_0^{(0)}.$$

There exists a σ-finite measure $\mathbb{N}_0^*$ on $\mathcal{S}_0$, which is supported on $\mathcal{S}_0^+$, and gives finite mass to the sets $\mathcal{S}_0^{(\delta)}$ for every $\delta > 0$, such that

$$\mathbb{N}_0^*(G) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\,\mathbb{N}_0(G(\mathrm{tr}_{-\varepsilon}(W))),$$

for every bounded continuous function G on $\mathcal{S}_0$ that vanishes on $\mathcal{S}_0 \backslash \mathcal{S}_0^{(\delta)}$ for some $\delta > 0$ (see [2, Theorem 23]). Under $\mathbb{N}_0^*$, each of the paths W_s, $0 < s < \sigma$, starts from 0, then stays positive during some time interval $(0, \alpha)$, and is stopped immediately when it returns to 0, if it does return to 0.

One can in fact make sense of the "quantity" of paths W_s that return to 0 under $\mathbb{N}_0^*$. To this end, one proves that the limit

$$\mathcal{Z}_0^* := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s < \varepsilon\}} \tag{4.3}$$

exists $\mathbb{N}_0^*$ a.e. See [2, Proposition 30] for a slightly weaker result – the stronger form stated above follows from the results of [71, Section 10]. Notice that replacing the limit by a liminf in (5.6) allows us to make sense of $\mathcal{Z}_0^*(\omega)$ for every $\omega \in \mathcal{S}_0^+$. The following conditional versions of the measure $\mathbb{N}_0^*$ play a fundamental role in the present work. According to [2, Proposition 33], there exists a unique collection $(\mathbb{N}_0^{*,z})_{z>0}$ of probability measures on $\mathcal{S}_0^+$ such that:

(i) We have

$$\mathbb{N}_0^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2}\,\mathbb{N}_0^{*,z}.$$

(ii) For every $z > 0$, $\mathbb{N}_0^{*,z}$ is supported on $\{\mathcal{Z}_0^* = z\}$.

(iii) For every $z, z' > 0$, $\mathbb{N}_0^{*,z'} = \theta_{z'/z}(\mathbb{N}_0^{*,z})$.

Informally, $\mathbb{N}_0^{*,z} = \mathbb{N}_0^*(\cdot \mid \mathcal{Z}_0^* = z)$.

4.2.4 Exit measures

Let $r \in \mathbb{R}$, $r \neq 0$. In a way similar to the definition of $\mathcal{Z}_0^*$ above, one can make sense of a quantity that measures the number of paths W_s that hit level r under $\mathbb{N}_0$. Precisely, the limit

$$\mathcal{Z}_r := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_r(W_s) \leqslant \zeta_s < \tau_r(W_s)+\varepsilon\}} \tag{4.4}$$

exists $\mathbb{N}_0$ a.e. Furthermore, $\mathcal{Z}_r > 0$ if and only if $r \in [W_*, W^*]$, $\mathbb{N}_0$ a.e. This definition of $\mathcal{Z}_r$ is a particular case of the theory of exit measures, see [65, Chapter V]. We note that $\mathcal{Z}_r$ is $\mathbb{N}_0$ a.e. equal to a measurable function of the truncated snake $\mathrm{tr}_r(W)$: When $r < 0$, this can be seen by observing that $\mathcal{Z}_r$ is the a.e. limit of the quantities $\tilde{\mathcal{Z}}_r^\varepsilon$ introduced in Remark (ii) after Proposition 7.5 below.

We now recall the special Markov property of the Brownian snake under $\mathbb{N}_0$ (see in particular the appendix of [70]).

Proposition 4.1 (Special Markov property). *Let (s_i, s_i'), $i \in I$ be the connected components of the open set $\{s \in [0,\sigma] : \tau_r(W_s) < \zeta_s\}$. For every $i \in I$, set $a_i := p_\zeta(s_i) = p_\zeta(s_i')$ and let ω_i be the subtrajectory of ω rooted at a_i. Then, under the probability measure $\mathbb{N}_0(\cdot \mid r \in [W_*, W^*])$, conditionally on $\mathrm{tr}_r(W)$, the point measure $\sum_{i\in I} \delta_{\omega_i}$ is Poisson with intensity $\mathcal{Z}_r\, \mathbb{N}_0(\cdot)$.*

Let us now explain the relations between exit measures and a certain continuous-state branching process. For $\lambda > 0$, we set

$$\phi(\lambda) := \sqrt{\frac{8}{3}}\, \lambda^{3/2}.$$

This notation will be used throughout this work. The continuous-state branching process with branching mechanism ϕ, or in short the ϕ-CSBP, is the Feller Markov process X in $\mathbb{R}_+$ whose transition kernels are given by the following Laplace transform,

$$\mathbb{E}[\exp(-\lambda X_t) \mid X_0 = x] = \exp\Big(- x \left(\lambda^{-1/2} + t\sqrt{2/3}\right)^{-2}\Big), \tag{4.5}$$

for every $x, t \geqslant 0$ and $\lambda > 0$. See e.g. [65, Chapter II] for basic facts about continuous-state branching processses.

For reasons that will appear later, we now concentrate on the variables $\mathcal{Z}_r$ with $r < 0$. According to [40, formula (6)], we have, for every $t > 0$,

$$\mathbb{N}_0(1 - e^{-\lambda \mathcal{Z}_{-t}}) = \left(\lambda^{-1/2} + t\sqrt{2/3}\right)^{-2}. \tag{4.6}$$

Using both the latter formula and the special Markov property, we get that the process $(\mathcal{Z}_{-r})_{r>0}$ is Markovian under $\mathbb{N}_0$ with the transition kernels of the ϕ-CSBP, with respect to the filtration $(\mathcal{G}_r)_{r>0}$, where $\mathcal{G}_r$ denotes the σ-field generated by $\mathrm{tr}_{-r}(W)$ and the $\mathbb{N}_0$-negligible sets (see [2, Section 2.5], for more details). Although $\mathbb{N}_0$ is an infinite measure, the preceding statement makes sense by considering the process $(\mathcal{Z}_{-\delta-r})_{r\geqslant 0}$ under the probability measure $\mathbb{N}_0(\cdot \mid W_* \leqslant -\delta)$, for every $\delta > 0$. As a consequence, the process $(\mathcal{Z}_{-r})_{r>0}$ has a càdlàg modification under $\mathbb{N}_0$, which we consider from now on.

The distribution of $(\mathcal{Z}_{-r})_{r>0}$ under $\mathbb{N}_0$ can be interpreted as an excursion measure for the ϕ-CSBP, in the following sense. Let $\alpha > 0$, and let

$$\sum_{i\in I} \delta_{\omega_i}$$

be a Poisson measure with intensity $\alpha\,\mathbb{N}_0$. Set $Y_0 = \alpha$ and for every $t > 0$,

$$Y_t = \sum_{i\in I} \mathcal{Z}_{-t}(\omega_i)$$

(note that this is a finite sum since $\mathbb{N}_0(W_* \leqslant r) < \infty$ if $r < 0$). Then the process $(Y_t)_{t\geqslant 0}$ is a ϕ-CSBP started from α. It is enough to verify that Y has the desired one-dimensional marginals and to this end we write, for every $t > 0$, $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(-\alpha\mathbb{N}_0(1 - e^{-\lambda\mathcal{Z}_{-t}}))$ and we use (7.8).

We note that, for every $z > 0$,

$$\lim_{\varepsilon\to 0} \downarrow \mathbb{N}_0\Big(\sup_{0<t\leqslant\varepsilon} \mathcal{Z}_{-t} \geqslant z\Big) = 0. \tag{4.7}$$

Indeed, assuming that this convergence does not hold, the preceding Poisson representation with $\alpha = z/2$ would imply that the probability of the event $\{\sup\{Y_t : 0 < t \leqslant \varepsilon\} \geqslant z\}$ is bounded below by a positive constant independent of ε, which contradicts the right-continuity of paths of Y at time 0. It follows from (4.7) that $\mathcal{Z}_{-t} \longrightarrow 0$ as $t \downarrow 0$, $\mathbb{N}_0$ a.e.

Exit measures allow us to state the following formula, which relates the measures $\mathbb{N}_0^*$ and $\mathbb{N}_0$ via a re-rooting procedure. Let G be a nonnegative measurable function on $\mathcal{S}_0$. Then,

$$\mathbb{N}_0^*\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big) = 2\int_{-\infty}^0 \mathrm{d}b\,\mathbb{N}_0\Big(\mathcal{Z}_b\, G(\mathrm{tr}_b(W))\Big). \tag{4.8}$$

See [2, Theorem 28].

In view of the subsequent developments, it will be important to have a uniform approximation of the exit measure process $(\mathcal{Z}_r)_{r<0}$ under $\mathbb{N}_0$. This is the goal of the next proposition. For $\mathrm{w} \in \mathcal{W}_0$ and $r \in \mathbb{R}$, we use the notation

$$T_r(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) < r\}.$$

Proposition 4.2. *For $r < 0$ and $\varepsilon > 0$, set*

$$\mathcal{Z}_r^\varepsilon := \varepsilon^{-2}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{T_r(W_s)=\infty,\widehat{W}_s<r+\varepsilon\}}.$$

Then, for every $\beta > 0$,

$$\sup_{r\in(-\infty,-\beta]} |\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r| \underset{\varepsilon\to 0}{\longrightarrow} 0\ , \qquad \mathbb{N}_0 \ \textit{a.e.}$$

Remarks. (i) The process $(\mathcal{Z}_r)_{r<0}$ is càglàd (left-continuous with right limits), and the same is true for the process $(\mathcal{Z}_r^\varepsilon)_{r<0}$ for every $\varepsilon > 0$: If $r_n \uparrow r < 0$, we have $\mathbf{1}_{\{T_{r_n}(W_s)=\infty\}} \downarrow \mathbf{1}_{\{T_r(W_s)=\infty\}}$ and $\mathbf{1}_{\{\widehat{W}_s<r_n+\varepsilon\}} \uparrow \mathbf{1}_{\{\widehat{W}_s<r+\varepsilon\}}$. The right limits of the process $(\mathcal{Z}_r^\varepsilon)_{r<0}$ are given by

$$\mathcal{Z}_{r+}^\varepsilon = \varepsilon^{-2}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\underline{W}_s>r,\widehat{W}_s\leqslant r+\varepsilon\}}. \tag{4.9}$$

(ii) The reader may notice that a slightly different approximation is used in [2, Lemma 14] or in [71, Proposition 34], where the quantities

$$\widetilde{\mathcal{Z}}_r^\varepsilon := \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leqslant \tau_r(W_s), \widehat{W}_s < r+\varepsilon\}}$$

are considered. If r is fixed, this makes no difference since $\widetilde{\mathcal{Z}}_r^\varepsilon = \mathcal{Z}_r^\varepsilon$ for every $\varepsilon > 0$, $\mathbb{N}_0$ a.e. (we may have $\widetilde{\mathcal{Z}}_r^\varepsilon \neq \mathcal{Z}_r^\varepsilon$ only if r is a local minimum of one of the paths W_s, and this occurs with zero $\mathbb{N}_0$-measure). The point in using $\mathcal{Z}_r^\varepsilon$ rather than $\widetilde{\mathcal{Z}}_r^\varepsilon$ is the fact that we want a uniform approximation of $(\mathcal{Z}_r)_{r<0}$ and to this end we are looking for càglàd approximating processes, which is the case for $r \mapsto \mathcal{Z}_r^\varepsilon$ but not for $r \mapsto \widetilde{\mathcal{Z}}_r^\varepsilon$.

We postpone the proof of Proposition 7.5 to the Appendix below. We note that, for every fixed value of $r < 0$, the convergence $\mathcal{Z}_r^\varepsilon \longrightarrow \mathcal{Z}_r$, $\mathbb{N}_0$ a.e., follows from [71, Proposition 34]. Unfortunately, the uniform convergence stated in the proposition requires more work.

4.2.5 A representation for the measure $\mathbb{N}_0^{*,z}$

For every $z > 0$, set

$$L_z := \inf\{r < 0 : \mathcal{Z}_r = z\}$$

if $\{r < 0 : \mathcal{Z}_r = z\}$ is not empty, and $L_z = 0$ otherwise.

Lemma 4.1. *We have*

$$\mathbb{N}_0(L_z < 0) = \frac{1}{2z}. \tag{4.10}$$

Proof. The fact that $\mathbb{N}_0(L_z < 0) = C/z$ for some positive constant C is easy by a scaling argument, but we need another argument to get the value of C. Let $\varepsilon > 0$. We have

$$\begin{aligned}
\mathbb{N}_0(L_z < 0) &= \mathbb{N}_0\Big(\sup_{t>0} \mathcal{Z}_{-t} \geqslant z\Big) \\
&= \mathbb{N}_0\Big(\sup_{0<t\leqslant\varepsilon} \mathcal{Z}_{-t} \geqslant z\Big) + \mathbb{N}_0\Big(\mathbf{1}_{\{\sup_{0<t\leqslant\varepsilon} \mathcal{Z}_{-t} < z\}} \mathbf{P}_{\mathcal{Z}_{-\varepsilon}}\Big(\sup_{t\geqslant 0} \mathscr{Y}_t \geqslant z\Big)\Big),
\end{aligned}$$

where we use the notation $(\mathscr{Y}_t)_{t\geqslant 0}$ for a ϕ-CSBP that starts from x under the probability measure $\mathbf{P}_x$, for every $x \geqslant 0$. By the classical Lamperti representation for CSBPs [61, 32], $(\mathscr{Y}_t)_{t\geqslant 0}$ can be written as a time change of a stable Lévy process with index $3/2$ and no negative jumps. The explicit solution of the two-sided exit problem for such Lévy processes (see [15, Theorem VII.8]) now gives

$$\mathbf{P}_{\mathcal{Z}_{-\varepsilon}}\Big(\sup_{t\geqslant 0} \mathscr{Y}_t \geqslant z\Big) = 1 - \sqrt{\Big(1 - \frac{\mathcal{Z}_{-\varepsilon}}{z}\Big)^+}.$$

Using also (4.7), we get that

$$\mathbb{N}_0(L_z < 0) = \mathbb{N}_0\left(1 - \sqrt{\Big(1 - \frac{\mathcal{Z}_{-\varepsilon}}{z}\Big)^+}\right) + o(1)\ , \quad \text{as } \varepsilon \to 0.$$

For every $\delta > 0$, $\mathbb{N}_0(\mathbf{1}_{\{\mathcal{Z}_{-\varepsilon}>\delta\}}\mathcal{Z}_{-\varepsilon}) = o(1)$ as $\varepsilon \to 0$. Indeed, this follows from the formula $\mathbb{N}_0(\mathcal{Z}_\varepsilon(1 - e^{-\lambda\mathcal{Z}_{-\varepsilon}})) = 1 - (1 + \sqrt{2/3}\,\varepsilon\lambda^{1/2})^{-3}$, which is a consequence of (7.8). Thanks to this observation and to the fact that $\sqrt{1-x} = 1 - \frac{x}{2} + o(x)$ when $x \to 0$, we find that

$$\mathbb{N}_0(L_z < 0) = \frac{1}{2}\mathbb{N}_0\Big(\frac{\mathcal{Z}_{-\varepsilon}}{z}\Big) + o(1)$$

as $\varepsilon \to 0$. The lemma follows since $\mathbb{N}_0(\mathcal{Z}_{-\varepsilon}) = 1$ for every $\varepsilon > 0$. □

The following proposition, which will play an important role, provides an analog of formula (4.8) where $\mathbb{N}_0^*$ is replaced by the conditional measure $\mathbb{N}_0^{*,z}$.

Proposition 4.3. *For any nonnegative measurable function G on $\mathcal{S}_0$, for every $z > 0$,*

$$z^{-2}\,\mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}s\, G(W^{[s]})\Big) = \mathbb{N}_0\Big(G(\mathrm{tr}_{L_z}(W))\,\Big|\,L_z < 0\Big).$$

Remark. When $G = 1$, one recovers the known formula $\mathbb{N}_0^{*,z}(\sigma) = z^2$, see the remark following Proposition 15 in [71].

Proof. We may and will assume that G is bounded and continuous. We use the same notation $(\mathscr{Y}_t, \mathbf{P}_x)$ as in the previous proof and we also set $\Lambda_z := \sup\{t \geqslant 0 : \mathscr{Y}_t = z\}$ with the convention $\sup\varnothing = 0$.

As a consequence of (4.8), the formula

$$\mathbb{N}_0^*\Big(\int_0^\sigma \mathrm{d}r\,\varphi(\mathcal{Z}_0^*)\,G(W^{[r]})\Big) = 2\int_{-\infty}^0 \mathrm{d}b\,\mathbb{N}_0\Big(\mathcal{Z}_b\,\varphi(\mathcal{Z}_b)\,G(\mathrm{tr}_b(W))\Big), \tag{4.11}$$

holds for any nonnegative measurable function φ on $[0,\infty)$. To derive (4.11) from (4.8), notice that (5.6) and Proposition 7.5 allow us to write $\mathcal{Z}_0^* = \Gamma(W^{[r]})$, $\mathbb{N}_0^*$ a.e., and $\mathcal{Z}_b = \Gamma(\mathrm{tr}_b(W))$, $\mathbb{N}_0$ a.e., with the *same* measurable function Γ on $\mathcal{S}_0$.

Let us fix $z_0 > 0$ and a continuous function φ on $\mathbb{R}_+$ which is supported on a compact subset of $(0,\infty)$ and such that $\varphi(z_0) > 0$. We observe that, for any $b < 0$, we have

$$\mathbb{N}_0\Big(\mathbf{1}_{\{b-\varepsilon\leqslant L_{z_0}<b\}}\,\mathcal{Z}_b\,\varphi(\mathcal{Z}_b)\,G(\mathrm{tr}_b(W))\Big) = \mathbb{N}_0\Big(h_\varepsilon(\mathcal{Z}_b, z_0)\,\mathcal{Z}_b\,\varphi(\mathcal{Z}_b)\,G(\mathrm{tr}_b(W))\Big), \tag{4.12}$$

where the function h_ε is defined for every $z > 0$ by

$$h_\varepsilon(z, z_0) = \mathbf{P}_z(0 < \Lambda_{z_0} \leqslant \varepsilon).$$

To get (4.12), we use the Markov property of the process $(\mathcal{Z}_{-r})_{r>0}$ (with respect to the filtration $(\mathcal{G}_r)_{r>0}$ introduced in Section 4.2.4) at time $-b$.

By combining (4.11) (with $\varphi(z)$ replaced by $h_\varepsilon(z, z_0)\,\varphi(z)$) and (4.12), we get

$$\begin{aligned}
&\mathbb{N}_0^*\Big(\int_0^\sigma \mathrm{d}r\,h_\varepsilon(\mathcal{Z}_0^*, z_0)\,\varphi(\mathcal{Z}_0^*)\,G(W^{[r]})\Big) \\
&\quad = 2\int_{-\infty}^0 \mathrm{d}b\,\mathbb{N}_0\Big(\mathbf{1}_{\{b-\varepsilon\leqslant L_{z_0}<b\}}\,\mathcal{Z}_b\,\varphi(\mathcal{Z}_b)\,G(\mathrm{tr}_b(W))\Big) \\
&\quad = 2\,\mathbb{N}_0\Big(\int_{L_{z_0}}^{(L_{z_0}+\varepsilon)\wedge 0} \mathrm{d}b\,\mathcal{Z}_b\,\varphi(\mathcal{Z}_b)\,G(\mathrm{tr}_b(W))\Big)
\end{aligned} \tag{4.13}$$

Let us multiply the right-hand side of (4.13) by ε^{-1} and study its limit as $\varepsilon \to 0$. By Lemma 11 in [2] we know that $\mathrm{tr}_b(W) \longrightarrow \mathrm{tr}_{L_{z_0}}(W)$ as $b \downarrow L_{z_0}$, $\mathbb{N}_0$ a.e. on $\{L_{z_0} < 0\}$. It follows that

$$\lim_{\varepsilon\to 0} \frac{2}{\varepsilon}\mathbb{N}_0\Big(\int_{L_{z_0}}^{(L_{z_0}+\varepsilon)\wedge 0} \mathrm{d}b\, \mathcal{Z}_b\, \varphi(\mathcal{Z}_b)\, G(\mathrm{tr}_b(W))\Big) \tag{4.14}$$
$$= 2z_0\, \varphi(z_0)\, \mathbb{N}_0\Big(G(\mathrm{tr}_{L_{z_0}}(W))\, \mathbf{1}_{\{L_{z_0}<0\}}\Big),$$

where dominated convergence is easily justified thanks to our assumptions on φ and the property $\mathbb{N}_0(L_{z_0} < 0) < \infty$. On the other hand, properties (i) and (ii) stated at the end of Section 5.2.2 allow us to rewrite the left-hand side of (4.13) as

$$\mathbb{N}_0^*\Big(\int_0^\sigma \mathrm{d}r\, h_\varepsilon(\mathcal{Z}_0^*, z_0)\, \varphi(\mathcal{Z}_0^*)\, G(W^{[r]})\Big) \tag{4.15}$$
$$= \sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{\mathrm{d}z}{z^{5/2}}\, h_\varepsilon(z, z_0)\, \varphi(z)\, \mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big).$$

Consider the special case $G = 1$. We deduce from the convergence (4.14), using also the formula $\mathbb{N}_0^{*,z}(\sigma) = z^2$ and the identities (4.13) and (4.15), that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{\mathrm{d}z}{z^{1/2}}\, h_\varepsilon(z, z_0)\, \varphi(z) = 2z_0\, \varphi(z_0)\mathbb{N}_0(L_{z_0} < 0). \tag{4.16}$$

For a general (bounded and continuous) function G, a simple scaling argument shows that the function $z \mapsto z^{-2}\, \mathbb{N}_0^{*,z}(\int_0^\sigma \mathrm{d}r\, G(W^{[r]}))$ is also bounded and continuous on $(0, \infty)$. We may thus apply (4.16) with $\varphi(z)$ replaced by the function

$$z \mapsto \varphi(z)\, z^{-2}\mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big)$$

and we get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{\mathrm{d}z}{z^{5/2}} h_\varepsilon(z, z_0)\varphi(z)\mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big) \tag{4.17}$$
$$= \frac{2\varphi(z_0)}{z_0}\, \mathbb{N}_0^{*,z_0}\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big)\, \mathbb{N}_0(L_{z_0} < \infty)$$

From the identities (4.13) and (4.15), the right-hand sides of (4.14) and (4.17) are equal, which gives the desired result. □

4.2.6 The exit measure process time-reversed at L_z

The goal of this section is to prove the following proposition. Recall that for a Lévy process ξ with only negative jumps we define its Laplace exponent $\psi(\lambda)$ by

$$\mathbb{E}[\exp(\lambda\xi(t))] = \exp(t\psi(\lambda))\ , \quad \lambda \geqslant 0.$$

We use the notation $\mathcal{Z}_{r+}$ for the right limit of $u \mapsto \mathcal{Z}_u$ at r.

Proposition 4.4. *Set $\mathcal{Z}_r = 0$ for $r \geqslant 0$. Under $\mathbb{N}_0(\cdot \mid L_z < 0)$, the process $(\mathcal{Z}_{(L_z+r)+})_{r\geqslant 0}$ is distributed as a self-similar Markov process $(X_r^\circ)_{r\geqslant 0}$ with index $\frac{1}{2}$ starting from z, which can be represented as*

$$X_t^\circ = z\,\exp(\xi^\circ(\chi^\circ(z^{-1/2}t))),$$

where $(\xi^\circ(s))_{s\geqslant 0}$ is the Lévy process with only negative jumps and Laplace exponent

$$\psi^\circ(\lambda) = \sqrt{\frac{3}{2\pi}}\int_{-\infty}^{0}(e^{\lambda y} - 1 - \lambda(e^y - 1))\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y,$$

and $(\chi^\circ(t))_{t\geqslant 0}$ is the time change

$$\chi^\circ(t) = \inf\Big\{s \geqslant 0 : \int_0^s e^{\xi^\circ(v)/2}\,\mathrm{d}v > t\Big\}.$$

We note that the Lévy process ξ° drifts to $-\infty$, and the quantity

$$H_0^\circ := z^{1/2}\int_0^\infty e^{\xi^\circ(v)/2}\,\mathrm{d}v$$

is finite a.s. For $t \geqslant H_0^\circ$, we have $\chi^\circ(z^{-1/2}t) = \infty$ and $\xi^\circ(\chi^\circ(z^{-1/2}t)) = -\infty$. Thus H_0° is the hitting time of 0 by X°, and X° is absorbed at 0.

Proof. Let $(U_t)_{t\geqslant 0}$ denote a stable Lévy process with index 3/2 and no negative jumps, whose distribution is characterized by the formula

$$\mathbb{E}[\exp(-\lambda U_t)] = \exp(t\phi(\lambda))\;,\quad \lambda > 0,\, t \geqslant 0$$

where $\phi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$ as previously. If $\underline{U}_t := \min\{U_s : 0 \leqslant s \leqslant t\}$, the process $U_t - \underline{U}_t$ is a strong Markov process for which 0 is a regular point. Furthermore, $-\underline{U}_t$ serves as a local time at 0 for $U - \underline{U}$. We refer to [15], especially Chapters VII and VIII, for these standard facts about Lévy processes. We denote the excursion measure of $U - \underline{U}$ away from 0, corresponding to the local time $-\underline{U}$, by $\mathbf{n}$. Then $\mathbf{n}$ is a σ-finite measure on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

For notational convenience, we write $\bar{\mathcal{Z}}_x = \mathcal{Z}_{-x}$ for $x > 0$ and $\bar{\mathcal{Z}}_0 = 0$. Notice that $\bar{\mathcal{Z}}$ has càdlàg sample paths. We also set

$$\bar{L}_z = -L_z = \sup\{x > 0 : \bar{\mathcal{Z}}_x = z\}$$

with the convention $\sup\varnothing = 0$.

Lemma 4.2. *For every $x \geqslant 0$, set*

$$\eta(x) := \inf\{y > 0 : \int_0^y \bar{\mathcal{Z}}_u\,\mathrm{d}u > x\}.$$

Let $\mathcal{Y}_x = \bar{\mathcal{Z}}_{\eta(x)}$ if $\eta(x) < \infty$ and $\mathcal{Y}_x = 0$ otherwise. Then the distribution of $(\mathcal{Y}_x)_{x\geqslant 0}$ under $\mathbb{N}_0$ is $\mathbf{n}$.

This lemma is basically a version for excursion measures of the Lamperti representation [61, 32] connecting continuous-state branching processes with Lévy processes. As we were unable to find a precise reference, we provide a proof in the Appendix below.

On the event $\{\bar{L}_z > 0\}$, set

$$\Lambda_z := \sup\{x \geqslant 0 : \mathcal{Y}_x = z\} = \int_0^{\bar{L}_z} \bar{\mathcal{Z}}_s \, \mathrm{d}s.$$

Still on the event $\{\bar{L}_z > 0\}$, we then introduce the time-reversed processes

$$\check{\mathcal{Z}}_u = \begin{cases} \bar{\mathcal{Z}}_{(\bar{L}_z - u)-} & \text{if } 0 \leqslant u < \bar{L}_z, \\ 0 & \text{if } u \geqslant \bar{L}_z, \end{cases}$$

and

$$\check{\mathcal{Y}}_u = \begin{cases} \mathcal{Y}_{(\Lambda_z - u)-} & \text{if } 0 \leqslant u < \Lambda_z, \\ 0 & \text{if } u \geqslant \Lambda_z. \end{cases}$$

We note that we have again the Lamperti representation

$$\check{\mathcal{Z}}_t = \check{\mathcal{Y}}_{\gamma(t)} \ , \ \text{with } \gamma(t) = \inf\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}v}{\check{\mathcal{Y}}_v} > t\}. \tag{4.18}$$

Next as a consequence of Lemma 4.2 and Theorem 4 in [36], we know that the process $(\check{\mathcal{Y}}_u)_{u\geqslant 0}$ is distributed under $\mathbb{N}_0(\cdot \mid L_z < 0)$ as the Lévy process $-U$ started from z and conditioned to hit zero continuously before hitting $(-\infty, 0)$, and stopped at that hitting time. We refer to Section 4 of [36] for a discussion of the latter process. Furthermore we can then use Corollary 3 of Caballero and Chaumont [31] to obtain that the process $(\check{\mathcal{Y}}_u)_{u\geqslant 0}$ under $\mathbb{N}_0(\cdot \mid L_z < 0)$ has the distribution of a self-similar Markov process $(X'_u)_{u\geqslant 0}$ which can be represented in the form

$$X'_u = z \, \exp(\xi^\circ(\chi'(z^{-3/2}u))),$$

where ξ° is the Lévy process in the statement of the proposition[1], and $(\chi'(t))_{t\geqslant 0}$ is the time change

$$\chi'(t) = \inf\Big\{s \geqslant 0 : \int_0^s e^{3\xi^\circ(v)/2} \, \mathrm{d}v > t\Big\}.$$

(Note that the self-similarity index of X' is $3/2$ as the one for U.)

Recalling (4.18), we see that $(\check{\mathcal{Z}}_t)_{t\geqslant 0}$ has the same distribution as $(X'_{\gamma'(t))})_{t\geqslant 0}$ where

$$\gamma'(t) = \inf\{u \geqslant 0 : \int_0^u \frac{\mathrm{d}v}{X'_v} > t\}$$

and $X'_\infty = 0$ by convention. Let $H'_0 := \inf\{t \geqslant 0 : X'_t = 0\}$ and $K'_0 := \int_0^{H'_0} (X'_v)^{-1} \, \mathrm{d}v$, so that $\gamma'(t) < H'_0$ if $t < K'_0$ and $\gamma'(t) = \infty$ if $t \geqslant K'_0$. Simple manipulations show that

$$\begin{aligned} \chi'(z^{-3/2}\gamma'(t)) = \int_0^t \frac{\mathrm{d}s}{\sqrt{X'_{\gamma'(s)}}} &= \inf\{u \geqslant 0 : z^{1/2} \int_0^u \exp(\frac{1}{2}\xi^\circ(v)) \, \mathrm{d}v > t\} \\ &= \chi^\circ(z^{-1/2}t) \end{aligned}$$

[1] In order for the reader to recover the exact form of the Laplace exponent ψ° in the proposition, we mention the following minor inaccuracy in [31]: In formula (23) of the latter paper, $+c_-$ should be replaced by $-c_-$.

if $t < K'_0 = z^{1/2}\int_0^\infty e^{\xi^\circ(u)/2}\,\mathrm{d}u$, whereas $\chi'(z^{-3/2}\gamma'(t)) = \infty$ if $t \geqslant K'_0$. In both cases we get $X'_{\gamma'(t)} = z\,\exp(\xi^\circ(\chi^\circ(z^{-1/2}t))) = X^\circ_t$, with the notation of the proposition. We conclude that $(\check{\mathcal{Z}}_t)_{t\geqslant 0}$ has the same distribution as $(X^\circ_t)_{t\geqslant 0}$. This is the desired result since by construction $\check{\mathcal{Z}}_t = \mathcal{Z}_{L_z+t}$. □

4.3 Special connected components of the genealogical tree

4.3.1 Components above a level

In this section and the next one, we formulate certain definitions and facts that make sense for a deterministic snake trajectory satisfying some regularity properties. We fix $\omega \in \mathcal{S}_0$ and consider the associated genealogical tree $\mathcal{T}_\zeta$.

Definition 4.2. *We say that $x \in \mathbb{R}$ is a local minimum of ω if there exist two distinct points $a_1, a_2 \in \mathcal{T}_\zeta$ and a point $b \in]\!]a_1, a_2[\![$ such that*

$$V_b = \min_{c\in[\![a_1,a_2]\!]} V_c = x.$$

We then also say that b is a point of local minimum.

Clearly the set of all local minima is countable.

We will assume the following regularity properties:

(i) local minima are distinct: if b, b' are two distinct points of local minimum, $V_b \neq V_{b'}$;

(ii) no branching point is a point of local minimum;

(iii) for every $x \in \mathbb{R}$, $\mathrm{vol}(\{c \in \mathcal{T}_\zeta : V_c = x\}) = 0$.

All these properties hold $\mathbb{N}_0$ a.e. and $\mathbb{N}_0^{*,z}$ a.e. For (i) under $\mathbb{N}_0$, one just uses the fact that the increments of V along disjoint segments of the tree $\mathcal{T}_\zeta$ are independent. As for (iii), the case of $\mathbb{N}_0$ follows from the fact that the push forward of $\mathrm{vol}(\mathrm{d}a)$ under the mapping $a \mapsto V_a$ has a continuous density [25], and one can then use Proposition 4.3 to deal with $\mathbb{N}_0^{*,z}$. Notice that (i) implies that the mapping $c \mapsto V_c$ cannot be constant on a nontrivial line segment of $\mathcal{T}_\zeta$.

In the remaining part of this section, we assume in addition that $\omega \in \mathcal{S}_0^+$. We set

$$\mathcal{T}_\zeta^\circ := \{a \in \mathcal{T}_\zeta : V_a > 0\}.$$

Let us fix $a \in \mathcal{T}_\zeta^\circ$. For every $r \in [0, V_a)$, let $\mathcal{C}_r^{(a)}$ denote the connected component of $\{b \in \mathcal{T}_\zeta : V_b > r\}$ that contains a. We note that $\mathcal{C}_{r'}^{(a)} \subset \mathcal{C}_r^{(a)}$ if $r < r' < V_a$. Let $\overline{\mathcal{C}}_r^{(a)}$ stand for the closure of $\mathcal{C}_r^{(a)}$ and, if $r \in (0, V_a)$, set

$$\mathcal{C}_{r-}^{(a)} := \bigcap_{r'\in[0,r)} \mathcal{C}_{r'}^{(a)}.$$

We always have $\overline{\mathcal{C}}_r^{(a)} \subset \mathcal{C}_{r-}^{(a)}$ and equality holds if and only if $r \notin D^{(a)}$, where the set $D^{(a)}$ is defined by

$$D^{(a)} := \{r \in (0, V_a) : \exists b \in \mathcal{T}_\zeta \backslash \{a\},\ V_b > r \text{ and } \min_{c \in [\![a,b]\!]} V_c = r\}.$$

Note that $D^{(a)}$ is a subset of the set of all local minima.

If $r \in D^{(a)}$ and $b \neq a$ is such that $V_b > r$ and $\min_{c \in [\![a,b]\!]} V_c = r$, then there exists a unique $c_0 \in]\!]a, b[\![$ such that $V_{c_0} = r$, and c_0 does not depend on the choice of b (because local minima are distinct by (i) above). Note that c_0 cannot be a branching point of the tree $\mathcal{T}_\zeta$, by (ii). We can then set

$$\check{\mathcal{C}}_r^{(a)} = \{b \in \mathcal{T}_\zeta : c_0 \in [\![a, b[\![\text{ and } V_c > r \text{ for every } c \in]\!]c_0, b]\!]\},$$

and $\mathcal{C}_{r-}^{(a)}$ is the closure of the union $\mathcal{C}_r^{(a)} \cup \check{\mathcal{C}}_r^{(a)}$. Notice that $\check{\mathcal{C}}_r^{(a)} = \mathcal{C}_r^{(b)}$ for any $b \in \check{\mathcal{C}}_r^{(a)}$. For future use, we note that the boundary of $\mathcal{C}_r^{(a)}$, or of $\check{\mathcal{C}}_r^{(a)}$, has zero volume (by (iii)).

4.3.2 Excursions above the minimum

Let us consider $\omega \in \mathcal{S}_0$, and assume that the conditions (i)–(iii) of the previous section hold. Recall our notation ρ for the root of $\mathcal{T}_\zeta$ and note that $V_\rho = 0$. In a way very similar to the definition of $D^{(a)}$ above we now set

$$D(\omega) = \{r < 0 : \exists a \in \mathcal{T}_\zeta, V_a > r \text{ and } \min_{c \in [\![\rho,a]\!]} V_c = r\}.$$

Let us fix $r \in D$. Then r is a local minimum and we let c_0 be the uniquely determined point of local minimum such that $V_{c_0} = r$. The same arguments as in the previous section allow us to single out a particular component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ by setting

$$\check{\mathcal{C}}_r = \{a \in \mathcal{T}_\zeta : c_0 \in [\![\rho, a[\![\text{ and } V_c > r \text{ for every } c \in]\!]c_0, a]\!]\}.$$

Note that this definition of $\check{\mathcal{C}}_r$ would correspond to $\check{\mathcal{C}}_r^{(\rho)}$ in the notation of the preceding section, but we are now considering negative values of r, instead of $r > 0$ in Section 4.3.1. It is convenient to represent $\check{\mathcal{C}}_r$ and the labels on this component by a snake trajectory ω^r, which may be defined as follows. Since the point c_0 has strict descendants in the tree $\mathcal{T}_\zeta$ and is not a branching point, we can make sense of the subtrajectory rooted at c_0, which we denote by $\tilde{\omega}^r$ (see Section 5.2.1). We are in fact only interested in those descendants of c_0 that lie in $\check{\mathcal{C}}_r$, and for this reason, we consider the truncation $\omega^r = \mathrm{tr}_0(\tilde{\omega}^r)$.

Write $\mathcal{T}_\zeta(\omega^r)$ for the genealogical tree of ω^r and, as previously, let $\mathcal{T}_\zeta^\circ(\omega^r)$ denote the subset of $\mathcal{T}_\zeta(\omega^r)$ consisting of points with positive labels. Then $\check{\mathcal{C}}_r$ is identified with $\mathcal{T}_\zeta^\circ(\omega^r)$ via a volume preserving isometry, in such a way that, for every $a \in \check{\mathcal{C}}_r$, we have $V_a(\omega) = r + V_{\tilde{a}}(\omega^r)$ if $\tilde{a}$ is the point of $\mathcal{T}_\zeta^\circ(\omega^r)$ corresponding to a. Consequently, for every $\delta \geqslant 0$, connected components of $\{a \in \mathcal{T}_\zeta : V_a(\omega) > r + \delta\}$ contained in $\check{\mathcal{C}}_r$ are in one-to-one correspondence with connected components of $\{a \in \mathcal{T}_\zeta(\omega^r) : V_a(\omega) > \delta\}$. The latter fact will be important for our applications in Section 4.8 below.

We call ω^r, $r \in D$, the excursions of ω above the minimum. We refer to [2, Section 3] for a (slightly different) more detailed presentation.

The following theorem, which is one of the main results of [2], identifies the conditional distribution of the excursions ω^r, $r \in D$, under $\mathbb{N}_0$ and conditionally on the exit measure process $(\mathcal{Z}_r)_{r<0}$.

Theorem 4.5. [2, Proposition 36, Theorem 40] $\mathbb{N}_0(\mathrm{d}\omega)$ *a.e., D coincides with the set of all discontinuity times of the process $(\mathcal{Z}_r)_{r<0}$. We can thus write $D = \{r_1, r_2, \ldots\}$ where $r_1, r_2, \ldots$ is the sequence of these discontinuity times ordered so that $|\Delta\mathcal{Z}_{r_1}| > |\Delta\mathcal{Z}_{r_2}| > \cdots$. Then, under $\mathbb{N}_0$ and conditionally on $(\mathcal{Z}_r)_{r<0}$, the random snake trajectories $\omega^{r_1}, \omega^{r_2}, \ldots$ are independent, and for every $i \geqslant 1$ the conditional distribution of ω^{r_i} is $\mathbb{N}_0^{*,|\Delta\mathcal{Z}_{r_i}|}$.*

4.4 Measuring the boundary size of components above a level

We will now argue under $\mathbb{N}_0^{*,z}$ for a fixed $z > 0$. The measure $\mathbb{N}_0^{*,z}$ is supported on $\mathcal{S}_0^+$, and so we may use the notation introduced in Section 4.3.1. Recall in particular that $\mathcal{T}_\zeta^\circ := \{a \in \mathcal{T}_\zeta : V_a > 0\}$.

For $a \in \mathcal{T}_\zeta^\circ$, $r \in [0, V_a)$ and $\varepsilon > 0$, we set

$$Z_r^{(a),\varepsilon} := \varepsilon^{-2}\,\mathrm{vol}(\{b \in \mathcal{C}_r^{(a)} : V_b \leqslant r + \varepsilon\}).$$

Proposition 4.5. *The following properties hold $\mathbb{N}_0^{*,z}$ a.e.*

(i) *For every $a \in \mathcal{T}_\zeta^\circ$, $(Z_r^{(a),\varepsilon})_{r\in[0,V_a)}$ converges as $\varepsilon \to 0$, uniformly on $[0, V_a - \beta]$ for every $\beta > 0$, to a limiting càdlàg function $(Z_r^{(a)})_{r\in[0,V_a)}$ with only negative jumps, which takes positive values on $[0, V_a)$ and is such that $Z_0^{(a)} = z$.*

(ii) *If $a, a' \in \mathcal{T}_\zeta^\circ$, we have $Z_r^{(a)} = Z_r^{(a')}$ for every $r \in [0, \min_{c\in[\![a,a']\!]} V_c)$.*

(iii) *Let $a \in \mathcal{T}_\zeta^\circ$. The set of discontinuities of $r \mapsto Z_r^{(a)}$ is $D^{(a)}$. If $r \in D^{(a)}$ we have*

$$Z_{r-}^{(a)} = Z_r^{(a)} + Z_r^{(b)}$$

where b is an arbitrary point of $\check{\mathcal{C}}_r^{(a)}$. Moreover $Z_r^{(a)} \neq Z_r^{(b)}$.

Proof. Recall from Proposition 7.5 the notation $\mathcal{Z}_r^\varepsilon(\omega)$ for $\omega \in \mathcal{S}_0$ and $r < 0$, and the fact that $r \mapsto \mathcal{Z}_r^\varepsilon(\omega)$ is càglàd. We let Θ_z be the set of all snake trajectories $\omega \in \mathcal{S}_0$ such that $W_*(\omega) < 0$ and:

(a) $(\mathcal{Z}_r^\varepsilon(\omega))_{r\in[W_*(\omega),0)}$ converges as $\varepsilon \to 0$ to a limiting càglàd function $(\mathcal{Z}_r(\omega))_{r\in[W_*(\omega),0)}$, uniformly on $[W_*(\omega), -\beta]$ for every $\beta \in (0, -W_*(\omega))$;

(b) the set of discontinuity times of this limiting function is $D(\omega) \cap (W_*(\omega), 0)$;

(c) the function $(\mathcal{Z}_r(\omega))_{r\in[W_*(\omega),0)}$ takes positive values on $[W_*(\omega), 0)$, and takes the value z for $r = W_*(\omega)$;

(d) $\mathcal{Z}_r(\omega) \neq |\Delta\mathcal{Z}_r(\omega)|$ for every $r \in D(\omega) \cap (W_*(\omega), 0)$.

It follows from Proposition 7.5 and the first assertion of Theorem 4.5 that $\mathrm{tr}_{L_z}(W)$ belongs to Θ_z, $\mathbb{N}_0$ a.e. on the event $\{L_z < 0\}$. We also use the fact that the process $(\mathcal{Z}_{-r}(\omega))_{r>0}$ evolves under $\mathbb{N}_0$ as a ϕ-CSBP (and therefore as the time change of a stable Lévy process) to obtain the property $\mathcal{Z}_r(\omega) \neq |\Delta\mathcal{Z}_r(\omega)|$ when $r \in D(\omega)$.

Taking $G = \mathbf{1}_{\Theta_z}$ in Proposition 4.3, we also get that, $\mathbb{N}_0^{*,z}(\mathrm{d}\omega)$ a.s., for $\mathrm{d}s$ a.e. $s \in [0,\sigma]$, the re-rooted snake trajectory $W^{[s]}$ belongs to Θ_z. So let us fix $\omega \in \mathcal{S}_0$ such that the preceding assertion holds. We can then take a sequence $s_1, s_2, \dots$ dense in $[0,\sigma]$ such that $\omega^{[s_i]}$ belongs to Θ_z for every $i = 1, 2, \dots$. Setting $a_i = p_\zeta(s_i)$, we also know that $a_1, a_2, \dots$ all belong to $\mathcal{T}_\zeta^\circ$ (otherwise $W_*(\omega^{[s_i]}) = 0$). We now observe that $W_*(\omega^{[s_i]}) = -V_{a_i}(\omega)$, and, for every $r \in [0, V_{a_i}(\omega))$,

$$Z_r^{(a_i),\varepsilon}(\omega) = \mathcal{Z}^\varepsilon_{(r-V_{a_i}(\omega))+}(\omega^{[s_i]}).$$

This is a simple consequence of our definitions and formula (4.9) for the right limits $\mathcal{Z}^\varepsilon_{r+}$.

Since $\omega^{[s_i]}$ belongs to Θ_z, we deduce from the last display and assertion (a) above that the convergence stated in part (i) of the proposition holds when $a = a_i$, and that, for every $r \in [0, V_{a_i}(\omega))$,

$$Z_r^{(a_i)}(\omega) = \mathcal{Z}_{(r-V_{a_i}(\omega))+}(\omega^{[s_i]}).$$

The function $r \mapsto Z_r^{(a_i)}(\omega)$ then satisfies the properties stated in (i). Moreover it is immediate that the set of discontinuity times of this function is

$$\{V_{a_i} + r : r \in D(\omega^{[s_i]})\} = D^{(a_i)}(\omega)$$

where the last equality is again a consequence of our definitions. Futhermore, if $r \in D^{(a_i)}(\omega)$ and if j is an index such that $a_j \in \check{\mathcal{C}}_r^{(a_i)}$, the fact that $\mathcal{C}_{r-}^{(a_i)}$ is the closure of the union $\mathcal{C}_r^{(a_i)} \cup \check{\mathcal{C}}_r^{(a_i)}$ implies that

$$Z_{r-}^{(a_i),\varepsilon}(\omega) = Z_r^{(a_i),\varepsilon}(\omega) + Z_r^{(a_j),\varepsilon}(\omega)$$

and by passing to the limit $\varepsilon \to 0$,

$$Z_{r-}^{(a_i)}(\omega) = Z_r^{(a_i)}(\omega) + Z_r^{(a_j)}(\omega).$$

Finally property (d) gives $Z_r^{(a_i)}(\omega) \neq Z_r^{(a_j)}(\omega)$.

The preceding discussion shows that properties (i) and (iii) of the proposition hold if we restrict our attention to points in the dense sequence $a_1, a_2, \dots$. However, it readily follows from our definitions that we have $\mathcal{C}_r^{(a)} = \mathcal{C}_r^{(a_i)}$, and thus also $Z_r^{(a),\varepsilon} = Z_r^{(a_i),\varepsilon}$ as soon as $r < \min_{c \in [\![a,a_i]\!]} V_c$. We infer that we can define $(Z_r^{(a)})_{r\in[0,V_a)}$ in a unique way so that

$$Z_r^{(a)} = Z_r^{(a_i)}\ ,\ \text{for every } r < \min_{c\in[\![a,a_i]\!]} V_c\ ,\ \text{for every } i \geqslant 1.$$

It is then a simple matter to verify that assertions (i) and (iii) hold in the stated form, and assertion (ii) is also immediate. □

4.5 The locally largest evolution

We say that a point $a \in \mathcal{T}_\zeta^\circ$ is regular if

$$\bigcap_{r\in[0,V_a)} \mathcal{C}_r^{(a)} = \{a\}.$$

Proposition 4.6. *There exists $\mathbb{N}_0^{*,z}$ a.e. a unique point $a^\bullet$ of $\mathcal{T}_\zeta^\circ$ such that the following two properties hold:*

(i) *We have $Z_r^{(a^\bullet)} > |\Delta Z_r^{(a^\bullet)}|$, for every $r \in [0, V_a)$.*

(ii) *The point $a^\bullet$ is regular.*

We will call $a^\bullet$ the terminal point of the locally largest evolution. Note that condition (i) is relevant only for $r \in D^{(a^\bullet)}$ since $Z^{(a^\bullet)}$ takes positive values.

Proof. We first establish uniqueness. Suppose that a_1 and a_2 are two distinct points of $\mathcal{T}_\zeta^\circ$ that satisfy the properties stated in (i) and (ii). We notice that we must have

$$\min_{c\in[\![a_1,a_2]\!]} V_c < V_{a_1} \wedge V_{a_2}$$

because if the latter minimum is equal say to V_{a_1} the whole segment $[\![a_1, a_2]\!]$ is contained in

$$\bigcap_{r\in[0,V_{a_1})} \mathcal{C}_t^{(a_1)}$$

contradicting the regularity of a_1. Set $u = \min_{c\in[\![a_1,a_2]\!]} V_c$. By definition, we have then $u \in D^{(a_1)} \cap D^{(a_2)}$ and, by property (iii) in Proposition 4.5, we get

$$Z_{u-}^{(a_1)} = Z_{u-}^{(a_2)} = Z_u^{(a_1)} + Z_u^{(a_2)}$$

and thus $|\Delta Z_u^{(a_1)}| = Z_u^{(a_2)}$, $|\Delta Z_u^{(a_2)}| = Z_u^{(a_1)}$. This shows that property (i) in Proposition 4.6 cannot hold simultaneously for a_1 and for a_2.

Let us turn to existence. We let r_∞ be the supremum of the set of all reals $r \geqslant 0$ such that there exists $a \in \mathcal{T}_\zeta^\circ$ with $V_a \geqslant r$ and $Z_s^{(a)} > |\Delta Z_s^{(a)}|$ for every $s < r$. By the definition of r_∞, we can then find a nondecreasing sequence $(r_n)_{n\geqslant 1}$ in $\mathbb{R}_+$ and a corresponding sequence $(a_n)_{n\geqslant 1}$ in $\mathcal{T}_\zeta^\circ$ such that $V_{a_n} \geqslant r_n$, $Z_s^{(a_n)} > |\Delta Z_s^{(a_n)}|$ for every $s < r_n$, and $r_n \uparrow r_\infty$ as $n \uparrow \infty$. By compactness, we may assume that $a_n \longrightarrow a_\infty \in \mathcal{T}_\zeta$ as $n \uparrow \infty$, and it is clear that $a_\infty \in \mathcal{T}_\zeta^\circ$ (the case $V_{a_\infty} = 0$ is excluded since it would imply that $r_\infty = 0$).

Using property (ii) in Proposition 4.5, we have, for every n,

$$Z_s^{(a_\infty)} > |\Delta Z_s^{(a_\infty)}| , \quad \text{for every } s < r_n \wedge \min_{c\in[\![a_n,a_\infty]\!]} V_c.$$

Since

$$\min_{c\in[\![a_n,a_\infty]\!]} V_c \underset{n\to\infty}{\longrightarrow} V_{a_\infty} \geqslant r_\infty$$

(the last inequality because $V_{a_n} \geqslant r_n$ for every n), it follows that

$$Z_s^{(a_\infty)} > |\Delta Z_s^{(a_\infty)}|\,, \quad \text{for every } s < r_\infty.$$

We next claim that $r_\infty = V_{a_\infty}$. Otherwise, we would have $r_\infty < V_{a_\infty}$ and then either r_∞ would be a continuity time of $Z^{(a_\infty)}$, which immediately gives a contradiction with the definition of r_∞, or r_∞ would be a discontinuity time of $Z^{(a_\infty)}$, and, by taking $a = a_\infty$ or $a \in \check{\mathcal{C}}_{r_\infty}^{(a_\infty)}$, we would again get a contradiction with the definition of r_∞.

It remains to verify that a_∞ is regular. If a_∞ is not regular, then we can choose

$$b \in \bigcap_{r \in [0, V_{a_\infty})} \mathcal{C}_r^{(a_\infty)}$$

with $V_b > V_{a_\infty} = r_\infty$. If r_∞ is a continuity point of $r \mapsto Z_r^{(b)}$, we get a contradiction with the definition of r_∞. If r_∞ is a discontinuity point of $r \mapsto Z_r^{(b)}$, then, by taking $a = b$ or $a \in \check{\mathcal{C}}_{r_\infty}^{(b)}$, we again get a contradiction. This completes the proof. □

Let $u \in (0, V_{a^\bullet})$. For future use, we note that, for every $a \in \mathcal{T}_\zeta^\circ$, we have $a \in \mathcal{C}_u^{(a^\bullet)}$ if and only if $V_a > u$ and $Z_r^{(a)} > |\Delta Z_r^{(a)}|$ for every $r \leqslant u$. The "only if" part is trivial. Conversely, assuming that $V_a > u$ and $Z_r^{(a)} > |\Delta Z_r^{(a)}|$ for every $r \leqslant u$, the property $a \notin \mathcal{C}_u^{(a^\bullet)}$ would lead to a contradiction by using Proposition 4.5 (iii) with $r = \min_{c \in [\![a^\bullet, a]\!]} V_c \leqslant u$.

4.6 The law of the locally largest evolution

Our next goal is to compute the distribution of $(Z_t^{(a^\bullet)})_{0 \leqslant t < V_{a^\bullet}}$ under $\mathbb{N}_0^{*,z}$.

Proposition 4.7. *The process $(Z_t^{(a^\bullet)})_{0 \leqslant t < V_{a^\bullet}}$ is distributed under $\mathbb{N}_0^{*,z}$ as $(X_t)_{0 \leqslant t < H_0}$, where $(X_t)_{t \geqslant 0}$ is the self-similar Markov process with index $\frac{1}{2}$ starting from z, which can be represented as*

$$X_t = z \exp(\xi(\chi(z^{-1/2}t))),$$

where $(\xi(s))_{s \geqslant 0}$ is the Lévy process with only negative jumps whose Laplace exponent ψ is given by formula (4.1) *and $(\chi(t))_{t \geqslant 0}$ is the time change defined in* (4.2)*, and $H_0 = \inf\{t \geqslant 0 : X_t = 0\}$.*

Proof. We fix $u > 0$ and consider a bounded measurable function F on the Skorokhod space $\mathbb{D}([0,u], \mathbb{R})$. We observe that

$$F\big((Z_t^{(a^\bullet)})_{0 \leqslant t \leqslant u}\big)\, \mathbf{1}_{\{V_{a^\bullet} > u\}} = \int \mathrm{vol}(\mathrm{d}a)\, F\big((Z_t^{(a)})_{0 \leqslant t \leqslant u}\big)\, \mathbf{1}_{\{a \in \mathcal{C}_u^{(a^\bullet)}\}} \frac{1}{\mathrm{vol}(\mathcal{C}_u^{(a)})},$$

simply because if $a \in \mathcal{C}_u^{(a^\bullet)}$ we have $Z_t^{(a)} = Z_t^{(a^\bullet)}$ for $0 \leqslant t \leqslant u$ and $\mathcal{C}_u^{(a)} = \mathcal{C}_u^{(a^\bullet)}$. From the definition of vol$(\cdot)$ and a previous observation, the right-hand side can also be written as

$$\int_0^\sigma \mathrm{d}s\, F\big((Z_t^{(s)})_{0 \leqslant t \leqslant u}\big)\, \mathbf{1}_{\{\widehat{W}_s > u;\, Z_t^{(s)} > |\Delta Z_t^{(s)}|, \forall t \leqslant u\}} \frac{1}{\mathrm{vol}(\mathcal{C}_u^{(p_\zeta(s))})},$$

where we have written $Z_t^{(s)} = Z_t^{(a)}$ if $a = p_\zeta(s)$ to simplify notation.

The preceding considerations show that

$$\begin{aligned}
&\mathbb{N}_0^{*,z}\Big(F\big((Z_t^{(a^\bullet)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{V_{a^\bullet}>u\}}\Big)\\
&= \mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}s\, F\big((Z_t^{(s)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{\widehat{W}_s>u;\, Z_t^{(s)}>|\Delta Z_t^{(s)}|,\forall t\leqslant u\}}\,\frac{1}{\mathrm{vol}(\mathcal{C}_u^{(p_\zeta(s))})}\Big)\\
&= z^2\,\mathbb{N}_0\Big(F\big(\mathcal{Z}_{(L_z+t)+}\big)_{0\leqslant t\leqslant u}\big)\\
&\qquad\times \mathbf{1}_{\{L_z<-u;\, \mathcal{Z}_{(L_z+t)+}>|\Delta\mathcal{Z}_{L_z+t}|,\forall t\leqslant u\}}\,\frac{1}{\mathrm{vol}(\mathcal{C}_{L_z+u})}\,\Big|\, L_z<0\Big),
\end{aligned}$$

where for $r<0$, we use (under $\mathbb{N}_0$) the notation $\mathcal{C}_r$ for the connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ containing the "root" $p_\zeta(0)$. The second equality of the last display is a consequence of Proposition 4.3 and the way the functions $Z_t^{(a)}$ were constructed in Section 4.4.

In the terminology of [2], $\mathcal{C}_{L_z+u}$ is (up to a set of zero volume) the union of the subsets of $\mathcal{T}_\zeta$ corresponding to the excursions above the minimum that start at a level greater than L_z+u. Using Theorem 4.5 and [2, Proposition 31], we obtain that the conditional distribution of $\mathrm{vol}(\mathcal{C}_{L_z+u})$ under $\mathbb{N}_0(\cdot \mid L_z<0)$ and knowing $(\mathcal{Z}_r)_{r<0}$ is the law of

$$\sum_{i=1}^{\infty} |\Delta\mathcal{Z}_{r_i}|^2\,\nu_i$$

where $r_1, r_2, \dots$ is an enumeration of the jumps of $\mathcal{Z}$ on $(L_z+u, 0)$, and the random variables $\nu_1, \nu_2, \dots$ are independent and distributed according to the density

$$\frac{1}{\sqrt{2\pi}}\,x^{-5/2}\,\exp(-\frac{1}{2x})\,\mathbf{1}_{\{x>0\}}.$$

Writing $\mathbb{E}^{(\nu)}[\cdot]$ for the expectation with respect to the variables $\nu_1, \nu_2, \dots$, we can thus also write

$$\begin{aligned}
&\mathbb{N}_0^{*,z}\Big(F\big((Z_t^{(a^\bullet)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{V_{a^\bullet}>u\}}\Big) &(4.19)\\
&= z^2\,\mathbb{N}_0\Big(F\big((\mathcal{Z}_{L_z+t})_{0\leqslant t\leqslant u}\big)\\
&\qquad\times \mathbf{1}_{\{L_z<-u;\, \mathcal{Z}_{(L_z+t)+}>|\Delta\mathcal{Z}_{L_z+t}|,\forall t\leqslant u\}}\,\mathbb{E}^{(\nu)}\Big[\frac{1}{\sum |\Delta\mathcal{Z}_{r_i}|^2\,\nu_i}\Big]\,\Big|\, L_z<0\Big),
\end{aligned}$$

and the right-hand side is an integral under $\mathbb{N}_0$ of a quantity depending only on the exit measure process $(\mathcal{Z}_r)_{r<0}$.

Thanks to Proposition 4.4, we can replace the right-hand side of (4.19) by

$$z^2\,\mathbb{E}\Big[F\big((X_t^\circ)_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{H_0^\circ>u;\, X_t^\circ>|\Delta X_t^\circ|,\forall t\leqslant u\}}\,\mathbb{E}^{(\nu)}\Big[\frac{1}{\sum |\Delta X_{s_i}^\circ|^2\,\nu_i}\Big]\Big], \tag{4.20}$$

where $s_1, s_2, \dots$ is an enumeration of the jump times of X° over $[u, H_0^\circ)$. Then, by the Markov property and the self-similarity of X°, the conditional expectation of the quantity

$$\mathbb{E}^{(\nu)}\Big[\frac{1}{\sum |\Delta X_{s_i}^\circ|^2\,\nu_i}\Big]$$

given $(X_t^\circ)_{0\leqslant t\leqslant u}$ is $C/(X_u^\circ)^2$, for some constant $C>0$ (the cases $C=0$ and $C=\infty$ are excluded since the preceding equalities would give an absurd statement). So the quantity (4.20) is also equal to C times

$$z^2\,\mathbb{E}\Big[F\big((X_t^\circ)_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{H_0^\circ>u;\,X_t^\circ>|\Delta X_t^\circ|,\forall t\leqslant u\}}\,(X_u^\circ)^{-2}\Big]. \tag{4.21}$$

We will rewrite this quantity in a different form. In the remaining part of the proof, we take $z=1$ for the sake of simplicity (of course the self-similarity of X° will then allow us to get a similar result for an arbitrary value of z). Using the representation in Proposition 4.4, we obtain that the quantity (4.21) is equal for $z=1$ to

$$\begin{aligned}\mathbb{E}\Big[F\Big(\big(\exp(\xi^\circ(\chi^\circ(t)))\big)_{0\leqslant t\leqslant u}\Big)\,\mathbf{1}_{\{\chi^\circ(u)<\infty\}}& \qquad (4.22)\\ \times\,\mathbf{1}_{\{\Delta\xi^\circ(s)>-\log 2,\,\forall s\in[0,\chi^\circ(u)]\}}\,\exp(-2\xi^\circ(\chi^\circ(u)))\Big].&\end{aligned}$$

Lemma 4.3. *For every $v\geqslant 0$, set*

$$M_v=\mathbf{1}_{\{\Delta\xi^\circ(s)>-\log 2,\forall s\in[0,v]\}}\,\exp(-2\xi^\circ(v)).$$

Then $(M_v)_{v\geqslant 0}$ is a martingale with respect to the canonical filtration of the process ξ°. Let ξ be as in Proposition 4.7. Then, for every fixed $v>0$ the process $(\xi^\circ(t))_{0\leqslant t\leqslant v}$ is distributed under the probability measure $M_v\cdot\mathbb{P}$ as $(\xi(t))_{0\leqslant t\leqslant v}$ under $\mathbb{P}$.

Proof. To simplify notation, we write $\alpha=\sqrt{3/2\pi}$. Thanks to the properties of Lévy processes, in order to verify that $(M_v)_{v\geqslant 0}$ is a martingale, it suffices to prove that $\mathbb{E}[M_v]=1$ for every $v>0$. It is convenient to set

$$\xi''(t)=\sum_{0\leqslant s\leqslant t}\Delta\xi^\circ(s)\,\mathbf{1}_{\{\Delta\xi^\circ(s)\leqslant -\log 2\}},$$

so that we can write $\xi^\circ(t)=\xi'(t)+\xi''(t)$, where ξ' and ξ'' are two independent Lévy processes. The Laplace exponent of ξ'' is

$$\psi''(\lambda)=\alpha\int_{-\infty}^{-\log 2}(e^{\lambda y}-1)\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y,$$

and the Laplace exponent of ξ' is

$$\begin{aligned}\psi'(\lambda)&=\psi^\circ(\lambda)-\psi''(\lambda) \qquad (4.23)\\ &=\alpha\bigg(\int_{-\log 2}^{0}(e^{\lambda y}-1-\lambda(e^y-1))\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y\\ &\qquad-\lambda\int_{-\infty}^{-\log 2}(e^y-1)\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y\bigg)\\ &=\alpha\bigg(\int_{-\log 2}^{0}(e^{\lambda y}-1-\lambda(e^y-1))\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y+2\lambda\bigg)\end{aligned}$$

using the simple calculation

$$\int_{-\infty}^{-\log 2} e^{y/2}\,(1-e^y)^{-3/2}\,\mathrm{d}y = \int_0^{1/2} (1-x)^{-3/2}\,\frac{\mathrm{d}x}{\sqrt{x}} = 2.$$

Note that ξ' has bounded jumps and therefore exponential moments of any order, so that $\psi'(\lambda)$ makes sense for every $\lambda \in \mathbb{R}$ and not only $\lambda \geqslant 0$.

We have then

$$\mathbb{E}[M_v] = \mathbb{P}(\xi''(v) = 0)\,\mathbb{E}[e^{-2\xi'(v)}].$$

On one hand, $\mathbb{E}[e^{-2\xi'(v)}] = \exp(\alpha K v)$, where

$$\begin{aligned} K &= -4 + \int_{-\log 2}^{0} (e^{-2y} - 1 + 2(e^y - 1))\,e^{y/2}\,(1-e^y)^{-5/2}\,\mathrm{d}y \\ &= -4 + \int_{1/2}^{1} (x^{-2} - 1 + 2(x-1))\,(1-x)^{-5/2}\,\frac{\mathrm{d}x}{\sqrt{x}} \\ &= -4 + 2\int_{1/2}^{1} x^{-3/2}\,(1-x)^{-1/2}\,\mathrm{d}x + \int_{1/2}^{1} x^{-5/2}\,(1-x)^{-1/2}\,\mathrm{d}x \\ &= \frac{8}{3} \end{aligned}$$

where the last equality follows from classical formulas for incomplete Beta functions. On the other hand,

$$\mathbb{P}(\xi''(v) = 0) = \exp\Big(-\alpha v \int_{-\infty}^{-\log 2} e^{y/2}\,(1-e^y)^{-5/2}\mathrm{d}y\Big) = \exp(-\frac{8}{3}\alpha v).$$

By combining the last two displays, we get the desired result $\mathbb{E}[M_v] = 1$.

Then, let us fix $v > 0$. It is straightforward to verify that the properties of stationarity and independence of the increments of ξ° are preserved under the probability measure $M_v \cdot \mathbb{P}$, so that $(\xi^\circ(t))_{0\leqslant t\leqslant v}$ remains a Lévy process under this probability measure. To evaluate the Laplace exponent of this Lévy process, we write

$$\begin{aligned} \mathbb{E}[M_v\, e^{\lambda\xi^\circ(v)}] = \exp(-\frac{8}{3}\alpha v)\,\mathbb{E}[e^{(\lambda-2)\xi'(v))}] &= \exp((\psi'(\lambda-2) - \frac{8}{3}\alpha)v) \\ &= \exp(v\psi(\lambda)), \end{aligned}$$

where ψ is as in (4.1). The last equality follows from formula (4.23) for ψ' and simple calculations left to the reader. This completes the proof of the lemma. □

Let us come back to (4.22). We first observe that, for every $v > 0$,

$$\begin{aligned} &\mathbb{E}\Big[F\Big(\big(\exp(\xi^\circ(\chi^\circ(t)))\big)_{0\leqslant t\leqslant u}\Big)\,\mathbf{1}_{\{\chi^\circ(u)\leqslant v\}}\,M_{\chi^\circ(u)}\Big] \\ &\quad = \mathbb{E}\Big[F\Big(\big(\exp(\xi^\circ(\chi^\circ(t)))\big)_{0\leqslant t\leqslant u}\Big)\,\mathbf{1}_{\{\chi^\circ(u)\leqslant v\}}\,M_v\Big] \\ &\quad = \mathbb{E}\Big[F\Big(\big(\exp(\xi(\chi(t)))\big)_{0\leqslant t\leqslant u}\Big)\,\mathbf{1}_{\{\chi(u)\leqslant v\}}\Big], \end{aligned} \tag{4.24}$$

where $\chi(t)$ is as in formula (4.2). As $v \to \infty$, the right-hand side of (4.24) converges to $\mathbb{E}[F(\exp(\xi(\chi(t))))_{0\leqslant t\leqslant u})\,\mathbf{1}_{\{\chi(u)<\infty\}}]$. Similarly the left-hand side of (4.24) converges to the quantity (4.22): Dominated convergence is easily justified by noting that

$$\mathbb{E}[M_{\chi^\circ(u)}\,\mathbf{1}_{\{\chi^\circ(u)<\infty\}}] \leqslant \liminf_{t\to\infty} \mathbb{E}[M_{\chi^\circ(u)\wedge t}\,\mathbf{1}_{\{\chi^\circ(u)\leqslant t\}}]$$

and $\mathbb{E}[M_{\chi^\circ(u)\wedge t}] = \mathbb{E}[M_t] = 1$ by the optional stopping theorem.

Hence the quantity (4.22) is equal to $\mathbb{E}[F\big((\exp(\xi(\chi(t))))_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{\chi(u)<\infty\}}]$. We conclude that

$$\mathbb{N}_0^{*,1}\Big(F\big((Z_t^{(a^\bullet)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{V_{a^\bullet}>u\}}\Big) = C\,\mathbb{E}\Big[F\big((\exp(\xi(\chi(t))))_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{\chi(u)<\infty\}}\Big].$$

At this stage, we can take $F = 1$ and let u tend to 0, and we find that $C = 1$. This gives the statement of Proposition 4.7 for $z = 1$, and it is easily extended by self-similarity. □

4.7 Excursions from the locally largest evolution

If $\omega \in \mathcal{S}_0$ satisfies the regularity properties stated in Section 4.3.1, we can define the excursions above the minimum ω^r, $r \in D$ in the way described in Section 4.3.2.

Let us now argue under $\mathbb{N}_0^{*,z}(\mathrm{d}\omega)$ and for $u \in [0,\sigma]$, consider the re-rooted snake trajectory $\omega^{[u]}$. Let $a = p_\zeta(u)$ and recall that $D^{(a)}$ is the set of the discontinuity times of $(Z_r^{(a)})_{r\in[0,V_a)}$. As we already noticed in the proof of Proposition 4.5, we have $D^{(a)}(\omega) = \{V_a + r : r \in D(\omega^{[u]})\}$. If $r \in D^{(a)}(\omega)$ we can thus associate with $r - V_a \in D(\omega^{[u]})$ an excursion of $\omega^{[u]}$ above the minimum, which we denote by $\omega^{a,r}$ (it is easy to see that this excursion only depends on a and not on u such that $a = p_\zeta(u)$). We already noticed that, if $a, a' \in \mathcal{T}_\zeta^\circ$ are such that $V_a \wedge V_{a'} > u$ and $\mathcal{C}_u^{(a)} = \mathcal{C}_u^{(a')}$, we have $D^{(a)} \cap [0,u] = D^{(a')} \cap [0,u]$, and it is also true that $\omega^{a,r} = \omega^{a',r}$ for $r \in D^{(a)} \cap [0,u]$.

We will now apply the preceding considerations to $a = a^\bullet$. We write $D^{(a^\bullet)} = \{r_1, r_2, \ldots\}$, where $|\Delta Z_{r_1}^{(a^\bullet)}| > |\Delta Z_{r_2}^{(a^\bullet)}| > \cdots$.

Proposition 4.8. *Under $\mathbb{N}_0^{*,z}$, conditionally on $(Z_r^{(a^\bullet)})_{0\leqslant r<V_{a^\bullet}}$, the excursions $\omega^{a^\bullet,r_i}$, $i = 1,2,\ldots$ are independent, and for every fixed $i \geqslant 1$ the conditional distribution of $\omega^{a^\bullet,r_i}$ is $\mathbb{N}_0^{*,|\Delta Z_{r_i}^{(a^\bullet)}|}$.*

Proof. We proceed in a way similar to the one used above to determine the law of $(Z_r^{(a^\bullet)}, 0 \leqslant r \leqslant V_{a^\bullet})$. We fix $u > 0$ and consider a bounded measurable function F on the Skorokhod space $\mathbb{D}([0,u],\mathbb{R}_+)$, and a bounded measurable function H on $\mathbb{R}_+ \times \mathcal{S}_0$ such that $H = 1$ on $\mathbb{R}_+ \times (\mathcal{S}_0 \backslash \mathcal{S}_0^{(\delta)})$ for some $\delta > 0$. The latter condition ensures that $H(r, \omega^{a^\bullet,r}) = 1$ except for finitely many values of $r \in D$. Then,

$$\begin{aligned} &\mathbb{N}_0^{*,z}\Big(F\big((Z_t^{(a^\bullet)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{V_{a^\bullet}>u\}}\prod_{r\in D^{(a^\bullet)}\cap[0,u]} H(r,\omega^{a^\bullet,r})\Big) \qquad (4.25)\\ &= \mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}s\,F\big((Z_t^{(s)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{\widehat{W}_s>u;\,Z_t^{(s)}>|\Delta Z_t^{(s)}|,\forall t\leqslant u\}}\\ &\quad\times\frac{1}{\mathrm{vol}(\mathcal{C}_u^{(p_\zeta(s))})}\prod_{r\in D^{(p_\zeta(s))}\cap[0,u]} H(r,\omega^{p_\zeta(s),r})\Big)\\ &= z^2\,\mathbb{N}_0\Big(F\big((\mathcal{Z}_{(L_z+t)+})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{L_z<-u;\,\mathcal{Z}_{(L_z+t)+}>|\Delta\mathcal{Z}_{L_z+t}|,\forall t\leqslant u\}}\\ &\quad\times\frac{1}{\mathrm{vol}(\mathcal{C}_{L_z+u})}\prod_{r\in D\cap[L_z,L_z+u]} H(r-L_z,\omega^r)\Big). \end{aligned}$$

We used the remarks preceding the statement of the proposition in the first equality, and Proposition 4.3 in the second one. At this stage, we use Theorem 4.5, which shows that conditionally on the exit measure process $(\mathcal{Z}_r)_{r<0}$ (whose set of discontinuities is D) the excursions ω_r, $r \in D$, are independent and the conditional distribution of ω^r is $\mathbb{N}_0^{*,|\Delta\mathcal{Z}_r|}$. Since the quantity $\mathrm{vol}(\mathcal{C}_{L_z+u})$ only depends on the excursions ω^r with $r > L_z + u$, we can rewrite the last line of the preceding display as

$$\begin{aligned} &z^2\,\mathbb{N}_0\Big(F\big((\mathcal{Z}_{(L_z+t)+})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{L_z<-u;\,\mathcal{Z}_{(L_z+t)+}>|\Delta\mathcal{Z}_{L_z+t}|,\forall t\leqslant u\}}\\ &\quad\times\frac{1}{\mathrm{vol}(\mathcal{C}_{L_z+u})}\prod_{r\in D\cap[L_z,L_z+u]} \mathbb{N}_0^{*,|\Delta\mathcal{Z}_r|}(H(r-L_z,\cdot))\Big) \end{aligned}$$

but then, we can re-use the same arguments “backwards” to see that the latter quantity is also equal to

$$\mathbb{N}_0^{*,z}\Big(F\big((Z_t^{(a^\bullet)})_{0\leqslant t\leqslant u}\big)\,\mathbf{1}_{\{V_{a^\bullet}>u\}}\prod_{r\in D^{(a^\bullet)}\cap[0,u]} \mathbb{N}_0^{*,|\Delta Z_r^{(a^\bullet)}|}(H(r,\cdot))\Big). \qquad (4.26)$$

The quantity (4.26) is thus equal to the left-hand-side of (4.25). Simple arguments show that this also implies that, for any bounded measurable function G on the appropriate space of càdlàg paths,

$$\begin{aligned} &\mathbb{N}_0^{*,z}\Big(G\big((Z_t^{(a^\bullet)})_{0\leqslant t<V_{a^\bullet}}\big)\prod_{r\in D^{(a^\bullet)}} H(r,\omega^{a^\bullet,r})\Big)\\ &= \mathbb{N}_0^{*,z}\Big(G\big((Z_t^{(a^\bullet)})_{0\leqslant t<V_{a^\bullet}}\big)\prod_{r\in D^{(a^\bullet)}} \mathbb{N}_0^{*,|\Delta Z_r^{(a^\bullet)}|}(H(r,\cdot))\Big). \end{aligned}$$

This gives the statement of the proposition. □

We will call $\omega^{a^\bullet,r_i}$, $i = 1, 2, \dots$ the excursions of ω from the locally largest evolution. The number r_i is called the starting level of $\omega^{a^\bullet,r_i}$. As previously, these excursions will always be listed in decreasing order of their “boundary sizes” $|\Delta Z_{r_i}^{(a^\bullet)}|$. To simplify notation, we set $\omega^{(i)} = \omega^{a^\bullet,r_i}$.

4.8 The growth-fragmentation process

In this section, we argue again under $\mathbb{N}_0^{*,z}(\mathrm{d}\omega)$. Recall that $\omega^{(1)}, \omega^{(2)}, \dots$ are the excursions of ω from the locally largest evolution and $r_1, r_2, \dots$ stand for the respective starting levels of these excursions.

Next, for every $i \geqslant 1$, since the conditional distribution of $\omega^{(i)}$ knowing $(Z_r^{(a^\bullet)})_{0 \leqslant r < V_{a^\bullet}}$ is $\mathbb{N}_0^{*,|\Delta Z_{r_i}^{(a^\bullet)}|}$, we can also define a point $a_i^\bullet$ as the terminal point of the locally largest evolution in $\omega^{(i)}$, and the excursions $\omega^{(i,1)}, \omega^{(i,2)}, \dots$ from the locally largest evolution in $\omega^{(i)}$ (ranked as explained at the end of the previous section). We write $r_{i,j}$ for the starting level of $\omega^{(i,j)}$.

Obviously we can continue the construction by induction. Assuming that we have defined $\omega^{(i_1,\dots,i_k)}$, we let $a^\bullet_{i_1,\dots,i_k}$ be the terminal point of the locally largest evolution in $\omega^{(i_1,\dots,i_k)}$, and we denote the excursions from the locally largest evolution in $\omega^{(i_1,\dots,i_k)}$ by $\omega^{(i_1,\dots,i_k,1)}, \omega^{(i_1,\dots,i_k,2)}, \dots$. For every $j \geqslant 1$, we let $r_{i_1,\dots,i_k,j}$ be the starting level of $\omega^{(i_1,\dots,i_k,j)}$.

We also set, for every $(i_1, \dots, i_k)$,

$$h_{i_1,\dots,i_k} = r_{i_1} + r_{i_1,i_2} + \cdots + r_{i_1,\dots,i_k},$$

and we let $\beta_{i_1,\dots,i_k}$ be the label of $a^\bullet_{i_1,\dots,i_k}$ (in $\omega^{(i_1,\dots,i_k)}$). Note that $h_{i_1,\dots,i_k} + \beta_{i_1,\dots,i_k}$ is the label of the point corresponding to $a^\bullet_{i_1,\dots,i_k}$ in ω.

Let $r \in [h_{i_1,\dots,i_k}, h_{i_1,\dots,i_k} + \beta_{i_1,\dots,i_k})$, and consider the connected component

$$\mathcal{C}_{r-h_{i_1,\dots,i_k}}^{(a^\bullet_{i_1,\dots,i_k})}(\omega^{(i_1,\dots,i_k)}).$$

As in Section 4.3.1, this is the connected component of

$$\{a \in \mathcal{T}_\zeta(\omega^{(i_1,\dots,i_k)}) : V_a(\omega^{(i_1,\dots,i_k)}) > r - h_{i_1,\dots,i_k}\}$$

that contains $a^\bullet_{i_1,\dots,i_k}$). As explained in Section 4.3.2, this connected component corresponds (via a volume-preserving isometry) to a connected component of $\{a \in \mathcal{T}_\zeta(\omega^{(i_1,\dots,i_{k-1})}) : V_a(\omega^{(i_1,\dots,i_{k-1})}) > r - h_{i_1,\dots,i_{k-1}}\}$ and inductively to a connected component of $\{a \in \mathcal{T}_\zeta(\omega) : V_a(\omega) > r\}$. The latter component is denoted by $\mathcal{D}_r^{(i_1,\dots,i_k)}$. Recall that this definition makes sense only if $r \in [h_{i_1,\dots,i_k}, h_{i_1,\dots,i_k} + \beta_{i_1,\dots,i_k})$ (otherwise we may take $\mathcal{D}_r^{(i_1,\dots,i_k)} = \varnothing$).

We set

$$\mathcal{U} = \bigcup_{k=0}^{\infty} \mathbb{N}^k$$

with the convention $\mathbb{N}^0 = \{\varnothing\}$. We define $h_\varnothing = 0$, $\beta_\varnothing = V_{a^\bullet}(\omega)$ and $\mathcal{D}_r^\varnothing = \mathcal{C}_r^{(a^\bullet)}$ if $0 \leqslant r < V_{a^\bullet}(\omega)$.

Lemma 4.4. *Let $r \geqslant 0$. The sets $\mathcal{D}_r^{(i_1,\dots,i_k)}$, for all $(i_1, \dots, i_k) \in \mathcal{U}$ such that $h_{i_1,\dots,i_k} \leqslant r < h_{i_1,\dots,i_k} + \beta_{i_1,\dots,i_k}$, are exactly the connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$.*

Proof. We already know that any of the sets $\mathcal{D}_r^{(i_1,\dots,i_k)}$, $(i_1, \dots, i_k) \in \mathcal{U}$, is a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, and we need to show that any connected component is of this type. Let $\mathcal{C}$

be a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, and choose any $a \in \mathcal{C}$. The process $(Z_t^{(a)})_{0\leqslant t\leqslant r}$ has only finitely many jump times $s \in [0,r]$ such that $|\Delta Z_s^{(a)}| > Z_s^{(a)}$ (note that $Z^{(a)}$ is bounded below by a positive constant on $[0,r]$). We denote these jump times by $0 < t_1 < t_2 < \cdots < t_k \leqslant r$, where $k \geqslant 0$.

• If $k = 0$, this means that $|\Delta Z_s^{(a)}| < Z_s^{(a)}$ for every $s \in [0,r]$, and we have already seen that this implies that $a \in \mathcal{C}_r^{(a^\bullet)}$. We have thus $\mathcal{C} = \mathcal{C}_r^{(a^\bullet)} = \mathcal{D}_r^{\varnothing}$ in that case.

• Suppose that $k \geqslant 1$. We have $Z_s^{(a)} = Z_s^{(a^\bullet)}$ if and only if $0 \leqslant s < t_1$. In particular $a \in \mathcal{C}_s^{(a^\bullet)}$ if and only if $0 \leqslant s < t_1$, so that a belongs to $a \in \mathcal{C}_{t_1-}^{(a^\bullet)}$, which is the closure of $\mathcal{C}_{t_1}^{(a^\bullet)} \cup \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$. Since points of the boundary of $\mathcal{C}_{t_1}^{(a^\bullet)} \cup \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$ have label t_1 whereas $V_a > t \geqslant t_1$, it follows that $a \in \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$, and $\mathcal{C} \subset \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$. Furthermore t_1 is a jump time of $Z^{(a^\bullet)}$, so that we have $t_1 = r_{i_1}$ for some $i_1 \geqslant 1$. As explained in Section 4.3.2, $\check{\mathcal{C}}_{t_1}^{(a^\bullet)}$ is identified to $\mathcal{T}_\zeta^\circ(\omega^{(i_1)})$, and through this identification $\mathcal{C}$ is identified to a connected component $\mathcal{C}'$ of $\{b \in \mathcal{T}_\zeta^\circ(\omega^{(i_1)}) : V_b(\omega^{(i_1)}) > r - r_{i_1}\}$ and a is identified to a point a' of $\mathcal{C}'$. We have then

$$(Z_t^{(a')}(\omega^{(i_1)}), 0 \leqslant t \leqslant r - t_1) = (Z_{r_1+t}^{(a)}, 0 \leqslant t \leqslant r - t_1).$$

In particular, if $k = 1$, there are no jump times $s \in [0, r-t_1]$ such that $|\Delta Z_s^{(a')}(\omega^{(i_1)})| > Z_s^{(a')}(\omega^{(i_1)})$, and we conclude that $\mathcal{C}' = \mathcal{C}_{r-r_{i_1}}^{(a_{i_1}^\bullet)}(\omega^{(i_1)})$, which means that $\mathcal{C} = \mathcal{D}_r^{(i_1)}$.

• Suppose $k \geqslant 2$. Then $t_2 - t_1$ is the first jump time of $(Z_t^{(a')}(\omega^{(i_1)}))_{0\leqslant t\leqslant r-t_1}$ such that $|\Delta Z_s^{(a')}(\omega^{(i_1)})| > Z_s^{(a')}(\omega^{(i_1)})$, we have $Z_s^{(a_{i_1}^\bullet)}(\omega^{(i_1)}) = Z_s^{(a')}(\omega^{(i_1)})$ for $0 \leqslant s < t_2 - t_1$, and there exists $i_2 \geqslant 1$ such that $t_2 - t_1 = r_{i_1,i_2}$. We have then $\mathcal{C}' \subset \check{\mathcal{C}}_{t_2-t_1}^{(a_{i_1}^\bullet)}(\omega^{(i_1)})$. It follows that $\mathcal{C}'$ is identified to a connected component $\mathcal{C}''$ of $\{b \in \mathcal{T}_\zeta^\circ(\omega^{(i_1,i_2)}) : V_b(\omega^{(i_1,i_2)}) > r - r_{i_1} - r_{i_1,i_2}\}$. If $k = 2$, we conclude as in the preceding step that $\mathcal{C}'' = \mathcal{C}_{r-h_{i_1,i_2}}^{(a_{i_1,i_2}^\bullet)}(\omega^{(i_1,i_2)})$, which means that $\mathcal{C} = \mathcal{D}_r^{(i_1,i_2)}$. The proof is easily completed by induction, and we omit the details. □

Proof of Theorem 4.2. The first part of Theorem 4.2 (concerning the definition and approximation of boundary sizes) is a consequence of Proposition 4.5, which in fact gives a stronger result. So we only need to prove the second part of the statement. If $\mathcal{C}$ is a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, we write $Z_{(\mathcal{C})} = Z_r^{(a)}$ where a is any point of $\mathcal{C}$ (this does not depend on the choice of a). To simplify notation, for every $(i_1,\ldots,i_k) \in \mathcal{U}$ and every $j \geqslant 1$, we write $\Delta_{(i_1,\ldots,i_k,j)}$ for the jump at time $h_{i_1,\ldots,i_k,j}$ of the function $r \mapsto Z_{(\mathcal{D}_r^{(i_1,\ldots,i_k)})}$ – from our construction this is also the jump at time $r_{i_1,\ldots,i_k,j}$ of the function $r \mapsto Z_r^{(a_{i_1,\ldots,i_k}^\bullet)}(\omega^{(i_1,\ldots,i_k)})$.

From the preceding lemma, we get that $\mathbf{Y}(r)$ is obtained as the (reordered) collection of the quantities $Z_{(\mathcal{D}_r^{(i_1,\ldots,i_k)})}$ for all $(i_1,\ldots,i_k) \in \mathcal{U}$ such that $h_{i_1,\ldots,i_k} \leqslant r < h_{i_1,\ldots,i_k} + \beta_{i_1,\ldots,i_k}$.

We know that the process $(Z_{(\mathcal{D}_r^\varnothing)})_{0\leqslant r<\beta_\varnothing} = (Z_r^{(a^\bullet)})_{0\leqslant r<V_{a^\bullet}}$ is distributed as the self-similar Markov process X of Proposition 4.7 started from z and killed when it hits 0. Thanks to Proposition 4.8, we then get that, conditionally on $(Z_{(\mathcal{D}_r^\varnothing)})_{0\leqslant r<\beta_\varnothing}$, the excursions $\omega^{(i)}$, $i = 1,2,\ldots$, are independent and for every fixed j the conditional distribution of $\omega^{(j)}$ is $\mathbb{N}_0^{*,|\Delta_{(j)}|}$. Consequently, under the same conditioning, the processes $(Z_{(\mathcal{D}^{(i)}_{h_i+r})})_{0\leqslant r\leqslant\beta_i}$, $i=1,2,\ldots$, are independent copies of X started respectively at $|\Delta_{(i)}|$, $i=1,2,\ldots$

We can continue by induction, using Proposition 4.8 at every step. We obtain that, conditionally on the processes

$$\Big(Z_{(\mathcal{D}^{(i_1,\ldots,i_\ell)}_r)}\Big)_{h_{(i_1,\ldots,i_\ell)}\leqslant r<h_{(i_1,\ldots,i_\ell)}+\beta_{(i_1,\ldots,i_\ell)}},\ 0\leqslant \ell\leqslant k,(i_1,\ldots,i_\ell)\in\mathbb{N}^\ell,$$

the excursions $\omega^{(i_1,\ldots,i_k,j)}$, $(i_1,\ldots,i_k,j)\in\mathbb{N}^{k+1}$, are independent and for every fixed $(i_1,\ldots,i_k,j)$ the conditional distribution of $\omega^{(i_1,\ldots,i_k,j)}$ is $\mathbb{N}_0^{*,|\Delta_{(i_1,\ldots,i_k,j)}|}$. Hence, under the same conditioning, the processes

$$\Big(Z_{(\mathcal{D}^{(i_1,\ldots,i_k,j)}_{h_{(i_1,\ldots,i_k,j)}+r})}\Big)_{0\leqslant r\leqslant\beta_{(i_1,\ldots,i_k,j)}},\ (i_1,\ldots,i_k,j)\in\mathbb{N}^{k+1}$$

are independent copies of X started respectively at $|\Delta_{(i_1,\ldots,i_k,j)}|$.

From these observations, we conclude that $(\mathbf{Y}(r))_{r\geqslant 0}$ is a growth fragmentation process whose Eve particle process is the self-similar Markov process X and with initial value $\mathbf{Y}(0)=(z,0,0,\ldots)$. □

Proof of Theorem 4.1. Under $\mathbb{N}_0$, the connected components $\mathcal{C}_1,\mathcal{C}_2,\ldots$ of $\{a\in\mathcal{T}_\zeta : V_a>0\}$, and the labels on these components can be represented by snake trajectories $\omega_1,\omega_2,\ldots$ defined in a way very similar to the excursions above the minimum in Section 4.3.2 (see also the introduction of [2]). By [2, Theorem 4], conditionally on the boundary sizes $(|\partial\mathcal{C}_1|,|\partial\mathcal{C}_2|,\ldots)$, $\omega_1,\omega_2,\ldots$ are independent and the conditional distribution of ω_i is $\mathbb{N}_0^{*,|\partial\mathcal{C}_i|}$. In the notation of Theorem 4.1, $\mathbf{X}(0)$ is just the (ranked) sequence $(|\partial\mathcal{C}_1|,|\partial\mathcal{C}_2|,\ldots)$, and we get that, conditionally on $\mathbf{X}(0)$, the process $(\mathbf{X}(r))_{r\geqslant0}$ is obtained by superimposing *independent* processes $(\mathbf{Y}_i(r))_{r\geqslant0}$ such that, for every $i\geqslant1$, $\mathbf{Y}_i$ is a growth-fragmentation process started from $(|\partial\mathcal{C}_i|,0,0,\ldots)$ (by Theorem 4.2). The desired result follows. □

4.9 Slicing the Brownian disk at heights

In this section, we prove Theorem 4.3. We rely on the construction of the free Brownian disk $\mathbb{D}_z$ from a random snake trajectory distributed according to $\mathbb{N}_0^{*,z}$. This construction is given in [71], to which we refer for additional details. Throughout this section, we argue under $\mathbb{N}_0^{*,z}$, and the following statements hold $\mathbb{N}_0^{*,z}$ a.s.

The free Brownian disk $\mathbb{D}_z$ is a random geodesic compact metric space, which is constructed (under $\mathbb{N}_0^{*,z}$) as a quotient space of $\mathcal{T}_\zeta$. The canonical projection, which is a continuous mapping from $\mathcal{T}_\zeta$ onto $\mathbb{D}_z$, is denoted by Π. We note that the push forward of the volume measure vol on $\mathcal{T}_\zeta$ is the volume measure $\mathbf{V}$ on $\mathbb{D}_z$.

Recall the notation $H(x)$, for the "height" of $x\in\mathbb{D}_z$ (the distance from x to the boundary $\partial\mathbb{D}_z$). We will not need the details of the construction of $\mathbb{D}_z$, but we record the following two facts:

(a) If $a\in\mathcal{T}_\zeta$ and $x=\Pi(a)$, we have $H(x)=V_a$.

(b) For every $a, b \in \mathcal{T}_\zeta$ such that $\Pi(a) = \Pi(b)$, we have $V_a = V_b = \min_{c\in[\![a,b]\!]} V_c$.

The following lemma is an analog for the Brownian disk of Proposition 3.1 of [63] for the Brownian map. The proof is similar, but we provide details because this result is the key to the derivation of Theorem 4.3.

Lemma 4.5. *Let $a, b \in \mathcal{T}_\zeta$ and let $(\gamma(t))_{0\leqslant t\leqslant T}$ be a continuous path in $\mathbb{D}_z$ such that $\gamma(0) = \Pi(a)$ and $\gamma(T) = \Pi(b)$. Then*

$$\min_{0\leqslant t\leqslant T} H(\gamma(t)) \leqslant \min_{c\in[\![a,b]\!]} V_c.$$

Proof. We may assume that

$$V_a \wedge V_b > \min_{c\in[\![a,b]\!]} V_c$$

since the result is trivial otherwise. Then we can find $c_0 \in]\!]a, b[\![$ such that

$$V_{c_0} = \min_{c\in[\![a,b]\!]} V_c.$$

The points a and b are in different connected components of $\mathcal{T}_\zeta\backslash\{c_0\}$. Let $\mathcal{C}_1$ be the connected component of $\mathcal{T}_\zeta\backslash\{c_0\}$ that contains a, and let $\mathcal{C}_2 = \mathcal{T}_\zeta\backslash\overline{\mathcal{C}}_1$, so that $b \in \mathcal{C}_2$. Set

$$t_0 := \inf\{t \in [0, T] : \gamma(t) \in \Pi(\overline{\mathcal{C}}_2)\}.$$

Since $\Pi(\overline{\mathcal{C}}_2)$ is closed, we have $\gamma(t_0) \in \Pi(\overline{\mathcal{C}}_2)$. Furthermore, $t_0 > 0$ because otherwise this would mean that $\Pi(a) \in \Pi(\overline{\mathcal{C}}_2)$, and thus that there would exist $a' \in \overline{\mathcal{C}}_2$ such that $\Pi(a) = \Pi(a')$: Noting that $c_0 \in [\![a, a']\!]$, property (b) above would imply that $V_a \leqslant V_{c_0}$, which is a contradiction.

We can then choose a sequence $(s_n)_{n\geqslant 1}$ in $[0, t_0)$ such that $s_n \uparrow t_0$ as $n \uparrow \infty$. Since $\gamma(s_n) \in \Pi(\mathcal{C}_1)$, there exists $a_n \in \mathcal{C}_1$ such that $\gamma(s_n) = \Pi(a_n)$. Up to extracting a subsequence, we can assume that $a_n \longrightarrow a_\infty \in \overline{\mathcal{C}}_1$. Then necessarily $\Pi(a_\infty) = \gamma(t_0) = \Pi(b')$ for some $b' \in \overline{\mathcal{C}}_2$. By properties (a) and (b), we must have

$$H(\gamma(t_0)) = V_{b'} = V_{a_\infty} = \min_{c\in[\![a_\infty,b']\!]} V_c \leqslant V_{c_0}.$$

This completes the proof. □

Proposition 4.9. *Let $r > 0$ and $a, b \in \mathcal{T}_\zeta$. Then $\Pi(a)$ and $\Pi(b)$ belong to the same connected component of $\{x \in \mathbb{D}_z : H(x) > r\}$ if and only if a and b belong to the same connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$.*

Proof. If a and b belong to the same connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$, then the line segment $[\![a, b]\!]$ is contained in $\{c \in \mathcal{T}_\zeta : V_c > r\}$, and $\Pi([\![a, b]\!])$ provides a path going from $\Pi(a)$ to $\Pi(b)$ that stays in $\{x \in \mathbb{D}_z : H(x) > r\}$, by property (a).

Conversely, if a and b belong to different connected components of $\{c \in \mathcal{T}_\zeta : V_c > r\}$, then

$$\min_{c\in[\![a,b]\!]} V_c \leqslant r,$$

and, by Lemma 8.40, any continuous path from $\Pi(a)$ to $\Pi(b)$ must visit a point x with $H(x) \leqslant r$. It follows that $\Pi(a)$ and $\Pi(b)$ belong to different connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$. □

Proof of Theorem 4.3. By Proposition 4.9, for every $r \geqslant 0$, the projection Π induces a one-to-one correspondence between connected components of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ and connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$. Furthermore, let $\mathcal{D}$ be a connected component of $\{x \in \mathbb{D}_z : H(x) > r\}$, and let $\mathcal{C}$ be the associated connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ (such that $\Pi(\mathcal{C}) = \mathcal{D}$). Together with property (a) above, the fact that Π maps the volume measure vol to $\mathbf{V}$ immediately shows that the boundary size $|\partial\mathcal{D}|$ can be defined by the approximation in Theorem 4.3, and that $|\partial\mathcal{D}| = |\partial\mathcal{C}|$. Theorem 4.3 is now a direct consequence of Theorem 4.2. □

4.10 The law of components above a fixed level

Our goal in this section is to prove Theorem 4.4. To this end, we will first state and prove a theorem about excursions "above a fixed height" for a snake trajectory distributed according to $\mathbb{N}_0^{*,z}$.

Let us fix $r \geqslant 0$. Let $\omega \in \mathcal{S}_0$ be chosen according to $\mathbb{N}_0$, or to $\mathbb{N}_0^{*,z}$, and consider all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$. If $\mathcal{C}$ is one of these connected components, we can represent $\mathcal{C}$ and the labels on $\mathcal{C}$ by a snake trajectory $\tilde{\omega}$, which is defined as follows. First we observe that there is a unique $a_0 \in \mathcal{T}_\zeta$ such that $a_0 \in \partial\mathcal{C}$ and every point of $\mathcal{C}$ is a descendant of a_0. Note that $V_{a_0} = r$, and that the point a_0 cannot be a branching point (no branching point can have label r, $\mathbb{N}_0$ a.e. or $\mathbb{N}_0^{*,z}$ a.s.). Hence we can make sense of the subtrajectory rooted at a_0, which we denote by $\breve{\omega}$. Finally, we let $\widetilde{\omega} = \mathrm{tr}_0(\breve{\omega})$.

We can define the boundary size $\mathcal{Z}_0^*(\widetilde{\omega})$ of $\widetilde{\omega}$, using Proposition 4.5 (setting $\mathcal{Z}_0^*(\widetilde{\omega}) = Z_r^{(a)}$, where a is an arbitrary point of $\mathcal{C}$) if ω is chosen according to $\mathbb{N}_0^{*,z}$, or the excursion theory of [2] if ω is chosen according to $\mathbb{N}_0$. We call the snake trajectories $\widetilde{\omega}$ obtained when varying $\mathcal{C}$ among the connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ the excursions of ω above level r.

Theorem 4.6. *Let $r > 0$. On the event $\{W^*(\omega) > r\}$, let $\widetilde{\omega}^1, \widetilde{\omega}^2, \ldots$ be the excursions of ω above level r, ranked in decreasing order of their boundary sizes. Write $\widetilde{z}_1 > \widetilde{z}_2 > \cdots$ for these boundary sizes. Then, under $\mathbb{N}_0^{*,z}(\cdot \mid W^* > r)$, conditionally on the collection $(\widetilde{z}_i)_{i\geqslant 1}$, the snake trajectories $\widetilde{\omega}^1, \widetilde{\omega}^2, \ldots$ are independent with respective distributions $\mathbb{N}_0^{*,\widetilde{z}_1}, \mathbb{N}_0^{*,\widetilde{z}_2}, \ldots$.*

Remark. It is not immediately obvious that the boundary sizes $\widetilde{z}^1, \widetilde{z}^2, \ldots$ are distinct a.s. This can however be deduced from the arguments of the proof below.

Proof. We will derive the theorem from the excursion theory of [2], and to this end we first need to argue under $\mathbb{N}_0(\mathrm{d}\omega)$. We write $\omega^1, \omega^2, \ldots$ for the excursions above 0 ranked in decreasing order of their boundary sizes $\mathcal{Z}_0^*(\omega^1), \mathcal{Z}_0^*(\omega^2), \ldots$. Theorem 4 in [2] then implies that, under $\mathbb{N}_0$ and conditionally on $\mathcal{Z}_0^*(\omega^1) = z_1, \mathcal{Z}_0^*(\omega^2) = z_2, \ldots$, the excursions $\omega^1, \omega^2, \ldots$ are independent and the conditional distribution of ω^i is $\mathbb{N}_0^{*,z_i}$.

Let A be the event where exactly one excursion above 0 hits r, and let ω^{i_0} be this excursion. It follows from the preceding observations that, under $\mathbb{N}_0(\cdot \mid A)$, the conditional distribution of

ω^{i_0} knowing $\mathcal{Z}_0^*(\omega^{i_0}) = z$ is $\mathbb{N}_0^{*,z}(\cdot \mid W^* > r)$. Hence, if φ is a bounded nonnegative measurable function on $\mathbb{R}_+$, and h is a nonnegative measurable function on $\mathcal{S}_0$, we have

$$\begin{aligned}\mathbb{N}_0\Big(\mathbf{1}_A\,\varphi(\mathcal{Z}_0^*(\omega^{i_0}))\,\exp\Big(-\sum_{i=1}^{\infty} h(\widetilde{\omega}^i)\Big)\Big) \\ = \mathbb{N}_0\Big(\mathbf{1}_A\,\varphi(\mathcal{Z}_0^*(\omega^{i_0}))\,\mathbb{N}_0^{*,\mathcal{Z}_0^*(\omega^{i_0})}\Big(\exp\Big(-\sum_{i=1}^{\infty} h(\widetilde{\omega}^i)\Big)\Big)\Big),\end{aligned} \tag{4.27}$$

where we use the notation $\widetilde{\omega}^1, \widetilde{\omega}^2, \ldots$ introduced in the theorem for the excursions above level r (notice that this makes sense both under $\mathbb{N}_0$ and under $\mathbb{N}_0^{*,z}$ and that, on the event A, the excursions of ω and of ω^{i_0} above level r are the same).

We will now rewrite the left-hand side of (4.27) in a different form. To this end (arguing under $\mathbb{N}_0(d\omega \mid W^* > r)$), it is convenient to introduce all excursions of ω away from r: each such excursion $\overline{\omega}^i$, $i = 1, 2, \ldots$, corresponds to one connected component of $\{a \in \mathcal{T}_\zeta : V_a \neq r\}$, but we exclude the connected component containing the root ρ (which may be represented by $\mathrm{tr}_r(\omega)$), and, apart from this fact, the definition of these excursions is exactly the same as that of excursions above level r. As previously, the excursions $\overline{\omega}^i$, $i = 1, 2, \ldots$ are listed in decreasing order of their boundary sizes $\overline{z}^i := \mathcal{Z}_0^*(\overline{\omega}^i)$, $i = 1, 2, \ldots$. For every $i \geqslant 1$, we also let $\overline{\varepsilon}^i$ be the sign of $\overline{\omega}^i$: $\overline{\varepsilon}^i = 1$ if the values of V on the corresponding connected component are greater than r and $\overline{\varepsilon}^i = -1$ otherwise. The sequence $\widetilde{\omega}^1, \widetilde{\omega}^2, \ldots$ is obtained by keeping only the excursions $\overline{\omega}^i$ with $\overline{\varepsilon}^i = 1$ in the sequence $\overline{\omega}^i$, $i = 1, 2, \ldots$. Let $\mathcal{B}$ be the σ-field generated by $\mathrm{tr}_r(\omega)$, the sequence $(\overline{\varepsilon}^i, \overline{z}^i)_{i=1,2\ldots}$ and the excursions $\overline{\omega}^i$ for all i such that $\overline{\varepsilon}^i = -1$ (in other words the excursions below level r). By combining the special Markov property with [2, Theorem 4], we get that conditionally on $\mathcal{B}$ the excursions $\widetilde{\omega}^i$, $i = 1, 2, \ldots$ are independent and the conditional distribution of $\widetilde{\omega}^i$ is $\mathbb{N}_0^{*,\tilde{z}_i}$, where we write $\widetilde{z}_i = \mathcal{Z}_0^*(\widetilde{\omega}^i)$ as in the statement of the theorem – notice that the quantities $\widetilde{z}_i$ are $\mathcal{B}$-measurable.

The point is now that the event A (and the variable $\mathbf{1}_A\,\varphi(\mathcal{Z}_0^*(\omega^{i_0}))$) is $\mathcal{B}$-measurable. In fact it is not hard to check that A is determined by the knowledge of $\mathrm{tr}_r(\omega)$ and of the excursions below level r (for A to hold, no such excursion is allowed to contain a path that comes back to 0 and then visits r again). Thanks to this observation, we can rewrite the left-hand side of (4.27) as

$$\mathbb{N}_0\Big(\mathbf{1}_A\,\varphi(\mathcal{Z}_0^*(\omega^{i_0}))\prod_{i=1}^{\infty}\mathbb{N}_0^{*,\tilde{z}_i}(e^{-h})\Big).$$

By the same argument that led us to (4.27), this is also equal to

$$\mathbb{N}_0\Big(\mathbf{1}_A\,\varphi(\mathcal{Z}_0^*(\omega^{i_0}))\,\mathbb{N}_0^{*,\mathcal{Z}_0^*(\omega^{i_0})}\Big(\prod_{i=1}^{\infty}\mathbb{N}_0^{*,\tilde{z}_i}(e^{-h})\Big)\Big). \tag{4.28}$$

Notice that the law of $\mathcal{Z}_0^*(\omega^{i_0})$ under $\mathbb{N}_0(\cdot \mid A)$ has a positive density with respect to Lebesgue measure (under $\mathbb{N}_0$, the boundary sizes of excursions away from 0 are the jumps of a ϕ-CSBP under its excursion measure, see [2, Theorem 4]). The equality between the quantity (4.28) and the right-hand side of (4.27) for every function φ implies that we have

$$\mathbb{N}_0^{*,z}\Big(\exp\Big(-\sum_{i=1}^{\infty} h(\tilde{\omega}^i)\Big)\Big) = \mathbb{N}_0^{*,z}\Big(\prod_{i=1}^{\infty} \mathbb{N}_0^{*,\tilde{z}_i}(e^{-h})\Big) \tag{4.29}$$

for Lebesgue a.a. $z > 0$. We claim that (4.29) in fact holds for *every* $z > 0$. To see this, we need a continuity argument. We restrict our attention to functions h of the type $h(\omega) = h_1(\mathcal{Z}_0^*(\omega))h_2(\omega)$, where h_1 and h_2 are both (nonnegative and) bounded and continuous on $\mathbb{R}_+$ and $\mathcal{S}_0$ respectively, and there exists $\delta > 0$ such that $h_1(x) = 0$ if $x \leqslant \delta$ and $h_2(\omega) = 0$ if $W^*(\omega) \leqslant \delta$. Under these assumptions on h, one can verify that both sides of (4.29) are left-continuous functions of z, which will yield our claim. Let us briefly explain this. We write $g_1(z)$ and $g_2(z)$ for the left-hand side and the right-hand side of (4.29) respectively. We use the scaling transformation $\theta_{z/z'}$ that maps $\mathbb{N}_0^{*,z'}$ to $\mathbb{N}_0^{*,z}$ (see Section 5.2.2) to check that $g_i(z') \longrightarrow g_i(z)$ as $z' \uparrow z$, for $i = 1$ or 2. We note that this scaling transformation maps excursions above level r to excursions above level $r\sqrt{z/z'}$, and, for the function g_2, we observe that the collection $(\tilde{z}_i)_{i\geqslant 1}$ is the value at time r of the growth-fragmentation process $\mathbf{X}$ of Theorem 4.2, and we use the continuity properties of this growth-fragmentation process (see in particular Corollary 4 in [17]). We omit a few details that are left to the reader.

Once we know that (4.29) holds for a fixed $z > 0$ and for a sufficiently large class of functions h, we obtain that the conditional distribution of the random point measure

$$\sum_{i=1}^{\infty} \delta_{\tilde{\omega}^i}$$

given $(\tilde{z}_i)_{i\geqslant 1}$ is as prescribed in the statement of the theorem. This completes the proof. □

Proof of Theorem 4.4. We can derive Theorem 4.4 from Theorem 4.6 by arguments very similar to those of the proof of Theorem 38 in [71] and, for this reason we only sketch the main steps of the proof. As we already noticed in the proof of Theorem 4.3 in the previous section, the connected components $\mathcal{C}_1, \mathcal{C}_2, \ldots$ (in the notation of Theorem 4.4) are in one-to-one correspondence with the excursions $\tilde{\omega}^1, \tilde{\omega}^2, \ldots$, in such a way that the boundary size of $\mathcal{C}_i$ is equal to the boundary size $\tilde{z}_i$ of $\tilde{\omega}_i$. Following [71, Section 8], we can associate a random compact metric space $\Theta(\tilde{\omega}^i)$ with each excursion $\tilde{\omega}^i$, and we know, by Theorem 4.6 and the main result of [71], that conditionally on $(\tilde{z}_i)_{i\geqslant 1}$, the random metric spaces $\Theta(\tilde{\omega}^i)$, $i \geqslant 1$, are independent free Brownian disks with respective perimeters $\tilde{z}_i$, $i \geqslant 1$. So, all that remains is to show that, for every $i \geqslant 1$, the random metric space $(\overline{\mathcal{C}}_i, d_i)$ can be constructed in the way explained in the statement of Theorem 4.4 and is isometric to $\Theta(\tilde{\omega}^i)$. This is exactly similar to the proof of the identity (60) in [71], to which we refer for additional details. □

4.11 Complements

4.11.1 The cumulant function.

It is known [95] that a (self-similar) growth-fragmentation process is characterized by a pair consisting of the self-similarity index (here $\alpha = -1/2$) and a cumulant function κ, which is a convex function defined on $(0,\infty)$ possibly taking the values $+\infty$. The cumulant function κ is given explicity in terms of the Laplace exponent ψ and the Lévy measure $\pi(\mathrm{d}y)$ of the Lévy process ξ appearing in the Lamperti representation of the self-similar process describing the evolution of the Eve particle, via the formula

$$\kappa(p) = \psi(p) + \int_{(-\infty,0)} (1-e^y)^p\,\pi(\mathrm{d}y), \quad p > 0.$$

This identity is used in [19] to give an explicit formula for $\kappa(p)$ (see formula (33) in [19]). We will now describe a different approach to the formula for κ, which is independent of the derivation of the Laplace exponent. This suggests that another proof of Theorem 4.2 should be possible without the identification of the law of the locally largest evolution, provided one knows a priori that the process $(\mathbf{Y}(r))_{r\geqslant 0}$ is a growth-fragmentation process – note that Theorem 4.6 does not provide enough information for this.

In view of recovering the expression of κ, we observe that the negative values of the cumulant function are given by the following formula [18, Section 3]. We consider the growth-fragmentation process $(\mathbf{Y}(r))_{r\geqslant 0}$ of Theorem 4.2 started at $(1,0,0,\ldots)$. For every $r \geqslant 0$, write $\mathbf{Y}(r) = (Y_r^1, Y_r^2, \ldots)$, and, for every $p \in \mathbb{R}$,

$$\|\mathbf{Y}(r)\|_p = \sum_{i=1}^{\infty} |Y_r^i|^p.$$

Then, for every $p > 1/2$, the quantity

$$\mathbb{N}_0^{*,1}\Big(\int_0^\infty \mathrm{d}r\,\|\mathbf{Y}(r)\|_{p-1/2}\Big) \tag{4.30}$$

is finite if and only if $\kappa(p) < 0$, and is then equal to $-1/\kappa(p)$ [18, Formula (16)].

For every $i \geqslant 1$, let σ_r^i be the volume of the i-th connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ (for our purposes here the way the connected components are ordered is irrelevant). Let $q \in (-1,1)$. As a consequence of (4.11), we have, for $r > 0$,

$$\mathbb{N}_0^*\Big(\sum_{i=1}^{\infty} \sigma_r^i (Y_r^i)^q\, e^{-\mathcal{Z}_0^*}\Big) = 2\,\mathbb{N}_0\Big(\int_{-\infty}^{-r} \mathrm{d}b\,\mathcal{Z}_b(\mathcal{Z}_{b+r})^q\, e^{-\mathcal{Z}_b}\Big). \tag{4.31}$$

Let us consider first the left-hand side of (4.31). Using Theorem 4.6 and the identity $\mathbb{N}_0^{*,z}(\sigma) = z^2$

in the second equality, we get

$$\begin{aligned}\mathbb{N}_0^*\Big(\sum_{i=1}^{\infty}\sigma_r^i(Y_r^i)^q\,e^{-\mathcal{Z}_0^*}\Big) &= \sqrt{\frac{3}{2\pi}}\int_0^\infty \mathrm{d}z\,z^{-5/2}\,e^{-z}\,\mathbb{N}_0^{*,z}\Big(\sum_{i=1}^{\infty}\sigma_r^i(Y_r^i)^q\Big)\\ &= \sqrt{\frac{3}{2\pi}}\int_0^\infty \mathrm{d}z\,z^{-5/2}\,e^{-z}\,\mathbb{N}_0^{*,z}\Big(\sum_{i=1}^{\infty}(Y_r^i)^{q+2}\Big)\\ &= \sqrt{\frac{3}{2\pi}}\int_0^\infty \mathrm{d}z\,z^{q-1/2}\,e^{-z}\,\mathbb{N}_0^{*,1}(\|\mathbf{Y}(rz^{-1/2})\|_{q+2}),\end{aligned}$$

by scaling. If we integrate with respect to $\mathrm{d}r$, we arrive at

$$\int_0^\infty \mathrm{d}r\mathbb{N}_0^*\Big(\sum_{i=1}^{\infty}\sigma_r^i(Y_r^i)^q\,e^{-\mathcal{Z}_0^*}\Big) = \sqrt{\frac{3}{2\pi}}\,\Gamma(q+1)\,\mathbb{N}_0^{*,1}\Big(\int_0^\infty \mathrm{d}r\,\|\mathbf{Y}(r)\|_{q+2}\Big). \tag{4.32}$$

Consider then the right-hand side of (4.31). Recalling formula (4.5) and the fact that the process $(\mathcal{Z}_{-r})_{r>0}$ is Markovian under $\mathbb{N}_0$ with the transition kernels of the ϕ-CSBP, we get

$$\begin{aligned}&\mathbb{N}_0\Big(\int_{-\infty}^{-r}\mathrm{d}b\,\mathcal{Z}_b(\mathcal{Z}_{b+r})^q\,e^{-\mathcal{Z}_b}\Big)\\ &= \int_{-\infty}^{-r}\mathrm{d}b\,\mathbb{N}_0\Big((\mathcal{Z}_{b+r})^{q+1}\times(1+r\sqrt{\frac{2}{3}})^{-3}\exp(-\mathcal{Z}_{b+r}(1+r\sqrt{\frac{2}{3}})^{-2})\Big)\\ &= \mathbb{N}_0\Big(\int_{-\infty}^{0}\mathrm{d}b\,(\mathcal{Z}_b)^{q+1}\times(1+r\sqrt{\frac{2}{3}})^{-3}\exp(-\mathcal{Z}_b(1+r\sqrt{\frac{2}{3}})^{-2})\Big).\end{aligned}$$

Integrating with respect to $\mathrm{d}r$, we find

$$\int_0^\infty \mathrm{d}r\,\mathbb{N}_0\Big(\int_{-\infty}^{-r}\mathrm{d}b\,\mathcal{Z}_b(\mathcal{Z}_{b+r})^q\,e^{-\mathcal{Z}_b}\Big) = \frac{1}{2\sqrt{2/3}}\,\mathbb{N}_0\Big(\int_{-\infty}^{0}\mathrm{d}b\,(\mathcal{Z}_b)^q(1-e^{-\mathcal{Z}_b})\Big). \tag{4.33}$$

To compute the right-hand side, write $x^{q-1} = \Gamma(1-q)^{-1}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-q}\,e^{-\lambda x}$ (for $x>0$), and recall (7.8), which gives $\mathbb{N}_0(\mathcal{Z}_b e^{-\lambda\mathcal{Z}_b}) = \lambda^{-3/2}(\lambda^{-1/2}-b\sqrt{2/3})^{-3}$ for $b<0$. It follows that

$$\begin{aligned}&\mathbb{N}_0\Big(\int_{-\infty}^{0}\mathrm{d}b\,(\mathcal{Z}_b)^q(1-e^{-\mathcal{Z}_b})\Big)\\ &= \frac{1}{\Gamma(1-q)}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-q}\,\mathbb{N}_0\Big(\int_{-\infty}^{0}\mathrm{d}b\,\mathcal{Z}_b\,(e^{-\lambda\mathcal{Z}_b}-e^{-(\lambda+1)\mathcal{Z}_b})\Big)\\ &= \frac{1}{\Gamma(1-q)}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-q}\int_{-\infty}^{0}\mathrm{d}b\,\Big(\lambda^{-3/2}(\lambda^{-1/2}-b\sqrt{2/3})^{-3}\\ &\qquad\qquad -(\lambda+1)^{-3/2}((\lambda+1)^{-1/2}-b\sqrt{2/3})^{-3}\Big)\\ &= \frac{1}{2\sqrt{2/3}\,\Gamma(1-q)}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-q}\,(\lambda^{-1/2}-(\lambda+1)^{-1/2}).\end{aligned}$$

The right-hand side is finite if and only if $-1/2<q<1/2$, and then an elementary calculation gives

$$\int_0^\infty \mathrm{d}\lambda\,\lambda^{-q}\,(\lambda^{-1/2}-(\lambda+1)^{-1/2}) = -\frac{1}{\sqrt{\pi}}\,\Gamma(1-q)\Gamma(q-\frac{1}{2}).$$

Coming back to (4.33), we see that

$$\int_0^\infty \mathrm{d}r\, \mathbb{N}_0\Big(\int_{-\infty}^{-r} \mathrm{d}b\, \mathcal{Z}_b(\mathcal{Z}_{b+r})^q\, e^{-\mathcal{Z}_b}\Big) = -\frac{3}{8\sqrt{\pi}}\Gamma(q-\frac{1}{2}).$$

Combining this equality with (4.31) and (4.32) leads to

$$\mathbb{N}_0^{*,1}\Big(\int_0^\infty \mathrm{d}r\, \|\mathbf{Y}(r)\|_{q+2}\Big) = -\sqrt{\frac{3}{8}}\,\frac{\Gamma(q-\frac{1}{2})}{\Gamma(q+1)}.$$

Replacing q by $p = q + 5/2$, we finally obtain that, for $2 < p < 3$, the quantity (4.30) is finite, and

$$\kappa(p) = \sqrt{\frac{8}{3}}\,\frac{\Gamma(p-\frac{3}{2})}{\Gamma(p-3)},$$

which is in agreement with formula (33) in [19] – note that the value of κ in the latter formula should be multiplied by the factor $\sqrt{3/2\pi}$ that appears in the formula for ψ in Theorem 4.1.

Finally, an argument of analytic continuation shows that the preceding formula for $\kappa(p)$ holds for every $p > 3/2$, whereas $\kappa(p) = +\infty$ for $p \in (0, 3/2]$. The function $p \mapsto \kappa(p)$ is (finite and) convex on $(3/2, \infty)$, and vanishes at $p = 2$ and $p = 3$ (with the notation of [18], we have $\omega_- = 2$ and $\omega_+ = 3$).

4.11.2 A growth-fragmentation process in the Brownian plane

In this section, we consider the random pointed metric space $(\mathcal{P}_\infty, \mathrm{d}_\infty)$ called the Brownian plane, which has been introduced and studied in [39]. The space $\mathcal{P}_\infty$ has a distinguished point ρ_∞, and, for every $r > 0$, we may define the boundary sizes of the connected components of $\{x \in \mathcal{P}_\infty : \mathrm{d}(\rho_\infty, x) > r\}$, via the same approximation as used above in Section 4.4: To see that this definition makes sense, one may argue that there exists a coupling of the Brownian plane and the Brownian map such that small balls centered at the distinguished point in the two spaces are isometric [39], then rely on Proposition 7.5 to treat the case when r is small enough, and finally use the scale invariance of the Brownian plane. Notice that there is exactly one unbounded component, whose boundary is also the boundary of the so-called hull of radius r in $\mathcal{P}_\infty$ (see in particular [40]).

We will relate this collection of boundary sizes to the growth-fragmentation process of Theorem 4.1 subject to a special conditioning. Precisely, we consider this growth-fragmentation process starting from 0 and conditioned to have indefinite growth (see [18, Section 4.2]). Let us briefly describe this process, referring to [18] for more details. We start with one Eve particle, whose mass process $(\widehat{X}_t)_{t\geqslant 0}$ evolves as the process X of Theorem 4.1 conditioned to start from 0 and to stay positive for all times. To be specific, the process $\widehat{X}$ is a self-similar Markov process with index 1/2, which can be obtained via the Lamperti representation from a Lévy process $\widehat{\xi}$ with no positive jumps and Laplace exponent

$$\widehat{\psi}(\lambda) := \kappa(3+\lambda) = \sqrt{\frac{8}{3}}\,\frac{\Gamma(\lambda+\frac{3}{2})}{\Gamma(\lambda)}, \quad \lambda > 0. \tag{4.34}$$

See [18, Lemma 2.1] for the fact that the function $\widehat{\psi}(\lambda)$ corresponds to the Laplace exponent of a Lévy process without positive jumps. Then, as previously, each jump time t of $\widehat{X}$ corresponds to the birth of a new particle (child of the Eve particle) with mass $|\Delta \widehat{X}_t|$, but the masses of these new particles evolve independently according to the distribution of the process X, and similarly for the children of these particles, and so on. We emphasize that only the mass process of the Eve particle evolves according to a different Markov process $\widehat{X}$, while the masses of its children, grandchildren, etc., evolve according to the law of the process X. As previously, we write $\widehat{\mathbf{X}}(r)$ for the collection of masses of all particles present at time r.

Theorem 4.7. *As a process indexed by the variable $r > 0$, the collection of the boundary sizes of all connected components of $\{x \in \mathcal{P}_\infty : \mathrm{d}_\infty(\rho_\infty, x) > r\}$ is distributed as the process $(\widehat{\mathbf{X}}(r))_{r>0}$.*

Proof. We first explain that the role of the Eve particle (for the process $\widehat{\mathbf{X}}$) is played by the evolution of the unbounded component of $\{x \in \mathcal{P}_\infty : \mathrm{d}_\infty(\rho_\infty, x) > r\}$. Let $\widehat{Z}_r$ be the boundary size of this component, with the convention that $\widehat{Z}_0 = 0$. The distribution of the process $(\widehat{Z}_r)_{r\geqslant 0}$ is described in [40, Proposition 1.2]. From this description, using also the arguments of [41, Section 4.4], one gets that $\widehat{Z}_r$ can be written as

$$\widehat{Z}_r = U^{\uparrow}_{\eta_r}$$

where $(U^{\uparrow}_t)_{t\geqslant 0}$ is the Lévy process with no positive jumps and Laplace exponent $\phi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$ conditioned to start from 0 and to stay positive for all times $t > 0$ – see [15, Chapter VII] for a rigorous definition of this process – and η_r is the time change

$$\eta_r = \inf\{t \geqslant 0 : \int_0^t \frac{\mathrm{d}s}{U^{\uparrow}_s} > r\}.$$

It follows from this representation that $(\widehat{Z}_r)_{r\geqslant 0}$ is a self-similar Markov process with index 1/2 with values in $[0, \infty)$. We will now verify that the Laplace exponent of the Lévy process arising in the Lamperti representation of this self-similar Markov process is equal to $\widehat{\psi}(\lambda)$, which will imply that the process $(\widehat{Z}_t)_{t\geqslant 0}$ has the same distribution as the mass process of the Eve particle in the description of the process $(\widehat{\mathbf{X}}(t))_{t\geqslant 0}$. We slightly abuse notation by introducing, for every $x \geqslant 0$, a probability measure $\mathbb{P}_x$ under which the Markov process $\widehat{Z}$ starts from x. By self-similarity, for every $a > 0$, the law of $(a^{-2}\widehat{Z}_{at})_{t\geqslant 0}$ under $\mathbb{P}_x$ coincides with the law of $(\widehat{Z}_t)_{t\geqslant 0}$ under $\mathbb{P}_{a^{-2}x}$. We recall from [40, Proposition 1.2] that, for every $r > 0$, $\widehat{Z}_r$ follows (under $\mathbb{P}_0$) a Gamma distribution with parameter 3/2 and mean r^2.

