Self-similar Markov trees and scaling limits

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This document presents the construction of self-similar Markov trees and explores several of their properties. The second part, covering discrete models and general results on scaling limits, will be available soon.

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Abstract

Self-similar Markov trees constitute a remarkable family of random compact real trees carrying a decoration function that is positive on the skeleton. As the terminology suggests, they are self-similar objects that further satisfy a Markov branching property. They are built from the combination of the recursive construction of real trees by gluing line segments with the seminal observation of Lamperti, which relates positive self-similar Markov processes and Lévy processes via a time change. They carry natural length and harmonic measures, which can be used to perform explicit spinal decompositions. Self-similar Markov trees encompass a large variety of random real trees that have been studied over the last decades, such as the Brownian CRT, stable Lévy trees, fragmentation trees, and growth-fragmentation trees. We establish general invariance principles for multi-type Galton–Watson trees with integer types and illustrate them with many combinatorial classes of random trees that have been studied in the literature, including (possibly dissipative) discrete fragmentation trees, peeling trees of Boltzmann (possibly $O(n)$-decorated) planar maps, or even the more recent fully parked trees.

Chapter 1

Introduction

Since the early 1990s and the introduction of the ubiquitous Brownian Continuum Random Tree (CRT) by Aldous [8], random real trees have become central objects in probability theory. Apart from their obvious applications to scaling limits of population models, they emerge in a variety of areas ranging from superprocesses, combinatorial optimization (minimal spanning trees), and analysis of algorithms to Liouville Quantum Gravity and the construction of the Brownian sphere [93]. For standard textbooks on the subject, see [69, 65, 96]. In particular, two important classes of random real trees have been known for a long time: the Lévy trees of Duquesne–Le Gall–Le Jan [63, 101], which are the scaling limits of discrete Bienaymé–Galton–Watson trees, and the fragmentation trees of Haas–Miermont [79], which are the genealogical trees underlying self-similar fragmentation processes [18]. The intersection of these two classes consists of the one-parameter family of stable trees, which generalize the Brownian CRT and have proved to be crucial objects in random geometry [52]. In this work, we considerably broaden the former and introduce the *self-similar Markov trees.* We prove that they appear as scaling limits of many natural discrete models of trees that already popped-up in the literature, and pave the way for their systematic study.

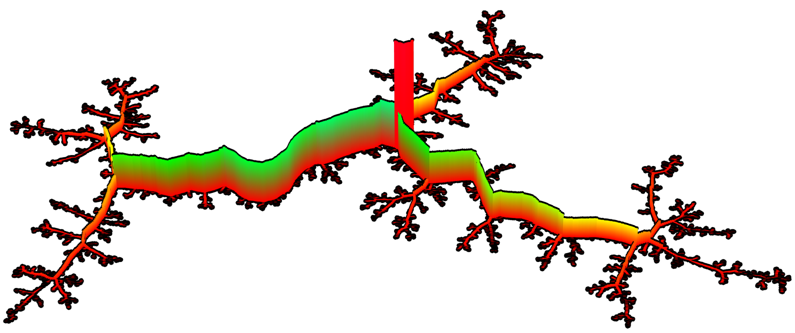

Figure 1.1: Illustration of a self-similar Markov tree (embedded in the plane $\mathbb{R}^2$) where its decoration function is represented in the third (vertical) dimension.

I. Self-Similar Markov Trees

The goal of this monography is to define and study the random rooted real trees (T, d_T, ρ), where $\rho \in T$ is a distinguished point called the **root**, which further support a function $g : T \to \mathbb{R}_+$, positive on its skeleton and referred to as a **decoration**. See Figure 1.1 for an illustration. Informally, a self-similar Markov tree (ssMt in short) is a family of laws $(\mathbb{Q}_x)_{x>0}$ on the Polish space of decorated compact random trees (see Chapter 2 for a presentation of the topology) together with a real number $\alpha > 0$, called the **self-similarity index**, so that the following properties hold:

- **Initial decoration.** Under $\mathbb{Q}_x$, the decoration at the root is $g(\rho) = x$.
- **Self-similarity**. For every $x > 0$, the law under $\mathbb{Q}_1$ of the rescaled version $(T, x^\alpha \cdot d_T, \rho, x \cdot g)$ is $\mathbb{Q}_x$.
- **Markov property**. For any height $h \geqslant 0$, conditionally on the subtree $\{u \in T : d_T(\rho, u) \leqslant h\}$ up to height h and on its decoration, the decorated subtrees above height h are independent and each has the law $\mathbb{Q}_y$, where y is the decoration at the root of the subtree (see Chapter 5 for proper statements and Figure 1.2 for an illustration).

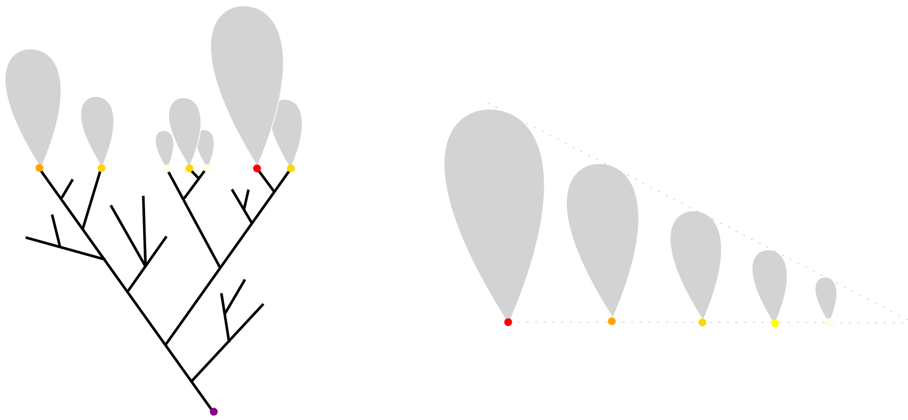

Figure 1.2: Illustration of the Markov property (left) and the self-similar property (right). The color of the root of a tree is meant for its decoration. Left: Conditionally on the black structure of the tree up to height h and on the decoration (colors) of the root vertices, the dangling subtrees (in gray) are independent.

A related Markov property (without decoration) was employed in [137] to characterize Lévy trees. Of course, random real trees carrying functions have been considered before in the literature. One may think e.g. of the genealogical trees underlying superprocesses [108] as constructed in [65]. One important difference though, is that usually for superprocesses, the spatial displacements is merely superposed on the branching structure, whereas in our case, the branching and the decoration are intimately tighten. Still, we borrow a lot from this theory, in particular in our rigorous treatment of the Markov property. When the tree is merely a segment, the decoration is given by a *positive self-similar Markov process* (pssMp). The latter have been studied in the pioneer work of Lamperti [91], who notably identified positive self-similar Markov processes as being time changed of exponential of Lévy processes, see [89, Chapter 5] and our Section 3.2. It follows from the well-known Lévy–Khintchine–Itô decomposition of Lévy processes that the distribution of pssMp is hence determined by a self-similarity exponent $\alpha \in \mathbb{R}$ together with the characteristic triplet $(\sigma^2 \geqslant 0, \mathrm{a} \in \mathbb{R}, \Lambda)$ of the underlying Lévy process ξ. More precisely, Λ is the so-called Lévy measure on $\mathbb{R} \cup \{-\infty\}$, it integrates $1 \wedge x^2$ and its mass at $-\infty$ denoted by $\mathrm{k} := \Lambda(\{-\infty\})$, serves as killing rate. The drift depends on the choice of cutoff in the Lévy–Khintchine formula, and in this work we use the standard cutoff $y \mapsto y\mathbf{1}_{|y|\leqslant 1}$. We have

$$\mathbb{E}\big(\exp(\gamma\xi_t)\big) = \exp\big(t\psi(\gamma)\big),$$

where

$$\psi(\gamma) := -\mathrm{k} + \frac{1}{2}\sigma^2\gamma^2 + \mathrm{a}\gamma + \int_{\mathbb{R}^*} \left(\mathrm{e}^{\gamma y} - 1 - \gamma y\mathbf{1}_{|y|\leqslant 1}\right)\Lambda(\mathrm{d}y).$$

Furthermore, to ensure that the pssMp gets absorbed at 0 in finite time a.s., α must be positive and the Lévy process ξ must either drift to $-\infty$ or be killed.

The first part of this work can be seen as the branching analog of the seminal contribution of Lamperti that we just sketched. We provide a rigorous setting (in particular a topology on the space decorated real trees, see Chapter 2) and construct what we believe are essentially the most general (positive) self-similar Markov trees. In a nutshell, we shall see a decorated tree (T, g) as a closed subset $\mathrm{Hyp}(g)$ of the space $T \times \mathbb{R}_+$ where the base space is the tree T, via its hypograph

$$\mathrm{Hyp}(g) := \big\{(u, x) \in T \times \mathbb{R}_+ : x \leqslant g(u)\big\}.$$

To ensure compactness of the latter, we shall impose that T is compact and g is upper-semi continuous. See the numerous illustrations below for a visualization of this concept. As for pssMp, ssMt are characterized by their self-similarity index $\alpha > 0$ together with a Gaussian coefficient $\sigma^2 \geqslant 0$, a drift coefficent $\mathrm{a} \in \mathbb{R}$, and what we call a **generalized Lévy measure** $\boldsymbol{\Lambda}$ which is a measure on the product space $\mathcal{S} = [-\infty, \infty) \times \mathcal{S}_1$, where $\mathcal{S}_1$ is the space of non-increasing sequences

$$\mathcal{S}_1 := \big\{\mathbf{y} = (y_i)_{i\geqslant 1} : y_1 \geqslant y_2 \geqslant \ldots \in \mathbb{R} \cup \{-\infty\}\big\}.$$

We also require that the image measure Λ_0 of $\boldsymbol{\Lambda}$ by the first projection $(y, \mathbf{y}) \to y$ is a usual Lévy measure, that is, integrates $1 \wedge y^2$. We then refer to $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ as a **characteristic quadruplet**. Roughly speaking, we extend the idea of Lamperti and construct ssMt through a time change of the exponential of **branching Lévy processes**, which have been introduced and studied recently by Bertoin–Mallein [26]. Heuristically, the evolution of the decoration along distinguished branches of the tree are pssMp given as the time-change exponential of the Lévy process ξ with characteristics $(\sigma^2, \mathrm{a}, \Lambda_0)$. Let us give an informal interpretation of the generalized Lévy measure. Recall first that in the pssMp case, if the process is at state $x > 0$, then it jumps to state $x \cdot \mathrm{e}^y$ with a rate $x^\alpha \cdot \Lambda_0(\mathrm{d}y)$. If one then imagines a ssMt as the genealogical tree of a cloud of independent particles, then each particle of mass $x > 0$ becomes a particle of $x \cdot \mathrm{e}^y$ and in the same time gives rise to a cloud of new particles of mass $x \cdot (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \ldots)$ at a rate $x^\alpha \cdot \boldsymbol{\Lambda}(\mathrm{d}y, \mathrm{d}(y_i)_{i \geqslant 1})$.

Many properties of ssMt are encapsulated by the so-called **cumulant function** defined by

$$\kappa(\gamma) := \psi(\gamma) + \int_{\mathcal{S}} \boldsymbol{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \left(\sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i} \right),$$

which can also be seen as the Biggins transform or moment generating function of the underlying branching Lévy process. To ensure non-explosion, we assume that κ takes negative values, which will sometimes be referred to as sub-criticality, and we leave open the construction of ssMt in the critical case $\min \kappa = 0$, see Section 3.4.

Informal Result (Construction of self-similar Markov trees)**.** *For every characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ such that $\kappa(\gamma) < 0$ for some $\gamma > 0$, there exists a family of laws $(\mathbb{Q}_x)_{x>0}$ of decorated random trees $\mathtt{T} = (T, d_T, \rho, g)$ which are self-similar with exponent $\alpha > 0$ and fulfill the Markov property.*

As already mentioned, the decoration g along branches of T evolves according to a pssMp with characteristics $(\sigma^2, \mathrm{a}, \Lambda_0; \alpha)$, and the generalized Lévy measure $\boldsymbol{\Lambda}$ induces a way to "explore" branches within T. However, contrary to the case of pssMp, different characteristic quadruplets can produce the same ssMt, and we identify precisely when this happens in Section 6.3, using a concept of bifurcators that is adapted from [123, 131].

Properties of ssMt and their random measures

Regarding the Markov property, we will in fact describe various Markov decompositions in Chapter 5. We then establish several basic properties of ssMt, including the computation of their Hausdorff dimension, by studying natural **finite measures** on ssMt. Real trees are naturally equipped with the length measure, i.e. the 1-dimensional Hausdorff measure, which may be thought of as the Lebesgue measure on T and is therefore denoted by λ. Although λ is not even locally finite in most cases of interest, the decoration function g enables us to circumvent this issue. We consider the measures $\mathrm{d}\lambda^\gamma := g^{\gamma-\alpha}\mathrm{d}\lambda$ supported by the skeleton of T and which we call **weighted length measures**. We show that the latter has a finite total mass provided κ takes negative values before γ, see Proposition 3.11. In particular, by self-similarity, we have

$$\lambda^\gamma(T) = \int_T \mathrm{d}\lambda \; g^{\gamma-\alpha} \quad \text{under } \mathbb{Q}_x \qquad \overset{(d)}{=} \qquad x^\gamma \int_T \mathrm{d}\lambda \; g^{\gamma-\alpha} = x^\gamma \lambda^\gamma(T) \quad \text{under } \mathbb{Q}_1,$$

so that the weighted length measures λ^γ are self-similar with exponent γ. In most situations, there is yet another natural finite measure on T, denoted by μ, which is supported by the leaves of T. Its construction requires a more stringent hypothesis on κ, that we call the first Cramér hypothesis. This requires the existence of $\omega_- \in (0, \infty)$ so that

$$\kappa(\omega_-) = 0 \quad \text{and} \quad \kappa(q) < 0 \text{ for some } q > \omega_-.$$

The formal slightly more restrictive condition is given by Assumption 3.12.

Informal Result (Harmonic measure)**.** *Suppose* $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *satisfies the first Cramér hypothesis. Then, as* $\gamma \to \omega_-$ *the renormalized length measure* $-\kappa(\gamma) \cdot \lambda^\gamma$ *converge in probability towards a measure* μ *supported by the leaves of* T*.*

The measure μ is called the **harmonic measure** since it is connected to the so-called harmonic or additive martingale in the branching random walk underlying our construction of $\mathtt{T} = (T, d_T, \rho, g)$. Alike the weighted length measures, the harmonic measure is self-similar with exponent ω_-. The harmonic measure is natural in many respects and it is for example used as a Frostman measure to compute the Hausdorff dimension of T:

Informal Result (Hausdorff dimension)**.** *Suppose* $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *satisfies the first Cramér hypothesis. Then almost surely,* T *has Hausdorff dimension*

$$\dim_H(T) = \left(1 \vee \frac{\omega_-}{\alpha}\right), \qquad a.s.$$

This result considerably extends the fragmentation case [78] or the growth-fragmentation case [124]. We then use those finite measures to provide **spine decompositions** of our decorated random trees. Roughly speaking, we use $-\kappa(\gamma) \cdot \lambda^\gamma$, for γ such that $\kappa(\gamma) < 0$, or μ, which can thought as the extremal case $\gamma = \omega_-$, to distinguish a point $\rho^\bullet$ at random in T. The branch $[\![\rho, \rho^\bullet]\!]$ connecting ρ and $\rho^\bullet$ is then called the spine. Using the Markov property, we shall see that the decorated trees dangling from the spine are, conditionally on their initial decoration, independent ssMt with characteristics $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. However, the evolution of the decoration along the spine is now governed by another set of characteristics $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ explicitly given in terms of $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$, and in particular, the Lévy Khintchine exponent ψ_γ of the Lévy process underlying the pssMp evolution along the tagged branch is simply given by

$$\psi_\gamma(z) := \kappa(\gamma + z).$$

As the reader may know, spinal decompositions are essential tools in branching process theory and are also instrumental in many of our proofs. The spinal decomposition can also be seen as a more intrinsic and geometric description of the law of the ssMt. Indeed, as we alluded to above, several characteristic quadruplets $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ can yield to the same ssMt. However we shall see in Corollary 6.13 that the quadruplet $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ generically uniquely specifies the law $(\mathbb{Q}_x)_{x>0}$ of the ssMt. Furthermore, although the pssMp associated to $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ is quite arbitrary, those appearing as the decoration along the tagged branch of a ssMt have special properties, see Section 6.5 for details.

II. Examples

Self-similar Markov trees include many examples of important random (undecorated) real trees that have appeared in the literature. They encompass in particular the fragmentations trees constructed by Haas and Miermont [78] as the genealogical trees underlying self-similar conservative fragmentations processes [17], and the generalized fragmentation trees introduced recently by Stephenson [134] to cover the dissipative case. If one considers ssMt as branching analogs of Lévy processes, then the (generalized) fragmentation trees would correspond to the subordinator case. More precisely, they consist of ssMt for which the decoration along branches is decreasing. The fragmentation trees of Haas and Miermont had "no erosion" and were conservative: the mass of particles is conserved in splitting events which in our case means that

$$\boldsymbol{\Lambda}\Big(\Big\{\big(y_0,(y_1,...)\big)\in\mathcal{S}:\sum_{j=0}^{\infty}\mathrm{e}^{y_j}\neq 1\Big\}\Big)=0.$$

In such situations, it is plain that the first Cramér hypothesis is always satisfied with $\omega_- = 1$, and in fact the decoration $g(u)$ corresponds to the harmonic mass of the fringe subtree above point $u \in T$. In particular, we have $\upmu(T) = x$ under $\mathbb{Q}_x$, so that the total harmonic mass is deterministic. Fragmentation trees already include many interesting examples such as the Brownian Continuum Random Tree (CRT) or more generally the stable trees. Specifically, it is well known that the **Brownian CRT** $\mathcal{T}_1$ can be seen as the real tree constructed from a standard Brownian excursion of length 1, say $(\mathrm{e}_1(s))_{0\leqslant s\leqslant 1}$ as in [66, 97]. We denote its contour measure by γ_{e_1} and endow $\mathcal{T}_1$ with the decoration which assigns to each vertex $v \in \mathcal{T}_1$ the contour-mass $\gamma_{\mathrm{e}_1}(\mathcal{T}_{1,v})$ of the fringe-subtree above point v, see Figure 1.3.

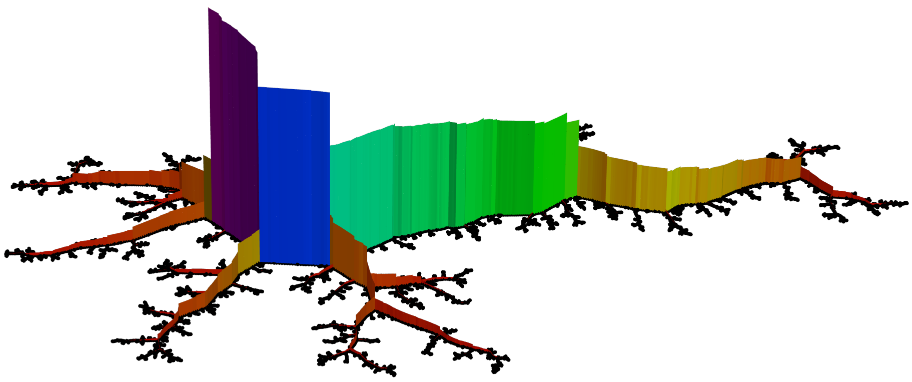

Figure 1.3: A simulation of a Brownian CRT. The tree is embedded (non-isometrically in $\mathbb{R}^2$) and the decoration function representing the $\upmu$-mass above each point is depicted in the vertical coordinate.

It follows then from [17, 135] that the decorated Brownian CRT is indeed a self-similar Markov tree with self-similar with index $\alpha = 1/2$, no erosion, no Gaussian part and generalized Lévy measure given by

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0},(\mathrm{e}^{y_1},\dots)\big)\ \boldsymbol{\Lambda}_{\mathrm{Bro}}(\mathrm{d}y_0,\mathrm{d}(y_i)_{i\geqslant 1}) := \sqrt{\frac{2}{\pi}}\int_{1/2}^{1} F(x,1-x,0,0,...)\frac{\mathrm{d}x}{(x(1-x))^{3/2}},$$

where F stands for a generic nonnegative functional on $\mathcal{S}$. See Example 4.6. This well-known and useful interpretation has already been used many times in the literature see e.g. [38].

Perhaps more surprisingly, there are other representations of (a small variant of the) Brownian CRT as a self-similar Markov tree which is not anymore a fragmentation tree. Consider this time the Brownian CRT $\mathcal{T}^{(1)}$ *of height* 1, i.e. the tree $\mathcal{T}_{\mathrm{e}^{(1)}}$ coded by an Brownian excursion $\mathrm{e}^{(1)}$ of height 1.

We can endow $\mathcal{T}^{(1)}$ with the deterministic decoration which assigns to each vertex $v \in \mathcal{T}^{(1)}$ the *height* of the fringe-subtree $\mathcal{T}_v^{(1)}$ rooted at v. See Figure 1.4 for an illustration. It follows

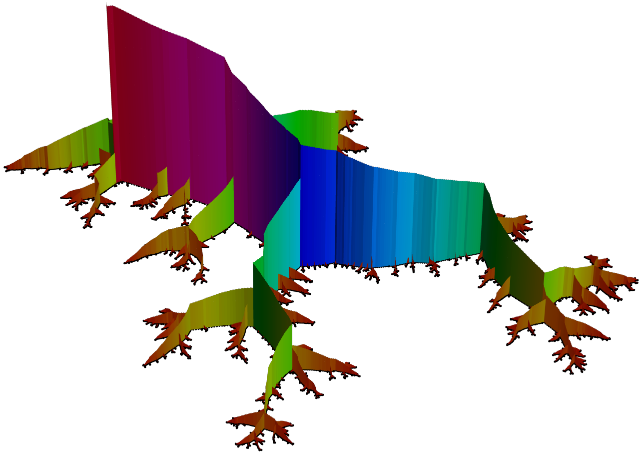

Figure 1.4: A simulation of a Brownian CRT normalized by the height. The tree is embedded non-isometrically in $\mathbb{R}^2$; the decoration function represents the height of fringe subtrees and is depicted in the vertical coordinate.

then by a classical decomposition of $\mathrm{e}^{(1)}$ due to David Williams, that the decorated tree $\mathcal{T}^{(1)}$ is indeed a self-similar Markov tree with index $\alpha = 1$, no Brownian part, constant erosion and generalized Lévy measure given by

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \dots)\big) \boldsymbol{\Lambda}_{\mathrm{Height}}(\mathrm{d}y, \mathrm{d}\mathbf{y}) = 2 \int_0^1 F(1, (x, 0, 0, \dots)) \frac{\mathrm{d}x}{x^2},$$

where F denotes a generic nonnegative functional on $\mathcal{S}$. The cumulant function is easy to compute and equals $\kappa_{\mathrm{Height}}(\gamma) = -\gamma + 2/(\gamma - 1)$ for $\gamma > 1$. In particular, Cramér assumption holds with $\omega_- = 2$. As the reader may expect, the harmonic measure μ then coincides with (a multiple of) the contour measure $\gamma_{\mathrm{e}^{(1)}}$ on $\mathcal{T}^{(1)}$, and in particular its total mass is now random. See Example 4.5. These are not the only representations of – variations of – the Brownian CRT that can be obtained using ssMt. In particular, in Example 4.12, we present another representation based on the recent work of Aidekon and Da Silva [6], which connects growth-fragmentations to planar Brownian excursions. In this example, the Brownian CRT is, roughly speaking, seen as a ssMt where the decoration processes along branches are variants of the symmetric Cauchy process, see Figure 1.5. We refer to Example 4.12 for details.

Luckily, there is more to life than variations on the Brownian CRT. The primary incentive for developing the general theory of ssMt is its connection with random planar geometry. Motivated

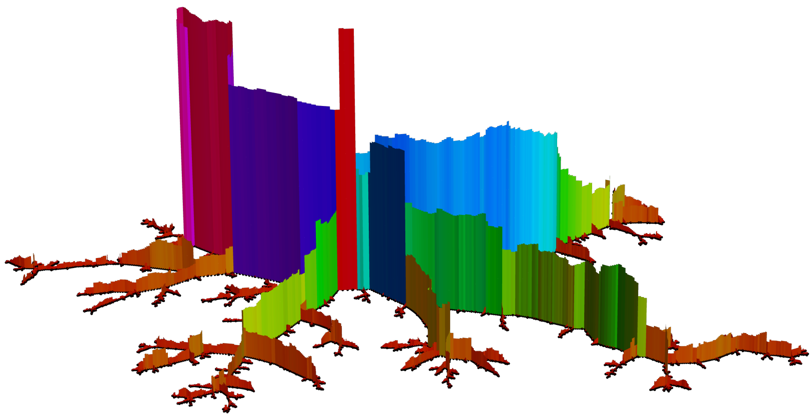

Figure 1.5: The decorated random tree $\mathcal{T}_e$ associated with a half-planar Brownian excursion. The tree is embedded non-isometrically in $\mathbb{R}^2$; the decoration function represents the horizontal X displacement in fringe subtrees and is depicted in the vertical coordinate.

in part by 2-dimensional quantum gravity, the last few decades have witness spectacular developments in this field, notably around the Brownian sphere. This is a random compact metric space almost surely homeomorphic to the 2-sphere [103] but with fractal dimension 4 [98]. It has first been constructed as the scaling limit of random planar quadrangulations by Le Gall [99] and Miermont [114] and since then appeared as a universal scaling limit model for many planar graphs models see e.g. [99, 54, 31, 7]. The Brownian sphere has also been shown to be the random metric induced by exponentiating a planar Gaussian Free Field (GFF) to the proper power, see the works of Miller & Sheffield [116, 117] yielding to a so-called Liouville Quantum Gravity metric [61, 75]. The Brownian sphere has a cousin, the Brownian disk [32], which, as the name suggests, has the topology of a disk and is better suited to make the connection with ssMt. Informally, the Brownian disk is a random compact metric space (S, d) homeomorphic to the closed unit disk of the complex plane, and can be obtained as the scaling limit of generic random planar maps with a large boundary [32]. In particular, we can define its boundary ∂S as the set of all points that have no neighborhood homeomorphic to the open unit disk. We can then consider, for every $r \geqslant 0$, the ball $B_r = \{x \in S : d(x, \partial S) \leqslant r\}$. The topological boundary ∂B_r of these balls generically has infinitely many connected components $(C_r^i : i \geqslant 1)$, which all have the topology of a circle and have fractal dimension 2. However it is possible to associate a natural notion of "boundary length" $(|C_r^i| : i \geqslant 1)$ to them extending the classical notion of perimeter, see [104]. Furthermore, as r grows, the connected components $(C_r^i : i \geqslant 1)$ describe a tree structure , see Figure 1.6 for an illustration and Example 4.9 for details. This tree is called the **Brownian cactus** and was first studied in [55]. We can then decorate each point of the Brownian cactus with the associated boundary length.