Let $q \in (-\frac{3}{2}, -\frac{1}{2})$. Then,

$$\mathbb{E}_0\Big[\int_1^\infty \widehat{Z}_t^q\,\mathrm{d}t\Big] = \int_1^\infty t^{2q}\,\mathbb{E}_0[\widehat{Z}_1^q]\,\mathrm{d}t = -\frac{\mathbb{E}_0[\widehat{Z}_1^q]}{2q+1} = -\left(\frac{2}{3}\right)^q \frac{1}{2q+1}\frac{\Gamma(q+\frac{3}{2})}{\Gamma(\frac{3}{2})}. \tag{4.35}$$

We may compute the left-hand side of (4.35) in a different manner by applying the Markov property at time 1. We get

$$\mathbb{E}_0\Big[\int_1^\infty \widehat{Z}_t^q\,\mathrm{d}t\Big] = \mathbb{E}_0\Big[\mathbb{E}_{\widehat{Z}_1}\Big[\int_0^\infty \widehat{Z}_t^q\,\mathrm{d}t\Big]\Big] = \mathbb{E}_0\Big[\widehat{Z}_1^{q+1/2}\Big] \times \mathbb{E}_1\Big[\int_0^\infty \widehat{Z}_t^q\,\mathrm{d}t\Big] \tag{4.36}$$

using the self-similarity of $\widehat{Z}$. Then $\mathbb{E}_0[\widehat{Z}_1^{q+1/2}] = (2/3)^{q+1/2}\Gamma(q+2)/\Gamma(3/2)$ and, on the other hand, if $\widehat{\xi}$ denotes the Lévy process (started from 0) arising in the Lamperti representation of the self-similar process $\widehat{Z}$, we have

$$\mathbb{E}_1\Big[\int_0^\infty \widehat{Z}_t^q \,\mathrm{d}t\Big] = \mathbb{E}\Big[\int_0^\infty e^{(q+\frac{1}{2})\hat{\xi}(t)}\,\mathrm{d}t\Big].$$

The quantity in the right-hand side must be finite, which implies that $\mathbb{E}[e^{(q+\frac{1}{2})\hat{\xi}(t)}] < \infty$ for every $t > 0$, and $\mathbb{E}[e^{(q+\frac{1}{2})\hat{\xi}(t)}] = \exp(t\psi^*(q+\frac{1}{2}))$ with $\psi^*(q+\frac{1}{2}) < 0$. From (4.35) and (4.36), we get

$$\frac{1}{\psi^*(q+\frac{1}{2})} = \frac{1}{2}\left(\frac{2}{3}\right)^{-1/2}\frac{\Gamma(q+\frac{1}{2})}{\Gamma(q+2)}.$$

Finally, we find that, for $-1 < q < 0$, we have

$$\psi^*(q) = \sqrt{\frac{8}{3}}\,\frac{\Gamma(q+\frac{3}{2})}{\Gamma(q)} = \widehat{\psi}(q).$$

An argument of analytic continuation now allows us to obtain that the Laplace exponent of the Lévy process $\widehat{\xi}$ is equal to $\widehat{\psi}(\lambda)$ as desired.

Once we have identified $(\widehat{Z}_t)_{t\geqslant 0}$ as the mass process of the Eve particle in the description of $(\widehat{\mathbf{X}}(t))_{t\geqslant 0}$, the remaining steps of the proof are very similar to those of the proof of Theorem 4.2, and we will only sketch the main ingredients. We first recall the relevant features of the construction of the Brownian plane $(\mathcal{P}_\infty, \mathrm{d}_\infty)$ which is developed in [40, Section 3.2], to which we refer for further details. The random metric space $\mathcal{P}_\infty$ is obtained as a quotient space of a (non-compact) random tree $\mathcal{T}_\infty$, which itself is constructed by grafting a Poisson collection of (compact) $\mathbb{R}$-trees to an infinite spine isometric to $[0,\infty)$. The point 0 of the spine corresponds to the distinguished point ρ_∞ of $\mathcal{P}_\infty$. Furthermore, every point a of $\mathcal{T}_\infty$ is assigned a nonnegative label Λ_a, and this label coincides with $\mathrm{d}_\infty(\rho_\infty, x)$, if x is the point of $\mathcal{P}_\infty$ corresponding to a. Then, as in the proof of Theorem 4.3, it is not hard to check that connected components of $\{x \in \mathcal{P}_\infty : \mathrm{d}_\infty(\rho_\infty, x) > r\}$ are in one-to-one correspondence with connected components of $\{a \in \mathcal{T}_\infty : \Lambda_a > r\}$, for every fixed $r > 0$.

For every $a \in \mathcal{T}_\infty$, let $[\![a,\infty[\![$ stand for the range of the unique geodesic path from a to ∞ in $\mathcal{T}_\infty$, and set $\underline{\Lambda}_a = \min\{\Lambda_b : b \in [\![a,\infty[\![\}$. If $\mathcal{C}$ is a (necessarily bounded) connected component of $\{a \in \mathcal{T}_\infty : \Lambda_a - \underline{\Lambda}_a > 0\}$, then both $\mathcal{C}$ and the labels $(\Lambda_a)_{a\in\mathcal{C}}$ can be represented by a snake trajectory $\omega_{\mathcal{C}}$, in a way very similar to what we did for $\check{\mathcal{C}}_r$ in Section 4.3.2.

Proposition 4.10. *Setting* $\inf\{\Lambda_a : a \in \mathcal{C}\} = r$ *yields a one-to-one correspondence between connected components* $\mathcal{C}$ *of* $\{a \in \mathcal{T}_\infty : \Lambda_a - \underline{\Lambda}_a > 0\}$ *and jump times* r *of the process* $(\widehat{Z}_t)_{t\geqslant 0}$*. Let* $r_1, r_2, \dots$ *be an enumeration of these jump times, which is measurable with respect to the* σ*-field generated by* $(\widehat{Z}_r)_{r\geqslant 0}$*, and for every* $i = 1, 2, \dots$*, let* $\mathcal{C}_i$ *be the connected component associated with* r_i*. Then, conditionally on the process* $(\widehat{Z}_r)_{r\geqslant 0}$*, the snake trajectories* $\omega_{\mathcal{C}_i}$*,* $i = 1, 2, \dots$*, are independent, and the conditional distribution of* $\omega_{\mathcal{C}_i}$ *is* $\mathbb{N}_0^{*,|\Delta\widehat{Z}_{r_i}|}$*.*

Proposition 4.10 is obviously an analog of Theorem 4.5, and can be derived from the latter result using the relations between the labeled tree $(\mathcal{T}_\infty, (\Lambda_a)_{a\in\mathcal{T}_\infty})$ and the pair $(\mathcal{T}_\zeta, (V_a)_{a\in\mathcal{T}_\zeta})$ under $\mathbb{N}_0$ (compare the decomposition of the Brownian snake at the minimum found in [40, Theorem 2.1] with the construction of $(\mathcal{T}_\infty, (\Lambda_a)_{a\in\mathcal{T}_\infty})$ in the same reference).

Given Proposition 4.10, the end of the proof of Theorem 4.7 follows the same general pattern as that of Theorem 4.2 in Section 4.8, and we leave the details to the reader. □

Remark. We could have used Corollary 2 of [31] to identify the Laplace exponent of the Lévy process $\widehat{\xi}$ in the preceding proof. This would still have required some calculations, and for this reason we preferred the more direct approach presented above.

4.11.3 Local times

In this section, we argue under $\mathbb{N}_0(\mathrm{d}\omega)$, but similar results hold under $\mathbb{N}_0^*(\mathrm{d}\omega)$. Recall from the introduction the definition of the local times $(\mathcal{L}_x, x \in \mathbb{R})$ of the process $(V_a)_{a\in\mathcal{T}_\zeta}$. In this section, we relate the values of $\mathcal{L}_x$ for $x > 0$ to the growth-fragmentation process of Theorem 4.1 or equivalently to the connected components of $\{a \in \mathcal{T}_\zeta : V_a > x\}$.

We fix $h > 0$. It is convenient to introduce the "local time exit process" $(\mathscr{X}_r^h)_{r\geqslant 0}$, which roughly speaking measures for every $r \geqslant 0$ the "quantity" of paths W_s that have hit level h and accumulated a local time equal to r at level h. The precise definition of this process fits in the general framework of exit measures [65, Chapter V] and we refer to the introduction of [2] for more details (only the case $h = 0$ is considered there, but the extension to the case $h > 0$ is straightforward). Note that $\mathscr{X}_0^h = \mathcal{Z}_h$ is just the usual exit measure from $(-\infty, r)$, which can be defined by formula (8.7). Furthermore, under $\mathbb{N}_0(\cdot \mid W^* > h)$, the process $(\mathscr{X}_r^h)_{r\geqslant 0}$ is a ϕ-CSBP started from $\mathcal{Z}_h$ – see again the introduction of [2]. Of course, on the event $\{W^* < h\}$, the process $(\mathscr{X}_r^h)_{r\geqslant 0}$ is identically equal to zero.

Let $\mathcal{C}_1^h, \mathcal{C}_2^h, \ldots$ be the connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$, ranked in decreasing order of their boundary sizes $|\partial\mathcal{C}_1^h| > |\partial\mathcal{C}_2^h| > \cdots$. For every $i = 1, 2, \ldots$, let H_i be the height of $\mathcal{C}_i^h$, defined by

$$H_i := \sup_{a\in\mathcal{C}_i^h} V_a - h.$$

Proposition 4.11. *We have $\mathbb{N}_0$ a.e.*

$$\mathcal{L}_h = \int_0^\infty \mathrm{d}r\, \mathscr{X}_r^h. \tag{4.37}$$

Moreover, $\mathbb{N}_0$ a.e.,

$$\delta^{3/2} \#\{i : |\partial\mathcal{C}_i^h| > \delta\} \underset{\delta\to 0}{\longrightarrow} \frac{1}{\sqrt{6\pi}}\, \mathcal{L}_h \tag{4.38}$$

and

$$\delta^3 \#\{i : H_i > \delta\} \underset{\delta\to 0}{\longrightarrow} \mathbf{c}\, \mathcal{L}_h \tag{4.39}$$

where $\mathbf{c} = \frac{3}{2}\pi^{-3/2}\Gamma(1/3)^3\Gamma(7/6)^3$.

REMARK. The proposition also holds for $h = 0$, but the proof of (4.37) requires a different argument in that case, see [77, Proposition 3].

Proof. The convergence (4.39) is already established in [68], in the slightly weaker form of convergence in measure (note that "upcrossings" from h to $h + \delta$, as defined in [68], exactly correspond to connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$ with height greater than δ). We will use this fact to prove the identity (4.37). To simplify notation, we set

$$\mathcal{L}^*_h = \int_0^\infty \mathrm{d}r\, \mathscr{X}^h_r.$$

As in the proof of Theorem 4.6, we consider all excursions away from h. It follows from [2, Proposition 3, Theorem 4] (and an application of the special Markov property) that these excursions are in one-to-one correspondence with the jumps of the process $(\mathscr{X}^h_r)_{r\geqslant 0}$, and that conditionally on the latter process, they are independent and the conditional distribution of the excursion corresponding to the jump $\Delta\mathscr{X}^h_r$ is

$$\frac{1}{2}\mathbb{N}_0^{*,\Delta\mathscr{X}^h_r} + \frac{1}{2}\check{\mathbb{N}}_0^{*,\Delta\mathscr{X}^h_r},$$

where we use the notation $\check{\mathbb{N}}_0^{*,z}$ for the push forward of $\mathbb{N}_0^{*,z}$ under the symmetry $\omega \mapsto -\omega$. We recall that the connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$ are in one-to-one correspondence with the excursions above level h, in such a way that the boundary size of a component is equal to the corresponding jump of $(\mathscr{X}^h_r)_{r\geqslant 0}$.

Write U for the (stopped) Lévy process obtained from the ϕ-CSBP $\mathscr{X}^h$ by the Lamperti transformation (note that U is stopped upon hitting 0 and that $U_0 = \mathcal{Z}_h$). Notice that the hitting time of 0 by U is $\mathcal{L}^*_h$. Since the jumps of $(\mathscr{X}^h_r,)_{r\geqslant 0}$ are also the jumps of U, we obtain the identity in distribution

$$\Big(\mathcal{L}^*_h, \sum_{i=1}^\infty \delta_{|\partial\mathcal{C}^h_i|}\Big) \overset{(\mathrm{d})}{=} \Big(\mathcal{L}^*_h, \sum_{i=1}^\infty \mathbf{1}_{\{\varepsilon_i=1\}}\delta_{\Delta U_{r_i}}\Big)$$

where $r_1, r_2, \dots$ are the jump times of U, and $\varepsilon_1, \varepsilon_2, \dots$ is a sequence of independent Bernoulli variables with parameter $1/2$, which is independent of U. Since the Lévy measure of U is

$$\pi(\mathrm{d}z) = \sqrt{\frac{3}{2\pi}}\, z^{-5/2}\,\mathrm{d}z,$$

so that $\pi((\delta,\infty)) = \sqrt{2/3\pi}\,\delta^{-3/2}$, it easily follows that, $\mathbb{N}_0$ a.e.,

$$\delta^{3/2}\#\{i : \varepsilon_i = 1 \text{ and } \Delta U_{r_i} > \delta\} \underset{\delta\to 0}{\longrightarrow} \frac{1}{\sqrt{6\pi}}\,\mathcal{L}^*_h.$$

This gives the convergence (4.38), except that we have not yet verified that $\mathcal{L}_h = \mathcal{L}^*_h$.

To this end, using again the conditional distribution of the excursions away from h given the process $(\mathscr{X}^h_r)_{r\geqslant 0}$, we observe that we have also

$$\Big(\mathcal{L}^*_h, \sum_{i=1}^\infty \delta_{H_i}\Big) \overset{(\mathrm{d})}{=} \Big(\mathcal{L}^*_h, \sum_{i=1}^\infty \mathbf{1}_{\{\varepsilon_i=1\}}\delta_{\sqrt{\Delta U_{r_i}}\, M_i}\Big)$$

where $M_1, M_2, \dots$ is a sequence of independent random variables distributed according to the law of W^* under $\mathbb{N}_0^{*,1}$, which is also independent of $(U, (\varepsilon_i)_{i\geqslant 1})$. Now observe that, if z is chosen according to $\pi(\mathrm{d}z)$ and M according to the law of W^* under $\mathbb{N}_0^{*,1}$, $\sqrt{z}\,M$ is distributed according to the "law" of W^* under $\mathbb{N}_0^*$, which satisfies

$$\mathbb{N}_0^*(W^* > \delta) = 2\mathbf{c}\,\delta^{-3}$$

by [2, Lemma 25]. It follows that, $\mathbb{N}_0$ a.e.,

$$\delta^3 \#\{i : H_i > \delta\} \underset{\delta\to 0}{\longrightarrow} \mathbf{c}\,\mathcal{L}_h^*.$$

By comparing this convergence with [68, Theorem 6], we get that $\mathcal{L}_h = \mathcal{L}_h^*$, which completes the proof. □

Appendix

This appendix is devoted to the proofs of Lemma 4.2 and Proposition 7.5.

Proof of Lemma 4.2. It is convenient to write $\mathbf{N}$ for the distribution of $(\mathcal{Z}_{-t})_{t\geqslant 0}$ under $\mathbb{N}_0$ (we agree that $\mathcal{Z}_0 = 0$). Then $\mathbf{N}$ is a σ-finite measure on the Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R})$. For $\varepsilon > 0$, let

$$\sum_{i\in I_\varepsilon} \delta_{w_i^\varepsilon}(\mathrm{d}w)$$

be a Poisson point measure on $\mathbb{D}([0,\infty), \mathbb{R})$ with intensity $\varepsilon\mathbf{N}$. As already noticed in Section 4.2.4, we can construct a ϕ-CSBP started from ε by setting, for $t > 0$,

$$Y_t^\varepsilon = \sum_{i\in I_\varepsilon} w_i^\varepsilon(t)$$

and $Y_0^\varepsilon = \varepsilon$. Set $T_0^\varepsilon := \inf\{t \geqslant 0 : Y_t^\varepsilon = 0\}$. The classical Lamperti transformation [61, 32] allows us to relate Y^ε to another process X^ε distributed as a stable Lévy process with no negative jumps and index 3/2, started from ε and stopped at the first time when it hits 0, via the formula

$$X_t^\varepsilon = Y_{\gamma_t^\varepsilon}^\varepsilon$$

where $\gamma_t^\varepsilon := \inf\{s \geqslant 0 : \int_0^s Y_u^\varepsilon \mathrm{d}u > t\}$ if $t < \int_0^\infty Y_u^\varepsilon \mathrm{d}u$ and $\gamma_t^\varepsilon = T_0^\varepsilon$ otherwise.

Let us fix $\delta > 0$ and assume from now on that $\varepsilon \in (0, \delta)$. Let B_ε stand for the event $\{\sup_{t\geqslant 0} Y_t^\varepsilon \geqslant \delta\} = \{\sup_{t\geqslant 0} X_t^\varepsilon \geqslant \delta\}$. By the solution of the two-sided exit problem already used in the proof of Lemma 4.1, we have

$$\mathbb{P}(B_\varepsilon) = 1 - \sqrt{\frac{\delta-\varepsilon}{\delta}} = \frac{\varepsilon}{2\delta} + O(\varepsilon^2)\ , \quad \text{as } \varepsilon \to 0.$$

On the other hand, let $A_\varepsilon \subset B_\varepsilon$ be the event where there is exactly one $i \in I_\varepsilon$ such that $\sup_{t\geqslant 0} w_i^\varepsilon(t) \geqslant \delta$. By Lemma 4.1,

$$\mathbb{P}(A_\varepsilon) = \frac{\varepsilon}{2\delta} \exp\Big(-\frac{\varepsilon}{2\delta}\Big) = \frac{\varepsilon}{2\delta} + O(\varepsilon^2)\ , \quad \text{as } \varepsilon \to 0.$$

If F is a bounded measurable function on the space $\mathbb{D}([0,\infty),\mathbb{R})$, we deduce from the last two displays that

$$\mathbb{E}[F((X^\varepsilon_t)_{t\geqslant 0}) \mid B_\varepsilon] = \mathbb{E}[F((X^\varepsilon_t)_{t\geqslant 0}) \mid A_\varepsilon] + O(\varepsilon)\ , \quad \text{as } \varepsilon \to 0. \tag{4.40}$$

We can associate with X^ε the process "reflected above the minimum" defined by

$$\tilde{X}^\varepsilon_t := X^\varepsilon_t - \inf\{X^\varepsilon_s : 0 \leqslant s \leqslant t\}.$$

We have obviously $0 \leqslant X^\varepsilon_t - \tilde{X}^\varepsilon_t \leqslant \varepsilon$ for every $t \geqslant 0$. If $\tilde{B}_\varepsilon \subset B_\varepsilon$ stands for the event $\{\sup_{t\geqslant 0} \tilde{X}^\varepsilon_t \geqslant \delta\}$, it is easily checked that $\mathbb{P}(B_\varepsilon \backslash \tilde{B}_\varepsilon) = O(\varepsilon^2)$, so that we can replace B_ε by $\tilde{B}_\varepsilon$ in (4.40). Furthermore, on the event $\tilde{B}_\varepsilon$ we can introduce the first excursion of $\tilde{X}^\varepsilon$ away from 0 that hits δ and denote this excursion by $(\mathcal{X}^\varepsilon_t)_{t\geqslant 0}$. Notice that the distribution of $(\mathcal{X}^\varepsilon_t)_{t\geqslant 0}$ is $\mathbf{n}_\delta(\mathrm{d}e) := \mathbf{n}(\mathrm{d}e \mid \sup\{e(t) : t \geqslant 0\} \geqslant \delta)$. Let d_{Sk} be a distance on $\mathbb{D}([0,\infty),\mathbb{R})$ that induces the Skorokhod topology. It is a simple matter to verify that, for every $\alpha > 0$,

$$\mathbb{P}(\mathrm{d}_{\mathrm{Sk}}((\tilde{X}^\varepsilon_t)_{t\geqslant 0}, (\mathcal{X}^\varepsilon_t)_{t\geqslant 0}) > \alpha \mid \tilde{B}_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Assume from now on that F is (bounded and) Lipschitz with respect to d_{Sk}. We deduce from (4.40) (with B_ε replaced by $\tilde{B}_\varepsilon$) and the last display that

$$\mathbb{E}[F((X^\varepsilon_t)_{t\geqslant 0}) \mid A_\varepsilon] - \mathbb{E}[F((\mathcal{X}^\varepsilon_t)_{t\geqslant 0}) \mid \tilde{B}_\varepsilon] \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.41}$$

Note that $\mathbb{E}[F((\mathcal{X}^\varepsilon_t)_{t\geqslant 0}) \mid \tilde{B}_\varepsilon]$ does not depend on ε and is equal to $\int \mathbf{n}_\delta(\mathrm{d}e)\, F(e)$.

On the other hand, conditionally on the event A_ε there is a unique index $i_0 \in I_\varepsilon$ such that $\sup_{t\geqslant 0} w^\varepsilon_{i_0}(t) \geqslant \delta$, and the distribution of $w^\varepsilon_{i_0}$ is $\mathbf{N}_\delta(\mathrm{d}w) := \mathbf{N}(\mathrm{d}w \mid \sup_{t\geqslant 0} w(t) \geqslant \delta)$. We then set $\mathcal{Y}^\varepsilon_t = w^\varepsilon_{i_0}(t)$, and

$$\eta^\varepsilon_t = \inf\{s \geqslant 0 : \int_0^s \mathrm{d}u\, \mathcal{Y}^\varepsilon_u > t\}$$

if $t < \int_0^\infty \mathrm{d}u\, \mathcal{Y}^\varepsilon_u$, and $\eta^\varepsilon_t = \inf\{s \geqslant 0 : \mathcal{Y}^\varepsilon_s = 0\}$ otherwise. Observing that the conditional distribution of $Y^\varepsilon - \mathcal{Y}^\varepsilon$ given A_ε is dominated by the law of a $\mathbf{\Phi}$-continuous state branching process started from ε, one verifies that, for every $\alpha > 0$,

$$\mathbb{P}(\mathrm{d}_{\mathrm{Sk}}((Y^\varepsilon_{\gamma^\varepsilon_t})_{t\geqslant 0}, (\mathcal{Y}^\varepsilon_{\eta^\varepsilon_t})_{t\geqslant 0}) > \alpha \mid A_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

(Here we omit a few details that are left to the reader.) Recalling that $Y^\varepsilon_{\gamma^\varepsilon_t} = X^\varepsilon_t$ and using (4.41), we get

$$\mathbb{E}[F((\mathcal{Y}^\varepsilon_{\eta^\varepsilon_t})_{t\geqslant 0}) \mid A_\varepsilon] - \int \mathbf{n}_\delta(\mathrm{d}e)\, F(e) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Since the conditional distribution of $(\mathcal{Y}^\varepsilon_{\eta^\varepsilon_t})_{t\geqslant 0}$ given A_ε is $\mathbf{N}_\delta$ (independently of ε), using the equalities $\mathbf{N}(\{w : \sup_{t\geqslant 0} w(t) \geqslant \delta\}) = \frac{1}{2\delta} = \mathbf{n}(\{e : \sup_{t\geqslant 0} e(t) \geqslant \delta\})$ (the first one by Lemma 4.1 and the second one as an easy consequence of the two-sided exit problem), we arrive at the result of the lemma. □

Proof of Proposition 7.5. *First step.* Recall that, for every $t > 0$, $\mathcal{G}_t$ denotes the σ-field on $\mathcal{S}_0$ generated by the mapping $\omega \mapsto \mathrm{tr}_{-t}(\omega)$ and completed by the collection of all $\mathbb{N}_0$-negligible sets. We also define $\mathcal{G}_0$ as the σ-field generated by the $\mathbb{N}_0$-negligible sets. For every $\eta > 0$, the process $(\mathcal{Z}_{-t})_{t \geqslant \eta}$ is Markov with respect to the filtration $(\mathcal{G}_t)_{t \geqslant \eta}$ under the probability measure $\mathbb{N}_0^{[\eta]} := \mathbb{N}_0(\cdot \mid W_* \leqslant -\eta)$. By the Feller property of the semigroup, the strong Markov property holds even for stopping times of the filtration $(\mathcal{G}_{t+})_{t \geqslant \eta}$.

We fix two reals $\eta \in (0,1)$ and $M > 1$. Let $\varepsilon \in (0, \eta)$. From the proof of Proposition 34 in the appendix of [71], we have, for every $r \leqslant -\eta$,

$$\mathbb{N}_0\Big((\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r)^2\Big) \leqslant 4\varepsilon^2.$$

We note that [71] deals with the quantity $\tilde{\mathcal{Z}}_r^\varepsilon$ defined in Remark (ii) after Proposition 7.5, rather than with $\mathcal{Z}_r^\varepsilon$, but as explained in this remark, this makes no difference for a fixed value of r. Furthermore, [71] gives the latter bound only for "truncated versions" of $\tilde{\mathcal{Z}}_r^\varepsilon$ and $\mathcal{Z}_r$, but an application of Fatou's lemma then yields the preceding display.

Let $\delta \in (0,1)$. By Markov's inequality, for $r \leqslant -\eta$,

$$\mathbb{N}_0(|\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r| > \delta) \leqslant \delta^{-2} \times 4\varepsilon^2.$$

We apply this to $r = -j\,\varepsilon^{3/2}$ for all integers j such that $\eta \leqslant j\,\varepsilon^{3/2} \leqslant M+1$. It follows that

$$\begin{aligned}&\mathbb{N}_0\Big(|\mathcal{Z}^\varepsilon_{-j\varepsilon^{3/2}} - \mathcal{Z}_{-j\varepsilon^{3/2}}| > \frac{\delta}{2}, \text{ for some } j \text{ s.t. } \eta \leqslant j\,\varepsilon^{3/2} \leqslant M+1\Big) \\ &\quad \leqslant 16(M+1)\delta^{-2}\,\varepsilon^{1/2}.\end{aligned} \tag{4.42}$$

Fix a real $K > 0$, and consider the random time

$$S := \inf\{t \geqslant \eta : \mathcal{Z}_{-t} < K \text{ and } |\mathcal{Z}^\varepsilon_{-t} - \mathcal{Z}_{-t}| > \delta\}.$$

Note that S is a stopping time of the filtration $(\mathcal{G}_{t+})_{t \geqslant \eta}$ (because both processes $(\mathcal{Z}^\varepsilon_{-t})_{t \geqslant \eta}$ and $(\mathcal{Z}_{-t})_{t \geqslant \eta}$ have càdlàg paths and are adapted to the filtration $(\mathcal{G}_t)_{t \geqslant \eta}$). On the event $\{S < \infty\}$, we have $|\mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}_{-S}| \geqslant \delta$ and $\mathcal{Z}_{-S} \leqslant K$.

Our goal is now to bound $\mathbb{N}_0(S \leqslant M)$. To this end, we will use (4.42). On the event $\{S < \infty\}$ write $[-S]_\varepsilon$ for the greatest number of the form $-j\varepsilon^{3/2}$ in the interval $(-\infty, -S)$. Then,

$$\{S \leqslant M\} = \{S \leqslant M, |\mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}_{[-S]_\varepsilon}| > \delta/2\} \cup \{S \leqslant M, |\mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}_{[-S]_\varepsilon}| \leqslant \delta/2\}.$$

By (4.42), the $\mathbb{N}_0$-measure of the first set in the right-hand side is bounded above by $c_1\varepsilon^{1/2}$ for some constant c_1 depending on M and δ. On the other hand, recalling that $|\mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}_{-S}| \geqslant \delta$ on $\{S < \infty\}$, we obtain that the second set is contained in

$$\{S \leqslant M,\, |\mathcal{Z}_{[-S]_\varepsilon} - \mathcal{Z}_{-S}| \geqslant \delta/4\} \cup \{S \leqslant M,\, |\mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}^\varepsilon_{-S}| \geqslant \delta/4\}.$$

Using the strong Markov property of $(\mathcal{Z}_{-t})_{t \geqslant \eta}$ at time S, the bound $\mathcal{Z}_{-S} \leqslant K$ on $\{S < \infty\}$, and the fact that a ϕ-CSBP can be written as a time change of a Lévy process, it is easy to verify that

$$\mathbb{N}_0(S \leqslant M,\, |\mathcal{Z}_{[-S]_\varepsilon} - \mathcal{Z}_{-S}| \geqslant \delta/4) \leqslant c_2\varepsilon^{3/2} \tag{4.43}$$

for some constant c_2 depending on δ and K.

In the second and the third step below, we will get similar estimates for the $\mathbb{N}_0$-measure of (appropriate subsets of) the event $\{S \leqslant M,\, |\mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}^\varepsilon_{-S}| \geqslant \delta/4\}$. We will explain in the fourth step how the proof of the proposition is completed by combining all these estimates.

Second step. We first study the quantity

$$\mathbb{N}_0(S \leqslant M,\, \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}^\varepsilon_{-S} \geqslant \delta/4).$$

From our definitions, on the event $\{S < \infty\}$, the quantity $\mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}^\varepsilon_{-S}$ is bounded above by

$$F_\varepsilon := \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{-S}(W_s)<\infty, T_{-S-\varepsilon^{3/2}}(W_s)=\infty\}}\, \mathbf{1}_{\{\widehat{W}_s < [-S]_\varepsilon + \varepsilon\}}.$$

For every integer $n \geqslant 1$, write $[-S]_{(n)}$ for the greatest number of the form $-j2^{-n}$ in $(-\infty, -S)$, and set, still on the event $\{S < \infty\}$,

$$F_{\varepsilon,n} := \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{[-S]_{(n)}}(W_s)<\infty, T_{[-S]_{(n)}-\varepsilon^{3/2}}(W_s)=\infty\}}\, \mathbf{1}_{\{\widehat{W}_s < [-S]_{(n)} + \varepsilon\}}.$$

Observing that $\mathbf{1}_{\{T_{-S}(W_s)<\infty\}} = \lim_{n\to\infty} \mathbf{1}_{\{T_{[-S]_{(n)}}(W_s)<\infty\}}$ and using Fatou's lemma, we have

$$\mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_\varepsilon) \leqslant \liminf_{n\to\infty} \mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_{\varepsilon,n}).$$

Then,

$$\mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_{\varepsilon,n}) = \varepsilon^{-2} \sum_{k=1}^{\infty} \mathbb{N}_0\Big(\mathbf{1}_{\{(k-1)2^{-n} \leqslant S < k2^{-n}\}} \times \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{-k2^{-n}}(W_s)<\infty, T_{-k2^{-n}-\varepsilon^{3/2}}(W_s)=\infty\}}\, \mathbf{1}_{\{\widehat{W}_s < -k2^{-n}+\varepsilon\}}\Big).$$

We can apply the special Markov property (Proposition 4.1) to each term of the sum in the right-hand side. Note that the variable $\mathbf{1}_{\{(k-1)2^{-n} \leqslant S < k2^{-n}\}}$ is measurable with respect to $\mathcal{G}_{k2^{-n}}$, whereas the subsequent integral is a function of the snake trajectories ω_i introduced in Proposition 4.1 when $r = -k2^{-n}$). We obtain

$$\mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_{\varepsilon,n}) = \varepsilon^{-2} \sum_{k=1}^{\infty} \mathbb{N}_0\Big(\mathbf{1}_{\{(k-1)2^{-n} \leqslant S < k2^{-n}\}}\, \mathcal{Z}_{k2^{-n}}\Big) \times \mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{-\varepsilon^{3/2}}(W_s)=\infty\}}\, \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}}\Big).$$

By the first-moment formula for the Brownian snake [65, Proposition 4.2], we have

$$\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{-\varepsilon^{3/2}}(W_s)=\infty\}}\, \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}}\Big) = \mathbb{E}_0\Big[\int_0^{\mathtt{t}_{-\varepsilon^{3/2}}} \mathrm{d}t\, \mathbf{1}_{\{B_t < \varepsilon\}}\Big] \leqslant c_3 \varepsilon^{5/2},$$

where $(B_t)_{t\geqslant 0}$ is a standard linear Brownian motion starting from x under the probability measure $\mathbb{P}_x$, $\mathtt{t}_r = \inf\{t \geqslant 0 : B_t = r\}$ for every $r \in \mathbb{R}$, and c_3 is a constant. We conclude that $\mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_{\varepsilon,n}) \leqslant c_3 \varepsilon^{1/2}\, \mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, \mathcal{Z}_{[S]_{(n)}}) \leqslant c_3 \varepsilon^{1/2}$, and the same bound holds for $\mathbb{N}_0(\mathbf{1}_{\{S<\infty\}}\, F_\varepsilon)$. Finally, Markov's inequality gives

$$\mathbb{N}_0(S \leqslant M,\, \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} - \mathcal{Z}^\varepsilon_{-S} \geqslant \delta/4) \leqslant \mathbb{N}_0(S < \infty,\, F_\varepsilon \geqslant \delta/4) \leqslant \frac{4}{\delta}\, c_3 \varepsilon^{1/2}.$$

Third step. We now consider the event $\{S \leqslant M,\, \mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} \geqslant \delta/4)\}$. We observe that, if $S < \infty$,

$$\mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} \leqslant \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_{-S}(W_s)=\infty,\, \widehat{W}_s \in [[-S]_\varepsilon+\varepsilon, -S+\varepsilon)\}}.$$

Notice that $T_{-S}(W_s) = \infty$ implies $T_{[-S]_\varepsilon}(W_s) = \infty$ and that $-S+\varepsilon \leqslant [-S]_\varepsilon + \varepsilon + \varepsilon^{3/2}$. Hence, on the event where $S \leqslant M$ and $\mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} \geqslant \delta/4$, we can find a real $r \in [-M-1, -\eta]$ of the form $r = j\varepsilon^{3/2}$ with $j \in \mathbb{Z}$, such that

$$\varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty,\, \widehat{W}_s \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}} \geqslant \frac{\delta}{4}.$$

Let us fix $r \in [-M-1, -\eta]$ in the following lines, and bound the probability of the event in the last display. From the first-moment formula for the Brownian snake, we have, with the same notation as in the second step,

$$\begin{aligned} &\mathbb{N}_0\Big(\varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty,\, \widehat{W}_s \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}}\Big) \\ &\quad = \varepsilon^{-2} \mathbb{E}_0\Big[\int_0^{\mathrm{t}_r} \mathrm{d}t\, \mathbf{1}_{\{B_t \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}}\Big] \leqslant c_4 \varepsilon^{1/2}, \end{aligned}$$

with some constant c_4. To get a better estimate, we use higher moments, but to this end we need to perform a suitable truncation. We fix $A > 0$, and we observe that, for every integer $k \geqslant 1$, for any nonnegative measurable function f on $\mathbb{R}$, we have

$$\begin{aligned} &\mathbb{N}_0\Big(\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty, \tau_A(W_s)=\infty\}} f(\widehat{W}_s)\Big)^k\Big) \\ &\quad \leqslant C_{k,A,M} \Big(\sup_{x \in [r,A]} \mathbb{E}_x\Big[\int_0^{\mathrm{t}_r \wedge \mathrm{t}_A} \mathrm{d}t\, f(B_t)\Big]\Big)^k, \end{aligned}$$

where $C_{k,A,M}$ is a constant depending only on k, A and M. The bound in the previous display can be derived in a straightforward way from the k-th moment formula for the Brownian snake [65, Proposition IV.2]. We omit the details. We apply this bound with $f = \mathbf{1}_{[r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]}$, and we arrive at the estimate

$$\mathbb{N}_0\Big(\Big(\varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty, \tau_A(W_s)=\infty\}} \mathbf{1}_{\{\widehat{W}_s \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}}\Big)^k\Big) \leqslant C_{k,A,M}\, (c_5 \varepsilon^{1/2})^k,$$

with a constant c_5 depending on A and M. From Markov's inequality, we then get

$$\begin{aligned} &\mathbb{N}_0\Big(W^* < A,\ \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty,\, \widehat{W}_s \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}} \geqslant \frac{\delta}{4}\Big) \\ &\quad \leqslant \Big(\frac{\delta}{4}\Big)^{-k} \mathbb{N}_0\Big(\Big(\varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_r(W_s)=\infty, \tau_A(W_s)=\infty\}} \mathbf{1}_{\{\widehat{W}_s \in [r+\varepsilon, r+\varepsilon+\varepsilon^{3/2}]\}}\Big)^k\Big) \\ &\quad \leqslant C_{k,A,M} \Big(\frac{\delta}{4}\Big)^{-k} (c_5 \varepsilon^{1/2})^k. \end{aligned}$$

We take $k = 4$ and sum the preceding estimate over possible values of $r = -j\varepsilon^{3/2}$ in $[-M-1, -\eta]$, and we arrive at the estimate

$$\mathbb{N}_0(W^* < A,\, S \leqslant M,\, \mathcal{Z}^\varepsilon_{-S} - \mathcal{Z}^\varepsilon_{[-S]_\varepsilon} \geqslant \delta/4) \leqslant c_6\, \varepsilon^{1/2}$$

with a constant c_6 depending on A, M and δ.

Fourth step. We deduce from the second and third steps that we have

$$\mathbb{N}_0(W^* < A,\, S \leqslant M,\, |\mathcal{Z}^{\varepsilon}_{[-S]_\varepsilon} - \mathcal{Z}^{\varepsilon}_{-S}| \geqslant \delta/4)| \leqslant c_7\, \varepsilon^{1/2}, \tag{4.44}$$

with a constant c_7 depending on δ, A, M, K. Combining (4.43) and (4.44) and recalling the considerations of the end of the first step, we arrive at the bound

$$\mathbb{N}_0(W^* < A,\, S \leqslant M) \leqslant (c_1 + c_2 + c_7)\, \varepsilon^{1/2}. \tag{4.45}$$

Let us write $S = S^{(\varepsilon)}$ to recall the dependence on ε. Let n_0 be the first integer such that $(n_0)^{-3} < \eta$. The bound (4.45) gives

$$\sum_{n=n_0}^{\infty} \mathbb{N}_0(W^* < A,\, S^{(n^{-3})} \leqslant M) < \infty.$$

Hence, $\mathbb{N}_0$ a.e. on the event $W^* < A$, we have $S^{(n^{-3})} > M$ for all large enough n. This means that, $\mathbb{N}_0$ a.e. on the event where $\sup\{\mathcal{Z}_{-t} : t > 0\} < K$ and $W^* < A$, we have for all large enough n,

$$\sup_{\eta \leqslant u \leqslant M} |\mathcal{Z}^{(n^{-3})}_{-u} - \mathcal{Z}_{-u}| \leqslant \delta.$$

Since δ, K and A are arbitrary, we obtain that, $\mathbb{N}_0$ a.e.,

$$\lim_{n\to\infty} \Big(\sup_{\eta \leqslant u \leqslant M} |\mathcal{Z}^{(n^{-3})}_{-u} - \mathcal{Z}_{-u}| \Big) = 0.$$

We can replace $\eta \leqslant u \leqslant M$ by $\eta \leqslant u < \infty$ since $\mathcal{Z}^{(\varepsilon)}_{-u} - \mathcal{Z}_{-u} = 0$ for $u > -W_* + \varepsilon$. The statement of the proposition then follows by a monotonicity argument. $\square$

Chapitre 5

Spine representations for non-compact models of random geometry

Les resultats de ce chapitre sont issus de l'article [79], écrit en collaboration avec Jean-François Le Gall et accepté pour publication dans *Probability Theory and Related Fields*.

We provide a unified approach to the three main non-compact models of random geometry, namely the Brownian plane, the infinite-volume Brownian disk, and the Brownian half-plane. This approach allows us to investigate relations between these models, and in particular to prove that complements of hulls in the Brownian plane are infinite-volume Brownian disks.

Contents

5.1 Introduction

In the recent years, much work has been devoted to the continuous models of random geometry that arise as scaling limits of planar maps, which are discrete graphs embedded in the sphere. The most famous model is the Brownian map or Brownian sphere, which is the limit in the Gromov-Hausdorff sense of large planar maps with n faces chosen uniformly at random in a suitable class and viewed as metric spaces for the graph distance rescaled by the factor $n^{-1/4}$, when $n \to \infty$ (see in particular [1, 4, **?**, 21, 67, 82, 85]). The rescaling factor $n^{-1/4}$ is relevant because the typical diameter of a random planar map with n faces is of order $n^{1/4}$ when n is large, and thus the rescaling leads to a compact limit. However, choosing a rescaling factor that tends to 0 at a slower rate than $n^{-1/4}$ yields a different limiting space, which can be interpreted as an infinite-volume version of the Brownian sphere and is called the Brownian plane [39, 40]. On the other hand, scaling limits of random planar maps with a boundary have also been investigated [7, 14, 20, 22, 55, 83]. In that case, assuming that the planar map has a fixed boundary size equal to n and a volume (number of faces) of order n^2, rescaling the graph distance by the factor $n^{-1/2}$ again leads to a compact limiting space called the Brownian disk. If however the volume grows faster than n^2, the same rescaling yields a non-compact limit which is the infinite-volume Brownian disk. The so-called Brownian half-plane arises when choosing a rescaling factor that tends to 0 at a slower rate than $n^{-1/2}$. A comprehensive discussion of all possible scaling limits of large random quadrangulations with a boundary, including the cases mentioned above, is given in the recent paper of Baur, Miermont and Ray [14]. In this discussion, the Brownian disk and the infinite-volume Brownian disk, the Brownian plane and the Brownian half-plane play a central role. It is worth noting that the Brownian plane is also the scaling limit [39, 29] in the local Gromov-Hausdorff sense of the random lattices called the uniform infinite planar triangulation (UIPT) and the uniform infinite planar quadrangulation (UIPQ), which have been studied extensively since the introduction of the UIPT by Angel and Schramm [13]. Similarly, the Brownian half-plane arises as the scaling limit [14, 53] of the uniform half-plane quadrangulation, which has been introduced and studied in [30, 43]. We finally mention that the preceding models of random geometry are closely related to Liouville quantum gravity surfaces, and the Brownian disk, the Brownian plane and the Brownian half-plane correspond respectively to the quantum disk, the quantum cone and the quantum wedge, see [86, Corollary 1.5], and [53] for the case of the Brownian half-plane.

The main goal of the present article is to provide a unified approach to the three most important non-compact models of random geometry, namely the Brownian plane, the infinite-volume Brownian disk and the Brownian half-plane. As we will discuss below, it is remarkable that these three models can all be constructed in a similar manner from the same infinite Brownian tree equipped with Brownian labels, subject to different conditionings – the precise definition of these conditionings however requires some care especially in the case of the infinite-volume Brownian disk. As an application of these constructions, we are able to get new relations between the different models of random geometry. In particular, we prove that the complement of a hull in the Brownian plane, equipped with its intrinsic metric, is an infinite-volume Brownian disk (this may be viewed as an infinite-volume counterpart of a property derived in [71] for the Brownian sphere). The latter property was in fact a strong motivation for the present study, as it plays a very important role in the forthcoming work [93] concerning isoperimetric bounds in the Brownian plane. We also prove that the "horohull" of radius r in the Brownian plane, corresponding to the connected component containing the root of the set of points whose "relative distance" to infinity is greater than $-r$, is a Brownian disk with height r (here, a Brownian disk with height r is obtained by conditioning a free pointed Brownian disk on the event that the distinguished point is at distance exactly r from the boundary).

Let us now explain our approach in more precise terms. The starting point of our construction is an infinite "Brownian tree" $\mathfrak{T}_*$ that consists of a spine isometric to $[0,\infty)$ and two Poisson collections of subtrees grafted respectively to the left side and to the right side of the spine. For our purposes, it is very important to distinguish the left side and the right side because we later need an order structure on the tree. The trees branching off the spine can be obtained as compact $\mathbb{R}$-trees coded by Brownian excursions (so they are scaled versions of Aldous' celebrated CRT). To be specific, in order to define the subtrees branching off the left side of the tree, one may consider a Poisson point measure

$$\sum_{i\in I}\delta_{(t_i,e_i)},$$

with intensity $2\,\mathbf{1}_{[0,\infty)}(t)\,\mathrm{d}t\,\mathbf{n}(\mathrm{d}e)$, where $\mathbf{n}(\mathrm{d}e)$ stands for the Itô measure of positive Brownian excursions, and then declare that, for every $i\in I$, the tree $\mathcal{T}_i$ coded by e_i is grafted to the left side of the spine at level t_i. For subtrees branching off the right side, we proceed in the same way, with an independent Poisson point measure. We equip $\mathfrak{T}_*$ with the obvious choice of a distance (see Section 5.2.4 below). Then $\mathfrak{T}_*$ is a non-compact $\mathbb{R}$-tree, and, for every $v\in\mathfrak{T}_*$, we can define the geodesic line segment $[[\rho,v]]$ between the root ρ (bottom of the spine) and v, and we use the notation $]]\rho,v[[=[[\rho,v]]\backslash\{\rho,v\}$. The tree $\mathfrak{T}_*$ may be viewed as an "infinite Brownian tree" corresponding to process 2 in Aldous [8].

We then introduce labels on $\mathfrak{T}_*$, that is, to each point v of $\mathfrak{T}_*$ we assign a real label Λ_v. We let the labels on the spine be given by a three-dimensional Bessel process $R=(R_t)_{t\geqslant 0}$ started from 0. Then, conditionally on R, the labels on the different subtrees are independent, and the labels on a given subtree $\mathcal{T}_i$ branching off the spine at level t_i are given by Brownian motion indexed by $\mathcal{T}_i$ and started from R_{t_i} at the root of $\mathcal{T}_i$ (which is the point of the spine at level t_i). In other words, labels evolve like linear Brownian motion when moving along a segment of a subtree branching off the spine.

We finally need a last operation, which ensures that we have only nonnegative labels. We let $\mathfrak{T}$ be the subset of $\mathfrak{T}_*$ that consists of all $v \in \mathfrak{T}_*$ such that labels do not vanish along $]\!]\rho, v[\![$. So the spine is contained in $\mathfrak{T}$, but some of the subtrees branching off the spine in $\mathfrak{T}_*$ are truncated at points where labels vanish. For each subtree $\mathcal{T}_i$, the theory of exit measures gives a way to define a quantity $\mathcal{Z}_0(\mathcal{T}_i)$ measuring the "number" of branches of $\mathcal{T}_i$ that are cut in the truncation procedure (or in a more precise manner, the "number" of points v of $\mathcal{T}_i$ such that $\Lambda_v = 0$ but $\Lambda_w > 0$ for every $w \in]\!]\rho, v[\![$), and we write $\mathcal{Z}_0$ for the sum of the quantities $\mathcal{Z}_0(\mathcal{T}_i)$ for all subtrees $\mathcal{T}_i$ branching off the spine. We have in fact $\mathcal{Z}_0 = \infty$ a.s., but a key point of the subsequent discussion is to discuss conditionings of the labeled tree $\mathfrak{T}$ that ensure that $\mathcal{Z}_0 < \infty$.

We are now in a position to define the random metric that will be used in the construction of the non-compact models of random geometry of interest in this work. Set $\mathfrak{T}^\circ = \{v \in \mathfrak{T} : \Lambda_v > 0\}$, and for $v, w \in \mathfrak{T}^\circ$,

$$D^\circ(v,w) = \Lambda_v + \Lambda_w - 2\max\Big(\inf_{u\in[v,w]} \Lambda_u, \inf_{u\in[w,v]} \Lambda_u\Big),$$

where $[v,w]$ stands for the set of points visited when going from v to w clockwise around the tree (see Section 5.2.4 for a more precise definition). We slightly modify $D^\circ(v,w)$ by setting $\Delta^\circ(v,w) = D^\circ(v,w)$ if the maximum in the last display is positive, and $\Delta^\circ(v,w) = \infty$ otherwise. Finally, we let $(\Delta(v,w); v,w \in \mathfrak{T}^\circ)$ be the maximal symmetric function of $(v,w) \in \mathfrak{T}^\circ \times \mathfrak{T}^\circ$ that is bounded above by Δ° and satisfies the triangle inequality. It turns out that the function $(v,w) \mapsto \Delta(v,w)$ takes finite values and can be extended by continuity to a pseudo-metric on $\mathfrak{T}$, and we may thus consider the quotient space of $\mathfrak{T}$ for the equivalence relation defined by setting $v \simeq w$ if and only if $\Delta(v,w) = 0$. The quotient space $\mathfrak{T}/\!\simeq$ equipped with the metric induced by Δ is:

1. the Brownian plane under the special conditioning $\mathcal{Z}_0 = 0$;
2. the infinite-volume Brownian disk with perimeter $z > 0$ under the special conditioning $\mathcal{Z}_0 = z$;
3. the Brownian half-plane under no conditioning (then $\mathcal{Z}_0 = \infty$ a.s.).

The really new contributions of the present work are cases 2 and 3, because case 1 corresponds to the construction of the Brownian plane in [40] (which is different from the one in [39]): in that case, the conditioning on $\mathcal{Z}_0 = 0$ turns the process of labels on the spine into a nine-dimensional Bessel process $X = (X_t)_{t\geqslant 0}$ started from 0, and the subtrees branching off the spine are conditioned to have positive labels (see Section 5.4.2 below).

A remarkable feature of the preceding constructions is the fact that labels on $\mathfrak{T}$ have a nice geometric interpretation in terms of the associated random metric spaces $\mathfrak{T}/\!\simeq$. Precisely, the label Λ_v of a point v of $\mathfrak{T}$ is equal to the distance from (the equivalence class of) v to the set of (equivalence classes of) points of zero label in $\mathfrak{T}/\simeq$. The latter set is either a single point in case 1, or a boundary homeomorphic to the circle in case 2, or a boundary homeomorphic to the line in case 3. Amongst other applications, this interpretation of labels allows us to prove the above-mentioned result about the complement of hulls in the Brownian plane. Write $\mathbb{BP}_\infty$ for the Brownian plane, and recall that the hull $B^\bullet(r)$ is defined by saying that its complement $\check{B}^\bullet(r) := \mathbb{BP}_\infty \backslash B^\bullet(r)$ is the unbounded component of the complement of the closed ball of radius r centered at the distinguished point (bottom of the spine) of $\mathbb{BP}_\infty$. Then Theorem 29 below states that (the closure of) $\check{B}^\bullet(r)$ equipped with its intrinsic metric is an infinite-volume Brownian disk whose perimeter is the boundary size $|\partial B^\bullet(r)|$ (see [40] for the definition of this boundary size).

Much of the technical work in the present paper is devoted to making sense of the conditioning $\mathcal{Z}_0 = z$ in case 2, which is not a trivial matter because $\mathcal{Z}_0 = \infty$ a.s. Our approach is motivated by the previously mentioned result concerning the distribution of $\check{B}^\bullet(r)$. At first, it would seem that our construction of the Brownian plane from an infinite tree $\mathfrak{T}$ made of a spine equipped with labels $(X_t)_{t\geqslant 0}$ (given by a nine-dimensional Bessel process), and labeled subtrees conditioned to have positive labels, would be suited perfectly to analyse the distribution of a hull or of its complement. In fact, it is observed in [40] that the set $\check{B}^\bullet(r)$ exactly corresponds to a subtree $\mathfrak{T}_{(r)}$ consisting of the part of the spine of $\mathfrak{T}$ above level $\mathrm{L}_r := \sup\{t \geqslant 0 : X_t = r\}$ and of the subtrees branching off the spine of $\mathfrak{T}$ above level L_r and truncated at points where labels hit r – furthermore the boundary size $|\partial B^\bullet(r)|$ is just the sum of the exit measures at level r of all these subtrees. However, this representation of $\check{B}^\bullet(r)$ seems to depend heavily on r, even if labels are shifted by $-r$: in particular, the distribution of the process $(X_{\mathrm{L}_r+t} - r)_{t\geqslant 0}$ depends on r. Nevertheless, and perhaps surprisingly, it turns out that, if we condition the boundary size $|\partial B^\bullet(r)|$ to be equal to a fixed $z > 0$, the conditional distribution of the labeled tree $\mathfrak{T}_{(r)}$ (with labels shifted by $-r$) does not depend on r, and this leads to the probability measure Θ_z which is used in our construction of the infinite-volume Brownian disk. The precise construction of the measures Θ_z, which involves an appropriate truncation procedure, is given in Section 5.3.3, where we also explain in which sense these measures correspond to the conditioning of case 2 above.

As the reader will have guessed from the preceding discussion, some of the technicalities in our proofs are made necessary by the problem of conditioning on events of zero probability. For instance, in order to define the free pointed Brownian disk with perimeter z and a given height $r > 0$, it is relevant to condition the Brownian snake excursion measure $\mathbb{N}_r$ (see Section 5.2 for a definition) on the event that the exit measure at 0 is equal to z. It is not immediately obvious how to make a canonical choice of these conditional distributions, so that they depend continuously on the pair (r, z). We deal carefully with these questions in Section 5.3.

Our proofs also rely on certain explicit distributions, which are of independent interest. In particular, we prove that, in a free pointed Brownian disk of perimeter 1, the density of the distribution of the distance from the distinguished point to the boundary is given by the function

$$p_1(r) := 9\,r^{-6}\Big(r + \frac{2}{3}\,r^3 - \Big(\frac{3}{2}\Big)^{1/2}\sqrt{\pi}\,(1+r^2)\,\exp\Big(\frac{3}{2r^2}\Big)\,\mathrm{erfc}\Big(\sqrt{\frac{3}{2r^2}}\Big)\Big),$$

with the usual notation $\mathrm{erfc}(\cdot)$ for the complementary error function. See Propositions 3 and 14 below for a short proof, which uses the representation of Brownian disks found in [71] (with some more work, the same formula could also be derived from the representation in [20, 22]).

The paper is organized as follows. Section 5.2 contains a number of preliminaries, and in particular we introduce the formalism of snake trajectories [2], and the associated Brownian snake excursion measures, to code compact continuous random trees equipped with real labels. We also introduce the notion of a "coding triple" for a non-compact continuous random tree. Such a coding triple consists of a random process representing the labels on the spine, and two random point measures on the space of all pairs (t, ω), where $t \geqslant 0$ and ω is a snake trajectory (the idea is that, for every such pair, the labeled tree coded by ω will be grafted to the left or to the right of the spine at level t). The main goal of Section 5.3 is to define the coding triple associated with the infinite-volume Brownian disk or, in other words, to make sense of the conditioning appearing in case 2 above. Section 5.4 then gives the construction of the random metric spaces of interest from the corresponding coding triples, starting from the construction of the Brownian plane in [40]. As an important ingredient of our discussion, we consider the free pointed Brownian disk $\mathbb{D}_z^{(a)}$ with perimeter z and height a (recall that the height refers to the distance from the distinguished point to the boundary). The infinite-volume Brownian disk with perimeter z is then obtained as the local limit of $\mathbb{D}_z^{(a)}$ in the Gromov-Hausdorff sense when $a \to \infty$. In an analogous manner, we construct the Brownian half-plane and we verify that it is the tangent cone in distribution of the free Brownian disk at a point chosen uniformly on its boundary. Finally, Section 5.5 is devoted to our applications to the complement of hulls and to horohulls in the Brownian plane, and Section 5.6 shows that our definitions of the infinite-volume Brownian disk and of the Brownian half-plane are consistent with previous work.

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5.2 Preliminaries

5.2.1 Snake trajectories

Continuous random trees whose vertices are assigned real labels play a fundamental role in this work. The formalism of snake trajectories, which has been introduced in [2], provides a convenient framework to deal with such labeled trees.

A (one-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})} \geqslant 0$ is called the lifetime of w. We let $\mathcal{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geqslant 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For $x \in \mathbb{R}$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x of $\mathbb{R}$. We also use the notation $\mathcal{W}^\infty$, resp. $\mathcal{W}_x^\infty$, for the space of all continuous functions $\mathrm{w} : [0,\infty) \longrightarrow \mathbb{R}$, resp. the set of all such functions with $\mathrm{w}(0) = x$.

Définition 1. *Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_x$ which satisfies the following two properties:*

(i) *We have $\omega_0 = x$ and the number $\sigma(\omega) := \sup\{s \geqslant 0 : \omega_s \neq x\}$, called the duration of the snake trajectory ω, is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = x$ for every $s \geqslant 0$).*

(ii) (Snake property) *For every $0 \leqslant s \leqslant s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s \leqslant r \leqslant s'} \zeta_{(\omega_r)}]$.*

We will write $\mathcal{S}_x$ for the set of all snake trajectories with initial point x and $\mathcal{S} = \bigcup_{x \in \mathbb{R}} \mathcal{S}_x$ for the set of all snake trajectories. If $\omega \in \mathcal{S}$, we often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geqslant 0$. The set $\mathcal{S}$ is a Polish space for the distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geqslant 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

A snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$: See [2, Proposition 8]. If ω is a snake trajectory, its time reversal $\check{\omega}$ defined by $\check{\omega}_s = \omega_{(\sigma(\omega)-s)^+}$ is also a snake trajectory.

Let $\omega \in \mathcal{S}$ be a snake trajectory and $\sigma = \sigma(\omega)$. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact $\mathbb{R}$-tree, which will be denoted by $\mathcal{T}_{(\omega)}$ and called the *genealogical tree* of the snake trajectory. This $\mathbb{R}$-tree is the quotient space $\mathcal{T}_{(\omega)} := [0,\sigma]/\sim$ of the interval $[0,\sigma]$ for the equivalence relation

$$s \sim s' \text{ if and only if } \zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s \wedge s' \leqslant r \leqslant s \vee s'} \zeta_r(\omega),$$

and $\mathcal{T}_{(\omega)}$ is equipped with the distance induced by

$$d_{(\omega)}(s, s') = \zeta_s(\omega) + \zeta_{s'}(\omega) - 2 \min_{s \wedge s' \leqslant r \leqslant s \vee s'} \zeta_r(\omega).$$

(notice that $d_{(\omega)}(s,s') = 0$ if and only if $s \sim s'$, and see e.g. [74, Section 3] for more information about the coding of $\mathbb{R}$-trees by continuous functions). We write $p_{(\omega)} : [0,\sigma] \longrightarrow \mathcal{T}_{(\omega)}$ for the canonical projection. By convention, $\mathcal{T}_{(\omega)}$ is rooted at the point $\rho_{(\omega)} := p_{(\omega)}(0)$, and the volume measure on $\mathcal{T}_{(\omega)}$ is defined as the pushforward of Lebesgue measure on $[0,\sigma]$ under $p_{(\omega)}$. If $u, v \in \mathcal{T}_{(\omega)}$, $[\![u,v]\!]$ denotes the geodesic segment between u and v in $\mathcal{T}_{(\omega)}$, and we also use the notation $[\![u,v[\![$ or $]\!]u,v[\![$ with an obvious meaning.

It will be useful to define also intervals on the tree $\mathcal{T}_{(\omega)}$. For $s, s' \in [0,\sigma]$, we use the convention $[s,s'] = [s,\sigma] \cup [0,s']$ if $s > s'$ (and of course, $[s,s']$ is the usual interval if $s \leqslant s'$). If $u, v \in \mathcal{T}_{(\omega)}$ are distinct, then we can find $s, s' \in [0,\sigma]$ in a unique way so that $p_{(\omega)}(s) = u$ and $p_{(\omega)}(s') = v$ and the interval $[s,s']$ is as small as possible, and we define $[u,v] := p_{(\omega)}([s,s'])$. Informally, $[u,v]$ is the set of all points that are visited when going from u to v in "clockwise order" around the tree. We take $[u,u] = \{u\}$.

By property (ii) in the definition of a snake trajectory, the condition $p_{(\omega)}(s) = p_{(\omega)}(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ can be viewed as defined on the quotient space $\mathcal{T}_{(\omega)}$. For $u \in \mathcal{T}_{(\omega)}$, we set $\ell_u(\omega) := \widehat{W}_s(\omega)$ whenever $s \in [0, \sigma]$ is such that $u = p_{(\omega)}(s)$ (by the previous observation, this does not depend on the choice of s). We can interpret $\ell_u(\omega)$ as a "label" assigned to the "vertex" u of $\mathcal{T}_{(\omega)}$. Notice that the mapping $u \mapsto \ell_u(\omega)$ is continuous on $\mathcal{T}_{(\omega)}$, and that, for every $s \geqslant 0$, the path $W_s(\omega)$ records the labels $\ell_u(\omega)$ along the "ancestral line" $[[\rho_{(\omega)}, p_{(\omega)}(s)]]$. We will use the notation $W_*(\omega) := \min\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$.

We now introduce two important operations on snake trajectories in $\mathcal{S}$. The first one is the re-rooting operation (see [2, Section 2.2]). Let $\omega \in \mathcal{S}$ and $r \in [0, \sigma(\omega)]$. Then $\omega^{[r]}$ is the snake trajectory in $\mathcal{S}_{\widehat{W}_r(\omega)}$ such that $\sigma(\omega^{[r]}) = \sigma(\omega)$ and for every $s \in [0, \sigma(\omega)]$,

$$\begin{aligned} \zeta_s(\omega^{[r]}) &= d_{(\omega)}(r, r \oplus s), \\ \widehat{W}_s(\omega^{[r]}) &= \widehat{W}_{r \oplus s}(\omega), \end{aligned}$$

where we use the notation $r \oplus s = r + s$ if $r + s \leqslant \sigma(\omega)$, and $r \oplus s = r + s - \sigma(\omega)$ otherwise. By a remark following the definition of snake trajectories, these prescriptions completely determine $\omega^{[r]}$. The genealogical tree $\mathcal{T}_{(\omega^{[r]})}$ may be interpreted as the tree $\mathcal{T}_{(\omega)}$ re-rooted at the vertex $p_{(\omega)}(r)$ (see [51, Lemma 2.2] for a precise statement) and vertices of the re-rooted tree receive the same labels as in $\mathcal{T}_{(\omega)}$. We sometimes write $W^{[t]}(\omega)$ instead of $\omega^{[t]}$.

The second operation is the truncation of snake trajectories. Let $x, y \in \mathbb{R}$ with $y < x$. For every $\mathrm{w} \in \mathcal{W}_x$, set

$$\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$$

with the usual convention $\inf \varnothing = \infty$ (this convention will be in force throughout this work unless otherwise indicated). Then, if $\omega \in \mathcal{S}_x$, we set, for every $s \geqslant 0$,

$$\eta_s(\omega) = \inf\Big\{t \geqslant 0 : \int_0^t \mathrm{d}u\, \mathbf{1}_{\{\zeta_{(\omega_u)} \leqslant \tau_y(\omega_u)\}} > s\Big\}.$$

Note that the condition $\zeta_{(\omega_u)} \leqslant \tau_y(\omega_u)$ holds if and only if $\tau_y(\omega_u) = \infty$ or $\tau_y(\omega_u) = \zeta_{(\omega_u)}$. Then, setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geqslant 0$ defines an element ω' of $\mathcal{S}_x$, which will be denoted by $\mathrm{tr}_y(\omega)$ and called the truncation of ω at y (see [2, Proposition 10]). The effect of the time change $\eta_s(\omega)$ is to "eliminate" those paths ω_s that hit y and then survive for a positive amount of time. We leave it as an exercise for the reader to check that the genealogical tree $\mathcal{T}_{(\mathrm{tr}_y(\omega))}$ is canonically and isometrically identified to the closed set $\{v \in \mathcal{T}_{(\omega)} : \ell_u(\omega) > y \text{ for every } u \in [[\rho_{(\omega)}, v[[\}$, and this identification preserves labels.

5.2.2 The Brownian snake excursion measure on snake trajectories

Let $x \in \mathbb{R}$. The Brownian snake excursion measure $\mathbb{N}_x$ is the σ-finite measure on $\mathcal{S}_x$ that satisfies the following two properties: Under $\mathbb{N}_x$,

(i) the distribution of the lifetime function $(\zeta_s)_{s\geqslant 0}$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x\Big(\sup_{s\geqslant 0}\zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon};$$

(ii) conditionally on $(\zeta_s)_{s\geqslant 0}$, the tip function $(\widehat{W}_s)_{s\geqslant 0}$ is a Gaussian process with mean x and covariance function

$$K(s,s') := \min_{s\wedge s'\leqslant r\leqslant s\vee s'} \zeta_r.$$

Informally, the lifetime process $(\zeta_s)_{s\geqslant 0}$ evolves under $\mathbb{N}_x$ like a Brownian excursion, and conditionally on $(\zeta_s)_{s\geqslant 0}$, each path W_s is a linear Brownian path started from x with lifetime ζ_s, which is "erased" from its tip when ζ_s decreases and is "extended" when ζ_s increases. The measure $\mathbb{N}_x$ can be interpreted as the excursion measure away from x for the Markov process in $\mathcal{W}_x$ called the Brownian snake. We refer to [65] for a detailed study of the Brownian snake. For every $y < x$, we have

$$\mathbb{N}_x(W_* \leqslant y) = \frac{3}{2(x-y)^2}. \tag{5.1}$$

See e.g. [65, Section VI.1] for a proof.

The measure $\mathbb{N}_x$ is invariant under the time-reversal operation $\omega \mapsto \check{\omega}$. Furthermore, the following scaling property is often useful. For $\lambda > 0$, for every $\omega \in \mathcal{S}_x$, we define $\theta_\lambda(\omega) \in \mathcal{S}_{x\sqrt{\lambda}}$ by $\theta_\lambda(\omega) = \omega'$, with

$$\omega'_s(t) := \sqrt{\lambda}\,\omega_{s/\lambda^2}(t/\lambda)\ , \quad \text{for } s \geqslant 0 \text{ and } 0 \leqslant t \leqslant \zeta'_s := \lambda\zeta_{s/\lambda^2}.$$

Then it is a simple exercise to verify that $\theta_\lambda(\mathbb{N}_x) = \lambda\,\mathbb{N}_{x\sqrt{\lambda}}$.

Exit measures. Let $x, y \in \mathbb{R}$, with $y < x$. Under the measure $\mathbb{N}_x$, one can make sense of a quantity that "measures the quantity" of paths W_s that hit level y. One shows [71, Proposition 34] that the limit

$$L^y_t := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^2}\int_0^t \mathrm{d}s\,\mathbf{1}_{\{\tau_y(W_s)=\infty,\,\widehat{W}_s<y+\varepsilon\}} \tag{5.2}$$

exists uniformly for $t \geqslant 0$, $\mathbb{N}_x$ a.e., and defines a continuous nondecreasing function, which is obviously constant on $[\sigma,\infty)$. The process $(L^y_t)_{t\geqslant 0}$ is called the exit local time at level y, and the exit measure $\mathcal{Z}_y$ is defined by $\mathcal{Z}_y = L^y_\infty = L^y_\sigma$. Then, $\mathbb{N}_x$ a.e., the topological support of the measure $\mathrm{d}L^y_t$ is exactly the set $\{s \in [0,\sigma] : \tau_y(W_s) = \zeta_s\}$, and, in particular, $\mathcal{Z}_y > 0$ if and only if one of the paths W_s hits y. The definition of $\mathcal{Z}_y$ is a special case of the theory of exit measures (see [65, Chapter V] for this general theory). We will use the formula for the Laplace transform of $\mathcal{Z}_y$: For $\lambda > 0$,

$$\mathbb{N}_x\Big(1-\exp(-\lambda\mathcal{Z}_y)\Big) = \Big((x-y)\sqrt{2/3}+\lambda^{-1/2}\Big)^{-2}. \tag{5.3}$$

See formula (6) in [40] for a brief justification.

It will be useful to observe that $\mathcal{Z}_y$ can be defined in terms of the truncated snake $\mathrm{tr}_y(\omega)$. To this end, recall the time change $(\eta_s(\omega))_{s\geqslant 0}$ used to define $\mathrm{tr}_y(\omega)$ at the end of Section 5.2.1, and set $\widetilde{L}^y_t = L^y_{\eta_t}$ for every $t \geqslant 0$. Then $\widetilde{L}^y_\infty = L^y_\infty = \mathcal{Z}_y$, whereas formula (8.7) implies that

$$\widetilde{L}^y_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} \int_0^t \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s(\mathrm{tr}_y(\omega))<y+\varepsilon\}} \tag{5.4}$$

uniformly for $t \geqslant 0$, $\mathbb{N}_x$ a.e.

5.2.3 The positive excursion measure

We now introduce another σ-finite measure on $\mathcal{S}_0$, which is supported on snake trajectories taking only nonnegative values. For $\delta \geqslant 0$, let $\mathcal{S}^{(\delta)}$ be the set of all $\omega \in \mathcal{S}$ such that $\sup_{s\geqslant 0}(\sup_{t\in[0,\zeta_s(\omega)]} |\omega_s(t)|) > \delta$. Also set

$$\mathcal{S}_0^+ = \{\omega \in \mathcal{S}_0 : \omega_s(t) \geqslant 0 \text{ for every } s \geqslant 0, t \in [0, \zeta_s(\omega)]\} \cap \mathcal{S}^{(0)}.$$

By [2, Theorem 23], there exists a σ-finite measure $\mathbb{N}^*$ on $\mathcal{S}$, which is supported on $\mathcal{S}_0^+$ and gives finite mass to the sets $\mathcal{S}^{(\delta)}$ for every $\delta > 0$, such that

$$\mathbb{N}^*(G) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\, \mathbb{N}_\varepsilon(G(\mathrm{tr}_0(\omega))),$$

for every bounded continuous function G on $\mathcal{S}$ that vanishes on $\mathcal{S}\backslash\mathcal{S}^{(\delta)}$ for some $\delta > 0$. Under $\mathbb{N}^*$, each of the paths W_s, $0 < s < \sigma$, starts from 0, then stays positive during some time interval $(0, \alpha)$, and is stopped immediately when it returns to 0, if it does return to 0.

The re-rooting formula. We can relate the measure $\mathbb{N}^*$ to the excursion measures $\mathbb{N}_x$ of the preceding section via a re-rooting formula which we now state [2, Theorem 28]. Recall the notation $\omega^{[t]}$ for the snake trajectory ω re-rooted at t. For any nonnegative measurable function G on $\mathcal{S}$, we have

$$\mathbb{N}^*\Big(\int_0^\sigma \mathrm{d}t\, G(\omega^{[t]})\Big) = 2\int_0^\infty \mathrm{d}x\, \mathbb{N}_x\Big(\mathcal{Z}_0\, G(\mathrm{tr}_0(\omega))\Big). \tag{5.5}$$

Conditioning on the exit measure at 0. In a way analogous to the definition of exit measures, one can make sense of the "quantity" of paths W_s that return to 0 under $\mathbb{N}^*$. To this end, one observes that the limit

$$\mathcal{Z}_0^* := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s<\varepsilon\}} \tag{5.6}$$

exists $\mathbb{N}^*$ a.e. (this indeed follows from (5.4), using (5.5) to relate $\mathbb{N}^*$ to the distribution of $\mathrm{tr}_0(\omega)$ under $\mathbb{N}_x$, $x > 0$).

According to [2, Proposition 33], there exists a unique collection $(\mathbb{N}^{*,z})_{z>0}$ of probability measures on $\mathcal{S}_0^+$ such that:

(i) We have

$$\mathbb{N}^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2}\, \mathbb{N}^{*,z}.$$

(ii) For every $z > 0$, $\mathbb{N}^{*,z}$ is supported on $\{\mathcal{Z}_0^* = z\}$.

(iii) For every $z, z' > 0$, $\mathbb{N}^{*,z'} = \theta_{z'/z}(\mathbb{N}^{*,z})$.

Informally, $\mathbb{N}^{*,z} = \mathbb{N}^*(\cdot \mid \mathcal{Z}_0^* = z)$.

It will be convenient to have a "pointed version" of the measures $\mathbb{N}^{*,z}$. We note that $\mathbb{N}^{*,z}(\sigma) = z^2$ (see the remark after [71, Proposition 15]) and define a probability measure on $\mathcal{S}_0 \times \mathbb{R}_+$ by setting

$$\overline{\mathbb{N}}^{*,z}(\mathrm{d}\omega\mathrm{d}t) = z^{-2}\, \mathbb{N}^{*,z}(\mathrm{d}\omega)\, \mathbf{1}_{[0,\sigma(\omega)]}(t)\, \mathrm{d}t.$$

5.2.4 Coding finite or infinite labeled trees

Many of the random compact (resp. non-compact) metric spaces that we discuss in the present work are coded by triples $(Z, \mathcal{M}, \mathcal{M}')$ where $Z = (Z_t)_{t\in[0,h]}$ (resp. $Z = (Z_t)_{t\in[0,\infty)}$) is a finite (resp. infinite) random path, and $\mathcal{M}$ and $\mathcal{M}'$ are random point measures on $[0,h] \times \mathcal{S}$ (resp. on $[0,\infty) \times \mathcal{S}$). Such a triple is called a coding triple, and we interpret it as coding a labeled tree, having a spine isometric to $[0,h]$ or to $[0,\infty)$, in such a way that the path Z corresponds to labels along the spine, and, for each atom (t_i, ω_i) of $\mathcal{M}$ (resp. of $\mathcal{M}'$), the genealogical tree of ω_i corresponds a subtree branching off the left side (resp. off the right side) of the spine at level t_i. The random metric spaces of interest are then obtained via some identification of vertices in the labeled trees, and equipped with a metric which is determined from the labels.

In this section we explain how coding triples are used to construct labeled trees. We follow closely the presentation given in [40] for a special case.

The infinite spine case. We consider a (deterministic) triple $(\mathrm{w}, \mathcal{P}, \mathcal{P}')$ such that:

(i) $\mathrm{w} \in \mathcal{W}^\infty$;

(ii) $\mathcal{P} = \sum_{i\in I} \delta_{(t_i,\omega_i)}$ and $\mathcal{P}' = \sum_{i\in J} \delta_{(t_i,\omega_i)}$ are point measures on $(0,\infty) \times \mathcal{S}$ (the indexing sets I and J are disjoint), and, for every $i \in I \cup J$, $\omega_i \in \mathcal{S}_{\mathrm{w}(t_i)}$ and $\sigma(\omega_i) > 0$;

(iii) all numbers t_i, $i \in I \cup J$, are distinct;

(iv) the functions

$$u \mapsto \beta_u := \sum_{i\in I} \mathbf{1}_{\{t_i \leqslant u\}}\, \sigma(\omega_i)\ , \quad u \mapsto \beta'_u := \sum_{i\in J} \mathbf{1}_{\{t_i \leqslant u\}}\, \sigma(\omega_i)\ .$$

take finite values and are monotone increasing on $\mathbb{R}_+$, and tend to ∞ at ∞ (in particular, the sets $\{t_i : i \in I\}$ and $\{t_i : i \in J\}$ are dense in $(0,\infty)$);

(v) for every $t > 0$ and $\varepsilon > 0$,

$$\#\Big\{ i \in I \cup J : t_i \leqslant t \text{ and } \sup_{0\leqslant s\leqslant\sigma(\omega_i)} |\widehat{W}_s(\omega_i) - \mathrm{w}(t_i)| > \varepsilon \Big\} < \infty. \tag{5.7}$$

Such a triple $(\mathrm{w}, \mathcal{P}, \mathcal{P}')$ will be called an infinite spine coding triple. Recall the notation $\mathcal{T}_{(\omega)}$ for the genealogical tree of the snake trajectory ω and $\rho_{(\omega)}$ for the root of $\mathcal{T}_{(\omega)}$. The tree $\mathcal{T}_\infty$ associated with the coding triple $(\mathrm{w}, \mathcal{P}, \mathcal{P}')$ is obtained from the disjoint union

$$[0, \infty) \cup \left(\bigcup_{i \in I \cup J} \mathcal{T}_{(\omega_i)} \right)$$

by identifying the point t_i of $[0, \infty)$ with the root $\rho_{(\omega_i)}$ of $\mathcal{T}_{(\omega_i)}$, for every $i \in I \cup J$. The metric $\mathrm{d}_{\mathcal{T}_\infty}$ on $\mathcal{T}_\infty$ is determined as follows. The restriction of $\mathrm{d}_{\mathcal{T}_\infty}$ to each tree $\mathcal{T}_{(\omega_i)}$ is the metric $d_{(\omega_i)}$ on $\mathcal{T}_{(\omega_i)}$, and the restriction of $\mathrm{d}_{\mathcal{T}_\infty}$ to the spine $[0, \infty)$ is the usual Euclidean distance. If $x \in \mathcal{T}_{(\omega_i)}$ and $t \in [0, \infty)$, we take $\mathrm{d}_{\mathcal{T}_\infty}(x, t) = d_{(\omega_i)}(x, \rho_{(\omega_i)}) + |t_i - t|$. If $x \in \mathcal{T}_{(\omega_i)}$ and $y \in \mathcal{T}_{(\omega_j)}$, with $i \neq j$, we take $\mathrm{d}_{\mathcal{T}_\infty}(x, y) = d_{(\omega_i)}(x, \rho_{(\omega_i)}) + |t_i - t_j| + d_{(\omega_j)}(\rho_{(\omega_j)}, y)$. We note that $\mathcal{T}_\infty$ is a non-compact $\mathbb{R}$-tree. By convention, $\mathcal{T}_\infty$ is rooted at 0. The tree $\mathcal{T}_\infty$ is equipped with a volume measure, which is defined as the sum of the volume measures on the trees $\mathcal{T}_{(\omega_i)}$, $i \in I \cup J$.

We can also define labels on $\mathcal{T}_\infty$. The label Λ_x of $x \in \mathcal{T}_\infty$ is defined by $\Lambda_x = \mathrm{w}(t)$ if $x = t$ belongs to the spine $[0, \infty)$, and $\Lambda_x = \ell_x(\omega_i)$ if x belongs to $\mathcal{T}_{(\omega_i)}$, for some $i \in I \cup J$. Note that the mapping $x \mapsto \Lambda_x$ is continuous (use property (5.7) to check continuity at points of the spine).

For our purposes, it is important to define an order structure on $\mathcal{T}_\infty$. To this end, we introduce a "clockwise exploration" of $\mathcal{T}_\infty$, which is defined as follows. Write β_{u-} and β'_{u-} for the respective left limits at u of the functions $u \mapsto \beta_u$ and $u \mapsto \beta'_u$ introduced in (iv) above, with the convention $\beta_{0-} = \beta'_{0-} = 0$. Then, for every $s \geqslant 0$, there is a unique $u \geqslant 0$ such that $\beta_{u-} \leqslant s \leqslant \beta_u$, and:

- Either we have $u = t_i$ for some $i \in I$ (then $\sigma(\omega_i) = \beta_{t_i} - \beta_{t_i-}$), and we set $\mathcal{E}^+_s := p_{(\omega_i)}(s - \beta_{t_i-})$.
- Or there is no such i and we set $\mathcal{E}^+_s = u$.

We define similarly $(\mathcal{E}^-_s)_{s \geqslant 0}$. For every $s \geqslant 0$, there is a unique $u \geqslant 0$ such that $\beta'_{u-} \leqslant s \leqslant \beta'_u$, and:

- Either we have $u = t_i$ for some $i \in J$, and we set $\mathcal{E}^-_s := p_{(\omega_i)}(\beta'_{t_i} - s)(= p_{(\check{\omega}_i)}(s - \beta'_{t_i-}))$.
- Or there is no such i and we set $\mathcal{E}^-_s = u$.

Informally, $(\mathcal{E}^+_s)_{s \geqslant 0}$ and $(\mathcal{E}^-_s)_{s \geqslant 0}$ correspond to the exploration of the left and right side of the tree $\mathcal{T}_\infty$ respectively. Noting that $\mathcal{E}^+_0 = \mathcal{E}^-_0 = 0$, we define $(\mathcal{E}_s)_{s \in \mathbb{R}}$ by setting

$$\mathcal{E}_s := \begin{cases} \mathcal{E}^+_s & \text{if } s \geqslant 0, \\ \mathcal{E}^-_{-s} & \text{if } s \leqslant 0. \end{cases}$$

It is straightforward to verify that the mapping $s \mapsto \mathcal{E}_s$ from $\mathbb{R}$ onto $\mathcal{T}_\infty$ is continuous. We also note that the volume measure on $\mathcal{T}_\infty$ is the pushforward of Lebesgue measure on $\mathbb{R}$ under the mapping $s \mapsto \mathcal{E}_s$.

This exploration process allows us to define intervals on $\mathcal{T}_\infty$, in a way similar to what we did in Section 5.2.1. Let us make the convention that, if $s > t$, the "interval" $[s, t]$ is defined by

$[s,t] = [s,\infty) \cup (-\infty,t]$. Then, for every $x,y \in \mathcal{T}_\infty$, such that $x \neq y$, there is a smallest interval $[s,t]$, with $s,t \in \mathbb{R}$, such that $\mathcal{E}_s = x$ and $\mathcal{E}_t = y$, and we define

$$[x,y] := \{\mathcal{E}_r : r \in [s,t]\}.$$

Note that we have typically $[x,y] \neq [y,x]$. Of course, we take $[x,x] = \{x\}$. We sometimes also use the self-evident notation $]x,y[$. For $x \in \mathcal{T}_\infty$, we finally define $[x,\infty) = \{\mathcal{E}_r : r \in [s,\infty)\}$, where s is the largest real such that $\mathcal{E}_s = x$, and we define $(-\infty,x]$ in a similar manner. Note that $[x,\infty) \cap (-\infty,x] = [[x,\infty[[$ is the range of the geodesic ray starting from x in $\mathcal{T}_\infty$.

The finite spine case. It will also be useful to consider the case where $\mathrm{w} = (\mathrm{w}(t))_{0\leqslant t\leqslant\zeta}$ is a finite path in $\mathcal{W}$ with positive lifetime ζ, and $\mathcal{P}$ and $\mathcal{P}'$ are now point measures supported on $(0,\zeta] \times \mathcal{S}$. We then assume that the obvious adaptations of properties (i)–(v) hold, and in particular (iv) is replaced by

(iv)' the functions $u \mapsto \beta_u := \sum_{i\in I} \mathbf{1}_{\{t_i\leqslant u\}}\, \sigma(\omega_i)$, $u \mapsto \beta'_u := \sum_{i\in J} \mathbf{1}_{\{t_i\leqslant u\}}\, \sigma(\omega_i)$ take finite values and are monotone increasing on $[0,\zeta]$.

The same construction yields a (labeled) compact $\mathbb{R}$-tree $(\mathcal{T},(\Lambda_v)_{v\in\mathcal{T}})$, with a spine represented by the interval $[0,\zeta]$. The distance on $\mathcal{T}$ is denoted by $\mathrm{d}_\mathcal{T}$ and the labels are defined in exactly the same way as in the infinite spine case. The tree $\mathcal{T}$ has a cyclic order structure induced by a clockwise exploration function $\mathcal{E}_s$, which is conveniently defined on the interval $[0,\beta_\zeta+\beta'_\zeta]$: Informally $(\mathcal{E}_s, s \in [0,\beta_\zeta])$ is obtained by concatenating the mappings $p_{(\omega_i)}$ for all atoms (t_i,ω_i) of $\mathcal{P}$, in the increasing order of the t_i's, and $(\mathcal{E}_s, s \in [\beta_\zeta,\beta_\zeta+\beta'_\zeta])$ is obtained by concatenating the mappings $p_{(\omega_j)}$ for all atoms (t_j,ω_j) of $\mathcal{P}'$, in the decreasing order of the t_j's (in particular, $\mathcal{E}_0 = \mathcal{E}_{\beta_\zeta+\beta'_\zeta}$ is the root or bottom of the spine and $\mathcal{E}_{\beta_\zeta}$ is the top of the spine). In order to define "intervals" on the tree $\mathcal{T}$, we now make the convention that, if $s,t \in [0,\beta_\zeta+\beta'_\zeta]$ and $t < s$, $[s,t] = [s,\beta_\zeta+\beta'_\zeta] \cup [0,t]$. In that setting, we again refer to $(\mathrm{w},\mathcal{P},\mathcal{P}')$ as a (finite spine) coding triple.

Here, in contrast with the infinite spine case, we can also represent the labeled tree $(\mathcal{T},(\Lambda_v)_{v\in\mathcal{T}})$ by a snake trajectory $\omega \in \mathcal{S}_{\mathrm{w}(0)}$ such that $\mathcal{T}_{(\omega)} = \mathcal{T}$, which is defined as follows. The duration $\sigma(\omega)$ is equal to $\beta_\zeta+\beta'_\zeta$, and, for every $s \in [0,\beta_\zeta+\beta'_\zeta]$, the finite path ω_s is such that $\zeta_{(\omega_s)} = \mathrm{d}_\mathcal{T}(\mathcal{E}_0,\mathcal{E}_s)$ and $\widehat{\omega}_s = \Lambda_{\mathcal{E}_s}$ (by a remark in Section 5.2.1, this completely determines ω). The snake trajectory ω obtained in this way will be denoted by $\omega = \Omega(\mathrm{w},\mathcal{P},\mathcal{P}')$. We note that the triple $(\mathrm{w},\mathcal{P},\mathcal{P}')$ contains more information than ω: Roughly speaking, in order to recover this triple from ω, we need to know $s_0 \in (0,\sigma(\omega))$, such that the ancestral line of $p_{(\omega)}(s_0)$ in the genealogical tree of ω corresponds to the spine.

It will be useful to consider the spine reversal operation on (finite spine) coding triples satisfying our assumptions, which is defined by

$$\mathbf{SR}\Big((\mathrm{w}(t))_{0\leqslant t\leqslant\zeta}, \sum_{i\in I}\delta_{(t_i,\omega_i)}, \sum_{j\in J}\delta_{(t'_j,\omega'_j)}\Big) := \Big((\mathrm{w}(\zeta-t))_{0\leqslant t\leqslant\zeta}, \sum_{j\in J}\delta_{(\zeta-t'_j,\omega'_j)}, \sum_{i\in I}\delta_{(\zeta-t_i,\omega_i)}\Big). \tag{5.8}$$

We note that the labeled trees associated with the coding triples in the left and right sides of (5.8) are identified via an isometry that preserves labels and intervals, but the roles of the top and the bottom of the spine are interchanged. Informally, the spine reversal operation corresponds to a re-rooting at the other end of the spine.

In Section 5.3 below we investigate relations between different distributions on coding triples, and in Section 5.4 we explain how to go from (random) coding triples to the random metric spaces of interest in this work.

Important remark. Later, when we speak about the tree associated with a coding triple (as we just defined both in the finite and in the infinite spine case), it will always be understood that this includes the labeling on the tree and the clockwise exploration, which is needed to make sense of intervals on the tree.

Spine decomposition under $\mathbb{N}_a$. Let $a > 0$. We conclude this section with a result connecting the measure $\mathbb{N}_a$ with a (finite spine) coding triple. Arguing under $\mathbb{N}_a(\mathrm{d}\omega)$, for every $r \in (0,\sigma)$, we can define two point measures $\mathcal{P}_{(r)}$ and $\mathcal{P}'_{(r)}$ that account for the (labeled) subtrees branching off the ancestral line of $p_{(\omega)}(r)$ in the genealogical tree $\mathcal{T}_{(\omega)}$. Precisely, if r is fixed, we consider all connected components (u_i, v_i), $i \in I$, of the open set $\{s \in [0,r] : \zeta_s(\omega) > \min_{t\in[s,r]} \zeta_t(\omega)\}$, and for each $i \in I$, we define a snake trajectory ω^i by setting $\sigma(\omega^i) = v_i - u_i$ and, for every $s \in [0, \sigma(\omega^i)]$,

$$\omega^i_s(t) := \omega_{u_i+s}(\zeta_{u_i}(\omega) + t) \;, \quad \text{for } 0 \leqslant t \leqslant \zeta_{(\omega^i_s)} := \zeta_{u_i+s}(\omega) - \zeta_{u_i}(\omega).$$

Note that $\omega^i \in \mathcal{S}_{\widehat{\omega}_{u_i}}$, and $\widehat{\omega}_{u_i} = \omega_r(\zeta_{u_i})$ by the snake property. We then set $\mathcal{P}_{(r)} = \sum_{i\in I} \delta_{(\zeta_{u_i},\omega^i)}$. To define $\mathcal{P}'_{(r)}$, we proceed in a very similar manner, replacing the interval $[0,r]$ by $[r,\sigma]$.

Recall our notation $(L^0_s)_{s\in[0,\sigma]}$ for the exit local time at level 0. Let $M_p(\mathbb{R}_+ \times \mathcal{S})$ stand for the set of all point measures on $\mathbb{R}_+ \times \mathcal{S}$.

Proposition 2. *Let $a > 0$ and let $Y = (Y_t)_{0\leqslant t\leqslant T^Y}$ stand for a linear Brownian motion started from a and stopped at its first hitting time of 0. Conditionally given Y, let $\mathcal{M}$ and $\mathcal{M}'$ be two independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity*

$$2\,\mathbf{1}_{[0,T^Y]}(t)\,\mathrm{d}t\,\mathbb{N}_{Y_t}(\mathrm{d}\omega).$$

Then, for any nonnegative measurable function F on $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$, we have

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L^0_r\, F(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)})\Big) = \mathbb{E}\Big[F(Y, \mathcal{M}, \mathcal{M}')\Big].$$

It is straightforward to verify that $\mathbb{N}_a(\mathrm{d}\omega)$ a.e., for every $r \in (0,\sigma)$, $(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)})$ is a coding triple in the sense of the previous discussion (finite spine case), and $\Omega(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)}) = \omega$.

Proof. We may assume that $F(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)}) = F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)})$ where F_1 is defined on $\mathcal{W}$ and F_2 and F_3 are defined on $M_p(\mathbb{R}_+ \times \mathcal{S})$. Let $(\tau^0_r)_{r\geqslant 0}$ be the inverse local time defined by $\tau^0_r = \inf\{s \geqslant 0 : L^0_s \geqslant r\}$. Then,

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L^0_r\, F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)})\Big) = \int_0^\infty dr\,\mathbb{N}_a\Big(\mathbf{1}_{\{\tau^0_r<\infty\}}\, F_1(W_{\tau^0_r})F_2(\mathcal{P}_{(\tau^0_r)})F_3(\mathcal{P}'_{(\tau^0_r)})\Big).$$

We may now apply the strong Markov property of the Brownian snake [65, Theorem IV.6], noting that $F_1(W_{\tau_r^0})F_2(\mathcal{P}_{\tau_r^0})$ is measurable with respect to the past up to time τ_r^0. Using also [65, Lemma V.5], we get, for every $r > 0$,

$$\mathbb{N}_a\Big(\mathbf{1}_{\{\tau_r^0<\infty\}}\,F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)})F_3(\mathcal{P}'_{(\tau_r^0)})\Big) = \mathbb{N}_a\Big(\mathbf{1}_{\{\tau_r^0<\infty\}}\,F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)})\,\mathbb{P}^{(W_{\tau_r^0})}(F_3)\Big),$$

where, for any finite path w, we write $\mathbb{P}^{(\mathrm{w})}$ for the distribution of a Poisson point measure on $\mathbb{R}_+ \times M_p(\mathcal{S})$ with intensity $2\,\mathbf{1}_{[0,\zeta_{(\mathrm{w})}]}(t)\,\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega)$. From the preceding two displays, we arrive at

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L_r^0\,F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)})\Big) = \mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L_r^0\,F_1(W_r)F_2(\mathcal{P}_{(r)})\,\mathbb{P}^{(W_r)}(F_3)\Big).$$

By the invariance of the excursion measure $\mathbb{N}_a$ under time-reversal, the right-hand side of the last display is also equal to

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L_r^0\,F_1(W_r)F_2(\check{\mathcal{P}}'_{(r)})\,\mathbb{P}^{(W_r)}(F_3)\Big),$$

where we write $\check{\mathcal{P}}'_{(r)}(\mathrm{d}s\mathrm{d}\omega)$ for the image of $\mathcal{P}'_{(r)}(\mathrm{d}s\mathrm{d}\omega)$ under the mapping $(s,\omega)\mapsto(s,\check{\omega})$. Then the same application of the strong Markov property shows that this equals

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L_r^0\,F_1(W_r)\,\mathbb{P}^{(W_r)}(F_2)\,\mathbb{P}^{(W_r)}(F_3)\Big).$$

Finally, the first-moment formula in [65, Proposition V.3] shows that this quantity is also equal to

$$\mathbb{E}\Big[F_1(Y)\,\mathbb{P}^{(Y)}(F_2)\,\mathbb{P}^{(Y)}(F_3)\Big]$$

with the notation of the proposition. This completes the proof. □

Proposition 2 will allow us to relate properties valid under $\mathbb{N}_a$ to similar properties for the triple $(Y,\mathcal{M},\mathcal{M}')$. Let us illustrate this on an example that will be useful later. Recall from (8.7) that the exit measure $\mathcal{Z}_0$ satisfies

$$\mathcal{Z}_0 = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^2}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\tau_0(W_s)=\infty,\,\widehat{W}_s<\varepsilon\}},\qquad \mathbb{N}_a \text{ a.e.}$$

Replacing the limit by a liminf, we may assume that $\mathcal{Z}_0(\omega)$ is defined for every $\omega\in\mathcal{W}$. It is a simple matter to verify that, $\mathbb{N}_a$ a.e. for every r such that $\zeta_r=\tau_0(W_r)$, we have

$$\mathcal{Z}_0 = \int \mathcal{P}_{(r)}(\mathrm{d}t\mathrm{d}\varpi)\,\mathcal{Z}_0(\varpi) + \int \mathcal{P}'_{(r)}(\mathrm{d}t\mathrm{d}\varpi)\,\mathcal{Z}_0(\varpi). \tag{5.9}$$

Then, if we define $\mathcal{Z}^Y := \int\mathcal{M}(\mathrm{d}t\mathrm{d}\varpi)\,\mathcal{Z}_0(\varpi)+\int\mathcal{M}'(\mathrm{d}t\mathrm{d}\varpi)\,\mathcal{Z}_0(\varpi)$, we deduce from Proposition 2 that, for any nonnegative measurable function φ on $\mathbb{R}_+$, we have $\mathbb{N}_a(\mathcal{Z}_0\,\varphi(\mathcal{Z}_0)) = \mathbb{E}[\varphi(\mathcal{Z}^Y)]$. More generally, set $\tau_u^Y=\inf\{t\geqslant 0 : Y_t=u\}$ and $\mathcal{Z}_u^Y := \int\mathcal{M}(\mathrm{d}t\mathrm{d}\varpi)\,\mathbf{1}_{\{t<\tau_u^Y\}}\,\mathcal{Z}_u(\varpi) + \int\mathcal{M}'(\mathrm{d}t\mathrm{d}\varpi)\,\mathbf{1}_{\{t<\tau_u^Y\}}\,\mathcal{Z}_0(\varpi)$, for every $u\in(0,a)$. Then we have

$$\mathbb{N}_a(\mathcal{Z}_0\,\varphi(\mathcal{Z}_0,\mathcal{Z}_{u_1},\ldots\mathcal{Z}_{u_p})) = \mathbb{E}[\varphi(\mathcal{Z}^Y,\mathcal{Z}^Y_{u_1},\ldots,\mathcal{Z}^Y_{u_p})], \tag{5.10}$$

for every $0<u_1<\cdots<u_p<a$ and any nonnegative measurable function φ on $\mathbb{R}_+^{p+1}$.

5.3 Distributional relations between coding processes

5.3.1 Some explicit distributions

Let us introduce the function

$$\psi(x) = \frac{2}{\sqrt{\pi}}(x^{1/2} + x^{-1/2}) - 2(x + \frac{3}{2})\, e^x \operatorname{erfc}(\sqrt{x}), \quad x > 0.$$

Note that $\psi(x) = x^{-1}\chi_3(x)$ in the notation of the Appendix below, and thus, by formula (A.3) there,

$$\int_0^\infty e^{-\lambda x}\, x\psi(x)\,\mathrm{d}x = (1+\sqrt{\lambda})^{-3}, \quad \lambda \geqslant 0. \tag{5.11}$$

Furthermore one checks from the explicit formula for ψ that $\psi(x) = \frac{2}{\sqrt{\pi}}x^{-1/2} + O(1)$ as $x \to 0$, and $\psi(x) = \frac{3}{2\sqrt{\pi}}x^{-5/2} + O(x^{-7/2})$ as $x \to \infty$.

Proposition 3. (i) *Let $a > 0$. The density of $\mathcal{Z}_0$ under $\mathbb{N}_a(\cdot \cap \{\mathcal{Z}_0 \neq 0\})$ is*

$$h_a(z) := \Big(\frac{3}{2a^2}\Big)^2 \psi(\frac{3z}{2a^2}), \quad z > 0. \tag{5.12}$$

(ii) *For every $z > 0$ and $a > 0$, set*

$$p_z(a) := 2\,\Big(\frac{3}{2}\Big)^{3/2} \sqrt{\pi}\, z^{3/2} a^{-4}\, \psi(\frac{3z}{2a^2}).$$

Then, $a \mapsto p_z(a)$ defines a probability density on $(0,\infty)$, and for every nonnegative measurable function g on $[0,\infty)$,

$$z^{-2}\mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s)\Big) = \int_0^\infty \mathrm{d}a\, p_z(a)\, g(a).$$

Remark. The construction of Brownian disks in [71] allows us to interpret (ii) by saying that p_z is the density of the distribution of the distance from the distinguished point to the boundary in a free pointed Brownian disk with perimeter z. See Proposition 14 below.

Proof. (i) From (5.3), we have for $\lambda \geqslant 0$,

$$\mathbb{N}_a\Big(1 - \exp(-\lambda\mathcal{Z}_0)\Big) = \Big(a\sqrt{2/3} + \lambda^{-1/2}\Big)^{-2} = \frac{3}{2a^2}\,\Big(1 + \Big(\frac{2a^2\lambda}{3}\Big)^{-1/2}\Big)^{-2},$$

and in particular $\mathbb{N}_a(\mathcal{Z}_0 \neq 0) = \frac{3}{2a^2}$ in agreement with (5.1). On the other hand, by formula (A.4) in the Appendix,

$$\int_0^\infty (1 - e^{-\lambda x})\, \psi(x)\,\mathrm{d}x = (1 + \lambda^{-1/2})^{-2}.$$

Part (i) follows by comparing the last two displays.

(ii) From (8.7) and (5.6), we get the existence of a measurable function Γ on $\mathcal{S}$ such that $\mathcal{Z}_0 = \Gamma(\mathrm{tr}_0(\omega))$, $\mathbb{N}_a(\mathrm{d}\omega)$ a.e., for any $a > 0$, and also $\mathcal{Z}_0^* = \Gamma(\omega) = \Gamma(\omega^{[t]})$ for every $t \in [0,\sigma(\omega)]$,

$\mathbb{N}^*(\mathrm{d}\omega)$ a.e. By applying the re-rooting formula (5.5) with $G(\omega) = f(\Gamma(\omega))g(\omega_0)$, where f and g are nonnegative real functions, we get

$$\mathbb{N}^*\Big(f(\mathcal{Z}_0^*)\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s)\Big) = 2\int_0^\infty \mathrm{d}a\, g(a)\, \mathbb{N}_a\Big(\mathcal{Z}_0 f(\mathcal{Z}_0)\Big).$$

The left-hand side can be written as

$$\sqrt{\frac{3}{2\pi}}\int_0^\infty \mathrm{d}z\, z^{-5/2} f(z)\, \mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s)\Big).$$

On the other hand, part (i) allows us to rewrite the right-hand side as

$$2\int_0^\infty \mathrm{d}a\, g(a)\int_0^\infty \mathrm{d}z\, h_a(z)\, z\, f(z) = 2\int_0^\infty \mathrm{d}z\, z\, f(z)\int_0^\infty \mathrm{d}a\, h_a(z)\, g(a).$$

By comparing the last two displays, we get, $\mathrm{d}z$ a.e.,

$$z^{-2}\mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s)\Big) = 2\sqrt{\frac{2\pi}{3}}\, z^{3/2}\int_0^\infty \mathrm{d}a\, h_a(z)\, g(a) = \int_0^\infty \mathrm{d}a\, p_z(a)\, g(a),$$

where $p_z(a)$ is as in the proposition. A scaling argument shows that this identity indeed holds for every $z > 0$. Since $z^{-2}\mathbb{N}^{*,z}(\sigma) = 1$, p_z is a probability density, which may also be checked directly. □

5.3.2 A distributional identity for coding triples

As we already explained, coding triples will be used to construct the random metric spaces of interest in this work. The relevant case for the forthcoming construction of the infinite-volume Brownian disk with perimeter $z > 0$ may be described as follows: we let $R = (R_t)_{t\in[0,\infty)}$ be a three-dimensional Bessel process started from 0, we assume that, conditionally on R, $\mathcal{P}$ and $\mathcal{P}'$ are independent Poisson measures with intensity $2\,\mathrm{d}t\,\mathbb{N}_{R_t}(\mathrm{d}\omega)$, and finally we condition on the event $\mathcal{Z} = z$, where $\mathcal{Z}$ denotes the total exit measure at 0 of the atoms of $\mathcal{P}$ and $\mathcal{P}'$. In Section 5.3.3, we will give a precise meaning to this conditioning and obtain a conditional distribution Θ_z on coding triples, which plays an important role in the next sections. Before doing that, we need to develop certain preliminary tools, and we first recall special cases of a well-known time-reversal property for Bessel processes. Let R be as above and let X be a Bessel process of dimension 9 started from 0. Then, for every $a > 0$,

(a) If $\mathbf{L}_a := \sup\{t \geqslant 0 : R_t = a\}$, the process $(R_{\mathbf{L}_a - t}, 0 \leqslant t \leqslant \mathbf{L}_a)$ is distributed as a linear Brownian motion started from a and stopped upon hitting 0.

(b) If $\mathrm{L}_a := \sup\{t \geqslant 0 : X_t = a\}$, the process $(X_{\mathrm{L}_a - t}, 0 \leqslant t \leqslant \mathrm{L}_a)$ is distributed as a Bessel process of dimension -5 started from a and stopped upon hitting 0.

Both (a) and (b) are special cases of a more general result for Bessel processes, which is itself a consequence of Nagasawa's time-reversal theorem (see [92, Theorem VII.4.5], and [92, Exercise

XI.1.23] for the case of interest here, and note that part (a) is due to Williams [98]). If $0 < a < b$, an application of the strong Markov property of the process $X_{L_b - t}$ at the first time it hits a shows that $(X_{\mathrm{L}_a - t}, 0 \leqslant t \leqslant \mathrm{L}_a)$, or equivalently $(X_t, 0 \leqslant t \leqslant \mathrm{L}_a)$, is independent of $(X_{(L_a+t)\wedge L_b}, t \geqslant 0)$. Letting b tend to infinity we get that $(X_{\mathrm{L}_a+t}, t \geqslant 0)$ is independent of $(X_t, 0 \leqslant t \leqslant \mathrm{L}_a)$, and similarly the process $(R_{\mathbf{L}_a+t}, t \geqslant 0)$ is independent of $(R_t, 0 \leqslant t \leqslant \mathbf{L}_a)$. These properties are used implicitly in what follows.

Let us introduce some notation needed for the technical results that follow. We fix $a > 0$ and consider a triple $(Y, \mathcal{M}, \mathcal{M}')$ distributed as in Proposition 2: $Y = (Y_t, 0 \leqslant t \leqslant T^Y)$ is a linear Brownian motion started from a and stopped at the first time it hits 0, and, conditionally on Y, $\mathcal{M}$ and $\mathcal{M}'$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $2\,\mathbf{1}_{[0,T^Y]}(t)\,\mathrm{d}t\,\mathbb{N}_{Y_t}(\mathrm{d}\omega)$. We also introduce the point measures $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ obtained by truncating the atoms of $\mathcal{M}$ and $\mathcal{M}'$ at level 0. More precisely, for any nonnegative measurable function Φ on $\mathbb{R}_+ \times \mathcal{S}$, we set

$$\int \widetilde{\mathcal{M}}(\mathrm{d}t\mathrm{d}\omega)\,\Phi(t,\omega) := \int \mathcal{M}(\mathrm{d}t\mathrm{d}\omega)\,\Phi(t, \mathrm{tr}_0(\omega))$$

and $\widetilde{\mathcal{M}}'$ is defined similarly from $\mathcal{M}'$. We will be interested in the triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$, which we may view as a coding triple in the sense of Section 5.2.4.

We define

$$\mathcal{Z}^Y = \int \mathcal{M}(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega) + \int \mathcal{M}'(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega)$$

in agreement with the end of Section 5.2.4. We saw that, for any nonnegative measurable function φ on $\mathbb{R}_+$, we have $\mathbb{E}[\varphi(\mathcal{Z}^Y)] = \mathbb{N}_a(\mathcal{Z}_0\,\varphi(\mathcal{Z}_0))$. It then follows from Proposition 3 (i) that the distribution of $\mathcal{Z}^Y$ has density $z\,h_a(z)$. In particular, we have

$$\mathbb{E}[e^{-\lambda \mathcal{Z}^Y}] = \int_0^\infty z\,h_a(z)\,e^{-\lambda z}\,\mathrm{d}z = \Big(1 + a\sqrt{2\lambda/3}\Big)^{-3}, \tag{5.13}$$

by (5.11) and (5.12).

We write $(\check{\Theta}^{(a)}_z)_{z>0}$ for a regular version of the conditional distributions of the triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ knowing that $\mathcal{Z}^Y = z$. The collection $(\check{\Theta}^{(a)}_z)_{z>0}$ is well defined only up to a set of values of z of zero Lebesgue measure, but we will see later how to make a canonical choice of this collection.

Let us also fix $r > 0$. We next consider a triple $(V, \mathcal{N}, \mathcal{N}')$, where

- $V = (V_t, 0 \leqslant t \leqslant T^V)$ is distributed as a Bessel process of dimension -5 started from $r + a$ and stopped at the first time it hits r.
- Conditionally on V, $\mathcal{N}$ and $\mathcal{N}'$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity
$$2\,\mathbf{1}_{[0,T^V]}(t)\,\mathrm{d}t\,\mathbb{N}_{V_t}(\mathrm{d}\omega \cap \{W_* > 0\}),$$
where we recall the notation $W_*(\omega) = \min\{\widehat{W}_s(\omega) : 0 \leqslant s \leqslant \sigma(\omega)\}$.

We write $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}'$ for the point measures obtained by truncating the atoms of $\mathcal{N}$ and $\mathcal{N}'$ at level r, in the same way as $\widetilde{\mathcal{M}}$ was defined above from $\mathcal{M}$ by truncation at level 0. We also introduce the exit measure

$$\mathcal{Z}^V = \int \mathcal{N}(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_r(\omega) + \int \mathcal{N}'(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_r(\omega).$$

As we will see in the next proof, the distributions of $\mathcal{Z}^V$ and $\mathcal{Z}^Y$ are related by the formula

$$\mathbb{E}[h(\mathcal{Z}^V)] = \Big(\frac{r+a}{r}\Big)^3\, \mathbb{E}[h(\mathcal{Z}^Y)\, e^{-\frac{3}{2r^2}\mathcal{Z}^Y}]. \tag{5.14}$$

In particular, the distribution of $\mathcal{Z}^V$ also has a positive density on $(0,\infty)$.

We let ϑ_r stand for the obvious shift that maps snake trajectories with initial point x to snake trajectories with initial point $x-r$. If $\mu = \sum_{i\in I}\delta_{(t_i,\omega_i)}$ is a point measure on $\mathbb{R}_+\times\mathcal{S}$, we also write $\vartheta_r\mu = \sum_{i\in I}\delta_{(t_i,\vartheta_r\omega_i)}$, by abuse of notation.

Proposition 4. *The collection $(\check{\Theta}^{(a)}_z)_{z>0}$ is a regular version of the conditional distributions of $(V-r,\vartheta_r\widetilde{\mathcal{N}},\vartheta_r\widetilde{\mathcal{N}}')$ knowing that $\mathcal{Z}^V = z$.*

In other words, the conditional distribution of $(V-r,\vartheta_r\widetilde{\mathcal{N}},\vartheta_r\widetilde{\mathcal{N}}')$ knowing that $\mathcal{Z}^V = z$ coincides with the conditional distribution of $(Y,\widetilde{\mathcal{M}},\widetilde{\mathcal{M}}')$ knowing that $\mathcal{Z}^Y = z$. In particular, the conditional distribution of $(V-r,\vartheta_r\widetilde{\mathcal{N}},\vartheta_r\widetilde{\mathcal{N}}')$ knowing that $\mathcal{Z}^V = z$ does not depend on r, which is by no means an obvious fact. This fact can be extended to Bessel processes of other dimensions, provided that the intensity measure of the Poisson measures $\mathcal{N}$ and $\mathcal{N}'$ is changed accordingly, but we shall leave this extension to the reader.

Proof. Recall our notation $M_p(\mathbb{R}_+\times\mathcal{S})$ for the set of all point measures on $\mathbb{R}_+\times\mathcal{S}$. As in the proof of Proposition 2, if w is a finite path taking nonnegative values, we write $\mathbb{P}^{(\mathrm{w})}(\mathrm{d}\mu)$ for the probability measure on $M_p(\mathbb{R}_+\times\mathcal{S})$ which is the distribution of a Poisson point measure on $\mathbb{R}_+\times\mathcal{S}$ with intensity $2\,\mathbf{1}_{[0,\zeta_{(\mathrm{w})}]}(t)\,\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega)$. Denoting the generic element of $M_p(\mathbb{R}_+\times\mathcal{S})\times M_p(\mathbb{R}_+\times\mathcal{S})$ by (μ,μ'), we have the formula

$$\mathbb{P}^{(\mathrm{w})}\otimes\mathbb{P}^{(\mathrm{w})}(\mu(W_*\leqslant 0)=\mu'(W^*\leqslant 0)=0) = \exp\Big(-4\int_0^{\zeta_{(\mathrm{w})}}\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(W_*\leqslant 0)\Big) = \exp\Big(-6\int_0^{\zeta_{(\mathrm{w})}}\frac{\mathrm{d}t}{\mathrm{w}(t)^2}\Big), \tag{5.15}$$

where in the left-hand side we abuse notation by writing $\mu(W_*\leqslant 0)$ instead of $\mu(\{(t,\omega)\in\mathbb{R}_+\times\mathcal{S} : W_*(\omega)\leqslant 0\})$.

We then introduce a random finite path $(U_t)_{0\leqslant t\leqslant T^U}$, which is distributed as a linear Brownian motion started from $r+a$ and stopped when hitting r (so $U-r$ has the same distribution as Y). Let $\mathcal{P}$ and $\mathcal{P}'$ be random elements of $M_p(\mathbb{R}_+\times\mathcal{S})$ such that the conditional distribution of the pair $(\mathcal{P},\mathcal{P}')$ given U is $\mathbb{P}^{(U)}\otimes\mathbb{P}^{(U)}(\mathrm{d}\mu\mathrm{d}\mu')$. Define

$$\mathcal{Z}^U = \int \mathcal{P}(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_r(\omega) + \int \mathcal{P}'(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_r(\omega),$$

and also write $\widetilde{\mathcal{P}}$, resp. $\widetilde{\mathcal{P}}'$, for the point measure $\mathcal{P}$, resp. $\mathcal{P}'$, truncated at level r. Then, the statement of the proposition reduces to showing that the conditional distribution of $(V,\widetilde{\mathcal{N}},\widetilde{\mathcal{N}}')$ knowing that $\mathcal{Z}^V = z$ coincides $\mathrm{d}z$ a.e. with the conditional distribution of $(U,\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}')$ knowing that $\mathcal{Z}^U = z$ (an obvious translation argument yields that $(U - r, \vartheta_r\widetilde{\mathcal{P}}, \vartheta_r\widetilde{\mathcal{P}}', \mathcal{Z}^U)$ has the same law as $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}', \mathcal{Z}^Y)$ above).

The first step of the proof is to verify that, for any nonnegative measurable functions F and G defined on $\mathcal{W}$ and on $M_p(\mathbb{R}_+ \times \mathcal{S})^2$ respectively,

$$\begin{aligned}&\mathbb{E}\Big[F(U)G(\mathcal{P},\mathcal{P}')\,\Big|\,\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0\Big]\\ &\quad = \mathbb{E}\Big[F(V)\,\mathbb{P}^{(V)} \otimes \mathbb{P}^{(V)}[G(\mu,\mu') \mid \mu(W_* \leqslant 0) = \mu'(W_* \leqslant 0) = 0]\Big].\end{aligned} \tag{5.16}$$

To prove (5.16), we first apply (5.15) to get

$$\mathbb{P}\Big(\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0\Big) = \mathbb{E}\Big[\exp\Big(-6\int_0^{T^U} \frac{\mathrm{d}t}{U_t^2}\Big)\Big] = \Big(\frac{r}{r+a}\Big)^3$$

where the last equality is easily derived by using Itô's formula to verify that $U_{t\wedge T^U}^{-3}\exp(-6\int_0^{t\wedge T^U} U_s^{-2}\,\mathrm{d}s)$ is a martingale. So we have

$$\begin{aligned}&\Big(\frac{r}{r+a}\Big)^3\mathbb{E}\Big[F(U)G(\mathcal{P},\mathcal{P}')\,\Big|\,\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0\Big]\\ &\quad = \mathbb{E}\Big[F(U)G(\mathcal{P},\mathcal{P}')\,\mathbf{1}_{\{\mathcal{P}(W_*\leqslant 0)=\mathcal{P}'(W_*\leqslant 0)=0\}}\Big]\\ &\quad = \mathbb{E}\Big[F(U)\,\mathbb{P}^{(U)}\otimes\mathbb{P}^{(U)}[G(\mu,\mu')\,\mathbf{1}_{\{\mu(W_*\leqslant 0)=\mu'(W_*\leqslant 0)=0\}}]\Big]\\ &\quad = \mathbb{E}\Big[F(U)\,\exp\Big(-6\int_0^{T^U}\frac{\mathrm{d}t}{U_t^2}\Big)\,\mathbb{P}^{(U)}\otimes\mathbb{P}^{(U)}[G(\mu,\mu') \mid \mu(W_* \leqslant 0) = \mu'(W_* \leqslant 0) = 0]\Big]\end{aligned}$$

using (5.15) in the last equality. To complete the proof of (5.16), we just observe that, by classical absolute continuity relations between Brownian motion and Bessel processes, the law of U under the probability measure

$$\Big(\frac{r+a}{r}\Big)^3\exp\Big(-6\int_0^{T^U}\frac{\mathrm{d}t}{U_t^2}\Big)\cdot\mathbb{P}$$

coincides with the law of V under $\mathbb{P}$ (see [69, Lemma 1] for a short proof).

Let us complete the proof of the proposition. By a standard property of Poisson measures and the definition of the pair $(\mathcal{N},\mathcal{N}')$, we have

$$\mathbb{E}[G(\mathcal{N},\mathcal{N}') \mid V] = \mathbb{P}^{(V)}\otimes\mathbb{P}^{(V)}[G(\mu,\mu') \mid \mu(W_* \leqslant 0) = \mu'(W_* \leqslant 0) = 0].$$

It thus follows from (5.16) that

$$\mathbb{E}\Big[F(U)G(\mathcal{P},\mathcal{P}')\,\Big|\,\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0\Big] = \mathbb{E}[F(V)\,\mathbb{E}[G(\mathcal{N},\mathcal{N}')\,|\,V]] = \mathbb{E}[F(V)\,G(\mathcal{N},\mathcal{N}')].$$

In particular, for any nonnegative measurable function h on $[0,\infty)$, we have

$$\mathbb{E}\Big[F(U)G(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}')h(\mathcal{Z}^U)\,\Big|\,\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0\Big] = \mathbb{E}[F(V)\,G(\widetilde{\mathcal{N}},\widetilde{\mathcal{N}}')\,h(\mathcal{Z}^V)].$$

The left-hand side of the last display is equal to

$$\Big(\frac{r+a}{r}\Big)^3\,\mathbb{E}\Big[F(U)G(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}')h(\mathcal{Z}^U)\,\exp(-\frac{3}{2r^2}\,\mathcal{Z}^U)\Big]$$

because, on one hand, we saw that $\mathbb{P}(\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0) = (\frac{r}{r+a})^3$ and, on the other hand, the special Markov property (see e.g. the appendix of [70]) and (5.1) show that

$$\mathbb{P}(\mathcal{P}(W_* \leqslant 0) = \mathcal{P}'(W_* \leqslant 0) = 0 \mid U, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}') = \exp(-\frac{3}{2r^2}\,\mathcal{Z}^U).$$

We can find nonnegative measurable functions φ_1 and φ_2 on $[0,\infty)$ such that

$$\mathbb{E}\Big[F(U)G(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}') \mid \mathcal{Z}^U\Big] = \varphi_1(\mathcal{Z}^U)\ , \quad \mathbb{E}[F(V)\,G(\widetilde{\mathcal{N}}, \widetilde{\mathcal{N}}') \mid \mathcal{Z}^V] = \varphi_2(\mathcal{Z}^V)\ ,$$

and it follows from the preceding considerations that, for any function h,

$$\Big(\frac{r+a}{r}\Big)^3\,\mathbb{E}[\varphi_1(\mathcal{Z}^U)\,h(\mathcal{Z}^U)\,\exp(-\frac{3}{2r^2}\,\mathcal{Z}^U)] = \mathbb{E}[\varphi_2(\mathcal{Z}^V)h(\mathcal{Z}^V)]. \tag{5.17}$$

By specializing this identity to the case $F = 1$, $G = 1$, we get the relation (5.14) between the distributions of $\mathcal{Z}^V$ and $\mathcal{Z}^Y$ (recall that $\mathcal{Z}^U$ has the same distribution as $\mathcal{Z}^Y$). It then also follows from (5.17) that (for arbitrary F and G) we have $\mathbb{E}[\varphi_1(\mathcal{Z}^V)h(\mathcal{Z}^V)] = \mathbb{E}[\varphi_2(\mathcal{Z}^V)h(\mathcal{Z}^V)]$ for any test function h, so that $\varphi_1(\mathcal{Z}^V) = \varphi_2(\mathcal{Z}^V)$ a.s. and $\varphi_1(z) = \varphi_2(z)$, $\mathrm{d}z$ a.e., which completes the proof. □

For every $z > 0$, we let $\Theta^{(a)}_z$ denote the image of $\check{\Theta}^{(a)}_z$ under the spine reversal transformation **SR** in (5.8). Since the process $(Y_t, 0 \leqslant t \leqslant T^Y)$ is mapped by time-reversal to a three-dimensional Bessel process started from 0 and stopped at its last passage at a (by property (a) stated at the beginning of the section), we could have defined $\Theta^{(a)}_z$ directly in terms of conditioning a coding triple whose first component is a three-dimensional Bessel process up to a last passage time. The connection with the discussion at the beginning of this section should then be clear: $\Theta^{(a)}_z$ is the analog of the probability measure Θ_z we are aiming at, when the three-dimensional Bessel process is truncated at a last passage time.

5.3.3 The coding triple of the infinite-volume Brownian disk

In this section, we define the probability measures Θ_z, $z > 0$, which were introduced informally at the beginning of the preceding section. Roughly speaking, the idea is to get Θ_z as the limit of $\Theta^{(a)}_z$ as $a \to \infty$. Proposition 6 below will also show that, for every $r > 0$, the collection $(\Theta_z)_{z>0}$ corresponds to conditional distributions of a coding triple whose first component is a nine-dimensional Bessel process considered after its last passage time at r (compare with Proposition 4). The latter fact is the key to the identification as infinite-volume Brownian disks of the complement of hulls in the Brownian plane.

We consider a triple $(X, \mathcal{L}, \mathcal{R})$ such that $X = (X_t)_{t\geqslant 0}$ is a nine-dimensional Bessel process started from 0 and, conditionally on X, $\mathcal{L}$ and $\mathcal{R}$ are two independent Poisson measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity

$$2\,\mathrm{d}t\,\mathbb{N}_{X_t}(\mathrm{d}\omega \cap \{W_* > 0\}).$$

As previously, we set, for every $r > 0$,

$$\mathrm{L}_r := \sup\{t \geqslant 0 : X_t = r\}. \tag{5.18}$$

In what follows, we fix $r > 0$, and we shall be interested in atoms (t,ω) of $\mathcal{L}$ or $\mathcal{R}$ such that $t > \mathrm{L}_r$. More precisely, we introduce a point measure $\mathcal{L}^{(r)}$ as the image of

$$\mathbf{1}_{(\mathrm{L}_r,\infty)}(t)\,\mathcal{L}(\mathrm{d}t\,\mathrm{d}\omega)$$

under the mapping $(t,\omega) \mapsto (t-\mathrm{L}_r, \vartheta_r\omega)$ (where ϑ_r is the shift operator already used in Proposition 4). In a way similar to the previous section, we define $\widetilde{\mathcal{L}}^{(r)}$ by truncating the atoms of $\mathcal{L}^{(r)}$ at level 0 (more precisely, $\widetilde{\mathcal{L}}^{(r)}$ is the image of $\mathcal{L}^{(r)}$ under the mapping $(t,\omega) \mapsto (t, \mathrm{tr}_0(\omega))$). We define similarly $\mathcal{R}^{(r)}$ and $\widetilde{\mathcal{R}}^{(r)}$ from the point measure $\mathcal{R}$. Finally, we set

$$\begin{aligned}\mathcal{Z}^{(r)} &= \int \mathcal{L}^{(r)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega)\\ &= \int \mathcal{L}(\mathrm{d}t\mathrm{d}\omega)\,\mathbf{1}_{(\mathrm{L}_r,\infty)}(t)\,\mathcal{Z}_r(\omega) + \int \mathcal{R}(\mathrm{d}t\mathrm{d}\omega)\,\mathbf{1}_{(\mathrm{L}_r,\infty)}(t)\,\mathcal{Z}_r(\omega)\end{aligned} \tag{5.19}$$

and we also consider the process $(X^{(r)}_t)_{t\geqslant 0}$ defined by

$$X^{(r)}_t = X_{\mathrm{L}_r+t} - r.$$

By [40, Proposition 1.2], $\mathcal{Z}^{(r)}$ is a *finite* random variable, with a density given by

$$k_r(z) := \frac{1}{\sqrt{\pi}}\,3^{3/2}2^{-1/2}\,r^{-3}\,z^{1/2}\,e^{-\frac{3z}{2r^2}}. \tag{5.20}$$

Our first goal is to verify that the conditional distribution of the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ knowing that $\mathcal{Z}^{(r)} = z$ does not depend on r. Note that, for instance, the unconditional distribution of $X^{(r)}$ depends on r.

We will deduce the preceding assertion from Proposition 4, but to this end a truncation argument is needed. So we consider $a > 0$, and we set

$$\mathrm{L}^{(r)}_a := \mathrm{L}_{r+a} - \mathrm{L}_r = \sup\{t \geqslant 0 : X^{(r)}_t = a\}.$$

We then set[1]

$$\begin{aligned}\mathcal{L}^{(r,r+a)}(\mathrm{d}t\mathrm{d}\omega) &= \mathbf{1}_{[0,\mathrm{L}^{(r)}_a]}(t)\,\mathcal{L}^{(r)}(\mathrm{d}t\mathrm{d}\omega),\\ \mathcal{L}^{(r+a,\infty)}(\mathrm{d}t\mathrm{d}\omega) &= \mathbf{1}_{(\mathrm{L}^{(r)}_a,\infty)}(t)\,\mathcal{L}^{(r)}(\mathrm{d}t\mathrm{d}\omega),\end{aligned}$$

and we define $\mathcal{R}^{(r,r+a)}$ and $\mathcal{R}^{(r+a,\infty)}$ in a similar way from $\mathcal{R}^{(r)}$. As previously, we let $\widetilde{\mathcal{L}}^{(r,r+a)}$, $\widetilde{\mathcal{L}}^{(r+a,\infty)}$, $\widetilde{\mathcal{R}}^{(r,r+a)}$, $\widetilde{\mathcal{R}}^{(r+a,\infty)}$ stand for these point measures truncated at level 0. We finally set

$$\begin{aligned}\mathcal{Z}^{(r,r+a)} &= \int \mathcal{L}^{(r,r+a)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r,r+a)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega)\\ \mathcal{Z}^{(r+a,\infty)} &= \int \mathcal{L}^{(r+a,\infty)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r+a,\infty)}(\mathrm{d}t\mathrm{d}\omega)\,\mathcal{Z}_0(\omega).\end{aligned}$$

Obviously $\mathcal{Z}^{(r)} = \mathcal{Z}^{(r,r+a)} + \mathcal{Z}^{(r+a,\infty)}$. Also, the random variables $\mathcal{Z}^{(r,r+a)}$ and $\mathcal{Z}^{(r+a,\infty)}$ are independent, as a consequence of the independence properties stated at the beginning of Section 5.3.2 after properties (a) and (b).