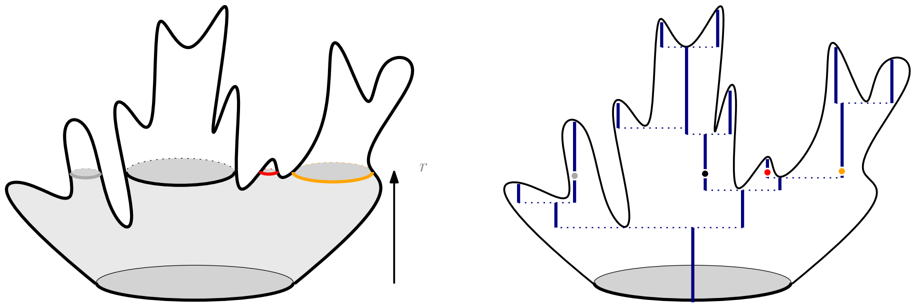

Figure 1.6: Illustration of the Cactus of a surface S with a boundary ∂S. The ball of radius r (measured from ∂S) is depicted in light gray and it has several boundary components. When each of these components has a "size", this enables us to decorated the cactus tree (on the right).

It follows from the recent work [104] (see e.g. [23, 22]) that the resulting decorated tree is[1] a (multiple of the) self-similar Markov tree called **Brownian growth-fragmentation tree**. This is the ssMt with exponent $\frac{1}{2}$, with no Brownian part and generalized Lévy measure given by

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \ldots)\big)\ \mathbf{\Lambda}_{\mathrm{BroGF}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \quad := \quad \frac{3}{4\sqrt{\pi}} \int_{1/2}^{1} F(x, 1-x, 0, 0, \ldots) \frac{\mathrm{d}x}{(x(1-x))^{5/2}}.$$

Notice that the Lévy measure $\Lambda_{\mathrm{BroGF},0}(\mathrm{d}y)$ does not integrate y in the vicinity of 0 and so compensation is involved in the definition. In particular, the drift term is non-trivial (and explicit) and the cumulant function is equal to $\kappa_{\mathrm{BroGF}}(\gamma) = \frac{\Gamma(\gamma-3/2)}{\Gamma(\gamma-3)}$, for $\gamma > 3/2$. So that $\omega_- = 2$ and the first Cramér hypothesis holds. We refer again to Example 4.9. The same ssMt but with self-similarity exponent $\alpha = 3/2$ also appears as the scaling limits of many labeled trees arising in the combinatorial literature associated with polynomial equations with one catalytic variable [40], we refer to the forthcoming Part II for examples. We also refer to Chapter 4 for other examples of ssMt related to α-stable processes for $\alpha \in (1, 3/2]$ and which appear in random planar geometry as well.

III. Invariance principles for multi-type Galton–Watson processes

The second main objective of our work is to develop robust invariance principles under which multi-type Galton–Watson trees converge towards a self-similar Markov tree. These are developed in Part II of this work which shall be available soon. More formally, suppose that we have a collection of particles which evolve as a multi-type Galton–Watson process with types in $\mathbb{N} = \{1, 2, \ldots\}$. We denote the law of the process, starting from a single particle of type $j \geqslant 1$,

[1] In the case when the base surface is a free Brownian disk, see [32]

Figure 1.7: A simulation of the Brownian growth-fragmentation tree. The self-similar Markov tree is binary and conservative: at each splitting event, the total mass is conserved and split between two children.

by $\mathbb{P}_j^{\mathrm{GW}}$ and suppose aperiodicity for simplicity. Here also we stress that the type is intimately tighten to the branching mechanism and should not be thought as a superposed spatial displacement as for many superprocesses. It is easy to interpret such a branching system as a random decorated tree $(T_{\mathrm{GW}}, d_{T_{\mathrm{GW}}}, \rho, g)$, see Figure 1.8 below.

In this direction, it will be convenient to systematically distinguish one child particle in each non-empty progeny, for instance the child with the largest type, and then gather the remaining children as a non-increasing sequence. Then, a reproduction event can be represented in the form $j \mapsto (k_0, (k_1, \dots, k_\ell))$, meaning that a particle of type j gives birth to $\ell + 1$ particles with types $k_0 \geqslant k_1 \geqslant \dots \geqslant k_\ell$, and the first one with type k_0 has been distinguished. The reproduction law of the Galton–Watson process induces a family of (sub)probability measures $(\pi_j)_{j \geqslant 1}$, where

$$\pi_j\left((k_0, (k_1, \dots, k_\ell)\right)$$

is the probability of the reproduction event $j \mapsto (k_0, (k_1, \dots, k_\ell))$ for any given non-increasing finite sequence $k_0 \geqslant k_1 \geqslant \dots \geqslant k_\ell$ in $\mathbb{N}$. We now present the conditions on the reproduction kernel $(\pi_j)_{j \geqslant 1}$ ensuring the convergence of the rescaled discrete decorated trees towards a $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$-ssMt. Recall that $\mathbb{P}_1$ is then the law of $\mathtt{T} = (T, d_T, \rho, g)$ when the decoration at the root is 1. The most obvious necessary condition is the vague convergence of the renormalized kernel towards the generalized Lévy measure $\mathbf{\Lambda}$, namely

$$\begin{aligned}&\lim_{n\to\infty} n^\alpha \cdot \sum_{k_0 \geqslant \dots \geqslant k_\ell} \pi_n\big(k_0, (k_1, \dots, k_\ell)\big) f\left(\log \frac{k_0}{n}, \left(\log \frac{k_1}{n}, \dots, \log \frac{k_\ell}{n}\right)\right)\\ &= \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) f(y_0, \mathbf{y}), \qquad (\heartsuit)\end{aligned}$$

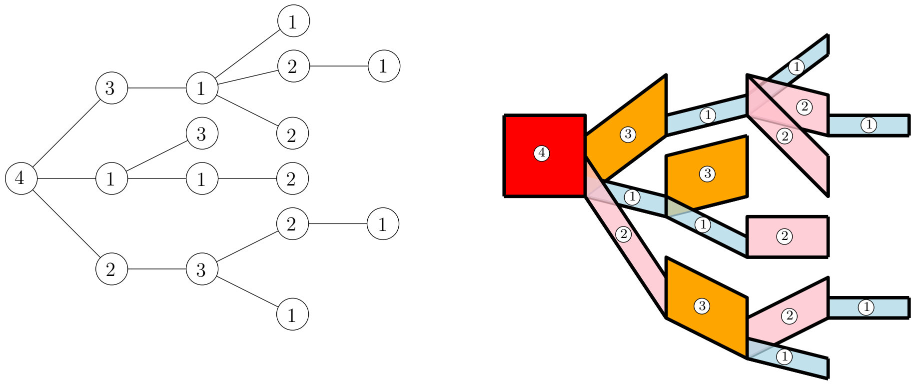

Figure 1.8: A representation of a Galton–Watson process with integer types as a random decorated tree, see the forthcoming Part II for more details.

for continuous functionals $f : \mathcal{S} \to \mathbb{R}_+$ with compact support avoiding $(0, (-\infty, \dots))$. In particular, the above assumption gives a clear meaning to the generalized Lévy measure: typically a particle of large type n gives birth to an essentially single particle of type close to n, but with probability of order $n^{-\alpha} \cdot \mathbf{\Lambda}(\mathrm{d}\mathbf{s})$ it gives rise to several particles $(ne^{y_0}, ne^{y_1}, \dots)$ of type comparable to n. We further need a control on the drift and variance of the first coordinate, see Part II for the proper definition and denote those assumptions by ($\spadesuit$) in this introduction. Extending a result of Bertoin & Kortchemski [24], the previous two conditions implies the convergence of rescaled decoration-reproduction processes over long branches of T_{GW} towards the Markov decoration-reproduction process X associated with the characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ defined in Section 3.2. However, these two conditions are only asymptotic in the particle-type and cannot account alone for the convergence of rescaled trees and does not even guarantee that the tree T_{GW} is finite under $\mathbb{P}^{\mathrm{GW}}_j$. The latter is ensured by requiring the existence of a super-harmonic function for the multi-type Galton–Watson process and the integrability condition

$$\limsup_{n\to\infty} n^{\alpha} \cdot \sum_{(k_1,\dots,k_\ell)} \pi_n(k_1, \dots, k_\ell) \left(\sum_{i=1}^{\ell} \left(\frac{k_i}{n}\right)^q - 1 \right) < 0, \tag{$\diamondsuit$}$$

in some open interval of $(0, \infty)$ for the parameter q. The latter assumption is the discrete counterpart to the sub-criticality assumption $\min \kappa < 0$ in the continuous. We prove that the previous set of three assumptions is sufficient to imply the convergence of the rescaled decorated tree in the Gromov–Hausdorff decorated sense, see Part II for the proper statement.

Informal Result (Scaling limits without mass measure)**.** *Suppose* ($\heartsuit$)*,*($\spadesuit$) *and* ($\diamondsuit$)*. Then as* $n \to \infty$*, we have the following convergence for the decorated Gromov–Hausdorff topology*

$$\left(T_{\mathrm{GW}}, \frac{d_{T_{\mathrm{GW}}}}{n^{\alpha}}, \rho, \frac{g}{n} \right) \quad \textit{under } \mathbb{P}^{\mathrm{GW}}_n \quad \xrightarrow[n\to\infty]{(d)} \quad (T, d_T, \rho, g) \quad \textit{under } \mathbb{P}_1.$$

The topology for this convergence is that developed in Chapter 2; it is adapted from the classical Gromov–Hausdorff topology to involve the decoration as well. Let us now consider more delicate versions of this result that incorporate measures. We fix a non-zero **weight function** $\varpi : \mathbb{N} \to \mathbb{R}_+$ regular varying with exponent $\gamma - \alpha$ and write $\upmu^\varpi$ for the measure on T_{GW} that assigns a mass $\varpi(k)$ to each particle with type k. We let $l^\varpi_{\mathrm{GW}}(n)$ denote the expected mass of a multi-type tree starting from a single particle of type n, i.e.

$$l^\varpi_{\mathrm{GW}}(n) := \mathbb{E}^{\mathrm{GW}}_n\big(\upmu^\varpi(T^{\mathrm{GW}})\big).$$

We naturally aim for a scaling limit result of the measure decorated tree $\left(\frac{T_{\mathrm{GW}}}{n^\alpha}, d_{T_{\mathrm{GW}}}, \rho, \frac{g}{n}, \frac{\mu^\varpi}{l^\varpi_{\mathrm{GW}}(n)}\right)$. As in the continuous case, there is a big difference between the case $\gamma > \omega_-$ and $\gamma \leqslant \omega_-$. More precisely, if $\gamma > \omega_-$, the measure μ^ϖ is essentially carried by the branches of T_{GW} and converge after renormalization towards the weighted length measure λ^γ (jointly with the converge in the above result). However, in the case when $\gamma \leqslant \omega_-$ the measure μ^ϖ is now carried by the "leaves" of T_{GW} and converges towards the harmonic measure $\upmu$. Actually, proving this convergence requires the discrete counterpart of the first Cramér hypothesis, that we denote by (♣), see the forthcoming Part II for details. Our discrete invariance principles recover those of Haas–Miermont [79] in the fragmentation case. This was actually one of the main source of inspiration for our convergence results. In particular, the requirement (♡) is the analog of the fundamental hypothesis (H) of Haas–Miermont [79]. But the most interesting, by far, applications of our invariance principles concern the discrete multi-type Galton–Watson trees converging to ssMt where the decoration can exhibit growth. An important example in this direction is given by the peeling trees underlying Boltzmann stable maps already considered in [22, Section 6] (see [23] in the triangulation case). Let us describe that setting more precisely. Given a non-zero sequence $\mathbf{q} := (q_k)_{k\geqslant 1}$ of non-negative numbers we define a measure w on the set of all bipartite planar maps $\mathbf{m}$ (finite graph embedded in the sphere up to homeomorphisms, given with a distinguished oriented edge) by the formula

$$w(\mathbf{m}) := \prod_{f \in \mathrm{Faces}(\mathbf{m})} q_{\deg(f)/2}.$$

We shall suppose that the weight sequence is admissible (the above measure is finite) and critical (roughly speaking, the weights cannot be increased while staying admissible), see [50, Chapter 5]. We shall furthermore suppose that $\mathbf{q}$ is non-generic, in the sense that it satisfies

$$q_k \sim c\,\kappa^{k-1}\,k^{-\beta} \qquad \text{as } k \to \infty, \qquad \text{for } \beta \in \left(\frac{3}{2}, \frac{5}{2}\right),$$

for some $c, \kappa > 0$.[2] For this choice of $\mathbf{q}$, a random $\mathbf{q}$-Boltzmann $\mathbf{M}_n$ conditioned on having n faces (provided that the conditioning make sense) possesses "large faces" and converge in the scaling limit

$$(\mathbf{M}_n, n^{-\frac{1}{2\beta-1}} \cdot d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{(d)} \mathcal{S}_\beta,$$

[2] Actually, we will establish the result for a slightly more general form of non-generic maps.

in law for the Gromov–Hausdorff distance, where $\mathcal{S}_\beta$ is the β-stable carpet/gasket introduced implicitly by Le Gall and Miermont in [102] and whose uniqueness has been very recently proved in [56]. In particular, the limiting case $\beta = 2$ corresponds to the Brownian sphere, see [112]. On the other hand, the dual maps $\mathbf{M}_n^\dagger$ (with large degree vertices) are much less understood[3]. Although the typical distances in $\mathbf{M}_n^\dagger$ are known to be of order $n^{\frac{\beta-2}{\beta-1/2}}$ when $\beta > 2$, see [42], it is not known whether the metrics spaces $(\mathbf{M}_n^\dagger, n^{-\frac{\beta-2}{\beta-1/2}} \cdot d_{\mathrm{gr}})$ are tight. One important tool in the theory of random planar maps is the so-called **peeling process**, which is a Markovian exploration procedure that discovers a map step-by-step, see [50]. This exploration actually encodes a planar map into a binary labeled plane tree, which under the **q**-Boltzmann measure is a multi-type Galton–Watson tree, see the forthcoming Part II for details. In the critical non-generic case above, we prove that those trees satisfy our standing assumptions ($\heartsuit, \spadesuit, \diamondsuit, \clubsuit$) with $\alpha = (\beta - 1)$ for an explicit subcritical characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. As an immediate corollary of this convergence, we deduce that the diameter of $(\mathbf{M}_n^\dagger, n^{-\frac{\beta-2}{\beta-1/2}} \cdot d_{\mathrm{gr}})$ is tight. The corresponding ssMt should play the role of the cactus tree in the potential scaling limits of $\mathbf{M}_n^\dagger$ and this convergence thus lays the foundations for the definition of their scaling limits when $\beta > 2$.

Relation to previous works

As we said above, the main source of inspiration for this monography is the work of Haas–Miermont [78, 79], where they constructed the self-similar fragmentation trees and established invariance principles, see [77] for a beautiful set of lecture notes. To be more specific, the fragmentation trees of Haas–Miermont [78], later generalized by Stephenson [134] correspond to the family of self-similar Markov trees where the decoration g is decreasing along branches. A powerful invariance principle for discrete fragmentation was established in [79] for the Gromov-Hausdorff topology (using quite different tools as ours) and proved to be very useful in the study of random trees [27, 129].

The definition of growth-fragmentation processes was given by Bertoin [20, 21] first in the binary conservative case. The properties of growth-fragmentation processes were then studied in much details, see e.g. [131, 136, 57, 29, 28] (the list is by no mean exhaustive). It was soon realized that these processes encode the genealogy of real trees, and they were constructed by Rembart–Winkel using the “string of beads” construction [124], which also inspired the construction of Chapter 3. In the self-similar case, those (decorated) trees are indeed examples of ssMt. Growth-fragmentation processes were shown to appear in peeling exploration of random planar maps [23, 22], directly within Brownian geometry [104] and in random plane Brownian excursions [6]. We revisit these results in light of the ssMt theory. Let us in particular point to the paper of Dadoun [58] which builds upon the techniques of [23] to establish an invariance principles in the Gromov–Hausdorff topology for discrete (binary, conservative) self-similar Markov trees,

[3]For readers familiar with the theory of random maps, this is due to the lack of suitable bijective encodings à la Schaeffer.

appropriately truncated. As stated above, much of these works focus on the binary conservative case until Bertoin–Mallein layed the foundations of general branching Lévy processes [25, 26] which are closely related to our construction of self-similar Markov trees.

An important caveat, is that in many of the previous works the random mass measures and the decorations on random continuous and discrete trees was neglected (it is "trivial" in the fragmentation case of Haas–Miermont). Establishing invariance principle for measured decorated random trees pushed us to introduce an appropriate topology, find the appropriate assumptions and develop new and wider proofs ideas. Most of our approach relies on construction of trees via "stick-breaking" construction, i.e. by the recursive gluing of (decorated) branches. This is old technique pioneered by Aldous [8] and which was revived recently [51, 124, 37] as an alternative to the "contour function approach" [66].

Acknowledgments. We thank Timothy Budd, Igor Kortchemski, Thomas Lehéricy, and Alejandro Rosales-Ortiz for valuable discussions at various stages of this project. We are also grateful to Elie Aidekon, Yueyun Hu, Bastien Mallein, and Zhan Shi for insightful discussions on branching random walks, particularly in the critical case. The second author is supported by SuPerGRandMa, the ERC CoG 101087572. Finally, we acknowledge the events *Décima Escuela de Probabilidad y Procesos Estocásticos de México: 2022* and *LMS Research School on Probability at the University of Liverpool: 2023*, where preliminary versions of this work were presented.

Index of important notations

To state our results, we use the standard Landau notation O, o and $\sim$ to compare the asymptotic behavior of sequences at infinity. To help the reader navigate through these pages, we make a list of some of the most important notation that are used across several sections.

Chapter 2

$\mathtt{T} = (T, d_T, \rho, g)$	a decorated tree where (T, d_T, ρ) is a real tree with root ρ and $g : T \to \mathbb{R}_+$ is the usc decoration.
$[[x, y]]$	segment between $x, y \in T$
T_x	fringe subtree above the point $x \in T$
∂T	leaves (points of degree 1) of T
$(f_u)_{u\in\mathbb{U}}, (t_u)_{u\in\mathbb{U}^*}$	building blocks serving for the construction of $\mathtt{T}$ in Theorem 2.5 $f_u : [0, z_u] \to \mathbb{R}_+$ given decoration along $[\rho(u, t) : 0 \leqslant t \leqslant z_u]$ identifications at $\rho(u, t_{ui}) = \rho(ui, 0) = \rho(ui)$
$(m_u)_{u\in\mathbb{U}}$	μ-mass of the subtree above $\rho(u)$ in Proposition 2.10
T^n	tree obtained by gluing the first n generations in Theorem 2.5
λ_T	Lebesgue measure on the skeleton of T
$\mathbf{T}$	a measured decorated tree, also denoted (T, d_T, ρ, g, ν)..
$\mathbb{T}_m$	space of equivalence classes of measured decorated compact metric spaces
$\mathbb{T}$	space of equivalence classes of (non-measured) decorated compact metric spaces
$\mathbb{T}^\bullet, \mathbb{T}^{\bullet I}$	space equivalence classes of pointed decorated compact metric spaces

Chapter 3

$(f_u, \eta_u)_{u\in\mathbb{U}}$	decoration-reproduction processes of the individuals
$(P_x)_{x>0}$	decoration-reproduction kernels
$(\chi(u))_{u\in\mathbb{U}}$	types of the individuals
$\mathbb{P}_x$	law of $(f_u, \eta_u)_{u\in\mathbb{U}}$ when $\chi(\varnothing) = x$
$(\mathcal{P})$	one can apply Theorem 2.5 to the building blocks $(f_u)_{u\in\mathbb{U}}, (t_u)_{u\in\mathbb{U}^*}$ constructed from $(f_u, \eta_u)_{u\in\mathbb{U}}$ in Section 3.1
$\mathcal{S}_1$	space of non-increasing sequences in $[-\infty, \infty)$
$\mathcal{S}$	space of elements $(y, \mathbf{y}) \in [-\infty, \infty) \times \mathcal{S}_1$
$\mathbf{\Lambda}$	generalized Lévy measure on $\mathcal{S}$
Λ_0	first marginal of $\mathbf{\Lambda}$ which is required to be a Lévy measure
$\mathbf{\Lambda}_1$	second marginal of $\mathbf{\Lambda}$
k	killing rate, i.e. $= \mathbf{\Lambda}(\{-\infty\} \times \mathcal{S}_1)$
$(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$	characteristic quadruplet
ψ	Lévy-Khintchine exponent associated to $(\sigma^2, \mathrm{a}, \Lambda_0)$ via (3.11)

$(\xi(t))_{t\geqslant 0}$	Lévy process with Lévy–Khintchine exponent ψ
$(X(t))_{t\geqslant 0}$	pssMp associated with ξ via Lamperti transformation with index $\alpha > 0$
κ	cumulant function defined via (3.19)
λ^γ	weighted length measure $g^{\gamma-\alpha}\lambda_T$ defined in Proposition 3.11
μ	harmonic measure defined in Lemma 3.13 under Assumption 3.12
(ω_-, ω_+)	intervalle over which κ is negative

Chapter 4

$\mathrm{a}_{\mathrm{can}} = \mathrm{a} - \int \Lambda_0(\mathrm{d}y) y \mathbf{1}_{\lvert y\rvert\leqslant 1}$	canonical drift, well-defined when $\Lambda_0(\mathrm{d}y)$ integrates $1 \wedge \lvert y\rvert$
finite branching activity	$\boldsymbol{\Lambda}_1\left(\mathcal{S}_1 \backslash \{(-\infty, -\infty, \ldots)\}\right) < \infty$
non-increasing	$\mathrm{a}_{\mathrm{can}} \leqslant 0$ and $\Lambda(\{(y_0, (y_i)_{i\geqslant 1}) : \exists y_j > 0\}) = 0$
conservative	$\boldsymbol{\Lambda}\left(\left\{(y_0, (y_1, \ldots)) : \sum_{j=0}^{\infty} \mathrm{e}^{y_j} \neq 1\right\}\right) = 0$
fragmentation	non-increasing conservative, $\mathrm{a}_{\mathrm{can}} = 0$ and $\mathrm{k} = 0$
binary	$\boldsymbol{\Lambda}\left(\{(y_0, (y_1, \ldots)) : y_3 \neq -\infty\}\right) = 0$
growth-fragmentation	conservative, binary and $\mathrm{k} = 0$

Chapter 5

$\mathbb{Q}_x$	laws of the equivalence class of $\mathtt{T} \in \mathbb{T}$ under $\mathbb{P}_x$
$(\tau)_{i\in I}$	subtrees dangling from a base subtree
$T^{[\varepsilon]}$	tree obtained by keeping only individuals for which $\chi(u) = f_u(0) \geqslant \varepsilon$
$B_a(T)$	ball of radius a centered at $\rho \in T$

Chapter 6

$(\sigma^2, \mathrm{a}_\gamma, \boldsymbol{\Lambda}_\gamma; \alpha)$	γ-tilted characteristic quadruplet defined in Lemma 6.4
(X_γ, η_γ)	decoration-reproduction process along the tagged branch
$\psi_\gamma(z) = \kappa(\gamma + z)$	Lévy–Khintichine exponent of X_γ
$\mathtt{T}^\bullet = (\mathtt{T}, \rho^\bullet)$	a pointed decorated tree
$(\widetilde{\mathbb{Q}}_x^\gamma)_{x>0}$	laws of pointed decorated tree defined in (6.9)
ord	the map $(y, (y_i)_{i\geqslant 1}) \in \mathcal{S} \mapsto (y_j : j \geqslant 0)^\downarrow \in \mathcal{S}_1$
$(\sigma^2, \mathrm{a}_*, \boldsymbol{\Lambda}_*; \alpha)$	the locally largest bifurcator

Part I

Self-similar Markov trees

Chapter 2

Decorated real trees and their topologies

Random decorated real trees are the central objects in this work. A decorated real tree consists of a compact real tree T equipped with a root ρ and nonnegative function g on T, which we refer to as a decoration and is typically discontinuous at branching points. Often, decorated trees will further be measured, i.e. endowed with a finite Borel measure.

After recalling some basic background on real trees, we present in the first section a natural gluing operation, using marks on the base tree to specify locations where the other trees are glued. This enables us in the second section to present a recursive construction of decorated real trees from so-called building blocks, indexed by the Ulam tree, by gluing iteratively line segments on which some functions have been defined (Theorem 2.5).

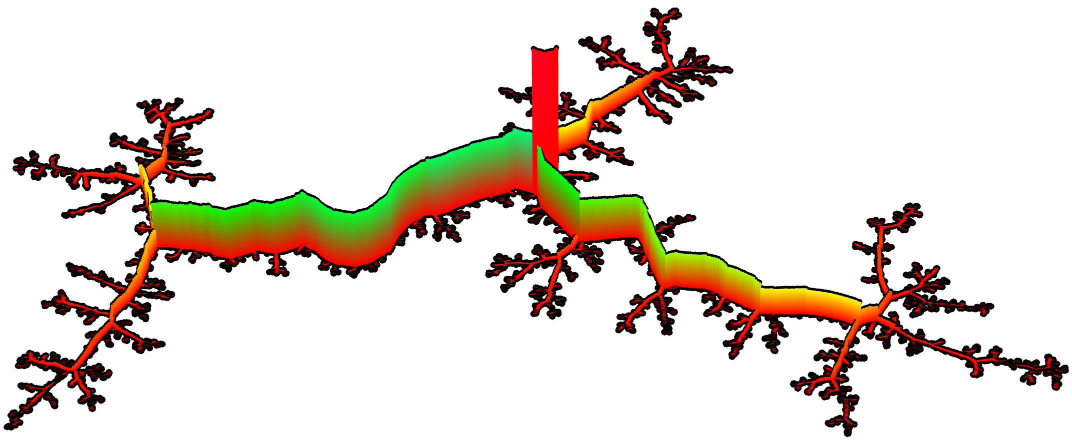

Figure 2.1: Hypograph of a tree (embedded in the plane) decorated by an usc function in the third (vertical) dimension.