[1] Our notation is somewhat misleading since $\mathcal{L}^{(r+a,\infty)}$ and $\mathcal{R}^{(r+a,\infty)}$ both depend on r and not only on $r+a$. Since r is fixed in most of this section, this should not be confusing.

Lemma 5. *The collection* $(\Theta^{(a)}_z)_{z>0}$ *is a regular version of the conditional distributions of the triple*

$$\Big((X^{(r)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(r)}_a}, \widetilde{\mathcal{L}}^{(r,r+a)}, \widetilde{\mathcal{R}}^{(r,r+a)}\Big)$$

knowing that $\mathcal{Z}^{(r,r+a)} = z$.

This lemma is merely a reformulation of Proposition 4. The point is that the time-reversed process $(X_{\mathrm{L}_{r+a}-t})_{0\leqslant t\leqslant \mathrm{L}^{(r)}_a}$ is distributed as a Bessel process of dimension -5 started from $r+a$ and stopped upon hitting r (by property (b) stated at the beginning of Section 5.3.2). Recalling our notation **SR** for the spine reversal operation defined in (5.8), it follows that

$$\Big(\mathbf{SR}\Big((X^{(r)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(r)}_a}, \widetilde{\mathcal{L}}^{(r,r+a)}, \widetilde{\mathcal{R}}^{(r,r+a)}\Big), \mathcal{Z}^{(r,r+a)}\Big)$$

has the same distribution as $((V-r, \vartheta_r\widetilde{\mathcal{N}}, \vartheta_r\widetilde{\mathcal{N}}'), \mathcal{Z}^V)$, with the notation introduced before Proposition 4. The result of the lemma now follows from Proposition 4.

Since the distribution of $\mathcal{Z}^{(r,r+a)}$ is the same as the distribution of $\mathcal{Z}^V$ in the preceding section, it has a positive density with respect to Lebesgue measure, which we denote by $g_{r,a}(z)$. Recalling that $\mathcal{Z}^Y$ has density $zh_a(z)$, (5.14) gives the explicit expression

$$g_{r,a}(z) = \Big(\frac{r+a}{r}\Big)^3 e^{-\frac{3z}{2r^2}}\, z\, h_a(z) \tag{5.21}$$

where h_a is defined in (5.12).

On the other hand, the distribution of $\mathcal{Z}^{(r+a,\infty)}$ may be written in the form

$$(1-\varepsilon_{r,a})\,\delta_0(\mathrm{d}z) + \Upsilon_{r,a}(\mathrm{d}z)$$

where $\varepsilon_{r,a} \in [0,1]$ and the measure $\Upsilon_{r,a}$ is supported on $(0,\infty)$. Note that

$$\varepsilon_{r,a} = \Upsilon_{r,a}((0,\infty)) = \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} > 0) = 1 - \Big(\frac{a}{r+a}\Big)^3$$

where the last equality follows from Lemma 4.2 in [40], using the fact that $\mathbb{N}_x(0 < W_* \leqslant r) = \frac{3}{2}((x-r)^{-2} - x^{-2})$ for $x > r$. In particular, $\varepsilon_{r,a} \longrightarrow 0$ as $a \to \infty$.

Recall that $k_r(z)$ denotes the density of $\mathcal{Z}^{(r)}$ (cf. (5.20)). Since $\mathcal{Z}^{(r)} = \mathcal{Z}^{(r,r+a)} + \mathcal{Z}^{(r+a,\infty)}$, and $\mathcal{Z}^{(r,r+a)}$ and $\mathcal{Z}^{(r+a,\infty)}$ are independent, the conditional distributions of $\mathcal{Z}^{(r+a,\infty)}$ knowing that $\mathcal{Z}^{(r)} = z$ are defined in a canonical manner by

$$\nu_{r,a}(\mathrm{d}z' \,|\, z) = \frac{1}{k_r(z)}\Big((1-\varepsilon_{r,a})g_{r,a}(z)\,\delta_0(\mathrm{d}z') + g_{r,a}(z-z')\,\Upsilon_{r,a}(\mathrm{d}z')\Big).$$

In particular, we have for every $z > 0$,

$$\nu_{r,a}(\{0\} \,|\, z) = \frac{(1-\varepsilon_{r,a})g_{r,a}(z)}{k_r(z)},$$

and the explicit expression (5.21) can be used to verify that $g_{r,a}(z) \longrightarrow k_r(z)$ as $a \to \infty$. It follows that

$$\nu_{r,a}(\{0\} \,|\, z) \underset{a\to\infty}{\longrightarrow} 1. \tag{5.22}$$

Recall the scaling transformations θ_λ on snake trajectories defined in Section 5.2.2. It will also be useful to consider restriction operators which are defined as follows. For every $a > 0$, $\mathfrak{R}_a$ acts both on $\mathcal{W}_0^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ and on $\mathcal{W}_0 \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ by

$$\mathfrak{R}_a : \Big(\mathrm{w}, \sum_{i\in I} \delta_{(t_i,\omega_i)}, \sum_{j\in J} \delta_{(t'_j,\omega'_j)}\Big) \mapsto \Big((\mathrm{w}(t))_{t\leqslant \lambda^{(a)}(\mathrm{w})}, \sum_{i\in I, t_i \leqslant \lambda^{(a)}(\mathrm{w})} \delta_{(t_i,\omega_i)}, \sum_{j\in J, t'_j\leqslant \lambda^{(a)}(\mathrm{w})} \delta_{(t_j,\omega'_j)}\Big), \tag{5.23}$$

where $\lambda^{(a)}(\mathrm{w}) = \sup\{t \geqslant 0 : \mathrm{w}(t) \leqslant a\}$ for $\mathrm{w} \in \mathcal{W}_0^\infty$ or $\mathrm{w} \in \mathcal{W}_0$.

Proposition 6. *We can find a collection $(\Theta_z)_{z>0}$ of probability measures on $\mathcal{W}^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ that does not depend on r and is such that, for every $r > 0$, $(\Theta_z)_{z>0}$ is a regular version of the conditional distributions of the triple*

$$(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$$

knowing that $\mathcal{Z}^{(r)} = z$. This collection is unique if we impose the additional scaling invariance property: for every $\lambda > 0$ and $z > 0$, $\Theta_{\lambda z}$ is the image of Θ_z under the scaling transformation

$$\Sigma_\lambda : \Big(\mathrm{w}, \sum_{i\in I} \delta_{(t_i,\omega_i)}, \sum_{j\in J} \delta_{(t'_j,\omega'_j)}\Big) \mapsto \Big(\sqrt{\lambda}\mathrm{w}(\cdot/\lambda), \sum_{i\in I} \delta_{(\lambda t_i,\theta_\lambda(\omega_i))}, \sum_{j\in J} \delta_{(\lambda t'_j,\theta_\lambda(\omega'_j))}\Big).$$

Proof. Let $r > 0$, and let $(\Theta_{z,r})_{z>0}$ be a regular version of the conditional distributions of $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ knowing $\mathcal{Z}^{(r)} = z$. Our first goal is to verify that $(\Theta_{z,r})_{z>0}$ does not depend on r, except possibly on a values of z of zero Lebesgue measure. To this end, let $c > 0$ and let G be a measurable function on $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ such that $0 \leqslant G \leqslant 1$. Then, for every $a \geqslant c$, and for every nonnegative measurable function f on $(0,\infty)$,

$$\begin{aligned}\int \mathrm{d}z\, k_r(z)\, f(z)\Theta_{z,r}(G\circ\mathfrak{R}_c) &= \mathbb{E}[G((X_t^{(r)})_{0\leqslant t\leqslant \mathrm{L}_c^{(r)}}, \widetilde{\mathcal{L}}^{(r,r+c)}, \widetilde{\mathcal{R}}^{(r,r+c)})\, f(\mathcal{Z}^{(r)})] \\ &= \mathbb{E}\Big[\mathbb{E}[G((X_t^{(r)})_{0\leqslant t\leqslant \mathrm{L}_c^{(r)}}, \widetilde{\mathcal{L}}^{(r,r+c)}, \widetilde{\mathcal{R}}^{(r,r+c)}) \,|\, \mathcal{Z}^{(r,r+a)}]\, f(\mathcal{Z}^{(r,r+a)} + \mathcal{Z}^{(r+a,\infty)})\Big]\end{aligned}$$

where we use the fact that $\mathcal{Z}^{(r+a,\infty)}$ is independent of $((X_t^{(r)})_{0\leqslant t\leqslant \mathrm{L}_c^{(r)}}, \widetilde{\mathcal{L}}^{(r,r+c)}, \widetilde{\mathcal{R}}^{(r,r+c)}), \mathcal{Z}^{(r,r+a)})$ to write the last equality. By Lemma 5, we have

$$\mathbb{E}[G((X_t^{(r)})_{0\leqslant t\leqslant \mathrm{L}_c^{(r)}}, \widetilde{\mathcal{L}}^{(r,r+c)}, \widetilde{\mathcal{R}}^{(r,r+c)}) \,|\, \mathcal{Z}^{(r,r+a)}] = \Phi(\mathcal{Z}^{(r,r+a)})$$

where $\Phi(z) = \Theta_z^{(a)}(G\circ\mathfrak{R}_c)$. Using the explicit distribution of $\mathcal{Z}^{(r,r+a)}$ and $\mathcal{Z}^{(r+a,\infty)}$, we thus get

$$\begin{aligned}\int \mathrm{d}z\, k_r(z)\, f(z)\Theta_{z,r}(G\circ\mathfrak{R}_c) &= \int \mathrm{d}y\, g_{r,a}(y) \int ((1-\varepsilon_{r,a})\delta_0 + \Upsilon_{r,a})(\mathrm{d}y')\, f(y+y')\, \Theta_y^{(a)}(G\circ\mathfrak{R}_c) \\ &= \int \mathrm{d}z\, k_r(z)\, f(z) \int \nu_{r,a}(\mathrm{d}z' \,|\, z)\, \Theta_{z-z'}^{(a)}(G\circ\mathfrak{R}_c).\end{aligned}$$

It follows that we have, $\mathrm{d}z$ a.e.,

$$\Theta_{z,r}(G\circ\mathfrak{R}_c) = \int \nu_{r,a}(\mathrm{d}z' \,|\, z)\, \Theta_{z-z'}^{(a)}(G\circ\mathfrak{R}_c) = \nu_{r,a}(\{0\} \,|\, z)\, \Theta_z^{(a)}(F) + \kappa_{r,a}(z)$$

where the "remainder" $\kappa_{r,a}(z)$ is nonnegative and bounded above by $1 - \nu_{r,a}(\{0\} \mid z)$. Specializing to integer values of a and using (5.22), we get, $\mathrm{d}z$ a.e.,

$$\lim_{\mathbb{N} \ni k \to \infty} \Theta_z^{(k)}(G \circ \mathfrak{R}_c) = \Theta_{z,r}(G \circ \mathfrak{R}_c).$$

Since the left-hand side does not depend on r, we conclude that, for every $r, r' > 0$, we must have $\Theta_{z,r}(G \circ \mathfrak{R}_c) = \Theta_{z,r'}(G \circ \mathfrak{R}_c)$, $\mathrm{d}z$ a.e., and since this holds for any $c > 0$ and any function G, we conclude that $\Theta_{z,r} = \Theta_{z,r'}$, $\mathrm{d}z$ a.e. So, if we take $\bar{\Theta}_z = \Theta_{z,1}$, the collection $(\bar{\Theta}_z)_{z>0}$ satisfies the first part of the statement.

It remains to obtain the scaling invariance property. To this end, we first observe that the process

$$X_t^{\{\lambda\}} := \sqrt{\lambda}\, X_{t/\lambda}$$

remains a nine-dimensional Bessel process started from 0. Furthermore, with an obvious notation, we have $\mathrm{L}^{\{\lambda\}}_{r\sqrt{\lambda}} = \lambda \mathrm{L}_r$ for every $r > 0$. Then, it is straightforward to verify that the image of $\mathcal{L}$ under the transformation

$$\sum_{i \in I} \delta_{(t_i, \omega_i)} \mapsto \sum_{i \in I} \delta_{(\lambda t_i, \theta_\lambda(\omega_i))} \tag{5.24}$$

is, conditionally on $X^{\{\lambda\}}$, a Poisson point measure with intensity

$$2\,\mathrm{d}t\, \mathbb{N}_{X_t^{\{\lambda\}}}(\mathrm{d}\omega \cap \{W_* > 0\}).$$

It follows that the image of $\mathcal{L}^{(r)}$ under the scaling transformation (5.24) has the same distribution as $\mathcal{L}^{(r\sqrt{\lambda})}$.

We also note that, for every $x > 0$, we have $\mathcal{Z}_0(\theta_\lambda(\omega)) = \lambda\, \mathcal{Z}_0(\omega)$, $\mathbb{N}_x(\mathrm{d}\omega)$ a.e. By combining the preceding observations, we get that, for every $r > 0$, the image of the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ under the scaling transformation Σ_λ has the same distribution as $(X^{(r\sqrt{\lambda})}, \widetilde{\mathcal{L}}^{(r\sqrt{\lambda})}, \widetilde{\mathcal{R}}^{(r\sqrt{\lambda})})$, and moreover the exit measure at 0 associated with $\Sigma_\lambda(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ is $\lambda \mathcal{Z}^{(r)}$. By considering conditional distributions with respect to $\mathcal{Z}^{(r)}$ and using the first part of the proof, we obtain that $\Sigma_\lambda(\bar{\Theta}_z) = \bar{\Theta}_{\lambda z}$ for a.e. $z > 0$. A Fubini type argument allows us to single out a real $z_0 > 0$ such that the equality $\bar{\Theta}_{\lambda z_0} = \Sigma_\lambda(\bar{\Theta}_{z_0})$ holds for a.e. $\lambda > 0$. We then define, for every $z > 0$,

$$\Theta_z = \Sigma_{z/z_0}(\bar{\Theta}_{z_0}).$$

Clearly the collection $(\Theta_z)_{z>0}$ is also a regular version of the conditional distributions of the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ knowing that $\mathcal{Z}^{(r)} = z$ (for any $r > 0$). Furthermore, by construction, the equality $\Sigma_\lambda(\Theta_z) = \Theta_{\lambda z}$ holds for every $z > 0$ and $\lambda > 0$. This completes the proof, except for the uniqueness statement, which is easy and left to the reader. □

From now on, $(\Theta_z)_{z>0}$ is the unique collection satisfying the properties stated in Proposition 6. Thanks to the scaling invariance property, we can in fact define this collection without appealing to any conditioning. We consider the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ as defined at the beginning of the section, and recall the notation $\mathcal{Z}^{(r)}$ and the scaling operators Σ_λ in Proposition 6.

Proposition 7. *Let $r > 0$ and $z > 0$. Then Θ_z is the distribution of $\Sigma_{z/\mathcal{Z}^{(r)}}(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$.*

Proof. Let F be a nonnegative measurable function on $\mathcal{W}^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$, and recall that the distribution of $\mathcal{Z}^{(r)}$ has density $k_r(z)$. Then, using Proposition 6,

$$\mathbb{E}\Big[F\Big(\Sigma_{z/\mathcal{Z}^{(r)}}(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})\Big)\Big] = \int_0^\infty \mathrm{d}y\, k_r(y)\, \Theta_y(F \circ \Sigma_{z/y}) = \Theta_z(F),$$

since the image of Θ_y under $\Sigma_{z/y}$ is Θ_z. ☐

Proposition 7 is useful to derive almost sure properties of coding triples distributed according to Θ_z. We give an important example.

Corollary 8. *Let $z > 0$, and let $\mathcal{T}$ be the labeled tree associated with a coding triple distributed according to Θ_z. Write $(\Lambda_v)_{v\in\mathcal{T}}$ for the labels on $\mathcal{T}$ and $(\mathcal{E}_s)_{s\in\mathbb{R}}$ for the clockwise exploration of $\mathcal{T}$. Then,*

$$\lim_{|s|\to\infty} \Lambda_{\mathcal{E}_s} = \infty\,, \qquad a.s.$$

Proof. By Proposition 7, it suffices to prove the similar statement for the labeled tree associated with $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$, or even for the labeled tree associated with $(X, \mathcal{L}, \mathcal{R})$. In the latter case this follows from [40, Lemma 3.3]. ☐

5.3.4 The coding triple of the Brownian disk with a given height

The fact that the collection $(\Theta_z)_{z>0}$ has been uniquely defined will now allow us to make a canonical choice for the conditional distributions $(\Theta_z^{(a)})_{z>0}$ (until now, these conditional distributions were only defined up to a set of values of z of zero Lebesgue measure). This will be important later as we use $\Theta_z^{(a)}$ to construct the free pointed Brownian disk with perimeter z and height a.

Recall the restriction operator $\mathfrak{R}_a$ introduced in (5.23), and the notation $\lambda^{(a)}(\mathrm{w}) = \sup\{t \geqslant 0 : \mathrm{w}(t) \leqslant a\}$ for $\mathrm{w} \in \mathcal{W}_0^\infty$ or $\mathrm{w} \in \mathcal{W}_0$.

Proposition 9. *Let $a > 0$, and define a function $W_{*,(a)} : \mathcal{W}_0^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2 \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by*

$$W_{*,(a)}\Big(\mathrm{w}, \sum_{i\in I} \delta_{(t_i,\omega_i)}, \sum_{j\in J} \delta_{(t'_j,\omega'_j)}\Big) = \min\Big(\inf_{i\in I, t_i > \lambda^{(a)}(\mathrm{w})} W_*(\omega_i), \inf_{j\in J, t'_j > \lambda^{(a)}(\mathrm{w})} W_*(\omega'_j)\Big).$$

Then, we have $\Theta_z(W_{*,(a)} > 0) = \sqrt{\pi}\, 2^{1/2} 3^{-3/2}\, a^3\, z^{1/2}\, h_a(z)$ and $\Theta_z(W_{*,(a)} > 0) \longrightarrow 1$ as $a \to \infty$. Furthermore, we can choose the collection $(\Theta_z^{(a)})_{z>0}$ so that, for every $z > 0$, $\Theta_z^{(a)}$ is the pushforward of $\Theta_z(\cdot \mid W_{*,(a)} > 0)$ under $\mathfrak{R}_a$.

Proof. Let $r > 0$ and $a > 0$. For test functions f and F defined on $\mathbb{R}_+$ and on $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{W})^2$

respectively, we have from Proposition 6,

$$\begin{aligned}&\int \mathrm{d}z\, k_r(z)\, f(z)\, \Theta_z\Big(F\circ \mathfrak{R}_a\, \mathbf{1}_{\{W_{*,(a)}>0\}}\Big)\\ &= \mathbb{E}\Big[f(\mathcal{Z}^{(r)})\, F\Big((X^{(r)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(r)}_a}, \widetilde{\mathcal{L}}^{(r,r+a)}, \widetilde{\mathcal{R}}^{(r,r+a)}\Big)\, \mathbf{1}_{\{\mathcal{Z}^{(r+a,\infty)}=0\}}\Big]\\ &= \mathbb{E}\Big[f(\mathcal{Z}^{(r,r+a)})\, F\Big((X^{(r)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(r)}_a}, \widetilde{\mathcal{L}}^{(r,r+a)}, \widetilde{\mathcal{R}}^{(r,r+a)}\Big)\Big] \times \mathbb{P}(\mathcal{Z}^{(r+a,\infty)}=0)\\ &= \int \mathrm{d}z\, g_{r,a}(z)\, f(z)\, \Theta^{(a)}_z(F)\, \times \mathbb{P}(\mathcal{Z}^{(r+a,\infty)}=0),\end{aligned}$$

using Lemma 5 in the last equality. It follows that we have $\mathrm{d}z$ a.e.,

$$k_r(z)\, \Theta_z\Big(F\circ \mathfrak{R}_a\, \mathbf{1}_{\{W_{*,(a)}>0\}}\Big) = g_{r,a}(z)\, \mathbb{P}(\mathcal{Z}^{(r+a,\infty)}=0)\; \Theta^{(a)}_z(F).$$

For $F = 1$, we get that the equality $k_r(z)\, \Theta_z(W_{*,(a)} > 0) = g_{r,a}(z)\, \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} = 0) = g_{r,a}(z)\, (\frac{a}{r+a})^3$ holds $\mathrm{d}z$ a.e., but then, by a scaling argument using also the monotonicity of $\Theta_z(W_{*,(a)} > 0)$ in the variable a, it must hold for every $z > 0$ and $a > 0$. It follows that $\Theta_z(W_{*,(a)} > 0) = (k_r(z))^{-1} g_{r,a}(z)\, (\frac{a}{r+a})^3$, and the explicit formulas for $k_r(z)$ and $g_{r,a}(z)$ give the first assertion of the proposition. Furthermore, the previous display gives

$$\Theta^{(a)}_z(F) = \frac{\Theta_z\Big(F\circ \mathfrak{R}_a\, \mathbf{1}_{\{W_{*,(a)}>0\}}\Big)}{\Theta_z(W_{*,(a)}>0)},$$

$\mathrm{d}z$ a.e. The second assertion follows. □

In what follows, we assume that, for every $a > 0$, the collection $(\Theta^{(a)}_z)_{z>0}$ is chosen as in the preceding proposition, and that the collection $(\check{\Theta}^{(a)}_z)_{z>0}$ is then derived from $(\Theta^{(a)}_z)_{z>0}$ via the spine reversal operation. From the scaling properties of $(\Theta_z)_{z>0}$, one checks that, for every $\lambda > 0$, the pushforward of $\Theta^{(a)}_z$ under the scaling operator Σ_λ is $\Theta^{(\sqrt{\lambda}a)}_{\lambda z}$.

The following corollary, which relates the measures $\Theta^{(a)}_z$ when a varies (and z is fixed) is an immediate consequence of Proposition 9. Before stating this corollary, we note that both $\mathfrak{R}_a$ and $W_{*,(a)}$ still make sense as mappings defined on $\mathcal{W}_0 \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$.

Corollary 10. *Let $0 < a < a'$. Then we have*

$$\Theta^{(a')}_z(W_{*,(a)} > 0) = \frac{a^3 h_a(z)}{a'^3 h_{a'}(z)},$$

and $\Theta^{(a)}_z$ is the pushforward of $\Theta^{(a')}_z(\cdot \mid W_{*,(a)} > 0)$ under $\mathfrak{R}_a$.

We now use the collection $(\check{\Theta}^{(a)}_z)_{z>0}$ to construct a regular version of the conditional distributions of $\mathrm{tr}_0(\omega)$ under $\mathbb{N}_a$ knowing $\mathcal{Z}_0 = z$, for every $a > 0$ and $z > 0$. This regular version is a priori unique up to sets of values of z of zero Lebesgue measure, but for our purposes it is important that the conditional distribution is defined for *every* $z > 0$.

We fix $a > 0$ and consider a triple $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ distributed according to $\check{\Theta}^{(a)}_z$. As explained at the end of Section 5.2.4 (finite spine case), we can use this triple to construct a snake trajectory, which belongs to $\mathcal{S}_a$ and is denoted by $\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$. We write $\mathbb{N}^{(z)}_a$ for the distribution of the snake trajectory $\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$.

Proposition 11. *The collection* $(\mathbb{N}_a^{(z)})_{z>0}$ *forms a regular version of the conditional distributions of* $\mathrm{tr}_0(\omega)$ *under* $\mathbb{N}_a$ *knowing that* $\mathcal{Z}_0 = z$.

Proof. Recall the notation introduced before Proposition 2: Under the measure $\mathbb{N}_a(\mathrm{d}\omega)$, we can consider, for every $s \in (0, \sigma(\omega))$, the point measure $\mathcal{P}_{(s)}$ (resp. $\mathcal{P}'_{(s)}$) that gives the snake trajectories associated with the subtrees branching off the left side (resp. off the right side) of the ancestral line of the vertex $p_{(\omega)}(s)$ in the genealogical tree of ω. Also use the notation $\widetilde{\mathcal{P}}_{(s)}$ (resp. $\widetilde{\mathcal{P}}'_{(s)}$) for the point measure $\mathcal{P}_{(s)}$ (resp. $\mathcal{P}'_{(s)}$) "truncated at level 0". This makes sense if s is such that $W_s(t) > 0$ for $0 \leqslant t < \zeta_s$, which is the case we will consider. From Proposition 2, and using also (5.9), we have, for every nonnegative measurable functions f and F defined on $\mathbb{R}_+$ and on $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ respectively,

$$\mathbb{N}_a\Big(\int_0^\sigma \mathrm{d}L_s^0\, f(\mathcal{Z}_0)\, F(W_s, \widetilde{\mathcal{P}}_{(s)}, \widetilde{\mathcal{P}}'_{(s)})\Big) = \mathbb{E}\Big[f(\mathcal{Z}^Y)\, F(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')\Big], \tag{5.25}$$

where $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ and $\mathcal{Z}^Y$ are as in Section 5.3.2. Notice that, $\mathbb{N}_a(\mathrm{d}\omega)\mathrm{d}L_s^0(\omega)$ a.e., we have $\mathrm{tr}_0(\omega) = \Omega(W_s, \widetilde{\mathcal{P}}_{(s)}, \widetilde{\mathcal{P}}'_{(s)})$. Hence the previous identity also gives, for every nonnegative measurable function H on $\mathcal{S}$,

$$\mathbb{N}_a(\mathcal{Z}_0 f(\mathcal{Z}_0)\, H(\mathrm{tr}_0(\omega))) = \mathbb{E}[f(\mathcal{Z}^Y)\, H(\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'))]. \tag{5.26}$$

Since the density of $\mathcal{Z}^Y$ is $zh_a(z)$ and $\check{\Theta}_z^{(a)}$ is the conditional distribution of $(Y, \mathcal{M}, \mathcal{M}')$ given $\mathcal{Z}^Y = z$, the right-hand side can be written as

$$\int \mathrm{d}z\, zh_a(z)\, f(z)\, \check{\Theta}_z^{(a)}(H \circ \Omega) = \int \mathrm{d}z\, zh_a(z)\, f(z)\, \mathbb{N}_a^{(z)}(H),$$

by the very definition of $\mathbb{N}_a^{(z)}$. The statement of the proposition follows. □

From the scaling properties of the measures $\Theta_z^{(a)}$, we immediately get that, for every $\lambda > 0$, the pushforward of $\mathbb{N}_a^{(z)}$ under the scaling transformation θ_λ is $\mathbb{N}_{a\sqrt{\lambda}}^{(\lambda z)}$.

In view of further applications, we also note that the definition of the exit local time at 0 makes sense under $\mathbb{N}_a^{(z)}$. Precisely, one gets that, $\mathbb{N}_a^{(z)}(\mathrm{d}\omega)$ a.e., the limit

$$\widetilde{L}_t^0 := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s(\omega) < \varepsilon\}} \tag{5.27}$$

exists uniformly for $t \geqslant 0$, and $\widetilde{L}_\infty^0 = \widetilde{L}_\sigma^0 = z$. If $\mathbb{N}_a^{(z)}$ is replaced by $\mathbb{N}_a$ (and $\widehat{W}_s(\omega)$ by $\widehat{W}_s(\mathrm{tr}_0(\omega))$) this is just formula (5.4) in Section 5.2.2. So (5.27) is a conditional version of (5.4), which must therefore hold $\mathbb{N}_a^{(z)}$ a.e., at least for a.e. value of z. Then a scaling argument, using also the way we have defined the conditional distributions $\mathbb{N}_a^{(z)}$ and Corollary 10, shows that (5.27) indeed holds for every $z > 0$. We omit the details.

5.4 From coding triples to random metric spaces

5.4.1 The pseudo-metric functions associated with a coding triple

Let $(\mathrm{w}, \mathcal{P}, \mathcal{P}')$ be a coding triple satisfying the assumptions of Section 5.2.4 in the infinite spine case, and let $(\mathcal{T}, (\Lambda_v)_{v\in\mathcal{T}})$ be the associated labeled tree. We suppose here that labels take nonnegative values, $\Lambda_v \geqslant 0$ for every $v \in \mathcal{T}$, and we set $\mathcal{T}^\circ := \{v \in \mathcal{T} : \Lambda_v > 0\}$ and $\partial\mathcal{T} = \mathcal{T}\backslash\mathcal{T}^\circ$. We assume that $\partial\mathcal{T}$ is not empty and that all points of $\partial\mathcal{T}$ are leaves (points whose removal does not disconnect $\mathcal{T}$). In particular, $\mathcal{T}^\circ$ is dense in $\mathcal{T}$. We denote the clockwise exploration of $\mathcal{T}$ by $(\mathcal{E}_t)_{t\in\mathbb{R}}$, and we assume that either $\Lambda_{\mathcal{E}_t} \longrightarrow \infty$ as $|t| \to \infty$, or the set $\{t \in \mathbb{R} : \Lambda_{\mathcal{E}_t} = 0\}$ intersects both intervals $[K, \infty)$ and $(-\infty, -K]$, for every $K > 0$. This ensures that $\inf_{w\in[u,v]} \Lambda_w$ is attained for every "interval" $[u, v]$ of $\mathcal{T}$.

We define, for every $u, v \in \mathcal{T}^\circ$,

$$\Delta^\circ(u,v) := \begin{cases} \Lambda_u + \Lambda_v - 2\max\Big(\inf\limits_{w\in[u,v]} \Lambda_w, \inf\limits_{w\in[v,u]} \Lambda_w\Big) & \text{if } \max\Big(\inf\limits_{w\in[u,v]} \Lambda_w, \inf\limits_{w\in[v,u]} \Lambda_w\Big) > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.28}$$

We then let $\Delta(u, v)$, $u, v \in \mathcal{T}^\circ$ be the maximal symmetric function on $\mathcal{T}^\circ \times \mathcal{T}^\circ$ that is bounded above by Δ° and satisfies the triangle inequality:

$$\Delta(u,v) = \inf_{u_0=u,u_1,\ldots,u_p=v} \sum_{i=1}^{p} \Delta^\circ(u_{i-1}, u_i) \tag{5.29}$$

where the infimum is over all choices of the integer $p \geqslant 1$ and of the finite sequence $u_0, u_1, \ldots, u_p$ in $\mathcal{T}$ such that $u_0 = u$ and $u_p = v$. Then $\Delta(u, v) < \infty$ for every $u, v \in \mathcal{T}^\circ$. Indeed, a compactness argument shows that we can find finitely many points $u_0 = u, u_1, \ldots, u_{p-1}, u_p = v$ belonging to the geodesic segment $[\![u, v]\!]$ of $\mathcal{T}$ and such that $\Delta^\circ(u_{i-1}, u_i) < \infty$ for every $1 \leqslant i \leqslant p$.

Furthermore, the mapping $(u, v) \mapsto \Delta(u, v)$ is continuous on $\mathcal{T}^\circ \times \mathcal{T}^\circ$ (observe that $\Delta^\circ(u_n, u) \longrightarrow 0$ if $u_n \to u$ in $\mathcal{T}^\circ$, and use the triangle inequality). We note the trivial bound $\Delta^\circ(u, v) \geqslant |\Lambda_u - \Lambda_v|$, which also implies

$$\Delta(u,v) \geqslant |\Lambda_u - \Lambda_v|. \tag{5.30}$$

We will call $\Delta^\circ(u, v)$ and $\Delta(u, v)$ the *pseudo-metric functions associated with the triple* $(\mathrm{w}, \mathcal{P}, \mathcal{P}')$. From now on, let us assume that the function $(u, v) \mapsto \Delta(u, v)$ has a continuous extension to $\mathcal{T} \times \mathcal{T}$, which is therefore a pseudo-metric on $\mathcal{T}$. We will be interested in the resulting quotient metric space $\mathcal{T}/\approx$ where the equivalence relation $\approx$ is defined by saying that $u \approx v$ if and only if $\Delta(u, v) = 0$. By abuse of notation, we will write $\mathcal{T}/\{\Delta = 0\}$ instead of $\mathcal{T}/\approx$. We write Π for the canonical projection from $\mathcal{T}$ onto $\mathcal{T}/\{\Delta = 0\}$. We also write $\Lambda_x = \Lambda_u$ when $x \in \mathcal{T}/\{\Delta = 0\}$ and $u \in \mathcal{T}$ are such that $x = \Pi(u)$ (this is unambiguous by (5.30)).

If $x \in \mathcal{T}/\{\Delta = 0\}$ is such that $\Lambda_x > 0$, we can define a geodesic path starting from x in the following way. We pick $u \in \mathcal{T}$ such that $\Pi(u) = x$ and then $s \in \mathbb{R}$ such that $\mathcal{E}_s = u$. We then define $\gamma^{(s)} = (\gamma^{(s)}_r)_{0\leqslant r\leqslant \Lambda_x}$ by setting $\gamma^{(s)}_r = \Pi(\mathcal{E}_{\eta^{(s)}_r})$, with

$$\eta^{(s)}_r := \begin{cases} \inf\{t \geqslant s : \Lambda_{\mathcal{E}_t} = \Lambda_x - r\} & \text{if } \inf\{\Lambda_{\mathcal{E}_t} : t \geqslant s\} \leqslant \Lambda_x - r, \\ \inf\{t \leqslant s : \Lambda_{\mathcal{E}_t} = \Lambda_x - r\} & \text{if } \inf\{\Lambda_{\mathcal{E}_t} : t \geqslant s\} > \Lambda_x - r. \end{cases}$$

It is then a simple matter to verify that $\gamma^{(s)}$ is a geodesic path in $(\mathcal{T}/\{\Delta = 0\}, \Delta)$, which starts from x and ends at a point belonging to $\Pi(\partial\mathcal{T})$. On the other hand, the bound (5.30) shows that $\Delta(x, y) \geqslant \Lambda_x$ if $y \in \Pi(\partial\mathcal{T})$. It follows that $\Delta(x, \Pi(\partial\mathcal{T})) = \Lambda_x$ for every $x \in \mathcal{T}/\{\Delta = 0\}$. The path $\gamma^{(s)}$ is called a simple geodesic (see e.g. [67, Section 2.6] for the analogous definition in the Brownian map).

We finally note that $\mathcal{T}/\{\Delta = 0\}$ is a length space, meaning that the distance between two points is equal to the infimum of the lengths of paths connecting these two points. To get this property, just notice that, if $u, v \in \mathcal{T}^\circ$ and $\Delta^\circ(u, v) < \infty$, then $\Delta^\circ(u, v)$ coincides with the length of a path from $\Pi(u)$ to $\Pi(v)$, that is obtained by concatenating two simple geodesics starting from $\Pi(u)$ and $\Pi(v)$ respectively, up to the time when they merge. More explicitly, if $\Delta^\circ(u, v) = \Lambda_u + \Lambda_v - \inf_{w\in[u,v]} \Lambda_w$, and if the reals s' and s'' are such that $\mathcal{E}_{s'} = u$, $\mathcal{E}_{s''} = v$ and $[u, v] = \{\mathcal{E}_r : r \in [s', s'']\}$, then the concatenation of $(\Pi(\gamma^{(s')}_r), 0 \leqslant r \leqslant \Lambda_u - \inf_{w\in[u,v]} \Lambda_w)$ and $(\Pi(\gamma^{(s'')}_r), 0 \leqslant r \leqslant \Lambda_v - \inf_{w\in[u,v]} \Lambda_w)$ gives a continuous path from $\Pi(u)$ to $\Pi(v)$ with length $\Delta^\circ(u, v)$, which furthermore is contained in $\Pi([u, v])$.

5.4.2 The Brownian plane

As an illustration of the procedure described in the previous section, and in view of further developments, we briefly recall the construction of the Brownian plane given in [40]. We consider a (random) coding triple $(X, \mathcal{L}, \mathcal{R})$ distributed as in Section 5.3.3:

- $X = (X_t)_{t\geqslant 0}$ is a nine-dimensional Bessel process started from 0.
- Conditionally on X, $\mathcal{L}$ and $\mathcal{R}$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity
$$2\,\mathrm{d}t\,\mathbb{N}_{X_t}(\mathrm{d}\omega \cap \{W_* > 0\}).$$

It is easy to verify that the assumptions of Section 5.2.4 hold a.s. for $(X, \mathcal{L}, \mathcal{R})$, and thus we can associate an infinite labeled tree $(\mathcal{T}^p_\infty, (\Lambda_v)_{v\in\mathcal{T}^p_\infty})$ with this coding triple. The assumptions of the beginning of Section 5.4.1 also hold (notice that the condition $\lim_{|s|\to\infty} \Lambda_{\mathcal{E}_s} = \infty$ holds by [40, Lemma 3.3]), and we introduce the two pseudo-metric functions $\Delta^{p,\circ}(u, v)$ and $\Delta^p(u, v)$ defined for $u, v \in \mathcal{T}^{p,\circ}_\infty := \{v \in \mathcal{T}^p_\infty : \Lambda_v > 0\}$ via formulas (8.15) and (5.29). In that case, since the root of $\mathcal{T}^p_\infty$ is the only point with zero label, it is easy to see that at least one of the two infima $\inf_{w\in[u,v]} \Lambda_w$ and $\inf_{w\in[v,u]} \Lambda_w$ is positive, for any $u, v \in \mathcal{T}^{p,\circ}_\infty$. Furthermore, it is immediate to obtain that $\Delta^{p,\circ}(u, v)$ and $\Delta^p(u, v)$ can be extended continuously to $\mathcal{T}^p_\infty$ – in fact in that case we can define $\Delta^{p,\circ}(u, v)$ for every $u, v \in \mathcal{T}^p_\infty$ by the quantity in the first line of (8.15), and use formula (5.29) to define $\Delta^p(u, v)$ for every $u, v \in \mathcal{T}^p_\infty$. One can prove [40, Section 3.2] that, for any $u, v \in \mathcal{T}^p_\infty$, $\Delta^p(u, v) = 0$ if and only if $\Delta^{p,\circ}(u, v) = 0$.

The Brownian plane $\mathbb{BP}_\infty$ is defined as the quotient space $\mathcal{T}^p_\infty/\{\Delta^p = 0\}$ equipped with the distance induced by Δ^p (for which we keep the same notation Δ^p) and with the volume measure which is the pushforward of the volume measure on $\mathcal{T}^p_\infty$ under the canonical projection. We note that $\mathbb{BP}_\infty$ comes with a distinguished point ρ, which is the image of the root of $\mathcal{T}^p_\infty$ under the canonical projection. Furthermore, we have $\Delta^p(\rho, x) = \Lambda_x$ for every $x \in \mathbb{BP}_\infty$.

The Brownian plane is scale invariant in the following sense. If E is a pointed measure metric space and $\lambda > 0$, we write $\lambda \cdot E$ for the same space E with the metric multiplied by the factor λ and the volume measure multiplied by the factor λ^4 (and the same distinguished point). Then, for every $\lambda > 0$, $\lambda \cdot \mathbb{BP}_\infty$ has the same distribution as $\mathbb{BP}_\infty$.

5.4.3 The pointed Brownian disk with given perimeter and height

In this section, we explain how a free pointed Brownian disk with perimeter z and height a is constructed from the measure $\mathbb{N}^{(z)}_a$ defined in Section 5.3.4. This is basically an adaptation of [71], but we provide some details in view of further developments.

We start with a preliminary result. Recall the notation $h_a(z)$ and $p_z(a)$ in Proposition 3.

Proposition 12. *For any nonnegative measurable functions G and f defined respectively on $\mathcal{S}$ and on $\mathbb{R}_+$, for every $z > 0$, we have*

$$z^{-2}\mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}t\, G(W^{[t]})\, f(\widehat{W}_t)\Big) = \int_0^\infty \mathrm{d}a\, p_z(a)\, f(a)\, \mathbb{N}^{(z)}_a(G).$$

Proof. We may assume that both G and f are bounded and continuous. Then the argument is very similar to the proof of Proposition 3 (ii) (which we recover when $G = 1$). Let g be a nonnegative measurable function on $\mathbb{R}_+$. We use the re-rooting formula (5.5), and then Proposition 3 (i), to get

$$\begin{aligned}\mathbb{N}^*\Big(\int_0^\sigma \mathrm{d}t\, G(W^{[t]})\, f(\widehat{W}_t) g(\mathcal{Z}^*_0)\Big) &= 2\int_0^\infty \mathrm{d}a\, \mathbb{N}_a\Big(\mathcal{Z}_0\, G(\mathrm{tr}_0(\omega))\, f(a)\, g(\mathcal{Z}_0)\Big)\\ &= 2\int_0^\infty \mathrm{d}a\, f(a) \int_0^\infty \mathrm{d}z\, h_a(z)\, zg(z)\, \mathbb{N}_a(G(\mathrm{tr}_0(\omega)) \mid \mathcal{Z}_0 = z)\\ &= 2\int_0^\infty \mathrm{d}z\, zg(z) \int_0^\infty \mathrm{d}a\, h_a(z)\, f(a)\, \mathbb{N}^{(z)}_a(G).\end{aligned}$$

On the other hand, the left-hand side is also equal to

$$\sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2} g(z)\, \mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}t\, G(W^{[t]})\, f(\widehat{W}_t)\Big).$$

Since this holds for any function g, we must have, $\mathrm{d}z$ a.e.,

$$z^{-2}\mathbb{N}^{*,z}\Big(\int_0^\sigma \mathrm{d}t\, G(W^{[t]})\, f(\widehat{W}_t)\Big) = \sqrt{\frac{8\pi}{3}} z^{3/2} \int_0^\infty \mathrm{d}a\, h_a(z)\, f(a)\, \mathbb{N}^{(z)}_a(G).$$

This is the identity of the proposition, except that we get it only $\mathrm{d}z$ a.e. However, a scaling argument shows that both sides of the preceding display are continuous functions of z, which gives the desired result for every $z > 0$. □

Recall from Section 8.2.1 the definition of the probability measure $\overline{\mathbb{N}}^{*,z}(\mathrm{d}\omega\mathrm{d}t)$ on $\mathcal{S} \times \mathbb{R}_+$, and, for $(\omega, t) \in \mathcal{S} \times \mathbb{R}_+$, write $U(\omega, t) = t$. We can then rewrite the identity of Proposition 12 in the form

$$\overline{\mathbb{N}}^{*,z}(f(\widehat{\omega}_U)G(\omega^{[U]})) = \int_0^\infty \mathrm{d}a\, p_z(a)\, f(a)\, \mathbb{N}_a^{(z)}(G). \tag{5.31}$$

Let us now come to Brownian disks. We write $\mathbb{K}$, resp. $\mathbb{K}^\bullet$, for the space of all compact measure metric spaces, resp. pointed compact measure metric spaces, equipped with the Gromov-Hausdorff-Prokhorov topology. Theorem 1 in [71] provides a measurable mapping $\Xi : \mathcal{S} \longrightarrow \mathbb{K}$ such that the distribution of $\Xi(\omega)$ under $\mathbb{N}^{*,z}(\mathrm{d}\omega)$ is the law of the free Brownian disk with perimeter z. Let us briefly recall the construction of this mapping, which is essentially an adaptation of the procedure of Section 5.4.1 to the finite spine case. Under $\mathbb{N}^{*,z}(\mathrm{d}\omega)$, each vertex u of the the genealogical tree $\mathcal{T}_{(\omega)}$ receives a nonnegative label $\ell_u(\omega)$, and we define the "boundary" $\partial\mathcal{T}_{(\omega)} := \{u \in \mathcal{T}_{(\omega)} : \ell_u(\omega) = 0\}$. We also set $\mathcal{T}^\circ_{(\omega)} := \mathcal{T}_{(\omega)} \backslash \partial\mathcal{T}_{(\omega)}$, and for every $u, v \in \mathcal{T}^\circ_{(\omega)}$, we define $\Delta^{d,\circ}(u, v)$ and $\Delta^d(u, v)$ by the exact analogs of formulas (8.15) and (5.29), where $\mathcal{T}$ and $\mathcal{T}^\circ$ are replaced by $\mathcal{T}_{(\omega)}$ and $\mathcal{T}^\circ_{(\omega)}$ respectively, and the labels $(\Lambda_u)_{u\in\mathcal{T}}$ are replaced by $(\ell_u)_{u\in\mathcal{T}_{(\omega)}}$ (recall the definition of intervals on $\mathcal{T}_{(\omega)}$ in Section 5.2.1).

A key technical point (Proposition 31 in [71]) is to verify that the mapping $(u, v) \mapsto \Delta^{\mathrm{d}}(u, v)$ can be extended continuously (in a unique way) to $\mathcal{T}_{(\omega)} \times \mathcal{T}_{(\omega)}$, $\mathbb{N}^{*,z}(\mathrm{d}\omega)$ a.s. We then define $\Xi(\omega)$ as the quotient space $\mathcal{T}_{(\omega)}/\{\Delta^d = 0\}$, which is equipped with the metric induced by Δ^d and with a volume measure which is the pushforward of the volume measure on $\mathcal{T}_{(\omega)}$ under the canonical projection. In the next definition, we use the notation $\mathbf{p}_\omega$ for the composition of the canonical projection $p_{(\omega)}$ from $[0, \sigma(\omega)]$ onto $\mathcal{T}_{(\omega)}$ with the canonical projection from $\mathcal{T}_{(\omega)}$ onto $\Xi(\omega)$.

Définition 13. *The distribution of the free pointed Brownian disk with perimeter z is the law of the random measure metric space $(\Xi(\omega), \Delta^d)$ pointed at the point $\mathbf{p}_\omega(t)$, under the probability measure $\overline{\mathbb{N}}^{*,z}(\mathrm{d}\omega\,\mathrm{d}t)$.*

The consistency of this definition of the free pointed Brownian disk with the one in [22] follows from the results in [71] (to be specific, [71, Theorem 1] identifies the law of $\Xi(\omega)$ under $\mathbb{N}^{*,z}$ as the distribution of the free Brownian disk of perimeter z, and one can then use formula (42) in [72] to see that Definition 13 is consistent with the definition of the free pointed Brownian disk in [22]). From [20, 22] one knows that, $\mathbb{N}^{*,z}(\mathrm{d}\omega)$ a.s., $\Xi(\omega)$ is homeomorphic to the unit disk. This makes it possible to define the boundary $\partial\Xi(\omega)$ of $\Xi(\omega)$, and this boundary is identified in [71] with the image of $\partial\mathcal{T}_{(\omega)}$ under the canonical projection. Furthermore, a.s. for every $u \in \mathcal{T}_{(\omega)}$, the label $\ell_u(\omega)$ is equal to the distance in $\Xi(\omega)$ from (the equivalence class of) u to the boundary $\partial\Xi(\omega)$.

We note that the random measure metric space $\Xi(\omega)$ pointed at $\mathbf{p}_\omega(t)$ is in fact a function of the re-rooted snake trajectory $\omega^{[t]}$, because $\Xi(\omega)$ is canonically identified to $\Xi(\omega^{[t]})$, and $\mathbf{p}_\omega(t)$ is mapped to $\mathbf{p}_{\omega^{[t]}}(0)$ in this identification. To simplify notation, we may thus write $\Xi^\bullet(\omega^{[t]})$ for the random metric space $\Xi(\omega)$ pointed at $\mathbf{p}_\omega(t)$, or equivalently for $\Xi(\omega^{[t]})$ pointed at $\mathbf{p}_{\omega^{[t]}}(0)$.

In the following developments, it will be convenient to write $\mathbb{D}_z^\bullet$ for a free pointed Brownian disk with perimeter z and $\partial\mathbb{D}_z^\bullet$ for the boundary of $\mathbb{D}_z^\bullet$. With a slight abuse of notation, we keep the notation Δ^d for the distance on $\mathbb{D}_z^\bullet$. By definition, the height H_z of $\mathbb{D}_z^\bullet$ is the distance from the distinguished point to the boundary. From the preceding interpretation of $\Xi(\omega)$ pointed at $\mathbf{p}_\omega(t)$ as a function of the re-rooted snake trajectory $\omega^{[t]}$, we get that, for any nonnegative measurable function Φ on $\mathbb{K}^\bullet$, for any nonnegative measurable function h on $\mathbb{R}_+$,

$$\mathbb{E}[\Phi(\mathbb{D}_z^\bullet)\, h(H_z)] = \overline{\mathbb{N}}^{*,z}\Big(\Phi(\Xi^\bullet(\omega^{[U]}))\, h(\widehat{\omega}_U)\Big)$$

using the same notation as in (5.31) and noting that $\ell_{p_{(\omega)}(U)}(\omega) = \widehat{\omega}_U$ by definition. By (5.31), we can rewrite this as

$$\mathbb{E}[\Phi(\mathbb{D}_z^\bullet)\, h(H_z)] = \int_0^\infty \mathrm{d}a\, p_z(a)\, h(a)\, \mathbb{N}_a^{(z)}(\Phi \circ \Xi^\bullet). \tag{5.32}$$

The next proposition readily follows from (5.32).

Proposition 14. *The height H_z of $\mathbb{D}_z^\bullet$ is distributed according to the density $p_z(a)$. Furthermore, the conditional distribution of $\mathbb{D}_z^\bullet$ knowing that $H_z = a$ is the law of $\Xi^\bullet(\omega)$ under $\mathbb{N}_a^{(z)}$. By definition, this is the distribution of the free pointed Brownian disk with perimeter z and height a.*

At this point, we should note that the definition of $\Xi^\bullet(\omega)$ requires the continuous extension of the mapping $(u,v) \mapsto \Delta^d(u,v)$ from $\mathcal{T}^\circ_{(\omega)} \times \mathcal{T}^\circ_{(\omega)}$ to $\mathcal{T}_{(\omega)} \times \mathcal{T}_{(\omega)}$. Proposition 31 in [71] and Proposition 11 above give the existence of this continuous extension $\mathbb{N}_a^{(z)}(\mathrm{d}\omega)$ a.s. for a.a. $z > 0$, for every fixed $a > 0$, and then one can use scaling arguments and Corollary 10 to get the same result for *every* $z > 0$ and $a > 0$ (this is used in Proposition 14). Similar considerations allow us to deduce the following property from Proposition 32 (iii) in [71]: $\mathbb{N}_a^{(z)}(\mathrm{d}\omega)$ a.s., for every $u, v \in \partial\mathcal{T}_{(\omega)}$, we have $\Delta^d(u,v) = 0$ if and only if $\ell_w(\omega) > 0$ for every $w \in]u,v[$, or for every $w \in]v,u[$. Finally, from [71, Proposition 30 (iv)], we also get that, $\mathbb{N}_a^{(z)}(\mathrm{d}\omega)$ a.s. for every $u, v \in \mathcal{T}^\circ_{(\omega)}$, we have $\Delta^d(u,v) = 0$ if and only if $\Delta^{d,\circ}(u,v) = 0$.

It will be useful to introduce the uniform measure on the boundary $\partial\mathbb{D}_z^\bullet$. There exists a measure μ_z on $\partial\mathbb{D}_z^\bullet$ with total mass equal to z, such that, a.s. for any continuous function φ on $\mathbb{D}_z^\bullet$, we have

$$\langle \mu_z, \varphi\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{D}_z^\bullet} \mathrm{Vol}(\mathrm{d}x)\, \mathbf{1}_{\{\Delta^d(x,\partial\mathbb{D}_z^\bullet)<\varepsilon\}}\, \varphi(x)$$

where $\mathrm{Vol}(\mathrm{d}x)$ denotes the volume measure on $\partial\mathbb{D}_z^\bullet$ (see [71, Corollary 37]). The preceding approximation and the definition of μ_z are also valid for the free pointed Brownian disk with perimeter z and height a: In fact, if this Brownian disk is constructed as $\Xi^\bullet(\omega)$ under $\mathbb{N}_a^{(z)}$ (as in Proposition 14), we define μ_z by setting

$$\langle \mu_z, \varphi\rangle = \int_0^\infty \mathrm{d}\widetilde{L}^0_s\, \varphi(\mathbf{p}_\omega(s)), \tag{5.33}$$

where the exit local time $\widetilde{L}^0_s$ is defined under $\mathbb{N}_a^{(z)}$ as explained at the end of Section 5.3.4. Note that the approximation formula for μ_z then reduces to formula (5.27), thanks to the interpretation of labels as distances to the boundary.

For our purposes in the next sections, it will be important to consider a free Brownian disk (with perimeter z and height a) equipped with a distinguished point chosen uniformly on the boundary. To this end, we proceed as follows. We start from a triple $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ distributed according to $\check{\Theta}_z^{(a)}$. As explained before Proposition 11, the random snake trajectory $\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ is then distributed according to $\mathbb{N}_a^{(z)}$, and (Proposition 14) we obtain a free pointed Brownian disk $\mathbb{D}_z^{\bullet,a}$ with perimeter z and height a by setting

$$\mathbb{D}_z^{\bullet,a} := \Xi^{\bullet}\Big(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})\Big). \tag{5.34}$$

In this construction, $\mathbb{D}_z^{\bullet,a}$ comes with a distinguished vertex of its boundary, namely the one corresponding to the top of the spine of the tree coded by $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$. We denote this special point by α.

In the next proposition, we verify that α is (in a certain sense) uniformly distributed over $\partial\mathbb{D}_z^{\bullet,a}$. To give a precise statement of this property, it is convenient to introduce the doubly pointed measure metric space $\mathbb{D}_z^{\bullet\bullet,a}$ which is obtained by viewing α as a second distinguished point of $\mathbb{D}_z^{\bullet,a}$.

Proposition 15. *Let F be a nonnegative measurable function on the space of all doubly pointed compact measure metric spaces, which is equipped with the Gromov-Hausdorff-Prokhorov topology. Then,*

$$\mathbb{E}[F(\mathbb{D}_z^{\bullet\bullet,a})] = \frac{1}{z}\,\mathbb{E}\Big[\int \mu_z(\mathrm{d}x)\, F\Big([\mathbb{D}_z^{\bullet,a}, x]\Big)\Big],$$

where μ_z is the uniform measure on $\partial\mathbb{D}_z^{\bullet,a}$, and we use the notation $[\mathbb{D}_z^{\bullet,a}, x]$ for the doubly pointed space obtained by equipping $\mathbb{D}_z^{\bullet,a}$ with the second distinguished point x.

Proof. Let $(Y, \mathcal{M}, \mathcal{M}')$ be distributed as in Proposition 2 (or in Section 5.3.2). As previously, let $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ be obtained by truncating the atoms of $\mathcal{M}$ and $\mathcal{M}'$ at level 0. Let us introduce the notation $\|\widetilde{\mathcal{M}}\|$ for the sum of the quantities $\sigma(\omega_i)$ over all atoms (t_i, ω_i) of $\widetilde{\mathcal{M}}$. Note that $\|\widetilde{\mathcal{M}}\|$ is the time at which the clockwise exploration of the tree associated with the triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ (which coincides with the genealogical tree of $\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$) visits the top of the spine. We start by observing that, for any nonnegative measurable function G on $\mathcal{S} \times \mathbb{R}_+$, for any nonnegative measurable function h on $\mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}\Big[G\Big(\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'), \|\widetilde{\mathcal{M}}\|\Big)\, h(\mathcal{Z}^Y)\Big] &= \mathbb{N}_a\Big(\int_0^\infty \mathrm{d}L_r^0\, G\Big(\mathrm{tr}_0(\omega), \int_0^r \mathrm{d}s\,\mathbf{1}_{\{\zeta_{(\omega_s)} \leqslant \tau_0(\omega_s)\}}\Big)\, h(\mathcal{Z}_0)\Big)\\ &= \mathbb{N}_a\Big(\int_0^\infty \mathrm{d}\widetilde{L}_r^0\, G(\mathrm{tr}_0(\omega), r)\, h(\mathcal{Z}_0)\Big)\end{aligned}$$

where $\widetilde{L}_r^0$ is defined under $\mathbb{N}_a$ as in formula (5.4). The first equality follows from Proposition 2 as in the derivation of (5.25) and (5.26) above, and the second equality is just a time change formula. By conditioning on $\mathcal{Z}^Y = z$ in the left-hand side and on $\mathcal{Z}_0 = z$ in the right-hand side, we get

$$\mathbb{E}\Big[G(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}), \|\widetilde{\mathcal{M}}^{(z)}\|)\Big] = \frac{1}{z}\,\mathbb{N}_a^{(z)}\Big(\int_0^\infty \mathrm{d}\widetilde{L}_r^0\, G(\omega, r)\Big).$$

Now recall that $\mathbb{D}_z^{\bullet,a} = \Xi^\bullet(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}))$ and that $\mathbb{D}_z^{\bullet\bullet,a}$ is obtained by assigning to $\mathbb{D}_z^{\bullet,a}$ a second distinguished point equal to $\alpha = \mathbf{p}_\Omega(\|\widetilde{\mathcal{M}}^{(z)}\|)$, where $\mathbf{p}_\Omega$ denotes the composition of the canonical projection onto the genealogical tree of $\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ with the projection from this genealogical tree onto $\Xi^\bullet(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}))$. Thanks to these observations, we obtain that

$$\mathbb{E}[F(\mathbb{D}_z^{\bullet\bullet,a})] = \frac{1}{z}\,\mathbb{N}_a^{(z)}\Big(\int_0^\infty \mathrm{d}\widetilde{L}_r^0\, F\Big([\Xi^\bullet(\omega), \mathbf{p}_\omega(r)]\Big)\Big),$$

and the desired result follows since (5.33) shows that μ_z is the pushforward of the measure $\mathrm{d}\widetilde{L}_r^0$ under the mapping $r \mapsto \mathbf{p}_\omega(r)$. □

In view of forthcoming limit results where the distinguished point of $\mathbb{D}_z^{\bullet,a}$ is "sent to infinity", it will be convenient to introduce the random pointed measure metric space $\bar{\mathbb{D}}_z^{\bullet,a}$ defined from the doubly pointed space $\mathbb{D}_z^{\bullet\bullet,a}$ by forgetting the first distinguished point. So $\bar{\mathbb{D}}_z^{\bullet,a}$ is pointed at a point which is uniformly distributed over its boundary.

5.4.4 Infinite-volume Brownian disks

For every $z > 0$ and $a > 0$, we keep the notation $\mathbb{D}_z^{\bullet,a}$ for a pointed Brownian disk with perimeter z and height a. We may and will assume that $\mathbb{D}_z^{\bullet,a}$ is constructed from a coding triple distributed according to $\check{\Theta}_z^{(a)}$ as in formula (5.34), or equivalently, using the spine reversal operation, from a coding triple distributed according to $\Theta_z^{(a)}$. The idea is now to let $a \to \infty$ and to use the "convergence" of $\Theta_z^{(a)}$ to Θ_z in order to get the convergence of $\mathbb{D}_z^{\bullet,a}$ as $a \to \infty$. As we already mentioned, for a precise statement of this convergence, it will be more convenient to replace $\mathbb{D}_z^{\bullet,a}$ by $\bar{\mathbb{D}}_z^{\bullet,a}$. The limit, which will be denoted by $\mathbb{D}_z^\infty$, is a random pointed locally compact measure metric space, which we call the infinite-volume Brownian disk with perimeter z.

Let us start by explaining the construction of $\mathbb{D}_z^\infty$, which follows the general pattern of Section 5.4.1. We consider a coding triple $((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}')$ with distribution Θ_z. From $((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}')$, Section 5.2.4 allows us to construct an infinite tree $\mathcal{T}_\infty^i$ equipped with nonnegative labels $(\Lambda_v)_{v\in\mathcal{T}_\infty^i}$, such that labels on the spine are given by the process $(\rho_t)_{t\geqslant 0}$. Note that the assumptions in Section 5.4.1 hold in particular thanks to Corollary 8.

We set $\mathcal{T}_\infty^{i,\circ} := \{v \in \mathcal{T}_\infty^i : \Lambda_v > 0\}$ and $\partial\mathcal{T}_\infty^i := \mathcal{T}_\infty^i \backslash \mathcal{T}_\infty^{i,\circ} = \{v \in \mathcal{T}_\infty^i : \Lambda_v = 0\}$. We define the pseudo-metric functions $\Delta^{i,\circ}(u,v)$ and $\Delta^i(u,v)$ on $\mathcal{T}_\infty^{i,\circ} \times \mathcal{T}_\infty^{i,\circ}$ as explained in Section 5.4.1.

Lemma 16. (i) *The mapping $(u,v) \mapsto \Delta^i(u,v)$ has a.s. a continuous extension to $\mathcal{T}_\infty^i \times \mathcal{T}_\infty^i$.*
(ii) *A.s., for every $u, v \in \mathcal{T}_\infty^{i,\circ}$, the property $\Delta^i(u,v) = 0$ holds if and only if $\Delta^{i,\circ}(u,v) = 0$.*
(iii) *A.s., for every $u, v \in \partial\mathcal{T}_\infty^i$, the property $\Delta^i(u,v) = 0$ holds if and only if $\Lambda_w > 0$ for every $w \in]u,v[$, or for every $w \in]v,u[$.*

Proof. Thanks to Proposition 9 and Corollary 8, it is enough to verify that properties analogous to (i),(ii),(iii) hold when $\mathcal{T}_\infty^i$ is replaced by the labeled tree associated with a coding triple distributed according to $\Theta_z^{(a)}$, for some fixed $a > 0$. But then this is a consequence of the similar results in [71], as it was explained in the discussion after the statement of Proposition 14. □

We let $\mathbb{D}_z^\infty$ denote the quotient space $\mathcal{T}_\infty^i/\{\Delta^i = 0\}$, which is equipped with the metric induced by Δ^i. The volume measure on $\mathbb{D}_z^\infty$ is (as usual) the pushforward of the volume measure on $\mathcal{T}_\infty^i$. We also distinguish a special point α_∞ of $\mathbb{D}_z^\infty$, which is the equivalence class of the root of $\mathcal{T}_\infty^i$.

Définition 17. *The random pointed measure metric space* $(\mathbb{D}_z^\infty, \Delta^i)$ *is the infinite-volume Brownian disk with perimeter* z.

As in Section 5.4.1, labels Λ_x make sense for $x \in \mathbb{D}_z^\infty$, and Λ_x is equal to the distance from x to the "boundary" $\partial\mathbb{D}_z^\infty$, which is defined as the set of all points of $\mathbb{D}_z^\infty$ with zero label. From the scaling properties of the collection $(\Theta_z)_{z>0}$, one also gets that $\lambda \cdot \mathbb{D}_z^\infty$ is distributed as $\mathbb{D}_{\lambda^2 z}^\infty$, for every $\lambda > 0$.

In Section 5.6 below, we will verify that this definition of the infinite-volume Brownian disk is consistent with [14]. It then follows from [14, Corollary 3.13] that $\mathbb{D}_z^\infty$ is homeomorphic to the complement of the open unit disk of the plane, so that the boundary $\partial\mathbb{D}_z^\infty$ can be understood in a topological sense. We will not use this result, which may also be derived from our interpretation of complements of hulls of the Brownian plane as infinite-volume Brownian disks in Section 5.5.1.

Remark. We could have defined the infinite-volume Brownian disk without distinguishing a special point of the boundary. The reason for distinguishing α_∞ comes from the use of the local Gromov-Hausdorff convergence in Theorem 21 below, which requires dealing with pointed spaces.

For the convergence result to follow, it is convenient to deal with the following definition of "balls": for every $h > 0$,

$$\mathcal{B}_h(\mathbb{D}_z^\infty) = \{v \in \mathbb{D}_z^\infty : \Delta^i(v, \partial\mathbb{D}_z^\infty) \leqslant h\},$$

and

$$\mathcal{B}_h(\bar{\mathbb{D}}_z^{\bullet,a}) = \{v \in \bar{\mathbb{D}}_z^{\bullet,a} : \Delta^{(a)}(v, \partial\bar{\mathbb{D}}_z^{\bullet,a}) \leqslant h\},$$

where we use the notation $\Delta^{(a)}$ for the metric on $\mathbb{D}_z^{\bullet,a}$. We view both $\mathcal{B}_h(\mathbb{D}_z^\infty)$ and $\mathcal{B}_h(\bar{\mathbb{D}}_z^{\bullet,a})$ as compact measure metric spaces, which are pointed at α_∞ and α respectively. The compactness of $\mathcal{B}_h(\mathbb{D}_z^\infty)$ is a consequence of the fact that the set $\{u \in \mathcal{T}_\infty^i : \Lambda_u \leqslant h\}$ is compact, by Corollary 8.

Proposition 18. *Let* $z > 0$ *and* $h > 0$. *There exists a function* $(\varepsilon(a), a > 0)$ *with* $\varepsilon(a) \longrightarrow 0$ *as* $a \to \infty$, *such that, for every* $a > 0$, *we can define on the same probability space both the infinite-volume Brownian disk* $\mathbb{D}_z^\infty$ *and the pointed Brownian disk* $\bar{\mathbb{D}}_z^{\bullet,a}$, *in such a way that*

$$\mathbb{P}(\mathcal{B}_{h'}(\bar{\mathbb{D}}_z^{\bullet,a}) = \mathcal{B}_{h'}(\mathbb{D}_z^\infty), \text{ for every } 0 \leqslant h' \leqslant h) \geqslant 1 - \varepsilon(a).$$

In other words, when a large, one can couple the spaces $\bar{\mathbb{D}}_z^{\bullet,a}$ and $\mathbb{D}_z^\infty$ so that their tubular neighborhoods of the boundary (of any fixed radius h) are isometric except on a set of small probability.

Proof. Let $a > 0$, and consider a triple $((\rho_t^{(a)})_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ with distribution $\Theta_z^{(a)}$. As it was explained before Proposition 15, this triple allows us to construct the Brownian disk $\bar{\mathbb{D}}_z^{\bullet,a}$ of perimeter z and height a pointed at a boundary point. To be specific, the triple $((\rho_t^{(a)})_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ codes a random compact tree $\mathcal{T}^{(a)}$ equipped with labels $(\Lambda_v^{(a)})_{v\in\mathcal{T}^{(a)}}$.

The pseudo-metric functions $\Delta^{(a),\circ}(u,v)$ and $\Delta^{(a)}(u,v)$ are then defined as in Section 5.4.1 for $u,v \in \mathcal{T}^{(a)}$ such that $\Lambda^{(a)}_u > 0$ and $\Lambda^{(a)}_v > 0$. The function $(u,v) \mapsto \Delta^{(a)}$ is extended by continuity to $\mathcal{T}^{(a)} \times \mathcal{T}^{(a)}$, and the resulting quotient metric space pointed at the root of $\mathcal{T}^{(a)}$ is the pointed Brownian disk $\bar{\mathbb{D}}^{\bullet,a}_z$ – here we observe that the distinguished point α corresponds to the root and not to the top of the spine of $\mathcal{T}^{(a)}$, because the effect of dealing with the triple $((\rho^{(a)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ instead of its image under the spine reversal transformation (5.8) interchanges the roles of the root and the top of the spine (see the comments after (5.8)).

Recall the restriction operator $\mathfrak{R}_a$ in (5.23).

Lemma 19. *We can couple the triple $((\rho^{(a)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ distributed according to $\Theta^{(a)}_z$ and the triple $((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}')$ distributed according to Θ_z so that the property*

$$\Big((\rho^{(a)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)}\Big) = \mathfrak{R}_a\Big((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}'\Big) \tag{5.35}$$

holds with probability tending to 1 as $a \to \infty$.

Proof. It suffices to verify that the variation distance between $\Theta^{(a)}_z$ and $\mathfrak{R}_a(\Theta_z)$ tends to 0 as $a \to \infty$. This is an easy consequence of Proposition 9. Indeed, let A be a measurable subset of $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$. Then,

$$\begin{aligned}\Theta_z(\mathfrak{R}_a^{-1}(A)) &= \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} > 0\}) + \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} = 0\}) \\ &= \Theta_z(W_{*,(a)} > 0)\, \Theta^{(a)}_z(A) + \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} = 0\}),\end{aligned}$$

by Proposition 9. It follows that the variation distance between $\Theta^{(a)}_z$ and $\mathfrak{R}_a(\Theta_z)$ is bounded above by $1 - \Theta_z(W_{*,(a)} > 0)$, which tends to 0 as $a \to \infty$. □

It will be convenient to write $\mathcal{T}^{i,(a)}_\infty$ for the (labeled) compact tree derived from $\mathcal{T}^i_\infty$ by removing the part of the spine above height $\lambda^{(a)}_\infty := \sup\{t \geqslant 0 : \rho_t = a\}$ (and of course the subtrees branching off this part of the spine). We can also view $\mathcal{T}^{i,(a)}_\infty$ as the labeled tree coded by $\mathfrak{R}_a((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}')$. On the event where (5.35) holds, we can therefore also identify $\mathcal{T}^{i,(a)}_\infty$ with the labeled tree $\mathcal{T}^{(a)}$ coded by $((\rho^{(a)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$, and this identification is used in the next lemma and its proof.

From now on, we assume that the triples $((\rho^{(a)}_t)_{0\leqslant t\leqslant \mathrm{L}^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ and $((\rho_t)_{t\geqslant 0}, \mathcal{Q}, \mathcal{Q}')$ are coupled as in Lemma 19, and that $\mathbb{D}^{\bullet,a}_z$ and $\mathbb{D}^\infty_z$ are constructed from these triples as explained above.

Lemma 20. *Let $h > 0$. Set*

$$A = \max\{\Delta^i(x,y) : x,y \in \mathbb{D}^\infty_z, \Lambda_x \leqslant h, \Lambda_y \leqslant h\}.$$

On the intersection of the event where (5.35) *holds with the event where*

$$\inf\{\Lambda_v : v \in \mathcal{T}^i_\infty \backslash \mathcal{T}^{i,(a)}_\infty\} \geqslant A + h + 1 \tag{5.36}$$

we have

$$\Delta^i(v,w) = \Delta^{(a)}(v,w)$$

for every $v, w \in \mathcal{T}^{(a)}$ *such that* $\Lambda_v \leqslant h$ *and* $\Lambda_w \leqslant h$.

Remark. The statement of the lemma makes sense because on the event where (5.35) holds, the trees $\mathcal{T}^{(a)}$ and $\mathcal{T}^{i,(a)}_\infty$ are identified (as explained before the statement of the lemma), and so $\Delta^{(a)}(v,w)$ and $\Delta^i(v,w)$ both make sense when $v, w \in \mathcal{T}^{(a)} = \mathcal{T}^{i,(a)}_\infty$.

The statement of the proposition follows from Lemma 20. Indeed, by Corollary 8,

$$\inf\{\Lambda_u : u \in \mathcal{T}^i_\infty \backslash \mathcal{T}^{i,(a)}_\infty\} \underset{a\to\infty}{\longrightarrow} +\infty \ , \ \text{a.s.}$$

and so, when a is large, the property (5.36) will hold except on a set of small probability. Also, by Lemma 19, we know that the property (5.35) holds outside an event of small probability. Note that, when (5.36) holds, labels do not vanish on $\mathcal{T}^i_\infty \backslash \mathcal{T}^{i,(a)}_\infty$, and so the "boundary" (set of points with zero label) of $\mathcal{T}^i_\infty$ is identified with the boundary of $\mathcal{T}^{(a)}$. When (5.35) and (5.36) both hold, the conclusion of Lemma 20 shows that $\mathcal{B}_{h'}(\mathbb{D}^\infty_z)$ and $\mathcal{B}_{h'}(\bar{\mathbb{D}}^{\bullet,a}_z)$ are isometric, for every $0 \leqslant h' \leqslant h$.

Proof of Lemma 20. Throughout the proof we assume that both (5.35) and (5.36) hold, so that $\mathcal{T}^{(a)}$ and $\mathcal{T}^{i,(a)}_\infty$ are identified. If $u, v \in \mathcal{T}^{i,(a)}_\infty = \mathcal{T}^{(a)}$, we use the notation $[u,v]_{\mathcal{T}^i_\infty}$ for the interval from u to v in $\mathcal{T}^i_\infty$, and similarly $[u,v]_{\mathcal{T}^{(a)}}$ for the same interval in $\mathcal{T}^{(a)}$. We note that either $[u,v]_{\mathcal{T}^i_\infty} \subset \mathcal{T}^{i,(a)}_\infty$, and then $[u,v]_{\mathcal{T}^i_\infty} = [u,v]_{\mathcal{T}^{(a)}}$, or $[u,v]_{\mathcal{T}^i_\infty} \not\subset \mathcal{T}^{i,(a)}_\infty$, and then $[u,v]_{\mathcal{T}^i_\infty}$ is the union of $[u,v]_{\mathcal{T}^{(a)}}$ and $\mathcal{T}^i_\infty \backslash \mathcal{T}^{i,(a)}_\infty$.

We use the notation $\mathcal{T}^{i,(a),\circ}_\infty = \{u \in \mathcal{T}^{i,(a)}_\infty : \Lambda_u > 0\}$. We first observe that, if $u, v \in \mathcal{T}^{i,(a),\circ}_\infty$, we have

$$\Delta^{(a),\circ}(u,v) \leqslant \Delta^{i,\circ}(u,v). \tag{5.37}$$

Let us explain this bound. Since labels do not vanish on $\mathcal{T}^i_\infty \backslash \mathcal{T}^{i,(a)}_\infty$, it is immediate that $\Delta^{(a),\circ}(u,v) = \infty$ if and only if $\Delta^{i,\circ}(u,v) = \infty$. So we may assume that both are finite and then (5.37) directely follows from the definition of these quantities and the fact that we have always $[u,v]_{\mathcal{T}^{(a)}} \subset [u,v]_{\mathcal{T}^i_\infty}$ (and labels are the same on $\mathcal{T}^{(a)}$ and $\mathcal{T}^{i,(a)}_\infty$).

Next suppose that $u, v \in \mathcal{T}^{i,(a),\circ}_\infty$ satisfy also $\Lambda_u \leqslant h$ and $\Lambda_v \leqslant h$. Recall the definition (5.29) of $\Delta^i(u,v)$ as an infimum involving all possible choices of $u_0, u_1, \ldots, u_p$ in $\mathcal{T}^{i,\circ}_\infty$ such that $u_0 = u$ and $u_p = v$. We claim that in this definition we can restrict our attention to the case when $u_0, u_1, \ldots, u_p$ satisfy $\Lambda_{u_j} < A + h + 1$ for every $1 \leqslant j \leqslant p$, and therefore $u_0, u_1, \ldots, u_p \in \mathcal{T}^{i,(a)}_\infty$, by (5.36). To see this, suppose that $\Lambda_{u_k} \geqslant A + h + 1$, for some $k \in \{1, \ldots, p-1\}$. Then, from the definition of A, we have also $\Lambda_{u_k} \geqslant \Delta^i(u,v) + h + 1$. Hence, using the bound (5.30), we get

$$\sum_{j=1}^{p} \Delta^{i,\circ}(u_{j-1}, u_j) \geqslant |\Lambda_{u_k} - \Lambda_u| \geqslant (\Delta^i(u,v) + h + 1) - h = \Delta^i(u,v) + 1,$$

so that we may disregard the sequence $u_0, u_1, \ldots, u_p$ in the infimum defining $\Delta^i(u,v)$.

By the previous considerations and (5.37), we get, for $u, v \in \mathcal{T}_\infty^{i,(a),\circ}$ such that $\Lambda_u \leqslant h$ and $\Lambda_v \leqslant h$,

$$\Delta^i(u,v) = \inf_{\substack{u_0=u,u_1,\dots,u_p=v\\ u_1,\dots,u_{p-1}\in\mathcal{T}_\infty^{i,(a)}}} \sum_{j=1}^{p} \Delta^{i,\circ}(u_{j-1},u_j) \geqslant \inf_{\substack{u_0=u,u_1,\dots,u_p=v\\ u_1,\dots,u_{p-1}\in\mathcal{T}^{(a)}}} \sum_{j=1}^{p} \Delta^{(a),\circ}(u_{j-1},u_j) = \Delta^{(a)}(u,v). \tag{5.38}$$

We now want to argue that we have indeed the equality $\Delta^{(a)}(u,v) = \Delta^i(u,v)$. To this end it is enough to show that, for any sequence $u_0 = u, u_1, \dots, u_p = v$ in $\mathcal{T}^{(a)}$ such that

$$\sum_{j=1}^{p} \Delta^{(a),\circ}(u_{j-1},u_j) < \Delta^{(a)}(u,v) + 1, \tag{5.39}$$

we have in fact $\Delta^{(a),\circ}(u_{j-1},u_j) = \Delta^{i,\circ}(u_{j-1},u_j)$ for every $j \in \{1,\dots,p\}$ (this will entail that the two infima in (5.38) are equal). We argue by contradiction and suppose that $\Delta^{(a),\circ}(u_{j-1},u_j) < \Delta^{i,\circ}(u_{j-1},u_j)$ for some $j \in \{1,\dots,p\}$. This means that we have

$$\inf_{w\in[u_{j-1},u_j]_{\mathcal{T}_\infty^i}} \Lambda_w < \inf_{w\in[u_{j-1},u_j]_{\mathcal{T}^{(a)}}} \Lambda_w,$$

or the same with $[u_{j-1},u_j]$ replaced by $[u_j,u_{j-1}]$. However, $[u_{j-1},u_j]_{\mathcal{T}_\infty^i}$ can be different from $[u_{j-1},u_j]_{\mathcal{T}^{(a)}}$ only if $[u_{j-1},u_j]_{\mathcal{T}_\infty^i}$ is the union of $[u_{j-1},u_j]_{\mathcal{T}^{(a)}}$ and $\mathcal{T}_\infty^i \backslash \mathcal{T}_\infty^{i,(a)}$, and we get

$$\inf_{w\in[u_{j-1},u_j]_{\mathcal{T}^{(a)}}} \Lambda_w \geqslant \inf_{w\in\mathcal{T}_\infty^i\backslash\mathcal{T}_\infty^{i,(a)}} \Lambda_w \geqslant A + h + 1.$$

This implies in particular that $\Lambda_{u_j} \geqslant A + h + 1$, and by the same argument as above this gives a contradiction with (5.39). This completes the proof of the lemma. $\square$

In the next statement, we use the local Gromov-Hausdorff-Prokhorov convergence for pointed locally compact measure length spaces, as defined in [3].

Theorem 21. *We have*

$$\bar{\mathbb{D}}_z^{\bullet,a} \xrightarrow[a\to\infty]{\text{(d)}} \mathbb{D}_z^\infty$$

in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.

Proof. The statement of the theorem is an immediate consequence of Proposition 18. In fact, it is enough to verify that, for every $h > 0$, the closed ball of radius h centered at the distinguished point α of $\bar{\mathbb{D}}_z^{\bullet,a}$ converges in distribution to the corresponding ball in $\mathbb{D}_z^\infty$ as $a \to \infty$, in the sense of the Gromov-Hausdorff-Prokhorov convergence for pointed compact measure metric spaces. However, this readily follows from the coupling obtained in Proposition 18, since the closed ball of radius h centered at α is obviously contained in $\mathcal{B}_h(\bar{\mathbb{D}}_z^{\bullet,a})$ and similarly in the limiting space. $\square$

We conclude this section with a couple of almost sure properties of the infinite-volume Brownian disk $\mathbb{D}_z^\infty$ that can be derived from our approach. First, from the analogous result for the disk

$\mathbb{D}_z^{\bullet,a}$ (see the remarks after Proposition 14) and the coupling in Proposition 18, one easily obtains the existence of the uniform measure μ_z^∞ on $\partial\mathbb{D}_z^\infty$, which is a measure of total mass z satisfying

$$\langle \mu_z^\infty, \varphi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{D}_z^\infty} \mathrm{Vol}(\mathrm{d}x)\, \mathbf{1}_{\{\Delta^i(x,\partial\mathbb{D}_z^\infty)<\varepsilon\}}\, \varphi(x),$$

for any continuous function φ on $\mathbb{D}_z^\infty$, a.s. In particular the volume of the tubular neighborhhood of radius ε of $\partial\mathbb{D}_z^\infty$ behaves like $z\varepsilon^2$ when $\varepsilon \to 0$.

Our construction of $\mathbb{D}_z^\infty$ is well suited to the analysis of geodesics to the boundary. Write $(\mathcal{E}_s^i)_{s\in\mathbb{R}}$ for the clockwise exploration of the tree $\mathcal{T}_\infty^i$, and set

$$s_0 := \min\{s \in \mathbb{R} : \Lambda_{\mathcal{E}_s^i} = 0\}\,,\ s_1 := \max\{s \in \mathbb{R} : \Lambda_{\mathcal{E}_s^i} = 0\}.$$

Also set $x_0 := \Pi^i(\mathcal{E}_{s_0}) = \Pi^i(\mathcal{E}_{s_1})$.

Proposition 22. *Almost surely, there exists $h_0 > 0$ such that, for every $x \in \mathbb{D}_z^\infty$ with $\Delta^i(x, \partial\mathbb{D}_z^\infty) > h_0$, any geodesic from x to $\partial\mathbb{D}_z^\infty$ hits $\partial\mathbb{D}_z^\infty$ at x_0.*

The proof shows more precisely that all geodesics to $\partial\mathbb{D}_z^\infty$ starting outside a sufficiently large ball coalesce before hitting the boundary.

Proof. Recall the notation $\mathcal{T}_\infty^{i,(a)}$ in the proof of Proposition 18. By Corollary 8, we may choose a large enough so that labels Λ_v do not vanish on $\mathcal{T}_\infty^i \backslash \mathcal{T}_\infty^{i,(a)}$. Then we may take $h_0 = \max\{\Lambda_v : v \in \mathcal{T}_\infty^{i,(a)}\}$. To verify this, fix $x \in \mathbb{D}_z^\infty$ such that $\Delta^i(x, \partial\mathbb{D}_z^\infty) > h_0$, then we may write $x = \Pi^i(v)$ with $v \in \mathcal{T}_\infty^i \backslash \mathcal{T}_\infty^{i,(a)}$, and we have $\Delta^i(x, \partial\mathbb{D}_z^\infty) = \Lambda_v$. Consider a simple geodesic $\gamma = (\gamma_r)_{0\leqslant r\leqslant \Lambda_v}$ from x to $\partial\mathbb{D}_z^\infty$ constructed as in Section 5.4.1. Then it is straightforward to verify that $\gamma_{\Lambda_v} = x_0$. To complete the proof, we just need the fact that any geodesic from x to $\partial\mathbb{D}_z^\infty$ is a simple geodesic. This follows via Theorem 29 below from the analogous result in the Brownian plane, which is itself a consequence of the study of geodesics in the Brownian map [63]. We omit the details. ☐

One may also consider geodesic rays in the infinite Brownian disk (a geodesic ray $\gamma = (\gamma_t)_{t\geqslant 0}$ is an infinite geodesic path). In a way analogous to the case of the Brownian plane (see [39, Theorem 18]) one obtains that any two geodesic rays in $\mathbb{D}_z^\infty$ coalesce in finite time. Again this can be deduced from the Brownian plane result via Theorem 29, but this also follows, with some more work, from the alternative construction of the infinite Brownian disk presented in Section 5.6 below.

5.4.5 The Brownian half-plane

In this section, we define the Brownian half-plane and show that it is the tangent cone in distribution of the free pointed Brownian disk at a point chosen uniformly on its boundary. Let us start with the definition. We consider a coding triple $(R, \mathcal{P}, \mathcal{P}')$, where $R = (R_t)_{t\in[0,\infty)}$ is a three-dimensional Bessel process started from 0, and, conditionally on R, $\mathcal{P}$ and $\mathcal{P}'$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $2\,\mathrm{d}t\,\mathbb{N}_{R_t}(\mathrm{d}\omega)$. For every $r > 0$, we set $\mathbf{L}_r := \sup\{t \geqslant 0 : R_t = r\}$ (as in Section 5.3.2), and we let $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$ stand for the point measures $\mathcal{P}$ and $\mathcal{P}'$ truncated at level 0.

Following Section 5.2.4, we can use the coding triple $(R, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$ to construct a tree $\mathcal{T}_\infty^{hp}$ equipped with nonnegative labels $(\Lambda_v)_{v\in\mathcal{T}_\infty^{hp}}$. In contrast with the measures Θ_z used to define the infinite-volume Brownian disk, there is no conditioning on the total exit measure at 0, which is here infinite a.s., as it can be seen from a scaling argument. There are subtrees carrying zero labels that branch off the right side or the left side of the spine at arbitrary high levels, so that labels along the clockwise exploration of $\mathcal{T}_\infty^{hp}$ vanish in both intervals $(-\infty, -K]$ and $[K, \infty)$, for any $K > 0$.

We then follow the general procedure of Section 5.4.1. We set $\mathcal{T}_\infty^{hp,\circ} := \{v \in \mathcal{T}_\infty^{hp} : \Lambda_v > 0\}$ and $\partial\mathcal{T}_\infty^{hp} := \mathcal{T}_\infty^{hp}\backslash\mathcal{T}_\infty^{hp,\circ}$, and we let $\Delta^{hp,\circ}(u,v)$ and $\Delta^{hp}(u,v)$, for $u, v \in \mathcal{T}_\infty^{hp,\circ}$, be the pseudo-metric functions associated with the triple $(R, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$ as in Section 5.4.1.

Lemma 23. (i) *The mapping $(u,v) \mapsto \Delta^{hp}(u,v)$ has a.s. a continuous extension to $\mathcal{T}_\infty^{hp} \times \mathcal{T}_\infty^{hp}$.*
(ii) *A.s., for every $u, v \in \mathcal{T}_\infty^{hp,\circ}$, the property $\Delta^{hp}(u,v) = 0$ holds if and only if $\Delta^{hp,\circ}(u,v) = 0$.*
(iii) *A.s., for every $u, v \in \partial\mathcal{T}_\infty^{hp}$, the property $\Delta^{hp}(u,v) = 0$ holds if and only if $\Lambda_w > 0$ for every $w \in]u,v[$, or for every $w \in]v,u[$.*

Proof. Property (i) can be derived by minor modifications of the proof of [71, Proposition 31], noting that we may restrict our attention to the bounded subtree obtained by truncating $\mathcal{T}_\infty^{hp}$ at height $\mathbf{L}_r$ for some $r > 0$. We omit the details. As for (ii) and (iii), there is an additional complication due to the fact that it is not immediately clear why we can restrict our attention to a bounded subtree. Let us explain the argument for (iii), which is the property we use below. The fact that $\Lambda_w > 0$ for every $w \in]u,v[$ implies $\Delta^{hp}(u,v) = 0$ is easy and left to the reader. Suppose then that u, v are distinct points of $\partial\mathcal{T}_\infty^{hp}$ are such that $\Delta^{hp}(u,v) = 0$. Without loss of generality we can assume that $[u,v]$ is compact, and we then have to check that $\Lambda_w > 0$ for every $w \in]u,v[$. Recall the notation $[[u,\infty[[$ for the unique geodesic ray from u in the tree $\mathcal{T}_\infty^{hp}$, and $]]u,\infty[[= [[u,\infty[[\backslash\{u\}$. We claim that, for every $\delta > 0$, we can find points $u' \in]]u,\infty[[$ and $v' \in]]v,\infty[[$ such that $\Lambda_{u'} < \delta$ and $\Lambda_{v'} < \delta$ and there exist $w_0 = u', w_1, \ldots, w_p = v' \in [u,v]$ such that

$$\sum_{i=1}^{p} \Delta^{hp,\circ}(w_{i-1}, w_i) < \delta. \tag{5.40}$$

If the claim holds, we can use [71, Proposition 32 (ii)] to see that necessarily $\Lambda_w > 0$ for every $w \in]u,v[$ (the point is the fact that all w_i's belong to $[u,v]$, and thus we are dealing with a compact subtree of $\mathcal{T}_\infty^{hp}$). So it remains to prove our claim. First note that we can find $u' \in]]u,\infty[[$ and $v' \in]]v,\infty[[$ such that $\Lambda_{u'} < \delta/2$, $\Lambda_{v'} < \delta/2$ and $\Delta^{hp}(u',v') < \delta/2$ and in particular there exist $w_0, w_1, \ldots, w_p \in \mathcal{T}_\infty^{hp,\circ}$ such that (5.40) holds with δ replaced by $\delta/2$. It may happen that some of the w_i's do not belong to $[u,\infty)$, but then we can replace u' by w_{j+1}, where $j = \max\{i : w_i \notin [u,\infty)\}$, noting that necessarily $w_{j+1} \in]]u,\infty[[$ (otherwise $\Delta^{hp,\circ}(w_j, w_{j+1})$ would be ∞) and $\Lambda_{w_{j+1}} < \delta$ by the bound $|\Lambda_{w_i} - \Lambda_{w_{i-1}}| \leqslant \Delta^{hp,\circ}(w_{i-1}, w_i)$ for $1 \leqslant i \leqslant j+1$. Therefore we can assume that all

w_i's belong to $[u, \infty)$, and then a symmetric argument shows that we can assume that they all belong to $[u, v]$ as desired. □

We set $\mathbb{H}_\infty = \mathcal{T}_\infty^{hp}/\{\Delta^{hp} = 0\}$, and we let Π^{hp} denote the canonical projection from $\mathcal{T}_\infty^{hp}$ onto $\mathbb{H}_\infty$. We equip $\mathbb{H}_\infty$ with the distance induced by Δ^{hp} and the volume measure which is the pushforward of the volume measure on $\mathcal{T}_\infty^{hp}$ under the canonical projection. We observe that $\mathbb{H}_\infty$ has a distinguished vertex, namely the root ρ of $\mathcal{T}_\infty^{hp}$ (or bottom of the spine). By Lemma 23 (iii), the equivalence class of ρ in the quotient $\mathcal{T}_\infty^{hp}/\{\Delta^{hp} = 0\}$ must be a singleton, since there are points of $\mathcal{T}_\infty^{hp}$ with zero label arbitrarily close to ρ, both on the left side and on the right side of the spine.

Définition 24. *The random pointed locally compact measure metric space* $\mathbb{H}_\infty$ *is called the Brownian half-plane.*

At the end of Section 5.6, we will explain why this definition is consistent with the one found in [14] or in [53]. The Brownian half-plane enjoys the same scale invariance property as the Brownian plane: Recalling the notation $\lambda \cdot E$ introduced in Section 5.4.2, $\lambda \cdot \mathbb{H}_\infty$ has the same distribution as $\mathbb{H}_\infty$, for every $\lambda > 0$. The boundary $\partial\mathbb{H}_\infty$ is defined by $\partial\mathbb{H}_\infty := \Pi^{hp}(\partial\mathcal{T}_\infty^{hp})$ (one can prove that $\mathbb{H}_\infty$ is homeomorphic to the usual half-plane and then $\partial\mathbb{H}_\infty$ is also the set of all points of $\mathbb{H}_\infty$ that have no neighborhood homeomorphic to an open disk, but we do not need these facts here). As noted in Section 5.4.1, for any $v \in \mathcal{T}_\infty^{hp}$, Λ_v is equal to the distance from $\Pi^{hp}(v)$ to the boundary $\partial\mathbb{H}_\infty$.

Let $r > 0$ and let $\mathcal{T}_{\infty,r}^{hp}$ be the closed subset of $\mathcal{T}_\infty^{hp}$ consisting of the part $[0, \mathbf{L}_r]$ of the spine and the subtrees branching off $[0, \mathbf{L}_r]$.

Lemma 25. *We have*

$$\inf_{v \notin \mathcal{T}_{\infty,r}^{hp}} \Delta^{hp}(\rho, v) > 0 \ , \quad a.s.$$

Proof. We argue by contradiction and assume that there is a sequence $(u_n)_{n\geqslant 1}$ in the complement of $\mathcal{T}_{\infty,r}^{hp}$ such that $\Delta^{hp}(\rho, u_n) \longrightarrow 0$ as $n \to \infty$. Suppose that infinitely many points of this sequence belong to $[\rho, \infty)$. Let $v_{(r)}$ be the last point of $\mathcal{T}_{\infty,r}^{hp} \cap \partial\mathcal{T}_\infty^{hp}$ visited by the exploration of $\mathcal{T}_\infty^{hp}$, and note that $\Pi^{hp}(\rho) \neq \Pi^{hp}(v_{(r)})$, by Lemma 23 (iii). Then an argument very similar to the proof of Lemma 23 (iii) shows that we can find another sequence $(v_n)_{n\geqslant 1}$ with $v_n \in [[v_{(r)}, \infty[[$ and such that we still have $\Delta^{hp}(\rho, v_n) \longrightarrow 0$ as $n \to \infty$. In particular, $\Lambda_{v_n} \longrightarrow 0$, and this implies that $v_n \longrightarrow v_{(r)}$ in $\mathcal{T}_\infty^{hp}$, and thus $\Delta^{hp}(v_{(r)}, v_n) \longrightarrow 0$ as $n \to \infty$. Finally we get $\Delta^{hp}(\rho, v_{(r)}) = 0$, which is a contradiction. The case when infinitely many points of this sequence belong to $(\infty, \rho]$ is treated in a symmetric manner. □

It follows from Lemma 25 that $\Pi^{hp}(\mathcal{T}_{\infty,r}^{hp})$ contains a ball of positive radius centered at ρ in $\mathbb{H}_\infty$. Then, by scale invariance, we have a.s.

$$\lim_{r\to\infty} \Big(\inf_{v \notin \mathcal{T}_{\infty,r}^{hp}} \Delta^{hp}(\rho, v) \Big) = +\infty.$$

This implies in particular that $\mathbb{H}_\infty$ is boundedly compact (any ball centered at ρ is contained in the image of a compact subtree of $\mathcal{T}_\infty^{hp}$ under Π^{hp}).

Our next goal is to prove that $\mathbb{H}_\infty$ is the tangent cone in distribution of the pointed Brownian disk at a point chosen uniformly on its boundary – this will eventually allow us to make the connection with previous definitions of the Brownian half-plane. Recall from the end of Section 5.4.3 the notation $\bar{\mathbb{D}}_z^{\bullet,a}$ for the pointed measure metric space obtained from $\mathbb{D}_z^{\bullet\bullet,a}$ by "forgetting" the first distinguished point (so $\bar{\mathbb{D}}_z^{\bullet,a}$ is pointed at a point chosen uniformly on its boundary).

Theorem 26. *Let $z > 0$ and $a > 0$. We have*

$$\lambda \cdot \bar{\mathbb{D}}_z^{\bullet,a} \xrightarrow[\lambda\to\infty]{\text{(d)}} \mathbb{H}_\infty,$$

in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.

We give below the proof of Theorem 26 for $a = 1$, but a scaling argument yields the general case. Before we proceed to the proof of Theorem 26, we start with some preliminary estimates. We consider again a triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ distributed as explained at the beginning of Section 5.3.2 with $a = 1$. Recall that the random path Y is defined on the interval $[0, T^Y]$, that $Y_0 = 1$ and $T^Y = \inf\{t \geqslant 0 : Y_t = 0\}$. For every $\varepsilon \in (0, 1)$, we also set

$$T_\varepsilon := \inf\{t \geqslant 0 : Y_t = \varepsilon\}.$$

We let $\widetilde{\mathcal{M}}^\varepsilon(\mathrm{d}t\mathrm{d}\omega)$, resp. $\widetilde{\mathcal{M}}'^\varepsilon(\mathrm{d}t\mathrm{d}\omega)$, be the image of $\mathbf{1}_{[T_\varepsilon,T^Y]}(t)\,\widetilde{\mathcal{M}}(\mathrm{d}t\mathrm{d}\omega)$, resp. of $\mathbf{1}_{[T_\varepsilon,T^Y]}(t)\,\widetilde{\mathcal{M}}'(\mathrm{d}t\mathrm{d}\omega)$, under the mapping $(t,\omega) \mapsto (t - T_\varepsilon, \omega)$. We also set $Y_t^\varepsilon := Y_{T_\varepsilon + t}$ for $0 \leqslant t \leqslant T^Y - T_\varepsilon$. Recall that

$$\mathcal{Z}^Y = \int \widetilde{\mathcal{M}}(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega) + \int \widetilde{\mathcal{M}}'(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega),$$

and also set

$$\mathcal{Z}^{Y,\varepsilon} := \int \widetilde{\mathcal{M}}^\varepsilon(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega) + \int \widetilde{\mathcal{M}}'^\varepsilon(\mathrm{d}t\mathrm{d}\omega)\mathcal{Z}_0(\omega).$$

Set $\Gamma^\varepsilon := (Y^\varepsilon, \widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{M}}'^\varepsilon)$, and observe that Γ^ε is a coding triple in the sense of Section 5.2.4. Moreover, the conditional distribution of Γ^ε knowing $\mathcal{Z}^{Y,\varepsilon} = z$ is $\check{\Theta}_z^{(\varepsilon)}$. Our first goal is to show that the conditional distribution of Γ^ε given $\mathcal{Z}^Y = z$ is close to its unconditional distribution when $\varepsilon \to 0$.

From (5.13), we have

$$\mathbb{E}[e^{-\lambda \mathcal{Z}^Y}] = \left(1 + \sqrt{2\lambda/3}\right)^{-3}, \quad \mathbb{E}[e^{-\lambda \mathcal{Z}^{Y,\varepsilon}}] = \left(1 + \varepsilon\sqrt{2\lambda/3}\right)^{-3}.$$

Furthermore, we may write $\mathcal{Z}^Y = \mathcal{Z}^{Y,\varepsilon} + \widehat{\mathcal{Z}}^{Y,\varepsilon}$, where $\mathcal{Z}^{Y,\varepsilon}$ and $\widehat{\mathcal{Z}}^{Y,\varepsilon}$ are independent (more precisely, $\widehat{\mathcal{Z}}^{Y,\varepsilon}$ is independent of Γ^ε). Hence

$$\mathbb{E}[e^{-\lambda \widehat{\mathcal{Z}}^{Y,\varepsilon}}] = \left(\frac{1 + \varepsilon\sqrt{2\lambda/3}}{1 + \sqrt{2\lambda/3}}\right)^3. \tag{5.41}$$

The distribution of $\widehat{\mathcal{Z}}^{Y,\varepsilon}$ can be written in the form

$$\varepsilon^3\,\delta_0(\mathrm{d}y) + \widehat{\Upsilon}_\varepsilon(\mathrm{d}y),$$

where the measure $\widehat{\Upsilon}_\varepsilon(\mathrm{d}y)$ is supported on $(0,\infty)$. To simplify notation, we also write $\varphi(y) = y\,h_1(y)$ for the density of $\mathcal{Z}^Y$ and $\varphi_\varepsilon(y) = y\,h_\varepsilon(y)$ for the density of $\mathcal{Z}^{Y,\varepsilon}$.

Lemma 27. *We have $\widehat{\Upsilon}_\varepsilon(\mathrm{d}y) = \widehat{\varphi}_\varepsilon(y)\,\mathrm{d}y$, where the functions $\widehat{\varphi}_\varepsilon(y)$ satisfy*

$$\lim_{\varepsilon\to 0}\widehat{\varphi}_\varepsilon(y) = \varphi(y)$$

uniformly on every interval of the form $[\delta,\infty)$, $\delta > 0$.

Proof. From (5.41), we have, for every $\lambda > 0$,

$$\int_0^\infty \widehat{\Upsilon}_\varepsilon(\mathrm{d}y)\,e^{-\lambda y} = \left(\frac{1+\varepsilon\sqrt{2\lambda/3}}{1+\sqrt{2\lambda/3}}\right)^3 - \varepsilon^3.$$

Now observe that

$$\left(\frac{1+\varepsilon\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^3 - \varepsilon^3 = \frac{(1-\varepsilon)^3 + 3\varepsilon(1-\varepsilon)^2(1+\sqrt{\lambda}) + 3\varepsilon^2(1-\varepsilon)(1+\sqrt{\lambda})^2}{(1+\sqrt{\lambda})^3} \tag{5.42}$$

where we have expanded $(1+\varepsilon\sqrt{\lambda})^3 = ((1-\varepsilon)+\varepsilon(1+\sqrt{\lambda}))^3$. It follows from formulas (A.1), (A.2), (A.3) in the Appendix that the Laplace transform of the function $\chi_{(\varepsilon)}$ defined by

$$\chi_{(\varepsilon)}(y) = (1-\varepsilon)^3\chi_3(y) + 3\varepsilon(1-\varepsilon)^2\chi_2(y) + 3\varepsilon^2(1-\varepsilon)\chi_1(y).$$

is the quantity in (5.42). Consequently, we have $\widehat{\Upsilon}_\varepsilon(\mathrm{d}y) = \widehat{\varphi}_\varepsilon(y)\,\mathrm{d}y$ with $\widehat{\varphi}_\varepsilon(y) = \frac{3}{2}\chi_{(\varepsilon)}(\frac{3y}{2})$. Furthermore, the explicit formulas for χ_1, χ_2, χ_3 show that $\chi_{(\varepsilon)}(y)$ converge to $\chi_3(y)$ as $\varepsilon \to 0$, uniformly on every interval of the form $[\delta,\infty)$, $\delta > 0$. The result of the proposition follows since $\varphi(y) = \frac{3}{2}\chi_3(\frac{3y}{2})$ by definition. □

Lemma 28. *Let $z > 0$. The total variation distance between the conditional distribution of Γ^ε knowing that $\mathcal{Z}^Y = z$ and the unconditional distribution of Γ^ε converges to 0 as $\varepsilon \to 0$.*

Remark. We have made a canonical choice for the conditional distribution $\check{\Theta}_z^{(1)}$ of $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ knowing $\mathcal{Z}^Y = z$, and so the conditional distribution of Γ^ε knowing that $\mathcal{Z}^Y = z$ is also well defined for every z.

Proof. The equality $\mathcal{Z}^Y = \mathcal{Z}^{Y,\varepsilon} + \widehat{\mathcal{Z}}^{Y,\varepsilon}$ gives

$$\varphi(x) = \varepsilon^3\varphi_\varepsilon(x) + \int_0^x \varphi_\varepsilon(y)\widehat{\varphi}_\varepsilon(x-y)\,\mathrm{d}y \tag{5.43}$$

for every $x > 0$. Let G and g be measurable functions defined respectively on $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ and on $\mathbb{R}_+$, such that $0 \leqslant G \leqslant 1$ and $0 \leqslant g \leqslant 1$. Then,

$$\begin{aligned}\mathbb{E}[G(\Gamma^\varepsilon) g(\mathcal{Z}^Y)] &= \mathbb{E}[G(\Gamma^\varepsilon) g(\mathcal{Z}^{Y,\varepsilon} + \widehat{\mathcal{Z}}^{Y,\varepsilon})] \\ &= \int \mathrm{d}z\, \widehat{\varphi}_\varepsilon(z)\, \mathbb{E}[G(\Gamma^\varepsilon) g(\mathcal{Z}^{Y,\varepsilon} + z)] + \varepsilon^3\, \mathbb{E}[G(\Gamma^\varepsilon) g(\mathcal{Z}^{Y,\varepsilon})] \\ &= \mathbb{E}\Big[G(\Gamma^\varepsilon) \int_{\mathcal{Z}^{Y,\varepsilon}}^\infty \mathrm{d}z\, g(z)\, \widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})\Big] + \varepsilon^3 \int \mathrm{d}z\, g(z)\, \varphi_\varepsilon(z)\, \mathbb{E}[G(\Gamma^\varepsilon) \,|\, \mathcal{Z}^{Y,\varepsilon} = z] \\ &= \int \mathrm{d}z\, g(z) \Big(\mathbb{E}[G(\Gamma^\varepsilon) \mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}}\, \widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})] + \varepsilon^3\, \varphi_\varepsilon(z)\, \mathbb{E}[G(\Gamma^\varepsilon) \,|\, \mathcal{Z}^{Y,\varepsilon} = z]\Big).\end{aligned}$$

Recalling that the density of $\mathcal{Z}^Y$ is φ, it follows that we have $\mathrm{d}z$ a.e.,

$$\mathbb{E}[G(\Gamma^\varepsilon) \,|\, \mathcal{Z}^Y = z] = \mathbb{E}\left[G(\Gamma^\varepsilon) \mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}}\, \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})}{\varphi(z)}\right] + \varepsilon^3 \mathbb{E}[G(\Gamma^\varepsilon) \,|\, \mathcal{Z}^{Y,\varepsilon} = z]\, \frac{\varphi_\varepsilon(z)}{\varphi(z)}, \quad (5.44)$$

where we observe that $\mathbb{E}[G(\Gamma^\varepsilon) \mid \mathcal{Z}^{Y,\varepsilon} = z] = \check{\Theta}_z^{(\varepsilon)}(G)$ is well defined for every z. We now want to argue that (5.44) holds for *every* $z > 0$ and not only $\mathrm{d}z$ a.e. To this end, it is enough to consider the special case $G(\mathrm{w}, \mu, \mu') = \exp(-f(\mathrm{w}) - \langle \mu, h \rangle - \langle \mu', h' \rangle)$ where, f, h, h' are nonnegative functions, f is bounded and continuous on $\mathcal{W}$, h and h' are bounded and continuous on $\mathbb{R}_+ \times \mathcal{S}$ and both h and h' vanish on $\{(t, \omega) : \sigma(\omega) \leqslant \delta\}$ for some $\delta > 0$. In that case, using a scaling argument and Corollary 10, one checks that both sides of (5.44) are continuous functions of z, so that they must be equal for every $z > 0$.

From (5.12), we have $\varphi_\varepsilon(z) = O(\varepsilon)$ as $\varepsilon \to 0$, hence, for every fixed $z > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^3 \mathbb{E}[G(\Gamma^\varepsilon) \mid \mathcal{Z}^{Y,\varepsilon} = z]\, \frac{\varphi_\varepsilon(z)}{\varphi(z)} = 0, \quad (5.45)$$

uniformly in the choice of G. On the other hand, using Lemma 27 and the fact that $\mathcal{Z}^{Y,\varepsilon} \longrightarrow 0$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} \mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}}\, \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})}{\varphi(z)} = 1 \quad (5.46)$$

almost surely. Moreover, using (5.43), we have

$$\mathbb{E}\left[\mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}}\, \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})}{\varphi(z)}\right] = \frac{1}{\varphi(z)} \int_0^z \mathrm{d}y\, \varphi_\varepsilon(y) \widehat{\varphi}_\varepsilon(z - y) = \frac{1}{\varphi(z)} (\varphi(z) - \varepsilon^3 \varphi_\varepsilon(z)),$$

which tends to 1 as $\varepsilon \to 0$. By Scheffé's lemma, the convergence (5.46) also holds in L^1. The statement of the lemma then follows from (5.44) and (5.45). □

Proof of Theorem 26. The proof is based on a coupling argument relying on Lemma 28. If E is a pointed metric space, we use the notation $B_r(E)$ for the closed ball of radius r centered at the distinguished point. The theorem will follow if we can prove that, for every $K > 0$ and every $\delta > 0$, if λ is large enough we can couple $\mathbb{H}_\infty$ and $\bar{\mathbb{D}}_z^{\bullet,1}$ in such a way that the balls $B_K(\lambda \cdot \bar{\mathbb{D}}_z^{\bullet,1})$ and $B_K(\mathbb{H}_\infty)$ are isometric with probability at least $1 - \delta$ (with an isometry preserving the volume measure and the distinguished point). Equivalently, recalling that $\lambda \cdot \mathbb{H}_\infty$ has the same distribution as $\mathbb{H}_\infty$, it suffices to prove that, for $\eta > 0$ small enough, $\mathbb{H}_\infty$ and $\bar{\mathbb{D}}_z^{\bullet,1}$ can be coupled so that $B_\eta(\bar{\mathbb{D}}_z^{\bullet,1})$ and $B_\eta(\mathbb{H}_\infty)$ are isometric with probability at least $1-\delta$ (again with an isometry preserving the volume measure and the distinguished point).

As explained at the end of Section 5.4.3, we may and will assume that $\bar{\mathbb{D}}_z^{\bullet,1}$ is constructed from a coding triple $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ distributed according to $\check{\Theta}_z^{(1)}$. The labeled tree associated with $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ is denoted by $\mathcal{T}^{(z)}$, and we write $\Delta^{(z),\circ}$ and $\Delta^{(z)}$ for the pseudo-distance functions on $\mathcal{T}^{(z)}$, so that $\Delta^{(z)}$ induces the metric on $\bar{\mathbb{D}}_z^{\bullet,1}$. The set of all points of $\mathcal{T}^{(z)}$ with positive label is denoted by $\mathcal{T}^{(z),\circ}$.

For $\varepsilon \in (0,1)$, let $\Gamma^{(z),\varepsilon}$ be defined as Γ^ε but replacing the triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ by $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ (so $\Gamma^{(z),\varepsilon}$ is distributed as Γ^ε conditioned on $\mathcal{Z}^Y = z$). Let $\check{\Gamma}^{(z),\varepsilon}$, resp. $\check{\Gamma}^\varepsilon$, denote the image of $\Gamma^{(z),\varepsilon}$, resp. Γ^ε, under the time reversal operation **SR** in (5.8). We fix $\delta > 0$ and claim that:

1. For $\varepsilon \in (0,1)$ small enough, the triples $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ and $(R, \mathcal{P}, \mathcal{P}')$ can be coupled in such a way that the equality
$$\check{\Gamma}^{(z),\varepsilon} = \Big((R_t)_{0 \leqslant t \leqslant \mathbf{L}_\varepsilon}, \mathbf{1}_{[0,\mathbf{L}_\varepsilon]}(t)\, \widetilde{\mathcal{P}}(\mathrm{d}t\mathrm{d}\omega), \mathbf{1}_{[0,\mathbf{L}_\varepsilon]}(t)\, \widetilde{\mathcal{P}}'(\mathrm{d}t\mathrm{d}\omega) \Big) \tag{5.47}$$
holds with probability at least $1 - \frac{\delta}{2}$.

2. For $\varepsilon \in (0,1)$ small enough, we can choose $\eta_0 > 0$ so that for every $0 < \eta \leqslant \eta_0$, we have
$$B_\eta(\bar{\mathbb{D}}_z^{\bullet,1}) = B_\eta(\mathbb{H}_\infty)$$
on the event where (5.47) holds, except possibly on an event of probability at most $\frac{\delta}{2}$.

Clearly the theorem follows from Properties 1 and 2. Property 1 is a consequence of Lemma 28: just note that the distribution of the coding triple in the right-hand side of (5.47) is the (unconditional) distribution of $\check{\Gamma}^\varepsilon$.

It remains to verify Property 2. We fix $\varepsilon > 0$ small enough so that we can apply Property 1. We then assume that the triples $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ and $(R, \mathcal{P}, \mathcal{P}')$ have been coupled in such a way that the event where (5.47) holds has probability greater than $1 - \frac{\delta}{2}$, and we denote the latter event by $\mathcal{F}$. We argue on the intersection $\mathcal{F} \cap \mathcal{F}'$, where $\mathcal{F}'$ denotes the event where $W_*(\omega_i) = 0$ for at least one atom (t_i, ω_i) of $\widetilde{\mathcal{M}}^{(z)}$ or of $\widetilde{\mathcal{M}}'^{(z)}$ such that $t_i < T_\varepsilon^{(z)} := \inf\{t \geqslant 0 : Y_t^{(z)} = \varepsilon\}$. Clearly we can also assume that $\mathcal{F}'$ has probability greater than $1 - \frac{\delta}{6}$ by choosing ε even smaller if necessary.

Recall the notation $\mathcal{T}_{\infty,r}^{hp}$ introduced before Lemma 25. From this lemma, we know that, for $\eta > 0$ small enough, the set $\{v \in \mathcal{T}_\infty^{hp} : \Delta^{hp}(\rho, v) \leqslant 4\eta\}$ will be contained in $\mathcal{T}_{\infty,\varepsilon}^{hp}$, except on an event of probability at most $\frac{\delta}{6}$. Moreover, if the latter property holds, we claim that we have also, for every $u, v \in \mathcal{T}_\infty^{hp,\circ}$ such that $\Delta^{hp}(\rho, u) \leqslant \eta$ and $\Delta^{hp}(\rho, v) \leqslant \eta$,
$$\Delta^{hp}(u,v) = \inf_{\substack{u_0 = u, u_1, \dots, u_p = v \\ u_1, \dots, u_{p-1} \in \mathcal{T}_{\infty,\varepsilon}^{hp} \cap \mathcal{T}_\infty^{hp,\circ}}} \sum_{i=1}^{p} \Delta^{hp,\circ}(u_{i-1}, u_i). \tag{5.48}$$

In other words, in formula (5.29) applied to $\Delta^{hp}(u,v)$, we may restrict the infimum to the case where all u_i's belong to $\mathcal{T}^{hp}_{\infty,\varepsilon}$. Let us justify (5.48). Assume that $\Delta^{hp}(\rho,u)\leqslant\eta$ and $\Delta^{hp}(\rho,v)\leqslant\eta$ (so that in particular $\Delta^{hp}(u,v)\leqslant 2\eta$) and $u_0=u,u_1,\ldots,u_q\in\mathcal{T}^{hp,\circ}_\infty$ are such that

$$\sum_{i=1}^{q}\Delta^{hp,\circ}(u_{i-1},u_i)<\Delta^{hp}(u,v)+\eta.$$

It then follows that $\Delta^{hp}(u,u_q)<3\eta$ and $\Delta^{hp}(\rho,u_q)<4\eta$ which implies $u_q\in\mathcal{T}^{hp}_{\infty,\varepsilon}$.

Furthermore, when applying formula (8.15) to compute the quantities $\Delta^{hp,\circ}(u_{i-1},u_i)$ in the right-hand side of (5.48), it is enough to consider the case when the interval $[u_{i-1},u_i]$ (resp. $[u_i,u_{i-1}]$) is contained in $\mathcal{T}^{hp}_{\infty,\varepsilon}$, because otherwise this interval contains $\mathcal{T}^{hp}_\infty\backslash\mathcal{T}^{hp}_{\infty,\varepsilon}$ and then the infimum of labels on $[u_{i-1},u_i]$ is 0. To summarize, on the event where (5.48) holds for every $u,v\in\mathcal{T}^{hp,\circ}_\infty$ such that $\Delta^{hp}(\rho,u)\leqslant\eta$ and $\Delta^{hp}(\rho,v)\leqslant\eta$, we get that the value of $\Delta^{hp}(u,v)$ for such points u and v is determined by the tree $\mathcal{T}^{hp}_{\infty,\varepsilon}$ and the labels on this tree.

On the event where (5.48) holds, we thus get that the ball $B_\eta(\mathbb{H}_\infty)$ can be written as a function of the coding triple

$$\Big((R_t)_{0\leqslant t\leqslant\mathbf{L}_\varepsilon},\mathbf{1}_{[0,\mathbf{L}_\varepsilon]}(t)\,\widetilde{\mathcal{P}}(\mathrm{d}t\mathrm{d}\omega),\mathbf{1}_{[0,\mathbf{L}_\varepsilon]}(t)\,\widetilde{\mathcal{P}}'(\mathrm{d}t\mathrm{d}\omega)\Big)$$

since the tree $\mathcal{T}^{hp}_{\infty,\varepsilon}$ and the labels on this tree are functions of this triple (and also the distinguished point of $B_\eta(\mathbb{H}_\infty)$ corresponds to the root of this coding triple). To complete the argument (recalling that we assume (5.47)), we need to justify that $B_\eta(\bar{\mathbb{D}}^{\bullet,1}_z)$ is given by the same function applied to the triple $\check{\Gamma}^{(z),\varepsilon}$, except possibly on a set of small probability. To get this, recall that $\mathbb{D}^{\bullet,1}_z$ is obtained by applying $\Xi^\bullet$ to the snake trajectory $\Omega(Y^{(z)},\widetilde{\mathcal{M}}^{(z)},\widetilde{\mathcal{M}}'^{(z)})$. With the coding triple $\check{\Gamma}^{(z),\varepsilon}$ we associate a labeled tree $\mathcal{T}^{(z)}_\varepsilon$, which is identified to a subtree of the labeled tree $\mathcal{T}^{(z)}$, and, modulo this identification, $\mathcal{T}^{(z)}_\varepsilon$ is rooted at the top of the spine of the tree $\mathcal{T}^{(z)}$, which corresponds to the distinguished point α of $\bar{\mathbb{D}}^{\bullet,1}_z$. We claim that the image of $\mathcal{T}^{(z)}_\varepsilon$ (viewed as a subset of $\mathcal{T}^{(z)}$) under the canonical projection from $\mathcal{T}^{(z)}$ onto $\bar{\mathbb{D}}^{\bullet,1}_z$ must contain a neighborhood of α. As in the proof of Lemma 25, this property holds because the equivalence class of α in $\mathcal{T}^{(z)}/\{\Delta^{(z)}=0\}$ is a singleton, which is a consequence of the fact that two points u and v of $\mathcal{T}^{(z)}$ with zero label are identified in $\bar{\mathbb{D}}^{\bullet,1}_z$ if and only if labels stay positive on the interval $]u,v[$, or on the interval $]v,u[$ (see the discussion after Proposition 14).

It follows from the preceding claim that, for η small enough, we have $\Delta^{(z)}(\alpha,v)>4\eta$ whenever $v\notin\mathcal{T}^{(z)}_\varepsilon$, except on an event of probability at most $\frac{\delta}{6}$. Discarding the latter event of small probability, the same argument as above shows that the analog of (5.48) holds for every $u,v\in\mathcal{T}^{(z),\circ}$ such that $\Delta^{(z)}(\alpha,u)<\eta$ and $\Delta^{(z)}(\alpha,v)<\eta$, provided we replace Δ^{hp} by $\Delta^{(z)}$, $\mathcal{T}^{hp,\circ}_\infty$ by $\mathcal{T}^{(z),\circ}$, and $\mathcal{T}^{hp}_{\infty,\varepsilon}$ by $\mathcal{T}^{(z)}_\varepsilon$. Furthermore, the quantities $\Delta^{(z),\circ}(u_{i-1},u_i)$ appearing in this analog can be computed from the labeled tree $\mathcal{T}^{(z)}_\varepsilon$ (here we use our definition of $\mathcal{F}'$, which implies that $\mathcal{T}^{(z)}\backslash\mathcal{T}^{(z)}_\varepsilon$ contains points with zero label).

It follows from the preceding discussion that, on the event $\mathcal{F}$ that has probability at least $1-\frac{\delta}{2}$, and except on an event of probability at most $\frac{\delta}{2}$, the ball $B_\eta(\bar{\mathbb{D}}^{\bullet,1}_z)$ is obtained from the triple $\check{\Gamma}^{(z),\varepsilon}$ by applying the same function that can be used to get the ball $B_\eta(\mathbb{H}_\infty)$ from the triple in the right-hand side of (5.47). The desired result follows. □

5.5 Applications

5.5.1 Infinite-volume Brownian disks in the Brownian plane

Recall the construction of the Brownian plane $(\mathbb{BP}_\infty, \Delta^p)$ from the coding triple $(X, \mathcal{L}, \mathcal{R})$ in Section 5.4.2 and note that the same triple was also considered in Section 5.3.3. We use the notation $(\mathcal{T}^p_\infty, (\Lambda_v)_{v\in\mathcal{T}^p_\infty})$ for the labeled tree associated with the triple $(X, \mathcal{L}, \mathcal{R})$, and we write Π^p for the canonical projection from $\mathcal{T}^p_\infty$ onto $\mathbb{BP}_\infty$. The distinguished point ρ of $\mathbb{BP}_\infty$ is the image of the root of $\mathcal{T}^p_\infty$ under Π^p.

To simplify notation, for every $r > 0$, we write $B(r) = B_r(\mathbb{BP}_\infty)$ for the closed ball of radius r centered at ρ in $\mathbb{BP}_\infty$. The hull $B^\bullet(r)$ is then the subset of $\mathbb{BP}_\infty$ defined by saying that $\mathbb{BP}_\infty \backslash B^\bullet(r)$ is the unique unbounded connected component of $\mathbb{BP}_\infty \backslash B(r)$ (this component is unique since $\mathbb{BP}_\infty$ is homeomorphic to the plane [39]). Informally, $B^\bullet(r)$ is obtained by filling in the (bounded) holes in $B(r)$. As in the introduction, it will be convenient to use the notation

$$\check{B}^\bullet(r) = \mathbb{BP}_\infty \backslash B^\bullet(r).$$

One can give an explicit description of $\check{B}^\bullet(r)$ in terms of the labeled tree $(\mathcal{T}^p_\infty, (\Lambda_v)_{v\in\mathcal{T}^p_\infty})$. For $v \in \mathcal{T}^p_\infty$, we recall that $[\![v, \infty[\![$ is the geodesic ray from v in $\mathcal{T}^p_\infty$. Then, $\check{B}^\bullet(r) = \Pi^p(F_r)$, where

$$F_r := \{v \in \mathcal{T}^p_\infty : \Lambda_w > r \text{ for every } w \in [\![v, \infty[\![\}. \tag{5.49}$$

Similarly, the topological boundary of $\check{B}^\bullet(r)$ (or of $B^\bullet(r)$) is $\partial\check{B}^\bullet(r) = \partial B^\bullet(r) = \Pi^p(\partial F_r)$, with

$$\partial F_r = \{v \in \mathcal{T}^p_\infty : \Lambda_w = r \text{ and } \Lambda_w > r \text{ for every } w \in]\!]v, \infty[\![\}, \tag{5.50}$$

with the obvious notation $]\!]v, \infty[\![$. See formulas (16) and (17) in [40]. We note that the intersection of the set F_r with the spine of $\mathcal{T}^p_\infty$ is just the interval (L_r, ∞), where we recall the notation L_r in (5.18) (as in Section 5.2.4, the spine is identified to $\mathbb{R}_+$). Following [40], we define the boundary size of $B^\bullet(r)$ to be $|\partial B^\bullet(r)| = \mathcal{Z}^{(r)}$, where the quantity $\mathcal{Z}^{(r)}$ is defined in (5.19): $\mathcal{Z}^{(r)}$ is the sum over all atoms (t, ω) of $\mathcal{L}$ and $\mathcal{R}$ such that $t > \mathrm{L}_r$ of the exit measures $\mathcal{Z}_r(\omega)$ at level r – see formula (18) in [40]. We write $\mathrm{cl}(\check{B}^\bullet(r)) = \check{B}^\bullet(r) \cup \partial B^\bullet(r)$ for the closure of $\check{B}^\bullet(r)$, and similarly $\mathrm{cl}(F_r) = F_r \cup \partial F_r$.

Recall that the intrinsic metric on an open subset O of $\mathbb{BP}_\infty$ is defined by declaring that the distance between two points x and y of O is the infimum of the lengths of all continuous curves $\gamma : [0, 1] \longrightarrow O$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here the lengths are of course computed with respect to the metric Δ^p of $\mathbb{BP}_\infty$.

Theorem 29. *Let $r > 0$. Then a.s. the intrinsic metric on $\check{B}^\bullet(r)$ has a unique continuous extension to $\mathrm{cl}(\check{B}^\bullet(r))$, which is a metric on this set. We write $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ for the resulting* *random locally compact metric space, which is equipped with the restriction of the volume measure on* $\mathbb{BP}_\infty$ *and pointed at* $\Pi^p(\mathrm{L}_r)$. *Then, conditionally on* $|\partial B^\bullet(r)|$, $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ *is an infinite-volume Brownian disk with perimeter* $|\partial B^\bullet(r)|$.

Proof. Recall the notation $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ introduced in Section 5.3.3, and the fact that, conditionally on $\mathcal{Z}^{(r)} = z$, this coding triple is distributed according to Θ_z (Proposition 6). The construction of Section 5.4.4 produces, from the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$, a random measure metric space $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ such that, conditionally on $\mathcal{Z}^{(r)} = z$, $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ is an infinite-volume Brownian disk with perimeter z. Furthermore, $\mathbb{D}^{\infty,(r)}$ is obtained as a quotient space of the labeled tree $\mathcal{T}^i_\infty$ coded by the triple $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$. Here we use the same notation $\mathcal{T}^i_\infty$ as in Section 5.4.4, where we were dealing with a different triple distributed according to Θ_z, but this should create no confusion. We write Π^i for the canonical projection from $\mathcal{T}^i_\infty$ onto $\mathbb{D}^{\infty,(r)}$.