We will be interested in the convergence of sequences of such decorated real trees, which incites us to define a notion of closeness for two decorations on two different domains. One of the purposes of this chapter is thus to introduce a convenient formalism and, to stay on safe ground as soon as randomness and measurability will be involved, we represent these as elements of some Polish space. The topological framework that is needed to define a notion of convergence for sequences of decorated real trees is developed in the third section. In this direction, we shall always impose that the decoration g is upper semi-continuous, i.e., the superlevel sets $\{x : g(x) \geqslant r\}$ are closed for every $r \geqslant 0$. The main idea there is to represent usc functions by their hypographs; see Figure 2.1. In the situation where the domains can be viewed as subsets of a same metric space, the Hausdorff metric yields a natural distance in terms of hypographs. We then adapt an idea of Gromov and consider isometric embeddings to measure the distance between the structures induced by two usc decorations with unrelated domains (Theorem 2.15).

2.1 Decorated real trees and a gluing operator

We start by recalling a few basic features on real compact trees. The reader is referred to the lecture notes by Evans [69], by Le Gall [96], or to the manuscript by Duquesne [64] for detailed accounts and further properties.

A **real tree** is a metric space (T, d_T) such that for any $x, y \in T$, there exists a unique isometry

$$\phi_{x,y} : [0, d_T(x, y)] \hookrightarrow T \quad \text{ with } \phi_{x,y}(0) = x \text{ and } \phi_{x,y}\big(d_T(x, y)\big) = y;$$

we shall refer to $\phi_{x,y}$ as the **path** from x to y in T. Furthermore, the image of any continuous injective map $\psi : [0, 1] \to T$ with $\psi(0) = x$ and $\psi(1) = y$ coincides with $\phi_{x,y}([0, d_T(x, y)])$. The image of $\phi_{x,y}$ is called the **segment** from x to y in T and is denoted by $[\![x, y]\!]$ in the sequel.

A useful characterization is that a connected metric space (T, d_T) is a real tree if and only if the so-called four point condition holds, that is:

$$d_T(x_1, x_2) + d_T(x_3, x_4) \leqslant (d_T(x_1, x_3) + d_T(x_2, x_4)) \vee (d_T(x_1, x_4) + d_T(x_2, x_3)), \tag{2.1}$$

for every x_1, x_2, x_3, x_4 in T. The real trees appearing in this work are typically **compact** and **rooted** at some distinguished point $\rho \in T$, even though for the sake of simplicity we shall often omit the distance d_T and the root ρ from the notation. The distance $d_T(\rho, x)$ of a point $x \in T$ to the root is called the **height** of x, and then the height of T, denoted by $\mathrm{Height}(T)$, is the maximal height of points in T. The root enables us to endow T with a partial order: for any two points $x, y \in T$ we write $x \preceq y$ and we say that x is an **ancestor** of y, or that y is a **descendant** of x, if x belongs to the segment $[\![\rho, y]\!]$. The **fringe subtree** $T_x = \{y \in T : x \preceq y\}$ induced by a point $x \in T$ is the subset of T consisting of all the descendants of x (including x itself). This fringe subtree is naturally equipped with the metric induced by the restriction of d_T, and rooted at x. We also use the following standard nomenclature and notation for points in a real tree. The **degree** of a point $x \in T$ is the number (possibly infinite) of connected components of $T\backslash\{x\}$, and then:

- A point $x \in T$ is a **leaf** if it has degree 1. We denote the set of leaves of T by ∂T.

- The **skeleton** of T is the subset $T \backslash \partial T$.
- A point $x \in T$ is a **branching point** if it has degree at least 3.

We will further equip real trees with some **upper semi-continuous** (usc in the sequel) functions[1].

Definition 2.1. *A* ***decorated real tree*** *is a quadruplet*

$$\mathtt{T} = (T, d_T, \rho, g)$$

where (T, d_T, ρ) is a rooted compact real tree and $g : T \to \mathbb{R}_+$ an usc function referred to as the ***decoration****.*

Let us give a couple of simple examples. Consider a rooted compact real tree (T, d_T, ρ). The function $g : T \to \mathbb{R}_+$ that assigns to every $x \in T$ its height, that is $g(x) = d_T(\rho, x)$, is continuous and can thus be taken as a decoration. If we further equip (T, d_T, ρ) with a finite measure ν, then the function $g' : T \to \mathbb{R}_+$ that assigns to every $x \in T$ the ν-mass of the fringe subtree induced by this vertex, $g'(x) = \nu(T_x)$, is clearly usc, and can also be used as a decoration of T.

In the sequel, it will sometimes be convenient to consider decorated compact trees not just with a single distinguished point (the root), but more generally endowed with a further finite or countable family of distinguished points in T, referred to as **marks**. Specifically, marks consist of a family $(r_i)_{i \in I}$ of points in T, where I is a finite or countable set of indices. We stress that we do not request $r_i \neq r_j$ for $i \neq j$, and the same mark may arise for different indices. In particular, the notion of marks allow us to introduce the **gluing operation** which will lie at the heart of the construction of decorated compact real trees from building blocks in the next section.

Let us clarify what we mean by gluing decorated trees. In this direction, consider a rooted compact real tree $(T', d_{T'}, \rho')$ with marks $(r_i)_{i \in I}$. The tree T' serves as a base, marks specify the locations on T' where gluing will be performed. Let further $(T_i, d_{T_i}, \rho_i)_{i \in I}$ be a family of rooted compact real trees which are pairwise disjoint, and also disjoint from T'. Roughly speaking, we will glue each T_i on T' by identifying the mark r_i and the root ρ_i of T_i, and equip the resulting space with the distance induced by $d_{T'}$ and the d_{T_i} on each component. See Figure 2.2 for an illustration.

In order to describe rigorously this operation, we first introduce the (disjoint) union of those trees

$$T^{\sqcup} := T' \sqcup \left(\bigsqcup_{i \in I} T_i \right).$$

We next define $d^\circ : T^{\sqcup} \times T^{\sqcup} \to \mathbb{R}_+$ by

$$d^\circ(x, y) := \begin{cases} d_{T'}(x, y) & \text{if } x, y \in T', \\ d_{T_i}(x, y) & \text{if } x, y \in T_i \text{ for some } i \in I, \\ d_{T'}(x, r_i) + d_{T_i}(\rho_i, y) & \text{if } x \in T' \text{ and } y \in T_i \text{ for some } i \in I, \\ d_{T_i}(x, \rho_i) + d_{T'}(r_i, y) & \text{if } x \in T_i \text{ for some } i \in I \text{ and } y \in T', \\ d_{T_i}(x, \rho_i) + d_{T'}(r_i, r_j) + d_{T_j}(\rho_j, y) & \text{if } x \in T_i \text{ and } y \in T_j \text{ for some } i \neq j \in I. \end{cases}$$

[1] We enforce this requirement because usc-functions are well-behaved when gluing trees (see below), and also for their compatibility with Gromov-type topologies (see Section 2.4).

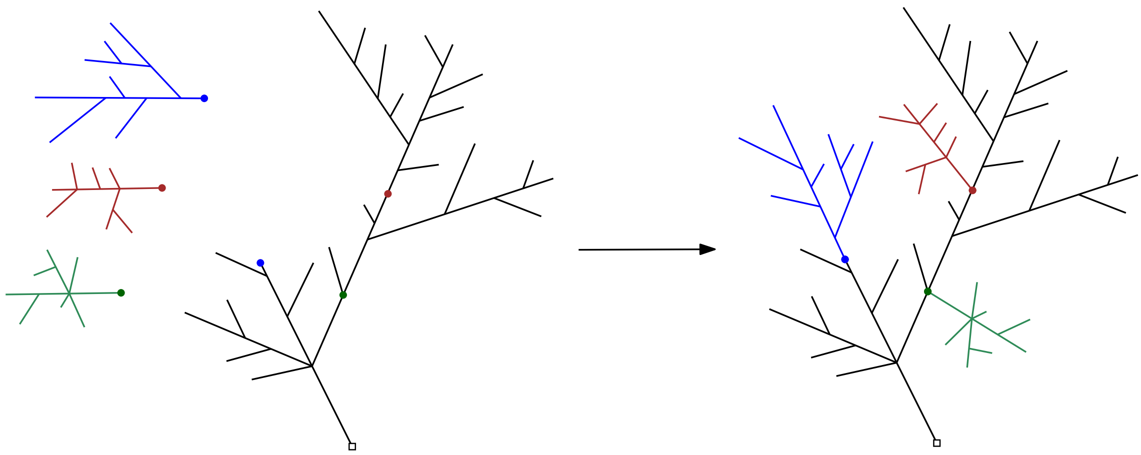

Figure 2.2: Gluing three subtrees on a base tree with three marked points.

It is immediately seen that d° is a pseudo-distance on $T^\sqcup$ such that for any distinct points x, y in $T^\sqcup$, we have $d^\circ(x,y) = 0$ if and only if, either $x = r_i$ and $y = \rho_i$ for some $i \in I$, or vice versa, or $x = \rho_i$ and $y = \rho_j$ for some $i \neq j \in I$ such that $r_i = r_j$.

We write T for the quotient space of $T^\sqcup$ for the equivalence relation

$$x \sim y \iff d^\circ(x,y) = 0,$$

and equip this quotient space with the metric d_T induced by d°, that is

$$d_T(\tilde{x}, \tilde{y}) = d^\circ(x,y), \qquad \tilde{x}, \tilde{y} \in T,$$

for any representatives $x \in \tilde{x}$ and $y \in \tilde{y}$ of these equivalence classes. In other words, (T, d_T) is obtained from $(T^\sqcup, d^\circ)$ by identifying r_i and ρ_i for every $i \in I$.

Lemma 2.2. *Suppose that*

$$(\mathrm{Height}(T_i))_{i\in I} \quad \textit{is a } \boldsymbol{\textit{null family}}, \tag{2.2}$$

meaning that for any $h > 0$, the set of indices $i \in I$ with $\mathrm{Height}(T_i) \geqslant h$ *is finite. Then (T, d_T) is a compact real tree. We further use the equivalence class of ρ' as the root ρ of T.*

Proof. We argued above that (T, d_T) is a metric space; let us now check that it is a real tree, which should be intuitively clear. We use the obvious notation $\tilde{z}$ for the equivalence class of a point z in $T^\sqcup$. Pick any x, y in $T^\sqcup$. If both points belong to the same tree before the gluing, say $x, y \in T_i$ for some $i \in I$, then $d^\circ(x,y) = d_{T_i}(x,y) = d_T(\tilde{x}, \tilde{y})$, and the path $\phi_i : [0, d_{T_i}(x,y)] \hookrightarrow T_i$ from $\phi_i(0) = x$ to $\phi_i(d_{T_i}(x,y)) = y$ in T_i yields in the obvious notation a path $\tilde{\phi}_i : [0, d_T(\tilde{x}, \tilde{y})] \hookrightarrow T$ from $\tilde{\phi}_i(0) = \tilde{x}$ to $\tilde{\phi}_i(d_T(\tilde{x}, \tilde{y})) = \tilde{y}$ in T. The case when, say $x \in T'$ and $y \in T_i$ (or *vice versa*), is treated by concatenating two paths, the first from x to r_i in T' and the second from ρ_i to y in T_i. Finally the case when $x \in T_i$ and $y \in T_j$ with $i \neq j$ involves the concatenation of three paths, the first from x to ρ_i in T_i, the second (possibly degenerate) from r_i to r_j in T', and the third from ρ_j to y in T_j.

The initial trees being pairwise disjoint, uniqueness of the path $\tilde{\phi}$ from $\tilde{x}$ to $\tilde{y}$ in T should be plain. Specifically, we may assume without loss of generality that $\tilde{x} \neq \tilde{y}$, pick an element in each equivalence class, say $x \in \tilde{x}$ and $y \in \tilde{y}$. Suppose first that both x and y can be chosen in the same tree, say for simplicity $x, y \in T'$. Write $\widetilde{T'} \subset T$ for the set of equivalence classes of points in T'. If $\tilde{\phi}$ entered $T \backslash \widetilde{T'}$, then by continuity we could find two times $0 \leqslant s < t \leqslant d_T(\tilde{x}, \tilde{y})$ such that $\tilde{\phi}(s) = \tilde{\phi}(t) = \tilde{\rho}_i$, which is absurd. Thus $\tilde{\phi}$ stays in $\widetilde{T'}$ and hence defines unambiguously a path ϕ from x to y in T'. By uniqueness of the latter, we conclude that $\tilde{\phi}$ is also unique. The case when x and y belong to different initial trees can be treated similarly. Essentially the same argument also shows that if $\psi : [0,1] \to T$ is a continuous injective map, then its image must coincides with the segment from $\psi(0)$ to $\psi(1)$ in T.

Finally, we check (sequential) compactness. Let $(\tilde{y}_n)_{n \geqslant 1}$ be a sequence in T; pick for each n an element $y_n \in \tilde{y}_n$. In the case when there exists some tree, say T_i, such that $y_n \in T_i$ for infinitely many n's, then, since T_i is compact, we can extract from $(y_n)_{n \geqslant 1}$ a subsequence which converges in T_i, say towards y, and it follows that there is a subsequence extract from $(\tilde{y}_n)_{n \geqslant 1}$ that converges towards the equivalence class $\tilde{y}$ of y. Next consider the complementary case when for all trees T_i, there are only finitely many n's with $y_n \in T_i$, and the same also holds for T'. We can then extract from $(y_n)_{n \geqslant 1}$ a subsequence such that each y_n (along this subsequence) belongs to a different tree, say $y_n \in T_{i(n)}$. The assumption (2.2) entails that $d_{T_{i(n)}}(y_n, \rho_{i(n)})$ converges to 0 as $n \to \infty$ along this subsequence. By compactness of T', we can extract a further subsequence from $(r_{i(n)})$ that converges to a point $x \in T'$. We conclude that $(\tilde{y}_n)_{n \geqslant 1}$ contains a subsequence that converges towards the equivalence class $\tilde{x}$ of x, and hence T is compact. □

We next extend the gluing operation to decorations; recall that these must be given by usc functions on trees. Specifically, we consider a decorated real tree $\mathtt{T}' = (T', d_{T'}, \rho', g')$ with marks $(r_i)_{i \in I}$ and a family of decorated real trees, $\mathtt{T}_i = (T_i, d_{T_i}, \rho_i, g_i)$ for $i \in I$, such that the T_i, $i \in I$, are pairwise disjoint and also disjoint from T'. We define a function g on the glued tree T which coincides of course with g' on $T' \backslash \{r_i : i \in I\}$, with g_i on $T_i \backslash \{\rho_i\}$, and is adjusted at the glued points in order to ensure upper semi-continuity. More precisely, define first the map $g^{\sqcup} : T^{\sqcup} \to \mathbb{R}_+$ by $g^{\sqcup}(y) = g'(y)$ if $y \in T'$ and $g^{\sqcup}(y) = g_i(y)$ if $y \in T_i$ for $i \in I$, and then

$$g(\tilde{y}) := \sup_{y \in \tilde{y}} g^{\sqcup}(y), \qquad \tilde{y} \in T.$$

Lemma 2.3. *Suppose* (2.2) *and further that*

$$\left(\max_{y \in T_i} g_i(y) \right)_{i \in I} \quad \textit{is a null family.} \tag{2.3}$$

Then $\mathtt{T} := (T, d_T, \rho, g)$ *is a decorated real tree.*

Proof. We have to verify that g is usc. Let $\tilde{y} \in T$ and $a > g(\tilde{y})$. We distinguish three possible situations.

The simplest is when $\tilde{y} = \{y_i\}$ for some $y_i \in T_i$, with $i \in I$. Then $g(\tilde{y}) = g_i(y_i)$, and since necessarily $y_i \neq \rho_i$, we can find a small neighborhood V_i of y_i in T_i such that $\rho_i \notin V_i$ and

$g_i(z) \leqslant a$ for all $z \in V_i$. In particular the equivalence class $\tilde{z}$ of any $z \in V_i$ is reduced to $\{z\}$, so $g(\tilde{z}) = g_i(z) \leqslant a$. In the obvious notation, $\widetilde{V}_i$ is a neighborhood of $\tilde{y}$ in T, and we conclude that g is usc at $\tilde{y}$. The same argument applies when $\tilde{y} = \{y\}$ for some $y \in T'$ which does not belong to the closure of $\{r_i : i \in I\}$ in T'.

Next suppose that $\tilde{y}$ is the equivalence class of a marked point in T', that is

$$\tilde{y} = \{x\} \sqcup \{\rho_j : j \in J\},$$

where x is one of the marked points in T' and $J = \{j \in I : r_j = x\} \neq \varnothing$. Let V be a neighborhood of x in T' such that $g'(z) \leqslant a$ for all $z \in V$. Similarly, for every $j \in J$, there is a neighborhood V_j of ρ_j in T_j such that $g_j(z) \leqslant a$ for all $z \in V_j$. The assumption (2.3) ensures that the set $J_a := \{j \in J : \max_{z \in T_j} g_j(z) > a\}$ is finite. We deduce that

$$G := V \sqcup \left(\bigsqcup_{j \in J_a} V_j \right) \sqcup \left(\bigsqcup_{j \in J \setminus J_a} T_j \right)$$

is a neighborhood of x in $T^{\sqcup}$ for the pseudo-distance d° and $g^{\sqcup}(z) \leqslant a$ for all $z \in G$. If we write $\widetilde{G}$ for the set of equivalence classes that G induces in T, then $\widetilde{G}$ is a neighborhood of $\tilde{y}$ and $g(\tilde{z}) \leqslant a$ for all $\tilde{z} \in \widetilde{G}$.

The last case is when $\tilde{y} = \{x\}$ for some $x \in T'$ which is not marked, but adherent in T' to the family of marked points. Recall from (2.3) that $I_a := \{i \in I : \max_{z \in T_i} g_i(z) > a\}$ is finite. Since x is not marked, we can find a neighborhood V' of x in T' that avoid the mark r_i for every $i \in I_a$, and then a possibly smaller neighborhood $V'' \subset V'$ such that $g'(z) \leqslant a$ for all $z \in V''$. The set

$$U := V'' \sqcup \left(\bigsqcup_{i \notin I_a} T_i \right)$$

is then a neighborhood of x in $T^{\sqcup}$ for the pseudo-distance d°, and we have $g^{\sqcup}(z) \leqslant a$ for all $z \in U$. If we write $\widetilde{U}$ for the set of equivalence classes $\tilde{z}$ for $z \in U$, then $\widetilde{U}$ is a neighborhood of $\tilde{y}$ in T, and $g(\tilde{z}) \leqslant a$ for all $\tilde{z} \in \widetilde{U}$.

We have thus verified that g is indeed usc on T. An appeal to Lemma 2.2 completes the proof. $\square$

When we want to record specifically all the elements involved in the gluing operator, we shall use the notation

$$\mathtt{T} = \mathrm{Gluing}\Big(\mathtt{T}', (r_i)_{i \in I}, \big(\mathtt{T}_i\big)_{i \in I}\Big).$$

It will be convenient to call **degenerate** a decorated real tree $\mathtt{T}_i = (T_i, d_{T_i}, \rho_i, g_i)$ such that $T_i = \{\rho_i\}$ is merely a singleton decorated with $g_i(\rho_i) = 0$; in that case we shall use the notation $\mathtt{T}_i \sim \dagger$. Plainly, gluing a degenerate tree $\mathtt{T}_i$ on a decorated real tree $\mathtt{T}'$ is a neutral operation without incidence on the outcome (apart from removing the associated mark), and we may as well discard degenerate elements in gluing operations. We also stress that if the decorated real trees $\mathtt{T}_i$, which are glued onto on $\mathtt{T}'$, are themselves marked, say $(r_{i,j})_{j \in J_i}$ on T_i for every $i \in I$, where we use pairwise disjoint set of indices J_i for $i \in I$, then slightly more generally the gluing process yields a decorated real tree with marks indexed by $J = \bigsqcup_{i \in I} J_i$. The latter is naturally denoted by

$$\mathrm{Gluing}\Big(\mathtt{T}', (r_i)_{i \in I}, \big(\mathtt{T}_i, (r_{i,j})_{j \in J_i}\big)_{i \in I}\Big).$$

In particular, this enables us to iterate the gluing operation, notably as it will be done in the next section.

2.2 Construction of decorated real trees by gluing building blocks

We first introduce some standard notation related to the **Ulam tree**, whose set of vertices[2] consists of finite sequence of positive integers,

$$\mathbb{U} \coloneqq \bigcup_{k \geqslant 0} \mathbb{N}^k,$$

with the convention $\mathbb{N}^0 \coloneqq \{\varnothing\}$. The vertices are sometimes referred to as individuals, since one may think of $\mathbb{U}$ as a population structured by its genealogy, where each individual has an infinite offspring. We use related vocabulary; for instance the edges of the Ulam tree connect parents to their children. We write $|u|$ for the generation of $u \in \mathbb{U}$ (the length of the sequence u), $u(j)$ for the ancestor at generation j of u (that is the prefix of u with length j) whenever $|u| \geqslant j$, and uv for the sequence of length $|u| + |v|$ resulting from appending v to u (in particular, for $i \in \mathbb{N}$, ui is viewed as the i-th child of u). We also write $\mathbb{U}^* = \mathbb{U} \backslash \{\varnothing\}$ for the set of nonempty finite sequences of positive integers and $u-$ for the parent of $u \in \mathbb{U}^*$, i.e. $u- = u(|u| - 1)$.

The Ulam tree will be used to index the building blocks appearing in the construction of decorated real trees through successive gluing. In order to unify the presentation, it is convenient to allow certain of these building blocks to be degenerate, where, as in Section 2.1, degenerate elements can always be discarded in the construction. For simplicity, we also call a vertex u **fictitious** if it indexes a degenerate block and then write $u \sim \dagger$ for a fictitious vertex. We agree that only fictitious vertices uv appear in the descent of a fictitious vertex u; as a consequence the subspace of non-fictitious vertices forms a subtree of $\mathbb{U}$.

For the sake of simplicity, we shall present first the construction of decorated real trees by recursive gluing without measures, and then later on discuss natural Borel measures on such spaces. The **building blocks** for the construction by gluing consist of a pair of families

$$(f_u)_{u \in \mathbb{U}} \quad , \quad (t_u)_{u \in \mathbb{U}^*}$$

that satisfy the following properties for any non-fictitious $u \in \mathbb{U}$:

- $f_u : [0, z_u] \to \mathbb{R}_+$ is a rcll (right-continuous with left limits) function with $z_u > 0$;
- either $t_{ui} \in [0, z_u]$ or $t_{ui} \sim \dagger$;

[2] $\mathbb{U}$ is not a *real* tree, but rather a tree in the sense of combinatorics.

- If $t_{ui} \sim \dagger$ then the vertex ui is also fictitious.

We further use the notation

$$\sup f_u := \sup_{0 \leqslant x \leqslant z_u} f_u(x) \quad \text{and} \quad \|f_u\| := z_u + \sup f_u.$$

We also agree for definitiveness that if $u \sim \dagger$ is fictitious, then f_u is degenerate (recall that this means $z_u = 0$ and $f_u(0) = 0$), $\|f_u\| = 0$, and $t_{ui} \sim \dagger$ for all $i \geqslant 1$ as well.

We always assume that

$$\big(\|f_u\| : u \in \mathbb{U}\big) \text{ is a null family.} \tag{2.4}$$

Before we formally define the gluing construction, let us provide an informal explanation of the roles of the building blocks. A (non-fictitious) interval $[0, z_u]$ will become a segment of the resulting real tree T, and the real number $t_{ui} \neq \dagger$ will be used in the gluing construction to specify the location on the parent segment where the i-th child segment labeled by ui is glued. In particular, t_{ui} will yield a branching point of the real tree whenever $0 < t_{ui} < z_u$ and $z_{ui} > 0$. When u has only fictitious children, the construction regarding the subtree rooted at u stops at that point, in the sense that no segments are glued on the segment labeled by u. Finally, the family $(f_u)_{u \in \mathbb{U}}$ will correspond to the usc-decoration. In this direction, we must address a technical issue regarding the regularity of trajectories. When dealing with most random processes in continuous time (e.g. a Feller process), one usually works with rcll versions, which is the reason why we requested the functions f_u above to be rcll. Nonetheless, we have to consider rather their ucs versions in order to conform to the framework introduced in Section 2.1. Formally, if $f : [0, z] \to \mathbb{R}_+$ is rcll function, then its usc version $\check{f}$ is defined by the relation:

$$\check{f}(t) := \max\big\{f(t-), f(t)\big\}, \quad \text{for } t \in [0, z], \tag{2.5}$$

with the convention $f(0-) = f(0)$. This only affects the values of the function at times when it has a negative jump (including possibly at lifetime).

We also agree to write simply d for the usual distance on any interval. For every non-fictitious $u \in \mathbb{U}$, we view f_u as a decorated compact interval $\big([0, z_u], d, 0, \check{f}_u\big)$, and the family of non-fictitious times t_{ui} for $i \geqslant 1$ as marks on $[0, z_u]$. The gluing operation described in the preceding section uses pairwise disjoint real trees, and we shall therefore introduce disjoint isomorphic copies of the preceding for different $u \in \mathbb{U}$. Namely, we use the Ulam tree to differentiate these segments, and, in the notation of Definition 2.1, we consider for every non-fictitious $u \in \mathbb{U}$ the decorated segment

$$\mathtt{S}_u := (S_u, d_u, \rho_u, \hat{f}_u),$$

where $S_u := \big\{(u, t) : \ t \in [0, z_u]\big\}$, $\rho_u := (u, 0)$, and the metric d_u and usc decoration $\hat{f}_u$ are defined by

$$d_u((u, s), (u, t)) := d(s, t) \quad \text{and} \quad \hat{f}_u(u, t) := \check{f}_u(t), \quad \text{for } s, t \in [0, z_u].$$

In particular, the segments S_u are pairwise disjoint sets and $\mathtt{S}_u$ is isomorphic to $\big([0, z_u], d, 0, \check{f}_u\big)$, in the sense that there is a bijective isometry $\varphi_u : [0, z_u] \to S_u$ with $\varphi_u(0) = \rho_u$, and $\hat{f}$ is the usc version of $f \circ \varphi_u^{-1}$. We stress that φ_u is actually unique, since we are dealing with oriented segments. Last but not least, we further mark each segment using the family of (non-fictitious) t_{ui} with $i \geqslant 1$. More precisely, if we let $I_u = \{uj : uj \neq \dagger\}$ for the non-fictitious offspring of the individual u (introducing this notation is needed as we want to use different set of indices for the marks arising from different blocks), then the family of marks on S_u is given by $(\hat{t}_v := (u, t_v))_{v \in I_u}$.