It is easy to verify that the tree $\mathcal{T}^i_\infty$ can be identified with $\mathrm{cl}(F_r)$. The spine of $\mathcal{T}^i_\infty$ is identified with the part $[\mathrm{L}_r, \infty)$ of the spine of $\mathcal{T}^p_\infty$, and we observe that, for each atom (t_i, ω_i) of $\mathcal{L}$ or $\mathcal{R}$ such that $t_i > \mathrm{L}_r$ (so that $\mathrm{tr}_r(\omega_i)$ shifted by $-r$ corresponds to an atom of $\widetilde{\mathcal{L}}^{(r)}$ or $\widetilde{\mathcal{R}}^{(r)}$), the genealogical tree $\mathcal{T}_{(\mathrm{tr}_r(\omega_i))}$ is identified with $\{v \in \mathcal{T}_{(\omega_i)} : \Lambda_w > r \text{ for every } w \in [[\rho_{(\omega_i)}, v[[\}$ (see the end of Section 5.2.1). The identification of $\mathcal{T}^i_\infty$ with $\mathrm{cl}(F_r)$ preserves labels, provided labels on $\mathrm{cl}(F_r)$ are shifted by $-r$. With a slight abuse of notation, if $u \in \mathcal{T}^i_\infty$, we will also write Λ_u for the label of the point of $\mathrm{cl}(F_r)$ corresponding to u in the identification of $\mathcal{T}^i_\infty$ with $\mathrm{cl}(F_r)$ (so the label of u in $\mathcal{T}^i_\infty$ is $\Lambda_u - r$).

Furthermore, two vertices of $\mathrm{cl}(F_r)$ are identified in the quotient space $\mathbb{BP}_\infty$ if and only if the corresponding vertices of $\mathcal{T}^i_\infty$ are identified in the quotient $\mathbb{D}^{\infty,(r)}$: to check this property in the case where the vertices belong to the boundary (the other case is immediate) we use the fact that two vertices u and v of $\mathcal{T}^i_\infty$ with zero label are identified if and only if labels remain positive on one of the two intervals $]u, v[$ and $]v, u[$ of the tree $\mathcal{T}^i_\infty$ (Lemma 16 (iii)). Thus we can identify $\mathbb{D}^{\infty,(r)}$ with the set $\Pi^p(\mathrm{cl}(F_r)) = \mathrm{cl}(\check{B}^\bullet(r))$, in such a way that $\partial\mathbb{D}^{\infty,(r)}$ is identified with $\partial\check{B}^\bullet(r)$, and this identification preserves the volume measures.

Modulo the preceding identification, both assertions of the theorem follow from the next lemma.

Lemma 30. *Let* x *and* y *be two points of* $\mathbb{D}^{\infty,(r)} \backslash \partial\mathbb{D}^{\infty,(r)}$, *and let* x' *and* y' *be the corresponding points in* $\check{B}^\bullet(r)$. *Then the intrinsic distance (relative to the open set* $\check{B}^\bullet(r)$*) between* x' *and* y' *coincides with* $\Delta^{\infty,(r)}(x, y)$.

Proof. Let $\Delta^{\infty,(r),\circ}(v, w)$ be defined as in (8.15) for the labeled tree $(\mathcal{T}^i_\infty, (\Lambda_u - r)_{u \in \mathcal{T}^i_\infty})$ (recall that the label in $\mathcal{T}^i_\infty$ of a point $u \in \mathcal{T}^i_\infty$ is equal to $\Lambda_u - r$), so that $\Delta^{\infty,(r)}(v, w)$ is then given from $\Delta^{\infty,(r),\circ}(v, w)$ by formula (5.29).

We first prove that the intrinsic distance between x' and y' is bounded above by $\Delta^{\infty,(r)}(x, y)$. To this end, let v and w be points of $\mathcal{T}^i_\infty$ such that $\Pi^i(v) = x$ and $\Pi^i(w) = y$. We claim that, if $\Delta^{\infty,(r),\circ}(v, w) < \infty$, then $\Delta^{\infty,(r),\circ}(v, w)$ is the length of a continuous curve in $\check{B}^\bullet(r)$ that connects

x' to y'. Let us explain this. Without loss of generality, we may assume that

$$\Delta^{\infty,(r),\circ}(v,w) = \Lambda_v + \Lambda_w - 2 \inf_{u\in[v,w]} \Lambda_u,$$

with $\inf_{u\in[v,w]} \Lambda_u > r$. We let v' and w' be the points of F_r corresponding to v and w in the identification of $\mathcal{T}^i_\infty$ with $\mathrm{cl}(F_r)$ (in particular $\Lambda_{u'} = \Lambda_u$ and $\Lambda_{v'} = \Lambda_v$). We note that the condition $\inf_{u\in[v,w]} \Lambda_u > r$ implies that the interval $[v,w]$ of $\mathcal{T}^i_\infty$ is also identified with the interval $[v',w']$ of $\mathcal{T}^p_\infty$ (in particular we have $\Delta^{p,\circ}(v',w') = \Delta^{\infty,(r),\circ}(v,w)$), and furthermore $\Pi^p([v',w'])$ is contained in $\check{B}^\bullet(r)$. By concatenating two simple geodesics starting from $\Pi^p(v') = x'$ and $\Pi^p(w') = y'$ respectively up to their merging time, as explained at the end of Section 5.4.1, we construct a path from x' to y' whose length is equal to $\Delta^{p,\circ}(v',w')$, and which stays in $\Pi^p([v',w']) \subset \check{B}^\bullet(r)$. This gives our claim.

From the definition of $\Delta^{\infty,(r)}$ as an infimum, we now get that $\Delta^{\infty,(r)}(x,y)$ is bounded below by the infimum of lengths of continuous curves connecting x' and y' that stay in $\check{B}^\bullet(r)$. We thus obtain that the intrinsic distance between x' and y' (with respect to the open set $\check{B}^\bullet(r)$) is bounded above by $\Delta^{\infty,(r)}(x,y)$.

It remains to prove the reverse bound. To this end, we need to verify that, if $\gamma : [0,1] \to \check{B}^\bullet(r)$ is a continuous curve such that $\gamma(0) = x'$ and $\gamma(1) = y'$, then the length of γ is bounded below by $\Delta^{\infty,(r)}(x,y)$. We write $\overline{\gamma}(t)$ for the point of $\mathbb{D}^{\infty,(r)}$ corresponding to $\gamma(t)$ in the identification of $\mathrm{cl}(\check{B}^\bullet(r))$ with $\mathbb{D}^{\infty,(r)}$. We may find $\delta > 0$ such that $\Lambda_{\gamma(t)} > r + \delta$ for every $t \in [0,1]$ (recall that $\Lambda_z = \Delta^p(\rho,z)$ for every $z \in \mathbb{BP}_\infty$). Then, we may choose n large enough so that $\Delta^p(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n})) < \delta/2$ for every $1 \leqslant i \leqslant n$. The length of γ is bounded below by $\sum_{i=1}^n \Delta^p(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n}))$, and so to get the desired result it suffices to verify that, for every $1 \leqslant i \leqslant n$,

$$\Delta^p\Big(\gamma\Big(\frac{i-1}{n}\Big),\gamma\Big(\frac{i}{n}\Big)\Big) \geqslant \Delta^{\infty,(r)}\Big(\overline{\gamma}\Big(\frac{i-1}{n}\Big),\overline{\gamma}\Big(\frac{i}{n}\Big)\Big).$$

Fix $1 \leqslant i \leqslant n$, and recall the definition (5.29) of $\Delta^p(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n}))$ as an infimum over possible choices of $u_0 = \gamma(\frac{i-1}{n}), u_1, \ldots, u_p = \gamma(\frac{i}{n})$ in $\mathcal{T}^p_\infty$, where we may restrict our attention to choices of $u_0, u_1, \ldots, u_p$ such that $\Lambda_{u_j} > r + \delta/2$ (use $\Delta^p(u,v) \geqslant |\Lambda_v - \Lambda_u|$) and $\Delta^{p,\circ}(u_{j-1},u_j) < \delta/2$ for every $1 \leqslant j \leqslant p$. It suffices to consider one such choice and to prove that

$$\sum_{j=1}^p \Delta^{p,\circ}(u_{j-1},u_j) \geqslant \Delta^{\infty,(r)}\Big(\overline{\gamma}\Big(\frac{i-1}{n}\Big),\overline{\gamma}\Big(\frac{i}{n}\Big)\Big). \tag{5.51}$$

For every $1 \leqslant j \leqslant p$, the properties $\Lambda_{u_j} > r + \delta/2$ and $\Delta^{p,\circ}(u_{j-1},u_j) < \delta/2$ imply that the minimal label on $[u_{j-1},u_j]$ is greater than r (or the same holds with $[u_{j-1},u_j]$ replaced by $[u_j,u_{j-1}]$). This shows in particular that there is a continuous curve from $\gamma(\frac{i}{n})$ to $\Pi^p(u_j)$ that stays in the complement of $B(r)$, so that $\Pi^p(u_j)$ belongs to $\check{B}^\bullet(r)$ and u_j must belong to F_r, which allows us to define $\overline{u}_j$ as the point of $\mathcal{T}^i_\infty$ corresponding to u_j. Furthermore the fact that the minimal label on $[u_{j-1},u_j]$ is greater than r also implies that the interval $[u_{j-1},u_j]$ in $\mathcal{T}^p_\infty$ is identified to the interval $[\overline{u}_{j-1},\overline{u}_j]$ in $\mathcal{T}^i_\infty$, and then that $\Delta^{p,\circ}(u_{j-1},u_j) = \Delta^{\infty,(r),\circ}(\overline{u}_{j-1},\overline{u}_j)$. The bound (5.51) follows, which completes the proof of the lemma and of Theorem 5.5.1. □

In view of applications to isoperimetric inequalities in the Brownian plane [93], we state another result which complements Theorem 29 by showing that, in some sense, the exterior of the hull $B^\bullet(r)$ is independent of this hull, conditionally on its boundary size. We keep the notation introduced at the beginning of this section, and in particular, we recall that the Brownian plane $\mathbb{BP}_\infty$ is constructed from the labeled tree $(\mathcal{T}^p_\infty, (\Lambda_v)_{v\in\mathcal{T}^p_\infty})$ associated with the coding triple $(X, \mathcal{L}, \mathcal{R})$. We fix $r > 0$ and write K_r for the complement of the set F_r defined in (5.49),

$$K_r := \{v \in \mathcal{T}^p_\infty : \Lambda_w \leqslant r \text{ for some } w \in [\![v, \infty[\![\}.$$

We have then $B^\bullet(r) = \Pi^p(K_r)$ (cf. formulas (16) and (17) in [40]). Recall that, for every $u, v \in \mathbb{BP}_\infty$,

$$\Delta^{p,\circ}(u,v) = \Lambda_u + \Lambda_v - 2\max\Big(\inf_{w\in[u,v]} \Lambda_w, \inf_{w\in[v,u]} \Lambda_w\Big). \tag{5.52}$$

We then set, for every $u, v \in K_r$,

$$\Delta^{p,(r)}(u,v) = \inf_{\substack{u_0=u,u_1,\dots,u_p=v\\ u_0,u_1,\dots,u_p\in K_r}} \sum_{i=1}^{p} \Delta^{p,\circ}(u_{i-1}, u_i) \tag{5.53}$$

where the infimum is over all choices of the integer $p \geqslant 1$ and of the finite sequence $u_0, u_1, \dots, u_p$ in K_r such that $u_0 = u$ and $u_p = v$. For every $u, v \in K_r$, we have $\Delta^p(u,v) \leqslant \Delta^{p,(r)}(u,v)$ (just note that $\Delta^p(u,v)$ is defined by the same formula (8.42) without the restriction to $u_0, \dots, u_p \in K_r$) and we also know that $\Delta^p(u,v) = 0$ implies $\Delta^{p,\circ}(u,v) = 0$ and a fortiori $\Delta^{p,(r)}(u,v) = 0$. It follows that $\Delta^{p,(r)}$ induces a metric on $\Pi^p(K_r) = B^\bullet(r)$, and we keep the notation $\Delta^{p,(r)}$ for this metric. For future use, we also observe that, in the right-hand side of formula (5.52) applied to $u, v \in K_r$, we may replace the infimum over $w \in [u,v]$ by an infimum over $w \in [u,v] \cap K_r$: The point is that, if the clockwise exploration going from u to v (or from v to u) intersects F_r, then it necessarily visits a point with label at most r, because otherwise u and v would have to be in F_r.

Theorem 31. *Conditionally on $|\partial B^\bullet(r)|$, the random compact measure metric space $(B^\bullet(r), \Delta^{p,(r)})$ and the space $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ in Theorem 29 are independent. Furthermore, the restriction of the metric $\Delta^{p,(r)}$ to the interior of $B^\bullet(r)$ coincides with the intrinsic metric induced by Δ^p on this open set.*

Proof. The general idea is to show that the space $(B^\bullet(r), \Delta^{p,(r)})$ can be constructed from random quantities that are independent of $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ conditionally on $|\partial B^\bullet(r)|$. We start by introducing the labeled tree $\mathcal{T}^{p,(r)}$ which consists of the part $[0, \mathrm{L}_r]$ of the spine of $\mathcal{T}^p_\infty$, and of the subtrees branching off $[0, \mathrm{L}_r]$. Equivalently, $\mathcal{T}^{p,(r)}$ is associated with the finite spine coding triple

$$\mathfrak{X}^{(r)} := \Big((X_t)_{0\leqslant t\leqslant \mathrm{L}_r}, \mathbf{1}_{[0,\mathrm{L}_r]}(t)\,\mathcal{L}(\mathrm{d}t\mathrm{d}\omega), \mathbf{1}_{[0,\mathrm{L}_r]}(t)\,\mathcal{R}(\mathrm{d}t\mathrm{d}\omega)\Big). \tag{5.54}$$

Clearly, $\mathcal{T}^{p,(r)}$ viewed as a subset of $\mathcal{T}^p_\infty$ is contained in K_r. If $(\mathcal{E}_s)_{s\in\mathbb{R}}$ is the clockwise exploration of $\mathcal{T}^p_\infty$, $\mathcal{T}^{p,(r)}$ corresponds to the points visited by $(\mathcal{E}_s)_{s\in\mathbb{R}}$ during an interval of the form $[-\sigma_r, \sigma'_r]$, with $\sigma_r, \sigma'_r > 0$. We also let $\check{\mathfrak{X}}^{(r)}$ be the image of $\mathfrak{X}^{(r)}$ under the spine reversal operation (5.8) and denote the associated labeled tree by $\check{\mathcal{T}}^{p,(r)}$ (replacing $\mathcal{T}^{p,(r)}$ by $\check{\mathcal{T}}^{p,(r)}$ just amounts to interchanging the roles of the root and the top of the spine).

We then consider all subtrees branching off the spine of $\mathcal{T}^p_\infty$ at a level higher than L_r, and, for each such subtree whose labels hit $[0,r]$, the "excursions outside" (r,∞). To make this precise, write

$$\mathcal{L} = \sum_{i\in I} \delta_{(t_i,\omega^i)} \ , \ \mathcal{R} = \sum_{i\in J} \delta_{(t_i,\omega^i)},$$

where the indexing sets I and J are disjoint. In the time scale of the clockwise exploration, each ω^i corresponds to an interval $[\alpha_i,\beta_i]$ contained in $(-\infty,0)$ if $i\in J$, or in $(0,\infty)$ if $i\in I$, and $\sigma(\omega^i)=\beta_i-\alpha_i$. Set $I_r := \{i\in I : t_i > \mathrm{L}_r \text{ and } W_*(\omega^i)\leqslant r\}$ and $J_r := \{i\in J : t_i > \mathrm{L}_r \text{ and } W_*(\omega^i)\leqslant r\}$. For each $i\in I_r\cup J_r$, we can make sense of the exit local time of ω^i at level r, as defined in Section 5.2.2, and we denote this local time by $(L^{i,r}_t)_{t\in[0,\sigma(\omega^i)]}$. We then set, for every $t\in\mathbb{R}$,

$$L^{*,r}_t = \sum_{i\in I_r\cup J_r} L^{i,r}_{t\wedge\beta_i - t\wedge\alpha_i},$$

so that, in some sense, $L^{*,r}_t$ represents the total exit local time accumulated at r by the clockwise exploration up to time t. We note that

$$L^{*,r}_\infty = \sum_{i\in I_r\cup J_r} L^{i,r}_{\sigma(\omega^i)} = \mathcal{Z}^{(r)},$$

and $|\partial B^\bullet(r)| = \mathcal{Z}^{(r)}$ by definition.

Then, for every $i\in I_r\cup J_r$, we consider the excursions $(\omega^{i,k})_{k\in\mathbb{N}}$ of ω^i outside (r,∞) (we refer to [2, Section 2.4] for more information about such excursions). These excursions $\omega^{i,k}$, $k\in\mathbb{N}$ are in one-to-one correspondence with the connected components $(a_{i,k},b_{i,k})$, $k\in\mathbb{N}$, of the open set $\{s\in[0,\sigma(\omega^i)] : \tau_r(\omega^i_s) < \zeta_s(\omega^i)\}$, in such a way that, for every $s\geqslant 0$,

$$\omega^{i,k}_s(t) := \omega^i_{(a_{i,k}+s)\wedge b_{i,k}}(\zeta_{a_{i,k}}(\omega^i)+t), \text{ for } 0\leqslant t\leqslant \zeta_s(\omega^{i,k}) := \zeta_{(a_{i,k}+s)\wedge b_{i,k}}(\omega^i) - \zeta_{a_{i,k}}(\omega^i).$$

In the time scale of the clockwise exploration, $\omega^{i,k}$ corresponds to the interval $[\alpha_{i,k},\beta_{i,k}]$, where $\alpha_{i,k}=\alpha_i+a_{i,k}$ and $\beta_{i,k}=\alpha_i+b_{i,k}$. In particular, the (labeled) tree $\mathcal{T}_{(\omega^{i,k})}$ coincides with the subtree of $\mathcal{T}^p_\infty$ consisting of the descendants of $\mathcal{E}_{\alpha_{i,k}}=\mathcal{E}_{\beta_{i,k}}$ (this set of descendants is $\{\mathcal{E}_s : s\in[\alpha_{i,k},\beta_{i,k}]\}$).

Recall the coding triple $(X^{(r)},\widetilde{\mathcal{L}}^{(r)},\widetilde{\mathcal{R}}^{(r)})$ which is used to construct the space $(\mathbb{D}^{\infty,(r)},\Delta^{\infty,(r)})$. An application of the special Markov property, in the form given in the appendix of [70], shows that, conditionally on $\mathcal{Z}^{(r)}$, the point measure

$$\mathcal{N}_{(r)}(\mathrm{d}t\mathrm{d}\omega) := \sum_{i\in I_r\cup J_r}\sum_{k\in\mathbb{N}} \delta_{(L^{*,r}_{\alpha_{i,k}},\omega^{i,k})}(\mathrm{d}t\mathrm{d}\omega)$$

is Poisson with intensity $\mathbf{1}_{[0,\mathcal{Z}^{(r)}]}(t)\,\mathrm{d}t\,\mathbb{N}_r(\mathrm{d}\omega)$, and is independent of $(X^{(r)},\widetilde{\mathcal{L}}^{(r)},\widetilde{\mathcal{R}}^{(r)})$. Note in particular that $\mathcal{Z}^{(r)}$ is a measurable function of $\mathcal{N}_{(r)}$. On the other hand, the coding triple $\mathfrak{X}^{(r)}$ in (5.54) is clearly independent of the pair $(\mathcal{N}_{(r)},(X^{(r)},\widetilde{\mathcal{L}}^{(r)},\widetilde{\mathcal{R}}^{(r)}))$. So the first assertion of the theorem would follow if we could prove that the space $(B^\bullet(r), \Delta^{p,(r)})$ is a function of $\mathcal{N}_{(r)}$ and $\mathfrak{X}^{(r)}$. This is not correct, but we will see that $(B^\bullet(r), \Delta^{p,(r)})$ is a function of $(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})$. Informally, the information given by $L_0^{*,r}$ is required to locate the root of the tree $\check{\mathcal{T}}^{p,r}$ associated with $\mathfrak{X}^{(r)}$ among the (roots of the) trees $\mathcal{T}_{(\omega^{i,k})}$ corresponding to the atoms of $\mathcal{N}_{(r)}$. Once we have written $(B^\bullet(r), \Delta^{p,(r)})$ as a function of $(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})$, we will verify that the conditional distribution of $(B^\bullet(r), \Delta^{p,(r)})$ knowing $(\mathcal{Z}^{(r)}, L_0^{*,r})$ only depends on $\mathcal{Z}^{(r)}$. This will suffice to get the first assertion of Theorem 31.

Let us explain how the space $(B^\bullet(r), \Delta^{p,(r)})$ can be written as a function of $(\mathcal{N}_{(r)}, L_0^{*,r})$ and $\mathfrak{X}^{(r)}$. To begin with, we introduce the right-continuous inverse of the process $(L_t^{*,r})_{t\in\mathbb{R}}$: for every $s \in [0, \mathcal{Z}^{(r)})$,

$$\tau_s^{*,r} := \inf\{t \in \mathbb{R} : L_t^{*,r} > s\},$$

and we also make the convention that $\tau^{*,r}_{\mathcal{Z}^{(r)}}$ is the left limit of $s \mapsto \tau_s^{*,r}$ at $s = \mathcal{Z}^{(r)}$. Then one verifies that $(\Pi^p(\mathcal{E}_{\tau_s^{*,r}}))_{0\leqslant t\leqslant \mathcal{Z}^{(r)}}$ is an injective loop whose range is precisely $\partial B^\bullet(r)$. Let us briefly justify this. Recall that $\partial B^\bullet(r) = \Pi^p(\partial F_r)$, with ∂F_r given by (5.50). We first observe that the mapping $s \mapsto \Pi^p(\mathcal{E}_{\tau_s^{*,r}})$ is continuous. Indeed, we already know that the function $s \mapsto \Pi^p(\mathcal{E}_s)$ is continuous. Furthermore, if $\tau_{s-}^{*,r} < \tau_s^{*,r}$, the support property of the exit local time (see the discussion following (8.7) in Section 5.2.2) implies that either all points of the form $\mathcal{E}_u$ with $u \in (\tau_{s-}^{*,r}, \tau_s^{*,r})$ are descendants of $\mathcal{E}_{\tau_{s-}^{*,r}}$ and necessarily $\mathcal{E}_{\tau_{s-}^{*,r}} = \mathcal{E}_{\tau_s^{*,r}}$, or the labels of all such points $\mathcal{E}_u$ are greater than r. In both cases, we have $\Pi^p(\mathcal{E}_{\tau_{s-}^{*,r}}) = \Pi^p(\mathcal{E}_{\tau_s^{*,r}})$. Then one easily deduces from the same support property that any point of the form $\Pi^p(\mathcal{E}_{\tau_s^{*,r}})$ belongs to $\partial B^\bullet(r)$. Conversely, using (5.50), any point x of $\partial B^\bullet(r)$, with the exception of the point L_r of the spine, must be of the form $\Pi^p(v)$ where v belongs to a subtree $\mathcal{T}_{(\omega^i)}$ with $i \in I_r \cup J_r$, and labels along the line segment between v and the root of $\mathcal{T}_{(\omega^i)}$ are greater than r except at v. From the support property of the exit local time, it follows that $v = \mathcal{E}_{\tau_s^{*,r}}$ for some $s \in [\alpha_i, \beta_i]$. The formula $v = \mathcal{E}_{\tau_s^{*,r}}$ also holds for $v = \mathrm{L}_r$ with $s = L_0^{*,r}$. Finally, from the description of the distribution of $\mathcal{N}_{(r)}$ and the fact that $\Delta^p(u,v) = 0$ holds if and only if $\Delta^{p,\circ}(u,v) = 0$, one checks that, for every $0 \leqslant s < s' \leqslant \mathcal{Z}^{(r)}$, the points $\Pi^p(\mathcal{E}_{\tau_s^{*,r}})$ and $\Pi^p(\mathcal{E}_{\tau_{s'}^{*,r}})$ are distinct, except in the case $s = 0$ and $s' = \mathcal{Z}^{(r)}$.

We let $\mathfrak{H}$ be derived from the disjoint union

$$[0, \mathcal{Z}^{(r)}] \cup \Big(\bigcup_{\substack{i\in I_r\cup J_r\\ k\in\mathbb{N}}} \mathcal{T}_{(\omega^{i,k})} \Big) \cup \check{\mathcal{T}}^{p,(r)}$$

by identifying 0 with $\mathcal{Z}^{(r)}$, the root of $\check{\mathcal{T}}^{p,(r)}$ with the point $L_0^{*,r}$ of $[0, \mathcal{Z}^{(r)}]$, and, for every $i \in I_r \cup J_r$ and $k \in \mathbb{N}$, the root of $\mathcal{T}_{(\omega^{i,k})}$ with the point $L^{*,r}_{\alpha_{i,k}}$ of $[0, \mathcal{Z}^{(r)}]$. We assign labels $(\Lambda_x^{(r)})_{x\in\mathfrak{H}}$ to the points of $\mathfrak{H}$: the label of any point of $[0, \mathcal{Z}^{(r)}]$ is equal to r, and points of the labeled trees $\mathcal{T}_{(\omega^{i,k})}$ and $\check{\mathcal{T}}^{p,(r)}$ keep their labels. We also define a volume measure on $\mathfrak{H}$ by summing the volume measures of the trees $\mathcal{T}_{(\omega^{i,k})}$ and of $\check{\mathcal{T}}^{p,(r)}$. The total volume of $\mathfrak{H}$ is

$$\Sigma^{(r)} := |\check{\mathcal{T}}^{p,(r)}| + \sum_{i\in I_r\cup J_r} \sum_{k\in\mathbb{N}} \sigma(\omega^{i,k}),$$

using the notation $|\check{\mathcal{T}}^{p,(r)}|$ for the total volume of $\check{\mathcal{T}}^{p,(r)}$.

We need to define a cyclic clockwise exploration of $\mathfrak{H}$, which will be denoted by $(\mathcal{E}^{(r)}_s)_{s\in[0,\Sigma^{(r)}]}$. Roughly speaking this exploration corresponds to concatenating the clockwise explorations of the trees $\mathcal{T}_{(\omega^{i,k})}$ and $\check{\mathcal{T}}^{p,(r)}$ in the order prescribed by the exploration of $\mathcal{T}^p_\infty$. To give a more precise definition, we first observe that we can write $K_r = K_r^\circ \cup \partial F_r$, where ∂F_r is as in (5.50), and

$$K_r^\circ := \{v \in \mathcal{T}^p_\infty : \Lambda_w \leqslant r \text{ for some } w \in]\!]v,\infty[\![\}.$$

If $v \in \partial F_r$, we know that $\Pi^p(v) \in \partial B^\bullet(r)$, so that, by previous observations, there is a unique $s \in [0, \mathcal{Z}^{(r)})$ such that $v = \Pi^p(\mathcal{E}_{\tau^{*,r}_s})$, and we set $\Phi_{(r)}(v) := s$.

We then define, for every $s \in \mathbb{R}$,

$$A^{(r)}_s := \int_{-\infty}^s \mathrm{d}t\, \mathbf{1}_{\{\mathcal{E}_t \in K_r\}}.$$

Note that $A^{(r)}_\infty = \Sigma^{(r)}$. We set $\eta^{(r)}_t := \inf\{s \in \mathbb{R} : A^{(r)}_s > t\}$ for every $t \in [0, \Sigma^{(r)})$. Then, for every $t \in [0, \Sigma^{(r)})$, either $\mathcal{E}_{\eta^{(r)}_t}$ belongs to K_r°, which implies that $\mathcal{E}_{\eta^{(r)}_t}$ is a point of one of the trees $\mathcal{T}_{(\omega^{i,k})}$ or of $\check{\mathcal{T}}^{p,(r)}$, and we let $\mathcal{E}^{(r)}_t$ be the "same" point in $\mathfrak{H}$, or $\mathcal{E}_{\eta^{(r)}_t}$ belongs to ∂F_r, and we set $\mathcal{E}^{(r)}_t = \Phi_{(r)}(\mathcal{E}_{\eta^{(r)}_t}) \in [0, \mathcal{Z}^{(r)})$. Finally we take $\mathcal{E}^{(r)}_{\Sigma^{(r)}} = \mathcal{E}^{(r)}_0 = 0$. Although this is not apparent in the preceding presentation, the reader will easily check that this exploration process $\mathcal{E}^{(r)}$ only depends on $(\mathcal{N}_{(r)}, L_0^{*,r})$ and $\mathfrak{X}^{(r)}$ (the reason why we need $L_0^{*,r}$ is because we have to rank the tree $\mathcal{T}^{p,(r)}$ among the trees $\mathcal{T}_{(\omega^{i,k})}$ – of course the order between the different trees $\mathcal{T}_{(\omega^{i,k})}$ is prescribed by the point measure $\mathcal{N}_{(r)}$).

The clockwise exploration of $\mathfrak{H}$ allows us to make sense of intervals on $\mathfrak{H}$. In turn, we can then define the function $D'^{(r),\circ}(u,v)$, for $u, v \in \mathfrak{H}$, by the right-hand side of (5.52), where we simply replace Λ by $\Lambda^{(r)}$. Similarly, we define $D'^{(r)}(u,v)$, for $u, v \in \mathfrak{H}$, by replacing $\Delta^{p,\circ}$ with $D'^{(r),\circ}$ in the right-hand side of (8.42) (and of course replacing $u_0, u_1, \ldots, u_p \in K_r$ by $u_0, u_1, \ldots, u_p \in \mathfrak{H}$). We now claim that the quotient space $\mathfrak{H}/\{D'^{(r)} = 0\}$, equipped with the metric induced by $D'^{(r)}$ and with the volume measure which is the pushforward of the volume measure on $\mathfrak{H}$, coincides with $(B^\bullet(r), \Delta^{p,(r)})$. This is a straightforward consequence of our construction (using the fact that one can replace $[u,v]$ by $[u,v] \cap K_r$ in the right-hand side of (5.52) when $u, v \in K_r$), and we omit the details.

We also observe that the conditional distribution of the space $\mathfrak{H}/\{D'^{(r)} = 0\}$ given $(\mathcal{Z}^{(r)}, L_0^{*,r})$ does not depend on $L_0^{*,r}$. This follows from the fact that the law of a Poisson point measure on $[0,z] \times \mathcal{S}$ with intensity $\mathrm{d}t\, \mathbb{N}_r(\mathrm{d}\omega)$ is invariant under the shift $t \mapsto t + a$ mod. z, for any fixed $a \in [0,z]$.

Finally, we can write $(B^\bullet(r), \Delta^{p,(r)}) = (\mathfrak{H}/\{D'^{(r)} = 0\}, D'^{(r)}) = \Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})$ with a $\mathbb{K}$-valued function Ψ, and we have for every nonnegative measurable function F on $\mathbb{K}$,

$$\begin{aligned}\mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid (X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})] &= \mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid (\mathcal{Z}^{(r)}, L_0^{*,r})]\\ &= \mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid \mathcal{Z}^{(r)}].\end{aligned}$$

The second equality follows from the preceding observation, and the first one holds because $\mathcal{N}_{(r)}$ and $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$ are conditionally independent given $\mathcal{Z}^{(r)}$ (and $L_0^{*,r}$ is a measurable function of $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$). Since $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$ is a function of $(X^{(r)}, \widetilde{\mathcal{L}}^{(r)}, \widetilde{\mathcal{R}}^{(r)})$, this gives the first assertion of Theorem 31.

The proof of the second assertion of Theorem 31 is very similar to the proof of Lemma 30, and we leave the details to the reader. □

Remark. The preceding proof gives a description of the distribution of the hull $B^\bullet(r)$ equipped with its intrinsic metric in terms of the space $\mathfrak{H}$. We note that the labeled tree $\check{\mathcal{T}}^{p,(r)}$ has the same distribution as the tree $\mathcal{T}_{(\omega)}$ under $\mathbb{N}_r(\mathrm{d}\omega \,|\, W_*(\omega) = 0)$ (see [69]). So the conditional distribution of $B^\bullet(r)$ knowing $\mathcal{Z}^{(r)} = z$ could as well be defined from a Poisson point measure $\sum_{i\in I} \delta_{(t_i,\omega_i)}$ with intensity $\mathbf{1}_{[0,z]}(t)\,\mathrm{d}t\,\mathbb{N}_r(\mathrm{d}\omega)$ conditioned on the event $\{\inf_{i\in I} W_*(\omega_i) = 0\}$. In this form, there is a striking analogy with the construction of the (free) Brownian disk with perimeter z found in [20] or [22] – see Section 5.6 below for a presentation within the formalism of the present work. The essential difference comes from the fact that the construction of the hull assigns constant labels equal to r to points of $\mathfrak{H}$ that belong to $[0, z]$, whereas, in the construction of the Brownian disk, labels along $[0, z]$ evolve like a scaled Brownian bridge.

5.5.2 Horohulls in the Brownian plane

In this section, we explain how pointed Brownian disks with a given height appear as horohulls in the Brownian plane. Let us first recall the definition of these horohulls. We consider the Brownian plane $(\mathbb{BP}_\infty, \Delta^p)$, with the distinguished point ρ. One can prove [39] that, a.s. for every $a, b \in \mathbb{BP}_\infty$, the limit

$$\lim_{x\to\infty} (\Delta^p(a, x) - \Delta^p(b, x))$$

exists in $\mathbb{R}$. Here the limit as $x \to \infty$ means that x tends to the point at infinity in the Alexandroff compactification of $\mathbb{BP}_\infty$. Clearly, the limit in the preceding display can be written in the form $\mathcal{H}_a - \mathcal{H}_b$, where the "horofunction" $a \mapsto \mathcal{H}_a$ is uniquely defined if we impose $\mathcal{H}_\rho = 0$. We interpret $\mathcal{H}_a$ as a (relative) distance from a to ∞, and call $\mathcal{H}_a$ the horodistance from a. Note the bound $|\mathcal{H}_a - \mathcal{H}_b| \leqslant \Delta^p(a, b)$.

For every $r > 0$, let $\mathfrak{B}^\circ(r)$ be the connected component of the open set $\{x \in \mathbb{BP}_\infty : \mathcal{H}_x > -r\}$ that contains ρ. So a point x belongs to $\mathfrak{B}^\circ(r)$ if and only if there is a continuous path from ρ to r that stays at horodistance greater than $-r$. The horohull $\mathfrak{B}^\bullet(r)$ is defined as the closure of $\mathfrak{B}^\circ(r)$. We view $\mathfrak{B}^\bullet(r)$ as a pointed compact measure metric space with distinguished point ρ. Note that the compactness of $\mathfrak{B}^\bullet(r)$ is not obvious a priori, but will follow from the description that we give in the proof of the next statements.

We write $\mathrm{Vol}(\cdot)$ for the volume measure on $\mathbb{BP}_\infty$. In the following two statements, we fix $r > 0$.

Proposition 32. *The limit*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-2}\,\mathrm{Vol}(\{x \in \mathfrak{B}^\bullet(r) : \mathcal{H}_x < -r + \varepsilon\})$$

exists a.s. This limit is called the boundary size of $\mathfrak{B}^\bullet(r)$ and denoted by $|\partial\mathfrak{B}^\bullet(r)|$.

Theorem 33. *The intrinsic metric on* $\mathfrak{B}^\circ(r)$ *has a.s. a continuous extension to* $\mathfrak{B}^\bullet(r)$*, which is denoted by* $\Delta_\infty^{horo,r}$*. Then conditionally on* $|\partial\mathfrak{B}^\bullet(r)| = z$*, the pointed measure metric space* $(\mathfrak{B}^\bullet(r), \Delta_\infty^{horo,r})$ *is a pointed Brownian disk with perimeter* z *and height* r*.*

Remark. In a way similar to Theorem 31, we could have stated an independence property: The closure of the horohull complement $\mathbb{BP}_\infty \backslash \mathfrak{B}^\bullet(r)$ equipped with its (extended) intrinsic distance is independent of the horohull $(\mathfrak{B}^\bullet(r), \Delta_\infty^{horo,r})$ conditionally on the boundary size $|\partial\mathfrak{B}^\bullet(r)|$. The distribution of this horohull complement can also be described, but it is *not* that of an infinite volume Brownian disk (all points of the boundary must be at the same horodistance). We shall leave these extensions to the reader.

The proof of both Proposition 32 and Theorem 33 relies on the construction of the Brownian plane found in [39], which is different from the one given in Section 5.4.2. Let us recall the construction of [39] using our formalism of coding triples (the presentation therefore seems to differ from the one in [39], but the relevant random objects are the same). We consider a coding triple $(B, \mathcal{P}, \mathcal{P}')$, such that:

- $B = (B_t)_{t \geqslant 0}$ is a linear Brownian motion started from 0.
- Conditionally on B, $\mathcal{P}$ and $\mathcal{P}'$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $2\,\mathrm{d}t\,\mathbb{N}_{B_t}(\mathrm{d}\omega)$.

Following Section 5.2.4, we then consider the infinite labeled tree $(\mathcal{T}_\infty'^p, (\Lambda_v')_{v \in \mathcal{T}_\infty'^p})$ associated with this coding triple. We define the functions $D^{\infty,\circ}(u,v)$ and $D^\infty(u,v)$, for $u, v \in \mathcal{T}_\infty'^p$, in a way similar to Section 5.4.1 (note however that labels are here of arbitrary sign):

$$D^{\infty,\circ}(u,v) = \Lambda_u' + \Lambda_v' - 2\max\Big(\inf_{w\in[u,v]} \Lambda_w', \inf_{w\in[v,u]} \Lambda_w'\Big), \tag{5.55}$$

and

$$D^\infty(u,v) = \inf_{u_0=u,u_1,\dots,u_p=v} \sum_{i=1}^{p} D^{\infty,\circ}(u_{i-1}, u_i) \tag{5.56}$$

where the infimum is over all choices of the integer $p \geqslant 1$ and of the finite sequence $u_0, u_1, \dots, u_p$ in $\mathcal{T}_\infty'^p$ such that $u_0 = u$ and $u_p = v$. We have $D^\infty(u,v) = 0$ if and only if $D^{\infty,\circ}(u,v) = 0$ [39, Proposition 11].

We let $\mathbb{BP}_\infty'$ be the quotient space $\mathcal{T}_\infty'^p / \{D^\infty = 0\}$, which is equipped with the distance induced by $D^\infty(u,v)$ and the volume measure which is the pushforward of the volume measure on $\mathcal{T}_\infty'^p$, and with the distinguished point which is the equivalence class of the root of $\mathcal{T}_\infty'^p$. We also let Π'^p stand for the canonical projection from $\mathcal{T}_\infty'^p$ onto $\mathbb{BP}_\infty'$.

Then the pointed measure metric space $\mathbb{BP}_\infty'$ is a Brownian plane, that is, it has the same distribution as $\mathbb{BP}_\infty$ (see [40, Theorem 3.4]). Therefore, we can replace $\mathbb{BP}_\infty$ by $\mathbb{BP}_\infty'$ in the proof of both Proposition 32 and Theorem 33. The point of this replacement is the fact that the horodistance from a point a of $\mathbb{BP}_\infty'$ is now equal to its label Λ_a' [40, Proposition 17]. Indeed, we can summarize the difference between the two constructions of the Brownian plane by saying that labels correspond to distances from the distinguished point in the first construction, and to horodistances in the second one. In the proofs below, we assume that $\mathfrak{B}^\circ(r)$ and $\mathfrak{B}^\bullet(r)$ are defined in $\mathbb{BP}'_\infty$, and without risk of confusion we use the notation ρ both for the root of $\mathcal{T}'^p_\infty$ and for the distinguished point of $\mathbb{BP}'_\infty$.

Proof of Proposition 32 and Theorem 33. The first step is to observe that we have $\mathfrak{B}^\circ(r) = \Pi'^p(G_r)$, where

$$G_r := \{v \in \mathcal{T}'^p_\infty : \Lambda'_w > -r \text{ for every } w \in [\![\rho, v]\!]\}, \tag{5.57}$$

and $\mathfrak{B}^\bullet(r) = \mathfrak{B}^\circ(r) \cup \partial\mathfrak{B}^\circ(r)$, with $\partial\mathfrak{B}^\circ(r) = \Pi'^p(\partial G_r)$, and

$$\partial G_r = \{v \in \mathcal{T}'^p_\infty : \Lambda'_v = -r \text{ and } \Lambda'_w > -r \text{ for every } w \in [\![\rho, v[\![\}. \tag{5.58}$$

Notice the similarity with (5.49) and (5.50). Let us justify the equality $\mathfrak{B}^\circ(r) = \Pi'^p(G_r)$. The inclusion $\mathfrak{B}^\circ(r) \supset \Pi'^p(G_r)$ is easy, because, if $v \in G_r$, the image under Π'^p of the geodesic segment $[\![\rho, v]\!]$ yields a continuous path from $\Pi'^p(v)$ to $\Pi'^p(\rho)$ along which labels (horodistances) stay greater than $-r$. The reverse inclusion comes from the so-called "cactus bound" which says than any continuous path between $\Pi'^p(\rho)$ and $\Pi'^p(v)$ must visit a point whose label is smaller than or equal to $\min_{u\in[\![\rho,v]\!]} \Lambda'_u$ (see formula (4) in [39] for a short proof in the case of the Brownian map, which is immediately extended to the present setting). Once the equality $\mathfrak{B}^\circ(r) = \Pi'^p(G_r)$ is established, the property $\partial\mathfrak{B}^\circ(r) = \Pi'^p(\partial G_r)$ is easy and we omit the details.

Write $\mathrm{cl}(G_r) = G_r \cup \partial G_r$, which we can view as a (compact) subtree of the tree $\mathcal{T}'^p_\infty$. In a way very similar to the proof of Theorem 29, we may interpret $\mathrm{cl}(G_r)$ as the (labeled) tree associated with a coding triple derived from the triple $(B, \mathcal{P}, \mathcal{P}')$. To this end, we set

$$T_r := \inf\{t \geqslant 0 : B_t = -r\},$$

and we note that $\mathrm{cl}(G_r)$ consists of the union of the part $[0, T_r]$ of the spine of $\mathcal{T}'^p_\infty$ with the subtrees branching off the spine between levels 0 and T_r and truncated at label $-r$. To make this more precise, if

$$\mathcal{P} = \sum_{i\in I} \delta_{(t_i,\omega_i)},$$

we define

$$\mathcal{P}^{(r)} = \sum_{i\in I, t_i<T_r} \delta_{(t_i,\mathrm{tr}_{-r}(\omega_i))}$$

and we similarly define $\mathcal{P}'^{(r)}$ from $\mathcal{P}'$. Let $B^{(r)}$ stand for the stopped path $(B_t)_{0\leqslant t\leqslant T_r}$. Then $\mathrm{cl}(G_r)$ is canonically and isometrically identified with the (labeled) tree coded by the triple $(B^{(r)}, \mathcal{P}^{(r)}, \mathcal{P}'^{(r)})$. This identification preserves the labels and the volume measures. The fact that the limit in Proposition 32 exists, and is in fact given by

$$|\partial\mathfrak{B}^\bullet(r)| = \int \mathcal{Z}_{-r}(\omega)\, \mathcal{P}^{(r)}(\mathrm{d}\omega) + \int \mathcal{Z}_{-r}(\omega)\, \mathcal{P}'^{(r)}(\mathrm{d}\omega)$$

now follows from the approximation formula (8.7) for exit measures, using also Proposition 2 and (5.9).

Recall the notation ϑ_r introduced before Proposition 4. In order to derive the statement of Theorem 33, we now notice that the triple $\mathfrak{T}^{(r)} := (B^{(r)} + r, \vartheta_{-r}\mathcal{P}^{(r)}, \vartheta_{-r}\mathcal{P}'^{(r)})$ has the same distribution as the coding triple $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ considered at the beginning of Section 5.3.2, provided we take $a = r$. It follows that the conditional distribution of $\mathfrak{T}^{(r)}$ knowing that $|\partial\mathfrak{B}^\bullet(r)| = z$ is $\check{\Theta}^r_z$. Recall the mapping Ω defined in Section 5.2.4. Then $\Omega(\mathfrak{T}^{(r)})$ is a random snake trajectory which, conditionally on $|\partial\mathfrak{B}^\bullet(r)| = z$, is distributed according to $\mathbb{N}^{(z)}_r$. Furthermore, by Proposition 14, the random metric space $\mathbb{D}^{\bullet,r} := \Xi^\bullet(\Omega(\mathfrak{T}^{(r)}))$ is a pointed Brownian disk with perimeter z and height r, conditionally on $|\partial\mathfrak{B}^\bullet(r)| = z$. To complete the proof, we just need to identify $\mathfrak{B}^\bullet(r)$ (equipped with the intrinsic distance) with $\mathbb{D}^{\bullet,r}$.

By a preceding observation, $\mathrm{cl}(G_r)$ is identified to the genealogical tree of $\Omega(\mathfrak{T}^{(r)})$ (which is the labeled tree associated with $\mathfrak{T}^{(r)}$) and this identification preserves labels, provided labels on $\mathrm{cl}(G_r)$ are shifted by r. One then verifies that two points of $\mathrm{cl}(G_r)$ are identified in $\Pi'^p(\mathrm{cl}(G_r)) = \mathfrak{B}^\bullet(r)$ if and only if the corresponding points of the genealogical tree of $\Omega(\mathfrak{T}^{(r)})$ are identified in $\mathbb{D}^{\bullet,r}$. It follows that $\mathfrak{B}^\bullet(r)$ and $\mathbb{D}^{\bullet,r}$ can be identified as sets. To complete the proof of Theorem 33, it then remains to show that the intrinsic distance between two points of $\mathfrak{B}^\circ(r)$ coincides with the distance between the corresponding points of the interior of $\mathbb{D}^{\bullet,r}$ (from the discussion in Section 5.4.3, this will imply first that the intrinsic distance on $\mathfrak{B}^\circ(r)$ can be extended to the boundary, and then that $\mathfrak{B}^\bullet(r)$ is isometric to $\mathbb{D}^{\bullet,r}$ as desired). This is derived by arguments very similar to the end of the proof of Theorem 29, and we omit the details. □

We conclude this section with some explicit distributional properties of the process of horohulls. It will be convenient to use the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions from $\mathbb{R}_+$ into $\mathbb{R}$. We write $(Z_t)_{t\geqslant 0}$ for the canonical process on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, and $(\mathcal{F}_t)_{t\geqslant 0}$ for the canonical filtration. We then introduce, for every $x \geqslant 0$, the probability measure P_x which is the law of the continuous-state branching process with branching mechanism Φ (in short, the Φ-CSBP), where $\Phi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$. We refer to [40, Section 2.1] for the definition and some properties of the Φ-CSBP.

The Φ-CSBP is critical, meaning that $E_x[Z_t] = x$ for every $t \geqslant 0$ and $x \geqslant 0$. Then, for every $x > 0$, we can define the law $P^\uparrow_x$ of the Φ-CSBP started from x and conditioned on non-extinction via the h-transform

$$\frac{\mathrm{d}P^\uparrow_x}{\mathrm{d}P_x}\Big|_{\mathcal{F}_t} = \frac{Z_t}{x}. \tag{5.59}$$

See [62, Section 4.1] for a discussion of continuous-state branching processes conditioned on non-extinction. The preceding formula does not make sense for $x = 0$. However, [60, Theorem 2] shows that the laws $P^\uparrow_x$ converge weakly as $x \downarrow 0$ to a limiting law denoted by $P^\uparrow_0$, which is characterized by the following two properties:

(i) for every $t > 0$, the law of Z_t under $P^\uparrow_0$ is given by

$$E^\uparrow_0[e^{-\lambda Z_t}] = \Big(1 + t\sqrt{2\lambda/3}\Big)^{-3}, \quad \lambda \geqslant 0,$$

so that in particular $Z_t > 0$, $P^\uparrow_0$ a.s.;

(ii) for every $t > 0$, under $P_0^{\uparrow}$, conditionally on $(Z_u)_{0\leqslant u\leqslant t}$, the process $(Z_{t+s})_{s\geqslant 0}$ is distributed according to $P_{Z_t}^{\uparrow}$.

From (5.13), property (i) is equivalent to saying that the density of $\mathcal{Z}_t$ is $z\, h_t(z)$.

In the next proposition, we take $|\partial\mathfrak{B}^\bullet(0)| = 0$ by convention.

Proposition 34. *The process $(|\partial\mathfrak{B}^\bullet(r)|)_{r\geqslant 0}$ has a càdlàg modification, which is distributed according to $P_0^{\uparrow}$.*

Proof. As a preliminary observation, we recall from [40, Section 2.2] that the exit measure process $(\mathcal{Z}_{-r})_{r>0}$ is Markovian under $\mathbb{N}_0$, with the transition kernels of the Φ-CSBP. In other words, we can find a càdlàg modification of $(\mathcal{Z}_{-r})_{r>0}$ such that, for every $t > 0$, the conditional distribution of $(\mathcal{Z}_{-t-r})_{r\geqslant 0}$ under $\mathbb{N}_0$ and knowing $(\mathcal{Z}_{-u})_{0<u\leqslant t}$ is $P_{\mathcal{Z}_{-t}}$.

In order to get the statement of the proposition, it suffices to verify that the finite-dimensional distributions of the process $(|\partial\mathfrak{B}^\bullet(r)|)_{r\geqslant 0}$ coincide with the finite-dimensional marginals under $\mathbb{P}_0^{\uparrow}$. So we need to verify that, for every $0 < t_1 < \cdots < t_p$, for every nonnegative measurable functions $\varphi_1, \dots, \varphi_p$ on $\mathbb{R}_+$, we have

$$\mathbb{E}\Big[\varphi_1(|\partial\mathfrak{B}^\bullet(r_1)|)\cdots\varphi_1(|\partial\mathfrak{B}^\bullet(r_p)|)\Big] = E_0^{\uparrow}[\varphi_1(Z_{r_1})\cdots\varphi_p(Z_{r_p})]. \tag{5.60}$$

Now recall from the preceding proof that, for every $j = 1, \dots, p$,

$$|\partial\mathfrak{B}^\bullet(r_j)| = \int \mathbf{1}_{\{t<T_{r_j}\}}\, \mathcal{Z}_{-r_j}(\omega)\, \mathcal{P}(\mathrm{d}t\,\mathrm{d}\omega) + \int \mathbf{1}_{\{t<T_{r_j}\}}\, \mathcal{Z}_{-r_j}(\omega)\, \mathcal{P}'(\mathrm{d}t\,\mathrm{d}\omega).$$

It then follows from (5.10) that

$$\begin{aligned}\mathbb{E}\Big[\varphi_1(|\partial\mathfrak{B}^\bullet(r_1)|)\cdots\varphi_1(|\partial\mathfrak{B}^\bullet(r_p)|)\Big] &= \mathbb{N}_0\Big(\mathcal{Z}_{-r_p}\,\varphi_1(\mathcal{Z}_{-r_1})\cdots\varphi_p(\mathcal{Z}_{-r_p})\Big)\\ &= \mathbb{N}_0\Big(\varphi_1(\mathcal{Z}_{-r_1})\,E_{\mathcal{Z}_{-r_1}}\Big[Z_{r_p-r_1}\,\varphi_2(Z_{r_2-r_1})\cdots\varphi_p(Z_{r_p-r_1})\Big]\Big),\end{aligned}$$

where we use the first observation of the proof in the last equality. Thanks to the h-transform relation (5.59), the right-hand side is also equal to

$$\mathbb{N}_0\Big(\varphi_1(\mathcal{Z}_{-r_1})\mathcal{Z}_{-r_1}E^{\uparrow}_{\mathcal{Z}_{-r_1}}\Big[\varphi_2(Z_{r_2-r_1})\cdots\varphi_p(Z_{r_p-r_1})\Big]\Big) = \int_0^\infty \mathrm{d}z\; z h_{r_1}(z)\varphi_1(z)E_z^{\uparrow}\Big[\varphi_2(Z_{r_2-r_1})\cdots\varphi_p(Z_{r_p-r_1})\Big]$$

since the density of $\mathcal{Z}_{-r_1}$ under $\mathbb{N}_0(\cdot\cap\{\mathcal{Z}_{-r_1}\neq 0\})$ is h_{r_1} (Proposition 3). Finally, properties (i) and (ii) above show that the right-hand side of the last display equals $E_0^{\uparrow}[\varphi_1(Z_{r_1})\cdots\varphi_p(Z_{r_p})]$, which completes the proof. $\square$

In the next proposition, we compute the joint distribution of the boundary size and the volume of the horohull $\mathfrak{B}^\bullet(r)$.

Proposition 35. *Let $r > 0$. We have, for every $\lambda \geqslant 0$ and $\mu > 0$,*

$$\mathbb{E}\Big[\exp\Big(-\lambda|\partial\mathfrak{B}^\bullet(r)| - \mu\mathrm{Vol}(\mathfrak{B}^\bullet(r))\Big)\Big] = \frac{(\frac{2}{3}+\frac{\lambda}{3}\sqrt{2/\mu}\,)^{-1/2}\sinh((2\mu)^{1/4}r)+\cosh((2\mu)^{1/4}r)}{\Big((\frac{2}{3}+\frac{\lambda}{3}\sqrt{2/\mu}\,)^{1/2}\sinh((2\mu)^{1/4}r)+\cosh((2\mu)^{1/4}r)\Big)^3}.$$

Proof. From the fact that $\mathfrak{B}^\bullet(r) = \Pi'^p(G_r \cup \partial G_r)$, with G_r and ∂G_r given by (5.57) and (5.58), we easily obtain that

$$\mathrm{Vol}(\mathfrak{B}^\bullet(r))) = \int \mathbf{1}_{\{t<T_r\}}\, \sigma(\mathrm{tr}_{-r}(\omega))\, \mathcal{P}(\mathrm{d}t\,\mathrm{d}\omega) + \int \mathbf{1}_{\{t<T_r\}}\, \sigma(\mathrm{tr}_{-r}(\omega))\, \mathcal{P}'(\mathrm{d}t\,\mathrm{d}\omega).$$

Thanks to the similar formula for $|\partial\mathfrak{B}^\bullet(r)|$, and to Proposition 2, we get

$$\mathbb{E}\Big[\exp(-\lambda|\partial\mathfrak{B}^\bullet(r)| - \mu\mathrm{Vol}(\mathfrak{B}^\bullet(r)))\Big] = \mathbb{N}_0\Big(\mathcal{Z}_{-r}\,\exp(-\lambda\mathcal{Z}_{-r} - \mu\mathcal{Y}_{-r})\Big),$$

where $\mathcal{Y}_{-r} = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_{-r}(W_s)=\infty\}}$. By Lemma 4.5 in [40], we have, for $\lambda > \sqrt{2\mu}$,

$$\mathbb{N}_0\Big(1 - \exp(-\lambda\mathcal{Z}_{-r} - \mu\mathcal{Y}_{-r})\Big) = \sqrt{\frac{\mu}{2}}\left(3\left(\coth\left((2\mu)^{1/4}r + \coth^{-1}\left(\sqrt{\tfrac{2}{3} + \tfrac{\lambda}{3}\sqrt{2/\mu}}\right)\right)\right)^2 - 2\right).$$

By differentiating with respect to λ, we get the formula of the proposition. The restriction to $\lambda > \sqrt{2\mu}$ can be removed by an argument of analytic continuation. □

Remark. Up to unimportant scaling constants, the formula of Proposition 35 already appears in [42, Proposition 4], which deals with asymptotics for the boundary size and volume of the (discrete) horohulls in the UIPT. This should not come as a surprise since the Brownian plane is known to be the scaling limit of the UIPT [29]. Note however that it would not be easy to deduce Proposition 35 from the corresponding discrete result.

Our last proposition characterizes the distribution of the process $(|\partial\mathfrak{B}^\bullet(r)|, \mathrm{Vol}(\mathfrak{B}^\bullet(r)))_{r>0}$. This is an analog of [40, Theorem 1.3], which is concerned with the usual hulls in the Brownian plane.

Proposition 36. *Let $U = (U_t)_{t\geqslant 0}$ be a random process distributed according to $P_0^\uparrow$, and let $s_1, s_2, \ldots$ be a measurable enumeration of jump times of U. Let $\xi_1, \xi_2, \ldots$ be an independent sequence of positive random variables distributed according to the density $(2\pi x^5)^{-1/2}\exp(-1/2x)$. Assume that the sequence $(\xi_1, \xi_2, \ldots)$ is independent of the process U. Then,*

$$\Big(|\partial\mathfrak{B}^\bullet(r)|, \mathrm{Vol}(\mathfrak{B}^\bullet(r))\Big)_{r>0} \overset{\text{(d)}}{=} \left(U_r, \sum_{i:s_i\leqslant r} \xi_i\,(\Delta U_{s_i})^2\right)_{r>0}.$$

From our presentation of the Brownian plane in terms of the triple $(B, \mathcal{P}, \mathcal{P}')$, and using Proposition 2 to relate this triple to the Brownian snake excursion measure, Proposition 36 follows as a straightforward application of the excursion theory developed in [2] (see in particular Theorem 40 and Proposition 32 in [2]). We omit the details of the proof.

5.5.3 Removing a strip from the Brownian half-plane

In this section, we give an analog of Theorem 29 showing that, if one removes a strip of width r from the Brownian half-plane, the resulting space equipped with its intrinsic metric is again a Brownian half-plane. We let $(\mathbb{H}_\infty, \Delta^{hp})$ stand for the Brownian half-plane constructed from a coding triple $(R, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$ as explained in Section 5.4.5. Recall that $\mathbb{H}_\infty$ is obtained as a quotient space of the labeled tree $\mathcal{T}_\infty^{hp}$ associated with $(R, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$, and that every $x \in \mathbb{H}_\infty$ thus has a label Λ_x, which is equal to the distance from x to the boundary $\partial\mathbb{H}_\infty$.

We fix $r > 0$ and set

$$\mathbb{H}_\infty^{(r)} := \{x \in \mathbb{H}_\infty : \Lambda_x \geqslant r\}.$$

The interior $\mathbb{H}_\infty^{(r),\circ}$ is $\{x \in \mathbb{H}_\infty : \Lambda_x > r\}$. We distinguish a special point $x_{(r)}$ of the boundary of $\mathbb{H}_\infty^{(r)}$, which corresponds to the point of the spine of $\mathcal{T}_\infty^{hp}$ at height $\mathbf{L}_r = \sup\{t \geqslant 0 : R_t = r\}$.

Theorem 37. *The intrinsic metric on $\mathbb{H}_\infty^{(r),\circ}$ has a unique continuous extension to $\mathbb{H}_\infty^{(r)}$, which is a metric on this space. Furthermore, the resulting random measure metric space pointed at $x_{(r)}$ is a Brownian half-plane.*

Proof. This is very similar to the proof of Theorem 29, and we only sketch the arguments. We first introduce the process $(R_t^{(r)})_{t\geqslant 0}$ defined by

$$R_t^{(r)} := R_{\mathbf{L}_r+t} - r,$$

and we note that $(R_t^{(r)})_{t\geqslant 0}$ is also a three-dimensional Bessel process started at 0. Recalling the point measures $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$ used in the construction of $\mathbb{H}_\infty$, we define two other point measures $\widetilde{\mathcal{P}}^{(r)}$ and $\widetilde{\mathcal{P}}'^{(r)}$ on $\mathbb{R}_+ \times \mathcal{S}$ by setting, for every nonnegative measurable function Φ on $\mathbb{R}_+ \times \mathcal{S}$,

$$\langle \widetilde{\mathcal{P}}^{(r)}, \Phi \rangle = \int \mathcal{P}(\mathrm{d}t\mathrm{d}\omega)\, \mathbf{1}_{(\mathbf{L}_r,\infty)}(t)\, \Phi(t - \mathbf{L}_r, \vartheta_r(\mathrm{tr}_r\omega)),$$

and similarly for $\widetilde{\mathcal{P}}'$, where we recall the notation ϑ_r for the shift on snake trajectories. Then it is straightforward to verify that the coding triples $(R, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$ and $(R^{(r)}, \widetilde{\mathcal{P}}^{(r)}, \widetilde{\mathcal{P}}'^{(r)})$ have the same distribution.

Consequently, the construction of Section 5.4.5 applied to the triple $(R^{(r)}, \widetilde{\mathcal{P}}^{(r)}, \widetilde{\mathcal{P}}'^{(r)})$ yields a pointed measure metric space $(\mathbb{H}'^{(r)}_\infty, \Delta'^{hp,(r)})$ which is a Brownian half-plane. To complete the proof we just have to identify $(\mathbb{H}'^{(r)}_\infty, \Delta'^{hp,(r)})$ with the space $\mathbb{H}_\infty^{(r)}$ equipped with its intrinsic metric. This is done in the same way as in the proof of Theorem 29 and we omit the details. ▫

Remark. We could also have derived an analog of Theorem 31 showing that the space $\mathbb{H}_\infty^{(r)}$ in Theorem 37 is independent of the strip $\mathbb{H}_\infty \backslash \mathbb{H}_\infty^{(r)}$ equipped with its intrinsic metric. We leave the precise formulation and proof of this result to the reader.

5.6 Consistency with previous definitions

In this section, we show that our definitions of the infinite-volume Brownian disk and of the Brownian half-plane are consistent with the previous definitions in [14] and [53]. This is relatively easy for the Brownian half-plane but somewhat more delicate for the infinite-volume Brownian disk.

We start by recalling the definition of the free pointed Brownian disk that can be found in [14, 20, 22]. Our presentation uses the notation introduced in the preceding sections and is therefore slightly different from the one in the previous papers.

We fix $z > 0$ and consider a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{(t_i,\omega_i)}$ on $\mathbb{R}_+ \times \mathcal{S}$ with intensity

$$2\,\mathbf{1}_{[0,z]}(t)\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega).$$

We then introduce the compact metric space $\mathcal{T}'$, which is obtained from the disjoint union

$$[0,z] \cup \Big(\bigcup_{i\in I} \mathcal{T}_{(\omega_i)}\Big) \tag{5.61}$$

by identifying 0 with z and, for every $i \in I$, the root $\rho_{(\omega_i)}$ of $\mathcal{T}_{(\omega_i)}$ with the point t_i of $[0,z]$. The metric on $\mathcal{T}'$ is defined in a very similar manner to Section 5.2.4. For instance, if $v \in \mathcal{T}_{(\omega_i)}$ and $w \in \mathcal{T}_{(\omega_j)}$, with $j \neq i$, the distance between v and w is

$$d_{(\omega_i)}(v,\rho_{(\omega_i)}) + \min\{(t_i \vee t_j) - (t_i \wedge t_j), z - (t_i \vee t_j) + (t_i \wedge t_j)\} + d_{(\omega_j)}(\rho_{(\omega_j)}, w),$$

and the reader will easily guess the formula in other cases. The volume measure on $\mathcal{T}'$ is just the sum of the volume measures on the trees $\mathcal{T}_{(\omega_i)}$, $i \in I$.

Set $\sigma' := \sum_{i\in I} \sigma(\omega_i)$. We can define a clockwise exploration $(\mathcal{E}'_t)_{0\leqslant t\leqslant \sigma'}$ of $\mathcal{T}'$, basically by concatenating the mappings $p_{(\omega_i)} : [0,\sigma(\omega_i)] \longrightarrow \mathcal{T}_{(\omega_i)}$ in the order prescribed by the t_i's. Note that, as in the finite spine case of Section 5.2.4, this exploration is cyclic (because 0 and z have been identified in $\mathcal{T}'$). The clockwise exploration allows us to define intervals in the space $\mathcal{T}'$, exactly as in Section 5.2.4.

We next assign real labels to the points of $\mathcal{T}'$. To this end we let $(\beta_t)_{0\leqslant t\leqslant z}$ be a standard Brownian bridge (starting and ending at 0) over the time interval $[0,z]$, which is independent of $\mathcal{N}$. For $t \in [0,z]$, we set $\ell'_t = \sqrt{3}\,\beta_t$, and for $v \in \mathcal{T}_{(\omega_i)}$, $i \in I$,

$$\ell'_v = \sqrt{3}\,\beta_{t_i} + \ell_v(\omega_i),$$

where $\ell_v(\omega_i)$ denotes the label of v in $\mathcal{T}_{(\omega_i)}$, as in Section 5.2.1. Then, $\min\{\ell'_v : v \in \mathcal{T}'\}$ is attained at a unique point v_* of $\mathcal{T}'$.

We may now define the pseudo-metric functions D'° and D' exactly as in (5.55) and (5.56),

$$D'^\circ(u,v) = \ell'_u + \ell'_v - 2\max\Big(\inf_{w\in[u,v]} \ell'_w, \inf_{w\in[v,u]} \ell'_w\Big), \tag{5.62}$$

and

$$D'(u,v) = \inf_{u_0=u,u_1,\dots,u_p=v} \sum_{i=1}^{p} D'^\circ(u_{i-1},u_i) \tag{5.63}$$

where the infimum is over all choices of the integer $p \geqslant 1$ and of the finite sequence $u_0, u_1, \dots, u_p$ in $\mathcal{T}'$ such that $u_0 = u$ and $u_p = v$. It is immediate to verify that, for every $u \in \mathcal{T}'$, $D'^\circ(u,v_*) = D'(u,v_*) = \ell'_u - \ell'_{v_*}$.

Let $\mathbb{D}'^{\bullet}_z$ denote the space $\mathcal{T}'/\{D'=0\}$, which is equipped with the metric induced by D', with the pushforward of the volume measure on $\mathcal{T}'$, and with the distinguished point which is the equivalence class of v_* (without risk of confusion, we will also write v_* for this equivalence class). Then $\mathbb{D}'^{\bullet}_z$ is a free pointed Brownian disk with perimeter z whose boundary $\partial\mathbb{D}'^{\bullet}_z$ is the image of $[0,z]$ under the canonical projection. This construction is basically the one in [22, Section 2.3], and it is consistent with Definition 13 as we already noted after this definition. We set

$$H'_z = D'(v_*, \partial\mathbb{D}'^{\bullet}_z) = \min\{\ell'_v : v \in [0,z]\} - \ell'_{v_*}. \tag{5.64}$$

A variant of the preceding construction yields the infinite-volume Brownian disk with perimeter z as considered[2] in [14]. We keep the same notation as before, and we introduce an infinite labeled tree $\mathcal{T}'_\infty$ which has the same distribution as the tree $\mathcal{T}'^{p}_\infty$ of Section 5.5.2 (so this is the labeled tree associated with a triple $(B,\mathcal{P},\mathcal{P}')$ whose distribution is specified in Section 5.5.2). We assume that $\mathcal{N}$ and $\mathcal{T}'_\infty$ are independent, and we also consider a random variable U uniformly distributed over $[0,z]$ and independent of the pair $(\mathcal{N},\mathcal{T}'_\infty)$. Then we let $\mathcal{T}'^{(\infty)}$ be derived from the disjoint union

$$[0,z] \cup \Big(\bigcup_{i\in I} \mathcal{T}_{(\omega_i)}\Big) \cup \mathcal{T}'_\infty \tag{5.65}$$

by the same identifications as in (5.61), and furthermore by identifying the root of $\mathcal{T}'_\infty$ with the point U of $[0,z]$. The metric on $\mathcal{T}'^{(\infty)}$ is defined as in the case of $\mathcal{T}'$. The clockwise exploration $(\mathcal{E}'^{(\infty)}_t)_{t\in\mathbb{R}}$ of $\mathcal{T}'^{(\infty)}$ is then defined in much the same way as in the infinite spine case of Section 5.2.4: We have $\mathcal{E}'^{(\infty)}_0 = 0 = z$, and the points $(\mathcal{E}'^{(\infty)}_t)_{t<0}$ now correspond to the right side of the tree $\mathcal{T}'_\infty$, to the trees $\mathcal{T}_{(\omega_i)}$ with $t_i > U$ and to the interval $[U,z)$, and similarly for the points $(\mathcal{E}'^{(\infty)}_t)_{t>0}$. The labels $\ell'^{(\infty)}_v$ on $\mathcal{T}'^{(\infty)}$ are obtained exactly as in the case of $\mathcal{T}'$, using the same Brownian bridge β and taking $\ell'^{(\infty)}_v = \sqrt{3}\,\beta_U + \Lambda'_v$ when $v \in \mathcal{T}'_\infty$, where Λ'_v stands for the label of v in $\mathcal{T}'_\infty$.

We may now define the pseudo-metric functions $D'^{\circ(\infty)}(u,v)$ and $D'^{(\infty)}(u,v)$ on $\mathcal{T}'^{(\infty)}$ by the very same formulas as in (5.62) and (5.63), just replacing the labels ℓ'_v by $\ell'^{(\infty)}_v$ (and noting that the clockwise exploration $(\mathcal{E}'^{(\infty)}_t)_{t\in\mathbb{R}}$ allows us to define intervals on $\mathcal{T}'^{(\infty)}$, exactly as in Section 5.2.4).

We then define $\mathbb{D}'^{\infty}_z$ as the quotient space $\mathcal{T}'^{(\infty)}/\{D'^{(\infty)}=0\}$, which is equipped with the metric induced by $D'^{(\infty)}$, with the volume measure which is the pushforward of the volume measure on $\mathcal{T}'^{(\infty)}$ and with the distinguished point which is the equivalence class of 0. In the terminology of [14], $\mathbb{D}'^{\infty}_z$ is an infinite-volume Brownian disk with perimeter z. The next proposition shows that this is consistent with Definition 17.

Proposition 38. *The pointed locally compact measure metric spaces $\mathbb{D}^{\infty}_z$ and $\mathbb{D}'^{\infty}_z$ have the same distribution.*

[2] Unfortunately, it seems that the definition given in [14] is slightly incorrect. We believe that the construction below is the correct way to define the infinite-volume Brownian disk as it appears in the limit theorems proved in [14].

We will deduce Proposition 38 from Proposition 39 below, which shows that $\mathbb{D}'^{\infty}_z$ is a limit of conditioned Brownian disks, in a way similar to Theorem 21 for $\mathbb{D}^{\infty}_z$. We note that [14] proves that the space $\mathbb{D}'^{\infty}$ is the limit in distribution of Brownian disks with perimeter z conditioned to have a large volume, but it is not so easy to verify that this conditioning has the same effect as the one in Theorem 21, which involves the height of the distinguished point.

Let us start with some preliminary observations. Since $\mathbb{D}'^{\bullet}_z$ has the same distribution as $\mathbb{D}^{\bullet}_z$, we know from the discussion after Proposition 14 that there exists a measure μ'_z on $\partial\mathbb{D}'^{\bullet}_z$ with total mass z, such that, a.s. for any continuous function φ on $\mathbb{D}'^{\bullet}_z$, we have

$$\langle \mu'_z, \varphi\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\mathbb{D}'^{\bullet}_z} \mathrm{Vol}(\mathrm{d}x)\,\mathbf{1}_{\{D(x,\partial\mathbb{D}'^{\bullet}_z)<\varepsilon\}}\,\varphi(x)$$

where $\mathrm{Vol}(\mathrm{d}x)$ denotes the volume measure on $\partial\mathbb{D}'^{\bullet}_z$. For our purposes, it is important to know that μ'_z is also the pushforward of Lebesgue measure on $[0,z]$ under the canonical projection from $\mathcal{T}'$ onto $\mathbb{D}'^{\bullet}_z$. This is proved in [72, Theorem 9].

We now note that, in addition to v_*, $\mathbb{D}'^{\bullet}_z$ has another distinguished point (belonging to its boundary) namely the point v_∂ which is the equivalence class of 0 in the quotient $\mathcal{T}'/\{D'=0\}$. We note that v_∂ is uniformly distributed over $\partial\mathbb{D}'^{\bullet}_z$, in the following sense. Similarly as in Proposition 15, we introduce the the doubly pointed measure metric space $\mathbb{D}'^{\bullet\bullet}_z$ which is obtained by viewing v_∂ as a second distinguished point of $\mathbb{D}'^{\bullet}_z$. We have then, for any nonnegative measurable function F on the space of all doubly pointed compact measure metric spaces,

$$\mathbb{E}[F(\mathbb{D}'^{\bullet\bullet}_z)] = \frac{1}{z}\,\mathbb{E}\Big[\int \mu'_z(\mathrm{d}x)\,F\Big([\mathbb{D}'^{\bullet}_z, x]\Big)\Big], \tag{5.66}$$

with the same notation as in Proposition 15. The proof of (5.66) is straightforward: For $r,t\in[0,z]$ use the notation $t\oplus r = t+r$ if $t+r\leqslant z$ and $t\oplus r = t+r-z$ if $t+r>z$, and note that, for every $r\in[0,z]$, the point measure $\sum_{i\in I}\delta_{(t_i\oplus r,\omega_i)}$ has the same distribution as $\mathcal{N}$, whereas $(\beta_{r\oplus t}-\beta_r)_{0\leqslant t\leqslant z}$ has the same distribution as $(\beta_t)_{0\leqslant t\leqslant z}$.

Consider a random doubly pointed space $\mathbb{D}^{\bullet\bullet}_z$ whose distribution is obtained by integrating the distribution of $\mathbb{D}^{\bullet\bullet,a}_z$ with respect to the probability density $p_z(a)$ in Proposition 3. By integrating the formula of Proposition 15 with respect to $p_z(a)\,\mathrm{d}a$, we get

$$\mathbb{E}[F(\mathbb{D}^{\bullet\bullet}_z)] = \frac{1}{z}\,\mathbb{E}\Big[\int \mu_z(\mathrm{d}x)\,F\Big([\mathbb{D}^{\bullet}_z, x]\Big)\Big]. \tag{5.67}$$

Since the pairs $(\mathbb{D}'^{\bullet}_z,\mu'_z)$ and $(\mathbb{D}^{\bullet}_z,\mu_z)$ have the same distribution, we obtain by comparing (5.66) and (5.67) that $\mathbb{D}'^{\bullet\bullet}_z$ and $\mathbb{D}^{\bullet\bullet}_z$ have the same distribution. Let $\bar{\mathbb{D}}^{\bullet}_z$, resp. $\bar{\mathbb{D}}'^{\bullet}_z$, be the pointed space obtained from $\mathbb{D}^{\bullet\bullet}_z$, resp. from $\mathbb{D}'^{\bullet\bullet}_z$, by forgetting the first distinguished point. Then $(\bar{\mathbb{D}}^{\bullet}_z, H_z)$ and $(\bar{\mathbb{D}}'^{\bullet}_z, H'_z)$ also have the same distribution.

Proposition 39. *For every $a>0$, let $\bar{\mathbb{D}}'^{\bullet,(a)}_z$ be distributed as $\bar{\mathbb{D}}'^{\bullet}_z$ conditioned on the event $\{H'_z\geqslant a\}$. Then*

$$\bar{\mathbb{D}}'^{\bullet,(a)}_z \xrightarrow[a\to\infty]{(\mathrm{d})} \mathbb{D}'^{\infty}_z$$

in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.

Before proving Proposition 39, let us explain why the statement of Proposition 38 follows from this proposition. Recall from Section 5.4.3 that H_z denotes the distance from the distinguished point of $\mathbb{D}^\bullet_z$ to the boundary. Since $(\bar{\mathbb{D}}^\bullet_z, H_z)$ and $(\bar{\mathbb{D}}'^\bullet_z, H'_z)$ have the same distribution, $\bar{\mathbb{D}}'^{\bullet,(a)}_z$ has the same distribution as $\bar{\mathbb{D}}^\bullet_z$ conditioned on $H_z \geqslant a$, whereas the pointed space $\bar{\mathbb{D}}^{\bullet,a}_z$ in Theorem 21 has the distribution of $\bar{\mathbb{D}}^\bullet_z$ conditioned on $H_z = a$. Hence, by comparing the convergences in Theorem 21 and in Proposition 39, we conclude that $\mathbb{D}^\infty_z$ and $\mathbb{D}'^\infty_z$ have the same distribution.

Proof of Proposition 39. Let E_a stand for the event $\{H'_z \geqslant a\}$. The idea of the proof is to study the effect on the pair $(\beta, \mathcal{N})$ (which determines $\mathbb{D}'^{\bullet\bullet}_z$) of conditioning on E_a. To this end, it will be useful to replace E_a by another event for which the conditioning will be easier to study. We first note that, by (5.64) and the definition of labels on $\mathcal{T}'$, we have

$$H'_z = \min_{0\leqslant t\leqslant z} (\sqrt{3}\,\beta_t) - \inf_{i\in I} \Big(\sqrt{3}\,\beta_{t_i} + W_*(\omega_i)\Big). \tag{5.68}$$

Set $\|\beta\| = \sup\{|\beta_t| : 0 \leqslant t \leqslant z\}$, and consider the events

$$\widetilde{E}_a := \Big\{\inf_{i\in I} W_*(\omega_i) \leqslant -a\Big\},\ E'_a := \Big\{\inf_{i\in I} W_*(\omega_i) \leqslant -a - 2\sqrt{3}\|\beta\|\Big\},\ E''_a := \Big\{\inf_{i\in I} W_*(\omega_i) \leqslant -a + 2\sqrt{3}\|\beta\|\Big\}.$$

From (5.68), we have $E'_a \subset E_a \subset E''_a$. On the other hand, it is an easy exercise to check that the ratio $\mathbb{P}(E'_a)/\mathbb{P}(E''_a)$ tends to 1 as $a \to \infty$ (in fact, it follows from (5.1) that both $\mathbb{P}(E'_a)$ and $\mathbb{P}(E''_a)$ are asymptotic to $3z/a^2$). Since we have also $E'_a \subset \widetilde{E}_a \subset E''_a$, we may condition on $\widetilde{E}_a$ instead of conditioning on $E_a = \{H'_z \geqslant a\}$ in order to get the convergence of the proposition.

Conditioning on $\widetilde{E}_a$ does not affect β. On the other hand, when a is large, the conditional distribution of $\mathcal{N}$ knowing $\widetilde{E}_a$ is close in total variation to the law of

$$\mathcal{N}'^{(a)} + \delta_{(\widetilde{U},\omega^{(a)})},$$

where $\mathcal{N}'^{(a)}$ is a Poisson point measure with intensity $2\,\mathbf{1}_{[0,z]}(t)\,\mathbf{1}_{\{W_*(\omega)>-a\}}\,\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega)$, $\omega^{(a)}$ is distributed according to $\mathbb{N}_0(\cdot \mid W_* \leqslant -a)$, and $\widetilde{U}$ is uniformly distributed over $[0,z]$ (and $\mathcal{N}'$, $\omega^{(a)}$ and $\widetilde{U}$ are independent). When a is large, $\mathcal{N}$ and $\mathcal{N}'^{(a)}$ can be coupled so that they are equal with high probability.

We then want to argue that, when a is large, we can couple $\omega^{(a)}$ and the labeled tree $\mathcal{T}'_\infty$ used to define $\mathbb{D}'_\infty$ so that $\mathcal{T}_{(\omega^{(a)})}$ and $\mathcal{T}'_\infty$, both viewed as labeled trees, are close in some appropriate sense. Recall that $\mathcal{T}'_\infty$ was constructed from a coding triple $(B, \mathcal{P}, \mathcal{P}')$ such that $B = (B_t)_{t\geqslant 0}$ is a linear Brownian motion started from 0 and, conditionally on B, $\mathcal{P}$ and $\mathcal{P}'$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $2\,\mathrm{d}t\,\mathbb{N}_{B_t}(\mathrm{d}\omega)$. On the other hand, the main results of [69] give the distribution of $\omega^{(a)}$. If $b \in [a,\infty)$, the conditional distribution of $\omega^{(a)}$ knowing that $W_*(\omega^{(a)}) = -b$ is that of the snake trajectory corresponding to a coding triple $(V, \mathcal{M}, \mathcal{M}')$ such that $V = (V'_t - b)_{0\leqslant t\leqslant T^{V'}}$, where $(V'_t)_{0\leqslant t\leqslant T^{V'}}$ is a Bessel process of dimension -5 started from b and stopped when it hits 0, and, conditionally on V, $\mathcal{M}$ and $\mathcal{M}'$ are independent Poisson measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity

$$2\,\mathbf{1}_{[0,T^{V'}]}(t)\,\mathbf{1}_{\{W_*(\omega)>-b\}}\,\mathrm{d}t\,\mathbb{N}_{V_t}(\mathrm{d}\omega).$$

From this description, we easily get that, for every $h > 0$ and $\varepsilon \in (0,1)$, we can for a large enough couple the coding triples $(B, \mathcal{P}, \mathcal{P}')$ and $(V, \mathcal{M}, \mathcal{M}')$ in such a way that the following two properties hold except on a set of probability smaller than ε:

- $V_t = B_t$ for $0 \leqslant t \leqslant h$;
- the restriction of $\mathcal{P}$, resp. of $\mathcal{P}'$, to $[0,h] \times \mathcal{S}$ coincides with the restriction of $\mathcal{M}$, resp. of $\mathcal{M}'$, to $[0,h] \times \mathcal{S}$.

Now recall that the construction of $\mathbb{D}'_\infty$ relies on the 4-tuple $(\beta, \mathcal{N}, U, \mathcal{T}'_\infty)$, whereas, up to an event of small probability when a is large, the space $\bar{\mathbb{D}}_z^{\prime\bullet,(a)}$ (which is $\bar{\mathbb{D}}_z^{\prime\bullet}$ conditioned on $\{H_z \geqslant a\}$) may be obtained from the 4-tuple $(\beta, \mathcal{N}'^{(a)}, \widetilde{U}, \mathcal{T}_{(\omega_{(a)})})$. It follows from the preceding considerations that, up to a set of small probability when a is large, we can couple these two 4-tuples in such a way that their first three components coincide and moreover the labeled trees $\mathcal{T}'_\infty$ and $\mathcal{T}_{(\omega_{(a)})}$ with their spines "truncated at height h" also coincide (in the case of $\mathcal{T}_{(\omega_{(a)})}$, the spine corresponds to the line segment between the root and the vertex with minimal label). Given $r > 0$, we deduce from the preceding observation (by choosing h large enough) that we can couple the spaces $\mathbb{D}_z^{\prime\infty}$ and $\bar{\mathbb{D}}_z^{\prime\bullet,(a)}$ so that the balls of radius r centered at the distinguished point are the same in both spaces, except on an event of small probability when a is large. We omit the detailed verification of this last coupling, which is very similar to the proof of Proposition 18 above or Theorem 1 in [39]. The convergence in distribution stated in Proposition 39 follows. □

Remark. The quantities H_z and H'_z have the same distribution, and thus the density of the random variable in the right-hand side of (5.68) is equal to $p_z(a)$. The reader is invited to give a direct proof of this fact, as a verification of the consistency of our definition of the free pointed Brownian disk with the one in [22].

To conclude this section, we explain why our definition of the Brownian half-plane $\mathbb{H}_\infty$ is consistent with the one given in [14] or [53]. We use the notation $\mathbb{H}'_\infty$ for the Brownian half-plane as defined in [14]. Then, from [14, Corollary 3.9], we get that $\mathbb{H}'_\infty$ is the tangent cone in distribution of the Brownian disk with perimeter z and volume r at a point uniformly distributed over its boundary – here "uniformly distributed" refers to the analog of the measure μ'_z (the construction of $\bar{\mathbb{D}}_z^{\prime\bullet}$ given above also works for the Brownian disk with fixed volume r, just by conditioning $\sum_{i \in I} \sigma(\omega_i)$ to be equal to r). By randomizing the volume r, we infer that we have also

$$\lambda \cdot \bar{\mathbb{D}}_z^{\prime\bullet} \xrightarrow[\lambda \to \infty]{\text{(d)}} \mathbb{H}'_\infty,$$

in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence. On the other hand, it follows from Theorem 26 that we have

$$\lambda \cdot \bar{\mathbb{D}}_z^{\bullet} \xrightarrow[\lambda \to \infty]{\text{(d)}} \mathbb{H}_\infty.$$

Since $\mathbb{D}_z^\bullet$ and $\mathbb{D}_z^{\prime\bullet}$ have the same distribution, we conclude that $\mathbb{H}_\infty$ and $\mathbb{H}'_\infty$ also have the same distribution as desired.

Appendix: Some Laplace transforms

Recall the standard notation

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t.$$

Then the function χ_1 defined for $x > 0$ by

$$\chi_1(x) = \frac{1}{\sqrt{\pi}} x^{-1/2} - e^x \operatorname{erfc}(\sqrt{x}) = \frac{1}{\sqrt{\pi}}\, e^x \int_{\sqrt{x}}^\infty \frac{1}{t^2}\, e^{-t^2}\,\mathrm{d}t, \tag{A.0}$$

satisfies, for every $\lambda > 0$,

$$\int_0^\infty \mathrm{d}x\, e^{-\lambda x}\, \chi_1(x) = (1+\sqrt{\lambda})^{-1}. \tag{A.1}$$

This is easily verified via an integration by parts which gives for $\lambda > 0$,

$$\int_0^\infty \operatorname{erfc}(\sqrt{x}) e^x\, e^{-\lambda x}\,\mathrm{d}x = \frac{1}{\sqrt{\lambda}(1+\sqrt{\lambda})}.$$

From the last two displays and an integration by parts, one checks that the function $\chi_2 = \chi_1 * \chi_1$, which satisfies

$$\int_0^\infty \mathrm{d}x\, e^{-\lambda x}\, \chi_2(x) = (1+\sqrt{\lambda})^{-2}, \tag{A.2}$$

is given for $x > 0$ by

$$\chi_2(x) = e^x \operatorname{erfc}(\sqrt{x}) - 2x\,\chi_1(x) = (2x+1)e^x \operatorname{erfc}(\sqrt{x}) - \frac{2}{\sqrt{\pi}} x^{1/2}.$$

Similar manipulations show that the function $\chi_3 = \chi_1 * \chi_1 * \chi_1$ satisfying

$$\int_0^\infty \mathrm{d}x\, e^{-\lambda x}\, \chi_3(x) = (1+\sqrt{\lambda})^{-3}. \tag{A.3}$$

is given by

$$\chi_3(x) = \frac{2}{\sqrt{\pi}}(x^{3/2} + x^{1/2}) - 2x(x+\frac{3}{2})\, e^x \operatorname{erfc}(\sqrt{x}).$$

We observe that $\chi_1(x) > 0$ for every $x > 0$ (this is obvious from (A.0)) and thus we have also $\chi_3(x) > 0$ for every $x > 0$. Finally, we note that

$$\int_0^\infty \frac{1-e^{-\lambda x}}{x}\, \chi_3(x)\,\mathrm{d}x = \int_0^\lambda \mathrm{d}\mu \int_0^\infty e^{-\mu x}\, \chi_3(x)\,\mathrm{d}x = \int_0^\lambda \mathrm{d}\mu (1+\sqrt{\mu})^{-3} = (1+\lambda^{-1/2})^{-2}. \tag{A.4}$$

Chapitre 6

Isoperimetric inequalities in the Brownian plane

Les resultats de ce chapitre sont issus de la pré-publication [93].

We consider the model of the Brownian plane, which is a pointed non-compact random metric space with the topology of the complex plane. The Brownian plane can be obtained as the scaling limit in distribution of the uniform infinite planar triangulation or the uniform infinite planar quadrangulation and is conjectured to be the universal scaling limit of many others random planar lattices. We establish sharp bounds on the probability of having a short cycle separating the ball of radius r centered at the distinguished point from infinity. Then we prove a strong version of the spatial Markov property of the Brownian plane. Combining our study of short cycles with this strong spatial Markov property we obtain sharp isoperimetric bounds for the Brownian plane.

Contents

6.1 Introduction

In recent years, much work and energy have been devoted to the study of discrete and continuous random geometry in dimension 2. In this paper we will study the Brownian plane $\mathcal{M}_\infty$, which appears as the scaling limit in distribution of the uniform infinite planar quadrangulation Q_∞, in the local Gromov-Hausdorff sense and can also be interpreted as a Brownian map with infinite volume, see [39]. The Brownian plane is a random pointed and weighted boundedly compact length space homeomorphic to $\mathbb{C}$ and is conjectured to be the universal scaling limit of other discrete models. The case of the uniform infinite planar triangulation of type I has been treated in [29]. We also mention that the Brownian plane is closely related to the Liouville quantum gravity surface called the quantum cone, see [86, Corollary 1.5].

The spaces Q_∞ and $\mathcal{M}_\infty$ have a distinguished point, also called the root. Our first goal is to understand the probability of having a short injective cycle separating the ball of radius r centered at the root in $\mathcal{M}_\infty$ from infinity. This will allow us to deduce isoperimetric inequalities for the Brownian plane. These results can then be extended to other models such as the Brownian sphere and the infinite Brownian disk. The study of short separating cycles starts in random planar geometry with the paper [59], where Krikun gave a construction of Q_∞ as the local limit of large finite planar quadrangulations. He also proved the existence, for every $r \in \mathbb{N}^*$, of cycles separating the ball of radius r of Q_∞ from infinity having length of order r. Krikun conjectured that it is not possible to find separating cycles with length of order smaller than r. In [73], Le Gall and Lehéricy confirmed Krikun's conjecture by proving that for every $\delta > 0$, there exists a constant $c_\delta > 0$ such that for every $r \in \mathbb{N}^*$:

$$\mathbb{P}(L_r(Q_\infty) < \varepsilon r) < c_\delta \varepsilon^{2-\delta}$$

where $L_r(Q_\infty)$ stands for the infimum of the lengths of injective cycles disconnecting the ball of radius r of Q_∞ from infinity. They also proved that the probability $\mathbb{P}(L_r(Q_\infty) > ur)$ decreases exponentially fast when u goes to infinity.

This work can be seen as a continuous counterpart of this study. We are aiming at similar results for the Brownian plane. Thanks to the geometric properties of $\mathcal{M}_\infty$ we get optimal results in the continuous setting. Since the Brownian plane is expected to be the universal scaling limit of random lattices such as the UIPQ and the UIPT, it is likely that these sharper results also have analogs for discrete models. Let us present our results more precisely. It should also be possible to adapt some of our techniques to the case of the UIPQ.

The Brownian plane $\mathcal{M}_\infty$ is equipped with a root, which we denote by 0, a distance Δ and a volume measure $|\cdot|$. The construction of $\mathcal{M}_\infty$ on a probability space $(\Omega, \mathcal{F}, \Theta_0)$ is recalled in Section 6.2.4. For every $r > 0$, let $B_r(\mathcal{M}_\infty)$ denote the closed ball of radius r centered at 0 in $\mathcal{M}_\infty$. For every path $\gamma : [t, t'] \to \mathcal{M}_\infty$, we denote its length by $\Delta(\gamma)$ i.e.:

$$\Delta(\gamma) := \sup_{t=t_1 \leqslant t_2, \dots \leqslant t_n = t'} \sum_{i=1}^{n-1} \Delta\big(\gamma(t_i), \gamma(t_{i+1})\big) \tag{6.1}$$

where the supremum is over all choices of the integer $n \geqslant 1$ and the finite sequence $t_1 \leqslant t_2 \leqslant \dots \leqslant t_n$ satisfying $(t_1, t_n) = (t, t')$. In this work a path has to be a continuous function. Moreover we say that a path $\gamma : [t, t'] \to \mathcal{M}_\infty$ is a separating cycle if:

- for every $t \leqslant s < s' \leqslant t'$ we have $\gamma(s) = \gamma(s')$ if and only if $(s, s') = (t, t')$;

- the distinguished point 0 does not belong to the range of γ and there exists $r > 0$ such that for any path $\tilde{\gamma} : [s, s'] \to \mathcal{M}_\infty$ with $\tilde{\gamma}(s) = 0$ and $\tilde{\gamma}(s') \notin B_r(\mathcal{M}_\infty)$ we have:

$$\gamma([t, t']) \cap \tilde{\gamma}([s, s']) \neq \varnothing.$$

We will say that a separating cycle γ separates $B_r(\mathcal{M}_\infty)$ from infinity if it takes values in the complement of $B_r(\mathcal{M}_\infty)$. Recall that $\mathcal{M}_\infty$ has a.s. the topology of $\mathbb{C}$ and consequently it has only one end. So for every $r > 0$, we can consider the hull of radius r, i.e. the complement of the unique unbounded connected component of the complement of the closed ball of radius r centered at the distinguished point. We denote the hull of radius r by $B_r^\bullet(\mathcal{M}_\infty)$. For every $r > 0$ and any separating cycle γ that separates $B_r(\mathcal{M}_\infty)$ from infinity, an application of Jordan's theorem shows that the path γ has to take values in the complement of $B_r^\bullet(\mathcal{M}_\infty)$. We say that such a path γ separates $B_r^\bullet(\mathcal{M}_\infty)$ from infinity and we introduce the set $\mathcal{C}_r$ of all cycles separating $B_r^\bullet(\mathcal{M}_\infty)$ from infinity, which is not empty since $B_r^\bullet(\mathcal{M}_\infty)$ is bounded. Remark that any separating cycle γ is in $\mathcal{C}_r$ for r small enough and set:

$$L_r := \inf\{\Delta(\gamma) : \gamma \in \mathcal{C}_r\}.$$

One of the benefits of working in the continuous setting is the fact that the Brownian plane is scale invariant in distribution, i.e. for every $r > 0$, $(\mathcal{M}_\infty, 0, \Delta, |\cdot|) \overset{(d)}{=} (\mathcal{M}_\infty, 0, r\Delta, r^4|\cdot|)$ (see Section 6.2.4). In particular, the scaling invariance implies that:

$$L_r \overset{(d)}{=} rL_1.$$

Therefore we will focus on the variable L_1. We will prove the following result:

Theorem 6.1.
(i) *We have*

$$\limsup_{u \to \infty} \frac{\log\big(\Theta_0(L_1 > u)\big)}{u} \leqslant -\sup_{s>1} \frac{1}{2(s-1)} \log\Big(\frac{s^2}{2s-1}\Big).$$

Consequently, $\Theta_0(L_1 > u)$ *decreases at least exponentially fast when* u *goes to* ∞*.*

(ii) *There exist two constants* $0 < c_1 \leqslant c_2$ *such that for every* $\varepsilon > 0$*:*

$$c_1(\varepsilon^2 \wedge 1) \leqslant \Theta_0(L_1 < \varepsilon) \leqslant c_2\varepsilon^2.$$

It may be possible to get a discrete version of Theorem 6.1 for the UIPQ by adapting our methods using the tree decomposition given in [35]. This decomposition is the discrete analog of the construction of the Brownian plane that we will present in the preliminaries. Let us mention that our methods also allow us to obtain upper and lower bounds on the probability of the event $\{L_1 < \varepsilon\}$ under various conditionings.

In Section 6.3.4 we prove a strong version of the spatial Markov property of the Brownian plane, which has been first derived in [79, Section 5.1]. The statement of this property requires some notation and we give the precise formulation of this property in Section 6.3.4. Combining Theorem 6.1 with this strong spatial Markov property we are able to study isoperimetric properties of the Brownian plane. Let us be more precise about this point.

We say that a closed subset A of $\mathcal{M}_\infty$ is a (closed) Jordan domain if it is homeomorphic to the closed disk of the complex plane $\mathbb{C}$. Let $\mathcal{K}$ be the set of all Jordan domains of $\mathcal{M}_\infty$ whose interior contains the distinguished point of $\mathcal{M}_\infty$. For every $A \in \mathcal{K}$, we can define the length $\Delta(\partial A)$ of its boundary, as follows. We consider an injective cycle $g : [0,1] \to \mathcal{M}_\infty$ such that $g([0,1]) = \partial A$ and we set:

$$\Delta(\partial A) := \Delta(g).$$

This definition does not depend on the parameterization g. We can now state our result concerning isoperimetric inequalities in the Brownian plane. We will prove in Section 6.4 that:

Theorem 6.2. *For any nondecreasing function* $f : \mathbb{R}_+ \to (0,\infty)$*:*

(i) *We have*

$$\inf_{A\in\mathcal{K}} \frac{\Delta\big(\partial A\big)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0\,,\ \Theta_0\text{-}a.s.\,,\ \textit{if}\ \sum_{m\in\mathbb{N}} \frac{1}{f(m)^2} = \infty.$$

(ii) *We have*

$$\inf_{A\in\mathcal{K}} \frac{\Delta\big(\partial A\big)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0\,,\ \Theta_0\text{-}a.s.\,,\ \textit{if}\ \sum_{m\in\mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

Theorem 6.2 can be extended to the infinite volume Brownian disk (see Corollary 6.4) and the Brownian map (since the Brownian map and the Brownian Plane are locally isometric [39, Theorem 1]). In [73], Le Gall and Lehéricy use their study of short cycles to get an analog for the UIPQ of

Theorem 6.2 for the special case $f(x) := x^{\frac{3}{4}+\delta}$ for any $\delta > 0$. We conclude this introduction by pointing out that the study of separating cycles appears naturally in other problems of random geometry; in the recent work [27] the authors use a class of separating cycles to obtain bijective enumerations of planar maps with three boundaries. They also discuss the statistics of the lengths of minimal separating loops in different discret models.

6.2 Preliminaries

The preliminaries are divided as follows. Section 8.2.1 gives a quick presentation of snake trajectories and the associated compact trees, we refer to [2, 50] for a more detailed description of these objects. Section 8.2.1 presents the Brownian snake excursion, which is the building block of the theory of Brownian geometry, and the special Markov property. Finally in Section 6.2.3 and 6.2.4 we introduce the notion of a coding triple and give the construction of the Brownian plane and the infinite volume Brownian disk; these last sections follow [79]. Before starting the preliminaries, let us introduce some standard notation.

Let (E, d) be a metric space.

• For $A \subset E$, we denote the closure (resp. the interior) of A in E by $\text{Cl}(A)$ (resp. $\text{Int}(A)$). Set $\partial A := \text{Cl}(A) \backslash \text{Int}(A)$.

• A path γ on E is a continuous function defined on an interval I of $\mathbb{R}$ taking values in E. We say that the path γ separates two subsets A and B of E if the range of γ does not intersect $A \cup B$ and if any path starting at A and ending at B intersects the range of γ. The path γ is a geodesic on E if for every $s, t \in I$, $d(\gamma(s), \gamma(t)) = |s - t|$.

• We denote the length of a path γ by $d(\gamma)$. The definition of $d(\gamma)$ is the same as defined in (6.1) replacing Δ by d. We say that (E, d) is a length space if, for every $x, y \in E$, the distance $d(x, y)$ is the infimum of the quantities $d(\gamma)$ over all the paths γ on E starting at x and ending at y.

• If (E, d) is a length space and U is a path-connected subset of E, the intrinsic distance induced by d on U is the distance d_U on U defined as follows:

$$\forall x, y \in U,\ d_U(x, y) := \inf \big\{ d(\gamma) :\ \gamma : [0, 1] \to U \text{ path with } (\gamma(0), \gamma(1)) = (x, y) \big\}.$$

Remark that d_U may take infinite values if U is not an open subset of E.

• We say that a compact (resp. boundedly compact) metric space is weighted if it is given with a finite (resp. finite on compact sets) measure, which is often called the volume measure. We denote by $\mathbb{K}$ (resp. $\mathbb{K}_\infty$) the set of all isometry classes of pointed and weighted compact (resp. boundedly compact) metric spaces equipped with the Gromov-Hausdorff-Prokhorov distance (resp.

the local Gromov-Hausdorff-Prokhorov distance). Both $\mathbb{K}$ and $\mathbb{K}_\infty$ are Polish spaces.

Finally, we write $s \vee t := \max(s,t)$, $s \wedge t := \min(s,t)$ and by convention $\inf \varnothing := \infty$.

6.2.1 Snake trajectories and labeled trees

Let $\mathcal{W}$ be the set of all continuous mappings $\mathrm{w} : [0,\zeta_{\mathrm{w}}] \to \mathbb{R}$, where $\zeta_{\mathrm{w}} \geqslant 0$ is called the lifetime of w. We will write $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{\mathrm{w}})$ for the endpoint of w. For every $x \in \mathbb{R}$, we identify x with the map starting from x with 0 lifetime. Set $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \ \mathrm{w}(0) = x\}$ and equip $\mathcal{W}$ with the distance:

$$d_{\mathcal{W}}(\mathrm{w},\mathrm{w}') = |\zeta_{\mathrm{w}} - \zeta_{\mathrm{w}'}| + \sup_{t \geqslant 0} |\mathrm{w}(t \wedge \zeta_{\mathrm{w}}) - \mathrm{w}'(t \wedge \zeta_{\mathrm{w}'})|.$$

Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $\omega : s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_x$ satisfying the following properties:

• $\omega_0 = x$ and the quantity $\sigma(\omega) := \sup\{s \geqslant 0 : \ \omega_s \neq x\}$ is finite. The quantity $\sigma(\omega)$ is called the lifetime of ω. By convention $\sigma(\omega) := 0$ if $\omega_s = x$ for every $s \geqslant 0$;
• For every $s, s' \in \mathbb{R}_+$ with $s \leqslant s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \leqslant \min_{r \in [s,s']} \zeta_{\omega_r}$. This property is called the snake property.

We denote the set of all snake trajectories starting at x by $\mathcal{S}_x$, and write $\mathcal{S} = \cup_{x \in \mathbb{R}} \mathcal{S}_x$ for the set of all snake trajectories. For every $\omega \in \mathcal{S}$ and $s \geqslant 0$, introduce the notation $W_s(\omega) := \omega_s$. The set $\mathcal{S}$ is equipped with the distance:

$$d_{\mathcal{S}}(\omega,\omega') := |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geqslant 0} d_{\mathcal{W}}\big(W_s(\omega), W_s(\omega')\big).$$

It is straightforward to verify that the space $(\mathcal{S}, d_{\mathcal{S}})$ is a Polish space. To simplify notation, for every $\omega \in \mathcal{S}$, we set

$$\omega_* := \inf\{\widehat{\omega}_s : \ s \geqslant 0\}.$$

It will be important for our study to associate a compact $\mathbb{R}$-tree $\mathcal{T}_\omega$ with every snake trajectory ω.

Let $\omega \in \mathcal{S}$ and define:

$$d_\omega(s,t) := \zeta_{\omega_s} + \zeta_{\omega_t} - 2 \inf_{r \in [s \wedge t, s \vee t]} \zeta_{\omega_r}$$

for every $s, t \in [0, \sigma(\omega)]$. Since $s \mapsto \zeta_{\omega_s}$ is continuous, d_ω is a continuous pseudo-distance on $[0,\sigma(\omega)]$. We define an equivalence relation $\approx_{d_\omega}$ by setting $s \approx_{d_\omega} t$ if $d_\omega(s,t) = 0$. The space $\mathcal{T}_\omega := [0,\sigma(\omega)]/\approx_{d_\omega}$ equipped with the distance induced by d_ω is a compact $\mathbb{R}$-tree. Let $p_\omega : [0,\sigma(\omega)] \to \mathcal{T}_\omega$ be the canonical projection and let V_ω be the pushforward of Lebesgue measure on $[0,\sigma(\omega)]$ under p_ω. We view the tree $\mathcal{T}_\omega$ as a pointed and weighted compact metric space, for which the volume measure is V_ω and the distinguished point is $\rho_\omega := p_\omega(0)$, which is called the root of $\mathcal{T}_\omega$. For every $u \in \mathcal{T}_\omega$, set $\Lambda_u^\omega := \widehat{\omega}_t$ where t is any element of $p_\omega^{-1}(u)$. The quantity Λ_u^ω

is well defined by the snake property and we interpret Λ_u^ω as a label assigned to u. The pair $\big(\mathcal{T}_\omega, (\Lambda_u^\omega)_{u\in\mathcal{T}_\omega}\big)$ is the labeled tree associated with the snake trajectory ω.

We will use the following standard nomenclature. Let $\mathcal{T}$ be a compact tree. The multiplicity of a point $a \in \mathcal{T}$ is the number of connected components of $\mathcal{T}\backslash\{a\}$. If the multiplicity of a is 1 (resp. > 2), a is called a leaf (resp. a branching point).

6.2.2 The Brownian snake excursion

To simplify notation, set $\widehat{W}_s(\omega) = \widehat{\omega}_s$ and $W_*(\omega) = \omega_*$ for every $\omega \in \mathcal{S}$. Fix $x \in \mathbb{R}$. The Brownian snake excursion measure $\mathbb{N}_x$ is the unique σ-finite measure on $\mathcal{S}_x$ that satisfies the following properties:

• The distribution of $s \mapsto \zeta_{\omega_s}$ is the Itô measure of positive excursions of linear Brownian motion, with the normalization:

$$\forall \varepsilon > 0,\ \mathbb{N}_x\Big(\sup_{s\in[0,\sigma(\omega)]} \zeta_{\omega_s} > \varepsilon\Big) = \frac{1}{2\varepsilon}\ ;$$

• Conditionally on $(\zeta_{\omega_s})_{s\geqslant 0}$, $(\widehat{W}_s(\omega))_{s\geqslant 0}$ is a Gaussian process with mean x and covariance function:

$$\forall s, s' \in [0, \sigma(\omega)],\ K(s,s') := \min_{r\in[s\wedge s', s\vee s']} \zeta_{\omega_r}.$$

Roughly speaking, conditionally on $(\zeta_s)_{s\geqslant 0}$, the process $(W_s)_{s\geqslant 0}$ evolves as follows. If ζ_s decreases, the path W_s is shortened from its tip, while if ζ_s increases, the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. We refer to [65] for a rigorous presentation. For every $x, y \in \mathbb{R}$ with $x < y$ we have:

$$\mathbb{N}_y\big(W_* < x\big) = \frac{3}{2(y-x)^2} \tag{6.2}$$

see [65, Chapter 6] for more details. To simplify notation, under $\mathbb{N}_x(d\omega)$ we will write σ for $\sigma(\omega)$ and $W_s(t)$ for $\omega_s(t)$.

Operations. We introduce a collection of elementary operations on $\mathcal{S}$.

• Translation:
For every snake trajectory ω and every $\lambda \in \mathbb{R}$, we will write $\omega + \lambda$ for the snake trajectory

$$(\omega+\lambda)_s(t) := \omega_s(t) + \lambda\,, \qquad 0 \leqslant t \leqslant \zeta_{(\omega+\lambda)_s} := \zeta_{\omega_s}.$$

By construction for every $x \in \mathbb{R}$ the pushforward measure of $\mathbb{N}_x$ under $\omega \mapsto \omega + \lambda$ is $\mathbb{N}_{x+\lambda}$.

• Scaling:

For every snake trajectory ω and every $\lambda \in \mathbb{R}_+^*$, we will write $\mathrm{hom}_\lambda(\omega)$ for the snake trajectory defined by

$$\mathrm{hom}_\lambda(\omega)_s(t) := \lambda\omega_{s\lambda^{-4}}(t\lambda^{-2})\,, \qquad 0 \leqslant t \leqslant \zeta_{\mathrm{hom}_\lambda(\omega)_s} := \lambda^2\zeta_{\omega_{s\lambda^{-4}}}.$$

It is also easy to deduce from the scaling property of Brownian motion that for every $x \in \mathbb{R}$ the pushforward measure of $\mathbb{N}_x$ under $\omega \mapsto \mathrm{hom}_\lambda(\omega)$ is $\lambda^2\mathbb{N}_{\lambda x}$. We will call this property the scaling property of the Brownian snake excursion.

• Truncation:
Let $(x, r) \in \mathbb{R}^2$ with $x > r$. For every $\mathrm{w} \in \mathcal{W}_x$, let:

$$\mathrm{hit}_r(\mathrm{w}) := \inf\{t \in [0, \zeta_{\mathrm{w}}] : \mathrm{w}(t) = r\}$$

with the usual convention $\inf \varnothing = \infty$. Consider $\omega \in \mathcal{S}_x$, and for every $s \geqslant 0$ set:

$$\eta_s^{(r)}(\omega) := \inf\Big\{t \geqslant 0 : \int_0^t \mathbb{1}_{\zeta_{\omega_u} \leqslant \mathrm{hit}_r(\omega_u)}\, du > s\Big\}.$$

The snake trajectory $\mathrm{tr}_r(\omega)$ defined by

$$\forall s \geqslant 0,\ \big(\mathrm{tr}_r(\omega)\big)_s := \omega_{\eta_s^{(r)}(\omega)}$$

is called the truncation of ω at level r. See [2, Proposition 10]. Roughly speaking, $\mathrm{tr}_r(\omega)$ is obtained by removing those paths ω that hit r and then survive for a positive amount of time. Let $\mathcal{Y}_r(\omega) := \sigma(\mathrm{tr}_r(\omega))$ which can be interpreted as the time spent by ω before hitting r and write $\mathcal{H}_r^x$ for the σ-field on $\mathcal{S}_x$ generated by $\mathrm{tr}_r(W)$ and the class of all $\mathbb{N}_x$-negligible sets.

We now discuss the special Markov property of the Brownian snake excursion, which will be crucial in our study. The set:

$$\{s \geqslant 0 :\ \mathrm{hit}_r(W_s) < \zeta_s\}$$

is open so it can be written as a union of disjoint open intervals $(a_i, b_i)_{i\in I}$ with I an indexing set that may be empty. For every $i \in I$, let $W^{(i)}$ be the snake trajectory defined by:

$$W_s^{(i)}(t) := W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)\ \text{ for } 0 \leqslant t \leqslant \zeta_{(a_i+s)\vee b_i} - \zeta_{a_i}$$

for every $s \geqslant 0$. By definition the snake trajectories $(W^{(i)})_{i\in I}$ are the excursions of W below r. Note that the information about the paths W_s before hitting r is contained in the sigma-field $\mathcal{H}_r^x$. The exit measure at level r is the quantity:

$$\mathcal{Z}_r(\omega) := \liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_0^\sigma ds\,\mathbb{1}_{\mathrm{hit}_r(\omega_s)=\infty,\ \widehat{\omega}_s < r+\varepsilon}.$$

The previous lim inf is a well defined finite limit $\mathbb{N}_x$-a.e. (we refer to [71, Proposition 28] for a proof) and it is $\mathcal{H}_r^x$-measurable by [64, Proposition 2.3]. We now can give a formal statement of the special Markov property.

Special Markov property. Let $x, r \in \mathbb{R}$, such that $x > r$. Under $\mathbb{N}_x$, conditionally on $\mathcal{H}_r^x$, the point measure:

$$\sum_{i \in I} \delta_{W^{(i)}}(d\omega)$$

is Poisson with intensity $\mathcal{Z}_r \mathbb{N}_r(d\omega)$.

We refer to [70, Corollary 21] for a proof. It will be useful to note that for $r' < r < x$, if we replace $\mathbb{N}_x(d\omega)$ by $\mathbb{N}_x(d\omega \,|\, W_* > r')$, the last statement remains valid up to the replacement of $\mathcal{Z}_r \mathbb{N}_r(d\omega)$ by $\mathcal{Z}_r \mathbb{N}_r(d\omega \cap \{W_* > r'\})$. The Laplace transform of $\mathcal{Z}_r$ is given by:

$$\mathbb{N}_x\big(1 - \exp(-\lambda \mathcal{Z}_r)\big) = \Big(\lambda^{-\frac{1}{2}} + \sqrt{\frac{2}{3}}(x - r)\Big)^{-2} \tag{6.3}$$

for every $\lambda \geqslant 0$. See e.g. formula (6) in [40] . Remark that the limit when λ goes to ∞ gives formula (8.6).

6.2.3 Coding triples and metric spaces

Infinite spine coding triples. An infinite spine coding triple is a triple $(\mathrm{w}, \mathfrak{N}^+, \mathfrak{N}^-)$ such that:

(i) $\mathrm{w} : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function;

(ii) $\mathfrak{N}^+ = \sum_{i \in I} \delta_{(t_i, \omega^i)}$ and $\mathfrak{N}^- = \sum_{i \in J} \delta_{(t_i, \omega^i)}$ are point measures on $(0, \infty) \times \mathcal{S}$ (I and J are two disjoint indexing sets) and for every $i \in I \cup J$, $\omega^i \in \mathcal{S}_{\mathrm{w}(t_i)}$;

(iii) the numbers $(t_i)_{i \in I \cup J}$ are distinct;

(iv) the functions

$$u \mapsto \beta_u^+ := \sum_{i \in I} \mathbb{1}_{t_i \leqslant u} \sigma(\omega^i), \quad u \mapsto \beta_u^- := \sum_{i \in J} \mathbb{1}_{t_i \leqslant u} \sigma(\omega^i)$$

take finite values, are monotone increasing on $\mathbb{R}_+$, and tend to ∞ at ∞;

(v) for every $t > 0$ and $\varepsilon > 0$:

$$\#\Big\{i \in I \cup J : \ t_i \leqslant t \text{ and } \sup_{s \in [0, \sigma(\omega^i)]} |\hat{\omega}_s^i - \mathrm{w}_{t_i}| > \varepsilon\Big\} < \infty.$$

We define a scaling operation for coding triples as follows; for every $\lambda > 0$

$$\mathrm{hom}_\lambda\Big(\mathrm{w}, \sum_{i \in I} \delta_{(t_i, \omega^i)}, \sum_{i \in J} \delta_{(t_i, \omega^i)}\Big) := \Big(\lambda \mathrm{w}(\cdot/\lambda^2), \sum_{i \in I} \delta_{(\lambda^2 t_i, \mathrm{hom}_\lambda(\omega^i))}, \sum_{i \in J} \delta_{(\lambda^2 t_i, \mathrm{hom}_\lambda(\omega^i))}\Big).$$

An infinite spine coding triple belongs to the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$, where $M(\mathcal{S})$ stands for the space of all σ-finite measures μ on $(0, \infty) \times \mathcal{S}$ putting no mass on the set $\{(t, \omega) : \sigma(\omega) = 0\}$

and satisfying $\mu\big([0,t] \times \{\omega \in \mathcal{S} : \ \sigma(\omega) > \delta\}\big) < \infty$, for every $t \geqslant 0$ and $\delta > 0$. We equip the space $M(\mathcal{S})$ with the distance:

$$d_{M(\mathcal{S})}(\mu, \mu') := \sum_{n \geqslant 0} d_{\text{Pro}}\big(\mu(\cdot \cap \mathcal{S}_{(n)}), \mu'(\cdot \cap \mathcal{S}_{(n)})\big) \wedge 2^{-n},$$

where $\mathcal{S}_{(n)} = [0, 2^n] \times \{\omega \in \mathcal{S} : \sigma(\omega) > 2^{-n}\}$, and d_{Pro} stands for the Prokhorov metric inducing the weak topology on finite measures on $\mathbb{R}_+ \times \mathcal{S}$.
We also equip $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$ with the product metric and the associated Borel sigma-field.

Let $(\mathrm{w}, \mathfrak{N}^+, \mathfrak{N}^-)$ be an infinite spine coding triple. We now introduce the infinite tree $\mathcal{T}_\infty$ associated with $(\mathrm{w}, \mathfrak{N}^+, \mathfrak{N}^-)$. For every $i \in I \cup J$, let (ζ^i_s) be the lifetime process associated with ω^i and $\sigma^i := \sigma(\omega^i)$. We write $\mathcal{T}^i$ for the tree coded by ζ^i, i.e. $\mathcal{T}^i = \mathcal{T}_{\omega^i}$, and p_{ζ^i} for the canonical projection from $[0, \sigma^i]$ onto $\mathcal{T}^i$. The tree $\mathcal{T}_\infty$ can be defined from the disjoint union:

$$[0, \infty) \cup \Big(\bigcup_{i \in I \cup J} \mathcal{T}^i \Big)$$

by identifying the point t_i of $[0, \infty)$ with $p_{\omega^i}(0)$ (that is, the root of $\mathcal{T}^i$) for every $i \in I \cup J$. The set $[0, \infty)$ is called the spine of $\mathcal{T}_\infty$.

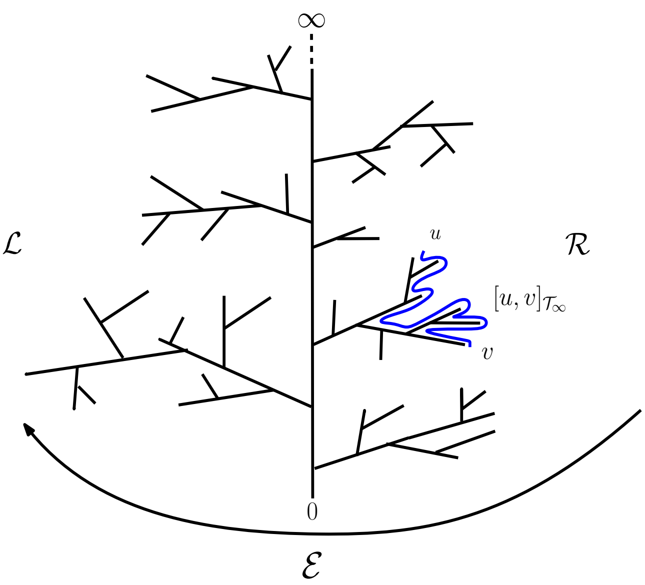

Figure 6.1 – A representation of the tree $\mathcal{T}_\infty$.

We equip $\mathcal{T}_\infty$ with a natural distance $d_{\mathcal{T}_\infty}$ as follows. The restriction of $d_{\mathcal{T}_\infty}$ to the spine is the Euclidean distance in $[0, \infty)$ and the restriction on each tree $\mathcal{T}^i$ is the tree distance d_{ω^i}. If $u \in \mathcal{T}^i$ and $t \in [0, \infty)$, we take $d_{\mathcal{T}_\infty}(u, t) = d_{\omega^i}(u, p_{\omega^i}(0)) + |t_i - t|$. If $u \in \mathcal{T}^i$ and $v \in \mathcal{T}^j$ with $i \neq j$, we take $d_{\mathcal{T}_\infty}(u, v) = d_{\omega^i}(u, p_{\omega^i}(0)) + |t_i - t_j| + d_{\omega^j}(v, p_{\omega^j}(0))$. Then $(\mathcal{T}_\infty, d_\infty)$ is a (non-compact) $\mathbb{R}$-tree. We can also assign a label Λ_u, to each u in $\mathcal{T}_\infty$ as follows. If $t \in [0, \infty)$, we take $\Lambda_t := \mathrm{w}(t)$.

If $u \in \mathcal{T}^i$, we take $\Lambda_u := \Lambda_u^{\omega^i}$. In particular, we have $\big(\mathcal{T}^i, (\Lambda_u)_{u\in\mathcal{T}^i}\big) = \big(\mathcal{T}_{\omega^i}, (\Lambda_u^{\omega^i})_{u\in\mathcal{T}_{\omega^i}}\big)$ for every $i \in I \cup J$. Moreover using property (v), one checks that the mapping $u \mapsto \Lambda_u$ is continuous on $\mathcal{T}_\infty$. Finally, we can define a natural volume measure $V_{\mathcal{T}_\infty}$ on $\mathcal{T}_\infty$ as follows, $V_{\mathcal{T}_\infty}$ gives no mass to the spine and its restriction to $\mathcal{T}^i$ is V_{ω^i}.

Roughly speaking, $\mathcal{T}_\infty$ is obtained by gluing the trees $\mathcal{T}^i$ along the spine and keeping their labels. It will be important for our purposes to equip $\mathcal{T}_\infty$ also with an order structure inherited from the coding triple. We define the left side of $\mathcal{T}_\infty$ as the subset:

$$\mathcal{L} := [0,\infty) \cup \Big(\bigcup_{i\in I} \mathcal{T}^i\Big)$$

where again the point t_i is identified with $p_{\omega^i}(0)$ for $i \in I$, and we define the right side $\mathcal{R}$ in the same way by replacing I by J. Remark that $\mathcal{L} \cap \mathcal{R} = [0,\infty)$. We write β^+_{u-} and β^-_{u-} for the left limits of β^+ and β^- at u (and we take $\beta^+_{0-} = \beta^-_{0-} = 0$ as convention). Note that if u is a discontinuity point of β^+ then there is a unique $i \in I$ such that $t_i = u$ and $\beta^+_u - \beta^+_{u-} = \sigma^i$ (and the same property is true for β^- replacing I by J).

We define the exploration process $\mathcal{E}^+$ of the left side of $\mathcal{T}_\infty$ as follows.
For every $s \geqslant 0$, there is a unique u such that $\beta^+_{u-} \leqslant s \leqslant \beta^+_u$. Then if there is an index $i \in I$ such that $t_i = u$, set

$$\mathcal{E}^+_s := p_{\omega^i}(s - \beta^+_{t_i-})$$

and if there is no such i, simply set $\mathcal{E}^+_s := u$. We define similarly the exploration of the right side $\mathcal{E}^-$ by replacing β^+ by β^- and I by J. Finally, let $\mathcal{E}$ be the function from $\mathbb{R}$ onto $\mathcal{T}_\infty$ defined by:

$$\mathcal{E}_s := \begin{cases} \mathcal{E}^+_s & \text{if } s \geqslant 0 \\ \mathcal{E}^-_{-s} & \text{if } s \leqslant 0 \end{cases}$$

Remark that $\mathcal{E}$ is continuous and the volume measure on $\mathcal{T}_\infty$ is the pushforward of Lebesgue measure on $\mathbb{R}$ under the mapping $s \mapsto \mathcal{E}_s$. Moreover the left side of $\mathcal{T}_\infty$ is $\{\mathcal{E}_s : \ s \geqslant 0\}$ and the right side is $\{\mathcal{E}_s : \ s \leqslant 0\}$. This exploration process allows us to define a notion of interval on $\mathcal{T}_\infty$. By convention, for every $s,t \in \mathbb{R}$ with $s < t$ we set $[t,s] := (-\infty, s] \cup [t,\infty)$. For every $u,v \in \mathcal{T}_\infty$ with $u \neq v$, let $[s,t]$ be the smallest interval such that $\mathcal{E}_s = u$ and $\mathcal{E}_t = v$. It is easy to check from the definition that there is always a smallest such interval. We put:

$$[u,v]_{\mathcal{T}_\infty} := \{\mathcal{E}_r : \ r \in [s,t]\}.$$

If $u = v$, take $[u,v]_{\mathcal{T}_\infty} = \{u\}$. Note that $[u,v]_{\mathcal{T}_\infty} \neq [v,u]_{\mathcal{T}_\infty}$ as long as $u \neq v$. See Figure 6.1 for an illustration.

By analogy with the case of compact trees, for every $u \in \mathcal{T}_\infty$, the multiplicity of u is the number of connected components of $\mathcal{T}_\infty \backslash \{u\}$. We will say that u is a leaf (resp. a branching point) if its multiplicity is 1 (resp. greater than 2). Remark that:

- 0 is the only leaf belonging to the spine.
- The branching points belonging to the spine are the points $(t_i)_{i\in I\cup J}$.
- For every $i \in I \cup J$, the multiplicity of $a \in \mathcal{T}^i\backslash\{t_i\}$ in $\mathcal{T}_\infty$ is its multiplicity in $\mathcal{T}^i$.

Finally, for every $u, v \in \mathcal{T}_\infty$, we denote the unique geodesic segment of $\mathcal{T}_\infty$ connecting u and v by $[[u,v]]_{\mathcal{T}_\infty}$. We write $u \preceq v$ for $u, v \in \mathcal{T}_\infty$ if and only if $u \in [[0,v]]_{\mathcal{T}_\infty}$. In this case we say that u is an ancestor of v. We also write $[[u,\infty[[_{\mathcal{T}_\infty}$ for the range of the unique geodesic from u to ∞ in $\mathcal{T}_\infty$.

From coding triples to metric spaces. Let $(\mathrm{w}, \mathfrak{N}^+, \mathfrak{N}^-)$ be a coding triple and let $\big(\mathcal{T}_\infty, (\Lambda_v)_{v\in\mathcal{T}_\infty}\big)$ be the associated labeled tree. We make the following assumption:

$$(H_1) : \begin{cases} \text{for every } v \in \mathcal{T}_\infty, \ \Lambda_v \geqslant 0; \\ \text{if } \Lambda_v = 0 \text{ then } v \text{ is a leaf;} \\ \Lambda_0 = 0; \\ \Lambda_{\mathcal{E}_t} \to \infty \text{ as } |t| \to \infty. \end{cases}$$

Set $\mathcal{T}_\infty^\circ := \{v \in \mathcal{T}_\infty : \Lambda_v > 0\}$ and $\partial\mathcal{T}_\infty := \mathcal{T}_\infty\backslash\mathcal{T}_\infty^\circ$. Remark that $\mathcal{T}_\infty^\circ$ is path connected and dense in $\mathcal{T}_\infty$ by (H_1). The last assumption in (H_1) implies that $\inf\limits_{[u,v]_{\mathcal{T}_\infty}} \Lambda$ is attained for every interval $[u,v]_{\mathcal{T}_\infty}$ of $\mathcal{T}_\infty$.

For every $u, v \in \mathcal{T}_\infty$ set:

$$\Delta^\circ(u,v) := \begin{cases} \Lambda_u + \Lambda_v - 2\max\big(\inf\limits_{[u,v]_{\mathcal{T}_\infty}} \Lambda, \inf\limits_{[v,u]_{\mathcal{T}_\infty}} \Lambda\big) & \text{if } \max\big(\inf\limits_{[u,v]_{\mathcal{T}_\infty}} \Lambda, \inf\limits_{[v,u]_{\mathcal{T}_\infty}} \Lambda\big) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

We then let

$$\forall u, v \in \mathcal{T}_\infty^\circ,\ \Delta(u,v) := \inf_{u_1=u,u_2,\dots,u_n=v} \sum_{i=1}^{n-1} \Delta^\circ(u_i, u_{i+1}) \tag{6.4}$$

where the infimum is over all choices of the integer $n \geqslant 1$ and of the finite sequence $u_1, \dots, u_n$ of elements of $\mathcal{T}_\infty$ verifying $u_1 = u$ and $u_n = v$. Using the continuity of $u \mapsto \Lambda_u$ one verifies that the mapping $(u,v) \mapsto \Delta(u,v)$ takes finite values and is continuous on $\mathcal{T}_\infty^\circ \times \mathcal{T}_\infty^\circ$. Since $\Delta^\circ(u,v) \geqslant |\Lambda_u - \Lambda_v|$, we have for every $u, v \in \mathcal{T}_\infty^\circ$

$$\Delta(u,v) \geqslant |\Lambda_u - \Lambda_v|. \tag{6.5}$$

It is important to remark that Δ defines a pseudo-distance on $\mathcal{T}_\infty^\circ$. From now on we make the extra assumption that:

(H_2): The map $(u,v) \mapsto \Delta(u,v)$ has a continuous extension to $\mathcal{T}_\infty \times \mathcal{T}_\infty$

and we consider this continuous extension in what follows. For simplicity we keep the notation Δ for this continuous extension, which defines a pseudo-distance on $\mathcal{T}_\infty$. The associated equivalence relation is defined by $u \approx v$ iff $\Delta(u,v) = 0$. By abuse of notation, we write $\mathcal{T}_\infty/\Delta$ for $\mathcal{T}_\infty/\approx$. Note that the definition of $u \approx v$ makes sense for $u, v \in \mathcal{T}_\infty^\circ$ even if (H_2) does not hold and so we can still consider the space $\mathcal{T}_\infty^\circ/\approx$ in that case. We denote the canonical projection by $\Pi : \mathcal{T}_\infty \to \mathcal{T}_\infty/\Delta$ and, for every $x \in \mathcal{T}_\infty/\Delta$, we set $\Lambda_x := \Lambda_u$ where u is any preimage of x under Π (remark that the definition is unambiguous by (8.17)). We write $|\cdot|$ for the pushforward of V under Π, which defines a volume measure on $\mathcal{T}_\infty/\Delta$, and for simplicity we write 0 for the equivalence class of 0 in $\mathcal{T}_\infty/\Delta$. The metric space $(\mathcal{T}_\infty/\Delta, 0, \Delta, |\cdot|)$ is a weighted locally compact length space which is pointed at 0, and we have:

$$\Delta(x, \Pi(\partial\mathcal{T}_\infty)) = \Lambda_x \tag{6.6}$$

for every $x \in \mathcal{T}_\infty/\Delta$. We refer [79, Subsection 4.1] for a proof of these two facts. For every $r \geqslant 0$, we write $B_r(\mathcal{T}_\infty/\Delta)$ for the set of all points $x \in \mathcal{T}_\infty/\Delta$ with $\Delta(x, \Pi(\partial\mathcal{T}_\infty)) \leqslant r$. By (8.19):

$$B_r(\mathcal{T}_\infty/\Delta) = \{x \in \mathcal{T}_\infty/\Delta : \Lambda_x \leqslant r\}.$$

It will also be useful to introduce for every $r > 0$, the set $\mathcal{T}_\infty^r$ of all points $u \in \mathcal{T}_\infty$ such that $\Lambda_u \geqslant r$ and $\Lambda_v > r$ for every $v \in [\![u, \infty[\![_{\mathcal{T}_\infty} \backslash \{u\}$. Remark that $\mathcal{T}_\infty^r$ is an $\mathbb{R}$-tree. We define:

$$\mathcal{T}_\infty^{r,\circ} := \Big\{u \in \mathcal{T}_\infty : \inf_{[\![u,\infty[\![_{\mathcal{T}_\infty}} \Lambda > r\Big\}$$

and we let the "boundary" $\partial\mathcal{T}_\infty^r$ be the set of all points $u \in \mathcal{T}_\infty$ such that $\Lambda_u = r$ and $\Lambda_v > r$ for every $v \in [\![u, \infty[\![_{\mathcal{T}_\infty} \backslash \{u\}$. Set $\check{B}_r^\bullet(\mathcal{T}_\infty/\Delta) := \Pi(\mathcal{T}_\infty^r)$, $\check{B}_r^\circ(\mathcal{T}_\infty/\Delta) := \Pi(\mathcal{T}_\infty^{r,\circ})$ and:

$$B_r^\bullet(\mathcal{T}_\infty/\Delta) := \Pi\Big(\Big\{u \in \mathcal{T}_\infty : \inf_{[\![u,\infty[\![_{\mathcal{T}_\infty}} \Lambda \leqslant r\Big\}\Big) = \Pi\Big(\mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ}\Big). \tag{6.7}$$

When there is no ambiguity, we will remove $\mathcal{T}_\infty/\Delta$ from the notation and write $B_r^\bullet, \check{B}_r^\bullet$ and $\check{B}_r^\circ$ instead. In the next section we explain the geometric meaning of these sets and we will see that the notation $B_r^\bullet$ is consistent with the one used in the introduction to designate the hull of the Brownian plane.

6.2.4 The Brownian plane and the infinite volume Brownian disk

In this section we give the construction of the Brownian plane and the infinite volume Brownian disk from random infinite spine coding triples. We also list some useful geometric properties of the Brownian plane.

The Brownian plane

We now consider a triple $(X, \mathfrak{L}, \mathfrak{R})$ such that:

- $X = (X_t)_{t \geqslant 0}$ is a nine-dimensional Bessel process started from 0;

• Conditionally on X, $\mathfrak{L}$ and $\mathfrak{R}$ are independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity:

$$2dt\mathbb{N}_{X_t}(d\omega \cap \{\omega_* > 0\}).$$

It is easy to verify that $(X, \mathfrak{L}, \mathfrak{R})$ is a.s. a coding triple in the sense of Section 6.2.3, and the root of $\mathcal{T}_\infty$ is the only point with zero label. We may assume that $(X, \mathfrak{L}, \mathfrak{R})$ is defined on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$ under the probability measure Θ_0. As previously, we write $(\mathcal{T}_\infty, (\Lambda_v)_{v \in \mathcal{T}_\infty})$ for the associated infinite labeled tree. We note that $(X, \mathfrak{L}, \mathfrak{R})$ satisfies assumptions (H_1, H_2), see [79, Section 4.2] . In fact, since the root of $\mathcal{T}_\infty$ is the only point with zero label, it is possible to define directly the continuous extension of Δ to $\mathcal{T}_\infty \times \mathcal{T}_\infty$, just replacing Δ° in formula (6.4) by

$$\Delta^{\circ,\prime}(u,v) := \Lambda_u + \Lambda_v - 2\max\big(\inf_{[u,v]_{\mathcal{T}_\infty}} \Lambda, \inf_{[v,u]_{\mathcal{T}_\infty}} \Lambda\big).$$

See [79, Section 4.2] for more details. The Brownian plane is the space $(\mathcal{T}_\infty/\Delta, 0, \Delta, |\cdot|)$ under Θ_0. To simplify notation, we denote this space, which is an element of $\mathbb{K}_\infty$, by $\mathcal{M}_\infty$. Remark that since $\partial\mathcal{T}_\infty = \{0\}$ we have $\Lambda_x = \Delta(0, x)$ for every $x \in \mathcal{M}_\infty$. Moreover, for every $\lambda > 0$, the pushforward of Θ_0 under hom_λ is Θ_0. Consequently, the space $(\mathcal{T}_\infty/\Delta, 0, \lambda\Delta, \lambda^4|\cdot|)$ is also distributed as a Brownian plane. Another important property is that , Θ_0-a.s., we have

(F): For every $u, v \in \mathcal{T}_\infty$, with $u \neq v$, we have $\Delta(u,v) = 0$ if and only if $\Delta^\circ(u,v) = \Delta^{\circ,\prime}(u,v) = 0$. Moreover if $u \neq v$ and $\Delta(u,v) = 0$ then u and v must be leaves.

This fact is a classical result in Brownian geometry. The first part of (F) is derived in [40, Section 3.2]. The second part follows from the first part and the known results for the Brownian map (see [76, Lemma 3.2]). By formulas (16) and (17) in [40] , the set $B_r^\bullet = B_r^\bullet(\mathcal{T}_\infty/\Delta)$ defined in (6.7) coincides with the hull of radius r of $\mathcal{M}_\infty$ as defined in the introduction (Section 6.1), $\check{B}_r^\circ$ is the complement of the hull, and $\check{B}_r^\bullet$ is the closure of $\check{B}_r^\circ$. We also have

$$\partial B_r^\bullet = \partial \check{B}_r^\bullet = \Pi\big(\partial\mathcal{T}_\infty^r\big) \tag{6.8}$$

which is the range of an injective cycle (see the proof of [79, Theorem 31] for more details). We will equip the hull $B_r^\bullet = \Pi\big(\mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ}\big)$ with the distance $\Delta^{(r)}$ defined as follows. First set

$$\forall u, v \in \mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ},\ \Delta^{(r)}(u,v) := \inf_{\substack{u=u_1,u_2,\dots u_n = v \\ u_2,\dots,u_{n-1} \in \mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ}}} \sum_{i=1}^{n-1} \Delta^{\circ,\prime}(u_i, u_{i+1}). \tag{6.9}$$

By (F), we see that for every $u, v \in \mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ}$ we have $\Delta(u,v) = 0$ iff $\Delta^{(r)}(u,v) = 0$. In particular, we can define $\Delta^{(r)}$ on the hull $B_r^\bullet$ taking for every $x, y \in B_r^\bullet$, $\Delta^{(r)}(x,y) := \Delta^{(r)}(u,v)$ where $u, v \in \mathcal{T}_\infty \backslash \mathcal{T}_\infty^{r,\circ}$ are any elements such that $(\Pi(u), \Pi(v)) = (x,y)$. By definition $\Delta^{(r)}$ is a distance on $B_r^\bullet$ and it is not hard to verify that the restriction of $\Delta^{(r)}$ on $\mathrm{Int}(B_r^\bullet)$ coincides with the intrinsic metric on $\mathrm{Int}(B_r^\bullet)$ viewed as a subset of the metric space $(\mathcal{T}_\infty/\Delta, \Delta)$ (one can directly adapt the proof of [79, Lemma 30]). In other words, $\Delta^{(r)}$ is the continuous extension to $B_r^\bullet$ of the intrinsic metric on $\mathrm{Int}(B_r^\bullet)$. In what follows, we will always view $B_r^\bullet$ as a (random) pointed and weighted compact metric space for the metric $\Delta^{(r)}$ (the volume measure is obviously the restriction of the volume measure on $\mathcal{M}_\infty$ and the distinguished point is the same as in $\mathcal{M}_\infty$).

Exit measures. We now introduce the exit measures of the infinite tree $\mathcal{T}_\infty$. For every $a \geqslant 0$, set

$$\tau_a := \sup\{t \geqslant 0 : X_t \leqslant a\} \tag{6.10}$$

which is Θ_0-a.s. finite since $X_t \to \infty$, Θ_0-a.s., when $t \to \infty$. We take $\tau_\infty := \infty$ by convention. For every $0 \leqslant s \leqslant t \leqslant \infty$, introduce the point measures $\mathfrak{L}^{s,t}$ and $\mathfrak{R}^{s,t}$ on $\mathbb{R}_+ \times \mathcal{S}$ defined as follows:

$$\int \Phi(\ell,\omega)\mathfrak{L}^{s,t}(d\ell\, d\omega) := \int_{\tau_s}^{\tau_t} \Phi(\ell-\tau_s,\omega)\mathfrak{L}(d\ell\, d\omega)$$

and

$$\int \Phi(\ell,\omega)\mathfrak{R}^{s,t}(d\ell\, d\omega) := \int_{\tau_s}^{\tau_t} \Phi(\ell-\tau_s,\omega)\mathfrak{R}(d\ell\, d\omega).$$

By the time reversal property of Bessel processes the process $(X_{(\tau_t-\ell)\vee 0})_{\ell\geqslant 0}$ is a Bessel process of dimension -5 started from t stopped when it hits 0 (see [98, Theorem 2.5]). Applying this property with t replaced by $t' > t$, we get that $(X_{(\tau_t-\ell)\vee 0})_{\ell\geqslant 0}$ and $(X_{(\tau_t+\ell)})_{\ell\geqslant 0}$ are independent. Consequently, for every $0 < t < \infty$:

$$\big((X_{(\tau_t+\ell)})_{\ell\geqslant 0}, \mathfrak{L}^{t,\infty}, \mathfrak{R}^{t,\infty}\big) \text{ and } \big((X_{(\tau_t-\ell)\vee 0})_{\ell\geqslant 0}), \mathfrak{L}^{0,t}, \mathfrak{R}^{0,t}\big) \text{ are independent.}$$

We call this property the spine independence property of $\mathcal{T}_\infty$. For every $0 < r \leqslant s \leqslant t$ set:

$$Z_r^{s,t} := \int \mathcal{Z}_r(\omega)\mathfrak{R}^{s,t}(d\ell d\omega) + \int \mathcal{Z}_r(\omega)\mathfrak{L}^{s,t}(d\ell d\omega)$$

which is the total exit measure at level r accumulated by the snakes glued on $[\tau_s, \tau_t]$. To simplify notation, write $Z_r := Z_r^{r,\infty}$. The proof of [40, Lemma 4.2] gives the following formula, for every $\lambda \geqslant 0$:

$$\Theta_0\big(\exp(-\lambda Z_r^{s,t})\big) = \Big(\frac{t}{s}\Big)^3 \cdot \Big(\frac{s-r+(r^{-2}+\frac{2}{3}\lambda)^{-\frac{1}{2}}}{t-r+(r^{-2}+\frac{2}{3}\lambda)^{-\frac{1}{2}}}\Big)^3. \tag{6.11}$$

Consequently, computing the limit when λ goes to infinity, we obtain:

Proposition 6.1. *For every* $0 \leqslant r \leqslant s \leqslant t < \infty$*:*

$$\Theta_0(Z_r^{s,t} = 0) = \big(\frac{t}{s}\big)^3 \cdot \big(\frac{s-r}{t-r}\big)^3. \tag{6.12}$$

The special Markov property of the Brownian snake excursion implies that conditionally on $Z_r^{s,t}$ the excursions outside r of the snake trajectories ω^i with $t_i \in [\tau_s, \tau_t]$ are distributed as the atoms of a Poisson point measure with intensity:

$$Z_r^{s,t}\mathbb{N}_r(\, d\omega\, \cap \{\omega_* > 0\}).$$

We will use this property throughout the article. It will be also useful to note that the Laplace transform of Z_r can be deduced from formula (6.11) taking the limit when t goes to infinity with $s = r$. More precisely, for every $r > 0$ and $\lambda \geqslant 0$ we get:

$$\Theta_0\big(\exp(-\lambda Z_r)\big) = \big(1 + \frac{2\lambda r^2}{3}\big)^{-\frac{3}{2}}. \tag{6.13}$$

Equivalently Z_r follows a Gamma distribution with parameter $\frac{3}{2}$ and mean r^2. The previous formula appears already in [40, Proposition 1.2], which also shows that $Z := (Z_r)_{r\geqslant 0}$ has a càdlàg modification, with only negative jumps, and from now on we consider this modification. Furthermore, [40, Proposition 4.3] states that, for every $0 \leqslant r \leqslant s$ and $\lambda \geqslant 0$, we have

$$\begin{aligned}\Theta_0\big(\exp(-\lambda Z_r)\,|Z_s\big) =& \Big(\frac{s}{r + (s-r)(1+\frac{2\lambda r^2}{3})^{\frac{1}{2}}}\Big)^3 \\ &\cdot \exp\Big(-\frac{3}{2}Z_s\big(\frac{1}{(s-r+(\frac{2\lambda}{3}+r^{-2})^{-\frac{1}{2}})^2} - \frac{1}{s^2}\big)\Big).\end{aligned} \tag{6.14}$$

We conclude this Subsection by giving a geometric interpretation of Z_r. One can derive from [77, Proposition 8] that

Lemma 6.1. Θ_0-*a.s. , for every* $r > 0$ *we have:*

$$Z_r = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|. \tag{6.15}$$

For the sake of completeness we give a proof of Lemma 6.1, but we postpone it to the Appendix below to avoid to weigh down the preliminaries. It will be important for us to know that the convergence holds simultaneously for every $r > 0$. Roughly speaking, Z_r represents the length or perimeter (in a generalized sense) of $\partial B_r^\bullet$.

The infinite volume Brownian disk

We keep the assumptions and notation of the preceding Subsection. Let $r > 0$ and set $\widetilde{X}_t^{(r)} := X_{\tau_r + t} - r$. Let us also introduce the point measures $\widetilde{\mathfrak{R}}_r$ and $\widetilde{\mathfrak{L}}_r$ on $\mathbb{R}_+ \times \mathcal{S}$ defined by:

$$\int \Phi(t,\omega)\widetilde{\mathfrak{L}}_r(dtd\omega) := \int \Phi(t, \mathrm{tr}_0(\omega - r))\mathfrak{L}^{r,\infty}(dt\, d\omega)$$

and

$$\int \Phi(t,\omega)\widetilde{\mathfrak{R}}_r(dtd\omega) := \int \Phi(t, \mathrm{tr}_0(\omega - r))\mathfrak{R}^{r,\infty}(dt\, d\omega).$$

One easily checks that the triple $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ is a random infinite spine coding triple satisfying (H_1). Moreover [79, Proposition 6] shows that there exists a unique collection of probability measures $(\Theta_z)_{z>0}$ on the space of coding triples such that for every $r > 0$:

$$\Theta_0\big(g(Z_r)F(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)\big) = \frac{3^{\frac{3}{2}}}{\sqrt{2\pi} r} \int_0^\infty dz z^{\frac{1}{2}} \exp(-\frac{3}{2r^2}z) g(z)\Theta_z(F). \tag{6.16}$$

and the pushforward of Θ_z by $\hom_\lambda$ is $\Theta_{\lambda^2 z}$ (for every $z, \lambda > 0$). In other words, conditionally on $Z_r = z$, the distribution of $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ is Θ_z. It is crucial that the preceding conditional distribution does not depend on r. Furthermore by [79, Lemma 16] an infinite coding triple distributed according to Θ_z satisfies a.s. (H_1, H_2). Consequently we can consider the associated metric space and according to [79, Proposition 38] this space is the infinite volume Brownian disk with perimeter z. The infinite volume Brownian disk is a random element of $\mathbb{K}_\infty$ and is a.s. homeomorphic to the complement of the open unit disk in the complex plane. It can also be obtained as scaling limit of random planar lattices with a boundary (see [14]). The boundary of the infinite volume Brownian disk is the set of points that have no neighborhood homeomorphic to the (open) disk. The infinite volume Brownian disk also satisfies a scale invariance property. More precisely since the pushforward of Θ_z by $\hom_\lambda$ is $\Theta_{\lambda^2 z}$, if $(E, \rho_E, \Delta_E, |\cdot|_E)$ is an infinite volume Brownian disk with perimeter z, then $(E, \rho_E, \lambda\Delta_E, \lambda^4|\cdot|_E)$ is an infinite volume Brownian disk with perimeter $\lambda^2 z$.

We now explain the geometric interpretation of $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ and the implications of (6.16) for the Brownian plane. First observe that the labeled tree associated with $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ can be identified with $\big(\mathcal{T}^r_\infty, (\Lambda_v - r)_{v\in\mathcal{T}^r_\infty}\big)$, see the beginning of the proof of Theorem 29 in [79]. For every $r > 0$, let $\check{\Delta}^{(r)}$ be the intrinsic distance induced by Δ on $\check{B}^\circ_r$ and also write $|\cdot|_{\check{\Delta}^{(r)}}$ for the restriction of the volume measure $|\cdot|$ to $\check{B}^\bullet_r$. The following lemma is then a consequence of [79, Lemma 30] and the identification of $\big(\mathcal{T}^r_\infty, (\Lambda_v - r)_{v\in\mathcal{T}^r_\infty}\big)$ with the labeled tree associated with $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$.

Lemma 6.2. Θ_0-*a.s., for every $r > 0$ such that $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ satisfies (H_2) the following properties hold:*

(i) *The intrinsic distance $\check{\Delta}^{(r)}$ has a unique continuous extension to $\check{B}^\bullet_r$;*

(ii) *The space $\check{B}^\bullet_r$ equipped with this continuous extension of $\check{\Delta}^{(r)}$, the measure $|\cdot|_{\check{\Delta}^{(r)}}$ and the distinguished point $\Pi(\tau_r)$ coincides as an element of $\mathbb{K}_\infty$ with the metric space associated with $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$.*

By (6.16), the coding triple $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ satisfies (H_2), Θ_0-a.s., for every fixed $r > 0$, and thus properties (i) and (ii) hold Θ_0-a.s. when $r > 0$ is fixed. However, we point out that we do not claim that $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ satisfies (H_2) simultaneously for every $r > 0$. Consequently it is not clear whether $\check{\Delta}^{(r)}$ has a unique continuous extension to $\check{B}^\bullet_r$ simultaneously for every $r > 0$, a.s.

On the other hand, we saw in Section 6.2.4 that the hull $B^\bullet_r$ (equipped with the distance $\Delta^{(r)}$ defined in (6.9)) can be viewed as a random element of the space $\mathbb{K}$. In the next statement, we also view $\check{B}^\bullet_r$ as an element of $\mathbb{K}_\infty$ as explained in property (ii) of Lemma 6.2. The following theorem is essentially a reformulation of Theorems 29 and 31 in [79].

Theorem 6.3. *Let $r > 0$. Then, conditionally on $Z_r = z$, the coding triple $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ is* *distributed according to* Θ_z *and is independent of* $B^\bullet_r$. *Consequently, conditionally on* $Z_r = z$, *the space* $\check{B}^\bullet_r$ *is an infinite Brownian disk with perimeter* z *and is independent of* $B^\bullet_r$.

The fact that conditionally on $Z_r = z$ the coding triple $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ is distributed according to Θ_z is just a reformulation of (6.16). Using property (ii) of Lemma 6.2, it follows that the conditional distribution of $\check{B}^\bullet_r$ knowing that $Z_r = z$ is the law of the infinite volume Brownian disk with perimeter z. The conditional independence of $\check{B}^\bullet_r$ and $B^\bullet_r$ given Z_r is stated in [79, Theorem 31] , and the slightly stronger conditional independence of $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ and $B^\bullet_r$ is also established at the end of the proof of this result.

We will refer to the last assertion of Theorem 6.3 as the spatial Markov property of $\mathcal{M}_\infty$. In Section 6.3.4 below, we will extend this property to the case of a random level r.

6.3 Separating cycles

In most of this section, we argue under Θ_0 and we use the following notation:

$$B^\circ_{r,s} := \mathrm{Int}(B^\bullet_s \backslash B^\bullet_r) \text{ and } B^\bullet_{r,s} := \mathrm{Cl}(B^\circ_{r,s}).$$

for every $r, s \in (0,\infty)$ with $r < s$. Our first goal is to study the quantity:

$$L_{r,s} = \inf\{\Delta(g) : \ g : [0,1] \to B^\circ_{r,s} \text{ cycle separating } B^\bullet_r \text{ from } \infty\}. \tag{6.17}$$

Since $\mathcal{M}_\infty$ has the topology of the complex plane $\mathbb{C}$, the quantity $L_{r,s}$ is well defined. Actually by construction it only depends on $B^\circ_{r,s}$ and the intrinsic distance induced by Δ on $B^\circ_{r,s}$. Let us briefly justify the measurability of the random variable $L_{r,s}$. We consider a dense sequence $(a_n : n \in \mathbb{N})$ in $\mathcal{M}_\infty$. Given $\alpha > 0$, we observe that $L_{r,s} < \alpha$ if and only if, for some $\delta > 0$, the following holds for every $\varepsilon > 0$: There exists a finite sequence $a_{n_1}, a_{n_2}, \dots, a_{n_p}, a_{n_{p+1}} = a_{n_1}$ such that $\Delta(a_{n_i}, (B^\circ_{r,s})^c) \geqslant \delta$, $\Delta(a_{n_i}, a_{n_{i+1}}) < \varepsilon$ for every $1 \leqslant i \leqslant p$, and

$$\sum_{i=1}^{p} \Delta(a_{n_i}, a_{n_{i+1}}) < \alpha - \delta,$$

and such that for any other sequence $a_{m_1}, \dots, a_{m_q}$ with $a_{m_1} \in B^\bullet_r$, $a_{m_q} \in \check{B}^\bullet_s$, and $\Delta(a_{m_j}, a_{m_{j+1}}) < \varepsilon$ for every $1 \leqslant j \leqslant q-1$, there exist $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$ such that $\Delta(a_{n_i}, a_{m_j}) < \varepsilon$.

6.3.1 Geometric properties

We argue under Θ_0 and use the notation introduced in Section 6.2.3. In particular, $(\mathcal{E}_s)_{s\in\mathbb{R}}$ is the exploration function of the tree $\mathcal{T}_\infty$. Our goal here is to identify a subclass of separating cycles and then to show that we can restrict our study to this collection of paths which is easier to study.

For every $t \geqslant 0$, set $p_\infty^{(\ell)}(t) := \mathcal{E}_{\sup\{s\in\mathbb{R}:\,\Lambda_s=t\}}$ and $p_\infty^{(r)}(t) := \mathcal{E}_{\inf\{s\in\mathbb{R}:\,\Lambda_s=t\}}$. Remark that $p_\infty^{(\ell)}$ (resp. $p_\infty^{(r)}$) takes values in $\mathcal{L}$ (resp. $\mathcal{R}$) since $\mathcal{E}_0 = 0$. By definition for every $s \leqslant t$, we have

$$\inf_{[p_\infty^{(\ell)}(s),p_\infty^{(\ell)}(t)]_{\mathcal{T}_\infty}} \Lambda = \inf_{[p_\infty^{(r)}(t),p_\infty^{(r)}(s)]_{\mathcal{T}_\infty}} \Lambda = s.$$

Consequently we have

$$\Delta^\circ\big(p_\infty^{(\ell)}(t), p_\infty^{(\ell)}(s)\big) = \Delta^\circ\big(p_\infty^{(r)}(t), p_\infty^{(r)}(s)\big) = |t-s| \tag{6.18}$$

for every $s, t > 0$. Moreover knowing that for every $t \geqslant 0$ and $u \in [p_\infty^{(\ell)}(t), p_\infty^{(r)}(t)]_{\mathcal{T}_\infty}$ we have $\Lambda_u \geqslant t$, we get that $\Delta^\circ(p_\infty^{(\ell)}(t), p_\infty^{(r)}(t)) = 0$ for every $t > 0$. We write

$$\gamma_\infty(t) := \Pi\big(p_\infty^{(\ell)}(t)\big) = \Pi\big(p_\infty^{(r)}(t)\big), \qquad t \in [0,\infty).$$

By (8.17) and (6.18), γ_∞ is a geodesic path connecting 0 and ∞. It can be shown that this geodesic path is the unique geodesic path connecting 0 and ∞ (see [39, Proposition 15]) but we will not use this result in this work. To simplify notation set $P^{(\ell)} := p_\infty^{(\ell)}(\mathbb{R}_+)$, $P^{(r)} := p_\infty^{(r)}(\mathbb{R}_+)$ and $P := P^{(\ell)} \cup P^{(r)}$. Remark that $\Pi(P)$ is the range of γ_∞.

We define the left (resp. right) side of $\mathcal{M}_\infty$ as the subset $\Pi(\mathcal{L})$ (resp. $\Pi(\mathcal{R})$).

Lemma 6.3. *The following properties hold Θ_0-a.s.*

(i) *The maps $\mathbb{R}_+ \ni t \mapsto \Pi(t)$ and $\mathbb{R}_+ \ni t \mapsto \gamma_\infty(t)$ are injective. Moreover $\Pi([0,\infty)) \cap \Pi(P) = \{0\}$.*

(ii) *The sets Int$(\Pi(\mathcal{L}))$ and Int$(\Pi(\mathcal{R}))$ are the connected components of the complement of $\Pi([0,\infty)) \cup \Pi(P)$.*

Proof.

(i) Since $\mathbb{R}_+ \ni t \mapsto \gamma_\infty(t)$ is a geodesic path it has to be injective. Moreover, as the only leaf on the spine $[0,\infty)$ is 0, we can apply (F) to deduce that $t \in \mathbb{R}_+ \mapsto \Pi(t)$ is also injective and that $\Pi([0,\infty)) \cap \Pi(P) \subset \{0\}$.

(ii) As a simple consequence of (F), a point x belongs to the boundary of $\Pi(\mathcal{L})$ iff it belongs to $\Pi([0,\infty))$ or to $\Pi(P^{(\ell)}) = \Pi(P)$, and similarly if $\mathcal{L}$ is replaced by $\mathcal{R}$. Consequently:

$$\mathrm{Int}\big(\Pi(\mathcal{L})\big) = \Pi(\mathcal{L})\backslash\big(\Pi([0,\infty)) \cup \Pi(P)\big)\ ,\ \mathrm{Int}\big(\Pi(\mathcal{R})\big) = \Pi(\mathcal{R})\backslash\big(\Pi([0,\infty)) \cup \Pi(P)\big).$$

Thanks again to (F) we have $\mathrm{Int}\big(\Pi(\mathcal{L})\big) \cap \mathrm{Int}\big(\Pi(\mathcal{R})\big) = \varnothing$. Since $\mathcal{M}_\infty$ has the topology of the complex plane and $\mathcal{M}_\infty\backslash\big(\Pi([0,\infty)) \cup \Pi(P)\big)$ is the union of $\mathrm{Int}\big(\Pi(\mathcal{L})\big)$ and $\mathrm{Int}\big(\Pi(\mathcal{R})\big)$ the desired result follows. □

Let us introduce the subclass of separating cycles that will play an important role. We define the set $\mathcal{A}$ of all paths $\gamma : [t,t'] \mapsto \mathcal{M}_\infty$ such that:

- $\gamma(t) = \gamma(t')$ is in $\Pi(P)$ and γ does not hit 0;
- For every $t \leqslant s < s' \leqslant t'$, we have $\gamma(s) = \gamma(s')$ if and only if $(s, s') = (t, t')$;
- There exist two times $t_1 \leqslant t_2$ in $[t, t']$, such that $\gamma(t_1), \gamma(t_2) \in \Pi([0, \infty))$, $(\gamma(t))_{t\in[t_1,t_2]}$ does not intersect $\Pi(P)$, and $\gamma(s) \in \Pi(\mathcal{R})$ (resp. $\gamma(s) \in \Pi(\mathcal{L})$) for every $s \in [t, t_1]$ (resp. for every $s \in [t_2, t']$).

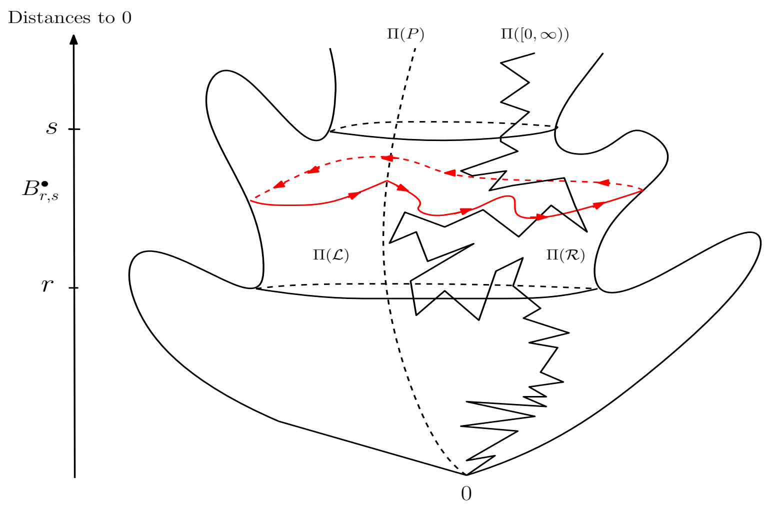

Figure 6.2 – Cactus representation of the Brownian plane. The vertical distances represent the distances to the root 0. In red a path of $\mathcal{A}$ taking values in $B^{\circ}_{r,s}$.

Using Lemma 6.3 and the fact that $\mathcal{M}_\infty$ is homeomorphic to $\mathbb{C}$, one easily verifies that any path γ in $\mathcal{A}$ is a separating cycle. So to give an upper bound for $L_{r,s}$ for $r < s$, it is sufficient to construct a path $\gamma \in \mathcal{A}$ taking values in $B^{\circ}_{r,s}$. See Figure 6.2 for an illustration. In the next lemma we explain why we can restrict our attention to the subclass $\mathcal{A}$.

Lemma 6.4. *Θ_0-a.s. for every separating cycle γ there exists a path γ' in $\mathcal{A}$ such that $\Delta(\gamma') \leqslant \Delta(\gamma)$. Moreover if γ takes values in $B^{\circ}_{r,s}$ then the path γ' also takes values in $B^{\circ}_{r,s}$.*

Proof. Let $(\gamma(t))_{t\in[0,1]}$ be a separating cycle. Since γ does not hit 0, there exist $r < s$ such that γ takes values in $B^{\circ}_{r,s}$. In what follows we fix $r < s$ such that γ stays in $B^{\circ}_{r,s}$. Our goal is to show that there exists $\gamma' \in \mathcal{A}$ taking values in $B^{\circ}_{r,s}$ such that $\Delta(\gamma') \leqslant \Delta(\gamma)$. First notice that since the path $(\gamma_\infty(t))_{t\geqslant 0}$ connects 0 and ∞, the range of γ has to intersect $\Pi(P) = \gamma_\infty(\mathbb{R}_+)$. Without loss of generality we may and will assume that $\gamma(0) = \gamma(1) \in \Pi(P)$. Let $(t_i, t'_i)_{i\in\mathcal{I}}$ be the connected components of $\{t \in [0,1] : \gamma(t) \notin \Pi(P)\}$ and to simplify notation set $\gamma^i := \gamma_{[t_i,t'_i]}$. Remark that γ^i hits $\Pi(P)$ only at times t_i and t'_i. In particular, since γ does not hit 0 we can use Lemma 6.3 to obtain that for every $i \in \mathcal{I}$ there exists $\varepsilon > 0$ such that:

$$\forall t \in [0, \varepsilon],\ \gamma^i(t_i + t) \in \Pi(\mathcal{R}) \text{ or } \forall t \in [0, \varepsilon],\ \gamma^i(t_i + t) \in \Pi(\mathcal{L}).$$

In the first case we say that γ^i starts in $\Pi(\mathcal{R})$ and in the second case that γ^i starts in $\Pi(\mathcal{L})$. Similarly by Lemma 6.3 there exists $\varepsilon > 0$ such that:

$$\forall t \in [0,\varepsilon],\ \gamma^i(t'_i - t) \in \Pi(\mathcal{R}) \text{ or } \forall t \in [0,\varepsilon],\ \gamma^i(t'_i - t) \in \Pi(\mathcal{L}).$$

In the first case we say that γ^i ends in $\Pi(\mathcal{R})$ and in the second case that γ^i ends in $\Pi(\mathcal{L})$. Then since γ is a separating cycle we claim that

(C): there exists $i \in \mathcal{I}$ such that γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$, or starts in $\Pi(\mathcal{L})$ and ends in $\Pi(\mathcal{R})$.

Let us explain why (C) holds. Thanks to Lemma 6.3, we can find a homeomorphism $h : \mathcal{M}_\infty \to \mathbb{C}$ such that $h(\Pi(\mathbb{R}_+)) = \mathbb{R}_-$, $h(\gamma_\infty(\mathbb{R}_+)) = \mathbb{R}_+$, and $h(\Pi(\mathcal{L}))$ and $h(\Pi(\mathcal{R}))$ are the upper and lower half-planes. In particular, since $h(0) = 0$, the injective cycle $h \circ \gamma$ separates 0 from infinity in the complex plane $\mathbb{C}$. Let $(\rho(t), \theta(t))_{t\in[0,1]}$ be two continuous functions from $[0,1]$ into $\mathbb{R}_+$ such that $h(\gamma(t)) = \rho(t)\exp(i\theta(t))$. Since $\gamma(0) \in \gamma_\infty(\mathbb{R}_+)$, we can take $\theta(0) = 0$ which determines in a unique way the pair $(\rho(t), \theta(t))_{t\in[0,1]}$. Since $h \circ \gamma$ separates 0 from infinity in $\mathbb{C}$, an application of Jordan's theorem shows that we must have $\theta(1) \in \{2\pi, -2\pi\}$. Suppose that $\theta(1) = 2\pi$ for definiteness and set $s_0 := \sup\{t \in [0,1] : \theta(0) = 0\}$ and $s_1 = \inf\{t \in [s_0,1] : \theta(t) = 2\pi\}$. Necessarily there exists i such that $\gamma^i = (\gamma(r))_{r\in[s_0,s_1]}$, and the "excursion" γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$ or conversely.

Let us derive the lemma from the claim (C). Up to replacing γ by $(\gamma(1-t))_{t\in[0,1]}$, we can assume that there exists $i \in \mathcal{I}$ such that γ^i starts in $\Pi(\mathcal{R})$ and ends in $\Pi(\mathcal{L})$. Since $\gamma(0), \gamma^i(t_i), \gamma^i(t'_i) \in \Pi(P)$, we can consider the geodesic path g (resp. g') taking values in $\Pi(P)$ starting at $\gamma(0)$ and ending at $\gamma(t_i)$ (resp. starting at $\gamma(t'_i)$ and ending at $\gamma(1) = \gamma(0)$). The concatenation of g', γ^i and g gives a path γ' in $\mathcal{A}$, which is shorter than γ. Moreover as γ takes values in $B^\circ_{r,s}$ we have $\gamma(0), \gamma^i(t_i), \gamma^i(t'_i) \in \gamma_\infty\big((r,s)\big)$ and consequently g and g' takes values in $\gamma_\infty\big((r,s)\big)$. We conclude that the concatenation of g', γ^i and g takes values in $B^\circ_{r,s}$. □

6.3.2 Lower bound for the tail of L_1 near 0

In this section, for every $0 < r < s$, we construct an explicit path in $\mathcal{A}$ taking values in $B^\circ_{r,s}$. This gives an upper bound for $L_{r,s}$. We will use this bound to obtain Theorem 6.1 (i) and the lower bound for point (ii) of Theorem 6.1.

We start with some notation. Let $u, v \in \mathcal{T}_\infty$ and let $t, t' \in \mathbb{R}$ be chosen in a unique way so that $\mathcal{E}_t = u$, $\mathcal{E}_{t'} = v$ and $[t,t']$ is as small as possible (recall our special convention for $[t,t']$ when $t > t'$). Suppose that $t \leqslant t'$. Recall that $[u,v]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : r \in [t,t']\}$. Let $M_{u,v} := \inf_{[u,v]_{\mathcal{T}_\infty}} \Lambda$ and, for every $0 \leqslant r \leqslant \Lambda_u - M_{u,v}$, set

$$\gamma_{u,v}(r) := \Pi\Big(\mathcal{E}_{\inf\left\{r'\in[t,t']:\, \Lambda_{r'}=\Lambda_u-r\right\}}\Big)$$

and for every $\Lambda_u - M_{u,v} < r \leqslant \Lambda_u + \Lambda_v - 2M_{u,v}$

$$\gamma_{u,v}(r) := \Pi\Big(\mathcal{E}_{\sup\left\{r'\in[t,t']:\,\Lambda_{r'}=r+2M_{u,v}-\Lambda_u\right\}}\Big).$$

By construction $\Delta^\circ(\gamma_{u,v}(r_1),\gamma_{u,v}(r_2)) = |r_1-r_2|$ as soon as $r_1,r_2 \in (0,\Lambda_u - M_{u,v})$ or $r_1,r_2 \in (\Lambda_u - M_{u,v}\ ,\ \Lambda_u+\Lambda_v-2M_{u,v})$. In particular, applying (8.17), we deduce that the restriction of $\gamma_{u,v}$ to $[0,\Lambda_u-M_{u,v}]$ or $[\Lambda_u-M_{u,v},\Lambda_u+\Lambda_v-2M_{u,v}]$ is a geodesic. Hence $\gamma_{u,v}$ is a path with length $\Lambda_u+\Lambda_v-2\inf_{[u,v]_{\mathcal{T}_\infty}}\Lambda$. Remark that the range of $\gamma_{u,v}$ is contained in $\Pi([u,v]_{\mathcal{T}_\infty})$. In particular, if $u,v\in\mathcal{R}$ (resp. $u,v\in\mathcal{L}$) the range of $\gamma_{u,v}$ is contained in $\Pi(\mathcal{R})$ (resp. $\Pi(\mathcal{L})$) since $[u,v]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : r\in[t,t']\}$.

Finally, for $r<s$ set:

$$K_r^s := \{u\in\partial\mathcal{T}_\infty^s :\ \text{there exists}\, v\in\mathcal{T}_\infty\,\text{such that}\ u\in[\![v,\infty[\![_{\mathcal{T}_\infty}\text{and}\ \Lambda_v\leqslant r\}.$$

The continuity of the label function $u\mapsto\Lambda_u$ from $\mathcal{T}_\infty$ into $\mathbb{R}$ and the last assumption in (H_1) imply that the set K_r^s is finite Θ_0-a.s. Recall the definition (6.10) of $(\tau_a)_{a\geqslant 0}$ and remark that τ_s is the unique element of K_r^s belonging to the spine $[0,\infty)$. Write $N_r^s := \#K_r^s - 1$ for the number of elements of K_r^s not belonging to the spine.

Proposition 6.2. *For every $r<s$, Θ_0-a.s., we have:*

$$L_{r,s} \leqslant 2(N_r^s+1)(s-r). \tag{6.19}$$

Proof. Denote the elements of K_r^s by $u_1,\dots,u_{N_r^s+1}$ in such a way that

$$\inf\{t\in\mathbb{R}:\ \mathcal{E}_s=u_i\} < \inf\{t\in\mathbb{R}:\ \mathcal{E}_s=u_j\}$$

if $i<j$. Recall that τ_s is the only point of the spine in K_r^s. Let $1\leqslant k\leqslant N_r^s+1$ be the unique index such that $u_k=\tau_s$. Fix $i\in\{1,\dots,N_r^s+1\}$ with $i\neq k$. Next set, for every $i\in\{1,\dots,N_r^s+1\}$,

$$f_i := \inf\{t\in\mathbb{R}:\ \mathcal{E}_t=u_i\}\ \text{and}\ \ell_i := \sup\{t\in\mathbb{R}:\ \mathcal{E}_t=u_i\}.$$

Since u_i cannot be a leaf we have $f_i<\ell_i$. Moreover, the sequence $(f_i)_{1\leqslant i\leqslant n}$ is increasing. Remark that, for every $1\leqslant i\leqslant N_r^s$, we have $\ell_i<f_{i+1}$ and $[\ell_i,f_{i+1}]$ is the smallest interval such that $\mathcal{E}_{\ell_i}=u_i$ and $\mathcal{E}_{f_{i+1}}=u_{i+1}$. Therefore we can consider the path $\gamma_{u_i,u_{i+1}}$ as defined before the proposition. By construction, labels of points of the form $\mathcal{E}_t$ with $t\in[\ell_i,f_{i+1}]$ are greater than r, and it follows that the length of the path $\gamma_{u_i,u_{i+1}}$ is smaller than $2(s-r)$. Finally set

$$R=\min\big(\inf_{(-\infty,f_1]}\Lambda;\ \inf_{[\ell_{N_r^s+1},\infty)}\Lambda\big)$$

which is greater than r by construction. Set $u_0 := p_\infty^{(r)}(R)$ and $u_{N_s^r+2} := p_\infty^{(\ell)}(R)$. Note in particular that $\Pi(u_0)=\Pi(u_{N_s^r+2})$. Again we can consider the paths γ_{u_0,u_1} and $\gamma_{u_{N_r^s+1},u_{N_r^s+2}}$. Let γ be the cycle obtained by concatenating the paths $(\gamma_{u_i,u_{i+1}})_{0\leqslant i\leqslant N_r^s+1}$. It is straightforward to verify using property (F) that γ is an injective cycle. The paths γ_{u_0,u_1} and $\gamma_{u_{N_r^s+1},u_{N_r^s+2}}$ have length $s-R \leqslant s-r$ and the paths $(\gamma_{u_i,u_{i+1}})_{1\leqslant i\leqslant N_r^s}$ have length smaller than $2(s-r)$. Hence the length of γ is smaller than $2(N_r^s+1)(s-r)$. Moreover, by a preceding remark, the range of $\gamma_{u_i,u_{i+1}}$ is contained in $\Pi(\mathcal{R})$ when $i<k$ and contained in $\Pi(\mathcal{L})$ when $i \geqslant k$. Consequently, if t_0 is the time at which γ visits the point $\Pi(u_k)$, we have $\gamma(t) \in \Pi(\mathcal{R})$ when $t \leqslant t_0$ and $\gamma(t) \in \Pi(\mathcal{L})$ when $t \geqslant t_0$. We conclude that the path γ is in $\mathcal{A}$ and in particular γ is a separating path. Finally, by construction the path γ visits only points v such that $\Lambda_v \in (r,s]$. Since $\gamma(0)=\gamma_\infty(R)$ does not belong to $B_r^\bullet$, it follows that γ takes values in $B_s\backslash B_r^\bullet$. Now remark that γ hits the boundary ∂B_s only at the times at which it visits $\Pi(u_1),\dots,\Pi(u_{N_r^s+1})$. Since N_r^s is, Θ_0-a.s., finite we deduce that γ hits the boundary ∂B_s a finite number of times. Consequently, by an approximation procedure, for every $\varepsilon>0$ we can find $\gamma' \in \mathcal{A}$ taking values in $\mathrm{Int}(B_s^\bullet)\backslash B_r^\bullet \subset B_{r,s}^\circ$ such that $\Delta(\gamma') < \Delta(\gamma)+\varepsilon$. Consequently by the definition of $L_{r,s}$ as an infimum (6.17) we deduce that

$$L_{r,s} \leqslant \Delta(\gamma) \leqslant 2(N_r^s+1)(s-r).$$

□

The proposition shows that it is enough to control N_r^s in order to get an upper bound for $L_{r,s}$. Moreover since the Brownian plane is scale invariant we have:

$$N_r^s \overset{(d)}{=} N_1^{\frac{s}{r}}.$$

So we consider only N_1^s with $s>1$. Thanks to (6.13), the law of N_1^s can be easily determined:

Proposition 6.3. *For $s>1$ and $\lambda \geqslant 0$ we have*

$$\Theta_0\big(\exp(-\lambda N_1^s)\big) = \Big(1+(1-\exp(-\lambda))\frac{2s-1}{(s-1)^2}\Big)^{-\frac{3}{2}}$$

Proof. Let $s>1$, and write $\mathfrak{L}^{s,\infty} := \sum_{i\in I_s}\delta_{s_i,\omega^i}$; $\mathfrak{R}^{s,\infty} := \sum_{i\in J_s}\delta_{s_i,\omega^i}$. Let $\tilde{I}_s$ (resp. $\tilde{J}_s$) be the set of all indices $i\in I_s$ (resp. $j \in J_s$) such that $\omega_*^i < s$. For every $i \in \tilde{I}_s \cup \tilde{J}_s$, write $(\omega^{i,k})_{k\in\mathbb{N}}$ for the excursions of ω^i below s. By the special Markov property, conditionally on Z_s,

$$\sum_{i\in \tilde{I}_s\cup\tilde{J}_s}\sum_{k\in\mathbb{N}}\delta_{\omega^{i,k}}$$

is a Poisson point measure with intensity $Z_s\mathbb{N}_s(\cdot \cap \{W_*>0\})$. Moreover by definition:

$$N_1^s = \#\{(i,k)\in(\tilde{I}_s\cup\tilde{J}_s)\times\mathbb{N} : \omega_*^{i,k}\leqslant 1\}.$$

So conditionally on Z_s, N_1^s is distributed as a Poisson variable with intensity $Z_s\mathbb{N}_s(0<W_*\leqslant 1)$. We can then apply (8.6) to obtain:

$$\mathbb{N}_s(0<W_*\leqslant 1) = \frac{3}{2}\Big(\frac{1}{(s-1)^2}-\frac{1}{s^2}\Big) = \frac{3}{2}\frac{2s-1}{s^2(s-1)^2}. \tag{6.20}$$

Using (6.13) we get that for $\lambda \geqslant 0$:

$$\begin{aligned}\Theta_0\big(\exp(-\lambda N_1^s)\big) &= \mathbb{E}[\exp(-(1-\exp(-\lambda))Z_s\mathbb{N}_s(0<W_*\leqslant 1))\big) \\ &= \Big(1+(1-\exp(-\lambda))\frac{2s-1}{(s-1)^2}\Big)^{-\frac{3}{2}}.\end{aligned}$$

□

Let us list some immediate properties of N_1^s.

Lemma 6.5.
(i) *For every $s>1$ we have $\Theta_0(N_1^s=0)=(\frac{s-1}{s})^3$.*
(ii) *The law of $(s-1)^2N_1^s$ under Θ_0 converges weakly to a Gamma distribution with parameter $\frac{3}{2}$ and mean $3/2$ when $s\downarrow 1$. Furthermore, N_1^s tends to 0 as $s\to\infty$, Θ_0-a.s.*
(iii) *For every $s>1$ and $q<\log(\frac{s^2}{2s-1})$ there exists a constant C_q such that for all $r>0$:*

$$\Theta_0(N_1^s>r)<C_q\exp(-qr).$$

Furthermore:

$$\limsup_{u\to\infty}\frac{\log\big(\Theta_0(N_1^s>u)\big)}{u}=-\log(\frac{s^2}{2s-1}).$$

(iv) *For every $s>1$ we have $\Theta_0\big(N_1^s\big)=\frac{3}{2}\frac{2s-1}{(s-1)^2}$.*

Proof.
(i) By Proposition 6.3, we have

$$\Theta_0(N_1^s=0)=\lim_{\lambda\to\infty}\Theta_0\big(\exp(-\lambda N_1^s)\big)=\big(\frac{s-1}{s}\big)^3$$

(we can also use the fact that $\Theta_0(N_1^s=0)=\Theta_0(Z_1^{s,\infty}=0)$ and Proposition 6.1).

(ii) Again using Proposition 6.3, we obtain:

$$\Theta_0\big(\exp(-\lambda(s-1)^2N_1^s)\big)=\Big(1+(1-\exp(-\lambda(s-1)^2))\frac{2s-1}{(s-1)^2}\Big)^{-\frac{3}{2}}.$$

When s goes to 1 the Laplace transform converges to $\lambda\mapsto(1+\lambda)^{-\frac{3}{2}}$ which is the Laplace transform of a Gamma distribution with parameter 3/2 and mean 3/2. The fact that N_1^s tends to 0 as $s\to\infty$, Θ_0 a.s., immediately follows from the property $\Lambda_{\mathcal{E}_t}\longrightarrow\infty$ as $|t|\to\infty$.

(iii) For every $\lambda>0$ we have:

$$\Theta_0\Big(\exp(\lambda N_1^s)\Big)=\Theta_0\Big(\exp\big(Z_s(\exp(\lambda)-1)\mathbb{N}_s(0<W_*\leqslant 1)\big)\Big)$$

because, conditionally on Z_s , the variable N_1^s is distributed as a Poisson random variable with intensity $Z_s\mathbb{N}_s(0<W_*\leqslant 1)$. But Z_s is a Gamma random variable with parameter 3/2 and mean s^2 so the previous expectation is finite if and only if

$$(\exp(-\lambda)-1)\mathbb{N}_s(0<W_*<1)<3/(2s^2)$$

or equivalently $\lambda < \log(\frac{s^2}{2s-1})$. The first part of (iii) then follows from the Markov inequality. On the other hand if we had:

$$\limsup_{u\to\infty} \frac{\log\big(\Theta_0(N_1^s > u)\big)}{u} \leqslant \alpha < -\log(\frac{s^2}{2s-1})$$

this would contradict $\mathbb{E}[\exp(\log(\frac{s^2}{2s-1})N_1^s)] = \infty$. This gives the second assertion of (iii).

(iv) We use again the fact that, under Θ_0, conditionally on Z_s, N_1^s is distributed as a Poisson random variable with intensity $Z_s\mathbb{N}_s(0 < W_* \leqslant 1)$. We have:

$$\Theta_0\big(N_1^s\big) = \mathbb{N}_s(0 < W_* \leqslant 1)\Theta_0\big(Z_s\big) = \frac{3}{2}\frac{2s-1}{s^2(s-1)^2}\Theta_0\big(Z_s\big) = \frac{3}{2}\frac{2s-1}{(s-1)^2}$$

where in the second equality we use (6.20) and in the last equality we use the fact that Z_s has mean s^2. □

As a direct consequence we derive Theorem 6.1 (i) from Lemma 6.5.

Proof of Theorem 6.1 (i). By (6.19), for every $s > 1$ we have $L_1 \leqslant 2(s-1)(N_1^s + 1)$ Θ_0-a.s. Let $s > 1$. Lemma 6.5 gives that :

$$\begin{aligned}\limsup_{u\to\infty} \frac{\log\big(\Theta_0(L_1 > u)\big)}{u} &\leqslant \limsup_{u\to\infty} \frac{\log\big(\Theta_0(N_1^s > \frac{u}{2(s-1)} - 1)\big)}{u} \\ &\leqslant -\frac{1}{2(s-1)}\log(\frac{s^2}{2s-1}).\end{aligned}$$

Since this holds for every $s > 1$, we obtain:

$$\limsup_{u\to\infty} \frac{\log\big(\Theta_0(L_1 > u)\big)}{u} \leqslant -\sup_{s>1} \frac{1}{2(s-1)}\log(\frac{s^2}{2s-1}).$$

□

The rest of this section is devoted to the proof of the lower bound appearing in Theorem 6.1 (ii). The proof relies again in (6.19) but in a more technical way. We state the following slightly stronger result:

Proposition 6.4. *There exists a positive constant, c_1, such that for every $\varepsilon \in [0,1]$ and $r > 0$:*

$$\Theta_0\big(L_{r,3r} < \varepsilon r\big) \geqslant c_1\varepsilon^2.$$

The factor 3 is arbitrary and we will see in the proof that it can be replaced by any constant greater than 1. It will be useful in what follows to note that for every $0 < r < t$, we have $\{N_r^t = 0\} = \{Z_r^{t,\infty} = 0\}$ Θ_0-a.s. We are going to deduce Proposition 6.4 from (6.19) and the following result:

Lemma 6.6. *There exists a positive constant c_1 such that for every $r > 0$ and $m \in \mathbb{N}^*$:*

$$\Theta_0\Big(\bigcup_{i=0}^{m-1}\{N_{(m+i)r}^{(m+i+1)r}=0\}\Big) \geqslant \frac{c_1}{m^2}$$

Proof. By the scaling invariance of $\mathcal{M}_\infty$ we can take $r=1$. For $m \geqslant 2$, we have:

$$\begin{aligned}\Theta_0\Big(\bigcup_{i=0}^{m-1}\{N_{m+i}^{m+i+1}=0\}\Big) &= \Theta_0(N_m^{m+1}=0) \\ &\quad+\sum_{k=0}^{m-2}\Theta_0\Big(\{N_{m+k+1}^{m+k+2}=0\}\cap\bigcap_{i=0}^{k}\{N_{m+i}^{m+i+1}>0\}\Big).\end{aligned} \tag{6.21}$$

Moreover, for every $k \in \{0,\dots m-2\}$:

$$\begin{aligned}&\Theta_0\Big(\{N_{m+k+1}^{m+k+2}=0\}\cap\bigcap_{i=0}^{k}\{N_{m+i}^{m+i+1}>0\}\Big) \\ &\qquad= \Theta_0\Big(\{Z_{m+k+1}^{m+k+2,\infty}=0\}\cap\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) \\ &\qquad= \Theta_0\big(Z_{m+k+1}^{m+k+2,\infty}=0\big)\,\Theta_0\Big(\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big)\end{aligned} \tag{6.22}$$

where the first equality comes from the fact that $N_{m+k+1}^{m+k+2}=0$, which is equivalent to $Z_{m+k+1}^{m+k+2,\infty}=0$, implies $Z_t^{m+k+2,\infty}=0$ for every $t \leqslant m+k+1$, so that, on the event $\{N_{m+k+1}^{m+k+2}=0\}$, we have $Z_{m+i}^{m+i+1,\infty}=Z_{m+i}^{m+i+1,m+k+2}$ for every $0 \leqslant i \leqslant k$. The second equality in (6.22) is a consequence of the spine independence property of $\mathcal{T}_\infty$ (see Section 6.2.4). The idea now is to prove that for every integer $k \geqslant 0$:

$$\Theta_0\Big(\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) \geqslant \prod_{i=0}^{k}\Theta_0\big(Z_{m+i}^{m+i+1,m+k+2}>0\big). \tag{6.23}$$

Let us explain how to obtain this inequality. Let $k>0$, then:

$$\begin{aligned}\Theta_0\Big(\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) &= \Theta_0\Big(\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) \\ &\quad-\Theta_0\Big(\{Z_{m+k}^{m+k+1,m+k+2}=0\}\cap\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) \\ &= \Theta_0\Big(\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big) \\ &\quad-\Theta_0\Big(\{Z_{m+k}^{m+k+1,m+k+2}=0\}\cap\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+1}>0\}\Big)\end{aligned}$$

where the second equality is a consequence of the fact that $Z_{m+k}^{m+k+1,m+k+2}=0$ implies that for every $i<k$ $Z_{m+i}^{m+i+1,m+k+2}=Z_{m+i}^{m+i+1,m+k+1}$. We now can apply the spine independence property

to obtain that $\Theta_0\Big(\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big)$ is equal to

$$\Theta_0\Big(\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big)-\Theta_0\big(Z_{m+k}^{m+k+1,m+k+2}=0\big)\Theta_0\Big(\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+1}>0\}\Big).$$

Now using the property $\{Z_{m+i}^{m+i+1,m+k+1}>0\}\subset\{Z_{m+i}^{m+i+1,m+k+2}>0\}$ for $i=0,\dots,k-1$, we derive

$$\Theta_0\Big(\bigcap_{i=0}^{k}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big)\geqslant\Theta_0\Big(\bigcap_{i=0}^{k-1}\{Z_{m+i}^{m+i+1,m+k+2}>0\}\Big)\Theta_0\big(Z_{m+k}^{m+k+1,m+k+2}>0\big).$$

We can then iterate this argument to obtain (6.23). By combining (6.22) and (6.23) we deduce that:

$$\Theta_0\Big(\{N_{m+k+1}^{m+k+2}=0\}\cap\bigcap_{i=0}^{k}\{N_{m+i}^{m+i+1}>0\}\Big)\geqslant\Theta_0\big(Z_{m+k+1}^{m+k+2,\infty}=0\big)\prod_{i=0}^{k}\Theta_0\big(Z_{m+i}^{m+i+1,m+k+2}>0\big).$$

On the other hand, Proposition 6.1 states that for $0<r<t<s$,

$$\Theta_0(Z_r^{t,s}=0)=\big(\frac{s}{t}\big)^3\big(\frac{t-r}{s-r}\big)^3$$

and taking the limit when s goes to ∞, we obtain $\Theta_0(Z_r^{t,\infty}=0)=\big(\frac{t-r}{t}\big)^3$. It follows that:

$$\Theta_0\big(\{N_{m+k+1}^{m+k+2}=0\}\cap\bigcap_{i=0}^{k}\{N_{m+i}^{m+i+1}>0\}\big)\geqslant\frac{1}{(m+k+2)^3}\prod_{i=0}^{k}\Big(1-\big(\frac{m+k+2}{m+i+1}\big)^3\frac{1}{(k+2-i)^3}\Big).$$

Then, for $m\geqslant 3$ and $k\in\{0,\dots,m-2\}$:

$$\begin{aligned}\prod_{i=0}^{k}\Big(1-\big(\frac{m+k+2}{m+i+1}\big)^3\frac{1}{(k+2-i)^3}\Big)&\geqslant\Big(1-\frac{1}{8}\big(\frac{m+k+2}{m+k+1}\big)^3\Big)\prod_{i=0}^{k-1}\Big(1-\big(\frac{2m}{m}\big)^3\frac{1}{(k+2-i)^3}\Big)\\&\geqslant\Big(1-\frac{1}{8}\Big(\frac{5}{4}\Big)^3\Big)\prod_{i=3}^{\infty}\big(1-\frac{8}{i^3}\big)\end{aligned}$$

which is a positive constant not depending on m. Let $\tilde{c}_1$ denote this constant. By applying the previous inequality to (6.21), we obtain that for every $m\geqslant 3$:

$$\begin{aligned}\Theta_0\Big(\bigcup_{i=0}^{m-1}\{N_{m+i}^{m+i+1}=0\}\Big)&\geqslant\sum_{k=0}^{m-2}\Theta_0\Big(\{N_{m+k+1}^{m+k+2}=0\}\cap\bigcap_{i=0}^{k}\{N_{m+i}^{m+i+1}>0\}\Big)\\&\geqslant\sum_{k=0}^{m-2}\frac{\tilde{c}_1}{(m+k+2)^3}\end{aligned}$$

which gives us the lower bound in the lemma. □

Proposition 6.4 follows now easily.

Proof of Proposition 6.4. By scaling we only need to prove the proposition for $r = 1$. Let $\varepsilon \in [0, 1)$ and set $s = \frac{\varepsilon}{2}$ and $m = \lceil \frac{2}{\varepsilon} \rceil$. Then the bound (6.19) and the fact that $L_{u',v'} \leqslant L_{u,v}$ if $[u', v'] \subset [u, v]$ give:

$$L_{1,3} \leqslant 2\big(1 + \min_{i \in \{0,\dots,m-1\}} N_{(m+i)s}^{(m+i+1)s}\big)s$$

Θ_0-a.s.. Consequently $\Theta_0(L_{1,3} \leqslant \varepsilon) \geqslant \Theta_0\Big(\bigcup_{i=0}^{m-1} \{N_{(m+i)s}^{(m+i+1)s} = 0\}\Big)$. The desired result follows directly from Lemma 6.4. □

6.3.3 Upper bound for the tail of L_1 near 0

We are going to deduce the upper bound for Theorem 6.1 (ii) from the following result

Proposition 6.5. *For every $\delta > 0$, there exists a constant α_δ such that for every $r \geqslant 1$:*

$$\Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) \leqslant \exp(1 - \alpha_\delta Z_{r+2\delta}).$$

We need to introduce some notation in order to prove Proposition 6.5 . Let $u \in \mathcal{T}_\infty \backslash [0, \infty)$ such that u is not a leaf. Set $f_u := \inf\{t \in \mathbb{R}_+ : \mathcal{E}_t = u\}$ and $\ell_u := \sup\{t \in \mathbb{R}_+ : \mathcal{E}_t = u\}$. We consider the subtree $\mathcal{T}_\infty^{(u)} := \{\mathcal{E}_t : t \in [f_u, \ell_u]\}$ which is also equal to the set of all points $v \in \mathcal{T}_\infty$ such that $u \preceq v$. Remark that for every $v_1, v_2 \in \mathcal{T}_\infty^{(u)}$ there are two possibilities, either $[v_1, v_2]_{\mathcal{T}_\infty} \subset \mathcal{T}_\infty^{(u)}$ and in this case $0 \notin [v_1, v_2]_{\mathcal{T}_\infty}$ or $0 \in [v_1, v_2]_{\mathcal{T}_\infty}$. Consequently $\Delta^\circ(v_1, v_2)$ only depends on the subtree $\big(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v \in \mathcal{T}_\infty^{(u)}}\big)$. For every $v, w \in \mathcal{T}_\infty^{(u)}$, set:

$$\widetilde{\Delta}_u(v, w) := \inf_{\substack{v = v_1, \dots, v_n = w \\ v_1, \dots, v_n \in \mathcal{T}_\infty^{(u)}}} \sum_{i=1}^{n-1} \Delta^\circ(v_i, v_{i+1}) \tag{6.24}$$

where the infimum is over all choices of the integer $n \geqslant 1$ and all the finite sequences $u_0, \dots, u_n$ of elements of $\mathcal{T}_\infty^{(u)}$ verifying $v_1 = v$ and $v_n = w$. Remark that $\widetilde{\Delta}_u$ defines a continuous pseudo-distance on $\mathcal{T}_\infty^{(u)}$ (since $v \mapsto \Lambda_v$ is continuous) and that $\Delta(v, w) \leqslant \widetilde{\Delta}_u(v, w)$ for every $v, w \in \mathcal{T}_\infty^{(u)}$. By the previous remark and since by (F), for every $v, w \in \mathcal{T}_\infty^{(u)}$, we have $\Delta(v, w) = 0$ if and only if $\Delta^\circ(v, w) = 0$, we see that $\widetilde{\Delta}_u$ defines a distance on $\Pi(\mathcal{T}_\infty^{(u)})$ (and we keep the notation $\widetilde{\Delta}_u$ for this distance). To simplify notation, we introduce the set $A_u := \Pi(\mathcal{T}_\infty^{(u)})$ and the paths $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ defined as follows. For every $0 \leqslant t \leqslant \Lambda_u - \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v$, take:

$$\gamma_u^{(1)}(t) := \Pi\big(\inf\{r \in [f_u, \ell_u] : \Lambda_r = \Lambda_u - t\}\big)$$

and

$$\gamma_u^{(2)}(t) := \Pi\big(\sup\{r \in [f_u, \ell_u] : \Lambda_r = \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v + t\}\big).$$

By construction and (8.17), $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ are two geodesic paths. We also observe that $\min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v < \Lambda_u$ as a consequence of [76, Lemma 3.2] (it is important to notice that this holds simultaneously for all $u \in \mathcal{T}_\infty \backslash [0, \infty)$ such that u is not a leaf). Moreover, we have $\gamma_u^{(1)}\big(\Lambda_u - \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v\big) = \gamma_u^{(2)}(0)$

and $\gamma_u^{(1)}(0) = \gamma_u^{(2)}\big(\Lambda_u - \min_{v\in\mathcal{T}_\infty^{(u)}} \Lambda_v\big)$. From property (F), we get that the concatenation of $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ is an injective cycle with length $2\Lambda_u - 2\min_{v\in\mathcal{T}_\infty^{(u)}} \Lambda_v$. We denote this path by γ_u. It will be useful to remark that γ_u^1 and γ_u^2 are also geodesic paths for $\widetilde{\Delta}_u$. Recall the notation Δ_{A_u} for the intrinsic distance induced by Δ on A_u.

Lemma 6.7. Θ_0*-a.s. for every* $u \in \mathcal{T}_\infty\backslash[0,\infty)$ *such that* u *is not a leaf, the set* A_u *is homeomorphic to the closed unit disk of* $\mathbb{C}$ *and its boundary is the range of* γ_u*. Moreover we have:*

$$\Delta_{A_u} = \widetilde{\Delta}_u. \tag{6.25}$$

The main interest of (6.25) is the fact that the function $\widetilde{\Delta}_u$ only depends on $\big(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v\in\mathcal{T}_\infty^{(u)}}\big)$. Recall the definition of Δ° above (6.4), and remark that Δ° on $\mathcal{T}_\infty^{(u)}$ does not change if we shift all the labels by any $a > -\min_{\mathcal{T}_\infty^{(u)}} \Lambda$. This gives that $\widetilde{\Delta}_u$ can also be defined from the labeled tree $\big(\mathcal{T}_\infty^{(u)}, (\Lambda_v)_{v\in\mathcal{T}_\infty^{(u)}} + a\big)$ for any $a > -\min_{\mathcal{T}_\infty^{(u)}} \Lambda$.

Proof. Let $u \in \mathcal{T}_\infty\backslash[0,\infty)$ such that u is not a leaf. Since γ_u is an injective cycle, Jordan's theorem implies that the complement of the range of γ_u has two connected components, namely a bounded connected component U_1 and an unbounded connected component U_2. Moreover, the closure of U_1 is homeomorphic to the closed unit disk. The first assertion of the lemma then follows from the fact that $\mathrm{Cl}(U_1) = A_u$, which is easy and left to the reader. Let us turn to the second part of the lemma. We start by showing that $\Delta_{A_u}(v,w) \leqslant \widetilde{\Delta}_u(v,w)$ for every $v,w \in \mathcal{T}_\infty^{(u)}$. Let $v,w \in \mathcal{T}_\infty^{(u)}$. Up to interchanging v and w, we can suppose that

$$\Delta^\circ(v,w) = \Lambda_v + \Lambda_w - 2\min_{[v,w]_{\mathcal{T}_\infty}} \Lambda,$$

and $v = \mathcal{E}_s$, $w = \mathcal{E}_t$ with $s \leqslant t$ and $[u,w]_{\mathcal{T}_\infty} = \{\mathcal{E}_r : s \leqslant r \leqslant t\}$. We can then consider the path $\gamma_{v,w}$ introduce at the beginning of Section 6.3.2. The length of $\gamma_{v,w}$ is $\Delta^\circ(v,w)$ and its range is a subset of $\Pi([v,w]_{\mathcal{T}_\infty})$. Since $[v,w]_{\mathcal{T}_\infty} \subset \mathcal{T}_\infty^{(u)}$, we deduce that $\gamma_{v,w}$ takes values in A_u. From the definition (6.24) we obtain that $\Delta_{A_u}(v,w) \leqslant \widetilde{\Delta}_u(v,w)$ for every $v,w \in \mathcal{T}_\infty^{(u)}$.

Let us prove the reverse inequality. Let $\gamma : [0,1] \to A_u$ be a path. It is enough to show that $\Delta(\gamma) \geqslant \widetilde{\Delta}_u(\gamma(0),\gamma(1))$. We deal with two separate cases.

• Case 1: We assume that, for every $t \in (0,1)$, $\gamma(t) \notin \partial A_u$. Let us start by showing that γ is also continuous for $\widetilde{\Delta}_u$. In order to prove this, remark that the identity function $(A_u,\widetilde{\Delta}_u) \mapsto (A_u,\Delta)$ is a bijection and that it is also continuous, since $\Delta \leqslant \widetilde{\Delta}_u$. Moreover, as $\mathcal{T}_\infty^{(u)}$ is compact, the continuity of Π implies that A_u is compact for the quotient topology. Since Δ and $\widetilde{\Delta}_u$ are continuous on $\mathcal{T}_\infty^{(u)} \times \mathcal{T}_\infty^{(u)}$ we derive that $(A_u,\widetilde{\Delta}_u)$ and (A_u,Δ) are both compact. So the identity function $(A_u,\widetilde{\Delta}_u) \mapsto (A_u,\Delta)$ is a continuous bijection between compact spaces which implies that it is also an homeomorphism. We deduce that γ is continuous for $\widetilde{\Delta}_u$. In particular, we have

$$\Delta(\gamma) = \lim_{\varepsilon\to 0} \Delta(\gamma|_{[\varepsilon,1-\varepsilon]}) \text{ and } \widetilde{\Delta}_u\big(\gamma(0),\gamma(1)\big) = \lim_{\varepsilon\to 0} \widetilde{\Delta}_u\big(\gamma(\varepsilon),\gamma(1-\varepsilon)\big).$$

By the previous display, we may and will restrict our attention to the case when $\gamma(t) \notin A_u$ for every $t \in [0,1]$. By compactness, the quantity $\delta := \inf_{(t,v)\in[0,1]\times\partial A_u} \Delta(\gamma(t), v)$ is positive. Let $n > 1$ be an integer such that for every $0 \leqslant i \leqslant n-1$:

$$\Delta\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big) < \frac{\delta}{3}.$$

We are going to show that $\Delta\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big) \geqslant \widetilde{\Delta}_u\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big)$ for every $0 \leqslant i \leqslant n-1$. By the triangle inequality this will imply that $\Delta(\gamma) \geqslant \widetilde{\Delta}_u(\gamma(0), \gamma(1))$. Fix $0 \leqslant i \leqslant n-1$ and consider $u_i, v_i \in \mathcal{T}_\infty$ such that $\big(\Pi(u_i), \Pi(v_i)\big) = \big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big)$. Remark that we must have $u_i, v_i \in \mathcal{T}_\infty^{(u)}$ and recall that:

$$\Delta(u_i, v_i) = \inf_{u_i = u_{i,1}, u_{i,2}\dots, u_{i,m} = v_i} \sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1})$$

where the infimum is over all choices of the integer $m \geqslant 1$ and all finite sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ with $(u_{i,1}, u_{i,m}) = (u_i, v_i)$. Since $\Delta(u_i, v_i) < \delta/3$ we can restrict the previous infimum to finite sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ with $(u_{i,1}, u_{i,m}) = (u_i, v_i)$ such that:

$$\sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1}) < \frac{\delta}{2}.$$

Consider such a sequence $u_{i,1}, \dots, u_{i,m} \in \mathcal{T}_\infty$ and remark that $\Delta(u_{i,1}, u_{i,k}) < \frac{\delta}{2}$ for every $1 \leqslant k \leqslant m$ by triangle inequality. This implies by the definition of δ that $u_{i,k} \in \mathcal{T}_\infty^{(u)}$ for every $1 \leqslant k \leqslant m$. We conclude from the definition of $\widetilde{\Delta}_u$ that:

$$\sum_{k=1}^{m-1} \Delta^\circ(u_{i,k}, u_{i,k+1}) \geqslant \widetilde{\Delta}_u(u_i, v_i).$$

Consequently, $\Delta\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big) \geqslant \widetilde{\Delta}_u\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big)$ for every $0 \leqslant i \leqslant n-1$ and thus $\Delta(\gamma) \geqslant \widetilde{\Delta}_u\big(\gamma(0), \gamma(1)\big)$.

• Case 2: We now assume that γ hits ∂A_u. Let $s := \inf\{r \in [0,1] : \gamma(r) \in \partial A_u\}$. Without loss of generality, we may assume that $\gamma(s)$ belongs to the range of $\gamma_u^{(1)}$. Let s' be the largest element of $[0,1]$ such that $\gamma(s')$ is in the range of $\gamma_u^{(1)}$. If $s' = 1$ or if $\gamma(t) \notin \partial A_u$ for every $s' < t \leqslant 1$, we have:

$$\Delta(\gamma) = \Delta(\gamma|_{[0,s]}) + \Delta(\gamma|_{[s,s']}) + \Delta(\gamma|_{[s',1]}) \geqslant \widetilde{\Delta}_u(\gamma(0), \gamma(s)) + \Delta(\gamma|_{[s,s']}) + \widetilde{\Delta}_u(\gamma(s'), \gamma(1))$$

since case 1 can be applied to $\gamma|_{[0,s]}$ and $\gamma|_{[s',1]}$. Moreover, since $\gamma_u^{(1)}$ is also a geodesic for $\widetilde{\Delta}_u$, we have $\Delta(\gamma|_{[s,s']}) \geqslant \Delta(\gamma(s), \gamma(s')) = \widetilde{\Delta}_u(\gamma(s), \gamma(s'))$ and by the triangle inequality we obtain $\Delta(\gamma) \geqslant \widetilde{\Delta}_u(\gamma(0), \gamma(1))$.

It remains to consider the case where $s' < 1$ and $\{t \in (s', 1] : \gamma(t) \in \partial A_u\}$ is not empty. Let t be the smallest element of $[s', 1]$ such that $\gamma(t) \in \partial A_u$. Then $\gamma(t)$ belongs to the range of

$\gamma_u^{(2)}$. Let t' be the largest element of $[t,1]$ such that $\gamma(t')$ belongs to the range of $\gamma_u^{(2)}$. Then we get that:

$$\Delta(\gamma) = \Delta(\gamma|_{[0,s]}) + \Delta(\gamma|_{[s,s']}) + \Delta(\gamma|_{[s',t]}) + \Delta(\gamma|_{[t,t']}) + \Delta(\gamma|_{[t',1]}).$$

Since $\gamma_u^{(1)}$ and $\gamma_u^{(2)}$ are two geodesics paths for $\widetilde{\Delta}_u$ we have:

$$\Delta(\gamma|_{[s,s']}) \geqslant \Delta\big(\gamma(s),\gamma(s')\big) = \widetilde{\Delta}_u\big(\gamma(s),\gamma(s')\big)$$

and

$$\Delta(\gamma|_{[t,t']}) \geqslant \Delta\big(\gamma(t),\gamma(t')\big) = \widetilde{\Delta}_u\big(\gamma(t),\gamma(t')\big).$$

On the other hand, $\gamma|_{[0,s]}$, $\gamma|_{[s',t]}$ and $\gamma|_{[t',1]}$ belong to case 1. Consequently, we obtain $\Delta(\gamma) \geqslant \widetilde{\Delta}_u(\gamma(0),\gamma(1))$. □

Let us deduce Proposition 6.5 from Lemma 6.7.

Proof of Proposition 6.5. Fix $\delta > 0$ and $r \geqslant 1$. We want to give a lower bound for $L_{r,r+\delta}$. By Lemma 6.4 it is enough to give a lower bound for $\Delta(\gamma)$ for every $\gamma \in \mathcal{A}$ taking values in $B^{\circ}_{r,r+\delta}$. Recall the notation of Section 6.3.2 and for every $u \in K_r^{r+2\delta}$ not belonging to the spine $[0,\infty)$ set:

$$M_u := \min_{v \in \mathcal{T}_\infty^{(u)}} \Lambda_v.$$

Let $u_1,\dots,u_{\tilde{N}_r^{r+2\delta}}$ be the elements of $K_r^{r+2\delta}$ that do not belong to the spine $[0,\infty)$ and such that $M_u > r - 1/2$. By Lemma 6.7, for every $1 \leqslant k \leqslant \tilde{N}_r^{r+2\delta}$ the set A_{u_k} is homeomorphic to the closed unit disk. Since $\partial B^\bullet_r$ and $\partial B^\bullet_{r+\delta}$ are injective cycles (see Section 6.2.4), it is straightforward to verify (using Jordan's theorem) that $A_{u_k} \cap B^\bullet_{r,r+\delta}$ is homeomorphic to the unit disk and its boundary is:

$$\big(A_{u_k} \cap \partial B^\bullet_{r,r+\delta}\big) \cup \gamma^{(1)}_{u_k}([\delta,2\delta]) \cup \gamma^{(2)}_{u_k}([r-M_u, r+\delta-M_u]).$$

This implies that any separating cycle $\gamma \in \mathcal{A}$ taking values in $B^\circ_{r,r+\delta}$ has to connect $\gamma^{(1)}_{u_k}([\delta,2\delta])$ and $\gamma^{(2)}_{u_k}([r-M_u, r+\delta-M_u])$ while staying in A_{u_k} see figure 6.3 for an illustration. In other words, for every $1 \leqslant k \leqslant \tilde{N}_r^{r+2\delta}$, there exist $t_k < t'_k$ such that $\gamma|_{[t_k,t'_k]}$ takes values in $A_{u_k} \cap B^\circ_{r,r+\delta}$, $\gamma(t_k) \in \gamma^{(1)}_{u_k}([\delta,2\delta])$ and $\gamma(t'_k) \in \gamma^{(2)}_{u_k}([r-M_u, r+\delta-M_u])$. In particular:

$$\Delta(\gamma) \geqslant \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \Delta(\gamma|_{[t_k,t'_k]}) \geqslant \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \Delta_{A_{u_k}}(\gamma(t_k),\gamma(t'_k)).$$

Set

$$D_k := \inf\big\{\Delta_{A_{u_k}}(x,y) :\ (x,y) \in \gamma^{(1)}_{u_k}([\delta,2\delta]) \times \gamma^{(2)}_{u_k}([r-M_u, r+\delta-M_u])\big\},$$

and note that $D_k > 0$ by property (F) (no point of $\gamma^{(1)}_{u_k}([\delta,2\delta])$ can be identified with a point of $\gamma^{(2)}_{u_k}([r-M_u, r+\delta-M_u])$). With this notation we have $\Delta(\gamma) \geqslant \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k$ for every $\gamma \in \mathcal{A}$ taking values in $B^\circ_{r,r+\delta}$. Consequently, we obtain that:

$$L_{r,r+\delta} \geqslant \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k.$$

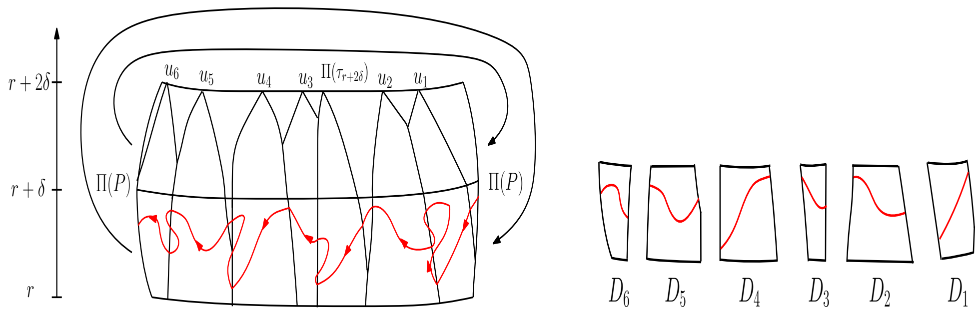

Figure 6.3 – Illustration of the inequality $L_{r,r+\delta} \geqslant \sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k$. The red path is an element of $\mathcal{A}$ taking values in $B^\circ_{r,r+\delta}$ and here $\tilde{N}_r^{r+2\delta} = 6$.

To conclude we use the following claim:

(C): Conditionally on $(Z_{r+2\delta}, \tilde{N}_r^{r+2\delta})$, the variables $(D_k)_{1\leqslant k\leqslant \tilde{N}_r^{r+2\delta}}$ are independent and identically distributed according to a distribution μ_δ that does not depend on r and is supported on $(0,\infty)$.

Before proving (C), let us explain why Proposition 6.5 follows.
We set $\chi_\delta := \int \mu_\delta(dx)\, e^{-x} \in (0,1)$, and then we have

$$\begin{aligned}\Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) &\leqslant \Theta_0\Big(\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k < 1 \mid Z_{r+2\delta}\Big)\\ &\leqslant e\,\Theta_0\Big(\exp(-\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} D_k) \mid Z_{r+2\delta}\Big)\\ &= e\,\Theta\big(\chi_\delta^{\tilde{N}_r^{r+2\delta}} \mid Z_{r+2\delta}\big)\end{aligned}$$

by our claim (C). By the special Markov property, conditionally on $Z_{r+2\delta}$, the variable $\tilde{N}_r^{r+2\delta}$ is distributed as a Poisson variable with parameter $Z_{r+2\delta}\mathbb{N}_{r+2\delta}(r-1/2 < W_* \leqslant r)$. It follows that

$$\Theta_0(L_{r,r+\delta} < 1 \mid Z_{r+2\delta}) \leqslant e \exp\big(-Z_{r+2\delta}\mathbb{N}_{r+2\delta}(r-1/2 < W_* \leqslant r)(1-\chi_\delta)\big)$$

and we obtain the desired result with

$$\alpha_\delta := \mathbb{N}_{r+2\delta}(r-1/2 < W_* \leqslant r)(1-\chi) = \mathbb{N}_{2\delta}(1/2 < W_* \leqslant 1)(1-\chi_\delta).$$

Let us explain why (C) holds in order to complete the proof. Let $(\omega^i)_{i\in I_{r+2\delta}\cup J_{r+2\delta}}$ be the atoms of $\mathfrak{L}^{r+2\delta,\infty}$ and $\mathfrak{R}^{r+2\delta,\infty}$. Let $\tilde{I}_{r+2\delta}\cup\tilde{J}_{r+2\delta}$ be the set of indices $i \in I_{r+2\delta}\cup J_{r+2\delta}$ such that $\omega^i_* < r+2\delta$. For every $i \in \tilde{I}_{r+2\delta}\cup\tilde{J}_{r+2\delta}$ we write $(\omega^{i,n})_{n\in\mathbb{N}}$ for the excursions of ω^i outside $r+2\delta$. By construction:

$$\tilde{N}_r^{r+2\delta} := \#\big\{(i,n) \in (\tilde{I}_{r+2\delta}\cup\tilde{J}_{r+2\delta})\times\mathbb{N} :\ r-1/2 < \omega^{i,n}_* < r\big\}.$$

For every $1 \leqslant k \leqslant \tilde{N}_r^{r+2\delta}$, there exists a unique (i,n) with $r-1/2 < \omega_*^{i,n} < r$ such that the labeled trees $\mathcal{T}_{\omega^{i,n}}$ and $\mathcal{T}_\infty^{(u_k)}$ can be identified and we write ω^k instead of $\omega^{i,n}$ to simplify notation. By the special Markov property, conditionally on $Z_{r+2\delta}$, the point measure:

$$\sum_{k=1}^{\tilde{N}_r^{r+2\delta}} \delta_{\omega^k}$$

is a Poisson measure with intensity:

$$Z_{r+2\delta}\mathbb{N}_{r+2\delta}\big(\cdot \cap \{r-\frac{1}{2} < W_* < r\}\big).$$

So conditionally on $(Z_{r+2\delta}, \tilde{N}_r^{r+2\delta})$, the sequence $(\omega^k - r + 1)_{1\leqslant k\leqslant \tilde{N}_r^{r+2\delta}}$ is an i.i.d. sequence with common distribution $\mathbb{N}_\delta(\cdot \cap \{\frac{1}{2} < W_* < 1\})$. In particular, this distribution does not depend on r. Moreover $\tilde{\Delta}_{u_k}$ depends only on the labeled tree $\mathcal{T}_\infty^{(u_k)} = \mathcal{T}_{\omega^k}$ and the definition (6.24) shows that $\tilde{\Delta}_{u_k}$ is not affected if labels are shifted by $(-r+1)$. So $\tilde{\Delta}_{u_k}$ is also a function of $\omega^k - r + 1$. Our claim (C) follows since by Lemma 6.7, we have $\Delta_{A_{u_k}} = \tilde{\Delta}_{u_k}$ for every $1 \leqslant k \leqslant \tilde{N}_r^{r+2\delta}$. □

We conclude this section with the proof of part (ii) of Theorem 6.1.

Proof of Theorem 6.1 (ii). We want to show that there exists c_2, such that for every $\varepsilon \geqslant 0$:

$$\Theta_0(L_1 < \varepsilon) \leqslant c_2\varepsilon^2.$$

To do so fix $\varepsilon \in (0, 1/2)$ and remark that:

$$\{L_1 < \varepsilon\} \subset \bigcup_{m=\lfloor\frac{1}{\varepsilon}\rfloor-1}^{\infty} \{L_{m\varepsilon,(m+3)\varepsilon} < \varepsilon\} \tag{6.26}$$

Let us explain why (6.26) holds. On the event $\{L_1 < \varepsilon\}$, let γ be a separating cycle in $\check{B}_1^\circ$ such that $\Delta(\gamma) < \varepsilon$. Since the sets $B^\bullet_{(m+1)\varepsilon}\backslash B^\bullet_{m\varepsilon}$, for $m \geqslant \lfloor\frac{1}{\varepsilon}\rfloor$, cover $\check{B}_1^\circ$, we can find $m_0 \geqslant \lfloor\frac{1}{\varepsilon}\rfloor$ such that $\gamma(0) \in B^\bullet_{(m_0+1)\varepsilon}\backslash B^\bullet_{m_0\varepsilon}$. Then notice that $\Delta(\gamma(0), B^\bullet_{(m_0-1)\varepsilon}) \geqslant \varepsilon$ and $\Delta(\gamma(0), \check{B}^\bullet_{(m_0+2)\varepsilon}) \geqslant \varepsilon$. Since the length of γ is smaller than ε, it follows that the path γ stays inside $B^\circ_{(m_0-1)\varepsilon,(m_0+2)\varepsilon}$, and consequently $\Delta(\gamma) \leqslant L_{(m_0-1)\varepsilon,(m_0+2)\varepsilon}$.

(6.26) implies that:

$$\Theta_0(L_1 < \varepsilon) \leqslant \sum_{m=\lfloor\frac{1}{\varepsilon}\rfloor-1}^{\infty} \Theta_0\big(L_{m\varepsilon,(m+3)\varepsilon} < \varepsilon\big) = \sum_{m=\lfloor\frac{1}{\varepsilon}\rfloor-1}^{\infty} \Theta_0\big(L_{m,m+3} < 1\big)$$

where to obtain the right equality we use the scale invariance of $\mathcal{M}_\infty$. We now can apply Proposition 6.5 to obtain that there exists $\alpha > 0$ such that :

$$\Theta_0(L_1 < \varepsilon) \leqslant e \sum_{m=\lfloor\frac{1}{\varepsilon}\rfloor-1}^{\infty} \Theta_0\big(\exp(-\alpha Z_{m+4})\big) = e \sum_{m=\lfloor\frac{1}{\varepsilon}\rfloor-1}^{\infty} (1 + \frac{2}{3}\alpha(m+4)^2)^{-\frac{3}{2}}$$

where we used (6.13) in the last equality. The desired result follows. □

6.3.4 Application to the infinite volume Brownian disk

The goal of this section is to extend Theorem 6.3 to random levels and then to derive some properties of injective cycles separating the boundary from infinity in the infinite volume Brownian disk. Let us recall the notation of Subsection 6.2.4, and in particular, the definition of the coding triple $(\widetilde{X}^{(r)}, \widetilde{\mathfrak{L}}_r, \widetilde{\mathfrak{R}}_r)$ for every $r \geqslant 0$. On the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$, for every $r \geqslant 0$, let $\mathcal{F}_r$ be the σ-field generated by $B_r^\bullet$ (view as a random variable with values in $\mathbb{K}$ as explained in Section 6.2.4) and the class of all Θ_0-negligible sets. The approximation property (6.15) implies that Z_r is $\mathcal{F}_{r+}$-measurable, for every $r \geqslant 0$. We write $\rho_r := \Pi(\tau_r)$ for every $r \geqslant 0$, where $(\tau_r)_{r\geqslant 0}$ is defined in (6.10).

Theorem 6.4. *Let T be a stopping time of the filtration $(\mathcal{F}_{r+})_{r\geqslant 0}$ such that we have $0 < T < \infty$, Θ_0-a.s. Then conditionally on $Z_T = z$, the coding triple $(\widetilde{X}^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T)$ is distributed according to Θ_z and is independent of $B_T^\bullet$. Furthermore, the intrinsic distance $\check{\Delta}^{(T)}$ on $\check{B}_T^\circ$ has a unique continuous extension to $\check{B}_T^\bullet$. The space $\check{B}_T^\bullet$ equipped with this continuous extension of $\check{\Delta}^{(T)}$, with the restriction of the volume measure and with the distinguished point ρ_T coincides (as an element of $\mathbb{K}_\infty$) with the metric space associated with $(\widetilde{X}^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T)$. In particular, conditionally on $Z_T = z$, the space $\check{B}_T^\bullet$ is an infinite Brownian disk with perimeter z and is independent of $B_T^\bullet$.*

Proof. Let T be as in the statement of the theorem. Recall the notation $M(\mathcal{S})$ and the distance $d_{M(\mathcal{S})}$ introduced in Section 6.2.3. Let F_1 and F_2 be two bounded nonnegative measurable functions on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times M(\mathcal{S}) \times M(\mathcal{S})$. Assume that F_1 is $\mathcal{F}_{T+}$-measurable and that F_2 is continuous. We will show that

$$\Theta_0\big(F_1 \times F_2\big(X^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T\big)\big) = \Theta_0\big(F_1\, \Theta_{Z_T}(F_2)\big). \tag{6.27}$$

Remark that (6.27) implies that, conditionally on $Z_T = z$, the coding triple $(\widetilde{X}^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T)$ is distributed according to Θ_z and is independent of $B_T^\bullet$ (the hull $B_T^\bullet$ is $\mathcal{F}_{T+}$ measurable, since the process $t \mapsto B_t^\bullet$ is adapted to $(\mathcal{F}_{t+})_{t\geqslant 0}$ and T is a stopping time). In particular, $(\widetilde{X}^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T)$ will a.s. verify (H_2). Then the different assertions of the theorem follow from Lemma 6.2. It remains to establish (6.27). For every integer $n \geqslant 1$ we have:

$$\Theta_0\big(F_1 \times F_2\big(X^{(\frac{[nT]}{n})}, \widetilde{\mathfrak{L}}_{\frac{[nT]}{n}}, \widetilde{\mathfrak{R}}_{\frac{[nT]}{n}}\big)\big) = \sum_{k=0}^{\infty} \Theta_0\big(F_1\, \mathbb{1}_{\frac{k}{n}\leqslant T<\frac{k+1}{n}} F_2\big(X^{(\frac{k+1}{n})}, \widetilde{\mathfrak{L}}_{\frac{k+1}{n}}, \widetilde{\mathfrak{R}}_{\frac{k+1}{n}}\big)\big) \tag{6.28}$$

For every atom (ℓ, ω) of $\mathfrak{R}$ or $\mathfrak{L}$ such that $\ell > \tau_T$, an application of [2, Lemma 11] shows that $\mathrm{tr}_{\frac{[nT]}{n}}(\omega) \to \mathrm{tr}_T(\omega)$ as $n \to \infty$. Using also the fact that $r \to \tau_r$ is càdlàg, we easily obtain that $\widetilde{\mathfrak{L}}_{\frac{[nT]}{n}} \to \widetilde{\mathfrak{L}}_T$ and $\widetilde{\mathfrak{R}}_{\frac{[nT]}{n}} \to \widetilde{\mathfrak{R}}_T$ when $n \to \infty$, with respect to the topology on $M(\mathcal{S})$. Since F_2 is bounded and continuous, we can take the limit when n goes to ∞ to obtain:

$$\lim_{n\to\infty} \Theta_0\big(F_1 \times F_2\big(X^{(\frac{[nT]}{n})}, \widetilde{\mathfrak{L}}_{\frac{[nT]}{n}}, \widetilde{\mathfrak{R}}_{\frac{[nT]}{n}}\big)\big) = \Theta_0\big(F_1 \times F_2\big(X^{(T)}, \widetilde{\mathfrak{L}}_T, \widetilde{\mathfrak{R}}_T\big)\big). \tag{6.29}$$

On the other hand, for every $k \geqslant 0$, $F_1\, \mathbb{1}_{\frac{k}{n}\leqslant T<\frac{k+1}{n}}$ is $\mathcal{F}_{\frac{k+1}{n}}$-measurable and is thus equal, Θ_0-a.s. , to a measurable function of $B^\bullet_{\frac{k+1}{n}}$. Hence we can apply the spatial Markov property of Theorem

6.3 to obtain:

$$\sum_{k=0}^{\infty} \Theta_0\big(F_1 \, \mathbb{1}_{\frac{k}{n}\leqslant T<\frac{k+1}{n}} \, F_2\big(X^{(\frac{k+1}{n})}, \widetilde{\mathfrak{L}}_{\frac{k+1}{n}}, \widetilde{\mathfrak{R}}_{\frac{k+1}{n}}\big)\big) = \sum_{k=0}^{\infty} \Theta_0\Big(F_1 \, \mathbb{1}_{\frac{k}{n}\leqslant T<\frac{k+1}{n}} \Theta_{Z_{\frac{k+1}{n}}}(F_2)\Big)$$
$$= \Theta_0\big(F_1 \, \Theta_{Z_{\frac{[nT]}{n}}}(F_2)\big).$$

Since the process Z is càdlàg, we have $Z_{\frac{[nT]}{n}} \to Z_T$ as $n \to \infty$. Moreover, the fact that F_2 is bounded and continuous and the scaling property of $(\Theta_l)_{l>0}$ imply that the mapping $l \mapsto \Theta_l(F_2)$ is also bounded and continuous. It follows that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \Theta_0\big(F_1 \, \mathbb{1}_{\frac{k}{n}\leqslant T<\frac{k+1}{n}} \, F_2\big(X^{(\frac{k+1}{n})}, \widetilde{\mathfrak{L}}_{\frac{k+1}{n}}, \widetilde{\mathfrak{R}}_{\frac{k+1}{n}}\big)\big) = \lim_{n\to\infty} \Theta_0\big(F_1 \, \Theta_{Z_{\frac{[nT]}{n}}}(F_2)\big)$$
$$= \Theta_0\big(F_1 \, \Theta_{Z_T}(F_2)\big). \tag{6.30}$$

The identity (6.27) then follows by passing to the limit $n \to \infty$ in (6.28), using (6.29) and (6.30). □

Let us state some direct consequences of Theorem 6.4 . For every $z > 0$, set

$$T_z := \inf\{r \geqslant 0 : \; Z_r \geqslant z\} \tag{6.31}$$

which is a stopping time of the filtration $(\mathcal{F}_{r+})_{r\geqslant 0}$. Note that $0 < T_z < \infty$, Θ_0-a.s. Moreover since Z does not have positive jumps we have $Z_{T_z} = z$. Applying Theorem 6.4 with $T = T_z$, we obtain the following result.

Corollary 6.1. *Let $z > 0$. Under Θ_0, $(\check{B}^{\bullet}_{T_z}, \rho_{T_z}, \check{\Delta}^{T_z}, |\cdot|_{\check{\Delta}^{T_z}})$ is an infinite volume Brownian disk with perimeter z and is independent of $(B^{\bullet}_{T_z}, 0, \Delta^{T_z}, |\cdot|_{\Delta^{T_z}})$.*

The next goal is to extend the definition of process Z under Θ_z. It will be useful to consider the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions from $\mathbb{R}_+$ into $\mathbb{R}$. We write $(\xi_t)_{t\geqslant 0}$ for the canonical process on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and $(\mathcal{D}_t)_{t\geqslant 0}$ for the canonical filtration. We introduce a probability measure $\mathbb{P}$ on $\big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}), \mathcal{D}\big)$ such that under $\mathbb{P}$, the process ξ is distributed as a Lévy process without positive jumps with Laplace exponent $\psi(\lambda) := \sqrt{\frac{8}{3}} \frac{\Gamma(\lambda+\frac{3}{2})}{\Gamma(\lambda)}$, i.e.:

$$\mathbb{E}[\exp(\lambda \xi_1)] = \exp(\psi(\lambda)), \quad \forall \lambda > -\frac{3}{2},$$

where $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$. We refer to [18, Lemma 2.1] for the existence of this Lévy process. Since $\psi'(0+) > 0$, standard properties of Levy processes imply that ξ drifts to ∞ (see for example [15, Chapter VII]). We also introduce the time change:

$$\kappa(r) := \inf\Big\{s \geqslant 0 : \int_0^s \exp(\frac{1}{2}\xi_t)\, dt \geqslant r\Big\}.$$

Theorem 24 in [77] states that the process Z under Θ_0 is a self-similar Markov process started at 0 with index $\frac{1}{2}$ and Laplace exponent ψ. In particular, the process $(Z_{T_z+t})_{t\geqslant 0}$ is distributed under Θ_0 as:

$$\Big(z \exp\big(\xi_{\kappa(z^{-\frac{1}{2}} r)}\big)\Big)_{r\geqslant 0}$$

under $\mathbb{P}$. As a consequence of (6.15) and Corollary 6.1 we obtain:

Corollary 6.2. *Fix $z > 0$. Then, for every $r \geqslant 0$,*

$$Z_r := \lim_{\varepsilon \to 0} \varepsilon^{-2} |\check{B}^\bullet_r \cap B_{r+\varepsilon}|$$

exists Θ_z-a.s. Moreover the process $(Z_r)_{r\geqslant 0}$ has a càdlàg modification under Θ_z, which is distributed as a $(\frac{1}{2}, \psi)$-self-similar Markov process started at z, i.e.

$$(Z_r)_{r\geqslant 0} \overset{(d)}{=} \Big(z \exp\big(\xi_{\kappa(z^{-\frac{1}{2}} r)}\big)\Big)_{r\geqslant 0} \tag{6.32}$$

where ξ is distributed according to $\mathbb{P}$.

Let $\mathcal{M}^{(z)}_\infty$ be the infinite volume Brownian disk with perimeter z defined under the probability measure Θ_z as explained in Section 6.2.4. We say that $\gamma : [s,t] \to \mathcal{M}^{(z)}_\infty$ is a separating cycle if it is an injective continuous cycle that does not hit the boundary of $\mathcal{M}^{(z)}_\infty$ and if any path connecting this boundary to ∞ has to cross the range of γ. We are going to use the "strong" spatial Markov property (Theorem 6.4) to study the separating cycles of the infinite volume Brownian disk. As in the previous sections, we consider $B^\circ_{r,s} = \mathrm{Int}(B^\bullet_s \backslash B^\bullet_r)$ and

$$L_{r,s} = \inf\{\Delta(g) : \ g : [0,1] \to B^\circ_{r,s} \text{ separating cycle}\},$$

for every $0 \leqslant r < s$, and we will now study $L_{r,s}$ under Θ_z. To simplify notation we write $L_r := L_{r,\infty}$ for every $r \geqslant 0$. Note that L_0 is the infimum of lengths of paths separating the boundary of $\mathcal{M}^{(z)}_\infty$ from infinity. We have the following analog of Theorem 6.1 for the infinite volume Brownian disk:

Proposition 6.6. *Fix z a positive real number.*

(i) *We have*

$$\limsup_{u\to\infty} \frac{\log\big(\Theta_z(L_0 > u)\big)}{u} \leqslant -\sup_{s>1} \frac{1}{2(s-1)} \log(\frac{s^2}{2s-1}).$$

Consequently, $\Theta_z(L_0 > u)$ decreases at least exponentially fast when u goes to ∞.

(ii) *There exist two constants $0 < \tilde{c}_1 \leqslant \tilde{c}_2$ such that:*

$$\forall \varepsilon \geqslant 0, \ \tilde{c}_1 (1 \wedge \varepsilon^2) \leqslant \Theta_z(L_0 < \varepsilon) \leqslant \tilde{c}_2 \varepsilon^2.$$

Proof. By scaling, it is enough to consider $z = 1$. The spatial Markov property (Theorem 6.3) and again a scaling argument show that the distribution of $Z_1^{-\frac{1}{2}} L_1$ under Θ_0 coincides with the distribution of L_0 under Θ_1 and moreover $Z_1^{-\frac{1}{2}} L_1$ is independent of Z_1 under Θ_0. We obtain:

$$\Theta_1(L_0 > u)\Theta_0(Z_1 > 1) = \Theta_0(L_1 > Z_1^{\frac{1}{2}} u,\, Z_1 > 1) \leqslant \Theta_0(L_1 > u).$$

Part (i) of the proposition then follows from Theorem 6.1 .

Let us prove (ii). The upper bound is a direct consequence of the beginning of the proof, noting that for every $\varepsilon \geqslant 0$:

$$\Theta_1(L_0 < \varepsilon)\Theta_0(Z_1 < 1) = \Theta_0(Z_1^{-\frac{1}{2}} L_1 < \varepsilon,\, Z_1 < 1) \leqslant \Theta_0(L_1 < \varepsilon)$$

so that, by Theorem 6.1 , if $\tilde{c}_2 := c_2/\Theta_0(Z_1 < 1) \in (0, \infty)$ we have:

$$\Theta_1(L_0 < \varepsilon) \leqslant \tilde{c}_2 \varepsilon^2 \tag{6.33}$$

for every $\varepsilon > 0$. We argue in a similar way to obtain the lower bound. Let $\varepsilon \leqslant 1$ and let $m_0 \geqslant 1$ be an integer. We have similarly:

$$\begin{aligned}\Theta_0(L_1 \leqslant \varepsilon) &= \Theta_0(Z_1^{\frac{1}{2}} Z_1^{-\frac{1}{2}} L_1 \leqslant \varepsilon)\\ &\leqslant \Theta_0(Z_1^{-\frac{1}{2}} L_1 \leqslant m_0\varepsilon,\, Z_1^{\frac{1}{2}} > \frac{1}{m_0}) + \sum_{m=m_0}^{\infty} \Theta_0\big(Z_1^{\frac{1}{2}} \in [\frac{1}{m+1}, \frac{1}{m}]\,,\, Z_1^{-\frac{1}{2}} L_1 \leqslant (m+1)\varepsilon\big).\end{aligned}$$

Now we can use the first observations of the proof and (6.33) to get:

$$\begin{aligned}\Theta_0(L_1 \leqslant \varepsilon) &\leqslant \Theta_0(Z_1^{-\frac{1}{2}} L_1 \leqslant m_0\varepsilon) + \tilde{c}_2\varepsilon^2 \sum_{m=m_0}^{\infty} (m+1)^2\, \Theta_0(Z_1 \in [(m+1)^{-2}, m^{-2}])\\ &= \Theta_1(L_0 \leqslant m_0\varepsilon) + \tilde{c}_2\varepsilon^2 \sum_{m=m_0}^{\infty} (m+1)^2\, \Theta_0(Z_1 \in [(m+1)^{-2}, m^{-2}])\end{aligned}$$

Under Θ_0, the density of Z_1 is $\frac{3^{\frac{3}{2}}}{\sqrt{2\pi}} \sqrt{x} \exp(-\frac{3}{2}x)\, dx$, so for every $0 < a < b$:

$$\Theta_0(Z_1 \in [a, b]) \leqslant \sqrt{\frac{6}{\pi}} (b^{\frac{3}{2}} - a^{\frac{3}{2}}).$$

Hence we can find a constant $c_3 > 0$, which does not depend on the choice of m_0, such that

$$\Theta_0(L_1 \leqslant \varepsilon) \leqslant \Theta_1(L_0 \leqslant m_0\varepsilon) + c_3\varepsilon^2 \sum_{m=m_0}^{\infty} \frac{1}{m^2}.$$

Then using Theorem 6.1 , we get for every $\varepsilon \in [0, 1]$:

$$(c_1 - c_3 \sum_{m=m_0}^{\infty} \frac{1}{m^2})\, \varepsilon^2 \leqslant \Theta_1(L_0 \leqslant m_0\varepsilon).$$

We obtain the lower bound in (ii) by choosing m_0 such that $\sum_{m=m_0}^{\infty} m^{-2} < \frac{c_1}{c_3}$. □

Recall that $0 < \tilde{c}_1 < \tilde{c}_2$ are the constants appearing in Proposition 6.6. The end of this section is devoted to the proof of the following result which will be crucial for the proof of Theorem 6.2 (i). Before stating the result, observe that the definition (6.31) of T_r for $r > 0$ also makes sense under Θ_z by Corollary 6.2.

Proposition 6.7. *There exists $\tilde{c}_3 > 0$ such that, for every $r > 2\tilde{c}_2/\tilde{c}_1$ and $\varepsilon > 0$,*

$$\Theta_1\big(L_{0,T_{2r}} \leqslant \varepsilon\big) \geqslant \tilde{c}_3(1 \wedge \varepsilon^2).$$

The proof of Proposition 6.7 is based on the next lemma:

Lemma 6.8. *Let $z > 0$. Then, for every $A > z$ and $\beta < 3$ we have:*

$$\Theta_z\big(\sup_{[0,\varepsilon]} Z \geqslant A\big) = o(\varepsilon^\beta)$$

as $\varepsilon \to 0$.

Proof. By a scaling argument, it is enough to prove the lemma with $z = 1$. Fix $A > 1$. Introduce the stopping time $T := \inf\{t \geqslant 0 : \ \xi_t \geqslant \log(A)\}$ which is finite $\mathbb{P}$-a.s. By Corollary 6.2 for every $\varepsilon > 0$:

$$\Theta_1\big(\sup_{t\in[0,\varepsilon]} Z_t \geqslant A\big) = \mathbb{P}\big(T \leqslant \kappa(\varepsilon)\big) = \mathbb{P}\Big(\int_0^T \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\Big).$$

Let $\alpha \in (0,1)$, we split $\mathbb{P}(\int_0^T \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon)$ as follows:

$$\mathbb{P}\Big(\int_0^T \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\Big) \leqslant \mathbb{P}(T \leqslant \varepsilon^\alpha) + \mathbb{P}\Big(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\Big) \tag{6.34}$$

and we study each term separately. We need to estimate $\mathbb{P}(T < \delta)$ for $\delta > 0$. As ξ is a Lévy process without positive jumps which drifts to ∞, we have by standard properties of Lévy processes

$$\mathbb{E}[\exp(-\psi(\lambda)T)] = \exp(-\lambda\log(A))$$

for every $\lambda > 0$. See for example [15, Chapter VII] for a proof. Remark that there exists $c > 0$ such that, for every $\lambda > 1$, we have $\psi(\lambda) < c\lambda^{\frac{3}{2}}$ and that an application of Markov's inequality gives:

$$\mathbb{P}(T < \delta) = \mathbb{P}(-\psi(\lambda)T > -\psi(\lambda)\delta) \leqslant \exp(\psi(\lambda)\delta - \lambda\log(A)).$$

So taking $\lambda = \delta^{-\frac{2}{3}}$ in the previous bound we obtain:

$$\mathbb{P}(T < \delta) = \underset{\delta\downarrow 0}{O}\big(\exp(-\delta^{-\frac{2}{3}}\log(A))\big)\,.$$

Consequently, for every $q > 0$, $\mathbb{P}(T < \delta) = o(\delta^q)$ as $\delta \downarrow 0$. Let us study the other term appearing in (6.34). Fix $\beta \in (0,3)$. Again by using Markov's inequality we have:

$$\mathbb{P}\Big(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\Big) \leqslant \varepsilon^\beta \mathbb{E}\Big[\Big(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r)\, dr\Big)^{-\beta}\Big].$$

But then an application of Jensen inequality gives

$$\begin{aligned}\mathbb{P}\Big(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\Big) &\leqslant \varepsilon^{(1-\alpha)\beta-\alpha}\mathbb{E}\Big[\int_0^{\varepsilon^\alpha} \exp(-\frac{\beta}{2}\xi_r)\, dr\Big] \\ &= \varepsilon^{(1-\alpha)\beta-\alpha}\frac{\exp(\psi(-\frac{\beta}{2})\varepsilon^\alpha) - 1}{\psi(-\frac{\beta}{2})}.\end{aligned}$$

We obtain that $\mathbb{P}\big(\int_0^{\varepsilon^\alpha} \exp(\frac{1}{2}\xi_r)\, dr \leqslant \varepsilon\big) = O(\varepsilon^{(1-\alpha)\beta})$ as $\varepsilon \downarrow 0$. Since this is true for every $\beta \in (0,3)$ and $\alpha \in (0,1)$, the lemma follows. □

Let us deduce Proposition 6.7 from Lemma 6.8.

Proof of Proposition 6.7. Fix $r > 2\tilde{c}_2/\tilde{c}_1 \geqslant 2$. Let γ be a path separating the boundary of $\mathcal{M}_\infty^{(1)}$ from infinity. If γ does not stay inside $B^\bullet_{T_{2r}}$ then it has to stay outside $B^\bullet_{T_r}$ or to connect $B^\bullet_{T_r}$ and $\check{B}^\bullet_{T_{2r}}$. Since the distance between $B^\bullet_{T_r}$ and $\check{B}^\bullet_{T_{2r}}$ is $T_{2r} - T_r$ we have:

$$L_0 \geqslant L_{0,T_{2r}} \wedge L_{T_r} \wedge (T_{2r} - T_r) \ \Theta_1\text{-a.s.}$$

Consequently, for every $\varepsilon > 0$:

$$\Theta_1(L_0 \leqslant \varepsilon) \leqslant \Theta_1\big(L_{0,T_{2r}} \leqslant \varepsilon\big) + \Theta_1\big(L_{T_r} \leqslant \varepsilon\big) + \Theta_1\big(T_{2r} - T_r \leqslant \varepsilon\big).$$

By Theorem 6.4 and Corollary 6.1 , the distribution of L_{T_r} under Θ_1 is the distribution of L_0 under Θ_r. Using a scaling argument, we obtain:

$$\Theta_1\big(L_{T_r} \leqslant \varepsilon\big) = \Theta_r\big(L_0 \leqslant \varepsilon\big) = \Theta_1\big(L_0 \leqslant \frac{\varepsilon}{\sqrt{r}}\big) \leqslant \tilde{c}_2 \frac{\varepsilon^2}{r}$$

and

$$\Theta_1\big(T_{2r} - T_r \leqslant \varepsilon\big) = \Theta_r(T_{2r} \leqslant \varepsilon) = \Theta_1\big(T_2 \leqslant \frac{\varepsilon}{\sqrt{r}}\big) = \Theta_1\big(\sup_{t \in [0, \frac{\varepsilon}{\sqrt{r}}]} Z_t \geqslant 2\big) = \underset{\varepsilon \downarrow 0}{o}(\varepsilon^2)$$

where the last equality comes from Lemma 6.8 (taking $A = 2$). We finally derive that:

$$\Theta_1\big(L_{0,T_{2r}} \leqslant \varepsilon\big) \geqslant \Theta_1\big(L_0 \leqslant \varepsilon\big) - \tilde{c}_2 \frac{\varepsilon^2}{r} + o(\varepsilon^2) \geqslant \tilde{c}_1 (1 \wedge \varepsilon^2) - \tilde{c}_2 \frac{\varepsilon^2}{r} + o(\varepsilon^2)$$

where in the second line we use Proposition 6.6 . Since $r > 2\tilde{c}_2/\tilde{c}_1$ we obtain the desired result. □

6.4 Isoperimetric inequalities

6.4.1 Preliminary results on the volume of the hulls

This section is devoted to preliminary results about the volume of hulls. This will simplify some arguments in the derivation of Theorem 6.2. We are going to use the following result [40, Theorem 1.4]

$$\begin{aligned} \Theta_0\big(\exp(-\lambda|B^\bullet_r|) \mid Z_r = l\big) = & r^3 (2\lambda)^{\frac{3}{4}} \frac{\cosh((2\lambda)^{\frac{1}{4}} r)}{\sinh^3((2\lambda)^{\frac{1}{4}} r)} \\ & \cdot \exp\Big(-l\big(\sqrt{\frac{\lambda}{2}}(3\coth^2((2\lambda)^{\frac{1}{4}} r) - 2) - \frac{3}{2r^2}\big)\Big) \end{aligned} \tag{6.35}$$

for every $\lambda > 0$. In particular, using (6.13), we obtain that for every $\lambda \geqslant 0$:

$$\Theta_0\big(\exp(-\lambda|B^\bullet_r|)\big) = 3^{\frac{3}{2}} \cosh((2\lambda)^{\frac{1}{4}} r) \, (\cosh^2((2\lambda)^{\frac{1}{4}} r) + 2)^{-\frac{3}{2}} \tag{6.36}$$

(this formula also appears in [40]).

Corollary 6.3. *There exists a constant $C > 0$ such that for every $z > 0$ and $r > 0$:*

$$\Theta_z\big(|B^\bullet_r|\big) \leqslant C(r+\sqrt{z})^4.$$

Proof. Fix $z > 0$ and $r > 0$. First remark that, under $\Theta_0(\cdot \,|\, T_z \leqslant \sqrt{z})$, the hull $B^\bullet_{T_z+r}$ is contained in $B^\bullet_{\sqrt{z}+r}$. So an application of Corollary 6.1 gives:

$$\begin{aligned}\Theta_0\big(|B^\bullet_{r+\sqrt{z}}|\big) &\geqslant \Theta_0\big(|B^\bullet_{r+\sqrt{z}}|\mathbb{1}_{T_z<\sqrt{z}}\big)\\ &\geqslant \Theta_0\big((|B^\bullet_{T_z+r}| - |B^\bullet_{T_z}|)\mathbb{1}_{T_z<\sqrt{z}}\big)\\ &= \Theta_z\big(|B^\bullet_r|\big)\,\Theta_0(T_z<\sqrt{z}).\end{aligned}$$

On the other hand, we have $\{Z_{\sqrt{z}} > z\} \subset \{T_z \leqslant \sqrt{z}\}$. By scaling we also have $\Theta_0(Z_{\sqrt{z}} > z) = \Theta_0(Z_1 > 1) > 0$ and $\Theta_0\big(|B^\bullet_{r+\sqrt{z}}|\big) = (r+\sqrt{z})^4\Theta_0\big(|B^\bullet_1|\big)$. Finally, it is easy to deduce from (6.36) that $\Theta_0\big(|B^\bullet_1|\big)$ is finite. This gives the statement of the corollary with $C = \Theta_0(|B^\bullet_1|)/\Theta_0(Z_1 > 1)$. ☐

We now give two lemmas that will be useful to control the fluctuations of the volume of hulls in the Brownian plane.

Lemma 6.9. *For every $\beta \in \mathbb{R}$, we have $\Theta_0(|B^\bullet_1|^\beta) < \infty$ if and only if $\beta < \frac{3}{2}$.*

Proof. To simplify notation write $F(\lambda) := \Theta_0(\exp(-\lambda|B^\bullet_1|))$ for every $\lambda \geqslant 0$. Remark that for every $\lambda > 0$, we have $F''(\lambda) = \Theta_0(|B^\bullet_1|^2\exp(-\lambda|B^\bullet_1|))$. If $\alpha > 0$, we have:

$$\int_{\mathbb{R}_+} \lambda^{\alpha-1}F''(\lambda)\,d\lambda = \Theta_0\Big(|B^\bullet_1|^2\int_{\mathbb{R}_+}\lambda^{\alpha-1}\exp(-\lambda|B^\bullet_1|)\,d\lambda\Big) = \Gamma(\alpha)\,\Theta_0\big(|B^\bullet_1|^{2-\alpha}\big).$$

From the explicit expression of F given in (6.36) we get:

$$F''(\lambda) = \frac{1}{9\sqrt{2}}\lambda^{-\frac{1}{2}} + O(1)$$

as $\lambda \downarrow 0$, and

$$F''(\lambda) = O\big(\exp(-2(2\lambda)^{\frac{1}{4}})\big)$$

as $\lambda \uparrow \infty$. So for $\alpha = \frac{1}{2}$, $\Theta_0(|B^\bullet_1|^{\frac{3}{2}}) = \int_{\mathbb{R}_+}\lambda^{-\frac{1}{2}}F''(\lambda)\,d\lambda = \infty$ and for every $\alpha > \frac{1}{2}$, $\Theta_0(|B^\bullet_1|^{2-\alpha}) < \infty$. ☐

We conclude this section with the following consequence of Lemma 6.9:

Lemma 6.10. *For every $\beta_1 > 0$ and $\beta_2 > 2/3$ we have Θ_0-a.s.*

$$\inf_{r>0}\frac{|B^\bullet_r|}{r^4(1+|\log(r)|)^{-\beta_1}} > 0$$

and

$$\sup_{r>0}\frac{|B^\bullet_r|}{r^4(1+|\log(r)|)^{\beta_2}} < \infty.$$

Proof. Fix β_1 and β_2 as in the statement. By Lemma 6.9, the quantities $\Theta_0(|B_1^\bullet|^{-1/\beta_1})$ and $\Theta_0(|B_1^\bullet|^{1/\beta_2})$ are finite. This implies by the scaling invariance of $\mathcal{M}_\infty$:

$$\sum_{m\in\mathbb{Z}} \Theta_0(|B_{2^m}^\bullet|^{-1} > |m|^{\beta_1}2^{-4m}) \leqslant 2\sum_{m=0}^{\infty}\Theta_0(|B_1^\bullet|^{-\frac{1}{\beta_1}} > m) < \infty$$

and

$$\sum_{m\in\mathbb{Z}} \Theta_0(|B_{2^m}^\bullet| > |m|^{\beta_2}2^{4m}) \leqslant 2\sum_{m=0}^{\infty}\Theta_0(|B_1^\bullet|^{\frac{1}{\beta_2}} > m) < \infty\,.$$

The result then follows by the Borel-Cantelli lemma. □

6.4.2 Proof of Part (i) of Theorem 6.2

The goal of this section is to prove the following slightly more precise form of Theorem 6.2 (ii).