We can now proceed with the formal construction of a decorated real tree from building blocks, where at first, measures are discarded. We introduce first the disjoint union of segments

$$T^{\sqcup} := \bigsqcup_{u \not\sim \dagger} S_u,$$

where the union is taken over the set of non-fictitious vertices $u \in \mathbb{U}$. We use $\rho_\varnothing$ as root, and write d^0 for the natural distance on $T^{\sqcup}$ which is given by d_u on each segment S_u and such that $d^0(x, y) = \infty$ when x and y belong to different segments (i.e. x and y are in different connected components). We also define unambiguously the map $g^{\sqcup} : T^{\sqcup} \to \mathbb{R}_+$ by

$$g^{\sqcup}(x) := \hat{f}_u(x), \qquad \text{for every } x \in S_u \text{ and } u \not\sim \dagger.$$

We then construct recursively[3] a sequence $(d^n)_{n \geqslant 1}$ of pseudo-distances on $T^{\sqcup}$, using the gluing operator of Section 2.1; note that (2.4) ensures the requirements (2.2) and (2.3) in this setting. Specifically, d^1 is the pseudo-distance on $T^{\sqcup}$ obtained by identifying ρ_i and the marked point $\hat{t}_i$ for each non-fictitious vertex i at the first generation, that is $i \in I_\varnothing$. Segments at generations 2 and more are not affected, in particular $d^1(x, y) = \infty$ for any $x \in S_u$ and $y \in S_v$ with $u \neq v$ and $|u| \vee |v| \geqslant 2$. Next d^2 is the pseudo-distance on $T^{\sqcup}$ obtained by further identifying ρ_{ij} and the marked point $\hat{t}_{ij}$ for each vertex $ij \in I_i$ and each $i \in I_\varnothing$. And so on, and so forth, generation after generation.

Then consider any $x, y \in T^{\sqcup}$, say $x \in S_u$ and $y \in S_v$. Plainly, if $u = v$, then $d^n(x, y) = d^0(x, y)$ for all $n \geqslant 0$. If $u \neq v$, then $d^n(x, y) = \infty$ when $n < |u| \vee |v|$, whereas $d^n(x, y) = d^{n'}(x, y) < \infty$ for all $n, n' \geqslant |u| \vee |v|$. We can then set

$$d^\circ(x, y) := \lim_{n \to \infty} d^n(x, y),$$

and d° defines a pseudo-distance on $T^{\sqcup}$ which is now everywhere finite (this can be interpreted as connectivity). Alternatively, we could also define directly d° as the largest pseudo-distance on $T^{\sqcup}$ which coincides with d_u on each segment S_u, and such that $d^\circ(\hat{t}_v, \rho_v) = 0$ for every non-fictitious $v \in \mathbb{U}^*$.

We next define T° as the quotient space for equivalence relation

$$x \sim y \iff d^\circ(x, y) = 0, \qquad x, y \in T^{\sqcup}.$$

[3] Instead of developing a recursive construction, one could have given directly an explicit formula for d^n in terms of distances d_w for all the vertices w on the segment from u to v in $\mathbb{U}$. However the formula would be a bit cumbersome to state precisely and not quite transparent, at least for the first reading.

We write d_{T° for the distance on T° induced by d° and also define the map $g^\circ : T^\circ \to \mathbb{R}_+$ given by

$$g^\circ(\tilde{x}) := \sup_{x \in \tilde{x}} g^\sqcup(x), \qquad \tilde{x} \in T^\circ.$$

Note that g° is actually bounded, thanks to (2.4).

The following claim can be viewed as the analog of Lemma 2.3 for infinite iterations of the gluing operation, and its proof relies heavily on this lemma. Recall that a metric space is called pre-compact if, for every $\varepsilon > 0$, it can be covered by finitely many balls with radius ε.

Lemma 2.4. *Assume* (2.4) *and*

$$\lim_{k \to \infty} \sup \left\{ \sum_{n=k}^{\infty} z_{\bar{u}(n)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \right\} = 0, \tag{2.6}$$

where $\bar{u}(n) \in \mathbb{U}$ stands for the prefix at generation n of an infinite sequence $\bar{u} \in \mathbb{N}^{\mathbb{N}}$ of positive integers. Then (T°, d_{T°) is a pre-compact real tree and g° an usc function.

We briefly postpone the proof of Lemma 2.4 and mention that since plainly

$$\sup \left\{ \sum_{n=k}^{\infty} z_{\bar{u}(n)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \right\} \leqslant \sum_{n=k}^{\infty} \sup \left\{ z_u : u \in \mathbb{N}^n \right\},$$

we shall often use in the sequel the simpler but stronger requirement

$$\sum_{n=0}^{\infty} \sup \left\{ z_u : u \in \mathbb{N}^n \right\} < \infty, \tag{2.7}$$

which implies (2.6).

Taking Lemma 2.4 for granted, we can easily finalize the construction of decorated real trees from building blocks as follows. We let (T, d_T) be the completion of the metric space (T°, d_{T°) and write simply ρ for the equivalence class of $\rho_\varnothing$ in T°. We also extend g° to the boundary $T \backslash T^\circ$ and define the map $g : T \to \mathbb{R}_+$ by $g(y) = g^\circ(\tilde{y})$ if $y = \tilde{y} \in T^\circ$, and $g(y) = 0$ if $y \in T \backslash T^\circ$.

Theorem 2.5. *Assume that the building blocks fulfill* (2.4) *and* (2.6)*. Then $\mathtt{T} = (T, d_T, \rho, g)$ is a decorated real tree.*

Proof. The completed space (T, d_T) plainly inherits connectivity and the four point condition (2.1) from (T°, d_{T°), and is therefore a real tree. We also know from Lemma 2.4 that (T°, d_{T°) is a pre-compact space, so its completion (T, d) is compact. We turn our attention to the extension g of g°. Take first $x \in T^\circ$ and a real number $a > g^\circ(x)$. Since g° is usc at x, we can choose $\varepsilon > 0$ small enough such that $g(y) = g^\circ(y) < a$ for all $y \in T^\circ$ such that $d_T(x, y) < \varepsilon$. Since $g(y) = 0$, for every $y \in T \backslash T^\circ$, it follows that g is usc at x. Suppose finally that $x \in T \backslash T^\circ$, and take any $a > 0$. The distance from x to any segment S_u with $\|f_u\| \geqslant a$ is bounded away from 0, since by (2.4), there is only finitely many such segments. We can again choose $\varepsilon > 0$ small enough such that $g(y) = g^\circ(y) < a$ for all $y \in T^\circ$ such that $d_T(x, y) < \varepsilon$, which shows that g is usc at x. So g is an usc function on T, and the proof is complete. □

Remark 2.6. Beware that, in spite of what the notation might suggest, T° should not be thought of as the interior of T. It should be plain that every boundary point of T° is always a leaf of T, but the converse usually fails. In other word, we have

$$T \backslash T^\circ \subseteq \partial T,$$

and the inclusion can be strict in general.

Let us now establish Lemma 2.4.

Proof of Lemma 2.4. By construction, (T°, d_{T°) is a connected metric space; let us show that it is a real tree by checking the four point condition (2.1). Pick any x_1, x_2, x_3, x_4 in $T^\sqcup$, and let k denote the maximal generation of the indices $u \in \mathbb{U}$ such that the segment S_u contains at least one of those points. Since we know by iteration from Lemma 2.2 that gluing the segments up to the k-th generation produces a tree and that trees satisfy the four point condition, we have

$$d^k(x_1, x_2) + d^k(x_3, x_4) \leqslant (d^k(x_1, x_3) + d^k(x_2, x_4)) \vee (d^k(x_1, x_4) + d^k(x_2, x_3)).$$

Recall that $d^\circ(x, y) = d^k(x, y)$ for any points x, y in segments indexed by vertices at generations at most k, so we have as well

$$d^\circ(x_1, x_2) + d^\circ(x_3, x_4) \leqslant (d^\circ(x_1, x_3) + d^\circ(x_2, x_4)) \vee (d^\circ(x_1, x_4) + d^\circ(x_2, x_3)).$$

Denote by $\tilde{x}_i$ the equivalence class of x_i for the pseudo-distance d°, with $i = 1, 2, 3, 4$. We can rewrite the above inequality as

$$d_{T^\circ}(\tilde{x}_1, \tilde{x}_2) + d_{T^\circ}(\tilde{x}_3, \tilde{x}_4) \leqslant \big(d_{T^\circ}(\tilde{x}_1, \tilde{x}_3) + d_{T^\circ}(\tilde{x}_2, \tilde{x}_4)\big) \vee \big(d_{T^\circ}(\tilde{x}_1, \tilde{x}_4) + d_{T^\circ}(\tilde{x}_2, \tilde{x}_3)\big),$$

and we conclude that (T°, d_{T°) is a real tree.

We next turn our attention the pre-compactness assertion. Fix $\varepsilon > 0$ arbitrarily small; the Assumption 2.6 allows us to pick $k \geqslant 1$ sufficiently large so that

$$\sum_{n=k+1}^{\infty} z_{\bar{u}(n)} < \varepsilon/2,$$

for any infinite sequence $\bar{u} \in \mathbb{N}^{\mathbb{N}}$. It follows from the construction by gluing that for every $y \in T^\sqcup$, we can find some $x(y) \in T^{\sqcup k} := \bigsqcup_{|u| \leqslant k} S_u$ such that

$$d^\circ(y, x(y)) \leqslant \varepsilon/2.$$

Indeed, this claim is trivial if $y \in S_v$ for some vertex v at generation $|v| \leqslant k$, and otherwise we can take for $x(y)$ the marked point $\hat{t}_{v(k+1)}$ on $S_{v(k)}$ (recall that $v(\ell)$ denotes the ancestor of the vertex $v \in \mathbb{U}$ at generation $\ell \leqslant |v|$) so that then

$$\begin{aligned} d^{\circ}(x(y), y) &\leqslant \sum_{n=k+1}^{|v|-1} d_{v(n)}(\rho_{v(n)}, \hat{t}_{v(n+1)}) + d_v(\rho_v, y) \\ &\leqslant \sum_{n=k+1}^{|v|-1} t_{v(n+1)} + z_v \\ &\leqslant \sum_{n=k+1}^{|v|} z_{v(n)}. \end{aligned}$$

Denote by $T^{\circ k} \subset T^{\circ}$ the subset of the equivalence classes of points in $T^{\sqcup k}$ and infer from above that for any $\tilde{y} \in T^{\circ}$, there is some $\tilde{x}(\tilde{y}) \in T^{\circ k}$ such that

$$d_{T^{\circ}}(\tilde{y}, \tilde{x}(\tilde{y})) \leqslant \varepsilon/2.$$

On the other hand, we deduce by induction from Lemma 2.2 that $T^{\circ k}$ endowed with the distance $d_{T^{\circ}}$ is compact. Thus there exists a finite sequence in $T^{\circ k}$, say $\tilde{x}_1, \ldots, \tilde{x}_n$ such that the sequence of balls with radius $\varepsilon/2$ centered at those points cover $T^{\circ k}$. We conclude from the triangle inequality that the sequence of balls with radius ε and centered at the $\tilde{x}_i$ now cover the whole T°, and T° is hence pre-compact.

We finally check that g° is an usc function on T°. Fix $\varepsilon > 0$; from (2.4) we can choose $k \geqslant 1$ sufficiently large so that $\sup f_u \leqslant \varepsilon$ for any $u \in \mathbb{U}$ at generation $|u| > k$. Define the function g_k° on $T^{\circ k}$ by

$$g_k^{\circ}(\tilde{x}) = \sup\left\{ g^{\sqcup}(y) : y \in \tilde{x} \cap T^{\sqcup k} \right\}.$$

We know from Lemma 2.3 and an iteration argument that g_k° is an usc function on $T^{\circ k}$. Therefore, for any $\tilde{x} \in T^{\circ k}$, there is $\varepsilon' > 0$ such that

$$g_k^{\circ}(\tilde{x}') \leqslant g_k^{\circ}(\tilde{x}) + \varepsilon, \quad \text{for every } \tilde{x}' \in T^{\circ k} \text{ with } d_{T^{\circ}}(\tilde{x}, \tilde{x}') < \varepsilon'.$$

It then follows from the choice of k that

$$g^{\circ}(\tilde{x}') \leqslant g^{\circ}(\tilde{x}) + \varepsilon, \quad \text{for every } \tilde{x}' \in T^{\circ} \text{ with } d_{T^{\circ}}(\tilde{x}, \tilde{x}') < \varepsilon',$$

proving that g° is indeed an usc function at any $\tilde{x} \in T^{\circ}$. $\square$

It will be convenient to introduce a notation for points in T° given by equivalence classes of marks (possibly fictitious) on the initial segments. For every vertex $v \in \mathbb{U}^*$, consider the mark $\hat{t}_v$ on the segment S_{v-} (recall that $v-$ stands for the parent of the vertex v). We then write $\mathsf{\rho}(v) \in T$ for the equivalence class of $\hat{t}_v$, that is also the equivalence class of the root ρ_v of the segment S_v as those two points are identified in T, and by convention set $\mathsf{\rho}(\varnothing) := \rho$. The reader will easily check that any branching point of T, say b, is of the form $b = \mathsf{\rho}(v)$ for some non-fictitious vertex $v \in \mathbb{U}$. We also stress that in the converse direction, $\mathsf{\rho}(v)$ is not necessarily a branching point of T (counter-examples arise in the situation where $t_v = z_{v-}$ and there are no other non-fictitious vertices aside v in the offspring of the parent $v-$). A bit more generally, we will sometimes use the following notation to identify points in T° even when they have not been marked. Consider any vertex $v \in \mathbb{U}$ and any $t \in [0, z_v]$. We write $\rho_v(t)$ for the unique point on the segment S_v at distance t from the root ρ_v, and then $\mathsf{\rho}(v,t)$ for the corresponding point in T° (strictly speaking, $\mathsf{\rho}(v,t)$ is the equivalence class of $\rho_v(t)$, which is actually reduced to the singleton $\{\rho_v(t)\}$ except if $\rho_v(t)$ has been marked).