Proposition 6.8. *Let $f:\mathbb{R}_+\to\mathbb{R}_+^*$ be a positive nondecreasing function such that $\sum_{m\in\mathbb{N}} f(m)^{-2} = \infty$. Then,*

$$\inf_{\substack{A\in\mathcal{K}\\ A\subset B_1^\bullet}} \frac{\Delta(\partial A)}{|A|^{\frac14}} f(|\log(|A|)|) = 0,\ \ \Theta_0\text{-}a.s. \tag{6.37}$$

and

$$\inf_{\substack{A\in\mathcal{K}\\ B_1^\bullet\subset A}} \frac{\Delta(\partial A)}{|A|^{\frac14}} f(|\log(|A|)|) = 0,\ \ \Theta_0\text{-}a.s. \tag{6.38}$$

Proof. Fix $r > 2\tilde{c}_2/\tilde{c}_1 \geqslant 2$, where $\tilde{c}_1$ and $\tilde{c}_2$ are as in Proposition 6.6, and let $f:\mathbb{R}_+\to\mathbb{R}_+^*$ be a positive nondecreasing function such that $\sum_{m\in\mathbb{N}} f(m)^{-2} = \infty$. We give a detailed proof of the (6.38) since (6.37) can be obtained, *mutatis mutandis*, by the same method. Recall the notation $T_r := \inf\{t\geqslant 0:\ Z_t\geqslant r\}$. Since Z does not have positive jumps, we have $Z_{T_r} = r$. For every $n\geqslant 1$, a separating cycle taking values in $B^\bullet_{T_{r^n},T_{r^{n+1}}}$ bounds a Jordan domain $A\in\mathcal{K}$ such that $B^\bullet_{T_{r^n}}\subset A$ and for n large enough we also have $B_1^\bullet\subset A$. Hence,

$$\inf_{\substack{A\in\mathcal{K}\\ B_1^\bullet\subset A}} \frac{\Delta(\partial A)}{|A|^{\frac14}} f(|\log(|A|)|) \leqslant \liminf_{n\to\infty} \frac{L_{T_{r^{2n+1}},T_{r^{2n+2}}}}{|B^\bullet_{T_{r^{2n+1}}}|^{\frac14}} f(\log(|B^\bullet_{T_{r^{2n+2}}}|)),\ \ \Theta_0\text{-a.s.}$$

So to obtain (6.38) it is enough to show that:

$$\liminf_{n\to\infty} \frac{L_{T_{r^{2n+1}},T_{r^{2n+2}}}}{|B^\bullet_{T_{r^{2n+1}}}|^{\frac14}} f(\log(|B^\bullet_{T_{r^{2n+2}}}|)) = 0,\ \ \Theta_0\text{-a.s.} \tag{6.39}$$

Let us study the growth of the sequence $(T_{r^n})_{n\in\mathbb{N}}$. First note that:

$$\Theta_0(T_1\geqslant u) \leqslant \Theta_0(Z_u\leqslant 1) = \Theta_0(Z_1\leqslant u^{-2}) = \frac{3^{\frac32}}{\sqrt{2\pi}}\int_0^{u^{-2}}\sqrt{x}\exp(-\frac32 x)\,dx \leqslant \sqrt{\frac{6}{\pi}}u^{-3}$$

where in the first equality we apply the scaling invariance of the Brownian plane and in the second one we use the density of Z_1 (see (6.13)). In particular, we have $\Theta_0(T_1) < \infty$. So by scaling invariance we obtain:

$$\sum_{n\in\mathbb{N}^*} \Theta_0(T_{r^n} > nr^{\frac{n}{2}}) = \sum_{n\in\mathbb{N}^*} \Theta_0(T_1 > n) < \infty.$$

The Borel-Cantelli lemma then implies that $\limsup_{n\to\infty}(nr^{n/2})^{-1}T_{r^n} \leqslant 1$, Θ_0-a.s. Since $\lim_{s\to\infty} T_s = \infty$, Θ_0-a.s. , Lemma 6.10 gives:

$$\limsup_{n\to\infty} \frac{\log(|B^\bullet_{T_{r^{2n+2}}}|)}{4\log(T_{r^{2n+2}})} \leqslant 1, \ \Theta_0\text{-a.s.}$$

and then we deduce:

$$\limsup_{n\to\infty} \frac{\log(|B^\bullet_{T_{r^{2n+2}}}|)}{2n\log(r)} \leqslant 1, \ \Theta_0\text{-a.s.}$$

Fix $h > 2\log(r)$. As f is nondecreasing we have:

$$\liminf_{n\to\infty} \frac{L_{T_{r^{2n+1}},T_{r^{2n+2}}}}{|B^\bullet_{T_{r^{2n+1}}}|^{\frac{1}{4}}} f(\log(|B^\bullet_{T_{r^{2n+2}}}|)) \leqslant \liminf_{n\to\infty} \frac{L_{T_{r^{2n+1}},T_{r^{2n+2}}}}{|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}|^{\frac{1}{4}}} f(hn). \tag{6.40}$$

We will use the Borel-Cantelli lemma to conclude. By Theorem 6.4, under Θ_0 :

$$\Big(\frac{1}{r^{4n}}|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}|, \frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}},T_{r^{2n+2}}}\Big)_{n\in\mathbb{N}}.$$

is an i.i.d. sequence of random variables and for every $n \geqslant 0$ the variables:

$$\frac{1}{r^{4n}}|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}|, \text{ and } \frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}},T_{r^{2n+2}}}$$

are independent. Moreover $\frac{1}{r^{4n}}|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}|$ (resp. $\frac{1}{r^{n+\frac{1}{2}}} L_{T_{r^{2n+1}},T_{r^{2n+2}}}$) is distributed under Θ_0 as $B^\bullet_{T_r}$ (resp. L_{0,T_r}) under Θ_1. Fix $\delta > 0$, such that $\Theta_1(B^\bullet_{T_r} > \delta) > 0$ and let $\varepsilon > 0$. By the previous remark we have:

$$\sum_{n=0}^{\infty} \Theta_0\Big(|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}| > \delta r^{4n}, L_{T_{r^{2n+1}},T_{r^{2n+2}}} < \frac{\varepsilon}{f(hn)} r^{n+\frac{1}{2}}\Big)$$
$$= \sum_{n=0}^{\infty} \Theta_1(|B^\bullet_{T_r}| > \delta)\, \Theta_1\big(L_{0,T_r} < \frac{\varepsilon}{f(hn)}\big).$$

By Proposition 6.7 the right-hand side of the last display is greater than

$$\tilde{c}_3\, \Theta_1(|B^\bullet_{T_r}| > \delta) \sum_{n=0}^{\infty} \left(\frac{\varepsilon^2}{f(hn)^2} \wedge 1\right)$$

which is infinite since $\sum_{m\in\mathbb{N}} f(m)^{-2} = \infty$. The Borel-Cantelli lemma then implies that:

$$\liminf_{n\to\infty} \frac{L_{T_{r^{2n+1}},T_{r^{2n+2}}}}{|B^\bullet_{T_{r^{2n}},T_{r^{2n+1}}}|^{\frac{1}{4}}} f(hn) \leqslant \varepsilon\delta^{-\frac{1}{4}}\sqrt{r}, \ \Theta_0\text{-a.s.}$$

This holds for every $\varepsilon > 0$, which together with (6.40) gives (6.39). □

6.4.3 Proof of Part (ii) of Theorem 6.2

We need to show that for any positive nondecreasing function f, the condition $\sum_{m\in\mathbb{N}} f(m)^{-2} < \infty$ implies $\inf_{A\in\mathcal{K}} \frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0$, Θ_0-a.s.

We begin with a technical lemma.

Lemma 6.11. *Let $\beta \in [0,1)$. There exists a constant $C_\beta > 0$, which only depends on β, such that for every $r > 0$ and $\varepsilon > 0$:*

$$\Theta_0\big(\mathbb{1}_{L_1<\varepsilon}|B^\bullet_r|^\beta\big) \leqslant C_\beta\, r^{4\beta}\varepsilon^2.$$

The reason for taking $\beta < 1$ is just technical and one can extend the result to $\beta < \frac{3}{2}$ but the proof will be more tedious. At an intuitive level, Lemma 6.11 states that if we know that there exists a small cycle separating the hull of radius 1 then the expected volume of the hull of radius r stays at most of order r^4 (with a uniform control).

Proof. Fix $\beta \in (0,1)$ and let $r > 0$ and $\varepsilon > 0$.

To simplify notation, set $m := \frac{1}{\varepsilon}$, $q := \frac{1}{\beta}$ and $p := \frac{q}{q-1} = \frac{1}{1-\beta}$. By the scaling property of $\mathcal{M}_\infty$:

$$\Theta_0\big(\mathbb{1}_{L_1<\varepsilon}|B^\bullet_r|^\beta\big) = \frac{1}{m^{4\beta}}\Theta_0\big(\mathbb{1}_{L_m<1}|B^\bullet_{mr}|^\beta\big).$$

If $L_m < 1$, there is a separating cycle of length smaller than 1 that is contained in $\check{B}^\circ_m$, and necessarily this separating cycle is contained in $B^\bullet_{m+k,m+k+2}$ for some integer $k \geqslant 0$. Hence,

$$\Theta_0\big(\mathbb{1}_{L_m<1}|B^\bullet_{mr}|^\beta\big) \leqslant \sum_{k=0}^{\infty} \Theta_0\big(\mathbb{1}_{L_{m+k,m+k+2}<1}|B^\bullet_{mr}|^\beta\big).$$

Applying the conditional version of the Hölder inequality with respect to Z_{m+k+4} we obtain:

$$\Theta_0\big(\mathbb{1}_{L_{m+k,m+k+2}<1}|B^\bullet_{mr}|^\beta\big) \leqslant \Theta_0\Big(\Theta_0\big(L_{m+k,m+k+2} < 1 \,\big|\, Z_{m+k+4}\big)^{\frac{1}{p}}\, \Theta_0\big(|B^\bullet_{mr}| \,\big|\, Z_{m+k+4}\big)^{\frac{1}{q}} \Big).$$

By Proposition 6.5, there exists $\alpha_1 > 0$ such that

$$\Theta_0\big(L_{m+k,m+k+2} < 1 \,\big|\, Z_{m+k+4}\big) \leqslant e \cdot \exp(-\alpha_1 Z_{m+k+4})$$

for every $k \geqslant 0$ and thus we get :

$$\Theta_0\big(\mathbb{1}_{L_m<1}|B^\bullet_{mr}|^\beta\big) \leqslant e \sum_{k=0}^{\infty} \Theta_0\Big(\exp(-\alpha Z_{m+k+4})\, \Theta_0\big(|B^\bullet_{mr}| \,\big|\, Z_{m+k+4}\big)^{\frac{1}{q}} \Big) \tag{6.41}$$

where $\alpha := \alpha_1/p$. Then again by the Hölder inequality:

$$\begin{aligned}\Theta_0\Big(\exp(-\alpha Z_{m+k+4})\ \Theta_0\big(|B^\bullet_{mr}| \,\big|\, Z_{m+k+4}\big)^{\frac{1}{q}} \Big) \\ &= \Theta_0\Big(\exp(-\frac{\alpha}{p}Z_{m+k+4})\exp(-\frac{\alpha}{q}Z_{m+k+4})\Theta_0\big(|B^\bullet_{mr}| \,\big|\, Z_{m+k+4}\big)^{\frac{1}{q}} \Big) \\ &\leqslant \Theta_0\big(\exp(-\alpha Z_{m+k+4})\big)^{\frac{1}{p}}\, \Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{mr}|\big)^{\frac{1}{q}}. \end{aligned} \tag{6.42}$$

By (6.13), Z_{m+k+4} follows the Gamma distribution with parameter $\frac{3}{2}$ and mean $(m+k+4)^2$. So we can find $d_1(\alpha) > 0$ independent of m and k such that:

$$\Theta_0(\exp(-\alpha Z_{m+k+4})) \leqslant \frac{d_1(\alpha)}{(m+k+4)^3}\,, \quad \Theta_0(Z_{m+k+4}\exp(-\alpha Z_{m+k+4})) \leqslant \frac{d_1(\alpha)}{(m+k+4)^3}. \quad (6.43)$$

Moreover, noting that $|\partial B^\bullet_{m+k+4}| = 0$, Θ_0-a.s. , we obtain:

$$\begin{aligned}\Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{mr}|\big) &= \Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{(m+k+4)\,\wedge\, mr}|\big)\\ &\quad + \Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{m+k+4,mr}|\big)\end{aligned}$$

where by convention $|B^\bullet_{s_1,s_2}| = 0$ if $s_2 \leqslant s_1$.

Let $0 < s_1 \leqslant s_2$. We observe that $|B^\bullet_{s_1}|$ is independent of Z_{s_2} conditionally on Z_{s_1}. This follows from the special Markov property and the spine independence property, using the fact that $|B^\bullet_{s_1}|$ is determined by the excursions of ω below s_1 for all atoms of $\mathfrak{L}$ and $\mathfrak{R}$ such that $t \geqslant \tau_{s_1}$, and by the atoms (t,ω) such that $t \leqslant \tau_{s_1}$. Thanks to this conditional independence property, we have

$$\Theta_0\Big(|B^\bullet_{s_1}| \,\Big|\, Z_{s_2}\Big) = \Theta_0\Big(\Theta_0\big(|B^\bullet_{s_1}| \,\big|\, Z_{s_1}\big) \,\Big|\, Z_{s_2}\Big)\,. \quad (6.44)$$

By differentiating the right-hand side of (6.35) at $\lambda = 0$ we get

$$\Theta_0\left(|B^\bullet_{s_1}| \,\middle|\, Z_{s_1}\right) = \frac{2}{15}s_1^4 + \frac{1}{5}s_1^2 Z_{s_1}$$

similarly we have from (6.14):

$$\Theta_0(Z_{s_1} \,|\, Z_{s_2}) = \frac{s_1^3}{s_2^3}Z_{s_2} + \frac{s_2 - s_1}{s_2}s_1^2.$$

So by (6.44), for every $0 < s_1 \leqslant s_2$

$$\Theta_0\Big(|B^\bullet_{s_1}| \,\Big|\, Z_{s_2}\Big) = \frac{1}{3}s_1^4 - \frac{s_1^5}{5s_2} + \frac{s_1^5}{5s_2^3}Z_{s_2}.$$

Taking $s_1 = (m+k+4) \wedge (mr)$ and $s_2 = m+k+4$, we deduce from the last two formulas and (6.43) that there exists $d_2(\alpha) > 0$ independent of m and k such that:

$$\Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{(m+k+4)\,\wedge\, mr}|\big) \leqslant \frac{d_2(\alpha)}{(m+k+4)^3}r^4 m^4.$$

Suppose that $m+k+4 < mr$. Then by the spatial Markov property and Corollary 6.3,

$$\begin{aligned}\Theta_0(\exp(-\alpha Z_{m+k+4})|B^\bullet_{m+k+4,mr}|) &= \Theta_0\Big(\exp(-\alpha Z_{m+k+4})\,\Theta_{Z_{m+k+4}}\big(|B^\bullet_{mr-m-k-4}|\big)\Big)\\ &\leqslant C'\,\Theta_0(\exp(-\alpha Z_{m+k+4})(m^4 r^4 + Z^2_{m+k+4})).\end{aligned}$$

where $C' = 16C$, if C is the constant appearing in Corollary 6.3. Using again the distribution of Z_{m+k+4} , we get that there exists a constant $d_3(\alpha) > 0$ independent of m and k such that, if $m+k+4 < mr$,

$$\Theta_0(\exp(-\alpha Z_{m+k+4})\,|B^\bullet_{m+k+4,mr}|\,) \leqslant \frac{d_3(\alpha)}{(m+k+4)^3}\,r^4 m^4.$$

Summarizing, we get in both cases $m+k+4 < mr$ and $m+k+4 \geqslant mr$,

$$\Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{mr}|\big) \leqslant \frac{d_2(\alpha)+d_3(\alpha)}{(m+k+4)^3}\, r^4 m^4.$$

Coming back to (6.41) and (6.42), using (6.43) once again, and recalling that $q = 1/\beta$ and $m = 1/\varepsilon$, we can find a constant $d(\alpha) > 0$ such that:

$$\begin{aligned}\Theta_0\big(\mathbb{1}_{L_1<\varepsilon}|B^\bullet_r|^\beta\big) &= \frac{1}{m^{4\beta}}\Theta_0(\mathbb{1}_{L_m<1}|B^\bullet_{mr}|^\beta)\\ &\leqslant \frac{c}{m^{4\beta}}\sum_{k=0}^{\infty}\Theta_0\big(\exp(-\alpha Z_{m+k+4})\big)^{\frac{1}{p}}\,\Theta_0\big(\exp(-\alpha Z_{m+k+4})|B^\bullet_{mr}|\big)^{\frac{1}{q}}\\ &\leqslant \sum_{k=0}^{\infty}\frac{d(\alpha)}{(m+k+4)^3} r^{4\beta}\end{aligned}$$

and the lemma follows since $m = \varepsilon^{-1}$. □

We now use Lemma 6.11 to prove that for any nondecreasing positive function f:

$$\sum_{m\in\mathbb{N}}\frac{1}{f(m)^2} < \infty \implies \inf_{A\in\mathcal{K}}\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}}f(|\log(|A|)|) > 0\,,\ \Theta_0\text{-a.s.} \tag{6.45}$$

Proof. Fix a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}^*_+$ such that $\sum_{m\in\mathbb{N}} f(m)^{-2} < \infty$. We begin by showing that for every $A \in \mathcal{K}$ we have:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}}f(|\log(|A|)|) \geqslant \Big(\inf_{m\in\mathbb{Z}}\frac{2^{m-1}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}}f\big(|\log(|B^\bullet_{2^m}|)|\big)\Big) \wedge \Big(\inf_{m\in\mathbb{Z}}\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}}f\big(|\log(|B^\bullet_{2^m}|)|\big)\Big). \tag{6.46}$$

Let $A \in \mathcal{K}$ and let m be the unique element of $\mathbb{Z}$ such that:

$$|B^\bullet_{2^m}| < |A| \leqslant |B^\bullet_{2^{m+1}}|.$$

We divide the proof of (6.46) in two cases:

• Case 1: Assume that ∂A intersects $\partial B^\bullet_{2^{m-1}}$. As $|B^\bullet_{2^m}| < |A|$, there exists $x \in A\backslash B^\bullet_{2^m}$. Consider a path p_∞ connecting x to ∞ that does not hit $B^\bullet_{2^m}$ and let y be the last point of p_∞ that belongs to A. By construction we have $y \in \partial A$ and $y \notin B^\bullet_{2^m}$. Finally let $z \in \partial A \cap \partial B^\bullet_{2^{m-1}}$. Since ∂A connects y and z we have

$$\Delta(\partial A) \geqslant \Delta(y,z) \geqslant 2^{m-1}.$$

This implies that:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}}f(|\log(|A|)|) \geqslant \frac{2^{m-1}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}}f(|\log(|B^\bullet_{2^m}|)|).$$

• Case 2: Assume ∂A does not intersect $\partial B^\bullet_{2^{m-1}}$. Since $|A| > |B^\bullet_{2^{m-1}}|$, the set A is not contained in $B^\bullet_{2^{m-1}}$. This implies that ∂A separates $B^\bullet_{2^{m-1}}$ from infinity, and consequently $\Delta(\partial A) \geqslant L_{2^{m-1}}$. It follows that:

$$\frac{\Delta(\partial A)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) \geqslant \frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} f(|\log(|B^\bullet_{2^m}|)|)$$

and this completes the proof of (6.46).

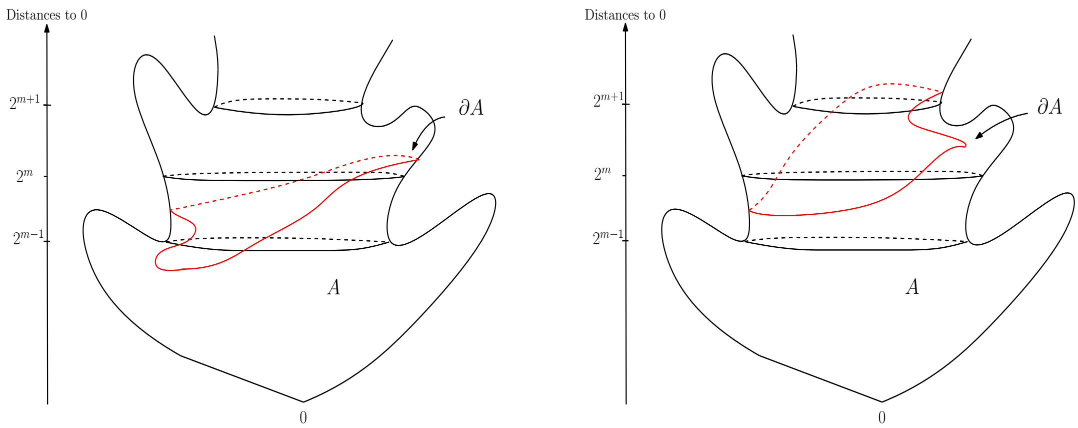

Figure 6.4 – Illustration of (6.46). In red we represent the boundary of A. On the left we are in case 1 and we have $\Delta(\partial A) \geqslant 2^{m-1}$. On the right we are in case 2 and we have $\Delta(\partial A) \geqslant L_{2^{m-1}}$.

Thanks to (6.46), the proof of (6.45) will be complete if we can verify that:

$$\inf_{m\in\mathbb{Z}} \frac{2^{m-1}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} f\big(|\log(|B^\bullet_{2^m}|)|\big) > 0, \ \ \Theta_0\text{-a.s.} \tag{6.47}$$

and

$$\inf_{m\in\mathbb{Z}} \frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} f\big(|\log(|B^\bullet_{2^m}|)|\big) > 0, \ \ \Theta_0\text{-a.s.} \tag{6.48}$$

Let us start by proving (6.47). By Lemma 6.10, Θ_0-a.s., there is a positive integer M such that for every $m \in \mathbb{Z}$ with $|m| \geqslant M$:

$$\frac{1}{|m|} 2^{4m} \leqslant |B^\bullet_{2^m}| \leqslant |m| 2^{4m} \tag{6.49}$$

In particular, we have:

$$\inf_{\substack{m\in\mathbb{Z}\\|m|>M}} \frac{2^{m-1}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} f(|\log(|B^\bullet_{2^m}|)|) \geqslant \frac{1}{4} \inf_{\substack{m\in\mathbb{Z}\\|m|>M}} \frac{f\big(|4\log(2)m - \log(|m|)|\big)}{(|m|+1)^{\frac{1}{4}}} \ \ \Theta_0\text{-a.s.}$$

On the other hand, by the Cauchy-Schwarz inequality:

$$\frac{1}{f(n)} \leqslant \frac{1}{n}\sum_{k=1}^{n} \frac{1}{f(k)} \leqslant \frac{1}{n^{\frac{1}{2}}}\Big(\sum_{k=1}^{n}\frac{1}{f(k)^2}\Big)^{\frac{1}{2}} \leqslant \frac{1}{n^{\frac{1}{2}}}\Big(\sum_{k=1}^{\infty}\frac{1}{f(k)^2}\Big)^{\frac{1}{2}}.$$

Consequently $\inf_{n\in\mathbb{N}} n^{-\frac{1}{2}} f(n) > 0$ and we obtain (6.47). Actually here it will be enough to have $\inf_{n\in\mathbb{N}} n^{-\frac{1}{4}} f(n) > 0$.

Let us prove (6.48). By (6.49), if we can verify that

$$\sum_{m\in\mathbb{Z}} \Theta_0\big(\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\big) < \infty$$

we will conclude by an application of the Borel-Cantelli lemma. Fix $\beta \in (\frac{3}{4}, 1)$. For every $m \in \mathbb{Z}$, by scaling we have:

$$\Theta_0\big(\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\big) = \Theta_0\big(\frac{L_1}{|B^\bullet_4|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\big).$$

Consequently, we get :

$$\Theta_0\big(\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\big) \leqslant \Theta_0\big(L_1 < \frac{1}{f(|m|)}\ ,\ |B^\bullet_4| \leqslant 1\big)$$
$$+ \sum_{n=1}^{\infty} \Theta_0\big(|B^\bullet_4|^{\frac{1}{4}} \in [n, n+1]\ ,\ L_1 < \frac{n+1}{f(|m|)}\big).$$

Now remark that the right term of the above display is bounded above by

$$\Theta_0\big(L_1 < \frac{1}{f(|m|)}\big) + \sum_{n=1}^{\infty} \Theta_0\big(|B^\bullet_4|^{\beta} \geqslant n^{4\beta},\ L_1 < \frac{n+1}{f(|m|)}\big).$$

We deduce by an application of Markov inequality that

$$\Theta_0\big(\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} < \frac{1}{f(|m|)}\big) \leqslant \Theta_0(L_1 < \frac{1}{f(|m|)}) + \sum_{n=1}^{\infty} \frac{1}{n^{4\beta}} \Theta_0(|B^\bullet_4|^{\beta} \mathbb{1}_{L_1 < \frac{n+1}{f(|m|)}})$$

Lemma 6.11 and Theorem 6.1 imply that there exist two constants $c_2 \in (0, \infty)$ and $C \in (0, \infty)$ such that:

$$\sum_{m\in\mathbb{Z}} \Theta_0(\frac{L_{2^{m-1}}}{|B^\bullet_{2^{m+1}}|^{\frac{1}{4}}} < \frac{1}{f(|m|)}) \leqslant 2c_2 \sum_{m\in\mathbb{N}}^{\infty} \frac{1}{f(m)^2} + C \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^{4\beta}} \sum_{m\in\mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

This completes the proof. □

We observe that the same proof will work mutatis mutandis if we replace $|A|$ by $\Delta(\partial A)$ inside the logarithm in theorem 6.2.

Recall that $\mathcal{M}^{(z)}_\infty$ stands for an infinite volume Brownian disk with perimeter z. With a slight abuse of terminology, we call Jordan domain of $\mathcal{M}^{(z)}_\infty$ the closure of the bounded component of the complement of an injective cycle of $\mathcal{M}^{(z)}_\infty$. As a direct consequence of Corollary 6.1, Proposition 6.8 (i) and Theorem 6.2 we obtain:

Corollary 6.4.
Fix $z > 0$. Let $\mathcal{M}_\infty^{(z)}$ be the infinite volume Brownian disk with perimeter z defined under the probability measure Θ_z. Consider the collection $\mathcal{K}^{(z)}$ of all Jordan domains of $\mathcal{M}_\infty^{(z)}$ whose interior contains the boundary of $\mathcal{M}_\infty^{(z)}$. For any nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+^*$

(i) *We have*

$$\inf_{A\in\mathcal{K}^{(z)}} \frac{\Delta\big(\partial A\big)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) = 0\,,\ \Theta_z\text{-a.s.}\,,\ \text{if}\ \sum_{m\in\mathbb{N}} \frac{1}{f(m)^2} = \infty.$$

(ii) *We have*

$$\inf_{A\in\mathcal{K}^{(z)}} \frac{\Delta\big(\partial A\big)}{|A|^{\frac{1}{4}}} f(|\log(|A|)|) > 0\,,\ \Theta_z\text{-a.s.}\,,\ \text{if}\ \sum_{m\in\mathbb{N}} \frac{1}{f(m)^2} < \infty.$$

Appendix

This appendix is devoted to the proof of Lemma 6.1, which relies on [77, Proposition 8]. We use the notation of Subsection 6.2.4.

Proof of Lemma 6.1.

First fix $0 < r_1 < r_2 < \infty$. The lemma will follow if we prove that, Θ_0-a.s. , for every $r_1 \leqslant r \leqslant r_2$, we have:

$$Z_r = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|.$$

In order to prove this, we introduce the event $A(s) := \{Z_{r_2}^{s,\infty} = 0\}$, for every $s > r_2$. In particular, under the event $A(s)$, we have $Z_r = Z_r^{r,s}$ for every $r \leqslant r_2$. Moreover, by Proposition 6.1 (taking the limit when $t \to \infty$) we also have:

$$\Theta(A(s)) = \big(\frac{s-r}{s}\big)^3,$$

which converges to 1 when $s \to \infty$. Consequently, to obtain the desired result it is sufficient to show that, for every $s > r_2$, under $A(s)$ we have:

$$Z_r^{r,s} = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} |\check{B}_r^\circ \cap B_{r+\varepsilon}|,$$

for every $r \in [r_1, r_2]$. Let us now introduce, for every $r \in \mathbb{R}$ and $\omega \in \mathcal{S}$ with $\omega_0 > r$, the quantity:

$$\mathcal{Z}_r^\varepsilon(\omega) := \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbb{1}_{\text{hit}_r(\omega_s)=\infty,\ \widehat{\omega}_s < r+\varepsilon}.$$

In particular, note that $\mathcal{Z}_r(\omega) = \liminf_{\varepsilon\to 0} \mathcal{Z}_r^\varepsilon(\omega)$. We set:

$$Z_r^{r,s}(\varepsilon) := \int \mathcal{Z}_r^\varepsilon(\omega)\mathfrak{R}^{r,s}(d\ell d\omega) + \int \mathcal{Z}_r^\varepsilon(\omega)\mathfrak{L}^{r,s}(d\ell d\omega).$$

Under $A(s)$, all the labels appearing after the point τ_s of the spine are greater than r_2. This implies that, under $A(s)$, the quantity $\varepsilon^2 Z_r^{r,s}(\varepsilon)$ is exactly $|\check{B}_r^\circ \cap B_{r+\varepsilon}|$ (since $|\cdot|$ is the pushforward of Lebesgue measure under $\Pi \circ \mathcal{E}$). To conclude, we are going to use [77, Proposition 8], which states that, for every $s > 0$ and $\beta > 0$, we have

$$\sup_{r \in (-\infty, s-\beta]} |\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r| \underset{\varepsilon \to 0}{\longrightarrow} 0 , \qquad \mathbb{N}_s\text{-a.e.} \tag{6.50}$$

To translate (6.50) in terms of $Z_r^{r,s}$ and $Z_r^{r,s}(\varepsilon)$, we recall that $(X_{(\tau_s - \ell)\vee 0})_{\ell \geqslant 0}$ is a Bessel process of dimension -5 started from s, and that, conditionally on $(X_{(\tau_s-\ell)\vee 0})_{\ell \geqslant 0}$, the measures $\mathfrak{R}^{0,s}$ and $\mathfrak{L}^{0,s}$ are two independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{S}$ with intensity:

$$2\mathbb{1}_{[0,\tau_s]}(\ell)\ d\ell\ \mathbb{N}_{X_\ell}(d\omega \cap \{\omega_* > 0\}).$$

In particular, the distribution $(X_{(\tau_s-\ell)\vee 0})_{0 \leqslant \ell \leqslant \tau_s - \tau_{r_1}}$ is absolute continuous with respect to the distribution of a Brownian motion started from s and stopped when it hits r_1. We can now apply [79, Proposition 2] to deduce that the distribution of $(Z_r^{r,s}, Z_r^{r,s}(\varepsilon))_{r \in [r_1, r_2]}$ is absolute continuous with respect to the distribution $(\mathcal{Z}_{r-r_1}, \mathcal{Z}_{r-r_1}^\varepsilon)_{r \in [r_1,r_2]}$ under $\mathbb{N}_{s-r_1}$. Consequently (6.50) gives that:

$$\sup_{r \in [r_1, r_2]} |Z_r^{r,s}(\varepsilon) - Z_r^{r,s}| \underset{\varepsilon \to 0}{\longrightarrow} 0 , \qquad \Theta_0\text{-a.s.}$$

which completes the proof. □

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PARTIE III

CALCULS EXPLICITES CONCERNANT DES LONGUEURS DE BORDS ET DES VOLUMES EN GÉOMÉTRIE BROWNIENNE

Chapitre 7

Explicit distributions for Brownian motion indexed by the Brownian tree

Les resultats de ce chapitre sont issus de l'article [78], écrit en collaboration avec Jean-François Le Gall et publié dans *Markov Processes and Related Fields*.

We derive several explicit distributions of functionals of Brownian motion indexed by the Brownian tree. In particular, we give a direct proof of a result of Bousquet-Mélou and Janson identifying the distribution of the density at 0 of the integrated super-Brownian excursion.

Contents

7.1 Introduction

The main purpose of the present work is to derive certain explicit distributions for the random process which we call Brownian motion indexed by the Brownian tree, which has appeared in a variety of different contexts. As a key tool for the derivation of our main results we use the excursion theory developed in [2] for Brownian motion indexed by the Brownian tree. In many respects, this excursion theory is similar to the classical Itô theory, which applies in particular to linear Brownian motion and has proved a powerful tool for the calculation of exact distributions of Brownian functionals.

Let us briefly describe the objects of interest in this work. We define the Brownian tree $\mathcal{T}_\zeta$ as the random compact $\mathbb{R}$-tree coded by a Brownian excursion $\zeta = (\zeta_s)_{s\geqslant 0}$ distributed according to the (infinite) Itô measure of positive excursions of linear Brownian motion. If σ stands for the duration of the excursion ζ, this coding means that $\mathcal{T}_\zeta$ is the quotient space of $[0,\sigma]$ for the equivalence relation defined by $s \sim s'$ if and only if $\zeta_s = \zeta_{s'} = m_\zeta(s,s')$, where $m_\zeta(s,s') := \min\{\zeta_r : s \wedge s' \leqslant r \leqslant s \vee s'\}$, and this quotient space is equipped with the metric induced by $\mathrm{d}_\zeta(s,s') = \zeta_s + \zeta_{s'} - 2m_\zeta(s,s')$. The volume measure $\mathrm{vol}(\mathrm{d}a)$ on $\mathcal{T}_\zeta$ is defined as the push forward of Lebesgue measure on $[0,\sigma]$ under the canonical projection, and the root ρ of $\mathcal{T}_\zeta$ is the equivalence class of 0. We note that under the conditioning by $\sigma = 1$ (equivalently the total volume is equal to 1) the tree $\mathcal{T}_\zeta$ is Aldous' Brownian Continuum Random Tree (also called the CRT, see [8, 9]), up to an unimportant scaling factor 2.

Let us turn to Brownian motion indexed by $\mathcal{T}_\zeta$. Informally, given $\mathcal{T}_\zeta$, this is the centered Gaussian process $(V_a)_{a\in\mathcal{T}_\zeta}$ such that $V_\rho = 0$ and $\mathrm{Var}(V_a - V_b) = \mathrm{d}_\zeta(a,b)$ for every $a, b \in \mathcal{T}_\zeta$. This definition is a bit informal since we are dealing with a random process indexed by a *random* set. These difficulties can be overcome easily by using the Brownian snake approach. We let $(W_s)_{s\geqslant 0}$ be the Brownian snake (whose spatial motion is linear Brownian motion started at 0) driven by the Brownian excursion $(\zeta_s)_{s\geqslant 0}$. Then, for every $s \geqslant 0$, W_s is a finite path started at 0 and with lifetime ζ_s, and for every $a \in \mathcal{T}_\zeta$ we may define V_a as the terminal point $\widehat{W}_s$ of the path W_s, for any $s \in [0,\sigma]$ such that a is the equivalence class of s in $\mathcal{T}_\zeta$. The Brownian snake approach thus reduces the study of a tree-indexed Brownian motion to that of a process indexed by the positive half-line, and we systematically use this approach in the next sections.

The total occupation measure $\Theta(\mathrm{d}x)$ of $(V_a)_{a\in\mathcal{T}_\zeta}$ is the push forward of $\mathrm{vol}(\mathrm{d}a)$ under the mapping $a \mapsto V_a$, or equivalently the push forward of Lebesgue measure on $[0,\sigma]$ under $s \mapsto \widehat{W}_s$. Under the special conditioning $\sigma = 1$, this random measure is known as ISE for Integrated Super-Brownian Excursion [10] (note that our normalization is different from the one in [10]).

At this point, we observe that both the Brownian tree (often under special conditionings) and Brownian motion indexed by the Brownian tree have appeared in different areas of probability theory. The Brownian snake is very closely related to the measure-valued process called super-Brownian motion and has proved an efficient tool to study this process (see [65] and the references therein). Super-Brownian motion and ISE arise in a number of limit theorems for discrete probability models, but also in the theory of interacting particle systems [28, 38, 49] and in a variety of models of statistical physics [46, 80, 81]. More recently, Brownian motion indexed by the Brownian tree has served as the essential building block in the construction of the universal model of random geometry called the Brownian map (see in particular [67, 71, 74, 85]). In this connection, we note that the distribution of certain functionals of Brownian motion indexed by the Brownian tree is investigated in the article [45], which was already motivated by asymptotics for random planar maps.

Let us now explain our main results more in detail. In agreement with the usual notation for the Brownian snake, we write $\mathbb{N}_0$ for the (infinite) measure under which $(\zeta_s)_{s\geqslant 0}$ and $(V_a)_{a\in\mathcal{T}_\zeta}$

are defined in the way we just explained – see Section 7.2 for more details. We are primarily interested in local times, which are the densities of the random measure $\Theta(\mathrm{d}x)$. It follows from the work of Bousquet-Mélou and Janson [24, 25] that $\Theta(\mathrm{d}x)$ has a continuous density $(\mathcal{L}^x)_{x\in\mathbb{R}}$ with respect to Lebesgue measure on $\mathbb{R}$, $\mathbb{N}_0$ a.e. (this fact could also be derived from the earlier work of Sugitani [96] dealing with super-Brownian motion, see in particular the introduction of [90]). We also consider the quantity $\sigma_+ = \Theta([0,\infty))$ (resp. $\sigma_- = \Theta((-\infty,0]))$ corresponding to the volume of the set of all points $a \in \mathcal{T}_\zeta$ such that $V_a \geqslant 0$ (resp. $V_a \leqslant 0$). One of our main technical results (Proposition 7.2) identifies the joint Laplace transform

$$\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu_1\sigma_+-\mu_2\sigma_-))\ ,\quad \lambda,\mu_1,\mu_2>0,$$

as the solution of the equation $h_{\mu_1,\mu_2}(v) = \sqrt{6}\,\lambda$, where, for every $v \geqslant 0$,

$$h_{\mu_1,\mu_2}(v) = \sqrt{\sqrt{2\mu_1}+v}\,\Big(2v-\sqrt{2\mu_1}\Big) + \sqrt{\sqrt{2\mu_2}+v}\,\Big(2v-\sqrt{2\mu_2}\Big).$$

In the special case $\mu_1 = \mu_2$, this equation can be solved explicitly and leads to the formula

$$\mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)\Big) = \begin{cases} \sqrt{2\mu}\,\cos\Big(\frac{2}{3}\,\arccos\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \leqslant 1, \\ \sqrt{2\mu}\,\cosh\Big(\frac{2}{3}\,\mathrm{arcosh}\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \geqslant 1. \end{cases} \tag{7.1}$$

We can extract the conditional distribution of $\mathcal{L}^0$ knowing σ from the preceding formula. In this way we obtain a short direct proof of a remarkable result of Bousquet-Mélou and Janson [25] stating that the local time $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = 1)$ (equivalently the density of ISE at 0) is distributed as $(2^{3/4}/3)\,T^{-1/2}$, where T is a positive stable variable with index 2/3, whose Laplace transform is $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{2/3})$ (Theorem 7.1). The original proof of Bousquet-Mélou and Janson relied on limit theorems for approximations of ISE by discrete labeled trees. Somewhat surprisingly, we are also able to obtain an analog of the latter result when instead of conditioning on $\sigma = 1$ we condition on $\sigma_+ = 1$. Precisely, we get that the local time $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma_+ = 1)$ is distributed as $(2^{9/4}/3)\,D\,T^{-1/2}$, where T is as previously and the random variable D is independent of T and has density $2x\,\mathbf{1}_{[0,1]}(x)$ (Theorem 7.2). Our proofs are computational and rely on explicit formulas for moments derived via the Lagrange inversion theorem. It would be interesting to have more probabilistic proofs and a better understanding of the reason why such simple distributions occur.

Because of the connections between the Brownian snake and super-Brownian motion, several of our results can be restated in terms of distributions of (one-dimensional) super-Brownian motion $(\mathbf{X}_t)_{t\geqslant 0}$ started from the Dirac measure δ_0. In particular, we get that the total local time at 0 (defined as the density at 0 of the measure $\int_0^\infty \mathrm{d}t\,\mathbf{X}_t$) is distributed as $3^{1/2}2^{-2/3}\,T$ where T is as previously a positive stable variable with index 2/3 (Corollary 7.2). This is by no means a difficult result (as pointed out to the authors by Edwin Perkins [91], the fact that the total local time is a stable variable with index 2/3 can also be derived by a scaling argument, see formula (2.13) in [90]), but it seems to have remained unnoticed by the specialists of super-Brownian motion. The fact that the same variable T occurs in the Bousquet-Mélou-Janson result suggests the existence of a direct connection between the two results, but we have been unable to find such a connection.

The present article is organized as follows. Section 7.2 gives a number of preliminaries concerning the Brownian snake. We have chosen to discuss the Brownian snake with a general spatial motion because it turns out to be useful to consider also the case where this spatial motion is the pair consisting of a linear Brownian motion and its local time at 0. In fact, Section 7.3 starts with a formula expressing the local time $\mathcal{L}^0$ in terms of certain exit measures of this two-dimensional Brownian snake (Proposition 7.1). This expression then leads to an easy calculation of the Laplace transform of $\mathcal{L}^0$, or more generally of $\mathcal{L}^x$ for any $x \in \mathbb{R}$, under $\mathbb{N}_0$ (Corollary 7.1). Section 7.4 gives the key Proposition 7.2 characterizing the joint Laplace transform of the triple $(\mathcal{L}^0, \sigma_+, \sigma_-)$ and establishes (7.1) as a consequence. Finally, Section 7.5 derives conditional distributions of the local time $\mathcal{L}^0$, and also discusses the interpretation of these distributions in continuous models of random geometry.

7.2 Preliminaries

7.2.1 The Brownian snake

In this section, we recall some basic facts about the Brownian snake with a general spatial motion. We let ξ stand for a Markov process with values in $\mathbb{R}^d$, which starts from $x \in \mathbb{R}^d$ under the probability measure $\mathbb{P}_x$. We assume that ξ has continuous sample paths, and moreover we require the following bound on the increments of ξ. There exist three positive constants C, $q > 2$ and $\chi > 0$ such that for every $t \in [0,1]$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\Big[\sup_{0 \leqslant s \leqslant t} |\xi_s - x|^q\Big] \leqslant C\, t^{2+\chi}. \tag{7.2}$$

Under this moment assumption, we may define the Brownian snake with spatial motion ξ as a strong Markov process with values in the space of d-dimensional finite paths (see [65, Section IV.4]). In this work, we will only need the Brownian snake excursion measures, which we now introduce within the formalism of snake trajectories [2].

First recall that a (d-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}^d$, where the number $\zeta = \zeta_{(\mathrm{w})} \geqslant 0$ is called the lifetime of w. We let $\mathcal{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geqslant 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For every $x \in \mathbb{R}^d$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x of $\mathbb{R}^d$.

Definition 7.1. *Let $x \in \mathbb{R}^d$. A snake trajectory with initial point x is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_x$ which satisfies the following two properties:*

(i) *We have* $\omega_0 = x$ *and the number* $\sigma(\omega) := \sup\{s \geqslant 0 : \omega_s \neq x\}$*, called the duration of the snake trajectory* ω*, is finite (by convention* $\sigma(\omega) = 0$ *if* $\omega_s = x$ *for every* $s \geqslant 0$*).*

(ii) *For every* $0 \leqslant s \leqslant s'$*, we have* $\omega_s(t) = \omega_{s'}(t)$ *for every* $t \in [0, \min_{s \leqslant r \leqslant s'} \zeta_{(\omega_r)}]$.

If ω is a snake trajectory, we will write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$. We denote the set of all snake trajectories with initial point x by $\mathcal{S}_x$. The set $\mathcal{S}_x$ is equipped with the distance

$$d_{\mathcal{S}_x}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geqslant 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega'))$$

and the associated Borel σ-field.

Let $\mathbf{n}(\mathrm{d}e)$ denote the classical Itô measure of positive excursions of linear Brownian motion (see e.g. [92, Chapter XII]). Then $\mathbf{n}(\mathrm{d}e)$ is a σ-finite measure on the space of all continuous functions $s \mapsto e_s$ from $\mathbb{R}_+$ into $\mathbb{R}_+$, and without risk of confusion, we will write $\sigma(e) = \sup\{s \geqslant 0 : e_s \neq 0\}$, in such a way that we have $0 < \sigma(e) < \infty$ and $e(s) > 0$ for every $0 < s < \sigma(e)$, $\mathbf{n}(\mathrm{d}e)$ a.e. We consider the usual normalization of $\mathbf{n}(\mathrm{d}e)$, so that, for every $\varepsilon > 0$,

$$\mathbf{n}\Big(\sup\{e_s : s \geqslant 0\} > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

We have then also, for every $\lambda > 0$,

$$\mathbf{n}(1 - \exp(-\lambda\sigma(e))) = \sqrt{\lambda/2}, \tag{7.3}$$

and equivalently the distribution of $\sigma(e)$ under $\mathbf{n}(\mathrm{d}e)$ is $(2\sqrt{2\pi})^{-1}\, s^{-3/2}\, \mathrm{d}s$.

Definition 7.2. *For every* $x \in \mathbb{R}^d$*, the Brownian snake excursion measure* $\mathbb{N}_x$ *is the* σ*-finite measure on* $\mathcal{S}_x$ *characterized by the following two properties:*

(i) *The distribution of* $(\zeta_s)_{s \geqslant 0}$ *under* $\mathbb{N}_x$ *is* $\mathbf{n}$*;*

(ii) *Under* $\mathbb{N}_x$ *and conditionally on* $(\zeta_s)_{s \geqslant 0}$*,* $(W_s)_{s \geqslant 0}$ *is a time-inhomogeneous Markov process whose transition kernels can be described as follows: For every* $0 \leqslant s \leqslant s'$*,*

- $W_{s'}(t) = W_s(t)$ *for all* $0 \leqslant t \leqslant m_\zeta(s, s') := \min\{\zeta_r : s \leqslant r \leqslant s'\}$*;*
- *conditionally on* W_s*, the random path* $(W_{s'}(m_\zeta(s, s') + t), 0 \leqslant t \leqslant \zeta_{s'} - m_\zeta(s, s'))$ *is distributed as the Markov process* ξ *started at* $W_s(m_\zeta(s, s'))$*.*

See again [65, Chapter IV] for more information about the measures $\mathbb{N}_x$. If F is a nonnegative function on $\mathcal{W}_x$, we have the first-moment formula

$$\mathbb{N}_x\Big(\int_0^\sigma F(W_s)\,\mathrm{d}s\Big) = \mathbb{E}_x\Big[\int_0^\infty F\Big((\xi_r)_{0 \leqslant r \leqslant t}\Big)\,\mathrm{d}t\Big]. \tag{7.4}$$

We now turn to exit measures. Let O be an open set in $\mathbb{R}^d$ such that $x \in O$. For every $\mathrm{w} \in \mathcal{W}_x$, set

$$\tau_O(\mathrm{w}) = \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin O\}$$

with the usual convention $\inf \varnothing = +\infty$. Then $\mathbb{N}_x$ a.e. there exists a random finite measure $\mathcal{Z}_O$ supported on ∂O such that, for every bounded continuous function φ on ∂O, we have

$$\langle \mathcal{Z}_O, \varphi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_O(W_s) \leqslant \zeta_s \leqslant \tau_O(W_s)+\varepsilon\}}\, \varphi(W_s(\tau_O(W_s))). \tag{7.5}$$

See [65, Chapter V]. Then, for every nonnegative measurable function φ on $\mathbb{R}^d$,

$$\mathbb{N}_x(\langle \mathcal{Z}_O, \varphi \rangle) = \mathbb{E}_x[\varphi(\xi_{\tau_O})\, \mathbf{1}_{\{\tau_O < \infty\}}], \tag{7.6}$$

where in the right-hand side $\tau_O = \inf\{t \geqslant 0 : \xi_t \notin O\}$.

Let us now recall the special Markov property of the Brownian snake, referring to the appendix of [70] for the proof of a slightly more precise statement. To this end we consider again the open set O such that $x \in O$, and fix a snake trajectory $\omega \in \mathcal{W}_x$. We observe that the set $\{s \geqslant 0 : \tau_O(W_s) < \infty\}$ is open and thus can be written as a disjoint union of open intervals (a_i, b_i), $i \in I$ (the indexing set I may be empty if none of the paths W_s exits O). For every $i \in I$, we may define a new snake trajectory $\omega^{(i)}$ by setting for every $s \geqslant 0$,

$$\omega_s^{(i)}(t) := \omega_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)\ , \text{ for every } 0 \leqslant t \leqslant \zeta_{(\omega_s^{(i)})} := \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}.$$

The snake trajectories $\omega^{(i)}$ represent the excursions of ω outside O (the word "outside" is somewhat misleading since these excursions typically come back into O though they start on ∂O). We also introduce a σ-field $\mathcal{E}_O$ corresponding informally to the information given by the paths W_s before they exit O (see [70] for a more precise definition), and note that $\mathcal{Z}_O$ is measurable with respect to $\mathcal{E}_O$. Then the special Markov property states that, under $\mathbb{N}_x$ and conditionally on $\mathcal{E}_O$, the point measure

$$\sum_{i \in I} \delta_{\omega^{(i)}}$$

is a Poisson random measure with intensity $\int \mathcal{Z}_O(\mathrm{d}y)\, \mathbb{N}_y(\cdot)$.

7.2.2 Specific properties when ξ is linear Brownian motion

We finally mention a few more specific properties that hold in the special case where $d = 1$ and ξ is standard linear Brownian motion. In that case, we have the following scaling property. If $\lambda > 0$ and

$$W'_s(t) = \lambda\, W_{s/\lambda^4}(t/\lambda^2)\ , \quad \text{for every } 0 \leqslant t \leqslant \zeta'_s := \lambda^2 \zeta_{s/\lambda^4}, \tag{7.7}$$

then the distribution of $(W'_s)_{s \geqslant 0}$ under $\mathbb{N}_x$ is $\lambda^2 \mathbb{N}_{\lambda x}$.

Suppose that the open set O is the interval $(-\infty, y)$ with $y > x$, or the interval (y, ∞) with $y < x$. In both cases, the exit measure $\mathcal{Z}_O$ can be written as $\mathcal{Z}_y\, \delta_y$, where $\mathcal{Z}_y \geqslant 0$ and δ_y denotes the Dirac measure at y, and we have, for every $\lambda > 0$,

$$\mathbb{N}_x\Big(1 - \exp(-\lambda \mathcal{Z}_y)\Big) = \Big(\lambda^{-1/2} + |y - x|\sqrt{2/3}\Big)^{-2}. \tag{7.8}$$

See formula (6) in [40].

Let $\mathcal{R} := \{\widehat{W}_s : s \geqslant 0\} = \{W_s(t) : s \geqslant 0, 0 \leqslant t \leqslant \zeta_s\}$ denote the range of the Brownian snake. Then, for every $y \in \mathbb{R}$, $y \neq x$,

$$\mathbb{N}_x(y \in \mathcal{R}) = \frac{3}{2(y-x)^2} = \mathbb{N}_x(\mathcal{Z}_y > 0). \tag{7.9}$$

See [65, Section VI.1] for the first equality, and note that the second one follows from (7.8).

Finally, it follows from the results of [25] that there exists $\mathbb{N}_0$ a.e. a continuous function $(\mathcal{L}^y)_{y\in\mathbb{R}}$, which is supported on $\mathcal{R}$, such that, for every nonnegative measurable function φ on $\mathbb{R}$,

$$\int_0^\sigma \mathrm{d}s\, \varphi(\widehat{W}_s) = \int_{\mathbb{R}} \mathrm{d}y\, \varphi(y)\, \mathcal{L}^y.$$

We call $\mathcal{L}^y$ the Brownian snake local time at y. Note that [25] deals with the case of ISE, that is, with the conditional measure $\mathbb{N}_0(\cdot \mid \sigma = 1)$, but then a scaling argument gives the desired result under $\mathbb{N}_0$. Next suppose that, for a given $\lambda > 0$, W' is defined from W as in (7.7). Then, with an obvious notation, we have $\sigma' = \lambda^4\sigma$ and $\mathcal{L}'^x = \lambda^3\mathcal{L}^{x/\lambda}$ for every $x \in \mathbb{R}$, $\mathbb{N}_0$ a.e. As a consequence, for every $s > 0$, the distribution of $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = s)$ is equal to the distribution of $s^{3/4}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = 1)$.

The scaling property also implies the existence of a constant C such that, for every $s > 0$ and $x \in \mathbb{R}$, we have $\mathbb{N}_0(\mathcal{L}^x \mid \sigma = s) \leqslant C\, s^{3/4}$ (the case $s = 1$ follows from [25, Corollary 11.3], or from a simple argument using Fatou's lemma and the approximation of $\mathcal{L}^x$ by $(2\varepsilon)^{-1}\int_0^\sigma \mathrm{d}r\, \mathbf{1}_{\{|\widehat{W}_r - x| < \varepsilon\}}$).

7.3 The local time at 0

In this section and the next ones, we consider the Brownian snake excursion measure $\mathbb{N}_0$ in the case where $\xi = B$ is linear Brownian motion. For every $a \in \mathbb{R}$ and $t \geqslant 0$, we use the notation $L^a_t(B)$ for the local time of the Brownian motion B at level a and at time t.

For every fixed $s \geqslant 0$, the path W_s is distributed under $\mathbb{N}_0$ and conditionally on ζ_s as a linear Brownian path started at 0 with lifetime ζ_s, and we can define its local time process at 0,

$$L^0_t(W_s) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[0,\varepsilon]}(W_s(r))\,\mathrm{d}r\ , \quad 0 \leqslant t \leqslant \zeta_s\ ,\ \text{a.s.}$$

We may view $L^0(W_s) = (L^0_t(W_s))_{0\leqslant t\leqslant\zeta_s}$ as a random element of $\mathcal{W}_0$ with lifetime ζ_s. Simple moment estimates show that we can choose a continuous modification of $(L^0(W_s))_{s\geqslant 0}$ (as a random process with values in $\mathcal{W}_0$). Moreover, the distribution under $\mathbb{N}_0$ of the two-dimensional process $(W_s, L^0(W_s))_{s\geqslant 0}$ is the Brownian snake excursion measure (from the point $(0,0)$ of $\mathbb{R}^2$) for the Markov process $\xi'_t = (B_t, L^0_t(B))$ (note that ξ' may be viewed as a Markov process with values in $\mathbb{R}^2$, which satisfies (7.2)).

The point of the preceding discussion is that, under $\mathbb{N}_0$, we can define exit measures for the process $(W_s, L^0(W_s))_{s\geqslant 0}$ from open subsets O of $\mathbb{R}^2$ containing $(0,0)$, in the way explained in Section 7.2 (e.g. from the approximation formula (7.5)). For every $r > 0$, we consider the exit measure from the open set $O = \mathbb{R} \times (-\infty, r)$ and denote its mass by $\mathscr{X}_r$ (as an application of the first-moment formula (7.6), this exit measure is a random multiple of the Dirac measure at $(0,r)$). By convention we also take $\mathscr{X}_0 = 0$.

On the other hand, as explained in Section 8 of [2], we can use a famous theorem of Lévy [92, Theorem VI.2.3] to give a different presentation of the process $(|W_s|, L^0(W_s))$. To this end, for every $s \geqslant 0$, write

$$W_s^\bullet(t) := W_s(t) - \min\{W_s(r) : 0 \leqslant r \leqslant t\}\;, L_s^\bullet(t) = -\min\{W_s(r) : 0 \leqslant r \leqslant t\}\;,\;\text{ for } 0 \leqslant t \leqslant \zeta_s.$$

Then the distribution of the pair $(W_s^\bullet, L_s^\bullet)_{s\geqslant 0}$ under $\mathbb{N}_0$ is equal to the distribution of $(|W_s|, L^0(W_s))_{s\geqslant 0}$ under the same measure.

Using the preceding identity in distribution of two-dimensional snake trajectories, and the approximation (7.5) of exit measures, we get that the process $(\mathscr{X}_r)_{r>0}$ has the same distribution under $\mathbb{N}_0$ as the process $(\mathcal{Z}_{-r})_{r>0}$, where we recall that, for every $x \in \mathbb{R}\backslash\{0\}$, $\mathcal{Z}_x$ denotes the (total mass of the) exit measure of $(W_s)_{s\geqslant 0}$ from the open interval (x, ∞) if $x < 0$, or $(-\infty, x)$ if $x > 0$ – of course, by symmetry, $(\mathcal{Z}_{-r})_{r>0}$ has the same distribution as $(\mathcal{Z}_r)_{r>0}$. In particular $\mathbb{N}_0(\mathscr{X}_r > 0) = \mathbb{N}_0(\mathcal{Z}_r > 0) < \infty$ by (7.9). The discussion in [77, Section 2.4] now shows that the process $(\mathscr{X}_r)_{r>0}$ has a càdlàg modification under $\mathbb{N}_0$, which we consider from now on. Furthermore the distribution of this càdlàg modification under $\mathbb{N}_0$ can be interpreted as the excursion measure of the continuous-state branching process with branching mechanism $\phi(u) = \sqrt{8/3}\,u^{3/2}$ (the ϕ-CSBP in short, see [65, Chapter II] for a brief presentation of continuous-state branching processes). This means that, if $\alpha > 0$, and $\sum_{i\in I} \delta_{\omega_i}$ is a Poisson point measure with intensity $\alpha\,\mathbb{N}_0$, the process Y defined by $Y_0 = \alpha$ and

$$Y_r = \sum_{i\in I} \mathscr{X}_r(\omega_i)$$

for every $r > 0$, is a ϕ-CSBP started from α. Note that the right-hand side of the last display is a finite sum since $\mathbb{N}_0(\mathscr{X}_r > 0) < \infty$.

Recall our notation $\mathcal{L}^0$ for the Brownian snake local time at 0.

Proposition 7.1. *We have*

$$\mathcal{L}^0 = \int_0^\infty \mathrm{d}r\, \mathscr{X}_r\;, \quad \mathbb{N}_0\ \textit{a.e.}$$

This proposition is obviously related to the identity (37) in [77, Proposition 25], which is however concerned with the local time $\mathcal{L}^x$ at a level $x > 0$. Unfortunately, the case $x = 0$ seems to require a different argument.

Proof. It will be convenient to write $\widehat{L}(W_s) = L^0_{\zeta_s}(W_s)$ and

$$L^* = \max\{\widehat{L}(W_s) : 0 \leqslant s \leqslant \sigma\}.$$

For every $\varepsilon > 0$, set

$$\mathcal{L}^{0,\varepsilon} := \varepsilon^{-1} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon\}}.$$

Then $\mathcal{L}^{0,\varepsilon} \longrightarrow \mathcal{L}^0$ as $\varepsilon \to 0$, $\mathbb{N}_0$ a.e. We also introduce, for every fixed $\delta > 0$,

$$\mathcal{L}^{0,\varepsilon,(\delta)} := \varepsilon^{-1} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, \hat{L}(W_s)>\delta\}}.$$

We observe that, for every $\varepsilon, \delta > 0$, we can use the first-moment formula (7.4) to compute

$$\begin{aligned}\mathbb{N}_0(\mathcal{L}^{0,\varepsilon} - \mathcal{L}^{0,\varepsilon,(\delta)}) &= \mathbb{N}_0\Big(\varepsilon^{-1} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, \hat{L}(W_s)\leqslant\delta\}}\Big) \\ &= \varepsilon^{-1}\mathbb{E}_0\Big[\int_0^\infty \mathrm{d}t\, \mathbf{1}_{\{0<B_t<\varepsilon, L_t^0(B)\leqslant\delta\}}\Big] = \delta, \end{aligned} \tag{7.10}$$

where the last equality follows from a standard Ray-Knight theorem for Brownian local times [92, Theorem IX.2.3]. We then need the following lemma.

Lemma 7.1. *For every $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0} \mathcal{L}^{0,\varepsilon,(\delta)} = \int_\delta^\infty \mathrm{d}r\, \mathscr{X}_r\,,$$

in probability under $\mathbb{N}_0(\cdot \mid L^* \geqslant \delta)$.

Let us postpone the proof of this lemma and complete that of Proposition 7.1. Write $\widetilde{\mathcal{L}}^0 = \int_0^\infty \mathrm{d}r\, \mathscr{X}_r$ and $\widetilde{\mathcal{L}}^{0,(\delta)} = \int_\delta^\infty \mathrm{d}r\, \mathscr{X}_r$ to simplify notation, and for $a > 0$ set $\mathbb{N}_0^{(a)} = \mathbb{N}_0(\cdot \mid L^* \geqslant a)$. Then, for every $\alpha > 0$,

$$\begin{aligned}\mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \widetilde{\mathcal{L}}^0| > \alpha) &\leqslant \mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \mathcal{L}^{0,\varepsilon}| > \alpha/4) + \mathbb{N}_0^{(a)}(|\mathcal{L}^{0,\varepsilon} - \mathcal{L}^{0,\varepsilon,(\delta)}| > \alpha/4) \\ &\quad + \mathbb{N}_0^{(a)}(|\mathcal{L}^{0,\varepsilon,(\delta)} - \widetilde{\mathcal{L}}^{0,(\delta)}| > \alpha/4) + \mathbb{N}_0^{(a)}(|\widetilde{\mathcal{L}}^{0,(\delta)} - \widetilde{\mathcal{L}}^0| > \alpha/4). \end{aligned} \tag{7.11}$$

Let $\gamma > 0$. We can fix $\delta > 0$ small enough so that, for every $\varepsilon > 0$, the second and the fourth term in the right-hand side of (7.11) are smaller than $\gamma/4$ (we use (7.10) for the second term). Then, if $\varepsilon > 0$ is small enough, the first and the third term are also smaller than $\gamma/4$ (using Lemma 7.1 for the third term). We conclude that $\mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \widetilde{\mathcal{L}}^0| > \alpha) \leqslant \gamma$ and since α and γ were arbitrary this gives the desired result $\widetilde{\mathcal{L}}^0 = \mathcal{L}^0$. □

Proof of Lemma 7.1. We keep the notation $\widetilde{\mathcal{L}}^{0,(\delta)}$ introduced in the previous proof. We first observe that

$$\widetilde{\mathcal{L}}^{0,(\delta)} = \lim_{\varepsilon\to 0} \varepsilon \sum_{k=0}^\infty \mathscr{X}_{\delta+k\varepsilon}\,,\ \mathbb{N}_0 \text{ a.e.} \tag{7.12}$$

and on the other hand,

$$\mathcal{L}^{0,\varepsilon,(\delta)} = \varepsilon^{-1} \sum_{k=0}^\infty \mathcal{H}_k^{\varepsilon,(\delta)}, \tag{7.13}$$

where

$$\mathcal{H}_k^{\varepsilon,(\delta)} = \int_0^\sigma \mathrm{d}s\ \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon,\ \delta+k\varepsilon<\hat{L}(W_s)\leqslant\delta+(k+1)\varepsilon\}}.$$

The idea of the proof is to bound $\mathbb{N}_0(|\varepsilon\mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-1}\mathcal{H}_k^{\varepsilon,(\delta)}|)$, for every fixed $k \geqslant 0$. To this end, we apply the special Markov property to the Brownian snake with spatial motion $(B_t, L_t^0(B))$ and the open set $O = \mathbb{R} \times (-\infty, \delta + k\varepsilon)$, noting that the event $\{L^* \geqslant \delta\}$ is then $\mathcal{E}_O$-measurable. It follows that, under $\mathbb{N}_0(\cdot \mid L^* \geqslant \delta)$ and conditionally on $\mathscr{X}_{\delta+k\varepsilon} = a$, the quantity $\mathcal{H}_k^{\varepsilon,(\delta)}$ is distributed as

$$\int \mathcal{N}(\mathrm{d}\omega)\, \mathcal{U}_\varepsilon(\omega)$$

where $\mathcal{N}(\mathrm{d}\omega)$ is a Poisson point measure with intensity $a\,\mathbb{N}_0$, and the random variable $\mathcal{U}_\varepsilon$ is defined under $\mathbb{N}_0$ by

$$\mathcal{U}_\varepsilon = \int_0^\sigma \mathrm{d}s\; \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, 0<\hat{L}(W_s)\leqslant\varepsilon\}}.$$

Hence, conditionally on $\mathscr{X}_{\delta+k\varepsilon} = a$, $\mathcal{H}_k^{\varepsilon,(\delta)}$ has the distribution of U_a^ε, where $(U_t^\varepsilon)_{t\geqslant 0}$ is the subordinator whose Lévy measure is the distribution of $\mathcal{U}_\varepsilon$ under $\mathbb{N}_0$. Note that $\mathbb{E}[U_1^\varepsilon] = \mathbb{N}_0(\mathcal{U}_\varepsilon) = \varepsilon^2$ by (7.10).

By a scaling argument, we get that $(U_t^\varepsilon)_{t\geqslant 0}$ has the same distribution as $(\varepsilon^4\, U^1_{\varepsilon^{-2}t})_{t\geqslant 0}$. Next the law of large numbers shows that

$$\lim_{t\to\infty} \sup_{s\leqslant t} \mathbb{E}\Big[\frac{|U_s^1 - s|}{t}\Big] = 0. \tag{7.14}$$

Fix $A > 0$ and consider the event $E_A := \{L^* \leqslant A\} \cap \{\sup\{\mathscr{X}_r : r \geqslant 0\} \leqslant A\}$. Notice that on this event we have $\mathscr{X}_{\delta+k\varepsilon} = 0$ and $\mathcal{H}_k^{\varepsilon,(\delta)} = 0$ as soon as $\delta + k\varepsilon > A$. It follows that

$$\begin{aligned}
&\mathbb{N}_0\Big(\mathbf{1}_{E_A}\, |\varepsilon \sum_{k=0}^{\infty} \mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-1} \sum_{k=0}^{\infty} \mathcal{H}_k^{\varepsilon,(\delta)}| \;\Big|\; L^* \geqslant \delta\Big) \\
&\quad \leqslant \varepsilon(\lfloor A/\varepsilon\rfloor + 1) \sup_{0\leqslant k\leqslant\lfloor A/\varepsilon\rfloor} \mathbb{N}_0\Big(\mathbf{1}_{\{\mathscr{X}_{\delta+k\varepsilon}\leqslant A\}}\, |\mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-2}\mathcal{H}_k^{\varepsilon,(\delta)}| \;\Big|\; L^* \geqslant \delta\Big) \\
&\quad \leqslant \varepsilon(\lfloor A/\varepsilon\rfloor + 1) \sup_{0\leqslant a\leqslant A} \mathbb{E}[|\varepsilon^{-2} U_a^\varepsilon - a|] \\
&\quad = \varepsilon(\lfloor A/\varepsilon\rfloor + 1) \sup_{0\leqslant s\leqslant A/\varepsilon^2} \mathbb{E}[\varepsilon^2 |U_s^1 - s|],
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$, by (7.14). The statement of the lemma follows, recalling (7.12) and (7.13). □

Corollary 7.1. *For every $\lambda > 0$,*

$$\mathbb{N}_0(1 - e^{-\lambda\mathcal{L}_0}) = \frac{3^{1/3}}{2}\,\lambda^{2/3}. \tag{7.15}$$

The distribution of $\mathcal{L}^0$ under $\mathbb{N}_0$ has density

$$h(\ell) = \frac{3^{-2/3}}{\Gamma(1/3)}\,\ell^{-5/3}$$

with respect to Lebesgue measure on $(0,\infty)$.

Proof. Let $(X_r)_{r\geqslant 0}$ denote a ϕ-CSBP started from 1, where we recall that $\phi(u) = \sqrt{8/3}\,u^{3/2}$. Using the interpretation of the distribution of $(\mathscr{X}_r)_{r>0}$ under $\mathbb{N}_0$ and the exponential formula for Poisson measures, we have

$$\mathbb{E}\Big[\exp\Big(-\lambda\int_0^\infty \mathrm{d}r\,X_r\Big)\Big] = \exp\Big(-\mathbb{N}_0\Big(1-\exp\Big(-\lambda\int_0^\infty \mathrm{d}r\,\mathscr{X}_r\Big)\Big)\Big).$$

It then follows from Proposition 7.1 that

$$\mathbb{N}_0(1-e^{-\lambda\mathcal{L}_0}) = -\log\mathbb{E}\Big[\exp\Big(-\lambda\int_0^\infty \mathrm{d}r\,X_r\Big)\Big].$$

The classical Lamperti transformation [32, 61] shows that $\int_0^\infty \mathrm{d}r\,X_r$ has the same distribution as $T_0 := \inf\{t\geqslant 0 : Y_t = 0\}$, where $(Y_t)_{t\geqslant 0}$ denotes a stable Lévy process with no negative jumps started from 1, whose distribution is characterized by the Laplace transform $\mathbb{E}[\exp(-\lambda(Y_t-1))] = \exp(t\,\phi(\lambda))$. It is then classical (see e.g. [15, Chapter VII]) that

$$\mathbb{E}[e^{-\lambda T_0}] = e^{-\phi^{-1}(\lambda)},$$

where $\phi^{-1}(\lambda) = (3/8)^{1/3}\,\lambda^{2/3}$ is the inverse function of ϕ. This completes the proof of the first assertion. The density of $\mathcal{L}^0$ is then obtained by inverting the Laplace transform. □

In the next corollary, we consider a one-dimensional super-Brownian motion $(\mathbf{X}_t)_{t\geqslant 0}$ with quadratic branching mechanism $\psi(u) = 2u^2$ (the choice of the constant 2 is only for convenience, since a scaling argument will give a similar result with a general quadratic branching mechanism). Then it is well known that we can define the associated (total) local times as the unique (random) continuous function $(\mathbf{L}^a)_{a\in\mathbb{R}}$ such that

$$\int_0^\infty \mathrm{d}t\,\langle \mathbf{X}_t, f\rangle = \int_{\mathbb{R}} \mathrm{d}a\,f(a)\,\mathbf{L}^a,$$

for every Borel function $f:\mathbb{R}\longrightarrow\mathbb{R}_+$. See in particular Sugitani [96].

Corollary 7.2. *Suppose that $\mathbf{X}_0 = \alpha\,\delta_0$ for some $\alpha>0$. Then, for every $a\in\mathbb{R}$ and $\lambda>0$,*

$$\mathbb{E}[e^{-\lambda\mathbf{L}^a}] = \exp\left(-\alpha\,\frac{3^{1/3}}{2}\Big(\lambda^{-1/3}+3^{-1/3}\,|a|\Big)^{-2}\right). \tag{7.16}$$

In particular,

$$\mathbb{E}[e^{-\lambda\mathbf{L}^0}] = \exp\left(-\alpha\frac{3^{1/3}}{2}\,\lambda^{2/3}\right), \tag{7.17}$$

so that $\mathbf{L}^0$ is a positive stable variable with index $2/3$.

Proof. We rely on the Brownian snake construction of super-Brownian motion (see in particular [65, Chapter 4]). We may assume that $(\mathbf{X}_t)_{t\geqslant 0}$ is constructed in such a way that there exists a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{\omega_i}$ with intensity $\alpha\mathbb{N}_0$, such that, for every Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}_+$,

$$\int_{\mathbb{R}} \mathrm{d}a\, f(a)\, \mathbf{L}^a = \int_0^\infty \mathrm{d}t\, \langle \mathbf{X}_t, f\rangle = \sum_{i\in I} \int_0^{\sigma(\omega_i)} \mathrm{d}s\, f(\widehat{W}_s(\omega_i)) = \sum_{i\in I} \int_{\mathbb{R}} \mathrm{d}a\, f(a)\, \mathcal{L}^a(\omega_i).$$

It follows that we have

$$\mathbf{L}^a = \sum_{i\in I} \mathcal{L}^a(\omega_i) \tag{7.18}$$

for Lebesgue a.e. $a \in \mathbb{R}$. The left-hand side is continuous in a, and the right-hand side is continuous on $\mathbb{R}\backslash\{0\}$ since, for every $\delta > 0$, there are only finitely many $i \in I$ such that $\mathcal{L}^a(\omega_i)$ is nonzero for some a with $|a| > \delta$. So (7.18) holds for every $a \in \mathbb{R}\backslash\{0\}$. In fact it is easy to see that (7.18) also holds for $a = 0$. First note that, by Fatou's lemma, $\mathbf{L}^0 \geqslant \sum_{i\in I} \mathcal{L}^0(\omega_i)$, so that it suffices to check that

$$\mathbb{E}[e^{-\mathbf{L}^0}] = \mathbb{E}\Big[\exp\Big(-\sum_{i\in I} \mathcal{L}^0(\omega^i)\Big)\Big].$$

The left-hand side is the limit when $a \to 0$ of $\mathbb{E}[e^{-\mathbf{L}^a}] = \exp(-\mathbb{N}_0(1 - e^{-\mathcal{L}^a}))$ and the right-hand side is equal to $\exp(-\mathbb{N}_0(1 - e^{-\mathcal{L}^0}))$. So we only need to verify that $\mathbb{N}_0(1 - e^{-\mathcal{L}^a})$ tends to $\mathbb{N}_0(1 - e^{-\mathcal{L}^0})$ as $a \to 0$, which is easy by conditioning on σ and then using the bound $\mathbb{N}_0(1 - e^{-\mathcal{L}^a} \,|\, \sigma = s) \leqslant C(s^{3/4} \wedge 1)$ to justify dominated convergence.

Formula (7.17) follows from the case $a = 0$ of (7.18) as an immediate application of (7.15) and the exponential formula for Poisson measures. As for formula (7.16), it is enough to verify that

$$\mathbb{N}_0(1 - e^{-\lambda\mathcal{L}^a}) = \frac{3^{1/3}}{2}\Big(\lambda^{-1/3} + 3^{-1/3}\,|a|\Big)^{-2}. \tag{7.19}$$

Fix $a > 0$ for definiteness, and recall our notation $\mathcal{Z}_a$ for the total mass of the exit measure from $(-\infty, a)$. Write $(\omega'_j)_{j\in J}$ for the excursions of the Brownian snake outside $(-\infty, a)$. By the special Markov property, under $\mathbb{N}_0$ and conditionally on $\mathcal{Z}_a$, the point measure $\sum_{j\in J} \delta_{\omega'_j}$ is Poisson with intensity $\mathcal{Z}_a\, \mathbb{N}_a$. Moreover, the first part of the proof shows that we have $\mathbf{L}^a = \sum_{j\in J} \mathcal{L}^a(\omega'_j)$, $\mathbb{N}_0$ a.e., and therefore

$$\mathbb{N}_0(1 - e^{-\lambda\mathcal{L}^a}) = \mathbb{N}_0\Big(1 - \exp\Big(-\mathcal{Z}_a\mathbb{N}_0(1 - \exp(-\lambda\mathcal{L}^0))\Big)\Big).$$

Then (7.19) follows from (7.15) and (7.8). □

Remark. An alternative way to derive the previous two corollaries would be to use the known connections between super-Brownian motion or the Brownian snake and partial differential equations. See formula (1.13) in [90], and note that, as a function of a, the right-hand side of (7.19) solves the differential equation $\frac{1}{2}u'' = 2u^2 - \lambda\delta_0$ in the sense of distributions. On the other hand, our method provides a better probabilistic understanding of the results and the derivation of (7.15) in particular relies on Proposition 7.1 which is of independent interest and will play a significant role in the proofs of the next section.

7.4 The joint distribution of the local time and the time spent above and below 0

Our next goal is to discuss the joint distribution of $(\mathcal{L}^0, \sigma_+, \sigma_-)$ under $\mathbb{N}_0$, where we write

$$\sigma_+ := \int_0^\sigma \mathbf{1}_{\{\widehat{W}_s > 0\}} \, \mathrm{d}s \ , \quad \sigma_- := \int_0^\sigma \mathbf{1}_{\{\widehat{W}_s < 0\}} \, \mathrm{d}s.$$

Proposition 7.2. *Let $\lambda, \mu_1, \mu_2 \geqslant 0$, and consider the function $h_{\mu_1,\mu_2} : [0, \infty) \longrightarrow \mathbb{R}$ defined by*

$$h_{\mu_1,\mu_2}(v) = \sqrt{\sqrt{2\mu_1} + v} \left(2v - \sqrt{2\mu_1}\right) + \sqrt{\sqrt{2\mu_2} + v} \left(2v - \sqrt{2\mu_2}\right).$$

Then the quantity

$$v(\lambda, \mu_1, \mu_2) := \mathbb{N}_0(1 - \exp(-\lambda \mathcal{L}^0 - \mu_1 \sigma_+ - \mu_2 \sigma_-))$$

is the unique solution of the equation $h_{\mu_1,\mu_2}(v) = \sqrt{6}\, \lambda$.

Proof. First note that the quantities $v(\lambda, \mu_1, \mu_2)$ are finite, since $v(\lambda, \mu, \mu) \leqslant \mathbb{N}_0(1 - \exp(-\lambda \mathcal{L}^0)) + \mathbb{N}_0(1 - \exp(-\mu\sigma)) < \infty$ by (7.3) and (7.15). Then, suppose that, under the probability measure $\mathbb{P}$, we are given a sequence $(\eta_i)_{i \geqslant 0}$ of independent Bernoulli variables with parameter 1/2, and a sequence $(U_i)_{i \geqslant 0}$ of i.i.d. nonnegative random variables with density $(2\pi u^5)^{-1/2} \exp(-1/2u)$ for $u > 0$. We note that, for every $\beta > 0$, we have

$$\mathbb{E}[\exp(-\beta U_1)] = (1 + \sqrt{2\beta}) \, \exp(-\sqrt{2\beta}). \tag{7.20}$$

The reason for introducing these two sequences is the following fact. If $(t_i)_{i \geqslant 0}$ is a measurable enumeration of the jump times of the process $(\mathscr{X}_t)_{t \geqslant 0}$ (under $\mathbb{N}_0$), the conditional distribution of the pair (σ_+, σ_-) under $\mathbb{N}_0$ and knowing $(\mathscr{X}_t)_{t \geqslant 0}$ is the law of

$$\Big(\sum_{i=0}^{\infty} \eta_i \, U_i \, (\Delta \mathscr{X}_{t_i})^2, \sum_{i=0}^{\infty} (1 - \eta_i) \, U_i \, (\Delta \mathscr{X}_{t_i})^2 \Big).$$

This fact is a consequence of the excursion theory developed in [2] (in particular Theorem 4 and Proposition 31 of [2]). In this theory, excursions away from 0 are in one-to-one correspondence with the jumps of $(\mathscr{X}_t)_{t \geqslant 0}$, so that in the preceding display η_i gives the sign of the associated excursion ($\eta_i = 1$ for a positive excursion and $\eta_i = 0$ for a negative one), and $U_i \, (\Delta \mathscr{X}_{t_i})^2$ corresponds to the duration of this excursion. We refer to [2] for more details.

Using also Proposition 7.1 and (7.20), it follows that

$$\mathbb{N}_0\Big(\exp(-\lambda \mathcal{L}^0 - \mu_1 \sigma_+ - \mu_2 \sigma_-) \, \Big| \, (\mathscr{X}_t)_{t \geqslant 0} \Big) = \exp\Big(-\lambda \int_0^\infty \mathrm{d}t \, \mathscr{X}_t \Big) \prod_{i=0}^{\infty} F(\mu_1, \mu_2, (\Delta \mathscr{X}_{t_i})^2),$$

where we have set, for every $x > 0$,

$$F(\mu_1, \mu_2, x) := \frac{1}{2}\Big((1 + \sqrt{2\mu_1 x}) \, \exp(-\sqrt{2\mu_1 x}) + (1 + \sqrt{2\mu_2 x}) \, \exp(-\sqrt{2\mu_2 x}) \Big).$$

Hence, with the notation of the theorem, we have

$$v(\lambda,\mu_1,\mu_2)=\mathbb{N}_0\Big(1-\exp\Big(-\lambda\int_0^\infty \mathrm{d}t\,\mathscr{X}_t\Big)\prod_{i=0}^\infty F(\mu_1,\mu_2,(\Delta\mathscr{X}_{t_i})^2)\Big).$$

We now recall that the distribution of $(\mathscr{X}_t)_{t\geqslant 0}$ is the excursion measure of the ϕ-CSBP in order to rewrite this equality in a slightly different form. Suppose that $\sum_{k\in K}\delta_{\omega_k}$ is a Poisson point measure with intensity $\mathbb{N}_0$. The process $(X_t)_{t\geqslant 0}$ defined by $X_0=1$ and $X_t=\sum_{k\in K}\mathscr{X}_t(\omega_k)$ if $t>0$ is then a ϕ-CSBP started at 1. Furthermore, the exponential formula for Poisson measures and the last display immediately give

$$\mathbb{E}\Big[\exp\Big(-\lambda\int_0^\infty \mathrm{d}t\,X_t\Big)\prod_{j=0}^\infty F(\mu_1,\mu_2,(\Delta X_{s_j})^2)\Big]=\exp(-v(\lambda,\mu_1,\mu_2)) \tag{7.21}$$

where we have written $(s_j)_{j\geqslant 0}$ for a measurable enumeration of the jumps of X.

Let $t\geqslant 0$. Using the Markov property of X at time t, the left-hand side of (7.21) is also equal to

$$\mathbb{E}\Big[\Big(\exp\Big(-\lambda\int_0^t \mathrm{d}s\,X_s\Big)\prod_{j:s_j\leqslant t}F(\mu_1,\mu_2,(\Delta X_{s_j})^2)\Big)\exp(-v(\lambda,\mu_1,\mu_2)X_t)\Big]. \tag{7.22}$$

To simplify notation, we write $v=v(\lambda,\mu_1,\mu_2)$ in the following calculations, which are very similar to the proof of Proposition 4.8 in [40]. We also set, for every $s\geqslant 0$,

$$V_s:=\exp\Big(-\lambda\int_0^s \mathrm{d}u\,X_u\Big)\prod_{j:s_j\leqslant s}F(\mu_1,\mu_2,(\Delta X_{s_j})^2).$$

From the form of the generator of the ϕ-CSBP, we have

$$e^{-vX_t}=e^{-v}+M_t+\phi(v)\int_0^t X_s\,e^{-vX_s}\,\mathrm{d}s,$$

where $(M_s)_{s\geqslant 0}$ is a martingale, which is bounded on every compact time interval. By using the integration by parts formula as in [40, formula (28)], we get

$$e^{-vX_t}V_t=e^{-v}+\int_0^t V_{s-}\,\mathrm{d}M_s+\phi(v)\int_0^t V_sX_s\,e^{-vX_s}\,\mathrm{d}s+\int_0^t e^{-vX_s}\,\mathrm{d}V_s.$$

From (7.21) and (7.22), we have $\mathbb{E}[e^{-vX_t}V_t]=e^{-v}$. Hence, taking expectations in the last display, we obtain

$$\phi(v)\,\mathbb{E}\Big[\int_0^t V_sX_s\,e^{-vX_s}\,\mathrm{d}s\Big]=-\mathbb{E}\Big[\int_0^t e^{-vX_s}\,\mathrm{d}V_s\Big].$$

Next we observe that

$$\int_0^t e^{-vX_s}\,\mathrm{d}V_s=-\lambda\int_0^t V_sX_s\,e^{-vX_s}\,\mathrm{d}s+\sum_{j:s_j\leqslant t}e^{-vX_{s_j}}\,V_{s_j-}\,(F(\mu_1,\mu_2,(\Delta X_{s_j})^2)-1),$$

and so we get

$$(\phi(v)-\lambda)\,\mathbb{E}\Big[\int_0^t V_sX_s\,e^{-vX_s}\,\mathrm{d}s\Big]=-\mathbb{E}\Big[\sum_{j:s_j\leqslant t}e^{-vX_{s_j}}\,V_{s_j-}\,(F(\mu_1,\mu_2,(\Delta X_{s_j})^2)-1)\Big].$$

We multiply both sides of this identity by $1/t$ and let $t \downarrow 0$. We have first

$$\lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}\Big[\int_0^t V_s X_s\, e^{-vX_s}\, \mathrm{d}s\Big] = e^{-v}.$$

On the other hand, as a consequence of the classical Lamperti representation of continuous-state branching processes [61, 32], we know that the dual predictable projection of the random measure

$$\sum_{i=0}^{\infty} \delta_{(s_j, \Delta X_{s_j})}(\mathrm{d}s, \mathrm{d}x)$$

is the measure $X_s\, \mathrm{d}s\, \kappa(\mathrm{d}x)$, where $\kappa(\mathrm{d}x) = \sqrt{3/2\pi}\, x^{-5/2}\, \mathbf{1}_{\{x>0\}}\mathrm{d}x$ is the Lévy measure of the Lévy process appearing in the Lamperti representation of X. This implies that

$$\mathbb{E}\Big[\sum_{j:s_j\leqslant t} e^{-vX_{s_j}}\, V_{s_j-}\, (F(\mu_1,\mu_2,(\Delta X_{s_j})^2)-1)\Big] = \mathbb{E}\Big[\int_0^t \mathrm{d}s\, e^{-vX_s}\, V_s X_s \int \kappa(\mathrm{d}x)\, e^{-vx}(F(\mu_1,\mu_2,x^2)-1)\Big].$$

Consequently,

$$\lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}\Big[\sum_{j:s_j\leqslant t} e^{-vX_{s_j}}\, V_{s_j-}\, (F(\mu_1,\mu_2,(\Delta X_{s_j})^2)-1)\Big] = e^{-v} \int \kappa(\mathrm{d}x)\, e^{-vx}(F(\mu_1,\mu_2,x^2)-1).$$

Finally, we have obtained

$$\phi(v) - \lambda = -\int \kappa(\mathrm{d}x)\, e^{-vx}(F(\mu_1,\mu_2,x^2)-1).$$

Using the equality $\phi(v) = \int \kappa(\mathrm{d}x)(e^{-vx} - 1 + vx)$, straightforward calculations left to the reader show that

$$\phi(v) + \int \kappa(\mathrm{d}x)\, e^{-vx}(F(\mu_1,\mu_2,x^2)-1) = \frac{1}{\sqrt{6}}\, h_{\mu_1,\mu_2}(v),$$

where h_{μ_1,μ_2} is as in the statement. This proves that $v = v(\lambda,\mu_1,\mu_2)$ solves $h_{\mu_1,\mu_2}(v) = \sqrt{6}\,\lambda$. Uniqueness is clear since the function h_{μ_1,μ_2} is monotone increasing over $[0,\infty)$. □

Corollary 7.3. *For every $\lambda \geqslant 0$ and $\mu > 0$, we have*

$$\mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0 - \mu\sigma)\Big) = \begin{cases} \sqrt{2\mu}\, \cos\Big(\frac{2}{3}\,\arccos\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \leqslant 1,\\ \sqrt{2\mu}\, \cosh\Big(\frac{2}{3}\,\mathrm{arcosh}\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \geqslant 1.\end{cases}$$

Proof. Set $w(\lambda,\mu) = \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))$. Note that $w(\lambda,\mu) \geqslant \mathbb{N}_0(1-\exp(-\mu\sigma)) = \sqrt{\mu/2}$ by (7.3). It follows from Proposition 7.2 applied with $\mu_1 = \mu_2 = \mu$ that $w(\lambda,\mu)$ is the unique solution of the equation

$$4w^3 - 6\mu w + (2\mu)^{3/2} = \frac{3}{2}\lambda^2$$

in $[\sqrt{\mu/2},\infty)$ (note that the left-hand side is a monotone increasing function of w on $[\sqrt{\mu/2},\infty)$). Set $\widetilde{w}(\lambda,\mu) = w(\lambda,\mu)/\sqrt{2\mu}$ and $a = \sqrt{3}\,\lambda/(2(2\mu)^{3/4})$. We immediately get that $\widetilde{w}(\lambda,\mu)$ is the unique solution of

$$4\widetilde{w}^3 - 3\widetilde{w} + 1 = 2a^2$$

in $[1/2,\infty)$. A simple calculation now shows that

$$\widetilde{w} = \begin{cases} \cos(\frac{2}{3}\arccos(a)) & \text{if } a \leqslant 1, \\ \cosh(\frac{2}{3}\operatorname{arcosh}(a)) & \text{if } a \geqslant 1, \end{cases}$$

solves the preceding equation. This completes the proof. □

We can also derive an explicit formula for $\mathbb{N}_x(1-\exp(-\lambda\mathcal{L}^0 - \mu\sigma))$, for every $x \in \mathbb{R}$, from Corollary 7.3. Fix $x > 0$ for definiteness and argue under the measure $\mathbb{N}_x$. Write $T_0(\mathrm{w}) = \inf\{t \in [0,\zeta_{(\mathrm{w})}] : \mathrm{w}(t) = 0\}$ for any finite path w and define

$$\mathcal{Y}_0 = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{T_0(W_s)=\infty\}}.$$

Also let $(\omega_i)_{i\in I}$ be the excursions outside $(0,\infty)$ defined as in Section 7.2. Then, we have $\mathbb{N}_x$ a.e.

$$\sigma = \mathcal{Y}_0 + \sum_{i\in I}\sigma(\omega_i)\ , \quad \mathcal{L}^0 = \sum_{i\in I}\mathcal{L}^0(\omega_i)$$

where the second equality follows from the proof of Corollary 7.2. Using the special Markov property (with the fact that $\mathcal{Y}_0$ is $\mathcal{E}_{(0,\infty)}$-measurable), we get

$$\mathbb{N}_x\Big(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)\Big) = \mathbb{N}_x\Big(1-\exp\Big(-\mu\mathcal{Y}_0 - \mathcal{Z}_0\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))\Big). \quad (7.23)$$

On the other hand, Lemma 4.5 in [40] shows that, for every $\mu,\theta > 0$ such that $\theta \geqslant \sqrt{\mu/2}$,

$$\mathbb{N}_x(1-\exp(-\mu\mathcal{Y}_0 - \theta\mathcal{Z}_0)) = \sqrt{\frac{\mu}{2}}\left(3\left(\coth\left((2\mu)^{1/4}x + \coth^{-1}\sqrt{\frac{2}{3}+\frac{1}{3}\sqrt{\frac{2}{\mu}}\,\theta}\right)\right)^2 - 2\right) \quad (7.24)$$

with the convention that the right-hand side equals $\sqrt{\mu/2}$ if $\theta = \sqrt{\mu/2}$.

Taking $\theta = \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)) \geqslant \sqrt{\mu/2}$ in (7.24), using the formula of Corollary 7.3, then yields a (complicated but explicit) expression for $\mathbb{N}_x(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))$.

Corollary 7.4. *For every $\mu_1,\mu_2 \geqslant 0$, we have*

$$\mathbb{N}_0(1-\exp(-\mu_1\sigma_+ - \mu_2\sigma_-)) = \frac{\sqrt{2}}{3}\,\frac{\mu_1^{3/2}-\mu_2^{3/2}}{\mu_1-\mu_2},$$

with the convention that the right-hand side equals $\sqrt{\mu_1/2}$ if $\mu_1 = \mu_2$. The distribution of the pair (σ_+,σ_-) under $\mathbb{N}_0$ has density

$$g(s_1,s_2) := \frac{1}{2\sqrt{2\pi}}\,(s_1+s_2)^{-5/2}$$

with respect to Lebesgue measure on $(0,\infty)^2$. In particular, the distribution of σ_+ (or of σ_-) under $\mathbb{N}_0$ has density $(3\sqrt{2\pi})^{-1}\,s^{-3/2}$ on $(0,\infty)$.

The form of the density $g(s_1, s_2)$ shows that the conditional distribution of σ_+ knowing that $\sigma = s$ is uniform over $[0, s]$. This is a well-known fact, which can be derived from the invariance of the CRT under uniform re-rooting (see e.g. [10, Section 3.2]).

Proof. The formula for $\mathbb{N}_0(1 - \exp(-\mu_1\sigma_+ - \mu_2\sigma_-))$ is obtained by solving the equation $h_{\mu_1,\mu_2}(v) = 0$. We can then verify that the function g satisfies

$$\int_0^\infty \int_0^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, g(s_1, s_2)\,(1 - e^{-\mu_1 s_1 - \mu_2 s_2}) = \frac{\sqrt{2}}{3}\,\frac{\mu_1^{3/2} - \mu_2^{3/2}}{\mu_1 - \mu_2},$$

which gives the second assertion. □

We finally give an application to super-Brownian motion in the spirit of Corollary 7.2.

Corollary 7.5. *Let* $\mathbf{X}$ *be a one-dimensional super-Brownian motion with branching mechanism* $\psi(u) = 2u^2$*, such that* $\mathbf{X}_0 = \alpha\delta_0$*. Set*

$$\mathbf{R}_+ = \int_0^\infty \mathrm{d}t\, \langle \mathbf{X}_t, \mathbf{1}_{[0,\infty)} \rangle\,, \quad \mathbf{R}_- = \int_0^\infty \mathrm{d}t\, \langle \mathbf{X}_t, \mathbf{1}_{(-\infty,0]} \rangle.$$

Then, for every $\mu_1, \mu_2 > 0$,

$$\mathbb{E}[\exp(-\mu_1\mathbf{R}_+ - \mu_2\mathbf{R}_-)] = \exp\Big(-\frac{\alpha\sqrt{2}}{3}\,\frac{\mu_1^{3/2} - \mu_2^{3/2}}{\mu_1 - \mu_2}\Big).$$

Given Corollary 7.4, the proof of Corollary 7.5 is an immediate application of the Brownian snake construction of super-Brownian motion along the lines of the proof of Corollary 7.2.

7.5 Conditional distributions of the local time at 0

We will now use the preceding results to recover the conditional distribution of $\mathcal{L}^0$ given σ, which was first obtained by Bousquet-Mélou and Janson [25] with a very different method.

Theorem 7.1. *Let* $s > 0$*. Under the probability measure* $\mathbb{N}_0(\cdot \mid \sigma = s)$*, the local time* $\mathcal{L}^0$ *is distributed as* $(2^{3/4}/3)\, s^{3/4}\, T^{-1/2}$*, where* T *is a positive stable variable with index* $2/3$*, whose Laplace transform is* $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{2/3})$*.*

Remark. In Corollary 3.4 of [25], the constant $2^{3/4}/3$ is replaced by $2^{1/4}/3$. This is due to a different normalization: In [25] (as in [10]) the random function coding the genealogy of ISE is twice the Brownian excursion, and it follows that our random variable $\mathcal{L}^0$ is distributed under $\mathbb{N}_0(\cdot \mid \sigma = 1)$ as $\sqrt{2}$ times the quantity $f_{\mathrm{ISE}}(0)$ considered in [25].

The occurence of a stable variable with index $2/3$ in Theorem 7.1 is of course reminiscent of Corollary 7.2 above. It would be very interesting to establish a direct connection between this corollary and Theorem 7.1.

Proof. From the scaling properties of the end of Section 7.2, it is enough to treat the case $s = 1$. Recall the notation $v(\lambda, \mu_1, \mu_2)$ in Proposition 7.2. For every $\lambda \geqslant 0$, set

$$F(\lambda) := 2\,\mathbb{N}_0\Big(e^{-\sigma/2}(1 - e^{-\lambda\mathcal{L}^0})\Big) = 2\,v(\lambda, \frac{1}{2}, \frac{1}{2}) - 1,$$

where the second equality holds because $\mathbb{N}_0(1 - \exp(-\sigma/2)) = 1/2$. The function F is continuous and vanishes at 0. As a straightforward consequence of Proposition 7.2, we have for every $\lambda \geqslant 0$,

$$F(\lambda) = \lambda\sqrt{\frac{3}{3 + F(\lambda)}}. \tag{7.25}$$

In particular, the right derivative of F at 0 is 1, and consequently $\mathbb{N}_0(\mathcal{L}^0 \exp(-\sigma/2)) = 1/2$. The fact that $\mathbb{N}_0(\mathcal{L}^0 \exp(-\sigma/2))$ is finite allows us to make sense of $F(\lambda)$ for every $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geqslant 0$, and the restriction of F to $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ is analytic.

Set $\psi(z) = \sqrt{3/(3+z)}$ so that ψ is analytic on a neighborhood of 0 in $\mathbb{C}$. Since $\psi(0) \neq 0$, we can find an analytic function G defined on a neighborhood of 0 such that $z\psi(G(z)) = G(z)$ for $|z|$ small enough. By (7.25), we must have $F(z) = G(z)$ for $\mathrm{Re}(z) > 0$ and $|z|$ small, and this means that F can be extended to an analytic function on a neighborhood of 0. By the Lagrange inversion theorem, we have then, for every integer $n \geqslant 1$,

$$[z^n]F(z) = \frac{1}{n}[z^{n-1}]\psi(z)^n = \frac{3^{n/2}}{n!}\,\frac{\mathrm{d}^{n-1}(3+z)^{-n/2}}{\mathrm{d}z^{n-1}}\Big|_{z\,=\,0} = \frac{(-1)^{n-1}3^{1-n}}{n!}\,\frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{n}{2})},$$

using the standard notation $[z^n]F(z)$ for the coefficient of z^n in the series expansion of $F(z)$ near 0. On the other hand, the fact that the function $z \mapsto F(z)$ is analytic in a neighborhood of 0 implies that all moments $\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2})$, $n \geqslant 1$, are finite and given by

$$\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2}) = \frac{1}{2}\,(-1)^{n-1} n! \times [z^n]F(z) = \frac{1}{2}\,3^{1-n}\,\frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{n}{2})}. \tag{7.26}$$

To complete the proof, we use a scaling argument. We recall that the distribution of $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = s)$ coincides with the distribution of $s^{3/4}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = 1)$. It follows that

$$\begin{aligned}\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2}) &= \int_0^\infty \frac{\mathrm{d}s}{2\sqrt{2\pi s^3}}\, e^{-s/2}\mathbb{N}_0\Big(s^{3n/4}(\mathcal{L}^0)^n \,\Big|\, \sigma = 1\Big)\\ &= \frac{2^{\frac{3n}{4}-2}}{\sqrt{\pi}}\,\Gamma(\frac{3n}{4} - \frac{1}{2}) \times \mathbb{N}_0\Big((\mathcal{L}^0)^n \,\Big|\, \sigma = 1\Big).\end{aligned}$$

By combining the last two displays and using the duplication formula for the Gamma function, we arrive at

$$\mathbb{N}_0\Big((\mathcal{L}^0)^n \,\Big|\, \sigma = 1\Big) = \frac{\sqrt{\pi}\,3^{1-n}}{2^{\frac{3n}{4}-1}}\,\frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{n}{2})\Gamma(\frac{3n}{4} - \frac{1}{2})} = \frac{2^{\frac{3n}{4}}}{3^n}\,\frac{\Gamma(\frac{3n}{4} + 1)}{\Gamma(\frac{n}{2} + 1)} = \Big(\frac{2^{3/4}}{3}\Big)^n \mathbb{E}[T^{-n/2}],$$

where T is as in the theorem. To check the last equality, the reader can write

$$T^{-n/2} = (\Gamma(n/2))^{-1}\int_0^\infty \mathrm{d}s\, s^{n/2-1}e^{-sT}.$$

The growth of the moments of the distribution of $T^{-1/2}$ ensures that this distribution is characterized by its moments, which completes the proof. □

Remark. Rather than using the Lagrange inversion theorem, we could have derived formula (7.26) for the moments $\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2})$ from a series expansion of the quantity $\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\sigma/2))$ as given in Corollary 7.3. This would still have required some calculations. We preferred to use the previous method because it also serves as a prototype for the proof of the (more delicate) Theorem 7.2 below.

Proposition 7.2 can also be used to derive the conditional distribution of $\mathcal{L}^0$ given σ_+. Perhaps surprisingly, this distribution turns out again to be remarkably simple.

Theorem 7.2. *Let $s>0$. Under the probability measure $\mathbb{N}_0(\cdot \mid \sigma_+ = s)$, the local time $\mathcal{L}^0$ is distributed as $(2^{9/4}/3)\, s^{3/4}\, D\, T^{-1/2}$, where the random variables D and T are independent, T is a positive stable variable with index $2/3$, whose Laplace transform is $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{2/3})$, and D has density $2x\,\mathbf{1}_{[0,1]}(x)$ with respect to Lebesgue measure on $\mathbb{R}_+$.*

Proof. It is enough to treat the case $s=1$. For every $\lambda \geqslant 0$, set

$$F_+(\lambda) := \mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0 - \frac{1}{2}\sigma_+)\Big) = v(\lambda,\frac{1}{2},0)$$

with the notation of Proposition 7.2. As for Theorem 7.1, the strategy of the proof is to compute the coefficient $[\lambda^n]F_+(\lambda)$ in two different ways. Unfortunately, the details of the argument are more involved than in the proof of Theorem 7.1.

By Proposition 7.2, we have

$$(2F_+(\lambda)-1)\sqrt{F_+(\lambda)+1} + 2\,F_+(\lambda)^{3/2} = \sqrt{6}\,\lambda.$$

We cannot apply directly the Lagrange inversion theorem, but the idea will be to find a rational parametrization of the preceding equation (see e.g. [22, Section 3]). It follows from the last display that we have $P(F_+(\lambda),\lambda)=0$, where

$$P(y,z) = 96\,y^3z^2 - 36\,z^4 - 36\,yz^2 + 12\,z^2 - 9\,y^2 + 6\,y - 1, \qquad y,z \in \mathbb{C}.$$

We now introduce[1] the rational functions

$$Q(z) = -\frac{1}{124416}\,z^3 + \frac{1}{48}\,z, \qquad R(z) = \frac{1}{3456}\,z^2 - \frac{1}{2} + \frac{216}{z^2},$$

which satisfy $P(R(z),Q(z)) = 0$ for every $z \in \mathbb{C}\backslash\{0\}$. We have $Q^{-1}(0) = \{-36\sqrt{2}, 0, 36\sqrt{2}\}$, and the derivative Q' does not vanish on $Q^{-1}(0)$. It follows that we can find $r_0>0$ and three analytic functions $\gamma_1,\gamma_2,\gamma_3$ defined on the disk $\mathbb{D}_{r_0} = \{z\in\mathbb{C} : |z|<r_0\}$ and with disjoint ranges, such that $\gamma_1(0) = -36\sqrt{2}$, $\gamma_2(0)=0$, $\gamma_3(0) = 36\sqrt{2}$ and for every $z\in\mathbb{D}_{r_0}$, $Q^{-1}(z) = \{\gamma_1(z),\gamma_2(z),\gamma_3(z)\}$. Note that $R(\gamma_1(0)) = 1/3 = R(\gamma_3(0))$ and $R'(\gamma_1(0)) = -\sqrt{2}/54 = -R'(\gamma_3(0))$. Also the fact that $Q(\gamma_i(z)) = z$ readily implies that $\gamma_1'(0) = \gamma_2'(0) = -24$.

Since $P(R(z),Q(z)) = 0$ for every $z\in\mathbb{C}\backslash\{0\}$, we get that $P(R(\gamma_i(z)),z) = 0$ for every $i\in\{1,2,3\}$ and $z\in\mathbb{D}_{r_0}\backslash\{0\}$. We claim that $F_+(\lambda) = R(\gamma_1(\lambda))$ for $\lambda>0$ small enough. To

[1]The functions Q and ψ have been found using the Maple package *algcurve*

see this, observe that for $z \neq 0$ and $|z|$ small enough, then the quantities $R(\gamma_i(z))$, $i \in \{1,2,3\}$, are distinct. Indeed, since $|R(y)| \longrightarrow \infty$ as $|y| \to 0$ it is clear that $R(\gamma_2(z))$ is distinct from $R(\gamma_1(z))$ and $R(\gamma_3(z))$ when $|z|$ is small, and on the other hand, the properties $\gamma_1'(0) = \gamma_3'(0) \neq 0$ and $R'(\gamma_1(0)) = -R'(\gamma_3(0)) \neq 0$ imply that $R(\gamma_1(z)) \neq R(\gamma_3(z))$ when $|z|$ is small. Hence, for $z \neq 0$ and $|z|$ small enough, the numbers $R(\gamma_i(z))$, $i \in \{1,2,3\}$, are three distinct roots of $P(y,z)$ viewed as a polynomial of degree 3 in y. Since we know that $P(F_+(\lambda),\lambda) = 0$, it follows that $F_+(\lambda) \in \{R(\gamma_1(\lambda)), R(\gamma_2(\lambda)), R(\gamma_3(\lambda))\}$ for $\lambda > 0$ small. The case $F_+(\lambda) = R(\gamma_2(\lambda))$ is clearly excluded for λ small, and since $F_+(\lambda)$ is a monotone increasing function of λ, noting that $\gamma_1'(0)R'(\gamma_1(0)) > 0$ whereas $\gamma_3'(0)R'(\gamma_3(0)) < 0$, we get our claim $F_+(\lambda) = R(\gamma_1(\lambda))$ for $\lambda > 0$ small.

In particular, we can extend F_+ to an analytic function in the neighborhood of 0, and we will then use the Lagrange inversion theorem to determine the coefficients of the Taylor expansion of F_+. To simplify notation, we set $\widetilde{F}_+(\lambda) = F_+(\lambda) - 1/3$, $\tilde{\gamma}(z) = \gamma_1(z) + 36\sqrt{2}$ and for every $\lambda \geqslant 0$,

$$\widetilde{R}(\lambda) = R(\lambda - 36\sqrt{2}) - 1/3.$$

Then, for $\lambda > 0$ small, we have

$$\widetilde{F}_+(\lambda) = F_+(\lambda) - \frac{1}{3} = R(\gamma_1(\lambda)) - \frac{1}{3} = \widetilde{R}(\tilde{\gamma}(\lambda)). \tag{7.27}$$

On the other hand, the property $Q(\gamma_1(z)) = z$ for $|z| < r_0$ shows that

$$\tilde{\gamma}(\lambda) = \lambda\, \widetilde{\psi}(\tilde{\gamma}(\lambda)), \tag{7.28}$$

with

$$\widetilde{\psi}(\lambda) = -\frac{124416}{(36\sqrt{2} - \lambda)(72\sqrt{2} - \lambda)}.$$

By (7.27), (7.28) and the Lagrange inversion theorem, we get for every $n \geqslant 1$,

$$[\lambda^n]F_+(\lambda) = [\lambda^n]\widetilde{F}_+(\lambda) = \frac{1}{n}[\lambda^{n-1}](\widetilde{R}'(\lambda)\widetilde{\psi}(\lambda)^n).$$

Note that

$$\begin{aligned}
\widetilde{R}'(72\sqrt{2}\lambda) &= R'(72\sqrt{2}(\lambda - 1)) = -\frac{\sqrt{2}}{48}(1 - 2\lambda) + \frac{1}{216\sqrt{2}}(1 - 2\lambda)^{-3} \\
\widetilde{\psi}(72\sqrt{2}\lambda) &= -\frac{24}{(1 - \lambda)(1 - 2\lambda)},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&[\lambda^{n-1}](\widetilde{R}'(72\sqrt{2}\lambda)\widetilde{\psi}(72\sqrt{2}\lambda)^n) \\
&= (-24)^n \times \left(\Big(\frac{-\sqrt{2}}{48}[\lambda^{n-1}]\Big((1 - 2\lambda)^{-n+1}(1 - \lambda)^{-n}\Big) + \frac{1}{216\sqrt{2}}[\lambda^{n-1}]\Big((1 - 2\lambda)^{-n-3}(1 - \lambda)^{-n}\Big)\right),
\end{aligned}$$

and finally

$$[\lambda^n]F_+(\lambda) = \frac{(-1)^n}{n}(3\sqrt{2})^{-n}\Big(-3[\lambda^{n-1}]\Big((1-2\lambda)^{-n+1}(1-\lambda)^{-n}\Big) + \frac{1}{3}[\lambda^{n-1}]\Big((1-2\lambda)^{-n-3}(1-\lambda)^{-n}\Big)\Big). \tag{7.29}$$

To compute the right-hand side, we observe that, for every integers $m \geqslant 0$, $k \geqslant 1$ and $\ell \geqslant 1$, we have

$$[\lambda^m](1-2\lambda)^{-k}(1-\lambda)^{-\ell} = 2^m\binom{m+k-1}{m}\,{}_2F_1(-m,\ell;-m-k+1;\frac{1}{2}),$$

where ${}_2F_1$ stands for the Gauss hypergeometric function. This equality is easily checked by a direct calculation, noting that the hypergeometric series reduces to a finite sum in the case we are considering It follows that, for every $n \geqslant 2$,

$$[\lambda^{n-1}]\Big((1-2\lambda)^{-n+1}(1-\lambda)^{-n}\Big) = 2^{n-1}\binom{2n-3}{n-1}\,{}_2F_1(-n+1,n;-2n+3;\frac{1}{2}) \tag{7.30}$$

$$[\lambda^{n-1}]\Big((1-2\lambda)^{-n-3}(1-\lambda)^{-n}\Big) = 2^{n-1}\binom{2n+1}{n-1}\,{}_2F_1(-n+1,n;-2n-1;\frac{1}{2}). \tag{7.31}$$

Fortunately, Bailey's theorem (see [12, Theorem 3.5.4 (ii)]) gives an explicit formula for ${}_2F_1(a,1-a;b;\frac{1}{2})$ in terms of a ratio of products of values of the Gamma function, which we can apply here. Using also Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to eliminate the poles of the Gamma function, we arrive at

$$\begin{aligned}
{}_2F_1(-n+1,n;-2n+3;\frac{1}{2}) &= \frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n-\frac{1}{2})\,\Gamma(n-1)} = \frac{2^{2n-3}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(2n-2)}\\
{}_2F_1(-n+1,n;-2n-1;\frac{1}{2}) &= \frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(n+\frac{3}{2})\,\Gamma(n+1)} = \frac{2^{2n+1}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(2n+2)},
\end{aligned}$$

where we applied the duplication formula for the Gamma function, and we recall that we assume $n \geqslant 2$. Using (7.30) and (7.31), we get from (7.29) that

$$\begin{aligned}
[\lambda^n]F_+(\lambda) &= \frac{(-1)^{n+1}}{n!}(3\sqrt{2})^{-n}\frac{2^{3n}}{\sqrt{\pi}}\Big(\frac{3}{16}\,\frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n-1)} - \frac{1}{3}\,\frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(n+3)}\Big)\\
&= \frac{(-1)^{n+1}}{n!}(3\sqrt{2})^{-n}\frac{2^{3n}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}+\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n)}\Big(\frac{3}{8} - \frac{1}{3}\times\frac{(\frac{3n}{2}-1)(\frac{n}{2}+\frac{1}{2})\frac{3n}{2}}{(n+2)(n+1)n}\Big)\\
&= \frac{(-1)^{n+1}}{n!}(3\sqrt{2})^{-n}\frac{2^{3n}}{\sqrt{\pi}}\,\frac{1}{n+2}\,\frac{\Gamma(\frac{n}{2}+\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n)}\\
&= \frac{(-1)^{n+1}}{n!}(3\sqrt{2})^{-n}\,2^{2n+1}\,\frac{1}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}.
\end{aligned}$$

We have assumed $n \geqslant 2$, but a direct calculation from (7.29) shows that the last line of the preceding display also gives the correct value $[\lambda]F_+(\lambda) = 4\sqrt{2}/9$ for $n = 1$. Similarly as in the proof of Theorem 7.1, we conclude that, for every $n \geqslant 1$,

$$\mathbb{N}_0\Big((\mathcal{L}^0)^n\,e^{-\sigma_+/2}\Big) = \Big(\frac{2\sqrt{2}}{3}\Big)^n\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}.$$

On the other hand, the same scaling argument as in the proof of Theorem 7.1 (using now the fact that the density of σ_+ under $\mathbb{N}_0$ is $(3\sqrt{2\pi})^{-1}s^{-3/2}$) gives

$$\begin{aligned}\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma_+/2}) &= \int_0^\infty \frac{\mathrm{d}s}{3\sqrt{2\pi s^3}}\, e^{-s/2}\mathbb{N}_0\Big(s^{3n/4}(\mathcal{L}^0)^n \,\Big|\, \sigma_+ = 1\Big)\\ &= \frac{2^{\frac{3n}{4}-1}}{3\sqrt{\pi}}\,\Gamma(\frac{3n}{4}-\frac{1}{2})\,\mathbb{N}_0\Big((\mathcal{L}^0)^n \,\Big|\, \sigma_+ = 1\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{N}_0\Big((\mathcal{L}^0)^n \,\Big|\, \sigma_+ = 1\Big) &= 3\sqrt{\pi}\Big(\frac{2\sqrt{2}}{3}\Big)^n 2^{-\frac{3n}{4}+1}\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})\Gamma(\frac{3n}{4}-\frac{1}{2})}\\ &= \Big(\frac{2^{9/4}}{3}\Big)^n\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{4}+1)}{\Gamma(\frac{n}{2}+1)}.\end{aligned}$$

The right-hand side is the n-th moment of $(2^{9/4}/3)\,D\,T^{-1/2}$, where the pair (D,T) is as in the theorem. This completes the proof. □

Interpretation in random geometry. We now explain briefly how both theorems of this section can be interpreted in the setting of continuous models of random geometry. It is best to start with the discrete picture of planar quadrangulations. For every integer $n \geqslant 1$, let Q_n be a uniformly distributed rooted and pointed quadrangulation with n faces. The fact that Q_n is pointed means that (in addition to the root edge) there is a distinguished vertex denoted by ∂. Write dg for the graph distance on the vertex set $V(Q_n)$ of Q_n. The Schaeffer bijection (see e.g. [74, Section 5]) allows us to code Q_n by a uniformly distributed labeled tree with n edges, which we denote by T_n, and a sign $\varepsilon_n \in \{-1,1\}$. Here a labeled tree is a (rooted) plane tree whose vertices are assigned integer labels ℓ_v, in such a way that the label of the root vertex ρ of the tree is $\ell_\rho = 0$ and the labels of two adjacent vertices differ by at most 1 in absolute value. Furthermore the set $V(Q_n)\backslash\{\partial\}$ is canonically identified with $V(T_n)$, where $V(T_n)$ denotes the vertex set of T_n. Through this identification, the graph distance $\mathrm{dg}(\partial,v)$ between ∂ and another vertex v of Q_n can be expressed as $\ell_v - \min\{\ell_w : w \in V(T_n)\} + 1$. Now consider the set $S_n = \{v \in V(Q_n) : \mathrm{dg}(\partial,v) = \mathrm{dg}(\partial,\rho)\}$ of all vertices v of Q_n that are at the same distance as ρ from the distinguished vertex ∂ (here we view ρ as a vertex of Q_n thanks to the preceding identification). From the previous observations, S_n is identified to $\{v \in V(T_n) : \ell_v = 0\}$. It then follows from [25, Theorem 3.6] that the distribution of $n^{-3/4}\#S_n$ converges as $n \to \infty$ to the distribution of $2^{-1/4}3^{-1/2}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \,|\, \sigma = 1)$, which is given in Theorem 7.1.

Consider then the (standard) Brownian map $(\mathbf{m}, D)$. This is a random compact metric space that can be constructed from Brownian motion indexed by the Brownian tree, which we denote here by $(V_a)_{a\in\mathcal{T}_\zeta}$ as in Section 7.1 above, under the probability measure $\mathbb{N}_0(\cdot \,|\, \sigma = 1)$ – see e.g. the introduction of [67] for details. In this construction, the space $\mathbf{m}$ is obtained as a quotient space of $\mathcal{T}_\zeta$, and comes with two distinguished points, namely the point ρ corresponding to the root of $\mathcal{T}_\zeta$, and another point denoted by x_* in [67], which corresponds to the point of $\mathcal{T}_\zeta$ where V_a achieves its minimum. Note that ρ and x_* can be viewed as independently and uniformly distributed on

$\mathbf{m}$. The “sphere” $\{x \in \mathbf{m} : D(x_*, x) = D(x_*, \rho)\}$ then corresponds to $\{a \in \mathcal{T}_\zeta : V_a = 0\}$, and so the local time $\mathcal{L}^0$ is naturally interpreted as the “measure” of this sphere (here the word measure should refer to a suitable Hausdorff measure, although this has not been justified rigorously). This interpretation is made very plausible by the discrete result for quadrangulations described above.

To get a similar interpretation for Theorem 7.2, we consider the free Brownian map (M, Δ), which is the scaling limit of quadrangulations distributed according to Boltzmann weights and can again be constructed from Brownian motion indexed by the Brownian tree, but now under the σ-finite measure $\mathbb{N}_0$ (see e.g. [71, Section 3]). As in the case of the standard Brownian map, the space M is defined as a quotient space of $\mathcal{T}_\zeta$ and comes with two distinguished points denoted by ρ and x_*. Furthermore, the sphere $\{x \in M : \Delta(x_*, x) = \Delta(x_*, \rho)\}$ corresponds to $\{a \in \mathcal{T}_\zeta : V_a = 0\}$, and the ball $\{x \in M : \Delta(x_*, x) \leqslant \Delta(x_*, \rho)\}$ corresponds to $\{a \in \mathcal{T}_\zeta : V_a \leqslant 0\}$. So Theorem 7.2 can be viewed as providing the conditional distribution of the measure of the sphere $\{x \in M : \Delta(x_*, x) = \Delta(x_*, \rho)\}$ given the volume of the ball it encloses.

Chapitre 8

The free Brownian disk with two distinguished boundary points

Les résultats de ce chapitre sont issus d'un travail en cours. Nous mentionnons que ce projet est encore dans une forme préliminaire. Il est de fait conçu davantage comme une présentation de résultats à venir que comme un article à part entière.

Nous étudierons une version du disque brownien, muni de deux points distingués sur le bord, avec volume et périmètre aléatoires. Ce chapitre sera divisé en deux parties. La première partie sera courte et présentera des résultats sur les cellules de Voronoï. La seconde partie sera quant à elle plus aboutie et portera sur les hulls par rapport à un point du bord dans le disque brownien.

Contents

8.1 Introduction

In recent years, the theory of scaling limits of random planar maps has seen many spectacular developments. The central object of this theory is the Brownian sphere [1, 4, 21, 67, 82, 85] which can be obtained as the scaling limit of random planar maps with small faces and without boundaries. Other models of random geometry related to the Brownian sphere have appeared recently [14, 39, 40, 79]. In this work we will focus on Brownian disks, which are random metric spaces homeomorphic to the closed disk of the complex plane. In particular, we can define the *boundary* of a Brownian disk as the set of all points that do not have a neighborhood homeomorphic to the open disk. Brownian disks can be constructed as scaling limits of random planar maps with small faces and with one boundary (see in particular [14, 22]), but they can also be obtained as special subsets of the Brownian sphere [71]. The goal of this work is to study special subsets of Brownian disks, using constructions related to labeled trees. More precisely, we are interested in Voronoï cells and hulls of Brownian disks. In particular, we will prove a special Markov property for Brownian disks when one explores them in a metric way from a point of the boundary. We will also obtain explicit formulas concerning these special subsets. We chose to gather these two questions since the techniques used to study them are very similar. Let us now give an overview of our results.

We are interested in the model of *free Brownian disks rooted on the boundary*. This is a version of the classical free Brownian disks [22, 71] but pointed at a uniform point of the boundary (as in [72]). Free Brownian disks rooted on the boundary form a one parameter family indexed by a positive number $z \in (0, \infty)$, which represents the boundary length or perimeter. The free Brownian disk rooted on the boundary with perimeter $z > 0$ is written in the form $(\mathbb{D}^z, \rho, \Delta^{\mathrm{d}}, \mathrm{Vol}^{\mathrm{d}})$, where $(\mathbb{D}^z, \Delta^{\mathrm{d}})$ is a compact metric space, ρ is a distinguished point of the boundary of $\mathbb{D}^z$ called the root and $\mathrm{Vol}^{\mathrm{d}}$ is a measure of finite mass supported on $\mathbb{D}^z$ called the volume measure. To simplify notation, we will write $\mathbb{D}^z$ for $(\mathbb{D}^z, \rho, \Delta^{\mathrm{d}}, \mathrm{Vol}^{\mathrm{d}})$ and we see it as a random element of $\mathbb{K}$, which is the set of all isometry classes of weighted pointed metric compact spaces equipped with the Gromov-Hausdorff-Prokhorov distance. The construction of $\mathbb{D}^z$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is recalled in Section 8.2.2. The total mass of $\mathrm{Vol}^{\mathrm{d}}$ i.e. the quantity $\mathrm{Vol}^{\mathrm{d}}(\mathbb{D}^z)$ is a random variable and its distribution is:

$$\mathbb{1}_{v\geqslant 0}\frac{1}{\sqrt{2\pi}}z^3v^{-\frac{5}{2}}\exp(-\frac{z}{2v})dv \ , \tag{8.1}$$

see [71, Theorem 1] and [2, Proposition 31]. We denote the boundary of $\mathbb{D}^z$ by $\partial\mathbb{D}^z$. In [71, Section 10] the volume measure $\mathrm{Vol}^{\mathrm{d}}$ is used to define a natural "uniform" measure $\mathrm{m}_{\mathbb{D}^z}$ supported on $\partial\mathbb{D}^z$ as follows:

$$< \mathrm{m}_{\mathbb{D}^z}, \phi >:= \lim_{\varepsilon\to 0}\varepsilon^{-2}\int_{\mathbb{D}^z}\mathrm{Vol}^{\mathrm{d}}(dx)\ \phi(x)\mathbb{1}_{\Delta^{\mathrm{d}}(x,\partial\mathbb{D}^z)<\varepsilon}\ , \tag{8.2}$$

where the limit holds, a.s., simultaneously for every continuous function $\phi : \mathbb{D}^z \to \mathbb{R}_+$. One can recover the boundary length by the formula $< \mathrm{m}_{\mathbb{D}^z}, 1 >= z$ a.s. We refer to [71, Corollary 37] for more details. Furthermore, the root ρ is a uniform point for the measure $\mathrm{m}_{\mathbb{D}^z}$. If we forget the root ρ, the space $\mathbb{D}^z$ is distributed as a free Brownian disk. These models benefit from the following scale invariance property: for every $\lambda > 0$, the space $(\mathbb{D}^z, \rho, \lambda\Delta^{\mathrm{d}}, \lambda^4\mathrm{Vol}^{\mathrm{d}})$ is a free Brownian disk rooted on the boundary with perimeter $\lambda^2 z$. It will be useful to introduce the measure $\mathbb{N}^*$ defined by the following relation:

$$\mathbb{N}^*(F(\mathbb{D})) := \sqrt{\frac{3}{2\pi}}\int_0^\infty dz\ z^{-\frac{5}{2}}\mathbb{E}[F(\mathbb{D}^z)]\ ,$$

for every measurable function $F : \mathbb{K} \mapsto \mathbb{R}_+$. Note that the measure $\mathbb{N}^*$ is no longer a finite measure but a σ-finite measure. Furthermore, formula (8.2) allows us to extend the definition of the uniform measure under $\mathbb{N}^*$ and we denote it by $\mathrm{m}_{\mathbb{D}}$. To simplify notation we write:

$$|\partial\mathbb{D}| =< \mathrm{m}_{\mathbb{D}}, 1 >$$

for the boundary length of $\mathbb{D}$. Under $\mathbb{N}^*$, since the boundary length is no longer fixed, we will call the space $\mathbb{D}$: *Brownian disk rooted on the boundary with free volume and free perimeter.* The variable $\mathbb{D}$ lives in the set $\mathbb{K}^\bullet := \{(\mathbb{M}, x) :\ \mathbb{M} \in \mathbb{K} \text{ and } x \in \mathbb{M}\}$, which we equip with the associated Gromov-Hausdorff-Prokhorov distance (see [71, Section 2.1] for more details). We now define a marked version of $\mathbb{N}^*$ by the relation:

$$\mathbb{N}^{*,\bullet}(F(\mathbb{D}^\bullet)) := \mathbb{N}^*\big(\int_{\partial\mathbb{D}}\mathrm{m}_{\mathbb{D}}(dx)F(\mathbb{D},x)\big) = \sqrt{\frac{3}{2\pi}}\int_0^\infty dz\ z^{-\frac{5}{2}}\mathbb{E}\big[\int_{\partial\mathbb{D}^z}\mathrm{m}_{\mathbb{D}^z}(dx)F(\mathbb{D}^z,x)\big]\ ,$$

for every measurable function $F : \mathbb{K}^\bullet \mapsto \mathbb{R}_+$. Under $\mathbb{N}^{*,\bullet}$, the variable $\mathbb{D}^\bullet$ can be understood as a Brownian disk $\mathbb{D}$, biased by the boundary length $|\partial\mathbb{D}|$ and with a uniform marked point $\rho^\bullet$ on the boundary $\partial\mathbb{D}$. To simplify notation, we write $\mathbb{D}^\bullet := (\mathbb{D}, \rho^\bullet)$ and $\partial\mathbb{D}^\bullet = \partial\mathbb{D}$. In particular, we can see that the distribution of $|\partial\mathbb{D}^\bullet|$, under $\mathbb{N}^{*,\bullet}$, is $\sqrt{3/2\pi}\cdot z^{-\frac{3}{2}}dz$. We call $\rho^\bullet$ the marked point of $\mathbb{D}^\bullet$. Remark that the two distinguished points ρ and $\rho^\bullet$ are in $\partial\mathbb{D}^\bullet$.

Let us describe the two classes of subsets of $\mathbb{D}^\bullet$ that we are interested in:

Voronoï cells. First, we will study Voronoï cells with respect to the boundary of $\mathbb{D}^\bullet$. More precisely, by removing the two distinguished points ρ and $\rho^\bullet$, the boundary $\partial\mathbb{D}^\bullet$ becomes the disjoint union of two sets, $\mathcal{L}_1^\circ$ and $\mathcal{L}_2^\circ$, both homeomorphic to the open interval $(0,1)$. Write $\mathcal{L}_1$ and $\mathcal{L}_2$ to denote the respective closures of $\mathcal{L}_1^\circ$ and $\mathcal{L}_2^\circ$. Let $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) be the set of points $x \in \mathbb{D}^\bullet$ such that $\Delta^{\mathrm{d}}(x, \mathcal{L}_1) = \Delta^{\mathrm{d}}(x, \partial\mathbb{D}^\bullet)$ (resp. $\Delta^{\mathrm{d}}(x, \mathcal{L}_2) = \Delta^{\mathrm{d}}(x, \partial\mathbb{D}^\bullet)$). In other words, $\mathcal{V}_1$ is the Voronoï cell of $\mathcal{L}_1$ with respect to $\mathcal{L}_2$ and conversely with $\mathcal{V}_1$ replaced by $\mathcal{V}_2$.

Hulls and their complements. The second goal of this work is to study hulls centered at ρ with respect to $\rho^\bullet$. For every $r > 0$, we write $\mathbb{B}_r := \{x \in \mathbb{D}^\bullet :\ \Delta(\rho, x) \leqslant r\}$. The set of all points $x \in \mathbb{D}^\bullet$ – such that any path connecting x and $\rho^\bullet$ has to hit the closed ball $\mathbb{B}_r$ – is called the *hull of radius* r. We denote this space by $\mathbb{B}_r^\bullet$. Equivalently, if $\Delta(\rho, \rho^\bullet) \leqslant r$, we have $\mathbb{B}_r^\bullet = \mathbb{D}^\bullet$ and if $\Delta(\rho, \rho^\bullet) > r$, the complement of $\mathbb{B}_r^\bullet$ is the unique connected component of $\mathbb{D}^\bullet\backslash\mathbb{B}_r$ containing $\rho^\bullet$.

Under the event $\{\Delta(\rho, \rho^\bullet) > r\}$, we denote the closure of $\mathbb{D}^\bullet \backslash \mathbb{B}_r^\bullet$ by $\check{\mathbb{B}}_r^\bullet$.

This work aims at the study of the pairs $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathbb{B}_r^\bullet, \check{\mathbb{B}}_r^\bullet)$. In particular, we will see that we can interpret all these spaces as elements of $\mathbb{K}$, and we will obtain explicit formulas concerning their volumes and their boundaries lengths. To this end, we will encode these spaces using labeled trees. Let us now give an informal presentation of this coding, which is highly inspired by [22, 79].

From labeled trees to metric spaces

Let $\mathrm{w} : [0, \zeta_{\mathrm{w}}] \mapsto \mathbb{R}_+$ be a continuous function such that $\mathrm{w}(t) > 0$, for every $t \in (0, \zeta_{\mathrm{w}})$. Under some assumptions, we can associate w with a labeled $\mathbb{R}$-tree as follows. We start from the segment $[0, \zeta_{\mathrm{w}}]$ that we call the *spine* and we assign the label $\Lambda_t := \mathrm{w}(t)$ to every point $t \in [0, \zeta_{\mathrm{w}}]$. Then, we consider a Poisson collection of continuous random trees (scaled versions of the celebrated Aldous Brownian CRT). The root of each tree is glued uniformly at random on the segment $[0, \zeta_{\mathrm{w}}]$. For every tree $\mathcal{T}$ and every point $u \in \mathcal{T}$, we assign a label Λ_u to u. The labels $(\Lambda_u)_{u \in \mathcal{T}}$ along the tree $\mathcal{T}$ are distributed according to a Brownian motion indexed by $\mathcal{T}$ and starting from the label of the root of $\mathcal{T}$. We then prune branches of $\mathcal{T}$ whenever they hit a label zero (if they do). Let $\mathfrak{H}_{\mathrm{w}}$ be the geodesic metric space consisting in the union of the segment $[0, \zeta_{\mathrm{w}}]$ and the forest of trees (after cutting their branches). In particular, every point $u \in \mathfrak{H}_{\mathrm{w}}$ has a nonnegative label Λ_u. Even if there generally are infinitely many points with zero label, we can introduce a random variable $Z(\mathfrak{H}_{\mathrm{w}})$ accounting for the "quantity" of points with zero label. Let us set $\partial_1 \mathfrak{H}_{\mathrm{w}} := \{u \in \mathfrak{H}_{\mathrm{w}} : \ \Lambda_u = 0\}$ and $\partial_2 \mathfrak{H}_{\mathrm{w}} := [0, \zeta_{\mathrm{w}}]$. We interpret the set $\partial \mathfrak{H}_{\mathrm{w}} := \partial_1 \mathfrak{H}_{\mathrm{w}} \cup \partial_2 \mathfrak{H}_{\mathrm{w}}$ as the "boundary" of $\mathfrak{H}_{\mathrm{w}}$ and we write $\mathfrak{H}_{\mathrm{w}}^\circ := \mathfrak{H}_{\mathrm{w}} \backslash \partial \mathfrak{H}_{\mathrm{w}}$. We will interpret the quantity ζ_{w} as the length of the boundary $\partial_2 \mathfrak{H}_{\mathrm{w}}$ and $Z(\mathfrak{H}_{\mathrm{w}})$ as the length of $\partial_1 \mathfrak{H}_{\mathrm{w}}$. We will also introduce a quantity $\mathcal{Y}(\mathfrak{H}_{\mathrm{w}})$, representing the total volume of points of $\mathfrak{H}_{\mathrm{w}}$. The quantities $\big(\zeta_{\mathrm{w}}, Z(\mathfrak{H}_{\mathrm{w}}), \mathcal{Y}(\mathfrak{H}_{\mathrm{w}})\big)$ are defined this way to have nice geometric interpretations in terms of $\mathbb{D}^\bullet$.

Let us now explain how to construct an element of $\mathbb{K}$ from $\mathfrak{H}_{\mathrm{w}}$. For every $u, v \in \mathfrak{H}_{\mathrm{w}}$ set:

$$\Delta_{\mathrm{w}}^\circ(u,v) := \begin{cases} \Lambda_u + \Lambda_v - 2 \max\big(\inf\limits_{[u,v]_{\mathfrak{H}_{\mathrm{w}}}} \Lambda, \inf\limits_{[v,u]_{\mathfrak{H}_{\mathrm{w}}}} \Lambda \big) & \text{if } \max\big(\inf\limits_{[u,v]_{\mathfrak{H}_{\mathrm{w}}}} \Lambda, \inf\limits_{[v,u]_{\mathfrak{H}_{\mathrm{w}}}} \Lambda \big) > 0 \\ \infty & \text{otherwise,} \end{cases}$$

where $[v,u]_{\mathfrak{H}_{\mathrm{w}}}$ is the interval of $\mathfrak{H}_{\mathrm{w}}$ consisting in all the points visited by going from u to v in “clockwise direction” along $\mathfrak{H}_{\mathrm{w}}$. Finally, we define Δ_{w} as the largest function on $\mathfrak{H}_{\mathrm{w}}^\circ \times \mathfrak{H}_{\mathrm{w}}^\circ$ bounded above by $\Delta_{\mathrm{w}}^\circ$ and verifying the triangle inequality. In the considered cases of this work, Δ_{w} will have a unique continuous extension to $\mathfrak{H}_{\mathrm{w}}$, which will define a pseudo-distance on $\mathfrak{H}_{\mathrm{w}}$. In the rest of this introduction assume that Δ_{w} has a continuous extension and, to simplify notation, write $\mathfrak{H}_{\mathrm{w}}/\Delta_{\mathrm{w}}$ for the quotient of $\mathfrak{H}_{\mathrm{w}}$ by the equivalence relation $\approx$ defined by $u \approx v$ if and only if $\Delta_{\mathrm{w}}(u,v) = 0$. By definition, the pseudo-distance Δ_{w} factorizes through $\approx$ and defines a distance on $\mathfrak{H}_{\mathrm{w}}/\Delta_{\mathrm{w}}$. We equip the space $\mathfrak{H}_{\mathrm{w}}/\Delta_{\mathrm{w}}$ with the distance Δ_{w} and we point it at the equivalence class of $0 \in [0, \zeta_{\mathrm{w}}]$. We will show in Section 8.2.2 that there exists a natural notion of volume measure on $\mathfrak{H}_{\mathrm{w}}/\Delta_{\mathrm{w}}$ and we will also equip $\mathfrak{H}_{\mathrm{w}}/\Delta_{\mathrm{w}}$ with this measure. Let us now apply this construction to random functions w, mainly to different variations of the Brownian excursion. To fix notation, we introduce $\mathbf{e}$, a Brownian excursion under the Itô measure $\mathbf{n}$, and denote its lifetime by $\zeta_{\mathbf{e}}$. The normalization is taken such that:

$$\mathbf{n}(\sup \mathbf{e} > \varepsilon) = \frac{1}{2\varepsilon}, \tag{8.3}$$

for every $\varepsilon > 0$. We also introduce the marked version of $\mathbf{n}$ defined by the relation:

$$\mathbf{n}^{\bullet}(F(t^{\bullet}, \mathbf{e})) := \mathbf{n}\Big(\int_0^{\sigma} \mathrm{d}t\, F(t, \mathbf{e})\Big).$$

Under $\mathbf{n}$ and $\mathbf{n}^{\bullet}$, and conditionally on $\mathbf{e}$, we write $\mathfrak{H}_{\mathbf{e}}$ (resp. $\mathfrak{H}_{\sqrt{3}\mathbf{e}}$) for the tree associated with $\mathbf{e}$ (resp. $\sqrt{3}\mathbf{e}$).

Voronoï cells on $\mathbb{D}^{\bullet}$ with respect to the boundary

Let us now present our results concerning the Voronoï cells $\mathcal{V}_1$ and $\mathcal{V}_2$. We write $\mathcal{V}_1^{\circ}$ and $\mathcal{V}_2^{\circ}$ for the interior of $\mathcal{V}_1 \backslash \partial\mathbb{D}^{\bullet}$ and $\mathcal{V}_2 \backslash \partial\mathbb{D}^{\bullet}$ respectively. In Section 8.3.1, we will show that the intrinsic distance Δ_1^{d} (resp. Δ_2^{d}) on $\mathcal{V}_1^{\circ}$ (resp. $\mathcal{V}_2^{\circ}$) has a continuous extension to $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). We will then equip the space $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) with this continuous extension, the restriction of the volume measure $\mathrm{Vol}^{\mathrm{d}}$ and the distinguished point ρ, leading us to see $\mathcal{V}_1$ and $\mathcal{V}_2$ as elements of $\mathbb{K}$. The main goal of Section 8.3.1 will be to show that we can encode spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ with two labeled trees as follows. Conditionally on the excursion $\mathbf{e}$, consider the tree $\mathfrak{H}_{\mathbf{e}}$ (associated with $\mathbf{e}$) and $\mathfrak{H}'_{\mathbf{e}}$ an independent copy of $\mathfrak{H}_{\mathbf{e}}$. We denote by $\Delta_{\mathbf{e}}$ and $\Delta'_{\mathbf{e}}$ the pseudo-distances respectively associated with $\mathfrak{H}_{\mathbf{e}}$ and $\mathfrak{H}'_{\mathbf{e}}$. We will show that $(\mathcal{V}_1, \mathcal{V}_2)$ is distributed as $(\mathfrak{H}_{\mathbf{e}}/\Delta_{\mathbf{e}}, \mathfrak{H}_{\mathbf{e}}/\Delta'_{\mathbf{e}})$ under $2\mathbf{n}$. We will also prove that if we glue the spaces $\mathfrak{H}_{\mathbf{e}}/\Delta_{\mathbf{e}}$ and $\mathfrak{H}_{\mathbf{e}}/\Delta'_{\mathbf{e}}$ along their spine in a metric manner, we recover the distribution of $\mathbb{D}^{\bullet}$, in such a way that $\mathfrak{H}_{\mathbf{e}}/\Delta_{\mathbf{e}}$ (resp. $\mathfrak{H}'_{\mathbf{e}}/\Delta'_{\mathbf{e}}$) becomes the Voronoï cell $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). In this identification, the spine is the interface between the two cells $\mathcal{V}_1$ and $\mathcal{V}_2$. In particular, we can use the variable $\zeta_{\mathbf{e}}$ to quantify the length of this interface. We also get that $Z(\mathfrak{H}_{\mathbf{e}})$ (resp. $Z(\mathfrak{H}'_{\mathbf{e}})$) is equal to the length $< \mathrm{m}_{\mathbb{D}}, \mathcal{L}_1 >$ (resp. $< \mathrm{m}_{\mathbb{D}}, \mathcal{L}_2 >$) and that $\mathcal{Y}(\mathfrak{H}_{\mathbf{e}})$ (resp. $\mathcal{Y}(\mathfrak{H}'_{\mathbf{e}})$) is the volume of $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). In Sections 8.3.2 and 8.3.3, we will study these random variables and we will obtain some explicit formulas for their Laplace transforms.

Hulls of $\mathbb{D}^{\bullet}$

Let us now present our results concerning the hulls. We fix $r > 0$, and in the rest of the introduction we argue under $\{\Delta(\rho, \rho^{\bullet}) > r\}$. Recall that $\mathbb{B}_r^{\bullet}$ stands for the hull of radius r and $\check{\mathbb{B}}_r^{\bullet}$ for the closure of the complement of $\mathbb{B}_r^{\bullet}$. One can deduce from Jordan's theorem that $\mathbb{B}_r^{\bullet}$ and $\check{\mathbb{B}}_r^{\bullet}$ are homeomorphic to the closed unit disk of the complex plane. As we did for the Brownian disk, we write $\partial\mathbb{B}_r^{\bullet}$ (resp. $\partial\check{\mathbb{B}}_r^{\bullet}$) for the set of all points of $\mathbb{B}_r^{\bullet}$ (resp. $\check{\mathbb{B}}_r^{\bullet}$) that do not have a neighborhood homeomorphic to the open disk. We interpret $\partial\mathbb{B}_r^{\bullet}$ and $\partial\check{\mathbb{B}}_r^{\bullet}$ as the boundaries of $\mathbb{B}_r^{\bullet}$ and $\check{\mathbb{B}}_r^{\bullet}$ respectively. Now remark that $\partial\check{\mathbb{B}}_r^{\bullet}$ is the union of $\partial_1\check{\mathbb{B}}_r^{\bullet} := \mathbb{B}_r^{\bullet} \cap \check{\mathbb{B}}_r^{\bullet}$ and $\partial_2\check{\mathbb{B}}_r^{\bullet} := \partial\mathbb{D}^{\bullet} \cap \check{\mathbb{B}}_r^{\bullet}$. By planarity, the intersection of $\partial_1\check{\mathbb{B}}_r^{\bullet}$ and $\partial_2\check{\mathbb{B}}_r^{\bullet}$ is reduced to two points. We finally introduce the set

$\check{\mathbb{B}}_r^\circ := \check{\mathbb{B}}_r^\bullet \backslash \partial \check{\mathbb{B}}_r^\bullet$. As previously seen for Voronoï cells, the intrinsic distance on $\check{\mathbb{B}}_r^\circ$ has a continuous extension to $\check{\mathbb{B}}_r^\bullet$ (see Theorem 8.3). We equip this space $\check{\mathbb{B}}_r^\bullet$ with this continuous extension and the restriction of the volume measure and we point it at one of the two points of $\partial_1\check{\mathbb{B}}_r^\bullet \cap \partial_2\check{\mathbb{B}}_r^\bullet$, uniformly at random. In Proposition 8.9, we will show that the limit:

$$|\partial_1\check{\mathbb{B}}_r^\bullet| = \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Vol}^d\big(\check{\mathbb{B}}_r^\bullet \cap \mathbb{B}_{r+\varepsilon}^\bullet\big),$$

exists in probability under $\mathbb{N}^{*,\bullet}\big(\,\cdot\mid \Delta(\rho,\rho^\bullet) > r\big)$. The quantity $|\partial_1\check{\mathbb{B}}_r^\bullet|$ can be interpreted as the boundary length of $\partial_1\check{\mathbb{B}}_r^\bullet$. Fix $z > 0$. In Section 8.4.4, we will show that, conditionally on $|\partial_1\check{\mathbb{B}}_r^\bullet| = z$, the space $\check{\mathbb{B}}_r^\bullet$ is distributed as $\mathfrak{H}_{\sqrt{3}\mathbf{e}}/\Delta_{\sqrt{3}\mathbf{e}}$ under $\mathbf{n}^\bullet(\cdot \mid Z(\mathfrak{H}_{\sqrt{3}\mathbf{e}}) = z)$. Here, $\mathfrak{H}_{\sqrt{3}\mathbf{e}}$ and $\Delta_{\sqrt{3}\mathbf{e}}$ are respectively the tree and the pseudo-distance associated with $\sqrt{3}\mathbf{e}$. One major consequence of this identity is that the distribution of $\check{\mathbb{B}}_r^\bullet$, conditionally on the boundary length $|\partial_1\check{\mathbb{B}}_r^\bullet|$, does not depend on r. We can perform the same kind of construction to encode the hull $\mathbb{B}_r^\bullet$ with a labeled tree (see Proposition 8.6). In particular, we can equip $\mathbb{B}_r^\bullet$ with the continuous extension of the intrinsic distance on the interior of $\mathbb{B}_r^\bullet$ (see (8.42) and the discussion below regarding the existence of this object) and with the restriction of the measure Vol^d. We also point $\mathbb{B}_r^\bullet$ at ρ. We will then show that, under $\mathbb{N}^{*,\bullet}\big(\,\cdot \mid |\partial_1\check{\mathbb{B}}_r^\bullet|\big)$, the two spaces $\mathbb{B}_r^\bullet$ and $\check{\mathbb{B}}_r^\bullet$ are independent. Moreover, these identifications will be used to obtain explicit formulas concerning the volume and the perimeter of $\mathbb{B}_r^\bullet$ and $\check{\mathbb{B}}_r^\bullet$.

Finally, let us conclude by conjecturing that, under the measure $4/\sqrt{3}\cdot\mathbf{n}$, the space $\mathfrak{H}_{\sqrt{3}\mathbf{e}}/\Delta_{\sqrt{3}\mathbf{e}}$, marked at the two extremities of the spine, is distributed as a free brownian disk – with free volume, free perimeter, and two distinguished boundary points. In other words, under $\mathbf{n}$, the space $\mathfrak{H}_{\sqrt{3}\mathbf{e}}/\Delta_{\sqrt{3}\mathbf{e}}$ should be distributed as $\mathbb{D}^\bullet$. The proof of this conjecture is a work in progress. Moreover, we will give some results supporting this conjecture in Section 8.4.2. Proving this conjecture will also allow us to extend our results concerning Voronoï cells to the Brownian sphere. In particular, it should be possible to show that if we take two uniform points on the Brownian sphere (with volume 1), then the volume of the Voronoï cell of the first point with respect to the second one is a uniform random variable on $[0,1]$. This will give a direct proof in the continuous of the simplest case of Chapuy's conjecture [33]. This special case was proved in [52] by Guitter using Miermont's bijection with delays [84] and taking the limit from the discrete.

8.2 Preliminaries

8.2.1 Snake trajectories and labeled trees

Framework

A (one-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0,\zeta_{\mathrm{w}}] \to \mathbb{R}$, where $\zeta_{\mathrm{w}} \geqslant 0$ is called the lifetime of w. We denote the set of all finite paths by $\mathcal{W}$. We write $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{\mathrm{w}})$ for the endpoint of w. The time reversal of w is the finite path $\mathrm{w}^\vee : [0,\zeta_{\mathrm{w}}] \to \mathbb{R}$ defined by the relation

$\mathrm{w}^{\vee}(t) = \mathrm{w}(\zeta_{\mathrm{w}} - t)$. For every $x \in \mathbb{R}$, we identify x with the finite path starting from x with 0 lifetime. Set $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$ and equip $\mathcal{W}$ with the distance:

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{\mathrm{w}} - \zeta_{\mathrm{w}'}| + \sup_{t \geqslant 0} |\mathrm{w}(t \wedge \zeta_{\mathrm{w}}) - \mathrm{w}'(t \wedge \zeta_{\mathrm{w}'})|.$$

Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $\omega : s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}$ satisfying the following properties:

• $\omega_0 = x$ and the quantity $\sigma(\omega) := \sup\{s \geqslant 0,\ \omega_s \neq x\}$ is finite. The quantity $\sigma(\omega)$ is called the lifetime of ω. By convention $\sigma(\omega) := 0$ if $\omega_s = x$ for every $s \geqslant 0$;
• For every $s, s' \in \mathbb{R}_+$ with $s \leqslant s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \leqslant \min_{r \in [s,s']} \zeta_{\omega_r}$. This property is called the snake property.

The set of all snake trajectories is denoted by $\mathcal{S}$, and the set of snake trajectories starting at x is denoted by $\mathcal{S}_x$. For every $\omega \in \mathcal{S}$ and $s \geqslant 0$, introduce the notation $W_s(\omega) := \omega_s$. The set $\mathcal{S}$ is equipped with the distance:

$$d_{\mathcal{S}}(\omega, \omega') := |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geqslant 0} d_{\mathcal{W}}\big(W_s(\omega), W_s(\omega')\big).$$

It is straightforward to verify that the space $(\mathcal{S}, d_{\mathcal{S}})$ is a Polish space. To simplify notation, for every $\omega \in \mathcal{S}$, we set $\omega_* := \inf\{\widehat{\omega}_s :\ s \geqslant 0\}$. We now introduce a collection of elementary operations on $\mathcal{S}$. For every snake trajectory ω and every $\lambda \in \mathbb{R}$, we will write $\omega + \lambda$ for the snake trajectory

$$(\omega + \lambda)_s(t) := \omega_s(t) + \lambda\,, \qquad 0 \leqslant t \leqslant \zeta_{(\omega+\lambda)_s} := \zeta_{\omega_s}$$

and if $\lambda > 0$ we write $\mathrm{hom}_\lambda(\omega)$ for the snake trajectory defined by

$$\mathrm{hom}_\lambda(\omega)_s(t) := \lambda \omega_{s\lambda^{-4}}(t\lambda^{-2})\,, \qquad 0 \leqslant t \leqslant \zeta_{\mathrm{hom}_\lambda(\omega)_s} := \lambda^2 \zeta_{\omega_{s\lambda^{-4}}}.$$

The snake trajectory $\omega + \lambda$ (resp. $\mathrm{hom}_\lambda(\omega)$) is called the snake trajectory ω translated (resp. scaled) by λ. If $\omega \in \mathcal{S}$, we can define its time reversal as the snake trajectory $\omega^{\vee}_s = \omega_{(\sigma(\omega)-s)_+}$. Remark that $\omega^{\vee}$ only reverse time and not the trajectories i.e. in general we do not have $\omega^{\vee}_s = (\omega_s)^{\vee}$. It will be also useful to introduce the truncation operation. Let $(x, r) \in \mathbb{R}^2$ with $x > r$, for every $\mathrm{w} \in \mathcal{W}_x$, let $\mathrm{hit}_r(\mathrm{w}) := \inf\{t \in (0, \zeta_{\mathrm{w}}] :\ \mathrm{w}(t) = r\}$. If $\omega \in \mathcal{S}_x$ then for every $s \geqslant 0$ set:

$$\eta^{(r)}_s(\omega) := \inf\Big\{t \geqslant 0 : \int_0^t \mathbb{1}_{\zeta_{\omega_u} \leqslant \mathrm{hit}_r(\omega_u)}\, du > s\Big\}.$$

The snake trajectory $\mathrm{tr}_r(\omega)$ defined by

$$\forall s \geqslant 0,\ \big(\mathrm{tr}_r(\omega)\big)_s := \omega_{\eta^{(r)}_s(\omega)}$$

is called the truncation of ω at level r. Let $\mathcal{Y}_r(\omega) := \sigma(\mathrm{tr}_r(\omega))$ which can be interpreted as the time spent by ω before hitting r. In this work we are also interested in the set $M(\mathcal{S})$ of all point measures on $\mathbb{R}_+ \times \mathcal{S}$. We equip the space $M(\mathcal{S})$ with the distance:

$$d_{M(\mathcal{S})}(\mu, \mu') := \sum_{n \geqslant 0} d_{\mathrm{Pro}}\big(\mu(\cdot \cap \mathcal{S}_{(n)}), \mu'(\cdot \cap \mathcal{S}_{(n)})\big) \wedge 2^{-n}, \tag{8.4}$$

where $\mathcal{S}_{(n)} = [0, 2^n] \times \{\omega \in \mathcal{S} : \sigma(\omega) > 2^{-n}\}$, and d_{Pro} stands for the Prokhorov metric inducing the weak topology on finite measures on $\mathbb{R}_+ \times \mathcal{S}$. Let us extend the previous operations to the set $M(\mathcal{S})$. Now for every $\lambda \in \mathbb{R}$ and $\mathcal{P} \in M(\mathcal{S})$ we define two elements $\mathcal{P} + \lambda$ and $\mathrm{hom}_\lambda \mathcal{P}(dtd\omega)$ of $M(\mathcal{S})$ by the relations

$$\int F(t,\omega)\ (\mathcal{P}+\lambda)(dtd\omega) := \int F(t,\omega+\lambda)\ \mathcal{P}(dtd\omega);$$
$$\int F(t,\omega)\ \mathrm{hom}_\lambda \mathcal{P}(dtd\omega) := \int F(\lambda t, \mathrm{hom}_\lambda \omega)\ \mathcal{P}(dtd\omega).$$

where the second measure is defined for $\lambda > 0$. If $\mathcal{P} \in M(\mathcal{S})$ is a point measure on $[a,b] \times \mathcal{S}$ we can also defined its time reversal by the relation:

$$\int F(t,\omega)\ \mathcal{P}^\vee(dtd\omega) := \int F(a+b-t, \omega^\vee)\ \mathcal{P}(dtd\omega).$$

Finally we define a truncation operation on $M(\mathcal{S})$. Let $r \in \mathbb{R}$ and $\mathcal{P} \in M(\mathcal{S})$. If for every atom (t,ω) of $\mathcal{P}$ we have $\omega_0 > r$, we set:

$$\int F(t,\omega)\ \mathrm{tr}_r(\mathcal{P})(dtd\omega) := \int F(t, \mathrm{tr}_r(\omega))\ \mathcal{P}(dtd\omega).$$

To simplify notation we will write

$$\widetilde{\omega} := \mathrm{tr}_0(\omega) \text{ and } \widetilde{\mathcal{P}} := \mathrm{tr}_0(\mathcal{P}), \tag{8.5}$$

for any $\omega \in \mathcal{S}$ and $\mathcal{P} \in M(\mathcal{S})$ (when the operation tr_0 is well defined). For every $\mathcal{P} := \sum_{i \in I} \delta_{t_i, \omega^i}$ element of $M(\mathcal{S})$ set $\mathcal{P}_* := \inf\{\omega^i_* :\ i \in I\}$.

Snake trajectories and labeled $\mathbb{R}$*-trees.*
It will be important for our study to associate a compact $\mathbb{R}$-tree, $\mathcal{T}_\omega$, with every snake trajectory ω. Let $\omega \in \mathcal{S}$ and define:

$$d_\omega(s,t) := \zeta_{\omega_s} + \zeta_{\omega_t} - 2 \inf_{r \in [s \wedge t, s \vee t]} \zeta_{\omega_r}$$

for every $s, t \in [0, \sigma(\omega)]$. Since $s \mapsto \zeta_{\omega_s}$ is continuous, d_ω is a continuous pseudo-distance on $[0,\sigma(\omega)]$. We define an equivalence relation by setting $s \approx_{d_\omega} t$ if $d_\omega(s,t) = 0$ and we denote the associated canonical projection by $p_\omega : [0,\sigma(\omega)] \to [0,\sigma(\omega)]/\approx_{d_\omega}$. The space $([0,\sigma(\omega)]/\approx_{d_\omega}, d_\omega)$ is an $\mathbb{R}$-tree. The volume measure V_ω on $[0,\sigma(\omega)]/\approx_{d_\omega}$ is the pushforward of Lebesgue measure on $[0,\sigma(\omega)]$ under p_ω. We denote the pointed weighted compact space $\big([0,\sigma(\omega)]/\approx_{d_\omega}, \rho_\omega, d_\omega, V_\omega\big)$, where ρ_ω is the equivalence class of 0, by $\mathcal{T}_\omega$. The point ρ_ω is called the root of $\mathcal{T}_\omega$. For every $u \in \mathcal{T}_\omega$, set $\Lambda^\omega_u := \widehat{\omega}_t$ where t is any element of $p_\omega^{-1}(u)$. The quantity Λ^ω_u is well defined by the snake property and we interpret Λ^ω_u as a label assigned to u. The pair $\big(\mathcal{T}_\omega, (\Lambda^\omega_u)_{u \in \mathcal{T}_\omega}\big)$ is the labeled tree associated with the snake trajectory ω.

We will use the following standard nomenclature. For every compact tree $\mathcal{T}$ and every point $a \in \mathcal{T}$, the multiplicity of a in $\mathcal{T}$ is the number of connected components of $\mathcal{T} \backslash \{a\}$. A point with multiplicity 1 (resp. bigger than 2) is called a leaf (resp. a branching point).

The Brownian snake excursion

To simplify notation, set $\widehat{W}_s(\omega) = \hat{\omega}_s$ and $W_*(\omega) = \omega_*$ for every $\omega \in \mathcal{S}$. Fix $x \in \mathbb{R}$. The Brownian snake excursion measure $\mathbb{N}_x$ is the unique σ-finite measure on $\mathcal{S}_x$ that satisfies the following properties:

• The process $s \mapsto \zeta_{\omega_s}$ is distributed according to the Itô measure of the Brownian excursion with the normalization

$$\mathbb{N}_x(\sup_{s\geqslant 0} \zeta_{\omega_s} > 1) = \frac{1}{2};$$

• Conditionally on $(\zeta_{\omega_s})_{s\in[0,\sigma(\omega)]}$, the process $\widehat{W}_s(\omega)$ is a Gaussian process with mean x and covariance function:

$$\forall s, s' \in [0, \sigma(\omega)],\ K(s, s') := \min_{r\in[s\wedge s', s\vee s']} \zeta_{\omega_r}.$$

Roughly speaking, conditionally on $(\zeta_s)_{s\geqslant 0}$, the process $(W_s)_{s\geqslant 0}$ evolves as follows. If ζ_s decreases, the path W_s is shortened from its tip, while if ζ_s increases, the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. We refer to [65] for a rigorous presentation. By construction for every $x \in \mathbb{R}$ the pushforward measure of $\mathbb{N}_x$ under $\omega \mapsto \omega + \lambda$ is $\mathbb{N}_{x+\lambda}$. It is also easy to deduce from the scaling property of the Brownian motion that for every $x \in \mathbb{R}$ the pushforward measure of $\mathbb{N}_x$ under $\omega \mapsto \hom_\lambda(\omega)$ is $\lambda^2\mathbb{N}_{\lambda x}$. We will call this property the scaling property of the Brownian snake excursion. For every $x, y \in \mathbb{R}$ with $x < y$ we have:

$$\mathbb{N}_y\big(W_* < x\big) = \frac{3}{2(y-x)^2} \tag{8.6}$$

see [65, Chapter 6] for more details. To simplify notation, under $\mathbb{N}_x(d\omega)$ we will write σ for $\sigma(W(\omega))$ and $W_s(t)$ for $\omega_s(t)$. Our goal now is to state the special Markov property of the Brownian snake excursion, which will be crucial in Section 8.4. First introduce for every $x > r$, the σ-field $\mathcal{H}_x^r$ generated by the process $\operatorname{tr}_r(W)$ and the class of all $\mathbb{N}_x$-negligible sets. The set:

$$\{s \geqslant 0,\ \operatorname{hit}_r(W_s) < \zeta_s\}$$

is open so it can be written as a union of disjoint open intervals $(a_i, b_i)_{i\in I}$ with I an indexing set that may be empty. For every $i \in I$, let $W^{(i)}$ be the snake trajectory defined by:

$$W_s^{(i)}(t) := W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t), \qquad 0 \leqslant t \leqslant \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$$

for every $s \geqslant 0$. By definition the snakes $(W^{(i)})_{i\in I}$ are the excursions of W outside r. Note that the information about the paths W_s before hitting r is contained in the sigma-field $\mathcal{H}_x^r$. The exit local time at level r is the process, $(\mathfrak{L}_t^r)_{t\geqslant 0}$, defined by the relation:

$$\mathfrak{L}_t^r(\omega) := \liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} \int_0^t ds \mathbb{1}_{\operatorname{hit}_r(\omega_s)=\infty,\ \hat{\omega}_s<r+\varepsilon}, \qquad 0 \leqslant t \leqslant \sigma. \tag{8.7}$$

The previous $\liminf$ is a well defined finite limit $\mathbb{N}_x$-a.e. (see [71, Proposition 34]). The process $\mathfrak{L}^r$ is a continuous nondecreasing process. To simplify notation set $\mathcal{Z}_r := \mathfrak{L}_\sigma^r$ which is called the exit measure at level r. Then, $\mathbb{N}_x$ a.e., the topological support of the measure $\mathrm{d}\mathfrak{L}^r$ is exactly the set $\{s \in [0,\sigma] : \mathrm{hit}_r(W_s) = \zeta_s\}$, and, in particular, $\mathcal{Z}_r > 0$ if and only if one of the paths W_s hits r. We can now give a formal statement of the special Markov property:

Special Markov property: Let $x, r \in \mathbb{R}$, such that $x > r$. Under $\mathbb{N}_x$ conditionally on $\mathcal{H}_x^r$, the point measure:

$$\sum_{i\in I} \delta_{\mathfrak{L}^r_{a_i}, W^{(i)}}(d\omega)$$

is Poisson with intensity $\mathbb{1}_{[0,\mathcal{Z}_r]}(t)dt\mathbb{N}_r(d\omega)$.

We refer to [70, Corollary 21] for a proof. It will be useful to note that for $r' < r < x$, if we replace $\mathbb{N}_x(d\omega)$ by $\mathbb{N}_x(d\omega \mid W_* > r')$ the last statement remains valid up to the replacement of $\mathbb{1}_{[0,Z_r]}\mathbb{N}_r(d\omega)$ by $\mathbb{1}_{[0,Z_r]}\mathbb{N}_r(d\omega \cap \{W_* > r'\})$.
Recall now that $\mathcal{Y}_0(W)$ stands for the lifetime of $\mathrm{tr}_0(W)$. The Laplace transform of $(\mathcal{Y}_0, \mathcal{Z}_0)$, i.e. the function $u_{\lambda,\mu}(x) := \mathbb{N}_x(1-\exp(-\lambda\mathcal{Z}_0 - \mu\mathcal{Y}_0))$, is given, for $x > 0$, by

$$u_{\lambda,\mu}(x) = \sqrt{\frac{\mu}{2}}\Big(3\, g^2_{\lambda,\mu}\Big((2\mu)^{\frac{1}{4}}x + g^{(-1)}_{\lambda,\mu}\big(\sqrt{\frac{2}{3}+\frac{2}{3}\frac{\lambda}{\sqrt{2\mu}}}\big)\Big) - 2\Big) \tag{8.8}$$

where $g_{\lambda,\mu}$ is the function defined as follows:

$$g_{\lambda,\mu}(x) := \begin{cases} \coth(x) & \text{if } \lambda > \sqrt{\frac{\mu}{2}} \\ 1 & \text{if } \lambda = \sqrt{\frac{\mu}{2}} \\ \tanh(x) & \text{if } \lambda < \sqrt{\frac{\mu}{2}} \end{cases}$$

and $g^{(-1)}_{\lambda,\mu}$ stands for the inverse of $g_{\lambda,\mu}$ with the convention $g^{(-1)}_{\sqrt{\mu/2},\mu}(x) := 0$ for every $\mu \geqslant 0$ and $x \in \mathbb{R}$. We refer to [40, Lemma 4.5.] for a proof.

In particular taking the limit when μ goes to 0, one obtain that

$$u_{\lambda,0}(x) = \mathbb{N}_x\big(1-\exp(-\lambda\mathcal{Z}_r)\big) = \Big(\lambda^{-\frac{1}{2}} + \sqrt{\frac{2}{3}}(x-r)\Big)^{-2} \tag{8.9}$$

for every $\lambda \geqslant 0$. Remark that the limit when λ goes to ∞ gives formula (8.6). It will be useful for every $\mathrm{w} : [a,b] \mapsto \mathbb{R}_+$ to introduce a probability measure $\mathbb{P}^{(\mathrm{w})}$ on $M(\mathcal{S})$ and a random point measures $\mathcal{P}$ on $\mathbb{R}_+ \times \mathcal{S}$ such that under $\mathbb{P}^{(\omega)}$, $\mathcal{P}$ is Poisson with intensity:

$$2\mathbb{1}_{[a,b]}(t)\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(d\omega).$$

If $\mathcal{P} \in M(\mathcal{S})$ such that all the atoms (t,ω) of $\mathcal{P}$ verify $\omega_0 > 0$ we set:

$$Z(\mathcal{P}) := \int \mathcal{Z}_0(\omega)\mathcal{P}(dtd\omega)\ ;\ \mathcal{Y}(\mathcal{P}) := \int \mathcal{Y}_0(\omega)\mathcal{P}(dtd\omega).$$

Moreover by the definition of tr_0 we have $Z(\mathcal{P}) = Z(\widetilde{\mathcal{P}})$ and $\mathcal{Y}(\mathcal{P}) = \mathcal{Y}(\widetilde{\mathcal{P}})$. It will be useful to remark that for every $\mathrm{w} : [a,b] \mapsto \mathbb{R}_+$ we have:

$$\mathbb{P}^{(\mathrm{w})}\big(\exp(-\lambda Z(\mathcal{P}) - \mu\mathcal{Y}(\mathcal{P}))\big) = \exp\big(-\int_a^b dt\ u_{\lambda,\mu}(\mathrm{w}(t))\big). \tag{8.10}$$

We write $\mathcal{P}_{\rm w}$ for a random element distributed as $\mathcal{P}$ under $\mathbb{P}^{(\rm w)}$. We will also write $\mathcal{P}'_{\rm w}$ for an independent copy of $\mathcal{P}_{\rm w}$ under $\mathbb{P}^{(\rm w)}$.

The positive excursion measure

As in the introduction, consider a σ-finite measure $\mathbf{n}$ and under this measure a process $(\mathbf{e}_t)_{t\in[0,\zeta_{\mathbf{e}}]}$ distributed as a Brownian excursion with normalization (8.3). We also introduce for every $h>0$, a probability measure $\mathbb{P}_h$ and a process B such that under $\mathbb{P}_h$ the process B is Brownian motion started at h stopped when it hits 0 for the first time. We see the processes $\mathbf{e}$ and B as random elements of $\mathcal{W}$. In particular, we have $\zeta_B = \inf\{s\geqslant 0:\ B_s=0\}$.

The Brownian snake.

We now introduce another σ-finite measure on $\mathcal{S}_0$, which is supported on snake trajectories taking only nonnegative values. For $\delta\geqslant 0$, let $\mathcal{S}^{(\delta)}$ be the set of all $\omega\in\mathcal{S}$ such that $\sup_{s\geqslant 0}(\sup_{t\in[0,\zeta_s(\omega)]}|\omega_s(t)|)>\delta$. Also set

$$\mathcal{S}_0^+ = \{\omega\in\mathcal{S}_0 : \omega_s(t)\geqslant 0 \text{ for every } s\geqslant 0, t\in[0,\zeta_s(\omega)]\}\cap\mathcal{S}^{(0)}.$$

By [2, Theorem 23], there exists a σ-finite measure $\mathbb{N}^*$ on $\mathcal{S}$, which is supported on $\mathcal{S}_0^+$ and gives finite mass to the sets $\mathcal{S}^{(\delta)}$ for every $\delta>0$, such that

$$\mathbb{N}^*(G) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\,\mathbb{N}_\varepsilon(G(\mathrm{tr}_0(\omega))),$$

for every bounded continuous function G on $\mathcal{S}$ that vanishes on $\mathcal{S}\backslash\mathcal{S}^{(\delta)}$ for some $\delta>0$. Under $\mathbb{N}^*$, each of the paths W_s, $0<s<\sigma$, starts from 0, then stays positive during some time interval, and is stopped immediately when it returns to 0, if it does return to 0.

A re-rooting formula. Arguing under $\mathbb{N}_0^*(\mathrm{d}\omega)$, for every $t\in(0,\sigma)$, we can define two point measures $\mathcal{N}_{(t)}$ and $\mathcal{N}'_{(t)}$ that account for the labeled subtrees branching off the ancestral line of $p_\omega(t)$ in the genealogical tree $\mathcal{T}_\omega$. More precisely, if t is fixed, we consider the connected components $\{(u_i,v_i):\ i\in I\}$ of the open set $\{s\in[0,t] : \zeta_s(\omega)>\min_{r\in[s,t]}\zeta_r(\omega)\}$, and for each $i\in I$, we define a snake trajectory ω^i by setting $\sigma(\omega^i)=v_i-u_i$ and, for every $s\in[0,\sigma(\omega^i)]$,

$$\omega^i_s(t) := \omega_{u_i+s}(\zeta_{u_i}(\omega)+t)\ ,\quad \text{for } 0\leqslant t\leqslant \zeta_{(\omega^i_s)} := \zeta_{u_i+s}(\omega)-\zeta_{u_i}(\omega).$$

Note that $\omega^i\in\mathcal{S}_{\widehat{\omega}_{u_i}}$, and $\widehat{\omega}_{u_i}=\omega_t(\zeta_{u_i})$ by the snake property. We then set $\mathcal{N}_{(t)}=\sum_{i\in I}\delta_{(\zeta_{u_i},\omega^i)}$. To define $\mathcal{N}'_{(t)}$, we proceed in a very similar manner, replacing the interval $[0,t]$ by $[t,\sigma]$. One can extend the proof of [79, Proposition 2] to derive the following re-rooting formula:

$$\mathbb{N}^*\Big(\int_0^\sigma dt F\big(W_t^\vee,\mathcal{N}_{(t)}^\vee,\mathcal{N}_{(t)}^{\prime,\vee}\big)\Big) = 2\int_0^\infty dr\ \mathbb{E}_r\big[F\big(B,\widetilde{\mathcal{P}}_B,\widetilde{\mathcal{P}}'_B\big)\big] \tag{8.11}$$

for any nonnegative measurable function F on $\mathcal{W}\times M(\mathcal{S})^2$. In this preliminary version, we leave this point to the reader.

Conditioning on the exit measure at 0. Recall the notation $\mathrm{hit}_r(\mathrm{w}) := \inf\{t \in (0, \zeta_{\mathrm{w}}] : \mathrm{w}(t) = r\}$. Under $\mathbb{N}^*$ formula we extend the definition of $\mathcal{Z}_0$ by setting

$$\mathcal{Z}_0 := \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}}$$

and the previous lim inf is a well defined finite limit $\mathbb{N}^*$-a.e. (this follows from the result under $\mathbb{N}_x$ for $x > 0$ using (8.11)). Under $\mathbb{N}^*$, the quantity $\mathcal{Z}_0$ can be interpreted as the "number" of paths W_s that return to 0 and the distribution of $(\mathcal{Z}_0, \sigma)$ under $\mathbb{N}^*$ is:

$$\frac{\sqrt{3}}{2\pi} \sqrt{z} s^{-\frac{5}{2}} \exp(-\frac{z^2}{2s}) dz ds. \tag{8.12}$$

One can derive (8.12) using the re-rooting equation (8.11) and the Laplace transform (8.8). We refer to [2, Proposition 31] for a full detailed proof of (8.12). According to [2, Proposition 33], there exists a unique collection $(\mathbb{N}^{*,z})_{z>0}$ of probability measures on $\mathcal{S}_0^+$ such that:

$$\mathbb{N}^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2}\, \mathbb{N}^{*,z}. \tag{8.13}$$

and for every $z > 0$, $\mathbb{N}^{*,z}$ is supported on $\{\mathcal{Z}_0 = z\}$. One can use the scaling invariance of $\mathbb{N}_0$ to derive that for every $z, z' > 0$, the pushforward of $\mathbb{N}^{*,z}$ under $\mathrm{hom}_{z'/z}$ is $\mathbb{N}^{*,z'}$.

8.2.2 Coding pairs and the Brownian disk.

From coding pairs to metric spaces

Coding pairs

A coding pair is a pair $(\mathrm{w}, \mathcal{P})$ such that:

(i) $\mathrm{w} : [a, b] \to \mathbb{R}$ is a continuous function;

(ii) $\mathcal{P} = \sum_{i \in I} \delta_{(t_i, \omega^i)}$ is a point measure on $[a, b] \times \mathcal{S}$ such that for every $i \in I$, $\omega^i \in \mathcal{S}_{\mathrm{w}(t_i)}$;

(iii) the numbers $(t_i)_{i \in I}$ are distinct;

(iv) the function $u \mapsto \beta_u := \sum_{i \in I} \mathbb{1}_{t_i \leqslant u} \sigma(\omega^i)$ takes finite values and is monotone increasing on $[a, b]$;

(v) for every $\varepsilon > 0$ we have $\#\big\{i \in I : \sup_{s \in [0, \sigma(\omega^i)]} |\hat{\omega}^i_s - \mathrm{w}(t_i)| > \varepsilon\big\} < \infty$.

We also define a scaling operation as follows:

$$\forall \lambda > 0,\ \mathrm{hom}_\lambda\Big(\mathrm{w}, \sum_{i \in I} \delta_{(t_i, \omega^i)}\Big) := \Big(\lambda \mathrm{w}(\cdot/\lambda^2), \sum_{i \in I} \delta_{(\lambda^2 t_i, \mathrm{hom}_\lambda(\omega^i))}\Big).$$

Let $(\mathrm{w},\mathcal{P})$ be a coding pair. Let us introduce the compact $\mathbb{R}$-tree $\mathcal{T}$ associated with $(\mathrm{w},\mathcal{P})$. For every $i\in I$, let (ζ^i_s) be the lifetime process associated with ω^i and set $\sigma^i := \sigma(\omega^i)$. We write $\mathcal{T}^i$ for the tree coded by ζ^i, i.e. $\mathcal{T}_{\omega^i}$, and p_{ζ^i} for the canonical projection from $[0,\sigma^i]$ onto $\mathcal{T}^i$. The tree $\mathcal{T}$ can be defined from the disjoint union:

$$[a,b]\cup\Big(\bigcup_{i\in I}\mathcal{T}^i\Big)$$

by identifying the point t_i with $p_{\zeta^i}(0)$ (the root of $\mathcal{T}^i$). The subset $[a,b]\subset\mathcal{T}$ is called the spine of $\mathcal{T}$. We can equip $\mathcal{T}$ with a natural distance $d_{\mathcal{T}}$ as follows. The restriction of $d_{\mathcal{T}}$ to the spine is the Euclidean distance in $[a,b]$ and the restriction of $d_{\mathcal{T}}$ to each tree $\mathcal{T}^i$ is the tree distance d_{ω^i}. If $x\in\mathcal{T}^i$ and $t\in[a,b]$, we take $d_{\mathcal{T}}(x,t)=d_{\omega^i}(x,p_{\zeta^i}(0))+|t_i-t|$. If $x\in\mathcal{T}^i$ and $y\in\mathcal{T}^j$ with $i\neq j$, we take $d_{\mathcal{T}}(x,y)=d_{\omega^i}(x,p_{\zeta^i}(0))+|t_i-t_j|+d_{(\omega^j)}(y,p_{\zeta^j}(0))$. We can also assign a label, Λ_x, to each x in $\mathcal{T}$ taking $\Lambda_x:=\mathrm{w}(x)$ if $x\in[a,b]$ and $\Lambda_x:=\Lambda_x^{\omega^i}$ if $x\in\mathcal{T}^i$. Finally, set $V_{\mathcal{T}}$ the measure on $\mathcal{T}$ which gives no mass to the spine and such that its restriction to $\mathcal{T}^i$ is V_{ω^i}.

The trees $\mathcal{T}^i$ are grafted on the spine of $\mathcal{T}$. It will be important for our purposes to equip $\mathcal{T}$ also with an order structure inherited from the coding triple. We write β_{u-} for the left limits of β and we take $\beta_{0-}=0$ by convention. Note that if u is a discontinuity point of β then there is a unique $i\in I$ such that $t_i=u$ and $\beta_u-\beta_{u-}=\sigma^i$. For every $s\in[0,\beta_b]$ there exists a unique u such that $\beta_{u-}\leqslant s\leqslant\beta_u$. Then if there exists an $i\in I$ such that $t_i=u$ we set $\mathcal{E}_s:=p_{\zeta^i}(s-\beta_{t_i-})$ and if it is not the case, we set $\mathcal{E}_s:=u$. The process $\mathcal{E}$ is called the contour exploration of $\mathcal{T}$. Remark also that $\mathcal{E}$ is continuous and the volume measure on $\mathcal{T}$ is the pushforward of Lebesgue measure on $[0,\beta_b]$ under the mapping $s\mapsto\mathcal{E}_s$. This exploration process allows us to define a notion of interval on $\mathcal{T}$. By convention, for every $s,t\in[0,\beta_b]$ with $s<t$ we write $[t,s]_{\mathcal{T}}=[0,s]\cup[t,\beta_b]$ (and $[s,t]_{\mathcal{T}}=[s,t]$). For every $u,v\in\mathcal{T}$ with $u\neq v$, let $[s,t]_{\mathcal{T}}$ be the smallest interval such that $\mathcal{E}_s=u$ and $\mathcal{E}_t=v$. It is easy to check from the definition that there is always a unique smallest interval. We put:

$$[u,v]_{\mathcal{T}}:=\big\{\mathcal{E}_r : r\in[s,t]_{\mathcal{T}}\big\}.$$

By convention if $u=v$, take $[u,v]_{\mathcal{T}}=\{u\}$. Note that $[u,v]_{\mathcal{T}}\neq[v,u]_{\mathcal{T}}$ if $u\neq v$. For every $u,v\in\mathcal{T}$, we denote the geodesic segment connecting u and v in $\mathcal{T}$ by $[\![u,v]\!]_{\mathcal{T}}$.

Let us close this subsection with a remark concerning notation. For $\mathrm{w}:[0,\zeta_{\mathrm{w}}]\mapsto\mathbb{R}$, we will write $\mathfrak{H}_{\mathrm{w}}$ for the tree associated with $(\mathrm{w},\widetilde{P}_{\mathrm{w}})$.

The associated metric space

Consider a coding pair $(\mathrm{w},\mathcal{P})$ and let $\big(\mathcal{T},(\Lambda_v)_{v\in\mathcal{T}}\big)$ be the associated labeled tree. Let $[a,b]$ be the interval of definition of w. We make the following assumption:

$$(H_1): \begin{cases} \min\{\Lambda_v : v \in \mathcal{T}\} = 0; \\ \text{if } \Lambda_v = 0 \text{ then } v \text{ is a leaf;} \\ \text{if } v \text{ is not a leaf then for every } \ w \in \mathcal{T}\backslash\{v\} \text{ we have } \max\Big(\inf_{[w,v]_{\mathcal{T}}} \Lambda, \inf_{w\in[v,w]_{\mathcal{T}}} \Lambda\Big) < \Lambda_v. \end{cases}$$

Set $\partial_1\mathcal{T} := \{v \in \mathcal{T} : \Lambda_v = 0\}$ and $\partial_2\mathcal{T} := [a, b]$. We interpret $\partial_1\mathcal{T}$ and $\partial_2\mathcal{T}$ as the "boundaries" of $\mathcal{T}$ and we write $\partial\mathcal{T}$ for the union $\partial_1\mathcal{T} \cup \partial_2\mathcal{T}$. The set $\partial_2\mathcal{T}$ is also called the spine of $\mathcal{T}$. From now on, set

$$|\partial_1\mathcal{T}| := Z(\mathcal{P}) \text{ and } |\partial_2\mathcal{T}| := b - a, \tag{8.14}$$

and we will interpret these quantities respectively as the boundary length of $\partial_1\mathcal{T}$ and $\partial_2\mathcal{T}$. We will also write $\mathcal{T}^\circ := \mathcal{T}\backslash\partial\mathcal{T}$ and remark that (H_1) implies that $\mathcal{T}^\circ$ is dense in $\mathcal{T}$. For every $u, v \in \mathcal{T}$, introduce the quantity:

$$\Delta^\circ(u, v) := \begin{cases} \Lambda_u + \Lambda_v - 2\max\big(\inf_{[u,v]_{\mathcal{T}}} \Lambda, \inf_{[v,u]_{\mathcal{T}}} \Lambda\big) & \text{if } \max\big(\inf_{[u,v]_{\mathcal{T}}} \Lambda, \inf_{[v,u]_{\mathcal{T}}} \Lambda\big) > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{8.15}$$

We then let

$$\forall u, v \in \mathcal{T}^\circ,\ \Delta(u, v) := \inf_{u_0=u,u_1,\dots u_n=v} \sum_{i=1}^{n} \Delta^\circ(u_{i-1}, u_i) \tag{8.16}$$

where the infimum is over all choices of the integer $n \geqslant 1$ and all the finite sequences $u_0, ..., u_n$ of elements of $\mathcal{T}$ verifying $u_0 = u$ and $u_n = v$. The mapping $(u, v) \mapsto \Delta(u, v)$ takes finite values and is continuous on $\mathcal{T}^\circ \times \mathcal{T}^\circ$. Indeed, a compactness argument shows that we can find finitely many points $u_0 = u, u_1, \dots, u_{n-1}, u_n = v$ belonging to the geodesic segment $[\![u, v]\!]_{\mathcal{T}}$ and such that $\Delta^\circ(u_i, u_{i+1}) < \infty$ for every $0 \leqslant i \leqslant n - 1$.

Since $\Delta^\circ(u, v) \geqslant |\Lambda_u - \Lambda_v|$, we have:

$$\forall u, v \in \mathcal{T}^\circ,\ \Delta(u, v) \geqslant |\Lambda_u - \Lambda_v|. \tag{8.17}$$

The function Δ is a pseudo-distance on $\mathcal{T}^\circ$. We write Π° for the canonical projection and we set $\Theta^\circ(\mathrm{w}, \mathcal{P}) := (\mathcal{T}^\circ/\Delta, \Delta, \mathrm{Vol})$ where Vol is the pushforward of $V_{\mathcal{T}}$ under the map Π°. For every $x \in \mathcal{T}^\circ/\Delta$ set $\Lambda_x := \Lambda_u$ where u is any preimage of x by Π° (remark that the definition is unambiguous by (8.17)). Let us introduce the following extra assumption:

$$(H_2): \begin{cases} (u, v) \mapsto \Delta(u, v) \text{ has a continuous extension on } \mathcal{T} \times \mathcal{T}; \\ \text{we have } V_{\mathcal{T}}(\partial\mathcal{T}) = 0. \end{cases}$$

Actually it is easy to see that if we replace the condition $u, v \in \mathcal{T}^\circ$ in (8.16) by $u, v \in \mathcal{T}^\circ \cup \partial_2\mathcal{T}$ we obtain a continuous extension of Δ to $\big(\mathcal{T}^\circ \cup \partial_2\mathcal{T}\big) \times \big(\mathcal{T}^\circ \cup \partial_2\mathcal{T}\big)$. So the important point in the first statement of (H_2) is that Δ can be extended by continuity to $\partial_1\mathcal{T}$. Assume in the rest of this section that (H_2) holds and keep the notation Δ for the continuous extension to $\mathcal{T} \times \mathcal{T}$, which defines a pseudo-distance on $\mathcal{T}$. Now we can consider the space $(\mathcal{T}/\Delta, \Delta)$ and we denote the canonical projection from $\mathcal{T}$ onto $\mathcal{T}/\Delta$ by Π. We indentify $\Theta^\circ(\mathrm{w}, \mathcal{P})$ with $\Pi(\mathcal{T}^\circ)$ and we extend the definition of Λ to $\mathcal{T}/\Delta$ taking $\Lambda_x := \Lambda_u$ where u is any preimage of x by Π (again the definition is unambiguous by continuity and (8.17)). Finally by the last assumption of (H_2) we may and will identify Vol with the pushforward of $V_{\mathcal{T}}$ under the map Π. The measure Vol is called the volume measure of $(\mathrm{w}, \mathcal{P})$.

Set $\Theta(\mathrm{w}, \mathcal{P}) := (\mathcal{T}/\Delta, \Pi(a), \Delta, \mathrm{Vol})$ which is an element of $\mathbb{K}$. We claim that the labels $(\Lambda_x)_{x\in\mathcal{T}/\Delta}$ correspond to the distances to $\Pi(\partial_1\mathcal{T})$. Let us explain this last point. Recall that $\mathcal{E} : [0, T] \mapsto \mathcal{T}$ is the exploration of $\mathcal{T}$ and for every $t \in [0, T]$ and $r \leqslant \Lambda_{\mathcal{E}_t}$ introduce the quantity:

$$\gamma_t(r) := \begin{cases} \mathcal{E}_{\inf\{s\geqslant t:\ \Lambda_{\mathcal{E}_s}=\Lambda_{\mathcal{E}_t}-r\}} & \text{if } \inf\limits_{[t,T]} \Lambda \leqslant r \\ \mathcal{E}_{\inf\{s\geqslant 0:\ \Lambda_{\mathcal{E}_s}=\Lambda_{\mathcal{E}_t}-r\}} & \text{otherwise.} \end{cases} \tag{8.18}$$

By construction we have $\Delta^\circ(\gamma_t(r), \gamma_t(r')) = |r - r'|$ for every $r, r' \in [0, \Lambda_t)$. Combining this with (8.17), we obtain that the path $\Pi \circ \gamma_t$ verifies $\Delta(\Pi(\gamma_t(r)), \Pi(\gamma_t(r'))) = |r - r'|$ for every $r, r' \in [0, \Lambda_t)$. Moreover $\Pi(\gamma_t(\Lambda_{\mathcal{E}_t})) \in \Pi(\partial_1\mathcal{T})$ and consequently $\Pi \circ \gamma_t$ is a geodesic path connecting $\Pi(\mathcal{E}_t)$ and $\Pi(\partial_1\mathcal{T})$ with length $\Lambda_{\mathcal{E}_t}$. Combining this with the bound (8.17) we get

$$\forall x \in \mathcal{T}/\Delta,\ \Delta(x, \Pi(\partial_1\mathcal{T})) = \Lambda_x. \tag{8.19}$$

A path of the form $\Pi \circ \gamma_t$ for some $t \in [0, T]$ is called a simple geodesic. One can also use the simple geodesics and the definition of Δ as an infimum given in (8.16) to show that the metric space $\Theta(\mathrm{w}, \mathcal{P})$ is also a length space. We refer to [79, Section 4.1] for a proof of this fact (a slightly different framework is considered in [79] but exactly the same argument works in our case without modifications). For every $r \geqslant 0$, we write $B_r(\mathcal{T}/\Delta)$ for the set of all points $x \in \mathcal{T}/\Delta$ such that $\Delta(x, \Pi(\partial_1\mathcal{T})) \leqslant r$ and note that by (8.19) we have:

$$B_r(\mathcal{T}/\Delta) = \{x \in \mathcal{T}/\Delta :\ \Lambda_x \leqslant r\}.$$

We introduce one last assumption that will be satisfied in all spaces considered in this work:

$$(H_3):\ \begin{cases} \text{– for every } u, v \in \mathcal{T},\ \text{if } u \neq v \text{ and } \Delta(u, v) = 0 \text{ then } u \text{ and } v \text{ must be leaves;} \\ \text{– for every } u, v \in \mathcal{T}^\circ,\ \text{the property } \Delta(u, v) = 0 \text{ holds if and only if } \Delta^\circ(u, v) = 0; \\ \text{– for every } u, v \in \partial_1\mathcal{T},\ \text{the property } \Delta(u, v) = 0 \text{ holds if and only if } \Lambda_w > 0 \\ \text{for every } w \in]u, v[_{\mathcal{T}},\ \text{or for every } w \in]v, u[_{\mathcal{T}}; \\ \text{–all the geodesics from a point } \Pi(\mathcal{T}\backslash\partial_1\mathcal{T}) \text{ to the boundary } \Pi(\partial_1\mathcal{T}) \\ \text{are simple geodesics.} \end{cases}$$

In particular remark that the only points of the spine $\partial_2\mathcal{T} := [a, b]$ that can be identified by Δ with another point are the extremities a and b since the other points of $[a, b]$ have at least multiplicity 2 in $\mathcal{T}$. By the first assumption of (H_3), we also deduce that under this assumption the simple geodesics can only hit a point $\Pi(u)$, where u is a point with multiplicity bigger than 1, at its extremities. In particular, a simple geodesic can only hit $\Pi(\partial\mathcal{T})$ at its extremities.

The coding pair associated with $\mathbb{D}$

We end the preliminaries by explaining how to construct $\mathbb{D}$ as the metric space associated with a coding pair. Under $\mathbb{N}^*$, the pair $(0, W)$ is a coding pair, if we see 0 as the constant path equal to 0 with 0 lifetime and we identified W with the point measure $\delta_{0,W}$. The coding pair $(0, W)$ clearly satisfies the two first assumptions of (H_1). It also satisfies the last assumption of (H_1) by [71, Proposition 32 (iii)]. So Section 8.2.2 allows us to construct the labeled tree $\mathcal{T}^d$ associated to $(0, W)$ and we denote its labels by $(\Lambda^d_v)_{v\in\mathcal{T}^d}$. We define the sets $\mathcal{T}^{d,\circ}$, $\partial\mathcal{T}^d$ and the pseudo-metric functions $\Delta^{d,\circ}(u, v)$, $\Delta^d(u, v)$ on $\mathcal{T}^{d,\circ} \times \mathcal{T}^{d,\circ}$ as explained in Section 8.2.2. We also introduce the exploration process $\mathcal{E}^d$ and the label process Λ^d of $\mathcal{T}^d$. It can be shown that $(0, W)$ satisfies (H_2) (we refer to [71, Proposition 28] for a proof) and consequently we can consider the continuous extension of Δ^d to $\mathcal{T}^d \times \mathcal{T}^d$ and the associated metric space $\Theta(0, W)$. The space $\Theta(0, W)$ is a Brownian disk rooted on the boundary with free volume and free perimeter, we refer to [71] for the connection with lattices models. To simplify notation we write $\mathbb{D} := \Theta(0, W)$ and Π^d (resp. $\mathrm{Vol}^{\mathrm{d}}$) for the canonical projection from $\mathcal{T}^d$ to $\mathbb{D}$ (resp. the volume measure of $\mathbb{D}$). As explained in the introduction, the space $\mathbb{D}$ is a.e. homeomorphic to the closed unit disk of the complex plane and we can define its *boundary* as the set of all points of $\mathbb{D}$ which does not have a neighborhood homeomorphic to the open disk. We denote this set by $\partial\mathbb{D}$ and we have $\Pi(\partial\mathcal{T}^d) = \partial\mathbb{D}$. In particular remark that the root $\Pi^d(0)$ belongs to $\partial\mathbb{D}$. The main interest of this construction of $\mathbb{D}$ is that the labels correspond to the distances to the boundary $\partial\mathbb{D}$. Thus we can rewrite the definition of $\mathcal{Z}_0$ under $\mathbb{N}^*$ as follows:

$$\mathcal{Z}_0 = \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Vol}^{\mathrm{d}}\big(\{x \in \mathbb{D} :\ \Delta^{\mathrm{d}}(x, \partial\mathbb{D}) \leqslant \varepsilon\}\big).$$

With the notation of the introduction one has $\mathcal{Z}_0 = |\partial\mathbb{D}|$. The quantity $\mathcal{Z}_0$ is called the perimeter (in a generalized sense) of $\partial\mathbb{D}$. One can fix the perimeter $\mathcal{Z}_0$ by disintegration using the measures $(\mathbb{N}^{*,z})_{z>0}$ and (8.13). Under $\mathbb{N}^{*,z}$, the space $\mathbb{D}$ is the free Brownian disk with perimeter z. We present now the uniform measure on $\partial\mathbb{D}$. Under $\mathbb{N}^*$, for every $s > 0$ the limit

$$\mathfrak{L}_s := \lim_{\varepsilon\to 0} \varepsilon^{-2} \int_0^s dt \mathbb{1}_{\widehat{W}_t \leqslant \varepsilon} \tag{8.20}$$

is well defined and we have $\mathfrak{L}_\sigma = \mathcal{Z}_0$. We refer to [71, Corollary 37] for a proof. Set $\Gamma(t) := \inf\{s \in [0, \sigma] :\ \mathfrak{L}_s > t\}$ for every $t \in [0, z)$ and $\Gamma\big(\mathcal{Z}_0\big) := \sigma$. It is shown in [71, Corollary 37] that the function $t \in [0, \mathcal{Z}_0] \mapsto \Pi^d(\mathcal{E}^d_{\Gamma(t)})$ is a continuous loop whose range is the boundary $\partial\mathbb{D}$. We have $\Gamma(0) = 0$ and $\Gamma(\mathcal{Z}_0) = \sigma$ and they are both continuity times of Γ. Let $m_{\mathbb{D}}$ be the random measure on $\partial\mathbb{D}$ defined by:

$$\int_{\partial\mathbb{D}} \mathrm{m}_{\mathbb{D}}(dx)\ F(x) := \int_0^{\mathcal{Z}_0} dt\ F(\Pi^d(\mathcal{E}^d_{\Gamma(t)})). \tag{8.21}$$

The measure $\mathrm{m}_{\mathbb{D}}$ is called the uniform measure on $\partial\mathbb{D}$. It is shown in [71, Corollary 37] that this measure coincides with the one defined in the introduction i.e. we have:

$$\int_{\partial\mathbb{D}} \mathrm{m}_{\mathbb{D}}(dx)\ F(x) := \lim_{\varepsilon\to 0} \varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}^{\mathrm{d}}(dx) F(x) \mathbb{1}_{\Delta^{\mathrm{d}}(x,\partial\mathbb{D})<\varepsilon} \tag{8.22}$$

where the limits holds $\mathbb{N}^*$-a.e. for every bounded continuous function $F : \mathbb{D} \mapsto \mathbb{R}_+$. It will be useful to introduce the measure

$$\mathbb{N}^{*,\bullet}(\mathrm{d}\omega\mathrm{d}t) = \mathbb{N}^*(\mathrm{d}\omega)\, \mathbb{1}_{[0,\mathcal{Z}_0(\omega)]}(t)\mathrm{d}t \tag{8.23}$$

which can be interpreted as a "pointed" version of $\mathbb{N}^*$. For every $(\omega, t) \in \mathcal{S} \times \mathbb{R}_+$, set $W(\omega, t) = \omega$ and $t^\bullet(\omega, t) = t$. Remark that we have

$$\mathbb{N}^{*,\bullet}\big(F(W, t^\bullet)\big) := \mathbb{N}^*\Big(\int_0^{\mathcal{Z}_0} \mathrm{d}t\ F(W, t)\Big). \tag{8.24}$$

The space $\mathbb{D}^\bullet = \big(\Theta(0, W), \Pi^d(\mathcal{E}^d_{\Gamma(t^\bullet)})\big)$ is a metric space with two distinguished points; namely the root $\rho := \Pi^d(0)$ and the marked point $\rho^\bullet := \Pi^d(\mathcal{E}^d_{\Gamma(t^\bullet)})$. The distribution of $\mathbb{D}^\bullet$ is characterized by the relation:

$$\mathbb{N}^{*,\bullet}\big(F(\mathbb{D}^\bullet)\big) = \mathbb{N}^*\Big(\int m_{\mathbb{D}}(dx)\ F(\mathbb{D}, x)\Big).$$

Under the σ-finite measure $\mathbb{N}^{*,\bullet}$, the space $\mathbb{D}^\bullet$ is a marked version of $\mathbb{D}$ as defined in the introduction.

8.3 Voronoï cells

Arguing under $\mathbb{N}^*$, the function $t \mapsto \Pi^d(\mathcal{E}^d_{\Gamma(t)})$ gives an orientation of $\partial\mathbb{D}$. We fix this orientation in the rest of this work. For every $s, t \in [0, \mathcal{Z}_0]$, we write $\mathcal{L}(s, t) := \Pi^d(\Gamma([s, t]))$, and let $\mathcal{V}(s, t)$ be the set of all points x verifying $\Delta^{\mathrm{d}}(x, \mathcal{L}(s, t)) = \Lambda^d_x$. As the label function Λ^d encodes the distances to $\partial\mathbb{D}$, the set $\mathcal{V}(s, t)$ is the collection of all the points that are closer to $\mathcal{L}(s, t)$ than to $\partial\mathbb{D}\backslash\mathcal{L}(s, t)$. We call $\mathcal{V}(s, t)$ the Voronoï cell of $\mathcal{L}(s, t)$, with respect to the rest of the boundary. We write $\mathcal{V}^\circ(s, t)$ to denote the interior of $\mathcal{V}(s, t)\backslash\partial\mathbb{D}$ and set $\partial\mathcal{V}(s, t) = \mathcal{V}(s, t)\backslash\mathcal{V}^\circ(s, t)$. In a coherent manner with the notation used in the introduction, under $\mathbb{N}^{*,\bullet}$, we may and will set $\mathcal{V}_1 := \mathcal{V}(0, t^\bullet)$ and $\mathcal{V}_2 := \mathcal{V}(t^\bullet, \mathcal{Z}_0)$.

Before to go into matter, let us prove the following lemma which will be useful throughout this work:

Lemma 8.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function and suppose that there exists a bounded twice differentiable function H_f solving the following differential equation:*

$$\frac{\partial^2}{\partial x^2} H_f(x) = 2f(x)H_f(x),$$

and taking values on $]0, \infty[$. We then have:

$$\mathbb{E}_h\Big[\exp\Big(-\int_0^{\zeta_B} dt f(B_t)\Big)\Big] = \frac{H_f(h)}{H_f(0)}, \tag{8.25}$$

for every $h > 0$, and:

$$\mathbf{n}\Big(1 - \exp\Big(-\int_0^{\zeta_{\mathbf{e}}} dt f(\mathbf{e}_t)\Big)\Big) = -\frac{1}{2H_f(0)} \cdot \frac{\partial}{\partial x} H_f(x)\big|_{x=0}. \tag{8.26}$$

Proof. Fix f and H_f as in the statement. Formula (8.25) is a particular case of the Feynman–Kac formula. Let us explain how to obtain (8.26). First, under $\mathbf{n}$, set $\tau_\varepsilon := \inf\{s \geqslant 0 : \mathbf{e}_s \geqslant \varepsilon\}$ with the convention $\inf \varnothing = \zeta_{\mathbf{e}}$. Then by monotone convergence we have

$$\mathbf{n}\big(1 - \exp\big(-\int_0^{\zeta_{\mathbf{e}}} \mathrm{d}t f(\mathbf{e}_t)\big)\big) = \lim_{\varepsilon \to 0} \mathbf{n}\big(1 - \exp\big(-\int_{\tau_\varepsilon}^{\zeta_{\mathbf{e}}} \mathrm{d}t f(\mathbf{e}_t)\big)\big).$$

We can then apply the Markov property to obtain that:

$$\mathbf{n}\big(1 - \exp\big(-\int_0^{\zeta_{\mathbf{e}}} \mathrm{d}t f(\mathbf{e}_t)\big)\big) = \lim_{\varepsilon \to 0} \mathbf{n}\big(\sup \mathbf{e} > \varepsilon\big) \cdot \big(1 - \mathbb{E}_\varepsilon\big[\exp(-\int_0^{\zeta_B} \mathrm{d}t f(B_t))\big]\big).$$

We can now apply (8.25) to get:

$$\mathbf{n}\big(1 - \exp\big(-\int_0^{\zeta_{\mathbf{e}}} dt f(\mathbf{e}_t)\big)\big) = \lim_{\varepsilon \to 0} \mathbf{n}(\sup \mathbf{e} > \varepsilon) \cdot \big(1 - \frac{H_f(\varepsilon)}{H_f(0)}\big) = \frac{1}{2} \lim_{\varepsilon \to 0} \varepsilon^{-1} \cdot \big(1 - \frac{H_f(\varepsilon)}{H_f(0)}\big).$$

This gives:

$$\mathbf{n}\big(1 - \exp\big(-\int_0^{\zeta_{\mathbf{e}}} dt f(\mathbf{e}_t)\big)\big) = -\frac{1}{2H_f(0)} \cdot \frac{\partial}{\partial x} H_f(x)\big|_{x=0}$$

where in the second equality we used (8.3). □

8.3.1 The coding pair of Voronoï cells

Recall that Δ_1^{d} (resp. Δ_2^{d}) stands for the intrinsic distance on the interior of $\mathcal{V}_1 \backslash \partial \mathbb{D}$ (resp. $\mathcal{V}_2 \backslash \partial \mathbb{D}$) and the notation of (8.5). The goal of this section is to prove the following result:

Theorem 8.1.

(i) $\mathbf{n}$*-a.e., the coding pair* $(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}})$ *satisfies* (H_2, H_3)*. In particular, we can consider the associated metric space* $\Theta(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}})$*.*

(ii) $\mathbb{N}^{*,\bullet}$*-a.e., the intrinsic distance* Δ_1^{d} *(resp.* Δ_2^{d}*) has a continuous extension to* $\mathcal{V}_1$ *(resp.* $\mathcal{V}_2$*). Equip the space* $\mathcal{V}_1$ *(resp.* $\mathcal{V}_2$*) with the continuous extension of* Δ_1^{d} *(resp.* Δ_2^{d}*), the restriction of* $\mathrm{Vol}^{\mathrm{d}}$ *to* $\mathcal{V}_1$ *(resp.* $\mathcal{V}_2$*) and the distinguished point* ρ*. Then the pair* $(\mathcal{V}_1\ ,\ \mathcal{V}_2)$*, under* $\mathbb{N}^{*,\bullet}$*, is distributed as* $(\Theta(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}}),\ \Theta(\mathbf{e}, \widetilde{\mathcal{P}}'_{\mathbf{e}}))$*, under* $2\mathbf{n}$*.*

We will also show in Lemma that one can recover the Brownian disk by gluing $\mathcal{V}_1$ and $\mathcal{V}_2$ along their common boundary in a metric way.

Let us start by identifying, under $\mathbb{N}^*$, the points of $\mathcal{V}(s,t)$ as the image by Π^d of a subtree of $\mathcal{T}^d$. Recall that $\Gamma(t) := \inf\{s \in [0,\sigma] : \mathfrak{L}_s > t\}$, where $\mathfrak{L}$ is defined in (8.20).

Lemma 8.2. *Under* $\mathbb{N}^*$*, the following property holds:*
Let $0 \leqslant s \leqslant t \leqslant \mathcal{Z}_0$*, such that the mapping* $r \mapsto \Gamma(r)$ *is continuous both at* s *and at* t*. We have* $\mathcal{V}(s,t) = \Pi^d\big([\mathcal{E}^d_{\Gamma(s)}, \mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d}\big)$ *and the intersection between* $\mathcal{V}(s,t)$ *and* $\mathcal{V}(0,s) \cup \mathcal{V}(t, \mathcal{Z}_0)$ *is the set* $\Pi^d([\![\mathcal{E}^d_{\Gamma(s)}, \mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d})$*.*

Proof. For every $\ell \in [0,\sigma]$, recall the notation γ_ℓ, introduced in (8.18) and defined for the labeled tree $\mathcal{T}^d$, and the fact that the map $r \mapsto \Pi^d(\gamma_\ell(r))$ is a geodesic path from $\Pi^d(\mathcal{E}^d_\ell)$ to the boundary $\partial\mathbb{D}$. Moreover by (H_3), any geodesic path to the boundary can be obtained as $\Pi^d \circ \gamma_\ell$ for some $\ell \in [0,\sigma]$. Fix $0 \leqslant s \leqslant t \leqslant \mathcal{Z}_0$ such that $r \mapsto \Gamma(r)$ is continuous both at s and at t. The measure $d\mathfrak{L}_s$ is supported on $\{s : \widehat{W}_s = 0\}$, and so by continuity we have $\Lambda^d_{\Gamma(s)} = \Lambda^d_{\Gamma(t)} = 0$. We deduce that for every $\ell \in [\Gamma(s),\Gamma(t)]$:

$$\Gamma(s) \leqslant \inf\left\{r \geqslant \ell :\ \Lambda^d_{\mathcal{E}^d_s} = 0\right\} \leqslant \Gamma(t).$$

Moreover, since $\Pi^d(\Gamma(s))$ and $\Pi^d(\Gamma(t))$ are leaves (by (H_1)), we have $\Pi^d([\Gamma(s),\Gamma(t)]) = [\mathcal{E}^d_{\Gamma(s)},\mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d}$. We get that, for every $\ell \in [\Gamma(s),\Gamma(t)]$, we must have $\gamma_\ell(\Lambda_\ell) \in [\mathcal{E}^d_{\Gamma(s)},\mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d}$. Consequently, for every $\ell \in [\Gamma(s),\Gamma(t)]$, the simple geodesic $\Pi^d \circ \gamma_\ell$ hits the boundary in $\mathcal{L}(s,t)$. In particular, we deduce that:

$$\Pi^d\big([\mathcal{E}^d_{\Gamma(s)},\mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d}\big) \subset \mathcal{V}(s,t).$$

Let us now explain why this inclusion is in fact an equality. Since s,t are continuity times of Γ, for every $\ell' \notin [\Gamma(s),\Gamma(t)]$ there exists $s' < s < t < t'$ such that $\ell' \in [0,\Gamma(s')] \cup [\Gamma(t'),\sigma]$. We can then apply the exact same argument used above to see that, for every such ℓ', the geodesic path $\Pi^d \circ \gamma_{\ell'}$ has to hit the boundary outside $\mathcal{L}(s,t)$. Knowing that all the geodesics to the boundary are simple geodesics, we deduce that $\mathcal{V}(s,t) = \Pi^d\big([\mathcal{E}^d_{\Gamma(s)},\mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d}\big)$ and that the intersection between $\mathcal{V}(s,t)$ and $\mathcal{V}(0,s) \cup \mathcal{V}(t,\mathcal{Z}_0)$ is precisely the image by $\Pi^d \circ \mathcal{E}^d$ of the set:

$$\Big\{\ell \in [\Gamma(s),\Gamma(t)] :\ \exists \ell' \notin [\Gamma(s),\Gamma(t)] \text{ such that } \mathcal{E}^d_\ell = \mathcal{E}^d_{\ell'}\Big\}.$$

This concludes the proof of the lemma, since the set above is exactly $[\![\mathcal{E}^d_{\Gamma(s)},\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}$. ☐

Arguing under $\mathbb{N}^*_0(\mathrm{d}\omega)$, for every $\ell \in (0,\sigma)$, recall the notation $\mathcal{N}_{(\ell)}$ and $\mathcal{N}'_{(\ell)}$, standing for the two point measures that account for the labeled subtrees branching off the ancestral line of $p_{(\omega)}(\ell)$ in the genealogical tree $\mathcal{T}_{(\omega)}$ (see 8.2.1). The pairs $(W_\ell,\mathcal{N}_{(\ell)})$ and $(W_t,\mathcal{N}'_{(\ell)})$ verify the assumptions made at the beginning of Section 8.2.2. Thus we can consider the labeled $\mathbb{R}$-trees $\mathcal{T}^d_{(\ell)}$ and $\mathcal{T}^{d,\prime}_{(\ell)}$ associated with $(W_\ell,\mathcal{N}_{(\ell)})$ and $(W_\ell,\mathcal{N}'_{(\ell)})$ respectively. Remark that we can identify $\mathcal{T}^d_{(\ell)}$ (resp. $\mathcal{T}^{d,\prime}_{(\ell)}$) with $\{\mathcal{E}^d_r :\ r \leqslant \ell\}$ (resp. $\{\mathcal{E}^d_r :\ r \geqslant \ell\}$). We will make this identification in what follows. We now introduce the pseudo-distances $\Delta^d_{(\ell)}$ and $\Delta^{d,\prime}_{(\ell)}$ associated with $(W_\ell,\mathcal{N}_{(\ell)})$ and $(W_\ell,\mathcal{N}'_{(\ell)})$ respectively. If $\Lambda^d_\ell = 0$, then it is straightforward to verify from the fact that $(0,W)$ verifies (H_2,H_3), that the two pairs $(W_\ell,\mathcal{N}_{(\ell)})$ and $(W_\ell,\mathcal{N}'_{(\ell)})$ also verify (H_2,H_3). Suppose from now on that $\Lambda^d_\ell = 0$ and keep the notation $\Delta^d_{(\ell)}$ and $\Delta^{d,\prime}_{(\ell)}$ for the continuous extensions of $\Delta^d_{(\ell)}$ and $\Delta^{d,\prime}_{(\ell)}$, respectively on $\mathcal{T}^d_{(\ell)}$ and $\mathcal{T}^{d,\prime}_{(\ell)}$. By convention we take $\Delta^d_{(\ell)}(u,v) = \infty$ (resp. $\Delta^{d,\prime}_{(\ell)}(u,v) = \infty$) if u or v are not both elements of $\mathcal{T}^d_{(\ell)}$ (resp. of $\mathcal{T}^{d,\prime}_{(\ell)}$).

Lemma 8.3. *Under $\mathbb{N}^*$ the following properties hold:*

Let $t \in (0,\mathcal{Z}_0)$, such that t is a continuous time of Γ and write $\widetilde{\Delta}^d_{(\Gamma(t))}$ and $\widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}$ for the intrinsic distances on $\mathcal{V}^\circ(0,t)$ and $\mathcal{V}^\circ(t,\mathcal{Z}_0)$ respectively. Then the intrinsic distances $\widetilde{\Delta}^d_{(\Gamma(t))}$ and

$\widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}$ *have (respectively) a continuous extension on* $\mathcal{V}(0,t)$ *and on* $\mathcal{V}(t,\mathcal{Z}_0)$. *We point the spaces* $\mathcal{V}(0,t)$ *and* $\mathcal{V}(t,\mathcal{Z}_0)$ *at* $\rho := \Pi^d(0)$ *and we equip the set* $\mathcal{V}(0,t)$ *(resp.* $\mathcal{V}(t,\mathcal{Z}_0)$ *) with the continuous extension of* $\widetilde{\Delta}^{d}_{(\Gamma(t))}$ *(resp.* $\widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}$*) and also with the restriction of* $\mathrm{Vol}^{\mathrm{d}}$ *to* $\mathcal{V}(0,t)$ *(resp.* $\mathcal{V}(t,\mathcal{Z}_0)$ *). We then have*

$$\mathcal{V}(0,t) = \Theta(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))}) \ ; \ \mathcal{V}(t,\mathcal{Z}_0) = \Theta(W_{\Gamma(t)},\mathcal{N}'_{(\Gamma(t))}), \tag{8.27}$$

and

$$\Delta^{\mathrm{d}}(x,y) := \inf_{x_0=u,x_1,\dots x_n=v} \sum_{i=1}^{n} \widetilde{\Delta}^{d}_{(\Gamma(t))}(x_{i-1},x_i) \wedge \widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}(x_{i-1},x_i). \tag{8.28}$$

where the infimum is over all choices of the integer $n \geqslant 1$ *and all the finite sequences* $x_0,...,x_n$ *of elements of* $\mathbb{D}$ *verifying* $x_0 = x$ *and* $x_n = y$.

Equation (8.28) shows that one can recover $\mathbb{D}$ from the two Voronoï cells $\mathcal{V}(0,t)$ and $\mathcal{V}(t,\mathcal{Z}_0)$ – provided that $r \mapsto \Gamma(r)$ is continuous at t – gluing these two spaces along their common boundary $\mathcal{V}(0,t) \cap \mathcal{V}(t,\mathcal{Z}_0)$. We will explain this point in more details in a future version of this work.

Proof. Fix a time $t \in (0,\mathcal{Z}_0)$ of continuity of Γ. By Lemma 8.2, we have $\mathcal{V}(0,t) = \Pi^d([\mathcal{E}^d_{\Gamma(0)},\mathcal{E}^d_{\Gamma(t)}]_{\mathcal{T}^d})$, $\mathcal{V}(t,\mathcal{Z}_0) = \Pi^d([\mathcal{E}^d_{\Gamma(t)},\mathcal{E}^d_{\Gamma(\mathcal{Z}_0)}]_{\mathcal{T}^d})$ and it is easy to see that:

$$\partial\mathcal{V}(0,t) = \mathcal{L}(0,t) \cup \Pi^d([\![0,\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}) \text{ and } \partial\mathcal{V}(t,\mathcal{Z}_0) = \mathcal{L}(t,\mathcal{Z}_0) \cup \Pi^d([\![0,\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}).$$

Now recall that $\mathcal{T}^d_{(\Gamma(t))}$ (resp. $\mathcal{T}^{d,\prime}_{(\Gamma(t))}$) is identified with $\{\mathcal{E}^d_r : r \leqslant \Gamma(t)\}$ (resp. $\{\mathcal{E}^d_r : r \geqslant \Gamma(t)\}$). As explained in the discussion above, the pairs $(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$ and $(W_{\Gamma(t)},\mathcal{N}'_{(\Gamma(t))})$ verify (H_2,H_3). Consider $v,w \in \mathcal{T}^d_{(\Gamma(t))}$ no belonging to the spine $[\![\mathcal{E}^d_0,\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}$. Now recall that $\Lambda_0 = \Lambda_{\Gamma(t)} = 0$, which implies that if $\Lambda^d_a > 0$ for every $a \in]v,w[_{\mathcal{T}^d}$, then the interval $[v,w]_{\mathcal{T}^d}$ is identified with the interval $[v,w]_{\mathcal{T}^d_{(\Gamma(t))}}$. We deduce by (H_3) that, for every $v,w \in \mathcal{T}^d_{(\Gamma(t))}$ no belonging to the spine $[\![\mathcal{E}^d_0,\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}$, we have $\Delta^{\mathrm{d}}(v,w) = 0$ iff $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}(v,w) = 0$. Moreover by the first assumption of (H_3), the only points of $[\![\mathcal{E}^d_0,\mathcal{E}^d_{\Gamma(t)}]\!]_{\mathcal{T}^d}$ that can be identify by Δ^{d} or $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}$ with another point of the tree $\mathcal{T}^d$ are the extremities $\mathcal{E}^d_0$ and $\mathcal{E}^d_{\Gamma(t)}$. But since 0 and t are continuity times of Γ, we see by the third assumption of (H_3), that either $\mathcal{E}^d_0$ nor $\mathcal{E}^d_{\Gamma(t)}$ are identify with others points of $\mathcal{T}^d$ (by Δ^{d} or $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}$). So we deduce the following fact:

For every $v,w \in \mathcal{T}^d_{(\Gamma(t))}$, we have $\Delta^{\mathrm{d}}(v,w) = 0$ if and only if $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}(v,w) = 0$.

The same argument works for $\mathcal{T}^d_{(\Gamma(t))}$ replaced by $\mathcal{T}^{d,\prime}_{(\Gamma(t))}$ and $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}$ replaced by $\Delta^{d,\prime}_{(\Gamma(t))}$. Consequently, we may and will identify the set of all points of $\Theta(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$ (resp. $\Theta(W_{\Gamma(t)},\mathcal{N}'_{(\Gamma(t))})$) with the set of all points of $\mathcal{V}(0,t)$ (resp. $\mathcal{V}(t,\mathcal{Z}_0)$). Moreover, by performing this identification, the set $\Theta^\circ(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$ is identified with $\mathcal{V}^\circ([0,t])$, and $\partial\Theta(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$ with $\partial\mathcal{V}(0,t)$. The same results hold if we replace $\mathcal{N}_{(\Gamma(t))}$ by $\mathcal{N}'_{(\Gamma(t))}$ and $\mathcal{V}(0,t)$ by $\mathcal{V}(t,\mathcal{Z}_0)$. In particular we may and will interpret $\Delta^d_{(\Gamma(t))}$ (resp. $\Delta^{d,\prime}_{(\Gamma(t))}$) as a distance on $\mathcal{V}(0,t)$ (resp. $\mathcal{V}(t,\mathcal{Z}_0)$). To obtain the first part of the lemma and (8.27) we need to show that the restriction of $\Delta^d_{(\Gamma(t))}$ to $\Theta^\circ(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$ coincides with the intrinsic distance $\widetilde{\Delta}^d_{(\Gamma(t))}$ on $\mathcal{V}^\circ(0,t)$ and that the same result for $\Delta^d_{(\Gamma(t))}$ replaced by $\Delta^{d,\prime}_{(\Gamma(t))}$ and $\widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}$ replaced by $\Theta^{\circ}(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))})$. Let us deal with the case of $\Delta^{d}_{(\Gamma(t))}$, since the exact same argument works for $\Delta^{d,\prime}_{(\Gamma(t))}$.

We first prove that the intrinsic distance between two points, x and y, of $\mathcal{V}^{\circ}([0,t])$ is bounded above by $\Delta^{d}_{(\Gamma(t))}(x,y)$. To this end, write $\partial\mathcal{T}^{d}_{(\Gamma(t))}$ for the boundary of $\mathcal{T}^{d}_{(\Gamma(t))}$ and set $\mathcal{T}^{d,\circ}_{(\Gamma(t))}=\mathcal{T}^{d}_{(\Gamma(t))}\backslash\partial\mathcal{T}^{d}_{(\Gamma(t))}$. We claim that, for every $v,w\in\mathcal{T}^{d,\circ}_{(\Gamma(t))}$ with $\inf_{[v,w]_{\mathcal{T}^{d}_{\Gamma(t)}}}\Lambda^{d}>0$, the quantity

$$\Lambda^{d}_{v}+\Lambda^{d}_{w}-2\inf_{[v,w]_{\mathcal{T}^{d}_{(\Gamma(t))}}}\Lambda^{d},$$

is the length of a path taking values in $\mathcal{V}^{\circ}([0,t])$ and connecting $\Pi^{d}(v)$ to $\Pi^{d}(w)$. This implies – by the previous display and the definition of $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}$ as an infimum (8.16) – that $\widetilde{\Delta}^{d}_{(\Gamma(t))}$ is bounded above by $\Delta^{\mathrm{d}}_{(\Gamma(\mathrm{t}))}$ on $\mathcal{V}^{\circ}([0,t])$.

Let us explain why this claim holds. Start by fixing $v,w\in\mathcal{T}^{d,\circ}_{(\Gamma(t))}$, such that $\inf_{[v,w]_{(\mathcal{T}^{d}_{\Gamma(t)})}}\Lambda^{d}>0$. In particular the interval $[v,w]_{\mathcal{T}^{d}_{\Gamma(t)}}$ is identified with the interval $[v,w]_{\mathcal{T}^{d}}$.

Now consider $[r_1,r_2]\subset[0,\sigma]$, the smallest interval verifying $(\Pi^{d}(r_1),\Pi^{d}(r_2))=(v,w)$, and set $M_{u,w}:=\inf_{[v,w]_{\mathcal{T}^{d}}}\Lambda^{d}$. By construction, we have $M_{u,w}>0$ and so by concatenating the two geodesics $\big(\Pi^{d}(\gamma_{r_1}(r))\big)_{r\leqslant M_{u,v}}$ and $\big(\Pi^{d}(\gamma_{r_2}(M_{u,v}-r))\big)_{r\leqslant M_{u,v}}$ we construct a path from $\Pi^{d}(u)$ to $\Pi^{\mathrm{d}}(v)$ whose length with respect to Δ^{d} is equal to

$$\Lambda^{d}_{v}+\Lambda^{d}_{w}-2\inf_{[v,w]_{\mathcal{T}^{d}_{(\Gamma(t))}}}\Lambda^{d}.$$

Moreover this path stays in $\Pi^{d}([v,w]_{\mathcal{T}^{d}})\subset\mathcal{V}(0,t)$ (the inclusion comes from Lemma 8.2) and does not hit $\partial\mathcal{V}(0,t)$ by the remark below (H_3). This gives our claim.

Let us now prove the reverse bound. To this end, we need to verify that, if $\gamma:[0,1]\to\mathcal{V}^{\circ}([0,t])$ is a path such that $\gamma(0)=x$ and $\gamma(1)=y$, then the length of γ (with respect to Δ^{d}) is bounded below by $\Delta^{d}_{(\Gamma(t))}(x,y)$. First let $\delta>0$ such that the distance between the range of γ and $\partial\mathcal{V}(0,t)$, with respect to Δ^{d}, is bigger than 2δ. We also introduce an integer $n\geqslant 1$ such that $\Delta^{\mathrm{d}}(\gamma(s),\gamma(t))<\delta$ for every s,t such that $|s-t|<n^{-1}$. Remark that the length of γ is bounded below by $\sum_{i=1}^{n}\Delta^{\mathrm{d}}(\gamma(\frac{i}{n}),\gamma(\frac{i+1}{n}))$, meaning that, in order to get the desired result, it is enough to verify that:

$$\Delta^{\mathrm{d}}\big(\gamma(\frac{i}{n}),\gamma(\frac{i+1}{n})\big)\geqslant\Delta^{d}_{(\Gamma(t))}\big(\gamma(\frac{i}{n}),\gamma(\frac{i+1}{n})\big),$$

for every $0\leqslant i\leqslant n-1$. Fix $0\leqslant i\leqslant n-1$ and recall the definition (8.16) of $\Delta^{\mathrm{d}}(\gamma(\frac{i}{n}),\gamma(\frac{i+1}{n}))$ as an infimum over possible choices of $u_0=\gamma(\frac{i}{n}),u_1,\dots,u_p=\gamma(\frac{i+1}{n})$ in $\mathcal{T}^{d}$. Remark that, by the definition of δ, we may restrict our attention to the choices of $u_0,u_1,\dots,u_p$ such that:

- for every $0\leqslant j\leqslant p$ the distance – with respect to Δ^{d} – between $\Pi^{d}(u_j)$ and the boundary $\partial\mathcal{V}(0,t)$ is bigger than δ;

- for every $0 \leqslant j \leqslant p-1$, we have $\Delta^{d,\circ}(u_{j-1}, u_j) < \delta$.

But, since $\Delta^{d,\circ}(w, w') = \infty$ for every $w \in \mathcal{T}^{d,\circ}_{(\Gamma(t))}$ and $w' \in \mathcal{T}^{d,\prime,\circ}_{(\Gamma(t))}$, we deduce that for any such choice $u_0, u_1, \dots, u_p$ we have:

$$\sum_{j=0}^{p-1} \Delta^{d,\circ}(u_j, u_{j+1}) = \sum_{j=0}^{p-1} \Delta^{d,\circ}_{(\Gamma(t))}(u_j, u_{j+1}) \geqslant \Delta^{d}_{(\Gamma(t))}\big(\gamma(\frac{i}{n}), \gamma(\frac{i+1}{n})\big),$$

and we get the reverse bound $\widetilde{\Delta}^d_{(\Gamma(t))}(x,y) \geqslant \Delta^d_{(\Gamma(t))}(x,y)$. Finally it remains to prove (8.28) in order to conclude the proof of the lemma. To this end remark that for every $v, w \in \mathcal{T}^d$ if $\inf_{[v,w]_{\mathcal{T}^d}} \Lambda^d > 0$, then the interval $[v,w]_{\mathcal{T}^d}$ is identified with an interval of $\mathcal{T}^d_{(\Gamma(t))}$ or an interval of $\mathcal{T}^{d,\prime}_{(\Gamma(t))}$. This implies that for every $v, w \in \mathcal{T}^d$:

$$\Delta^{d,\circ}(v,w) \geqslant \Delta^d_{(\Gamma(t))}(v,w) \wedge \Delta^{d,\prime}_{(\Gamma(t))}(v,w) = \widetilde{\Delta}^d_{(\Gamma(t))}(v,w) \wedge \widetilde{\Delta}^{d,\prime}_{(\Gamma(t))}(v,w).$$

Formula (8.28) then follows directly by the definition of the pseudo-distance Δ^d as an infimum (8.16) and triangle inequality. □

We now apply Lemma 8.3 to the study of $\mathcal{V}_1$ and $\mathcal{V}_2$ under $\mathbb{N}^{*,\bullet}$. First remark that the mapping $t \mapsto \Gamma(t)$ only has countably many discontinuity times and consequently, $\mathbb{N}^{*,\bullet}$-a.e., the map Γ is continuous at $t^\bullet$. We can now apply Lemma 8.3 at $\Gamma(t^\bullet)$ to deduce that the intrinsic distances Δ^d_1 and Δ^d_2 (with the notation of Lemma 8.3: $\widetilde{\Delta}^d_{(\Gamma(t^\bullet))}$ and $\widetilde{\Delta}^{d,\prime}_{(\Gamma(t^\bullet))}$) have respectively a continuous extension on $\mathcal{V}_1$ and on $\mathcal{V}_2$. This gives the first part of Theorem 8.1 (*ii*). As in the statement of Theorem 8.1, we point $\mathcal{V}_1$ and $\mathcal{V}_2$ at $\Pi^d(0)$ and we equip the set $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) with the continuous extension of Δ^d_1 (resp. Δ^d_2) and with the restriction of $\mathrm{Vol}^{\mathrm{d}}$ to $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). By Lemma 8.3 we have

$$\mathcal{V}_1 = \Theta\big(W_{\Gamma(t^\bullet)}, \mathcal{N}_{(\Gamma(t^\bullet))}\big) \ ; \ \mathcal{V}_2 = \Theta\big(W_{\Gamma(t^\bullet)}, \mathcal{N}'_{(\Gamma(t^\bullet))}\big). \tag{8.29}$$

We are going to deduce Theorem 8.1 from the following result:

Proposition 8.1. *For every nonnegative measurable function $F : \mathcal{W} \times M(\mathcal{S}) \times M(\mathcal{S}) \to \mathbb{R}_+$, we have:*

$$\mathbb{N}^{*,\bullet}\big(F(W_{\Gamma(t^\bullet)}, \mathcal{N}_{(\Gamma(t^\bullet))}, \mathcal{N}'_{(\Gamma(t^\bullet))})\big) = 2\mathbf{n}(F(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}}, \widetilde{\mathcal{P}}'_{\mathbf{e}})).$$

Before giving the proof of Proposition 8.1, let us explain why Theorem 8.1 follows from (8.29) and Proposition 8.1. First, by Proposition 8.1, the distribution of $\big(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}}, \widetilde{\mathcal{P}}'_{\mathbf{e}}\big)$ under $2\mathbf{n}$ is the same as the distribution of $\big(W_{\Gamma(t^\bullet)}, \mathcal{N}_{(\Gamma(t^\bullet))}, \mathcal{N}'_{(\Gamma(t^\bullet))}\big)$ under $\mathbb{N}^{*,\bullet}$. In particular, the coding pairs $\big(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}}\big)$ and $\big(\mathbf{e}, \widetilde{\mathcal{P}}'_{\mathbf{e}}\big)$ verify (H_2, H_3) and we obtain Theorem 8.1 (*i*). Then, an application of (8.29) gives us Theorem 8.1 (*ii*).

Proof of Proposition 8.1. We want to show that, for every nonnegative measurable function F on $\mathcal{W} \times M(\mathcal{S}) \times M(\mathcal{S})$, we have:

$$\mathbb{N}^*\Big(\int_0^{\mathcal{Z}_0} dt F(W_{\Gamma(t)}, \mathcal{N}_{(\Gamma(t))}, \mathcal{N}'_{(\Gamma(t))})\Big) = 2\mathbf{n}(F(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}}, \widetilde{\mathcal{P}}'_{\mathbf{e}})). \tag{8.30}$$

We are going to show (8.30), for F of the form

$$F(W_{\Gamma(t)},\mathcal{N}_{(\Gamma(t))},\mathcal{N}'_{(\Gamma(t))}) \\ = \Big(1-\exp\big(-\lambda\big(Z(\mathcal{N}^{\vee}_{(\Gamma(t))})+Z(\mathcal{N}'^{,\vee}_{(\Gamma(t))})\big)\big)\Big)F_1(W^{\vee}_{\Gamma(t)})F_2(\mathcal{N}^{\vee}_{(\Gamma(t))})F_3(\mathcal{N}'^{,\vee}_{(\Gamma(t))}),$$

where $\lambda>0$ and the functions $F_1:\mathcal{W}\mapsto\mathbb{R}_+$ and $F_2,F_3:M(\mathcal{S})\mapsto\mathbb{R}_+$ are all continuous and bounded above by 1 (recall that the topology on $M(\mathcal{S})$ is defined by the distance (8.4)). We also assume that there exists $\delta>0$ such that, for every $\mathrm{w}\in\mathcal{W}$, $F_1(\mathrm{w})=0$ if $\sup\mathrm{w}<\delta$. An application of monotone convergence, taking the limit when $\lambda\to\infty$, gives (8.30) for F of the form $F_1\cdot F_2\cdot F_3$ satisfying the previous assumptions. The general case follows using standard approximation procedures. Fix $\lambda>0$ and three functions $F_1:\mathcal{W}\mapsto\mathbb{R}_+$ and $F_2,F_3:M(\mathcal{S})\mapsto\mathbb{R}_+$ as above and, to simplify notation, set $G(\mathcal{N}^{\vee},\mathcal{N}'^{,\vee})=1-\exp\big(-\lambda Z(\mathcal{N}^{\vee}_{(\Gamma(t))})-\lambda Z(\mathcal{N}'^{,\vee}_{(\Gamma(t))})\big)$. The reason why we introduce the function G is the fact that the quantities $\mathbb{N}_0^{*,\bullet}(G)$ and $\mathbf{n}(G)$ are finite even though the measures $\mathbb{N}_0^{*,\bullet}$ and $\mathbf{n}$ have infinite mass. Let us start by showing that we have:

$$\mathbb{N}_0^{*,\bullet}(G)=2\mathbf{n}(G)=\sqrt{6\lambda}. \tag{8.31}$$

First remark that under $\mathbb{N}^*$, we have $\mathcal{Z}_0=Z(\mathcal{N}^{\vee}_{(\Gamma(t))})+Z(\mathcal{N}'^{,\vee}_{(\Gamma(t))})$ for every $t\in[0,\mathcal{Z}_0]$ and that, in particular, an application of (8.13) gives:

$$\mathbb{N}^*\big(\mathcal{Z}_0(1-\exp(-\lambda\mathcal{Z}_0))\big)=\sqrt{\frac{3}{2\pi}}\int_0^{\infty}dz\ z^{-\frac{3}{2}}(1-\exp(-\lambda z))=\sqrt{6\lambda}.$$

On the other hand, recalling the notation $u_{\lambda,\mu}$ from (8.8) , we have:

$$\mathbf{n}\big(1-\exp(-Z(\mathcal{P}_{\mathbf{e}})-Z(\mathcal{P}'_{\mathbf{e}}))\big)=\mathbf{n}\big(1-\exp\big(-2\int_0^{\zeta_{\mathbf{e}}}dt\ u_{\lambda,0}(\mathbf{e}_t)\big)\big) \\ =\mathbf{n}\big(1-\exp\big(-3\int_0^{\zeta_{\mathbf{e}}}dt\ \big(\sqrt{\frac{3}{2}}\lambda^{-\frac{1}{2}}+\mathbf{e}_t\big)^{-2}\big)\big).$$

Now remark that the function $H(x)=\big(\sqrt{3/2}\lambda^{-\frac{1}{2}}+x\big)^{-2}$ is bounded and solves the differential equation $H''(x)=6(\sqrt{3/2}\lambda^{-\frac{1}{2}}+x)^{-2}H(x)$ so by (8.26) we get:

$$\mathbf{n}\big(1-\exp(-Z(\mathcal{P}_{\mathbf{e}})-Z(\mathcal{P}'_{\mathbf{e}}))\big)=-\frac{1}{2}\frac{H'(0)}{H(0)}=\sqrt{\frac{3}{2}}\lambda^{\frac{1}{2}},$$

which gives (8.31). Let us now turn to the case $(*)$. First remark that, by (8.20) and an application of Fatou's lemma, we have:

$$\mathbb{N}^*\Big(\int_0^{\mathcal{Z}_0}dt\ G(\mathcal{N}^{\vee}_{(\Gamma(t))},\mathcal{N}'^{,\vee}_{(\Gamma(t))})F_1(W^{\vee}_{\Gamma(t)})F_2(\mathcal{N}^{\vee}_{(\Gamma(t))})F_3(\mathcal{N}'^{,\vee}_{(\Gamma(t))})\Big) \\ =\mathbb{N}^*\Big(\int_0^{\sigma}d\mathfrak{L}_s\ \exp(-\lambda\mathcal{Z}_0)F_1(W^{\vee}_{\Gamma(t)})F_2(\mathcal{N}^{\vee}_{(\Gamma(t))})F_3(\mathcal{N}'^{,\vee}_{(\Gamma(t))})\Big) \\ \leqslant\liminf_{\varepsilon\to0}\varepsilon^{-2}\mathbb{N}^*\Big(\int_0^{\sigma}dt\ G(\mathcal{N}^{\vee}_{(t)},\mathcal{N}'^{,\vee}_{(t)})F_1(W_t^{\vee})F_2(\mathcal{N}^{\vee}_{(t)})F_3(\mathcal{N}'^{,\vee}_{(t)})\mathbb{1}_{W_t^{\vee}(0)\leqslant\varepsilon}\Big).$$

We are going to conclude by showing that:

$$2\mathbf{n}\big(G(\mathcal{P}_{\mathbf{e}},\mathcal{P}'_{\mathbf{e}})F_1(\mathbf{e})F_2(\widetilde{\mathcal{P}}_{\mathbf{e}})F_3(\widetilde{\mathcal{P}}'_{\mathbf{e}})\big) = \liminf_{\varepsilon\to 0}\varepsilon^{-2}\mathbb{N}^*\Big(\int_0^\sigma dt\; G(\mathcal{N}^{\vee}_{(t)},\mathcal{N}'^{,\vee}_{(t)})F_1(W^{\vee}_t)F_2(\mathcal{N}^{\vee}_{(t)})F_3(\mathcal{N}'^{,\vee}_{(t)})\mathbb{1}_{W^{\vee}_t(0)\leqslant\varepsilon}\Big). \quad (8.32)$$

Let us explain why the proposition follows from (8.32). First (8.32) gives:

$$\mathbb{N}^*\Big(\int_0^{\mathcal{Z}_0} dt\, G(\mathcal{N}^{\vee}_{(\Gamma(t))},\mathcal{N}'^{,\vee}_{(\Gamma(t))}))F_1(W^{\vee}_{\Gamma(t)})F_2(\mathcal{N}^{\vee}_{(\Gamma(t))})F_3(\mathcal{N}'^{,\vee}_{(\Gamma(t))})\Big) \leqslant 2\mathbf{n}\big(G(\mathcal{P}_{\mathbf{e}},\mathcal{P}'_{\mathbf{e}})F_1(\mathbf{e})F_2(\widetilde{\mathcal{P}}_{\mathbf{e}})F_2(\widetilde{\mathcal{P}}'_{\mathbf{e}})\big).$$

But we have the same inequality if we replace F_1, F_2 and F_3 by $(1-F_1),(1-F_2)$ and $(1-F_3)$. Then (8.31) implies that the previous inequality is an equality and gives the desired result. Let us now focus on the proof of (8.32). An application of the re-rooting formula (8.11) to the right term of (8.32) gives:

$$\mathbb{N}^*\Big(\int_0^\sigma dt\; G(\mathcal{N}^{\vee}_t,\mathcal{N}'^{,\vee}_t)F_1(W^{\vee}_t)F_2(\mathcal{N}^{\vee}_{(t)})F_3(\mathcal{N}'^{,\vee}_{(t)})\mathbb{1}_{W^{\vee}_t(0)\leqslant\varepsilon}\Big) = 2\int_0^\varepsilon dr\; \mathbb{E}_r\big[G(\widetilde{\mathcal{P}}_B,\widetilde{\mathcal{P}}'_B)F_1(B)F_2(\widetilde{\mathcal{P}}_B)F_3(\widetilde{\mathcal{P}}'_B)\big],$$

or every $\varepsilon>0$. Since conditionally on B, the distribution of $\big(\mathcal{P}_B,\mathcal{P}'_B\big)$ is $\mathbb{P}^{(B)}\otimes\mathbb{P}^{(B)}$, we obtain:

$$2\int_0^\varepsilon dr\; \mathbb{E}_r\big[G(\widetilde{\mathcal{P}}_B,\widetilde{\mathcal{P}}'_B)F_1(B)F_2(\widetilde{\mathcal{P}}_B)F_3(\widetilde{\mathcal{P}}'_B)\big] = 2\int_0^\varepsilon dr\; \mathbb{E}_r\big[\overline{F}_1(B)\big],$$

with $\overline{F}_1(B) := F_1(B)\mathbb{E}^{(B)}\otimes\mathbb{E}^{(B)}\big[G(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}')F_2(\widetilde{\mathcal{P}})F_3(\widetilde{\mathcal{P}}')\big]$. The functions $\overline{F}_1$ is continuous on $\mathcal{W}_+$ and there exists $\delta>0$ such that $\overline{F}_1(\mathrm{w})=0$ if $\sup\mathrm{w}<\delta$. By standard properties of the Itô measure, we have:

$$\lim_{r\to 0}\frac{1}{2r}\mathbb{E}_r[\overline{F}_1(B)] = \mathbf{n}\big(\overline{F}_1(\mathbf{e})\big).$$

□

8.3.2 Perimeter and volume of the Brownian Voronoï cell

In this section we study the Brownian Voronoï cell i.e. the space $\Theta(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}})$ under the measure $\mathbf{n}$. As previously seen we denote the tree associated with $(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}})$ by $\mathfrak{H}_{\mathbf{e}}$. Recall that the "boundary" $\partial\mathfrak{H}_{\mathbf{e}}$ is divide in two parts, namely the set $\partial_1\mathfrak{H}_{\mathbf{e}}$ of all points with zero label and the spine $\partial_2\mathfrak{H}_{\mathbf{e}} := [0,\zeta_{\mathbf{e}}]$. By Lemmas 8.2 and 8.3 and Proposition 8.1, the space $\Theta(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}})$ is a.e. homeomorphic to the open disk and its boundary is the image of $\partial\mathcal{T}_{\mathbf{e}}$ under the canonical projection from $\mathfrak{H}_{\mathbf{e}}$ onto $\Theta(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}})$. We interpret $Z(\mathcal{P}_{\mathbf{e}})$ as the boundary length of the projection of $\partial_1\mathfrak{H}_{\mathbf{e}}$ and $\zeta_{\mathbf{e}}$ as the boundary length of the projection of $\partial_2\mathfrak{H}_{\mathbf{e}}$. Actually, by (8.19) and (8.20), the last assumption of (H_3) gives that the quantity $Z(\mathcal{P}_{\mathbf{e}})$ has the following nice geometric interpretation:

$$Z(\mathcal{P}_{\mathbf{e}}) := \lim_{\varepsilon\to 0}\varepsilon^{-2}\cdot\mathrm{Vol}_{\mathbf{e}}\big(\{x\in\Theta(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}}):\ \Delta_{\mathbf{e}}(x,\partial_1\mathfrak{H}_{\mathbf{e}}/\Delta_{\mathbf{e}})\leqslant\varepsilon\}\big)$$

where the distance (resp. the volume measure) of $\Theta(\mathbf{e},\widetilde{\mathcal{P}}_{\mathbf{e}})$ is denoted by $\Delta_{\mathbf{e}}$ (resp. by $\mathrm{Vol}_{\mathbf{e}}$). Remark that the total volume of $\mathrm{Vol}_{\mathbf{e}}$ is $\mathcal{Y}(\mathcal{P}_{\mathbf{e}})$. The goal of this section is to study the triplet $\big(\zeta_{\mathbf{e}},Z(\mathcal{P}_{\mathbf{e}}),\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)$. Let us start by determining its Laplace transform

Lemma 8.4. *For every $\lambda, \gamma, \mu \in \mathbb{R}_+$, we have:*

$$\mathbf{n}\Big(1 - \exp\big(-(\gamma-\sqrt{\mu})\zeta_{\mathbf{e}} - (\lambda-\sqrt{\mu})Z(\mathcal{P}_{\mathbf{e}}) - \mu/2\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\Big) = \sqrt{\frac{2}{3}} \cdot \frac{2\lambda^{\frac{3}{2}} + 2\sqrt{3}(\lambda-\sqrt{\mu})\sqrt{\gamma} + 3(\gamma-\sqrt{\mu})\sqrt{\lambda} + \sqrt{3}\gamma^{\frac{3}{2}}}{2\lambda + 2\gamma - \sqrt{\mu} + 2\sqrt{3}\sqrt{\gamma\lambda}}.$$

Proof. By a scaling argument, for every $r > 0$, the distribution of $\big(r\zeta_{\mathbf{e}}, rZ(\mathcal{P}_{\mathbf{e}}), r^2\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)$ under $\mathbf{n}$ is the same as the distribution of $\big(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\mathbf{e}}), \mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)$ under $\sqrt{r}\mathbf{n}$. So it is sufficient to prove the lemma for $\mu = 1$. Fix $\lambda, \gamma > 0$. By (8.10) we have:

$$\mathbf{n}\big(1 - \exp\big(-(\gamma-1)\zeta_{\mathbf{e}} - (\lambda-1)Z(\mathcal{P}_{\mathbf{e}}) - 1/2\ \mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\big) = \mathbf{n}\big(1 - \exp\big(-\int_0^{\zeta_{\mathbf{e}}} dt(\gamma - 1 + 2u_{\lambda-1,\frac{1}{2}}(\mathbf{e}_t))\big)\big).$$

Recall the definition of $g_{\lambda,\mu}$ after (8.8), and observe that $u_{\lambda-1,\frac{1}{2}}(x) = 3/2 \cdot g^2\big(x\big) - 1$ with the notation $g(x) := g_{\lambda-1,\frac{1}{2}}(x + g^{(-1)}_{\lambda-1,\frac{1}{2}}(\sqrt{\frac{2}{3}\lambda}))$. So by (8.26) to compute the quantity above we need to find a two times differentiable bounded function H solving the differential equation:

$$\frac{\partial^2}{\partial x^2}H(x) = \big(6g^2(x) + 2\gamma - 6\big)\cdot H(x). \tag{8.33}$$

Using the fact that $g'(x) = 1 - g^2(x)$ and performing some tedious calculations one can verify that we can take

$$H(x) = (3g^2(x) + 3\sqrt{2\gamma}g(x) + 2\gamma - 1)\cdot\exp(-\sqrt{2\gamma}\cdot x).$$

The lemma then follows from (8.26) and a straightforward computation. □

We now give some consequences of Lemma 8.4, starting by the study of the boundary lengths $\big(Z(\mathcal{P}_{\mathbf{e}}), \zeta_{\mathbf{e}}\big)$.

Lemma 8.5. *The distribution of $(Z(\mathcal{P}_{\mathbf{e}}), \zeta_{\mathbf{e}})$ is:*

$$\sqrt{\frac{3}{8\pi}}\cdot\Big(\exp\big(i\frac{\pi}{3}\big)\big(x+\exp\big(i\frac{\pi}{3}\big)\cdot y\big)^{-\frac{5}{2}} + \exp\big(-i\frac{\pi}{3}\big)\big(x+\exp\big(-i\frac{\pi}{3}\big)\cdot y\big)^{-\frac{5}{2}}\Big)dxdy.$$

Proof. By Lemma (8.4) taking $\mu = 0$ we obtain:

$$\mathbf{n}\big(1-\exp(-\lambda Z(\mathcal{P}_{\mathbf{e}}) - \gamma\zeta_{\mathbf{e}})\big) = \sqrt{\frac{2}{3}}\cdot\frac{2\lambda^{\frac{3}{2}} + 2\sqrt{3}\lambda\sqrt{\gamma} + 3\gamma\sqrt{\lambda} + \sqrt{3}\gamma^{\frac{3}{2}}}{2\lambda + 2\gamma + 2\sqrt{3}\sqrt{\gamma\lambda}}. \tag{8.34}$$

One can then make two Laplace inversions using the partial fraction decomposition of (8.34) noting that $\lambda + \sqrt{3\gamma\lambda} + \gamma = (\sqrt{\lambda} + \exp\big(i\frac{\pi}{6}\big)\sqrt{\gamma})\cdot(\sqrt{\lambda} + \exp\big(-i\frac{\pi}{6}\big)\sqrt{\gamma})$. Although it is achievable, it remains very technical. To bypass this issue, remark that we also have:

$$\begin{aligned}2\lambda^{\frac{3}{2}} + 2\sqrt{3\gamma}\lambda + 3\gamma\sqrt{\lambda} + \sqrt{3}\gamma^{\frac{3}{2}} &= \big(\sqrt{\lambda} + \exp\big(i\frac{\pi}{6}\big)\sqrt{\gamma}\big)\cdot\big(\lambda + \exp\big(-i\frac{\pi}{6}\big)\sqrt{\lambda\gamma} + \exp\big(-i\frac{\pi}{3}\big)\gamma\big)\\ &\quad + \big(\sqrt{\lambda} + \exp\big(-i\frac{\pi}{6}\big)\sqrt{\gamma}\big)\cdot\big(\lambda + \exp\big(i\frac{\pi}{6}\big)\sqrt{\lambda\gamma} + \exp\big(i\frac{\pi}{3}\big)\gamma\big).\end{aligned}$$

Consequently we can rewrite (8.34) as follows:

$$\mathbf{n}\big(1-\exp(-\lambda Z(\mathcal{P}_{\mathbf{e}})-\gamma\zeta_{\mathbf{e}})\big)$$
$$=\sqrt{\frac{2}{3}}\cdot\Big(\frac{\lambda+\exp\big(-i\frac{\pi}{6}\big)\sqrt{\lambda\gamma}+\exp\big(-i\frac{\pi}{3}\big)\gamma}{\sqrt{\lambda}+\exp\big(-i\frac{\pi}{6}\big)\sqrt{\gamma}}+\frac{\lambda+\exp\big(i\frac{\pi}{6}\big)\sqrt{\lambda\gamma}+\exp\big(i\frac{\pi}{3}\big)\gamma}{\sqrt{\lambda}+\exp\big(i\frac{\pi}{6}\big)\sqrt{\gamma}}\Big).$$

We can then use formula (A.3) in the Appendix – and the remark below this formula – to obtain:

$$\mathbf{n}\big(1-\exp(-\lambda Z(\mathcal{P}_{\mathbf{e}})-\gamma\zeta_{\mathbf{e}})\big)=\sqrt{\frac{3}{8\pi}}\cdot\int_{\mathbb{R}_+^2}\frac{dxdy}{(x+y)^{\frac{5}{2}}}\Big(1-\exp\big(-\lambda x-\exp\big(-i\frac{\pi}{3}\big)\gamma y\big)\Big)$$
$$+\sqrt{\frac{3}{8\pi}}\cdot\int_{\mathbb{R}_+^2}\frac{dxdy}{(x+y)^{\frac{5}{2}}}\Big(1-\exp\big(-\lambda x-\exp\big(i\frac{\pi}{3}\big)\gamma y\big)\Big)$$

and we obtain the desired result by a simple change of variables. □

In this preliminary version, we have not been able to determine the density of $(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\mathbf{e}}), \mathcal{Y}(\mathcal{P}_{\mathbf{e}}))$. For the time being we give some explicit conditional Laplace transform.

Proposition 8.2. *For every $\ell>0$ and $\mu>0$, we have:*

$$\mathbf{n}\Big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\big|\,Z(\mathcal{P}_{\mathbf{e}})+\zeta_{\mathbf{e}}=\ell\Big)$$
$$=\exp(-\sqrt{2\mu}\ell)\cdot\Big(1+\sqrt{2\mu}\cdot\ell+\sqrt{\frac{\pi}{2(2+\sqrt{3})}}(2\mu)^{\frac{3}{4}}\ell^{\frac{3}{2}}\exp\big(\frac{\sqrt{2\mu}\cdot\ell}{2(2+\sqrt{3})}\big)\cdot erf\Big(\sqrt{\frac{\sqrt{2\mu}\cdot\ell}{2(2+\sqrt{3})}}\Big)\Big)$$

Proof. An application of the scaling property shows that it is sufficient to prove the proposition for $\mu=\frac{1}{2}$. To simplify notation, we set $K_{\mathbf{e}}:=Z(\mathcal{P}_{\mathbf{e}})+\zeta_{\mathbf{e}}$ and we note that, for every $\lambda>0$, we have:

$$\mathbf{n}\Big(1-\exp\big(-(\lambda-1)K_{\mathbf{e}}-\frac{1}{2}\cdot\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\Big)-\mathbf{n}\Big(1-\exp\big(-\lambda K_{\mathbf{e}}\big)\Big)$$
$$=\mathbf{n}\Big(\exp\big(-\lambda K_{\mathbf{e}}\big)\cdot\Big(1-\exp\big(K_{\mathbf{e}}-\frac{1}{2}\cdot\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\Big)\Big).$$

To simplify notation set $G(\ell):=\mathbf{n}\big(1-\exp\big(\ell-1/2\cdot\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\mid K_{\mathbf{e}}=\ell\big)$. Remark that by an application of the scaling property the function $\ell\mapsto G(\ell)$ is continuous. We can now use Lemma 8.4 to derive the following formulas:

$$\mathbf{n}\big(1-\exp(-\lambda K_{\mathbf{e}})\big):=\frac{1+\sqrt{3}}{\sqrt{6}}\lambda^{\frac{1}{2}}\text{ and }\mathbf{n}\big(\exp(-\lambda K_{\mathbf{e}})G(K_{\mathbf{e}})\big)=-\frac{5+3\sqrt{3}}{\sqrt{6}}\cdot\frac{\lambda^{\frac{1}{2}}}{2(2+\sqrt{3})\lambda-1}.$$

The left formula gives that the distribution of $K_{\mathbf{e}}$ under $\mathbf{n}$ is $\frac{1+\sqrt{3}}{2\sqrt{6\pi}}\ell^{-\frac{3}{2}}\mathrm{d}\ell$. We derive that:

$$\frac{1+\sqrt{3}}{2\sqrt{6\pi}}\int_0^\infty\mathrm{d}\ell\exp(-\lambda\ell)\ell^{-\frac{3}{2}}G(\ell)=-\frac{1+\sqrt{3}}{2\sqrt{6}}\lambda^{-\frac{1}{2}}+\frac{1+\sqrt{3}}{2\sqrt{6}}\cdot\frac{1}{\lambda^{\frac{1}{2}}\big(2(2+\sqrt{3})\lambda-1\big)},$$

and by performing a Laplace inversion, we get:

$$\frac{1+\sqrt{3}}{2\sqrt{6\pi}}\ell^{-\frac{3}{2}}G(\ell)=-\frac{1+\sqrt{3}}{2\sqrt{6\pi}}\ell^{-\frac{1}{2}}+\frac{1+\sqrt{3}}{4\cdot\sqrt{6+3\sqrt{3}}}\exp\big(\frac{\ell}{2(2+\sqrt{3})}\big)\mathrm{erf}\cdot\Big(\sqrt{\frac{\ell}{2(2+\sqrt{3})}}\Big)$$

for Lebesgue almost every $\ell > 0$. By continuity the previous equation holds for every $\ell > 0$ and allows us to obtain the desired result. $\square$

We can give a similar result when conditioning on the boundary length $Z(\mathcal{P}_{\mathbf{e}})$ or on $\zeta_{\mathbf{e}}$. Using the notation of the Appendix:

$$\chi_1(x) := \frac{1}{\sqrt{\pi}} x^{-1/2} - \operatorname{erfc}(\sqrt{x}) \exp(x),$$

we have:

Proposition 8.3. *For every $\ell > 0$ and $\mu > 0$, we have:*

$$\begin{aligned}\mathbf{n}\Big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\Big|\, Z(\mathcal{P}_{\mathbf{e}}) = \ell\Big) \\ = \exp(-\sqrt{2\mu\ell})\cdot\Big(1 + \frac{\sqrt{\pi}}{2}\cdot(2\mu)^{\frac{3}{4}}\ell^{\frac{3}{2}}\cdot\Big(\chi_1\Big(\frac{2-\sqrt{3}}{2}\sqrt{2\mu\ell}\Big) - \chi_1\Big(\frac{2+\sqrt{3}}{2}\sqrt{2\mu\ell}\Big)\Big)\Big)\end{aligned}$$

and

$$\begin{aligned}\mathbf{n}\Big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\Big|\, \zeta_{\mathbf{e}} = \ell\Big) \\ = \exp(-\sqrt{2\mu\ell})\cdot\Big(1 + \frac{\sqrt{\pi}}{2\sqrt{3}}\cdot(2\mu)^{\frac{3}{4}}\ell^{\frac{3}{2}}\cdot\Big(\chi_1\Big(\frac{2-\sqrt{3}}{2}\sqrt{2\mu\ell}\Big) + \chi_1\Big(\frac{2+\sqrt{3}}{2}\sqrt{2\mu\ell}\Big)\Big)\Big).\end{aligned}$$

Proof. The proof is similar as the one of Proposition 8.2. We focus on the computation of

$$\mathbf{n}\Big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\Big|\, Z(\mathcal{P}_{\mathbf{e}}) = \ell\Big)$$

since the same method works to compute $\mathbf{n}\big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\big|\,\zeta_{\mathbf{e}} = \ell\big)$. By an application of the scaling property, it is enough to compute the quantity $\mathbf{n}\big(\exp\big(-\mu\mathcal{Y}(\mathcal{P}_{\mathbf{e}})\big)\,\big|\, Z(\mathcal{P}_{\mathbf{e}}) = \ell\big)$ for $\mu = 1/2$. To simplify notation, introduce the function $G(\ell) := \mathbf{n}\big(1 - \exp(\ell - \mathcal{Y}_{\mathbf{e}})\,|\, Z(\mathcal{P}_{\mathbf{e}}) = \ell\big)$ (which is a continuous function by scaling). Let us now take $\lambda > 0$ and remark that, by Lemma 8.4, we have:

$$\mathbf{n}\Big(1 - \exp\big(-\lambda Z(\mathcal{P}_{\mathbf{e}})\big)\Big) = \sqrt{\frac{2}{3}\cdot\lambda} \text{ and } \mathbf{n}\Big(\exp\big(-\lambda Z(\mathcal{P}_{\mathbf{e}})\big)G(Z(\mathcal{P}_{\mathbf{e}}))\Big) = -\sqrt{\frac{2}{3}}\cdot\frac{\sqrt{\lambda}+\sqrt{3}}{2\lambda + 2\sqrt{3\lambda} + 1}$$

The left formula gives that the distribution of $Z(\mathcal{P}_{\mathbf{e}})$ under $\mathbf{n}$ is $(\sqrt{6\pi})^{-1}\ell^{-\frac{3}{2}}\mathrm{d}\ell$. We now remark that:

$$\frac{\sqrt{\lambda}+\sqrt{3}}{2\lambda + 2\sqrt{3\lambda} + 1} = \frac{1}{2}\Big(\frac{(\sqrt{3}+1)/2}{\sqrt{\lambda} + (\sqrt{3}-1)/2} - \frac{(\sqrt{3}-1)/2}{\sqrt{\lambda} + (\sqrt{3}+1)/2}\Big)$$

which gives:

$$\frac{1}{\sqrt{6\pi}}\int_0^\infty \mathrm{d}\ell \exp(-\lambda\ell)\ell^{-\frac{3}{2}}G(\ell) = -\frac{1}{\sqrt{6}}\Big(\frac{(\sqrt{3}+1)/2}{\sqrt{\lambda} + (\sqrt{3}-1)/2} - \frac{(\sqrt{3}-1)/2}{\sqrt{\lambda} + (\sqrt{3}+1)/2}\Big).$$

We can now use formula (A.2) in the Appendix to get

$$\frac{1}{\sqrt{6\pi}}\ell^{-\frac{3}{2}}G(\ell) = \frac{1}{2\sqrt{6}}\cdot\Big(\chi_1\Big(\frac{2+\sqrt{3}}{2}\ell\Big) - \chi_1\Big(\frac{2-\sqrt{3}}{2}\ell\Big)\Big).$$

$\square$

8.3.3 Explicit computations of Voronoï cells of $\mathbb{D}$

For simplicity we introduce the notation:

$$\alpha_{\lambda,\mu} := g^{(-1)}_{\lambda,\mu}\Big(\sqrt{\frac{2}{3}+\frac{2}{3}\frac{\lambda}{\sqrt{2\mu}}} \Big). \tag{8.35}$$

for every $\lambda, \mu \geqslant 0$. We argue under $\mathbf{n}$ and the goal of this section is to compute the Laplace transform of $(Z(\mathcal{P}_{\mathbf{e}}), Z(\mathcal{P}'_{\mathbf{e}}), \mathcal{Y}(\mathcal{P}_{\mathbf{e}}), \mathcal{Y}(\mathcal{P}'_{\mathbf{e}}))$ i.e. to give a formula for the function:

$$(\lambda_1,\lambda_2,\mu_1,\mu_2) \in \mathbb{R}_+^4 \mapsto \mathbf{n}\Big(1-\exp\Big(-\lambda_1\mathcal{Z}(\mathcal{P}_{\mathbf{e}})-\lambda_2\mathcal{Z}(\mathcal{P}'_{\mathbf{e}})-\mu_1\mathcal{Y}(\mathcal{P}_{\mathbf{e}})-\mu_2\mathcal{Y}(\mathcal{P}'_{\mathbf{e}})\Big)\Big).$$

Let us introduce some notation:

$$A(a) := \frac{1}{2a^2+1}\begin{pmatrix} \frac{1}{3}(a^2-1)^2 & \frac{1}{2}a(2a^2+1)\sqrt{2a^2+2} & a^2(a^2+2) \\ \frac{1}{2}(a^2+2)\sqrt{2a^2+2} & 3a(a^2+1) & \frac{3}{2}a^2\sqrt{2a^2+2} \\ 2a^2+1 & \frac{3}{2}a\sqrt{2a^2+2} & 0 \end{pmatrix}$$

and

$$P_a(x,y) := \sum_{i=1}^{3}\sum_{j=1}^{3} A_{i,j}(a)x^{i-1}y^{j-1}.$$

The polynomial P_a is constructed so that the function

$$\begin{aligned} H(\lambda_1,\mu_1,\lambda_2,\mu_2,x) :=& P_{\mu_1^{-\frac{1}{4}}\mu_2^{\frac{1}{4}}}\Big(g_{\lambda_1,\mu_1}\big((2\mu_1)^{\frac{1}{4}}x+\alpha_{\lambda_1,\mu_1}\big), g_{\lambda_2,\mu_2}\big((2\mu_2)^{\frac{1}{4}}x+\alpha_{\lambda_2,\mu_2}\big)\Big) \\ &\cdot \exp\big(-\sqrt{2\sqrt{2\mu_1}+2\sqrt{2\mu_2}}\cdot x\big) \end{aligned}$$

defined for $(\lambda_1,\mu_1,\lambda_2,\mu_2,x) \in \mathbb{R}_+^5$ is a solution of the differential equation:

$$\frac{\partial^2}{\partial x^2}H(\lambda_1,\mu_1,\lambda_2,\mu_2,x) = 4\Big(u_{\lambda_1,\mu_1}(x)+u_{\lambda_2,\mu_2}(x)\Big)H(\lambda_1,\mu_1,\lambda_2,\mu_2,x). \tag{8.36}$$

The verification of (8.36) is straightforward (but tedious) using the fact that $g'_{\lambda,\mu}(x) = 1-g^2_{\lambda,\mu}(x)$ for every $\lambda,\mu \geqslant 0$. It is important to note that $x \mapsto H(\lambda_1,\mu_1,\lambda_2,\mu_2,x)$ is bounded. So combining this with (8.25) and (8.26) we get:

Proposition 8.4. *For every $(\lambda_1,\mu_1,\lambda_2,\mu_2,h) \in \mathbb{R}_+^5$ we have*

$$\mathbb{E}_h\big[\exp(-\lambda_1\mathcal{Z}(\mathcal{P}_B)-\lambda_2\mathcal{Z}(\mathcal{P}'_B)-\mu_1\mathcal{Y}(\mathcal{P}_B)-\mu_2\mathcal{Y}(\mathcal{P}'_B))\big] = \frac{H(\lambda_1,\mu_1,\lambda_2,\mu_2,h)}{H(\lambda_1,\mu_1,\lambda_2,\mu_2,0)}$$

and

$$\mathbf{n}\Big(1-\exp\big(-\lambda_1\mathcal{Z}(\mathcal{P}_{\mathbf{e}})-\lambda_2\mathcal{Z}(\mathcal{P}'_{\mathbf{e}})-\mu_1\mathcal{Y}(\mathcal{P}_{\mathbf{e}})-\mu_2\mathcal{Y}(\mathcal{P}'_{\mathbf{e}})\big)\Big) = -\frac{1}{2}\frac{\frac{\partial}{\partial x}H(\lambda_1,\mu_1,\lambda_2,\mu_2,x)|_{x=0}}{H(\lambda_1,\mu_1,\lambda_2,\mu_2,0)}.$$

8.4 The distribution of hulls

8.4.1 Geometric properties

In this section, we will use the construction of the free Brownian disk introduced in [71]. This construction is different from the one we used in Section 8.3.1 but it also involves the metric space associated with a coding pair. To simplify notation, for every $a \in \mathbb{R}$, we write $\bar{a} := \sqrt{3}a$ and, for every $\mathrm{w} \in \mathcal{W}$, we set:

$$\overline{\mathrm{w}}(t) := \sqrt{3} \cdot \mathrm{w}(t)\,, \qquad 0 \leqslant t \leqslant \zeta_{\mathrm{w}}.$$

We also write $\mathcal{P}^{+}_{\mathrm{w}}$ for a Poisson measure with intensity:

$$2\mathbb{1}_{[0,\zeta_{\mathrm{w}}]}(t) dt \mathbb{N}_{\mathrm{w}(t)}\big(\,\cdot \cap \{W_* > 0\}\big),$$

under the probability measure $\mathbb{P}^{(\mathrm{w})}$. Let us now recall the construction of the Brownian disk given in [72]. Let $z > 0$ and, under $\mathbf{n}(\cdot \,|\, \zeta_{\mathbf{e}} = z)$, consider R a 5-dimensional Bessel excursion of duration z. The distribution of R can be characterizes by the relation:

$$\mathbf{n}\big(F(R) \,\big|\, \zeta_{\mathbf{e}} = z\big) = 3 \lim_{\varepsilon \to 0} \varepsilon^{-2} \mathbf{n}\Big(F(\mathbf{e}) \exp(-\int_0^z \frac{dt}{(\mathbf{e}_t + \varepsilon)^2}) \,\Big|\, \zeta_{\mathbf{e}} = z\Big), \tag{8.37}$$

where the limit holds for every bounded continuous function $F : \mathcal{C}([0,z],\mathbb{R}_+) \to \mathbb{R}_+$ (see [72, Proposition 4]). It will be convenient to assume that, under $\mathbf{n}$ and conditionally on $\zeta_{\mathbf{e}}$, the processes $\mathbf{e}$ and R are independent, and also to remark that $\zeta_{\mathbf{e}} = \zeta_R$. Let us now consider conditionally on R, the point measure $\mathcal{P}^{+}_{\overline{R}}$. It is easy to verify that the assumptions at the beginning of Subsection 8.2.2 hold $\mathbf{n}$-a.e. for the pair $(\overline{R}, \mathcal{P}^{+}_{\overline{R}})$, and thus, that we can associate a labeled tree $(\mathcal{T}^+, (\Lambda^+_v)_{v \in \mathcal{T}^+})$ with this pair. The only points with zero label in $\mathcal{T}^+$ are the two extremities of its spine i.e. $\partial_1 \mathcal{T}^+ = \{0, \zeta_R\}$. Thus, $(\overline{R}, \mathcal{P}^{+}_{\overline{R}})$ also verifies (H_1), and we can consider the two pseudo-metric functions $\Delta^{+,\circ}(u,v)$ and $\Delta^{+}(u,v)$, defined on $\mathcal{T}^{+,\circ} \times \mathcal{T}^{+,\circ}$, by using formulas (8.15) and (8.16). Since 0 and ζ_R are the only points of zero label in $\mathcal{T}^+$, for any $u, v \in \mathcal{T}^+$ at least one of the two infima $\inf_{[u,v]_{\mathcal{T}^+}} \Lambda^+$ and $\inf_{[v,u]_{\mathcal{T}^+}} \Lambda^+$ is positive. We can then extend $\Delta^+(u,v)$ continuously to $\mathcal{T}^+ \times \mathcal{T}^+$ by replacing Δ° in formula (8.16) by

$$D^{+,\circ}(u,v) := \Lambda^+_u + \Lambda^+_v - 2 \max\big(\inf_{[u,v]_{\mathcal{T}^+}} \Lambda^+, \inf_{[v,u]_{\mathcal{T}^+}} \Lambda^+\big). \tag{8.38}$$

In particular, $(\overline{R}, \mathcal{P}^{+}_{\overline{R}})$ is a coding pair satisfying (H_1, H_2). Therefore we can consider the space $\mathbb{D}_+ := \Theta\big(\overline{R}, \mathcal{P}^{+}_{\overline{R}}\big)$ under the measure $\mathbf{n}$ and denote the canonical projection from $\mathcal{T}^+$ onto $\mathbb{D}_+$ by Π^+. The main result of [72] is that, under the measure $\mathbf{n}\big(\cdot \,|\zeta_R = z\big)$, the space $\mathbb{D}_+$ has the distribution of a free Brownian disk with perimeter z. Moreover, we clearly have $\Delta^+(0,\zeta_R) = 0$. Labels in this construction correspond to distances from the root $\rho_+ := \Pi^+(0)$. The set $\Pi^+(\partial \mathcal{T}^+)$ is the boundary of $\mathbb{D}_+$ and we denote it by $\partial \mathbb{D}_+$. Recall that in Subsection 8.2.2 we defined the uniform measure on $\partial \mathbb{D}_+$ by the relation:

$$\int_{\partial \mathbb{D}_+} \mathrm{m}_{\mathbb{D}_+}(dx)\; F(x) := \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{D}_+} \mathrm{Vol}^+(dx) F(x) \mathbb{1}_{\Delta^+(x, \partial \mathbb{D}_+) < \varepsilon}.$$

Besides we can consider another natural measure on $\partial\mathbb{D}_+$ by taking the pushforward of the Lebesgue measure on $[0,\zeta_R]$ by Π^+. Formula (42) in [72] shows that this measure coincides with $\mathrm{m}_{\mathbb{D}_+}$. Let us now introduce a uniform marked version of $\mathbf{n}$. Let $\mathbf{n}^\bullet$ be the σ-finite measure defined by:

$$\mathbf{n}^\bullet(F(\mathbf{e},R,t^\bullet)) := \mathbf{n}\Big(\int_0^{\zeta_{\mathbf{e}}} dt\, F(\mathbf{e},R,t)\Big).$$

The measure $\mathbf{n}^\bullet$ is a version of $\mathbf{n}$ biased by $\zeta_{\mathbf{e}} = \zeta_R$. The quantity $t^\bullet$ can be interpreted as a uniform point on $[0,\zeta_{\mathbf{e}}]$. Under $\mathbf{n}^\bullet$, the doubly marked space $\mathbb{D}_+^\bullet := \big(\Theta(\overline{R},\mathcal{P}^+_{\overline{R}}),\Pi^+(t^\bullet)\big)$ is distributed as $\mathbb{D}^\bullet$. Recall that the spine $[0,\zeta_R]$ is a subset of $\mathcal{T}^+$. We write $\rho_+^\bullet := \Pi^+(t^\bullet)$ and $|\partial\mathbb{D}^+|$ for the total mass of $\mathrm{m}_{\mathbb{D}_+}$ i.e. $|\partial\mathbb{D}^+| = \zeta_R$. One can derive from Lemmas 12 and 18 in [22], using classical absolute continuity properties, that the pair $(\overline{R},\mathcal{P}^+_{\overline{R}})$ also satisfies (H_3). In this preliminary version we leave the details to the reader.

For every $r>0$, $\mathbb{B}_r$ stands for the closed ball of radius r and centered at ρ_+ i.e. $\mathbb{B}_r = \{x\in\mathbb{D}_+ :\ \Delta^+(\rho_+,x)\leqslant r\}$. In the rest of this section we argue under the event $\{r<\Delta^+(\rho_+,\rho_+^\bullet)\}$. The notation $\mathbb{B}_r^\bullet$ stands for the hull of radius r centered at ρ_+ with respect to $\rho_+^\bullet$. The closure of $\mathbb{D}_+^\bullet\backslash\mathbb{B}_r^\bullet$ is denoted by $\check{\mathbb{B}}_r^\bullet$. As explained in the introduction, the sets $\mathbb{B}_r^\bullet$ and $\check{\mathbb{B}}_r^\bullet$ are homeomorphic to the closed unit disk of the complex plane. We define the boundary $\partial\check{\mathbb{B}}_r^\bullet$, resp. $\partial\mathbb{B}_r^\bullet$, as the set of all points of $\check{\mathbb{B}}_r^\bullet$, resp. $\mathbb{B}_r^\bullet$, that have no neighborhood homeomorphic to the open unit disk of the complex plane. We finally introduce the set $\check{\mathbb{B}}_r^\circ := \check{\mathbb{B}}_r^\bullet\backslash\partial\check{\mathbb{B}}_r^\bullet$. Let us explain how to identify these sets in the labeled tree $\mathcal{T}^+$. First, by using the interpretation of labels as distances from ρ_+, we see that $\mathbb{B}_r$ is the image by Π^+ of the points of $u\in\mathcal{T}^+$ with $\Lambda_u^+\leqslant r$. Now write:

$$\overline{\mathcal{T}}_r^{+,\circ} = \{u\in\mathcal{T}^+ :\ \inf_{[\![u,t^\bullet]\!]_{\mathcal{T}^+}}\Lambda^+ > r\},$$

where we recall that the notation $[\![u,t^\bullet]\!]_{\mathcal{T}^+}$ stands for the geodesic segment in $\mathcal{T}^+$ connecting u and $t^\bullet$. Let $\mathcal{T}_r^{+,\circ} := \overline{\mathcal{T}}_r^{+,\circ}\cap\mathcal{T}^{+,\circ}$ which is the set of all points of $\overline{\mathcal{T}}_r^{+,\circ}$ that do not belong to the spine $[0,\zeta_R]$. Finally we also introduce the set $\mathcal{T}_r^+$ of all points $u\in\mathcal{T}^+$ such that for every $v\in[\![u,t^\bullet]\!]_{\mathcal{T}^+}\backslash\{u\}$ we have $\Lambda_v^+>r$. Remark $\mathcal{T}_r^+$ is the closure of the set $\mathcal{T}_r^{+,\circ}$ in $\mathcal{T}^+$.

Lemma 8.6. *Fix $r>0$, under $\mathbf{n}^\bullet\big(\,\cdot\mid\overline{R}_{t^\bullet}>r\big)$, we have the following relation:*

$$\check{\mathbb{B}}_r^\circ = \Pi^+\big(\mathcal{T}_r^{+,\circ}\big)\,;\ \check{\mathbb{B}}_r^\bullet = \Pi^+\big(\mathcal{T}_r^+\big)\ \text{and}\ \mathbb{B}_r^\bullet = \Pi^+(\mathcal{T}^+\backslash\overline{\mathcal{T}}_r^{+,\circ}). \tag{8.39}$$

Proof. Let us show that $\Pi^+\big(\overline{\mathcal{T}}_r^{+,\circ}\big) = \mathbb{D}_+^\bullet\backslash\mathbb{B}_r^\bullet$ and $\mathbb{B}_r^\bullet = \Pi^+(\mathcal{T}^+\backslash\overline{\mathcal{T}}_r^{+,\circ})$. First consider $u\in\overline{\mathcal{T}}_r^{+,\circ}$ and note that $\Pi^+([\![u,t^\bullet]\!]_{\mathcal{T}^+})$ is the range of a path connecting $\Pi^+(u)$ and $\rho_+^\bullet$. Moreover, since $u\in\overline{\mathcal{T}}_r^{+,\circ}$, all the points in the geodesic segment $[\![u,t^\bullet]\!]_{\mathcal{T}^+}$ have a label greater than r. Consequently, the set $\Pi^+\big(\overline{\mathcal{T}}_r^{+,\circ}\big)$ is contained in $\mathbb{D}_+^\bullet\backslash\mathbb{B}_r^\bullet$. On the other hand, $\mathbb{D}_+^\bullet$ satisfies the so-called cactus-bound stating that, for every path $\gamma:[0,1]\mapsto\mathbb{D}_+^\bullet$, we have:

$$\inf_{t\in[0,1]}\Lambda^+_{\gamma(t)}\leqslant\inf_{[\![\gamma(0),\gamma(1)]\!]_{\mathcal{T}^+}}\Lambda^+. \tag{8.40}$$

The cactus-bound is a classical bound in Brownian geometry. We refer to [63, Proposition 3.1] for a proof of (8.40) in the context of the Brownian Sphere that is easily extended to our setting. An application of the cactus bound (8.40) gives that, for every $u \notin \overline{\mathcal{T}}_r^{+,\circ}$, and every path $\gamma : [0,1] \mapsto \mathbb{D}_+^\bullet$ with $\big(\gamma(0), \gamma(1)\big) = \big(\Pi^+(u), \rho_+^\bullet\big)$, we have:

$$\inf_{t\in[0,1]} \Lambda^+_{\gamma(t)} \leqslant \inf_{[\![\gamma(0),\gamma(1)]\!]_{\mathcal{T}^+}} \Lambda^+ \leqslant r.$$

This implies that the image of the complement of $\overline{\mathcal{T}}_r^{+,\circ}$ is inside $\mathbb{B}_r^\bullet$, and consequently we deduce that $\mathbb{D}_+^\bullet \backslash \mathbb{B}_r^\bullet = \Pi^+\big(\overline{\mathcal{T}}_r^{+,\circ}\big)$ and $\mathbb{B}_r^\bullet = \Pi^+(\mathcal{T}^+ \backslash \overline{\mathcal{T}}_r^{+,\circ})$. It then takes a simple verification to complete the proof of (8.39). □

We can divide the boundary $\partial \check{\mathbb{B}}_r^\bullet$ in two parts. On one hand, we have $\mathbb{B}_r^\bullet \cap \check{\mathbb{B}}_r^\bullet \subset \partial \check{\mathbb{B}}_r^\bullet$ which is the image by Π^+ of :

$$\partial_1 \mathcal{T}_r^+ := \big\{u \in \mathcal{T}_{\overline{R}} :\ \Lambda_u^+ = r \text{ and } \Lambda_v^+ > r \text{ for every } v \in [\![u, t^\bullet]\!]_{\mathcal{T}^+} \backslash \{u\}\big\}.$$

We denote this set by $\partial_1 \check{\mathbb{B}}_r^\bullet$. On the other hand, we also have $\partial \mathbb{D}_+^\bullet \cap \check{\mathbb{B}}_r^\bullet \subset \partial \check{\mathbb{B}}_r^\bullet$, and (8.39) gives that $\partial \mathbb{D}_+^\bullet \cap \check{\mathbb{B}}_r^\bullet$ is the image by Π^+ of:

$$\partial_2 \mathcal{T}_r^+ := [\tau_r, \tau_r'],$$

where we write $\tau_r := \sup\{s \in [0, t^\bullet] :\ \overline{R}_s = r\}$ and $\tau_r' := \inf\{s \geqslant t^\bullet :\ \overline{R}_s = r\}$. To simplify notation, we will now write $\partial_2 \check{\mathbb{B}}_r^\bullet$ for the set $\partial \mathbb{D}_+^\bullet \cap \check{\mathbb{B}}_r^\bullet$. Remark that $\partial \check{\mathbb{B}}_r^\bullet$ is the union of $\partial_1 \check{\mathbb{B}}_r^\bullet$ and $\partial_2 \check{\mathbb{B}}_r^\bullet$ (and that this notation is consistent with the notation used in the introduction). Moreover, the intersection of the boundaries $\partial_1 \check{\mathbb{B}}_r^\bullet$ and $\partial_2 \check{\mathbb{B}}_r^\bullet$ consists in the points $\Pi^+(\tau_r)$ and $\Pi^+(\tau_r')$.

Let us now introduce the following process

$$\overline{R}^{(r)}_t := \overline{R}_{t+\tau_r} - r\,, \qquad 0 \leqslant t \leqslant \tau_r' - \tau_r \tag{8.41}$$

and the point measure $\mathcal{P}^{+,r}_{\overline{R}}$ on $\mathbb{R}_+ \times \mathcal{S}$, defined by:

$$\int \Phi(t,\omega)\ \mathcal{P}^{+,r}_{\overline{R}}(dtd\omega) := \int_{\tau_r}^{\tau_r'} \Phi(t-\tau_r, \omega - r)\ \mathcal{P}^{+}_{\overline{R}}(dtd\omega).$$

The pair $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ is clearly a coding pair verifying (H_1). We can now remark that, by the very construction of $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$, the tree associated with $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ is directly identified with $\mathcal{T}_r^+$ as a metric space and that this identification preserves the labels – provided that the labels of $\mathcal{T}_r^+$ are shifted by $-r$. From now on we will make this identification. We will later show that $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ also verifies (H_2, H_3). We equip the space $\check{\mathbb{B}}_r^\circ$ with the intrinsic distance, denoted by $\Delta^{+,r}$ and the restriction of Vol^+, denoted by $\mathrm{Vol}^{+,r}$.

Proposition 8.5. *Fix $r > 0$. Under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$, there exists an isometry*

$$\Phi : \Theta^\circ\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big) \to \check{\mathbb{B}}_r^\circ,$$

such that the pushforward of the volume measure of $\Theta^\circ\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ under Φ is $\mathrm{Vol}^{+,r}$.

Proof. Fix $r>0$. In this proof we argue under $\mathbf{n}^\bullet(\cdot\mid \overline{R}_{t^\bullet}>r)$ and we denote the labeled tree associated with $\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ by $\mathcal{T}_{\overline{R}^{(r)}}$. We add the index $\overline{R}^{(r)}$ to the quantities associated with the coding pair $\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$. In particular, we denote the pseudo-distances associated with $\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$, defined in (8.15) and (8.16), by $\Delta^\circ_{\overline{R}^{(r)}}$ and $\Delta_{\overline{R}^{(r)}}$. We also write $\Lambda^{\overline{R}^{(r)}}$ for the label function on $\mathcal{T}_{\overline{R}^{(r)}}$. Now recall that $\mathcal{T}_{\overline{R}^{(r)}}$ is identified with $\mathcal{T}_r^+$ (after shifting the labels of $\mathcal{T}_r^+$ by $-r$). For every $u\in\mathcal{T}_{\overline{R}^{(r)}}$, we write u' for the point of $\mathcal{T}_r^+$ that is identified with u. In particular, for every $u\in\mathcal{T}_{\overline{R}^{(r)}}$ we have $\Lambda_u^{R^{(r)}}=\Lambda^+_{u'}-r$. We are going to show that this identification induces an isometry from $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ onto $\check{\mathbb{B}}^\circ_r$ such that the pushforward of the volume measure of $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ under this isometry is $\mathrm{Vol}^{+,r}$. We start by observing that, for every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$ such that $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}}\Lambda^{R^{(r)}}>0$, the interval $[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}$ is identified with the interval $[u',v']_{\mathcal{T}^+}$. In particular we obtain

(C): For every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$, we have $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}}\Lambda^{R^{(r)}}>0$ if and only if $\inf_{[u',v']_{\mathcal{T}^+}}\Lambda^+>r$.

Using definition (8.15), this implies that, for every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$, we have $\Delta^\circ_{\overline{R}^{(r)}}(u,v)=\Delta^{+,\circ}(u,v)$ or $\Delta^\circ_{\overline{R}^{(r)}}(u,v)=\infty$. Consequently, by the definition of $\Delta_{\overline{R}^{(r)}}$ and Δ^+ as an infimum (8.16), we deduce that $\Delta_{\overline{R}^{(r)}}(u,v)\geqslant\Delta^+(u',v')$ for every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$. This bound gives that if $\Delta_{\overline{R}^{(r)}}(u,v)=0$ then, by the previous bound, we also have $\Delta^+(u',v')=0$. Moreover since $(\overline{R},\mathcal{P}^+_{\overline{R}})$ satisfies (H_3), we know that for every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$, the condition $\Delta^+(u',v')=0$ holds only if $\Delta^{+,\circ}(u',v')=0$. Moreover $\Delta^{+,\circ}(u',v')=0$ implies that

$$r<\Lambda^+_{u'}=\Lambda^+_{v'}=\inf_{[u',v']_{\mathcal{T}^+}}\Lambda^+\ \text{ or }\ r<\Lambda^+_{u'}=\Lambda^+_{v'}=\inf_{[v',u']_{\mathcal{T}^+}}\Lambda^+.$$

In both cases we note that we must have $\Delta^\circ_{\overline{R}^{(r)}}(u,v)=0$. We have obtained that, for every $u,v\in\mathcal{T}^\circ_{\overline{R}^{(r)}}$, the condition $\Delta_{\overline{R}^{(r)}}(u,v)=0$ is equivalent to $\Delta^+(u',v')=0$. Consequently the identification between $\mathcal{T}_{\overline{R}^{(r)}}$ and $\mathcal{T}_r^+$ induces an identification between the points of $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ and the points of $\check{\mathbb{B}}^\circ_r$. To simplify notation, for every $x\in\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$, we write x' to denote the point identified with x in $\check{\mathbb{B}}^\circ_r$. Let us now show that this identification satisfies the properties of the proposition. The fact that $\mathrm{Vol}^{+,r}$ is the pushforward of the volume measure of $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ under this identification is a direct consequence of the construction. In order to complete the proof, we need to show that $\Delta_{\overline{R}^{(r)}}(x,y)=\Delta^{+,r}(x',y')$ for every points x and y of $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$. To this end, we fix x and y in $\Theta^\circ\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ and we start by explaining why $\Delta_{\overline{R}^{(r)}}(x,y)$ is bounded above by $\Delta^{+,r}(x',y')$. By the definition of $\Delta_{\overline{R}^{(r)}}(x,y)$ as an infimum (8.16), it is sufficient to show that, for every $u,v\in\mathcal{T}_{\overline{R}^{(r)}}$ with $\Delta^\circ_{\overline{R}^{(r)}}(u,v)<\infty$, there exists a path γ' taking values in $\check{\mathbb{B}}^\circ_r$, starting at $\Pi^+(u')$ and ending at $\Pi^+(v')$, such that the length of γ' with respect to Δ^+ is $\Delta^\circ_{\overline{R}^{(r)}}(u,v)$. We now use the same argument as above. First remark that, without loss of generality, we may assume that:

$$\Delta^\circ_{\overline{R}^{(r)}}(u,v)=\Lambda_u^{\overline{R}^{(r)}}+\Lambda_v^{\overline{R}^{(r)}}-2\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}}\Lambda^{\overline{R}^{(r)}},$$

with $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}} \Lambda^{\overline{R}^{(r)}} > 0$. Since $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}} \Lambda^{\overline{R}^{(r)}} > 0$, we note that the interval $[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}$ is identified with the interval $[u',v']_{\mathcal{T}^+}$. In particular, the condition $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}} \Lambda^{\overline{R}^{(r)}} > 0$ implies that the interval $[u',v']_{\mathcal{T}^+}$ is contained in $\mathcal{T}^{+,r}$. So by concatenating two simple geodesics starting respectively from $\Pi^+(u')$ and $\Pi^+(v')$ up to their merging time, as in the proof of Lemma 8.3, we can construct a path from $\Pi^+(u')$ to $\Pi^+(v')$ whose length equal to $\Delta^\circ_{\overline{R}^{(r)}}(u,v)$, and that stays in $\Pi^{+,r}([u',v']_{\mathcal{T}^+}) \subset \check{\mathbb{B}}^\bullet_r$. Moreover, since $\inf_{[u,v]_{\mathcal{T}_{\overline{R}^{(r)}}}} \Lambda^{\overline{R}^{(r)}} > 0$ we can apply (C) to see that the path γ' does not hit $\mathbb{B}^\bullet_r$. Finally, since $(\overline{R}, \mathcal{P}^+_{\overline{R}})$ satisfies (H_3) – and more precisely the remark below assumption (H_3) – we deduce that γ' does not hit $\Pi^+([0,\zeta_R])$ and we obtain that $\Delta_{\overline{R}^{(r)}}(x,y)$ is bounded above by $\Delta^{+,r}(x',y')$.

Let us conclude by showing the reverse bound. The proof is similar to the end of the proof of Lemma 8.3. We want to show that if $\gamma' : [0,1] \to \check{\mathbb{B}}^\circ_r$ is a path such that $\gamma'(0) = x'$ and $\gamma'(1) = y'$, then the length of γ' is bounded below by $\Delta_{\overline{R}^{(r)}}(x,y)$. To simplify notation, set $\gamma(t)$ for the point of $\Theta^\circ\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ corresponding to $\gamma'(t)$ in the identification of $\Theta^\circ\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ with $\check{\mathbb{B}}^\circ_r$. Recall that the boundary of $\check{\mathbb{B}}^\circ_r$ is the image by Π^+ of $\partial \mathcal{T}^+_r$. By compactness we may find $\delta > 0$ such that, for every $t \in [0,1]$, the point $\gamma'(t)$ is at least at distance 2δ from $\Pi^+\big(\partial \mathcal{T}^+_r\big)$ with respect to the metric Δ^+. In particular, we have $\Lambda^+_{\gamma'(t)} > r + 2\delta$ for every $t \in [0,1]$. Then, we may choose n large enough so that $\Delta^+(\gamma'(\frac{i-1}{n}), \gamma'(\frac{i}{n})) < \delta$ for every $1 \leqslant i \leqslant n$. The length of γ' is bounded below by $\sum_{i=1}^n \Delta^+(\gamma'(\frac{i-1}{n}), \gamma'(\frac{i}{n}))$. Thus, to get the desired result, we only have to verify that, for every $0 \leqslant i \leqslant n-1$:

$$\Delta^+\big(\gamma'\big(\frac{i-1}{n}\big), \gamma'\big(\frac{i}{n}\big)\big) \geqslant \Delta_{\overline{R}^{(r)}}\big(\gamma\big(\frac{i-1}{n}\big), \gamma\big(\frac{i}{n}\big)\big).$$

Fix $0 \leqslant i \leqslant n-1$. In the definition (8.16) of $\Delta^+(\gamma'(\frac{i-1}{n}), \gamma'(\frac{i}{n}))$ as an infimum, we can restrict to the choices of $u'_0 = \gamma(\frac{i-1}{n}), u'_1, \ldots, u'_p = \gamma'(\frac{i}{n})$ in $\mathcal{T}^+$, such that u'_j is at Δ^+-distance at least δ from $\partial \mathcal{T}^{+,r}$ and $\Delta^{+,\circ}(u'_j, u'_{j+1}) < \delta$. In particular the infimum of the labels over $[u'_j, u'_{j+1}]_{\mathcal{T}^+}$ must be bigger than $r + \delta$, meaning that $[u'_j, u'_{j+1}]_{\mathcal{T}^+}$ is contained in $\mathcal{T}^+_r$. We then note that the interval $[u'_j, u'_{j+1}]_{\mathcal{T}^+}$ is identified with an interval $[u_j, u_{j+1}]_{\mathcal{T}_{\overline{R}^{(r)}}}$ of $\mathcal{T}_{\overline{R}^{(r)}}$. We deduce that $\Delta^{+,\circ}(u'_j, u'_{j+1}) = \Delta^\circ_{\overline{R}^{(r)}}(u_j, u_{j+1})$ and we obtain the desired bound using the triangle inequality. □

We will later show that $\Delta^{+,r}$ has a continuous extension to $\check{\mathbb{B}}^\bullet_r$ and that the space $\check{\mathbb{B}}^\bullet_r$ – equipped with this distance, the restriction of the volume measure, and pointed at $\Pi^+(\tau_r)$ – is equal to $\Theta\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ (we will show that $\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ satisfies (H_2, H_3)).

Let us now explain how to encode the hull $\mathbb{B}^\bullet_r$ with a coding pair. Fix $r > 0$ and write K_r for the complement of the set $\overline{\mathcal{T}}^{+,\circ}_r$ on $\mathcal{T}^+$. Equivalently, K_r is defined by the relation:

$$K_r := \{v \in \mathcal{T}^+ : \Lambda^+_w \leqslant r \text{ for some } w \in [\![v, t^\bullet]\!]_{\mathcal{T}^+}\}.$$

By (8.39), the hull $\mathbb{B}^\bullet_r$ is the image of K_r by Π^+. We are going to consider $\mathbb{B}^\bullet_r$ as an element of $\mathbb{K}$. To do so, we point $\mathbb{B}^\bullet_r$ at $\rho_+ = \Pi^+(0)$ and we equip $\mathbb{B}^\bullet_r$ with the restriction of the volume measure Vol^+, that we denote by $\mathrm{Vol}^{+,(r)}$. We now introduce a good distance on $\mathbb{B}_r^\bullet$. First recall the definition of $D^{+,\circ}$ and note that, for every $u, v \in K_r$, we have

$$D^{+,\circ}(u,v) := \Lambda_u^+ + \Lambda_v^+ - 2\max\Big(\inf_{[u,v]_{\mathcal{T}^+}} \Lambda^+, \inf_{[v,u]_{\mathcal{T}^+}} \Lambda^+\Big)$$
$$= \Lambda_u^+ + \Lambda_v^+ - 2\max\Big(\inf_{[u,v]_{\mathcal{T}^+}\cap K_r} \Lambda^+, \inf_{[u,v]_{\mathcal{T}^+}\cap K_r} \Lambda^+\Big).$$

In other words, we only need to know the set K_r and the labels on K_r to be able to compute the quantities $D^{+,\circ}(u,v)$ for every $u, v \in K_r$. We set:

$$\Delta^{+,(r)}(u,v) = \inf_{\substack{u_0,u_1,\dots,u_p \in K_r \\ (u_0,u_p)=(u,v)}} \sum_{i=1}^{p} D^{+,\circ}(u_{i-1}, u_i), \tag{8.42}$$

where the infimum is over all choices of the integer $p \geqslant 1$ and of the finite sequence $u_0, u_1, \dots, u_p$ in K_r such that $u_0 = u$ and $u_p = v$. Of course, since the inf is the same as the one appearing in the formula of Δ^+ but with the restriction $u_0, \dots, u_p \in K_r$, we get that $\Delta^+(u,v) \leqslant \Delta^{+,(r)}(u,v) \leqslant D^{+,\circ}(u,v)$ for every $u, v \in K_r$. As by (H_3), the condition $\Delta^+(u,v) = 0$ only holds if $D^{+,\circ}(u,v) = 0$ we deduce that for every $u, v \in K_r$ we have $\Delta^+(u,v) = 0$ if and only if $\Delta^{+,(r)}(u,v) = 0$. This equivalence implies that $\Delta^{+,(r)}$ induces a metric on $\Pi^+(K_r) = \mathbb{B}_r^\bullet$. We keep the notation $\Delta^{+,(r)}$ for this metric. One can use the family of simple geodesics and the definition of Δ^+ as an infimum (8.16) to obtain that the restriction of $\Delta^{+,(r)}$ coincides with the intrinsic distance induced by Δ^+ in the interior of $\mathbb{B}_r^\bullet$. This can be derived by adapting the proof of Proposition 8.5 (we leave the details to the reader). We equip the space $\mathbb{B}_r^\bullet$ with the distance $\Delta^{+,(r)}$ and the restriction of the volume measure Vol^+, denoted by $\mathrm{Vol}^{+,(r)}$. We also point $\mathbb{B}_r^\bullet$ at $\rho_+ = \Pi^+(0)$. Let us now explain how to obtain this space as the metric space associated with a coding pair. This is a similar construction as the one appearing in the proof of [79, Theorem 31] concerning the hull of the Brownian plane and we follow the presentation therein.

First set $Z_r := Z(\mathcal{P}_{\overline{R}}^{+,r})$ (we will give a geometric interpretation of Z_r in Proposition 8.9) and introduce the following process:

$$X_t^{(r)} := \begin{cases} \overline{R}_t & \text{if } t \in [0, \tau_r] \\ r & \text{if } t \in [\tau_r, \tau_r + Z_r] \\ \overline{R}_{t+\tau_{r'}-(\tau_r+Z_r)} & \text{if } t \in [\tau_r + Z_r, \tau_r + Z_r + \zeta_R - \tau_r'], \end{cases}$$

and the point measures $\mathcal{X}_1^{(r)}$ and $\mathcal{X}_1^{(r),\prime}$ defined by the relation:

$$\int F(t,\omega)\, \mathcal{X}_1^{(r)}(dtd\omega) := \int_0^{\tau_r} F(t,\omega)\, \mathcal{P}_{\overline{R}}^{+}(dtd\omega)\, ;\, \int F(t,\omega)\, \mathcal{X}_1^{(r),\prime}(dtd\omega) := \int_{\tau_r'}^{\zeta_R} F(t-\tau_r',\omega)\, \mathcal{P}_{\overline{R}}^{+}(dtd\omega). \tag{8.43}$$

Let us consider all the subtrees branching off the subset $[\tau_r, \tau_r']$ of the spine and for each such subtree whose labels hit $[0, r]$ we consider the "excursions outside" (r, ∞). More precisely, we write $\mathcal{P}^+_{\overline{R}} = \sum_{i\in I}\delta_{(t_i,\omega^i)}$. In the time scale of the contour exploration $\mathcal{E}^+$ of $\mathcal{T}^+$, each snake trajectory ω^i corresponds to an interval $[\alpha_i,\beta_i]$ and we have $\sigma(\omega^i)=\beta_i-\alpha_i$. Set $I_r := \{i\in I : t_i\in[\tau_r,\tau'_r]$ and $W_*(\omega^i)\leqslant r\}$. For each $i\in I_r$, we can consider the exit local time of ω^i at level r, as defined in (8.7), and denote it by $(\mathfrak{L}^{i,r}_t)_{t\in[0,\sigma(\omega^i)]}$. We then set, for every $t\in\mathbb{R}$:

$$\mathfrak{L}^{*,r}_t = \sum_{i\in I_r}\mathfrak{L}^{i,r}_{t\wedge\beta_i - t\wedge\alpha_i}, \tag{8.44}$$

so that $\mathfrak{L}^{*,r}_t$ can be understood as the total exit local time accumulated at r by the contour exploration up to time t (by the subtrees glued to the subset $[\tau_r,\tau'_r]$ of the spine of $\mathcal{T}^+$). Note that the total exit local time $\mathfrak{L}^{*,r}_\infty$ is exactly the quantity Z_r. Now, for every $i\in I_r$, consider the excursions $(\omega^{i,k})_{k\in\mathbb{N}}$ of ω^i outside (r,∞). Every $\omega^{i,k}$ with $(i,k)\in I_r\times\mathbb{N}$ corresponds to a connected component $(a_{i,k},b_{i,k})$ of $\{s\in[0,\sigma(\omega^i)] : \mathrm{hit}_r(\omega^i_s)<\zeta_s(\omega^i)\}$, in such a way that, for every $s\geqslant 0$:

$$\omega^{i,k}_s(t) := \omega^i_{(a_{i,k}+s)\wedge b_{i,k}}(\zeta_{a_{i,k}}+t) \text{ for } 0\leqslant t\leqslant \zeta_{a_{i,k}+s}(\omega^i)-\zeta_{a_{i,k}}(\omega^i).$$

Set $\alpha_{i,k} := \alpha_i + a_{i,k}$ and $\beta_{i,k} := \beta_i + a_{i,k}$ and introduce the point measure:

$$\mathcal{X}^{(r)}_2(dtd\omega) := \sum_{i\in I_r}\sum_{k\in\mathbb{N}}\delta_{\mathfrak{L}^{*,r}_{\alpha_{i,k}},\omega^{i,k}}(dtd\omega). \tag{8.45}$$

By the special Markov property, conditionally on Z_r, $\mathcal{X}^{(r)}_2$ is a Poisson point measure with intensity:

$$\mathbb{1}_{[0,Z_r]}(t)dt\ \mathbb{N}_r\big(d\omega\cap\{\omega_*>0\}\big),$$

and is independent of $(X^{(r)},\mathcal{X}^{(r)}_1,\mathcal{X}^{(r),\prime}_1)$ and $\big(\overline{R}^{(r)},\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$. We write $\mathcal{X}^{(r)}$ for the point measure defined by:

$$\int F(t,\omega)\ \mathcal{X}^{(r)}(dtd\omega) := \int F(t,\omega)\ \mathcal{X}^{(r)}_1(dtd\omega) + \int F(\tau_r+t,\omega)\ \mathcal{X}^{(r)}_2(dtd\omega) + \int F(\tau_r+Z_r+t,\omega)\ \mathcal{X}^{(r),\prime}_1(dtd\omega).$$

It is easy to verify from the construction that the pair $(X^{(r)},\mathcal{X}^{(r)})$ is a coding pair. Let $\mathcal{T}_{X^{(r)}}$ denote the labeled tree associated with $(X^{(r)},\mathcal{X}^{(r)})$. The only points of $\mathcal{T}_{X^{(r)}}$ with zero label are the extremities of the spine of $\mathcal{T}_{X^{(r)}}$ i.e. the points 0 and $\zeta_{X^{(r)}} = \tau_r + Z_r + \zeta_R - \tau'_r$. The same arguments we used for the pair $\big(\overline{R},\mathcal{P}^+_{\overline{R}}\big)$ can then be applied to deduce that $(X^{(r)},\mathcal{X}^{(r)})$ satisfies (H_1,H_2). We can now assert that the space $\Theta\big(X^{(r)},\mathcal{X}^{(r)}\big)$ is well defined.

Proposition 8.6. *Fix $r>0$. Under $\mathbf{n}^\bullet\big(\cdot\mid\overline{R}_{t^\bullet}>r\big)$, the spaces $\Theta\big(X^{(r)},\mathcal{X}^{(r)}\big)$ and $\mathbb{B}^\bullet_r$, viewed as random elements of $\mathbb{K}$ are equal a.s.*

Proof. In this proof we argue under $\mathbf{n}^\bullet\big(\cdot\mid\overline{R}_{t^\bullet}>r\big)$. To show that we can identify $\mathcal{T}_{X^{(r)}}$ with a subset of K_r. We use the notation $\mathcal{P}^+_{\overline{R}} = \sum_{i\in I}\delta_{(t_i,\omega^i)}$ introduced above and, to simplify notation, we write $\mathcal{X}^{(r)} := \sum_{j\in J} \delta_{t_j,\omega^j}$ where J is an indexing set disjoint of I. We can then write $\mathcal{T}_{X^{(r)}}$ as the disjoint union

$$[0,\zeta_{X^{(r)}}] \cup \Big(\bigcup_{j\in J} \mathcal{T}_{\omega^j} \Big),$$

where the point t_j is glued with the root of $\mathcal{T}_{\omega^j}$. Let us now identify the subset $[0,\tau_r]$ (resp. $[\tau_r + Z_r, \zeta_{X^{(r)}}]$) of the spine of $\mathcal{T}_{X^{(r)}}$ with the subset $[0,\tau_r]$ (resp. $[\tau_r', \zeta_R]$) of the spine of $\mathcal{T}_+$, which are also subsets of K_r. By the very definition of $X^{(r)}$, this identification preserves the labels. Now remark that, as the point measures $\mathcal{X}_1^{(r)}$ and $\mathcal{X}_1^{(r),\prime}$ are defined by keeping parts of the point measure $\mathcal{P}_R^+$ (after performing a time translation for $\mathcal{X}_1^{(r),\prime}$), we can identify every tree $\mathcal{T}_{\omega^j}$ for $t_j \in [0,\tau_r]$ (resp. $t_j \in [\tau_r + Z_r, \zeta_{X^{(r)}}]$) with a subtree of $\mathcal{T}^+$ glued to a point of the spine $[0,\tau_r]$ (resp. $[\tau_r', \zeta_R]$) of $\mathcal{T}^+$. This identification also preserves the labels. We now need to identify the set:

$$[\tau_r, \tau_r + Z_r] \cup \Big(\bigcup_{j\in J_r} \mathcal{T}_{\omega^j} \Big),$$

where $J_r := \{j \in J :\ t_j \in [\tau_r, \tau_r + Z_r]\}$, with a subset of K_r. To this end let us introduce the right-continuous inverse of the process $\mathfrak{L}^{*,r}$ i.e. for every $s \in [0, Z_r)$ set:

$$\eta_s^{*,r} := \inf\{t \in \mathbb{R} : \mathfrak{L}_t^{*,r} > s\},$$

where, by convention, $\eta_{Z_r}^{*,r}$ is the left limit of $s \mapsto \eta_s^{*,r}$ at $s = Z_r$. We now use the process $\eta^{*,r}$ to identify $[\tau_r, \tau_r + Z_r] \subset \mathcal{T}_{X^{(r)}}$ with a subset of $K_r \cap \mathcal{T}_r^+$. First remark that:

$$K_r \cap \mathcal{T}_r^+ = \big\{ u \in \mathcal{T}^+ :\ \Lambda_u^+ = r \text{ and } \Lambda_v^+ > r \text{ for every } v \in [\![u, t^\bullet]\!]_{\mathcal{T}^+} \backslash \{u\} \big\}.$$

It is then straightforward to verify from standard properties of the support of the exit local time, that:

$$K_r \cap \mathcal{T}_r^+ = \big\{ \mathcal{E}^+_{\eta_s^{*,r}} :\ s \in [0, Z_r] \big\} \cup \big\{ \mathcal{E}^+_{\eta_{s-}^{*,r}} :\ s \in [0, Z_r] \big\}.$$

If $s \in [\tau_r, \tau_r + Z_r] \subset \mathcal{T}_{X^{(r)}}$, we identify the point s with the point $\mathcal{E}^+_{\eta^{*,r}_{s-\tau_r}}$. Knowing that all the points of $K_r \cap \mathcal{T}_r^+$ have label r and that $X_s^{(r)} = r$ for every $s \in [\tau_r, \tau_r + Z_r]$, we see that this identification preserves the labels. It remains to identify the trees $\big(\mathcal{T}_{\omega^j}\big)_{j\in J_r}$. Recall the notation $(\omega^{i,k})_{(i,k)\in I_r\times\mathbb{N}}$, introduced above for the excursion outside (r,∞) of the snakes trajectories ω^i with $t_i \in [\tau_r, \tau_r']$, and also that, in the time scale of the clockwise exploration $\mathcal{E}^+$ of $\mathcal{T}^+$, the snake trajectory $\omega^{i,k}$ corresponds to the interval $[\alpha_{i,k}, \beta_{i,k}]$. Remark that $\mathcal{T}_{\omega^{i,k}}$ coincides with the subtree $\{\mathcal{E}_s^+ : s \in [\alpha_{i,k}, \beta_{i,k}]\}$ of $\mathcal{T}^+$. By construction, for $j \in J_r$, there exists $\omega^{i,k}$ such that $\omega^{i,k} = \omega^j$ and $\mathfrak{L}^{*,r}_{\alpha_{i,k}} = t_j - \tau_r$. Moreover, by the support property of the exit local time, this identification is a bijection between $(\omega^{i,k})_{(i,k)\in I_r\times\mathbb{N}}$ and $(\omega^j)_{j\in J_r}$. This way we have identified $\mathcal{T}_{X^{(r)}}$ with a subset of K_r. To fix notation, let us write $\phi : \mathcal{T}_{X^{(r)}} \mapsto K_r$ for the injective function induced by this identification. The complement of $\phi(\mathcal{T}_{X^{(r)}})$ in K_r is the set:

$$\big\{ \mathcal{E}^+_{\eta_{s-}^{*,r}} :\ s \in [0, Z_r] \text{ such that } \mathcal{E}^+_{\eta_s^{*,r}} \neq \mathcal{E}^+_{\eta_{s-}^{*,r}} \big\} \subset K_r \cap \mathcal{T}_r^+.$$

It is then an easy verification to show that, for every $u, v \in \mathcal{T}_{X^{(r)}}$, we have:

$$\inf_{[u,v]_{\mathcal{T}_{X^{(r)}}}} \Lambda^{X^{(r)}} = \inf_{[\phi(u),\phi(v)]_{\mathcal{T}^+}} \Lambda^+ \text{ and } \inf_{[v,u]_{\mathcal{T}_{X^{(r)}}}} \Lambda^{X^{(r)}} = \inf_{[\phi(v),\phi(u)]_{\mathcal{T}^+}} \Lambda^+$$

where we write $\Lambda^{X^{(r)}}$ for the labels on $\mathcal{T}_{X^{(r)}}$. Consequently by the definition of $\Delta_{X^{(r)}}$ and $\Delta^{+,(r)}$ as an infimum, we deduce that we have $\Delta_X(u,v) = \Delta^{+,(r)}(\phi(u),\phi(v))$, for every $u, v \in \mathcal{T}_{X^{(r)}}$. In particular ϕ induces an isometry between $\Theta\big(X^{(r)}, \mathcal{X}^{(r)}\big)$ and $\Pi^+\big(\phi(\mathcal{T}_{X^{(r)}})\big)$. To conclude remark that if $\tau^{*,r}_{s-} < \tau^{*,r}_{s}$, by the support property of the exit local time implies that either all points of the form $\mathcal{E}^+_u$ with $u \in (\tau^{*,r}_{s-}, \tau^{*,r}_{s})$ are descendants of $\mathcal{E}^+_{\tau^{*,r}_{s-}}$ and necessarily $\mathcal{E}^+_{\tau^{*,r}_{s-}} = \mathcal{E}^+_{\tau^{*,r}_{s}}$, or the labels of all such points $\mathcal{E}^+_u$ are greater than r. In both cases we have $\Pi^+(\mathcal{E}^+_{\eta^{*,r}_{s-}}) = \Pi^+(\mathcal{E}^+_{\eta^{*,r}_{s}})$ and consequently $\Pi^+\big(\phi(\mathcal{T}_{X^{(r)}})\big) = \mathbb{B}^\bullet_r$. Finally, since $\mathrm{Vol}^+(\partial_1 \mathbb{B}^\bullet_r) = 0$, it is easy to see, from the definition of ϕ, that the pushforward of the volume measure of $\Theta\big(X^{(r)}, \mathcal{X}^{(r)}\big)$ under $\Pi^+ \circ \phi$ is $\mathrm{Vol}^{+,(r)}$. This completes the proof of the proposition. □

8.4.2 A technical result

The goal of this section is to give an explicit formula for the Laplace transform of $\big(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\bar{\mathbf{e}}}), \mathcal{Y}(\mathcal{P}_{\bar{\mathbf{e}}})\big)$ under $\mathbf{n}$ and the Laplace transform of $\big(\zeta_B, Z(\mathcal{P}_{\overline{B}}), \mathcal{Y}(\mathcal{P}_{\overline{B}})\big)$ under $\mathbb{P}_h$. These formulas will be helpful to obtain the spatial Markov property of the Brownian disk and to derive explicit computations. Recall the notation $\alpha_{\lambda,\mu}$ defined in (8.35).

Proposition 8.7. *For every $\gamma, \lambda, \mu \geqslant 0$ we have:*

$$\mathbb{E}_h\Big[\exp\big(-\gamma\zeta_B - \lambda\mathcal{Z}(\mathcal{P}_{\overline{B}}) - \mu\mathcal{Y}(\mathcal{P}_{\overline{B}})\big)\Big] = \frac{\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{3/2}\cdot(2\mu)^{\frac14}\cdot g_{\lambda,\mu}\big(\sqrt{3}(2\mu)^{\frac14}h + \alpha_{\lambda,\mu}\big)}{\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{\lambda+\sqrt{2\mu}}} \cdot \exp\Big(-\sqrt{2\gamma+\sqrt{8\mu}}\cdot h\Big)$$

and $\mathbf{n}\big(1-\exp\big(-\gamma\zeta_{\mathbf{e}} - \lambda\mathcal{Z}(\mathcal{P}_{\bar{\mathbf{e}}}) - \mu\mathcal{Y}(\mathcal{P}_{\bar{\mathbf{e}}})\big)\big) = \frac{(2\gamma-\sqrt{2\mu})\sqrt{\gamma+\sqrt{2\mu}} - (2\lambda-\sqrt{2\mu})\sqrt{\lambda+\sqrt{2\mu}}}{2\sqrt{2\gamma}-2\sqrt{2\lambda}}$.

Proof. Set

$$F(\gamma,\lambda,\mu,x) := \Big(\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{3/2}\cdot(2\mu)^{\frac14} g_{\lambda,\mu}\big(\sqrt{3}(2\mu)^{\frac14}x + \alpha_{\lambda,\mu}\big)\Big)\cdot\exp\Big(-\sqrt{2\gamma+2\sqrt{2\mu}}\cdot x\Big).$$

A direct computation shows that F solves the differential equation:

$$\frac{\partial^2}{\partial x^2}F(\gamma,\lambda,\mu,x) = \Big(2\gamma + 4u_{\lambda,\mu}(\sqrt{3}x)\Big)\cdot F(\gamma,\lambda,\mu,x).$$

We can now apply Lemma 8.1 to obtain

$$\mathbb{E}_h\Big[\exp\big(-\gamma\zeta_B - \lambda\mathcal{Z}(\mathcal{P}_{\overline{B}}) - \mu\mathcal{Y}(\mathcal{P}_{\overline{B}})\big)\Big] = \frac{F(\gamma,\lambda,\mu,h)}{F(\gamma,\lambda,\mu,0)}$$

and

$$\mathbf{n}\big(1-\exp(-\gamma\zeta_{\mathbf{e}} - \lambda\mathcal{Z}(\mathcal{P}_{\bar{\mathbf{e}}}) - \mu\mathcal{Y}(\mathcal{P}_{\bar{\mathbf{e}}}))\big) = -\frac{1}{2}\frac{1}{F(\gamma,\lambda,\mu,0)}\cdot\frac{\partial}{\partial x}F(\gamma,\lambda,\mu,x)\Big|_{x=0}.$$

It is then a standard computation to derive the proposition. □

Thanks to Proposition 8.7, we can determine the distribution of the triplet $\big(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\bar{\mathbf{e}}}), \mathcal{Y}(\mathcal{P}_{\bar{\mathbf{e}}})\big)$ under $\mathbf{n}$.

Lemma 8.7. *Under* $\mathbf{n}$ *the density of* $\big(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\bar{\mathbf{e}}}), \mathcal{Y}(\mathcal{P}_{\bar{e}})\big)$ *is:*

$$\mathbb{1}_{(\ell,z,v)\in\mathbb{R}_+^3} \frac{3}{8\pi}\frac{(z+\ell)^{\frac{1}{2}}}{v^{\frac{5}{2}}} \exp\big(-\frac{(z+\ell)^2}{2v}\big) d\ell dz dv.$$

Proof. By Proposition 8.7 the quantity:

$$\mathbf{n}\Big(1-\exp\Big(-(\gamma-\sqrt{2\mu})\zeta_{\bar{e}}-(\lambda-\sqrt{2\mu})Z\big(\mathcal{P}_{\bar{\mathbf{e}}}\big)-\mu\mathcal{Y}(\mathcal{P}_{\bar{\mathbf{e}}})\Big)\Big)$$

is equal to

$$\frac{(2\gamma-3\sqrt{2\mu})\sqrt{\gamma}-(2\lambda-3\sqrt{2\mu})\sqrt{\lambda}}{2\sqrt{2}\gamma-2\sqrt{2}\lambda}=\frac{\gamma^{\frac{3}{2}}-\lambda^{\frac{3}{2}}}{\sqrt{2}\gamma-\sqrt{2}\lambda}-\sqrt{2\mu}\cdot\frac{3\sqrt{\gamma}+3\sqrt{\lambda}}{2\sqrt{2}\gamma-2\sqrt{2}\lambda}.$$

for every $\mu,\gamma,\lambda\in\mathbb{R}_+$. Taking $\mu=0$ we obtain:

$$\mathbf{n}\Big(1-\exp\big(-\gamma\zeta_{\bar{e}}-\lambda Z\big(\mathcal{P}_{\bar{\mathbf{e}}}\big)\big)\Big)=\frac{\gamma^{\frac{3}{2}}-\lambda^{\frac{3}{2}}}{\sqrt{2}\gamma-\sqrt{2}\lambda}=\frac{1}{\sqrt{2}}\frac{\gamma+\lambda+\sqrt{\gamma\lambda}}{\sqrt{\gamma}+\sqrt{\lambda}}.$$

We can now apply formula (A.3) to derive that the distribution of $\big(Z(\mathcal{P}_{\bar{\mathbf{e}}}),\zeta_{\mathbf{e}}\big)$ is $3/4\cdot(2\pi)^{-\frac{1}{2}}(z+\ell)^{-\frac{5}{2}}dzd\ell$. Let us now come back to the case $\mu>0$. First remark that:

$$\mathbf{n}\Big(\exp\big(-\gamma\zeta_{\bar{e}}-\lambda Z\big(\mathcal{P}_{\bar{\mathbf{e}}}\big)\big)\cdot\big(1-\exp\Big(\sqrt{2\mu}\zeta_{\bar{e}}+\sqrt{2\mu}Z\big(\mathcal{P}_{\bar{\mathbf{e}}}\big)-\mu\mathcal{Y}\big(\mathcal{P}_{\bar{\mathbf{e}}}\big)\Big)\Big)$$

is equal to

$$-\sqrt{2\mu}\cdot\frac{3\sqrt{\gamma}+3\sqrt{\lambda}}{2\sqrt{2}\gamma-2\sqrt{2}\lambda}=-\sqrt{2\mu}\cdot\frac{3}{2\sqrt{2}\sqrt{\gamma}+2\sqrt{2}\sqrt{\lambda}}.$$

We then observe that for every $\gamma,\lambda>0$ we have:

$$\frac{3}{4\sqrt{2\pi}}\int_0^\infty\int_0^\infty \mathrm{d}z\mathrm{d}\ell\ (z+\ell)^{-\frac{3}{2}}\exp(-\gamma z-\lambda\ell)=\frac{3}{2\sqrt{2}\sqrt{\gamma}+2\sqrt{2}\sqrt{\lambda}}$$

and consequently, recalling that the distribution of $\big(Z(\mathcal{P}_{\bar{\mathbf{e}}}),\zeta_{\mathbf{e}}\big)$ is $3/4\cdot(2\pi)^{-\frac{1}{2}}(z+\ell)^{-\frac{5}{2}}dzd\ell$, we derive that:

$$\mathbf{n}\big(\exp(-\mu\mathcal{Y}(\mathcal{P}_{\bar{e}}))\mid Z(\mathcal{P}_{\bar{\mathbf{e}}})=z,\zeta_{\mathbf{e}}=\ell\big)=\big(1+\sqrt{2\mu}(z+\ell)\big)\exp(-\sqrt{2\mu}(z+\ell))$$

for Lebesgue almost every $(z,\ell)\in\mathbb{R}_+^2$. But the right term is the Laplace transform of

$$\frac{1}{\sqrt{2\pi}}\frac{(z+\ell)^3}{v^{\frac{5}{2}}}\exp(-\frac{(z+\ell)^2}{2v})\ dv,$$

and the desired result follows. $\square$

We conclude this section with a remark and a conjecture. Lemma 8.7 gives that the distribution of $\big(\zeta_{\mathbf{e}}, Z(\mathcal{P}_{\bar{\mathbf{e}}}), \mathcal{Y}(\mathcal{P}_{\bar{e}})\big)$ under $4/\sqrt{3}\cdot\mathbf{n}$ is the same as the distribution of $\big(L_1, L_2, \mathrm{Vol}^{\mathrm{d}}(\mathbb{D})\big)$ under $\mathbb{N}^{*,\bullet}$ (use (8.12) and the fact that, conditionally on σ, the variable L_1 is uniform on $[0,\sigma]$).

We claim that the pair $(\bar{\mathbf{e}}, \widetilde{\mathcal{P}}_{\bar{\mathbf{e}}})$ is a coding pair verifying (H_1, H_2) under $\mathbf{n}$. Let us explain why $(\bar{\mathbf{e}}, \widetilde{\mathcal{P}}_{\bar{\mathbf{e}}})$ satisfies (H_2). Introduce under $\mathbf{n}$ and conditionally on $\mathbf{e}$, a Poisson point measure $\mathcal{N}$ with intensity $2/3\mathbb{1}_{[0,\zeta_{\mathbf{e}}]}(t)\mathrm{d}t\,\mathbb{N}_{\mathbf{e}_t}(d\omega)$. By a scaling argument it is enough to show that $(\mathbf{e},\mathcal{N})$ satisfies (H_2). Recall now that the coding pair $(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}})$ satisfies (H_2) (refer to Proposition 8.1 and the discussion below) and note, that without loss of generality, we may assume that $\mathcal{P}_{\mathbf{e}}$ is of the form $\mathcal{P}_{\mathbf{e}} = \mathcal{N} + \mathcal{N}'$, where $\mathcal{N}'$ is conditionally on $\mathbf{e}$ a Poisson point measure independent of $\mathcal{N}$ and intensity $4/3\mathbb{1}_{[0,\zeta_{\mathbf{e}}]}(t)\mathrm{d}t\,\mathbb{N}_{\mathbf{e}_t}(d\omega)$. We denote the tree associated with $(\mathbf{e},\mathcal{N})$ by $\mathcal{G}_{\mathbf{e}}$ and we write $\mathcal{G}^{\circ}_{\mathbf{e}}$ for the set of all points in $\mathcal{G}_{\mathbf{e}}$ with positive label that do not belong to the spine. We may and will see $\mathcal{G}_{\mathbf{e}}$ as a subtree of $\mathfrak{H}_{\mathbf{e}}$ (the tree associated with $(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}})$). We write $\Delta_{\mathbf{e}}$ for the pseudo-distance associated with $(\mathbf{e}, \widetilde{\mathcal{P}}_{\mathbf{e}})$ and $D_{\mathbf{e}}$ for the pseudo-distance associated with $(\mathbf{e},\mathcal{N})$. Then, for every $u, v \in \mathcal{G}^{\circ}_{\mathbf{e}} \subset \mathfrak{H}_{\mathbf{e}}$, we have $D_{\mathbf{e}}(u,v) \leqslant \Delta_{\mathbf{e}}(u,v)$. Since the pseudo-distance $\Delta_{\mathbf{e}}$ has a continuous extension to $\mathfrak{H}_{\mathbf{e}}$, it is easy to derive that $D_{\mathbf{e}}$ has a continuous extension to $\mathcal{G}_{\mathbf{e}}$. Consequently $(\mathbf{e},\mathcal{N})$ satisfies (H_2). We conjecture that

Conjecture. *Under $4/\sqrt{3}\cdot\mathbf{n}$, the space $\Theta(\bar{\mathbf{e}}, \widetilde{\mathcal{P}}_{\bar{\mathbf{e}}})$, rooted at the equivalence class of 0 and market at the equivalence class of ζ_e, has the same distribution as $\mathbb{D}^{\bullet}$.*

The proof of this conjecture is a work in progress.

8.4.3 The distribution of $\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ under $\mathbf{n}^{\bullet}$

In this section we consider different conditionings. The definition of a canonical choice of these conditionings is straightforward imposing the scaling property and using a Fubini type argument (in this preliminary version we leave the details to the reader). The goal of this section is to show the following result:

Theorem 8.2. *Fix $z > 0$. Under $\mathbf{n}^{\bullet}$, the distribution of $\big(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}}\big)$ conditionally on $Z(\widetilde{\mathcal{P}}^{+,r}_{\overline{R}}) = z$ is the same as the distribution of $\big(\bar{\mathbf{e}}, \widetilde{\mathcal{P}}_{\bar{\mathbf{e}}}\big)$ conditionally on $Z(\mathcal{P}_{\bar{\mathbf{e}}}) = z$.*

In particular, Theorem 8.2 gives that the distribution of $\overline{R}^{(r)}$ conditionally on $Z(\mathcal{P}^{+}_{\overline{R}^{(r)}}) = z$ does not depend on r. Recall that under $\mathbb{P}_h$ the process B is a Brownian motion started at h and stopped when it hits 0 for the first time. Also suppose that we are given a process Y distributed under $\mathbb{P}_h$ as a (-1)-dimensional Bessel process started at h and stopped when reaching 0 for the first time. In order to obtain Theorem 8.2 we start with the following lemma:

Lemma 8.8.
(i) The distribution of $\mathbf{e}_{t^{\bullet}}$ under $\mathbf{n}^{\bullet}$ is dh. Moreover, conditionally on $\mathbf{e}_{t^{\bullet}} = h$, the processes $(\mathbf{e}_{t^{\bullet}-t})_{t\in[0,t^{\bullet}]}$ and $(\mathbf{e}_{t^{\bullet}+t})_{t\in[0,\sigma-t^{\bullet}]}$ are independent and distributed as B under $\mathbb{P}_h$.

(ii) The distribution of $R_{t^\bullet}$ under $\mathbf{n}^\bullet$ is $3 \cdot h^{-2}dh$. Moreover, conditionally on $R_{t^\bullet} = h$, the processes $(R_{t^\bullet - t})_{t\in[0,t^\bullet]}$ and $(R_{t^\bullet + t})_{t\in[0,\sigma - t^\bullet]}$ are independent and distributed as Y under $\mathbb{P}_h$.

Proof. The first point is a classical result and we refer to [23]. Let us derive (ii) form (i). First remark that it is enough to show that, for any bounded continuous functions $g : \mathbb{R} \mapsto \mathbb{R}_+$, $F_1 : \mathcal{W} \mapsto \mathbb{R}_+$ and $F_2 : \mathcal{W} \mapsto \mathbb{R}_+$, we have

$$\mathbf{n}\Big(\int_0^{\zeta_R} dt\ g(R_t) F_1\big((R_{t-s})_{s\in[0,t]}\big) F_2\big((R_{t+s})_{s\in[0,\zeta_R - t]}\big)\Big) = 3\int_0^\infty dh\ h^{-2} g(h) \mathbb{E}_h\big[F_1(Y)\big]\mathbb{E}_h\big[F_2(Y)\big].$$

Without loss of generality we may assume that $\int_0^\infty dh\ h^{-2}g(h) < \infty$ and that there exists $\delta > 0$ such that $F_1(\mathrm{w}) = 0$ for every $\mathrm{w} \in \mathcal{W}$ with $\zeta_{\mathrm{w}} < \delta$. Then since $\mathbf{n}(\zeta_{\mathbf{e}} > \delta) < \infty$, by (8.37) and an application of dominated convergence we get:

$$\begin{aligned}
&\mathbf{n}\Big(\int_0^{\zeta_R} dt\ g(R_t) F_1\big((R_{t-s})_{s\in[0,t]}\big) F_2\big((R_{t+s})_{s\in[0,\sigma - t]}\big)\Big) \\
&\qquad = 3 \lim_{\varepsilon\to 0} \varepsilon^{-2} \mathbf{n}\Big(\int_0^{\zeta_{\mathbf{e}}} dt\ g(\mathbf{e}_t) F_1\big((\mathbf{e}_{t-s})_{s\in[0,t]}\big) F_2\big((\mathbf{e}_{t+s})_{s\in[0,\zeta_{\mathbf{e}} - t]}\big) \cdot \exp\big(-\int_0^{\zeta_{\mathbf{e}}} \frac{ds}{(\mathbf{e}_s+\varepsilon)^2}\big)\Big).
\end{aligned}$$

Moreover using (i) we derive that:

$$\begin{aligned}
&\mathbf{n}\Big(\int_0^{\zeta_{\mathbf{e}}} dt\ g(\mathbf{e}_t) F_1\big((\mathbf{e}_{t-s})_{s\in[0,t]}\big) F_2\big((\mathbf{e}_{t+s})_{s\in[0,\sigma - t]}\big) \cdot \exp\big(-\int_0^{\zeta_{\mathbf{e}}} \frac{ds}{(\mathbf{e}_s+\varepsilon)^2}\big)\Big) \\
&\qquad = \int_0^\infty dh\ g(h) \mathbb{E}_h\big[F_1(B)\exp(-\int_0^{\zeta_B}\frac{ds}{(B_s+\varepsilon)^2})\big] \cdot \mathbb{E}_h\big[F_2(B)\exp(-\int_0^{\zeta_B}\frac{ds}{(B_s+\varepsilon)^2})\big].
\end{aligned}$$

Now remark that:

$$\mathbb{E}_h\big[F_1(B)\exp(-\int_0^{\zeta_B}\frac{ds}{(B_s+\varepsilon)^2})\big] = \mathbb{E}_{h+\varepsilon}\big[F_1\big((B_s-\varepsilon)_{s\leqslant T_\varepsilon(B)}\big)\exp(-\int_0^{T_\varepsilon(B)}\frac{ds}{B_s^2})\big],$$

where $T_\varepsilon(B) := \inf\{s \geqslant 0 :\ B_s = \varepsilon\}$. The formula above also holds with F_1 replaced by F_2. By classical absolute continuity relations between Brownian motion and Bessel processes, the law of $(B_s)_{s\leqslant T_\varepsilon(B)}$ under the probability measure

$$\frac{h+\varepsilon}{\varepsilon}\cdot \exp(-\int_0^{T_\varepsilon(B)}\frac{ds}{B_s^2})\cdot \mathbb{P}_{h+\varepsilon}$$

coincides with the law of $(Y_t)_{s\leqslant T_\varepsilon(Y)}$ under $\mathbb{P}_{h+\varepsilon}$, with the notation $T_\varepsilon(Y) := \inf\{s \geqslant 0 :\ Y_s = \varepsilon\}$. Consequently

$$\begin{aligned}
&\mathbf{n}\Big(\int_0^{\zeta_{\mathbf{e}}} dt\ g(\mathbf{e}_t) F_1\big((\mathbf{e}_{t-s})_{s\in[0,t]}\big) F_2\big((\mathbf{e}_{t+s})_{s\in[0,\zeta_{\mathbf{e}} - t]}\big) \cdot \exp\big(-\int_0^{\zeta_{\mathbf{e}}} \frac{ds}{(\mathbf{e}_s+\varepsilon)^2}\big)\Big) \\
&\qquad = \varepsilon^2 \int_0^\infty dh\ (h+\varepsilon)^{-2} g(h) \mathbb{E}_{h+\varepsilon}\big[F_1\big((Y_s-\varepsilon)_{s\leqslant T_\varepsilon(Y)}\big)\big]\mathbb{E}_{h+\varepsilon}\big[F_2\big((Y_s-\varepsilon)_{s\leqslant T_\varepsilon(Y)}\big)\big].
\end{aligned}$$

The desired result follows then by dominated convergence, using $\int_0^\infty dh\ h^{-2}g(h) < \infty$ and taking the limit when ε goes to 0. $\square$

Now remark that, by the special Markov property, the distribution of the point measure $\mathcal{P}^{+,r}_{\overline{R}}$ conditionally on $\overline{R}^{(r)}$ is $\mathbb{P}^{(\overline{R}^{(r)})}\big(\cdot \mid \mathcal{P}_* > -r\big)$. The following result gives an identity between coding pairs with a (-1)-dimensional Bessel spine (with some label constraints) and coding pairs with a Brownian spine (without label constraints). For every $h, r > 0$ with $h > 3^{-1/2}r$, we set under $\mathbb{P}_h$, with $h > 3^{-1/2}r$:

$$\overline{Y}^{(r)}_t := \overline{Y}_t - r\,, \qquad 0 \leqslant t \leqslant \zeta_{\overline{Y}} \inf\{s \geqslant 0 :\ \overline{Y}_s = r\}.$$

Proposition 8.8. *For every nonnegative measurable functions F, G defined respectively on $\mathcal{W}$ and on $M(\mathcal{S})$, we have :*

$$\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\Big[F\big(\overline{Y}^{(r)}\big)\mathbb{P}^{(\overline{Y}^{(r)})}\big(G(\widetilde{\mathcal{P}})\,|\,\mathcal{P}_* > -r\big)\Big] = \big(\frac{h+r}{r}\big)\cdot \mathbb{E}_{\frac{h}{\sqrt{3}}}\Big[F(\overline{B})G(\widetilde{\mathcal{P}}_{\overline{B}})\exp(-\frac{3}{2r^2}Z(\mathcal{P}_{\overline{B}}))\Big].$$

Proof. Fix $F : \mathcal{W} \to \mathbb{R}_+$ and $G : \mathcal{M}(\mathcal{S}) \to \mathbb{R}_+$, two measurable functions. Recall that $\zeta_{\overline{Y}^{(r)}} := \inf\{s \geqslant 0 :\ \overline{Y}_t = r\}$. Now remark that for every $\mathrm{w} \in \mathcal{W}$, such that $\inf \mathrm{w} \geqslant -3^{-\frac{1}{2}}r$, an application of the special Markov property combined with equation (8.6) gives

$$\mathbb{P}^{(\overline{\mathrm{w}})}\big(\mathcal{P}_* > -r\big) = \exp\big(-\int_0^{\zeta_{\mathrm{w}}} dt\ (\mathrm{w}(t) + 3^{-1/2}r)^{-2}\big).$$

Consequently we derive:

$$\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\Big[F\big(\overline{Y}^{(r)}\big)\mathbb{P}^{(\overline{Y}^{(r)})}\big(G(\widetilde{\mathcal{P}})\,|\,\mathcal{P}_* > -r\big)\Big] = \mathbb{E}_{\frac{h+r}{\sqrt{3}}}\Big[F\big(\overline{Y}^{(r)}\big)\mathbb{P}^{(\overline{Y}^{(r)})}\Big(G(\widetilde{\mathcal{P}})\mathbb{1}_{\mathcal{P}_*>-r}\Big)\exp\big(\int_0^{\zeta_{\overline{Y}^{(r)}}}\frac{dt}{Y_t^2}\big)\Big].$$

Remark that we have $\zeta_{\overline{Y}^{(r)}} = \inf\{s \geqslant 0 :\ Y_s > 3^{-\frac{1}{2}}r\}$. As in the previous proof, we use the fact that the law of $(B_s + \frac{r}{\sqrt{3}})_{s\leqslant\zeta_B}$ under the probability measure:

$$\frac{h+r}{r}\cdot\exp\big(-\int_0^{\zeta_B}\frac{ds}{B_s + 3^{-\frac{1}{2}}r}\big)\cdot\mathbb{P}_{\frac{h}{\sqrt{3}}}\ ,$$

coincides with the law of $(Y_s)_{s\leqslant\zeta_{\overline{Y}^{(r)}}}$ under $\mathbb{P}_{\frac{h+r}{\sqrt{3}}}$, to deduce that:

$$\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\Big[F\big(\overline{Y}^{(r)}\big)\mathbb{P}^{(\overline{Y}^{(r)})}\big(G(\widetilde{\mathcal{P}})\mathbb{1}_{\mathcal{P}_*>-r}\big)\exp\big(\int_0^{\zeta_{\overline{Y}^{(r)}}}\frac{dt}{Y_t^2}\big)\Big] = \frac{h+r}{r}\mathbb{E}_{\frac{h}{\sqrt{3}}}\Big[F\big(\overline{B}\big)\mathbb{P}^{(\overline{B})}\big(G(\widetilde{\mathcal{P}})\mathbb{1}_{\mathcal{P}_*>-r}\big)\Big].$$

Finally by the special Markov property and (8.6) we have:

$$\mathbb{P}^{(\overline{B})}\big(G(\widetilde{\mathcal{P}})\mathbb{1}_{\mathcal{P}_*>-r}\big) = \mathbb{P}^{(\overline{B})}\big(G(\widetilde{\mathcal{P}})\exp(-\frac{3}{2r^2}Z(\mathcal{P}))\big),$$

and the proposition follows. □

We need one last result to be able to prove Theorem 8.2. Let us use the notation of the Appendix:

$$\chi_2(x) = (2x+1)e^x\mathrm{erfc}(\sqrt{x}) - \frac{2}{\sqrt{\pi}}x^{1/2}.$$

We are going to apply the identify in distribution between coding pairs in Proposition 8.8 to obtain the following Lemma

Lemma 8.9.
(i) For $h>0$, the distribution of $Z(\mathcal{P}_{\bar{\mathbf{e}}})$ under $\mathbf{n}^\bullet(\cdot \mid \bar{\mathbf{e}}_{t^\bullet} = h)$ is:

$$\frac{3}{2}\ h^{-2}\chi_2(\frac{3z}{2h^2})dz.$$

(ii) For every $h, r>0$, the distribution of $Z(\mathcal{P}^{+,(r)}_{\overline{R}})$ under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} = h+r)$ is:

$$\frac{3}{2}\ \frac{(h+r)^2}{h^2r^2}\ \chi_2(\frac{3z}{2h^2})\exp(-\frac{3}{2r^2}z)dz.$$

Proof. Let $\lambda>0$. By Lemma 8.8 (i), we have:

$$\begin{aligned}\mathbf{n}^\bullet\Big(\exp\big(-\lambda Z(\mathcal{P}_{\bar{\mathbf{e}}})\big)\,\Big|\,\bar{\mathbf{e}}_{t^\bullet}=h\Big) &= \mathbf{n}^\bullet\Big(\exp\big(-\lambda Z(\mathcal{P}_{(\bar{\mathbf{e}}_{t^\bullet-s})_{s\leqslant t^\bullet}})-\lambda Z(\mathcal{P}_{(\bar{\mathbf{e}}_{t^\bullet+s})_{s\leqslant \zeta_{\mathbf{e}}-t^\bullet}})\big)\,\Big|\,\mathbf{e}_{t^\bullet}=\frac{h}{\sqrt{3}}\Big)\\ &= \mathbb{E}_{\frac{h}{\sqrt{3}}}\Big[\exp\big(-\lambda Z(\mathcal{P}_{\overline{B}})\big)\Big]^2. \qquad (8.46)\end{aligned}$$

By Proposition 8.7, taking the limit when $\mu\to 0$ and using $\coth(x)\sim_{x\to 0} x^{-1}$, we have:

$$\mathbb{E}_{\frac{h}{\sqrt{3}}}[\exp(-\lambda Z(\mathcal{P}_{\overline{B}}))] = (1+\sqrt{2/3\lambda}\cdot h)^{-1},$$

and so we derive the formula:

$$\mathbf{n}^\bullet\Big(\exp(-\lambda Z(\mathcal{P}_{\bar{\mathbf{e}}}))\,\big|\,\bar{\mathbf{e}}_{t^\bullet}=h\Big) = \big(1+\sqrt{2/3\lambda}\cdot h\big)^{-2}.$$

Point (i) is then a consequence of formula (A.2).

Let us prove point (ii), which is a consequence of (i) and the identity between coding pairs given in Proposition 8.8. First recall that, conditionally on $\overline{R}^{(r)}$, the distribution of the point measure $\mathcal{P}^{+,r}_{\overline{R}}$ is $\mathbb{P}^{(\overline{R}^{(r)})}\big(\cdot\,\big|\,\mathcal{P}_*>-r\big)$. Thus we can apply Lemma 8.8 (ii) to obtain:

$$\mathbf{n}^\bullet\Big(\exp(-\lambda Z(\mathcal{P}^{+,(r)}_{\overline{R}}))\,\big|\,\overline{R}_{t^\bullet}=h+r\Big) = \mathbb{E}_{\frac{h+r}{\sqrt{3}}}\big[\mathbb{P}^{(\overline{Y}^{(r)})}\big(\exp(-\lambda Z(\mathcal{P}))\,\big|\,\mathcal{P}_*>-r\big)\big]^2.$$

But by Proposition 8.8, we have :

$$\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\Big[\mathbb{P}^{(\overline{Y}^{(r)})}\Big(\exp\big(-\lambda Z(\mathcal{P})\big)\,\Big|\,\mathcal{P}_*>-r\Big)\Big] = (\frac{h+r}{r})\cdot\mathbb{E}_{\frac{h}{\sqrt{3}}}\Big[\exp\big(-(\lambda+\frac{3}{2r^2})Z(\mathcal{P}_{\overline{B}})\big)\Big].$$

Consequently, by (8.46):

$$\mathbf{n}^\bullet\Big(\exp\big(-\lambda Z(\mathcal{P}^{+,(r)}_{\overline{R}})\big)\,\Big|\,\overline{R}_{t^\bullet}=h+r\Big) = (\frac{h+r}{r})^2\cdot\mathbf{n}^\bullet\Big(\exp\big(-(\lambda+\frac{3}{2r^2})Z(\mathcal{P}_{\bar{\mathbf{e}}})\big)\,\Big|\,\bar{\mathbf{e}}_{t^\bullet}=h\Big),$$

and we derive (ii) from (i). □

We now can give a proof of Theorem 8.2.

Proof of Theorem 8.2. We argue under $\mathbf{n}^\bullet$. Let us introduce the point measures $\mathcal{H}^1_{\overline{R}}$ and $\mathcal{H}^2_{\overline{R}}$ defined by:

$$\int F(t,\omega)\ \mathcal{H}^1_{\overline{R}}(dtd\omega) := \int_0^{t^\bullet-\tau_r} F(t,\omega)\ \mathcal{P}^{+,(r)}_{\overline{R}}(dtd\omega)\,;$$
$$\int F(t,\omega)\ \mathcal{H}^2_{\overline{R}}(dtd\omega) := \int_{t^\bullet-\tau_r}^{\tau'_r-\tau_r} F(t-t^\bullet+\tau_r,\omega)\ \mathcal{P}^{+,(r)}_{\overline{R}}(dtd\omega).$$

We also introduce the processes $R^{(r),1}_t := R^{(r)}_{t^\bullet-\tau_r-t}$ for $t \in [0, t^\bullet - \tau_r]$ and $R^{(r),2}_t := R^{(r)}_{t^\bullet-\tau_r+t}$ for $t \in [0, \tau'_r - t^\bullet]$. Similarly we also write $\mathcal{H}^1_{\bar{\mathbf{e}}}$ and $\mathcal{H}^2_{\bar{\mathbf{e}}}$ for the point measures:

$$\int F(t,\omega)\ \mathcal{H}^1_{\bar{\mathbf{e}}}(dtd\omega) := \int_0^{t^\bullet} F(t,\omega)\ \mathcal{P}_{\bar{\mathbf{e}}}(dtd\omega)\,;$$
$$\int F(t,\omega)\ \mathcal{H}^2_{\bar{\mathbf{e}}}(dtd\omega) := \int_{t^\bullet}^{\zeta_{\mathbf{e}}} F(t-t^\bullet,\omega)\ \mathcal{P}_{\bar{\mathbf{e}}}(dtd\omega),$$

and we set $\bar{\mathbf{e}}^1_t := \bar{\mathbf{e}}_{t^\bullet-t}$ for $t \in [0, t^\bullet]$ and $\bar{\mathbf{e}}^2_t := \bar{\mathbf{e}}_{t^\bullet+t}$ for $t \in [0, \zeta_{\mathbf{e}} - t^\bullet]$. Let F_1, F_2 (resp. G_1, G_2) be two nonnegative measurable functions defined on $\mathcal{W}$ (resp. $M(S)$). We can find nonnegative measurable functions φ_1 and φ_2 on $[0,\infty)\times[0,\infty)$, such that:

$$\mathbf{n}^\bullet\Big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\big|\mathcal{Z}(\mathcal{P}^{+,(r)}_{\overline{R}}),\overline{R}^{(r)}_{t^\bullet-\tau_r}\Big) = \varphi\big(\mathcal{Z}(\mathcal{P}^{+,(r)}_{\overline{R}}),\overline{R}^{(r)}_{t^\bullet-\tau_r}\big),$$

and

$$\mathbf{n}^\bullet\Big(F_1(\bar{\mathbf{e}}^1)F_2(\bar{\mathbf{e}}^2)G_1(\mathcal{H}^1_{\bar{\mathbf{e}}})G_2(\mathcal{H}^2_{\bar{\mathbf{e}}})\big|\mathcal{Z}(\mathcal{P}_{\mathbf{e}}),\bar{\mathbf{e}}_{t^\bullet}\Big) = \varphi'\big(\mathcal{Z}(\mathcal{P}_{\mathbf{e}}),\bar{\mathbf{e}}_{t^\bullet}\big).$$

We point out that the scaling property does not give directly a canonical choice of φ and φ'. Let us show that $\varphi(z,h) = \varphi'(z,h)$, $\mathrm{d}z\mathrm{d}h$-a.e. In order to proof this equality fix $h > 0$ and remark that, by Lemma 8.8 (ii), we have:

$$\begin{aligned}\mathbf{n}^\bullet&\Big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\ \big|\ \overline{R}_{t^\bullet} = h+r\Big)\\ &= \mathbb{E}_{\frac{h+r}{\sqrt{3}}}\big[F_1(\overline{Y}^{(r)})\mathbb{P}^{(\overline{Y}^{(r)})}\big(G_1(\widetilde{\mathcal{P}})\ |\ \mathcal{P}_* > -r\big)\big]\cdot\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\big[F_2(\overline{Y}^{(r)})\mathbb{P}^{(\overline{Y}^{(r)})}\big(G_2(\widetilde{\mathcal{P}})\ |\ \mathcal{P}_* > -r\big)\big].\end{aligned}$$

Then, by an application of the identity between coding pairs given in Proposition 8.8 we get:

$$\mathbb{E}_{\frac{h+r}{\sqrt{3}}}\big[F_1(\overline{Y}^{(r)})\mathbb{P}^{(\overline{Y}^{(r)})}\big(G_1(\widetilde{\mathcal{P}})\ |\ \mathcal{P}_* > -r\big)\big] = \frac{h+r}{r}\cdot\mathbb{E}_{\frac{h}{\sqrt{3}}}\big[F_1(\overline{B})G_1(\widetilde{\mathcal{P}})\exp\big(-\frac{3}{2r^2}Z(\mathcal{P}_{\overline{B}})\big)\big],$$

and the same holds with (F_1, G_1) replaced by (F_2, G_2). Putting all together we obtain that:

$$\mathbf{n}^\bullet\Big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\ \big|\ \overline{R}_{t^\bullet} = h+r\Big)$$

is equal to

$$\big(\frac{h+r}{r}\big)^2\cdot\mathbb{E}_{\frac{h}{\sqrt{3}}}\big[F_1(\overline{B})G_1(\widetilde{\mathcal{P}})\exp\big(-\frac{3}{2r^2}Z(\mathcal{P}_{\overline{B}})\big)\big]\cdot\mathbb{E}_{\frac{h}{\sqrt{3}}}\big[F_2(\overline{B})G_2(\widetilde{\mathcal{P}})\exp\big(-\frac{3}{2r^2}Z(\mathcal{P}_{\overline{B}})\big)\big].$$

We can now apply Lemma 8.8 (i) to deduce the following relation:

$$\mathbf{n}^{\bullet}\Big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\,\big|\,\overline{R}_{t^{\bullet}} = h+r\Big)$$
$$= \big(\frac{h+r}{r}\big)^2\mathbf{n}^{\bullet}\Big(F_1(\bar{\mathbf{e}}^1)F_2(\bar{\mathbf{e}}^2)G_1(\mathcal{H}^1_{\bar{\mathbf{e}}})G_2(\mathcal{H}^2_{\bar{\mathbf{e}}})\exp\big(-\frac{3}{2r^2}Z(\mathcal{P}_{\bar{\mathbf{e}}})\big)\,\big|\,\bar{\mathbf{e}}_{t^{\bullet}} = h\Big). \tag{8.47}$$

Since we can recover $\mathcal{P}^{+,(r)}_{\overline{R}}$ from $(\mathcal{H}^1_{\overline{R}}, \mathcal{H}^2_{\overline{R}})$ and $\mathcal{P}_{\bar{\mathbf{e}}}$ from $(\mathcal{H}^1_{\bar{\mathbf{e}}}, \mathcal{H}^2_{\bar{\mathbf{e}}})$, we can use definition (8.7) to see that $Z(\mathcal{P}^{+,(r)}_{\overline{R}})$ (resp. $Z(\mathcal{P}_{\bar{\mathbf{e}}})$) is a nonnegative measurable function of $(\mathcal{H}^1_{\overline{R}}, \mathcal{H}^2_{\overline{R}})$ (resp. $(\mathcal{H}^1_{\bar{\mathbf{e}}}, \mathcal{H}^2_{\bar{\mathbf{e}}})$). It follows from (8.47) that, for any nonnegative measurable function ϕ defined on $\mathbb{R}_+$, we have:

$$\mathbf{n}^{\bullet}\Big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\phi\big(Z(\mathcal{P}^{+,(r)}_{\overline{R}})\big)\,\big|\,\overline{R}_{t^{\bullet}} = h+r\Big)$$
$$= \big(\frac{h+r}{r}\big)^2\mathbf{n}^{\bullet}\Big(F_1(\bar{\mathbf{e}}^1)F_2(\bar{\mathbf{e}}^2)G_1(\mathcal{H}^1_{\bar{\mathbf{e}}})G_2(\mathcal{H}^2_{\bar{\mathbf{e}}})\phi\big(Z(\mathcal{P}_{\mathbf{e}})\big)\exp\big(-\frac{3}{2r^2}Z(\mathcal{P}_{\bar{\mathbf{e}}})\big)\,\big|\,\bar{\mathbf{e}}_{t^{\bullet}} = h\Big).$$

By Lemma 8.8, the distribution $\bar{\mathbf{e}}_{t^{\bullet}}$ and $\overline{R}_{t^{\bullet}}$ are equivalent to the Lebesgue measure on $\mathbb{R}_+$. We can now apply Lemma 8.9 to obtain that, for Lebesgue almost every $h > 0$:

$$\int_0^{\infty} dz\ \varphi(z,h)\phi(z)\chi_2(\frac{3z}{2h^2})\exp(-\frac{3}{2r^2}z) = \int_0^{\infty} dz\ \varphi'(z,h)\phi(z)\chi_2(\frac{3z}{2h^2})\exp(-\frac{3}{2r^2}z).$$

Consequently, we deduce that $\varphi(z,h) = \varphi'(z,h)$, for Lebesgue almost every $z, h \in \mathbb{R}_+$. To conclude, we can now remark that, by Lemmas 8.8 and 8.9, the conditional distribution of $\overline{R}_{t^{\bullet}} - r$, knowing $Z(\mathcal{P}^{+,(r)}_{\overline{R}}) = z$, is the same as the conditional distribution of $\bar{\mathbf{e}}_{t^{\bullet}}$, knowing $Z(\mathcal{P}_{\bar{\mathbf{e}}}) = z$, and this distribution is precisely:

$$C(z)h^{-2}\chi_2(\frac{3z}{2h^2})\ dh$$

where $C(z)$ is a constant only depending on z. We get:

$$\mathbf{n}^{\bullet}\big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\,\big|\,Z(\mathcal{P}^{+,(r)}_{\overline{R}}) = z\big) = C(z)\int_0^{\infty} h^{-2}\chi_2(\frac{3z}{2h^2})\varphi(z,h)\ dh;$$

and

$$\mathbf{n}^{\bullet}\big(F_1(\bar{\mathbf{e}}^1)F_2(\bar{\mathbf{e}}^2)G_1(\mathcal{H}^1_{\bar{\mathbf{e}}})G_2(\mathcal{H}^2_{\bar{\mathbf{e}}})\,\big|\,Z(\mathcal{P}_{\bar{\mathbf{e}}}) = z\big) = C(z)\int_0^{\infty} h^{-2}\chi_2(\frac{3z}{2h^2})\varphi'(z,h)\ dh$$

for Lebesgue almost every $z > 0$. Using the equality $\varphi(z,h) = \varphi'(z,h)$, d$zdh$-a.e. We finally deduce :

$$\mathbf{n}^{\bullet}\big(F_1(\overline{R}^{(r),1})F_2(\overline{R}^{(r),2})G_1(\mathcal{H}^1_{\overline{R}})G_2(\mathcal{H}^2_{\overline{R}})\,\big|\,Z(\mathcal{P}^{+,(r)}_{\overline{R}}) = z\big)$$
$$= \mathbf{n}^{\bullet}\big(F_1(\bar{\mathbf{e}}^1)F_2(\bar{\mathbf{e}}^2)G_1(\mathcal{H}^1_{\bar{\mathbf{e}}})G_2(\mathcal{H}^2_{\bar{\mathbf{e}}})\,\big|\,Z(\mathcal{P}_{\bar{\mathbf{e}}}) = z\big)$$

where the equality holds dz-a.e. The theorem follows since the previous equality can be extended to every $z > 0$ by using the scaling property. □

8.4.4 Spatial Markov property and explicit formulas

Let us explain why $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ verifies assumptions (H_2, H_3). The fact that it verifies (H_2) comes from Theorem 8.2 since $(\bar{\mathbf{e}}, \mathcal{P}_{\bar{\mathbf{e}}})$ verifies (H_2). On the other hand, using that the pair $\big(\overline{R}, \widetilde{\mathcal{P}}^{+}_{\overline{R}}\big)$ verifies (H_3) and the identification of the tree encoded by $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ with the tree $\mathcal{T}^{+}_{r}$, one can easily deduce that $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$ also verifies (H_3).

Recall now the definition $Z_r := Z(\widetilde{\mathcal{P}}^{+,r}_{\overline{R}})$.

Proposition 8.9.
(i) Under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$, we have $Z_r = \lim_{\varepsilon \to 0} \varepsilon^{-2} \mathrm{Vol}^{+}(\check{\mathbb{B}}^\bullet_r \cap \mathbb{B}_{r+\varepsilon})$, in probability.

(ii) The distribution of Z_r under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$ is :

$$\sqrt{\frac{3}{2\pi}}\frac{z^{-\frac{1}{2}}}{r}\exp(-\frac{3}{2r^2}z)\mathrm{d}z.$$

Proof. Point (i) can be obtained by a direct adaptation of [40, Proposition 1.1] (we leave the details to the reader). Let us explain why (ii) holds. First remark that, by Lemma 8.8 (ii), the distribution of $\overline{R}_{t^\bullet}$ is $3h^{-2}\mathrm{d}h$. In particular, we have $\mathbf{n}^\bullet(\overline{R}_{t^\bullet} > r) = 3^{\frac{3}{2}} r^{-1}$. Applying Lemma 8.9 we obtain:

$$\begin{aligned}
\mathbf{n}^\bullet\big(F(Z_r) \mid \overline{R}_{t^\bullet} > r\big) &= 3^{-\frac{3}{2}} r \int_r^\infty \mathrm{d}h\ h^{-2} \mathbf{n}^\bullet\big(F(Z_r) \mid \overline{R}_{t^\bullet} = h\big) \\
&= 3^{-\frac{3}{2}} r \int_0^\infty \mathrm{d}h\ (h+r)^{-2} \mathbf{n}^\bullet\big(F(Z_r) \mid \overline{R}_{t^\bullet} = h + r\big) \\
&= \frac{r^{-1}}{2\sqrt{3}} \int_0^\infty \mathrm{d}z\ F(z) \exp\big(-\frac{3}{2r^2} z\big) \int_0^\infty dh\ h^{-2} \chi_2\big(\frac{3z}{2h^2}\big).
\end{aligned}$$

We can now perform a change of variables to get:

$$\mathbf{n}^\bullet\big(F(Z_r) \mid \overline{R}_{t^\bullet} > r\big) = C(r) \int_0^\infty dz\ z^{-\frac{1}{2}} \exp\big(-\frac{3}{2r^2} z\big) F(z),$$

where $C(r)$ is a positive constant. Since $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$ is a probability measure and taking $F = 1$, we derive that $C(r) = \sqrt{\frac{3}{2\pi}} r^{-1}$. □

Recall that $\mathbb{B}^\bullet_r$ is an element of $\mathbb{K}$. It will be useful to remark that Proposition 8.9 shows that the quantity Z_r can be obtained as measurable function of $\check{\mathbb{B}}^\bullet_r$ (since $\check{\mathbb{B}}^\bullet_r \cap \mathbb{B}_{r+\varepsilon}$ is the set of all the points $x \in \check{\mathbb{B}}^\bullet_r$ at distance smaller than ε from the boundary $\partial_1 \check{\mathbb{B}}^\bullet_r$). For this reason we will interpret Z_r as the boundary length of $\partial_1 \check{\mathbb{B}}^\bullet_r$ and we set $|\partial_1 \check{\mathbb{B}}^\bullet_r| := Z_r$.

Theorem 8.3. *Under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$, the intrinsic distance on $\check{\mathbb{B}}^\circ_r$ has a continuous extension to $\check{\mathbb{B}}^\bullet_r$. Equip the space $\check{\mathbb{B}}^\bullet_r$ with this continuous extension and the restriction of the volume measure and point it at $\Pi^{+}(\tau_r)$. Then, conditionally on $|\partial_1 \mathbb{B}^\bullet_r|$, the spaces $\check{\mathbb{B}}^\bullet_r$ and $\mathbb{B}^\bullet_r$ are independent. Furthermore, for every $z > 0$, the distribution of $\check{\mathbb{B}}^\bullet_r$, under $\mathbf{n}^\bullet(\cdot \mid |\partial_1 \check{\mathbb{B}}^\bullet_r| = z)$, is the distribution of $\Theta\big(\bar{\mathbf{e}}, \mathcal{P}_{\bar{\mathbf{e}}}\big)$, under $\mathbf{n}^\bullet(\cdot \mid Z(\mathcal{P}_{\bar{\mathbf{e}}}) = z)$.*

Proof. In this proof we argue under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} > r)$. Lemma 8.8 (ii) gives that, conditionally on $|\partial_1 \mathbb{B}_r^\bullet|$, the processes $R^{(r)}$ and $X^{(r)}$ are independent. We can now apply the special Markov property to deduce that, conditionally on $|\partial_1 \mathbb{B}_r^\bullet|$, the pairs $(X^{(r)}, \mathcal{X}^{(r)})$ and $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ are independent. Recall that, by Proposition 8.6, the hull $\mathbb{B}_r^\bullet$ coincides – as an element of $\mathbb{K}$ – with $\Theta(X, \mathcal{X}^{(r)})$. Moroever the labeled tree $\mathcal{T}_r^+$ is identified with $\mathcal{T}_{\overline{R}^{(r)}}$, the tree associated with the coding pair $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$. In Proposition 8.5, we showed that this identification induces an isometry Φ from $\Theta^\circ(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ to $\check{\mathbb{B}}_r^\circ$ and that the pushforward of the volume measure of $\Theta^\circ(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ under Φ is $\mathrm{Vol}^{+,r}$ (the restriction of Vol^+ to $\check{\mathbb{B}}_r^\circ$). Since $(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ satisfies (H_2), the space $\Theta(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ is well defined. Write $\Delta_{\overline{R}^{(r)}}$ for the metric of $\Theta(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ and remark that $\Delta_{\overline{R}^{(r)}}$ is the continuous extension of the metric of $\Theta^\circ(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$. To conclude, it is sufficient to show that the isometry Φ can be extended to a bijection from $\Theta(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ to the set of all points of $\check{\mathbb{B}}_r^\bullet$. Let us explain how to extend Φ. For every $u \in \mathcal{T}_{\overline{R}^{(r)}}$, write $u' \in \mathcal{T}_r^+$ for the point associated with u in the identification between $\mathcal{T}_{\overline{R}^{(r)}}$ and $\mathcal{T}_r^+$. Note that by construction we have:

(C') For every $u, v \in \mathcal{T}_{\overline{R}^{(r)}}$, $\Delta_{\overline{R}^{(r)}}(u,v) \geqslant \Delta^+(u',v')$.

Let us conclude. The boundary $\partial_2 \mathcal{T}_{\overline{R}^{(r)}}$ is identified with $\partial_2 \mathcal{T}_r^+$ and we have $\Pi^+(\partial_2 \mathcal{T}_r^+) = \partial_2 \check{\mathbb{B}}_r^\bullet$. Since $(\overline{R}, \mathcal{P}_{\overline{R}}^+)$ satisfies (H_3), we have $\Delta^+(u',v') > 0$ for every $(u',v') \in \partial_2 \mathcal{T}_r^+ \times \mathcal{T}_r^+$ with $u' \neq v'$. In particular, the projection Π^+ realizes a bijection between $\partial_2 \mathcal{T}_r^+$ and $\partial_2 \check{\mathbb{B}}_r^\bullet$. We also deduce by (C') that the canonical projection from $\partial_2 \mathcal{T}_{\overline{R}^{(r)}}$ onto $\partial_2 \Theta(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ is a bijection. Furthermore since $(\overline{R}, \mathcal{P}_{\overline{R}}^+)$ satisfies (H_3), and by standard properties of the exit local time $\mathfrak{L}^{*,r}$ defined on (8.44), it is easy to see that for every $u', v' \in \partial_1 \mathcal{T}_r^+$ we have $\Delta^+(u',v') = 0$ iff $\Lambda_{w'}^+ > r$ for every $w' \in]u', v'[_{\mathcal{T}^+}$ or for every $w' \in]v', u'[_{\mathcal{T}_+}$. Then by applying (C') we obtain that, for every $u', v' \in \partial_1 \mathcal{T}_r^+$, if we have $\Delta^+(u',v') = 0$ then $\Delta_{\overline{R}^{(r)}}(u,v) = 0$. This gives that the identification between $\mathcal{T}_{\overline{R}^{(r)}}$ and $\mathcal{T}_r^+$ induces a bijection between $\Theta(\overline{R}^{(r)}, \widetilde{\mathcal{P}}_{\overline{R}}^{+,r})$ and $\mathbb{B}_r^\bullet$. □

We conclude this work by giving some explicit formulas for the $\mathbb{D}^\bullet$. Recall that, under $\mathbf{n}^\bullet$, the quantity $\overline{R}_{t^\bullet}$ stands for the distance between the root ρ_+ and the marked point $\rho_+^\bullet$.

We start with the Laplace transform of $(|\partial \mathbb{D}_+^\bullet|, \mathrm{Vol}^+(\mathbb{D}_+^\bullet))$ under $\mathbf{n}^\bullet(\cdot \mid \overline{R}_{t^\bullet} = h)$, for every $h > 0$. We also recall that under $\mathbb{P}_{3^{-\frac{1}{2}}h}$ the notation Y stands for a (-1)-dimensional Bessel process started at $3^{-\frac{1}{2}}h$ and stopped when it hits 0. We write $\mathcal{P}_Y^+$ to denote a point measure which is distributed conditionally on Y as a Poisson point measure with intensity $2\mathbb{1}_{[0,\zeta_Y]}(t)dt\mathbb{N}_{\overline{Y}}\big(\cdot \cap \{W_* > 0\}\big)$.

Proposition 8.10. *For every $\mu \geqslant 0$ and $\gamma \geqslant 0$, we have:*

$$\mathbf{n}^\bullet\big(\exp\big(-\gamma|\partial \mathbb{D}_+^\bullet| - \mu \mathrm{Vol}^+(\mathbb{D}_+^\bullet)\big) \,\big|\, \overline{R}_{t^\bullet} = h\big)$$
$$= \frac{2}{3}h^2 \cdot \Big(\sqrt{\gamma + \sqrt{2\mu}} + \sqrt{\frac{3}{2}} \cdot (2\mu)^{\frac{1}{4}} \coth\big((2\mu)^{\frac{1}{4}} h\big)\Big)^2 \cdot \exp\Big(-\sqrt{\frac{8}{3}} \cdot \sqrt{\gamma + \sqrt{2\mu}} \cdot h\Big). \tag{8.48}$$

Proof. Fix $h > 0$ and $\mu, \gamma \geqslant 0$. The boundary length $|\partial \mathbb{D}_+^\bullet|$ is ζ_R. So using the notation above and Lemma 8.8 (*ii*), we get:

$$\mathbf{n}^\bullet\big(\exp\big(-\gamma|\partial\mathbb{D}_+^\bullet| - \mu\mathrm{Vol}^+\big(\mathbb{D}_+^\bullet\big)\big)\,\big|\,\overline{R}_{t^\bullet} = h\big) = \mathbb{E}_{\frac{h}{\sqrt{3}}}\big[\exp\big(-\gamma\zeta_Y - \mu\mathcal{Y}(\mathcal{P}^+_{\overline{Y}})\big)\big]^2.$$

Now recall for every $\varepsilon > 0$ the notation

$$\overline{Y}^{(\varepsilon)}_t = \overline{Y}_t - \varepsilon\,, \qquad 0 \leqslant t \leqslant \zeta_{\overline{Y}^{(\varepsilon)}} := \inf\{t \geqslant 0 :\ \overline{Y}_t = \varepsilon\},$$

and introduce $\mathcal{P}^{+,(\varepsilon)}_{\overline{Y}}$ the point measure defined by the relation:

$$\int F(t,\omega)\ \mathcal{P}^{+,(\varepsilon)}_{\overline{Y}}(dtd\omega) := \int_0^{\zeta_{\overline{Y}^{(\varepsilon)}}} F(t,\omega-\varepsilon)\ \mathcal{P}^+_{\overline{Y}}(dtd\omega).$$

Then the point measure $\mathcal{P}^{+,(\varepsilon)}_{\overline{Y}}$ is distributed, conditionally on $\overline{Y}^{(\varepsilon)}$, as a Poisson point measure with intensity $2\mathbb{1}_{[0,\zeta_{\overline{Y}^{(\varepsilon)}}]}(t)dt\ \mathbb{N}_{\overline{Y}^{(\varepsilon)}}(\cdot\cap\{W_* > -\varepsilon\})$. In other words, conditionally on $\overline{Y}^{(\varepsilon)}$, the law of $\mathcal{P}^{+,(\varepsilon)}_{\overline{Y}}$ is $\mathbb{P}^{(\overline{Y}^{(\varepsilon)})}(\cdot\,|\,\mathcal{P}_* > -\varepsilon)$. Moroever since $\zeta_Y = \inf\{s \geqslant 0 :\ Y_s = 0\}$, we have:

$$\zeta_{Y^{(\varepsilon)}} \to \zeta_Y \ \text{ and } \ \mathcal{Y}(\mathcal{P}^{+,(\varepsilon)}_{\overline{Y}}) \to \mathcal{Y}(\mathcal{P}^+_{\overline{Y}})$$

when ε tends to 0. We can now apply monotone convergence, to get

$$\mathbf{n}^\bullet\big(\exp\big(-\gamma|\partial\mathbb{D}_+^\bullet| - \mu\mathrm{Vol}^+\big(\mathbb{D}_+^\bullet\big)\big)\,\big|\,\overline{R}_{t^\bullet} = h\big) = \lim_{\varepsilon\to 0}\mathbb{E}_{\frac{h}{\sqrt{3}}}\big[\exp\big(-\gamma\zeta_{\overline{Y}^{(\varepsilon)}} - \mu\mathcal{Y}(\mathcal{P}^{+,(\varepsilon)}_{\overline{Y}})\big)\big]^2$$

and then, applying the coding identity of Proposition 8.8, we deduce that the previous display is equal to

$$\lim_{\varepsilon\to 0}\Big(\frac{h}{\varepsilon}\Big)^2\cdot\mathbb{E}_{\frac{h-\varepsilon}{\sqrt{3}}}\Big[\exp\Big(-\gamma\zeta_B - \frac{3}{2\varepsilon^2}Z(\mathcal{P}_{\overline{B}}) - \mu\mathcal{Y}(\mathcal{P}_{\overline{B}})\Big)\Big]^2.$$

We can now apply Proposition 8.7 to derive:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}_{\frac{h-\varepsilon}{\sqrt{3}}}\Big[\exp\Big(-\gamma\zeta_B - \frac{3}{2\varepsilon^2}Z(\mathcal{P}_{\overline{B}}) - \mu\mathcal{Y}(\mathcal{P}_{\overline{B}})\Big)\Big]\\ &\qquad = \sqrt{\frac{2}{3}}\Big(\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{3/2}\cdot(2\mu)^{\frac14}\coth\big((2\mu)^{\frac14}h\big)\Big)\cdot\exp\Big(-\sqrt{\frac{2}{3}}\sqrt{\gamma+\sqrt{2\mu}}\cdot h\Big).\end{aligned}$$

and the desired result follows. □

As we did for the boundary of $\check{\mathbb{B}}_r^\bullet$, we divide the boundary of the hull $\mathbb{B}_r^\bullet$ in two parts, noting that $\partial\mathbb{B}_r^\bullet = \big(\mathbb{B}_r^\bullet\cap\check{\mathbb{B}}_r^\bullet\big)\cup\big(\mathbb{B}_r^\bullet\cap\partial\mathbb{D}_+\big)$ and taking:

$$\partial_1\mathbb{B}_r^\bullet := \mathbb{B}_r^\bullet\cap\check{\mathbb{B}}_r^\bullet\ ;\ \partial_2\mathbb{B}_r^\bullet := \mathbb{B}_r^\bullet\cap\partial\mathbb{D}_+.$$

We have a natural notion of boundary length for $\partial_1\check{\mathbb{B}}_r^\bullet$ and $\partial_2\check{\mathbb{B}}_r^\bullet$; indeed we can set

$$|\partial_1\mathbb{B}_r^\bullet| = |\partial_1\check{\mathbb{B}}_r^\bullet| = Z_r \text{ and } |\partial_2\check{\mathbb{B}}_r^\bullet| = \mathrm{m}_{\mathbb{D}_+}(\mathbb{B}_r^\bullet\cap\partial\mathbb{D}_+) = \zeta_R - (\tau_r' - \tau_r).$$

Proposition 8.11. *For every $\gamma, \mu \geqslant 0$ and $r, z > 0$ we have:*

$$\begin{aligned}\mathbf{n}^\bullet\big(\exp\big(-\gamma|\partial_2\mathbb{B}_r^\bullet| - \mu\mathrm{Vol}^+\big(\mathbb{B}_r^\bullet\big)\big)\,\big|\,|\partial_1\mathbb{B}_r^\bullet| = z\big)\\ = \frac{2}{3}r^2\cdot\Big(\sqrt{\gamma+\sqrt{2\mu}} + \sqrt{\frac{3}{2}}\cdot(2\mu)^{\frac14}\coth\big((2\mu)^{\frac14}r\big)\Big)^2\cdot\exp\Big(-\sqrt{\frac{8}{3}}\cdot\sqrt{\gamma+\sqrt{2\mu}}\cdot r\Big)\\ \cdot\exp\Big(-z\Big(\sqrt{\frac{\mu}{2}}\big(3\coth^2\big((2\mu)^{\frac14}r\big)-2\big) - \frac{3}{2r^2}\Big)\Big).\end{aligned}$$

Proof. Recall the notation $\mathcal{X}_1^{(r)}$, $\mathcal{X}_1^{(r),\prime}$ and $\mathcal{X}_r^{(2)}$ defined in 8.43 and 8.45. The quantity τ_r (resp. $\zeta_R - \tau_r'$) is a measurable function of $\mathcal{X}_1^{(r)}$ (resp. $\mathcal{X}_1^{(r),\prime}$). To see it set

$$|\mathcal{X}_1^{(r)}| := \inf\{t \geqslant 0 : \ \mathcal{X}_1^{(r)}([t,\infty)\times\mathcal{S}) = 0\} \text{ and } |\mathcal{X}_1^{(r),\prime}| := \inf\{t \geqslant 0 : \ \mathcal{X}_1^{(r),\prime}([t,\infty)\times\mathcal{S}) = 0\},$$

and remark that we have $|\mathcal{X}_1^{(r)}| = \tau_r$ and $|\mathcal{X}_1^{(r),\prime}| = \zeta_R - \tau_r$. By Proposition 8.6 we have:

$$\begin{aligned}\mathbf{n}^\bullet\big(\exp(-\gamma|\partial_2\mathbb{B}_r^\bullet| - \mu\mathrm{Vol}^+(\mathbb{B}_r^\bullet))\,\big|\,|\partial_1\mathbb{B}_r^\bullet|\big)\\ = \mathbf{n}^\bullet\Big(\exp\Big(-\gamma\big(|\mathcal{X}_1^{(r)}| + |\mathcal{X}_1^{(r),\prime}|\big) - \mu\Big(\mathcal{Y}(\mathcal{X}_1^{(r)}) + \mathcal{Y}(\mathcal{X}_r^{(1)\prime}) + \mathcal{Y}(\mathcal{X}_r^{(2)})\Big)\Big)\,\Big|\,|\partial_1 B_r^\bullet|\Big).\end{aligned}$$

As we already noted below (8.45), an application of the special Markov property gives that, conditionally on $|\partial_1\mathbb{B}_r^\bullet|$, the point measure $\mathcal{X}_2^{(r)}$ is independent of $\big(\mathcal{X}_1^{(r)}, \mathcal{X}_1^{(r),\prime}\big)$ and is a Poisson point measure with intensity $2\mathbb{1}_{[0,|\partial_1\mathbb{B}_r^\bullet|]}(t)dt\ \mathbb{N}_r\big(\cdot\cap\{W_* > 0\}\big)$. In particular we have:

$$\mathbf{n}^\bullet\big(\exp\big(-\mu\mathcal{Y}(\mathcal{X}_r^{(2)})\big)\,\big|\,|\partial_1\mathbb{B}_r^\bullet|\big) = \exp\Big(-|\partial_1\mathbb{B}_r^\bullet|\mathbb{N}_r\big((1-\exp(-\mu\sigma))\mathbb{1}_{W_*>0}\big)\Big).$$

We can now apply [45, Lemma 7] to get:

$$\mathbb{N}_r\big((1-\exp(-\mu\sigma))\mathbb{1}_{W_*>0}\big) = \sqrt{\frac{\mu}{2}}\Big(3\coth^2\big((2\mu)^{\frac14}r\big) - 2\Big) - \frac{3}{2r^2}.$$

Finally, to obtain the desired formula, remark that by Lemma 8.8 (ii), the processes $(\overline{R}_{\tau_r - t})_{t\leqslant\tau_r}$ and $(\overline{R}_{\tau_r' + t})_{t\leqslant\zeta_R-\tau_r'}$ are independent of $|\partial_1\mathbb{B}_r^\bullet|$ and are distributed as two independent copies of $\overline{Y}$ under $\mathbb{P}_{3^{-\frac12}r}$. This implies that:

$$\begin{aligned}\mathbf{n}^\bullet\big(\exp\big(-\gamma|\partial_2\mathbb{B}_r^\bullet| - \mu\mathrm{Vol}^+\big(\mathbb{B}_r^\bullet\big)\big)\,\big|\,|\partial_1\mathbb{B}_r^\bullet|\big)\\ = \exp\Big(-|\partial_1\mathbb{B}_r^\bullet|\Big(\sqrt{\frac{\mu}{2}}\big(3\coth^2\big((2\mu)^{\frac14}r\big)-2\big) - \frac{3}{2r^2}\Big)\Big)\cdot\mathbb{E}_{\frac{r}{\sqrt3}}\big[\exp\big(-\gamma\zeta_Y - \mu\mathcal{Y}(\mathcal{P}_{\overline{Y}}^{+})\big)\big]^2.\end{aligned}$$

Finally since $\mathbb{E}_{3^{-\frac12}r}\big[\exp\big(-\gamma\zeta_Y - \mu\mathcal{Y}(\mathcal{P}_{\overline{Y}}^{+})\big)\big]^2$ is equal to

$$\mathbf{n}^\bullet\big(\exp(-\gamma|\partial\mathbb{D}_+^\bullet| - \mu\mathrm{Vol}^+(\mathbb{D}_+^\bullet))\,|\,\overline{R}_{t^\bullet} = r\big)$$

by Lemma 8.8 (ii), we can use (8.48) to get the desired result. □

Appendix: Some Laplace transforms

Recall the standard notation

$$\mathrm{erf}(x) = 1 - \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t,$$

and set $\psi(x) := \mathrm{erfc}(x) \cdot \exp(x)$. An application of an integration by parts gives the formula:

$$\int_0^\infty \mathrm{erfc}(\sqrt{x}) e^x\, e^{-\lambda x}\,\mathrm{d}x = \frac{1}{\sqrt{\lambda}(1+\sqrt{\lambda})}\ , \quad \lambda > 0.$$

We introduce now the function χ_1 defined for $x > 0$ by

$$\chi_1(x) := \frac{1}{\sqrt{\pi}} x^{-1/2} - \psi(x),$$

and remark that χ_1 satisfies, for every $\lambda > 0$,

$$\int_0^\infty \mathrm{d}x\, e^{-\lambda x}\,\chi_1(x) = (1+\sqrt{\lambda})^{-1}. \tag{A.1}$$

From the last two displays and an integration by parts, one checks that the function $\chi_2 = \chi_1 * \chi_1$, which satisfies

$$\int_0^\infty \mathrm{d}x\, e^{-\lambda x}\,\chi_2(x) = (1+\sqrt{\lambda})^{-2}, \tag{A.2}$$

is given for $x > 0$ by

$$\chi_2(x) = e^x \mathrm{erfc}(\sqrt{x}) - 2x\,\chi_1(x) = (2x+1)e^x \mathrm{erfc}(\sqrt{x}) - \frac{2}{\sqrt{\pi}} x^{1/2}.$$

It is a standard verification to see that $\chi_1(x) > 0$ for every $x > 0$ and consequently we also have $\chi_2(x) > 0$ for every $x > 0$. This remark is useful to define some conditionings. Finally note that

$$\begin{aligned}\int_0^\infty \int_0^\infty dxdy\ (x+y)^{-\frac{5}{2}}(1 - \exp(-\lambda x - \gamma y)) &= \int_0^1 du \int_0^\infty d\ell\ \ell^{-\frac{3}{2}}(1 - \exp(-(u\lambda + (1-u)\gamma)\ell)) \\ &= 2\sqrt{\pi} \int_0^1 du\ (u\lambda + (1-u)\gamma)^{\frac{1}{2}} \\ &= \frac{4\sqrt{\pi}}{3} \cdot \frac{\gamma + \lambda + \sqrt{\lambda\gamma}}{\sqrt{\lambda} + \sqrt{\gamma}}. \end{aligned} \tag{A.3}$$

It is useful to remark that formula $(A.3)$ can be extended to $\mathbb{C}\backslash\mathbb{R}_-$ using the complex logarithm, log, defined on $\mathbb{C}\backslash\mathbb{R}_-$, and verifying $\log(1) = 0$.

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université PARIS-SACLAY

ÉCOLE DOCTORALE
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Hadamard (EDMH)

Titre: Géométrie brownienne

Mots clés: Cartes, géométrie aléatoire, serpent brownien, processus de Bessel, inégalités isopérimétriques, théorie des excursions

Résumé: Cette thèse porte sur l'étude des modèles continus de surfaces aléatoires, émergeant comme limites d'échelle de modèles de cartes planaires aléatoires. Par analogie avec le mouvement brownien, nous parlerons de géométrie brownienne.

Nous commençons par explorer métriquement le disque brownien en suivant les distances par rapport à son bord. Nous montrons en particulier que le disque brownien bénéficie d'une propriété de Markov spatiale encodée par un processus de croissance-fragmentation explicite. Nous dérivons ensuite des résultats similaires pour la sphère brownienne et le plan brownien.

Dans un deuxième temps, nous donnons une construction unifiée pour les trois principaux modèles non-compacts de géométrie brownienne. Cette construction permet de contrôler finement les distances de ces modèles à leurs bords. Ces contrôles nous aident à obtenir plusieurs résultats géométriques et, en particulier, à montrer que les complémentaires des hulls dans le plan brownien sont des disques browniens de volume infini. Cette identification nous amène finalement à établir une propriété de Markov spatiale forte pour le plan brownien et à donner le profil isopérimétrique optimal de ce dernier.

Enfin, nous nous intéressons à des résultats plus quantitatifs en établissant des formules explicites concernant des objets géométriques. Ainsi, nous étudions certaines masses de sphères, et établissons – pour des cellules de Voronoï ou des hulls – des formules explicites concernant leurs volumes et périmètres dans le disque brownien.

Title: Brownian geometry

Keywords: Maps, random geometry, brownien snake, Bessel processes, isoperimetric inequalities, excursion theory

Abstract: This thesis is devoted to the study of continuous models arising as the scaling limit of different models of random planar maps. By analogy with Brownian motion, this theory is called Brownian geometry.

We start by exploring the Brownian disk in a metric way, by following the distances to its boundary. In particular, we establish that the Brownian disk satisfies a spatial Markov property encoded by an explicit growth-fragmentation process. We then extend our results to the Brownian sphere and to the Brownian plane.

We also provide a unified construction of the three main non-compact models of Brownian geometry. This construction allows us to control precisely the distances to the boundary in these models. We then derive several geometric results and we manage to prove that the complements of hulls in the Brownian plane are infinite volume Brownian disks. This identification leads us to determine a strong spatial Markov property for the Brownian plane and to obtain its optimal isoperimetric profile.

Finally, we focus on quantitative results by establishing explicit formulas concerning geometric objects. In particular, we study some sphere masses and give explicit computations concerning the volumes and perimeters of Voronoï cells and hulls in the Brownian disk.

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