We also point out that the proof of Lemma 2.4 also yields some useful information, notably about the height of T and the maximum of the decoration g, which we record in the next statement. In this direction, recall that we wrote there $T^{\circ k}$ for the subset of T° given by the equivalence classes of points in $T^{\sqcup k} = \bigsqcup_{|u| \leqslant k} S_u$. On the other hand, we infer from Lemma 2.2 that $T^{\sqcup k}$ is sequentially compact for the pseudo-distance d^k, that is we can always extract from any sequence in $T^{\sqcup k}$ a subsequence which converges in $T^{\sqcup k}$ for the pseudo-distance d°. *A fortiori*, this holds as well for the pseudo-distance d^k, and we deduce that $T^{\circ k}$ is actually closed for the distance d_{T°. We shall henceforth use the simpler notation $T^k = T^{\circ k}$, and view T^k as a closed subset of T.

Proposition 2.7. *Assuming (2.4) and (2.6), we have for every $k \geqslant 0$ that*

$$d_T(y, T^k) \leqslant \sup\Big\{ \sum_{n=k+1}^{\infty} z_{\bar{u}(n)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \Big\}, \qquad \text{for all } y \in T$$

and

$$\text{Height}(T) \leqslant \sup\Big\{ \sum_{n=0}^{\infty} z_{\bar{u}(n)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \Big\}.$$

Moreover, we have also

$$\sup_{y \in T \backslash T^k} g(y) \leqslant \max\big\{ \sup f_u : u \in \mathbb{U}, |u| \geqslant k+1 \big\}.$$

We also note the following direct consequence concerning Hausdorff dimensions.

Lemma 2.8. *Assume (2.4) and (2.6), and write*

$$\partial_0 T \coloneqq \{x \in \partial T : g(x) = 0\}, \tag{2.8}$$

for the set of leaves with decoration 0. Then, we have

$$\dim_H(T) \leqslant \dim_H(\partial_0 T) \vee 1.$$

Proof. Recall that for any $n \geqslant 0$, the subset $T^n \subset T$ induced by the countable collection of segments S_u with generation $|u| \leqslant n$ is a closed subtree. Therefore its Hausdorff dimension cannot exceed 1. Moreover, $T^\circ = \bigcup_{n \geqslant 0} T^n$ obviously contains the skeleton $T \backslash \partial T$ of T, and by definition we have $g(x) = 0$ for every $x \in T \backslash T^\circ$. □

We now conclude this section and record notation for two important families of (decorated) subtrees. The second and third have already appeared in this chapter, whereas the first will be used later.

Notation 2.9. Let $\mathtt{T} = (T, d_T, \rho, g)$ denote the decorated real tree constructed from the building blocks $(f_u)_{u\in\mathbb{U}}$ and $(t_u)_{u\in\mathbb{U}^*}$ in Theorem 2.5.

(i) For every non-fictitious vertex $u \in \mathbb{U}$, the building blocks indexed by descendants of u, namely $(f_{uv})_{v\in\mathbb{U}}$, $(t_{uv})_{v\in\mathbb{U}^*}$ and $(m_{uv})_{v\in\mathbb{U}}$ also fulfill the requirements of Theorem 2.5. We write $\mathtt{T}_u = (T_u, d_{T_u}, \rho(u), g_u)$ for the decorated real tree constructed from the latter by gluing[4].

(ii) For every generation $n \geqslant 0$, we write $\mathtt{T}^n = (T^n, d_{T^n}, \rho, g^n)$ for the decorated real tree constructed by gluing the building blocks up to generation n only, that is from the building blocks $(f^n_u)_{u\in\mathbb{U}}$ and $(t^n_u)_{u\in\mathbb{U}^*}$, where $f^n_u = f_u$ and $t^n_u = t_u$ when $|u| \leqslant n$, whereas these quantities are fictitious when $|u| > n$.

(iii) We write $T^\circ = \bigcup_{n\geqslant 1} T^n$. The set $T\backslash T^\circ$ of adherence points of T° is included into the set of leaves ∂T.

We stress that T_u is always a subtree of the fringe subtree $T_{\rho(u)}$ rooted at $\rho(u) \in T$, and that the inclusion is strict in general. Last, note also that although g^n is always dominated by the restriction of g to T^n, these two functions may be different only at marked points.

2.3 Measured decorated trees

We will now equip the compact real tree T constructed in Theorem 2.5 with some finite measures. First, as any compact real tree, our tree T is naturally equipped with the length measure λ_T, which is given by the one-dimensional Hausdorff measure on the skeleton, see [70, Section 2.4]. In words, λ_T mirrors on each segment of T the Lebesgue measure on intervals, where we recall that segments of T are isometric to intervals of $\mathbb{R}_+$. We stress that λ_T is usually only a sigma-finite measure. However, we can consider a density function $f : T \to \mathbb{R}_+$ in $L^1(\lambda_T)$ and then take $\nu(\mathrm{d}x) = f(x)\lambda_T(\mathrm{d}x)$. Densities that will appear naturally in this work depend on the decoration, that is, they are of the type $f = \varpi \circ g$, where $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function. The resulting measure on T is then denoted by $\varpi\circ g\cdot\lambda_T$ and often called the **weighted length measure**. We stress that the length measure, and therefore *a fortiori* $\varpi \circ g \cdot \lambda_T$ as well, gives no mass to the set of leaves ∂T. We also stress that, since the set of points $\{\rho(v) : v \in \mathbb{U}\}$

[4] We make here a slight abuse in order to view T_u as a subtree of T: we have previously defined $\rho(u) \in T$ as the equivalence class of the root ρ_u of the segment S_u, which is actually larger than the equivalence class obtained when the gluing construction restricted to descendants of u. A similar minor abuse is made for the same reason in (ii).

is countable and thus receives no mass from λ_T, we have

$$\int_T \varpi \circ g(x) \lambda_T(\mathrm{d}x) = \sum_u \int_0^{z_u} \varpi \circ f_u(t) \mathrm{d}t, \tag{2.9}$$

where the sum in the right-hand side is implicitly taken over non-fictitious vertices. In particular, the requirement $\varpi \circ g \in L^1(\lambda_T)$ is equivalent to the finiteness of the previous display.

We shall now construct another class of measures on T, which, at the opposite, are typically carried by ∂T. This requires to introduce, on top of the building blocks, a further family

$$(m_u)_{u \in \mathbb{U}}$$

where $m_u \geqslant 0$, $m_u = 0$ when $u \sim \dagger$ is a fictitious vertex, and most importantly,

$$\sum_{i=1}^{\infty} m_{ui} = m_u \qquad \text{for all } u \in \mathbb{U}. \tag{2.10}$$

Note that if $m_u > 0$, then (2.10) implies that u has at least one infinite line of descent with only non-fictitious vertices. Next recall that $\rho(u)$ denotes the point in T associated to a mark t_u. We then define for every $n \geqslant 0$ the purely atomic measure μ^n on T given by

$$\mu^n := \sum_{|u|=n} m_u \cdot \delta_{\rho(u)}, \tag{2.11}$$

where the notation δ_x is used for the Dirac point mass at x, and we implicitly agree to ignore fictitious vertices in the sum. It is immediately seen by induction from the requirement (2.10) that $\mu^n(T) = m_\varnothing$ for any $n \geqslant 0$.

Proposition 2.10. *Assume that the building blocks fulfill* (2.4), (2.6) *and* (2.10). *The sequence* $(\mu^n)_{n \geqslant 1}$ *converges in the sense of Prokhorov towards a Borel measure on* T *denoted by* μ. *One has* $\mu(T) = m_\varnothing$.

Proof. Recall that for any $n \geqslant 0$, the set T^n denotes the subtree of T obtained by gluing the collection of segments S_u with generation $|u| \leqslant n$. We write $p^n : T \to T^n$ for the projection on T^n, that is for any $y \in T$, $p^n(y)$ is the right endpoint of the segment $[\![\rho, y]\!] \cap T^n$. On the one hand, the first claim in Proposition 2.7 entails

$$d_T\big(y, p^n(y)\big) \leqslant \sup\Big\{ \sum_{k=n+1}^{\infty} z_{\bar{u}(k)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \Big\}.$$

On the other hand, we infer by iteration from (2.10) that for any $n' \geqslant n$, the push-forward of $\mu^{n'}$ by p^n coincides with μ^n.

We deduce from these two observations that the Prokhorov distance between μ^n and $\mu^{n'}$ is at most

$$\mathrm{d}_{\mathrm{Prok}}(\mu^n, \mu^{n'}) \leqslant \sup\Big\{ \sum_{k=n+1}^{\infty} z_{\bar{u}(k)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \Big\}.$$

Since (2.6) requests the right-hand side to converge to 0 as $n \to \infty$, the sequence $(\mu^n)_{n \geqslant 1}$ is Cauchy on the space of measures on T with total mass $m_\varnothing$, and our claim follows. □

We also point out that, under a minor additional hypothesis, the measure μ constructed above is carried by the boundary points of the pre-compact tree T°. Recall that the latter denotes the equivalence class of $T^\sqcup$; see also Remark 2.6.

Proposition 2.11. *Assume (2.4), (2.6) and (2.10). Suppose further that for any non-fictitious vertex $v \in \mathbb{U}^*$, the mark t_v is strictly positive. Then we have $\mu(T^\circ) = 0$, and as a consequence, μ is carried by the subset of leaves $T \backslash T^\circ$.*

Proof. We shall check first that the root ρ is not an atom of μ, which should be intuitively clear. In this direction, write $\tilde{S}_\varnothing := \{\rho(\varnothing, t) : \ t \in [0, z_\varnothing]\}$ for the segment in T induced by the equivalence class of the ancestral segment $S_\varnothing$. Then for any $a > 0$, consider the subset $T_{<a}$ of points $x \in T$ such that the length (i.e. the λ_T measure) of the segment $\tilde{S}_\varnothing \cap [[\rho, x]]$ is less than a. By (2.10) and the construction by gluing, we have

$$\mu(T_{<a}) = \sum_{j \geqslant 1} \mathbf{1}_{t_j < a} m_j,$$

and since we assumed that the (non-fictitious) marks t_j are strictly positive, we have

$$\lim_{a \to 0+} \mu(T_{<a}) = 0.$$

Also by construction, the open ball in T centered at ρ and with radius a is contained in $T_{<a}$. Letting $a \to 0+$, we conclude that $\mu(\{\rho\}) = 0$.

Next, recall that for every non-fictitious vertex $j \geqslant 1$ at the first generation, $\rho(j) \in \tilde{S}_\varnothing$ denotes the equivalence class of the mark $\hat{t}_j$ on the ancestral segment $S_\varnothing$. On the one hand, the argument above shows that we have also $\mu(\{\rho(j)\}) = 0$. On the other hand, the requirement (2.10) entails that the entire mass of μ is carried by the union for $j \geqslant 1$ of the fringe subtrees rooted at the $\rho(j)$. This shows that μ assigns zero mass to the equivalence class of ancestral segment, $\mu(\tilde{S}_\varnothing) = 0$. We conclude by iteration on generations that μ assigns zero mass to T°, and *a fortiori* to the skeleton $T \backslash \partial T$. $\square$

2.4 Hypographs, topologies, and isomorphic identifications

We have developed so far general material on (measured) decorated real trees and their construction, and roughly speaking, we now would like to compare two different decorated real trees one with the other. More precisely, our main motivation is to give a rigorous definition of a notion of convergence for sequences of these objects. Actually, although dealing with usc functions on compact metric spaces is fundamental to our approach, tree structures are essentially irrelevant for this question, and we shall develop first a more general framework that could also be used for other purposes. It will be only at the end of this section that the special case of real trees will be addressed more specifically.

Let (Y, d_Y) be a Polish space and $\mathcal{K}(Y)$ denote the set of non-empty compact subspaces in Y. Consider an usc function $g : K \to \mathbb{R}_+$ for some $K \in \mathcal{K}(Y)$ which we refer to as the domain of

g. Upper semi-continuity enables us to view g as a compact subspace of the larger Polish space $Y \times \mathbb{R}_+$ by introducing the **hypograph**

$$\mathrm{Hyp}(g) := \{(x, r) : x \in K \text{ and } 0 \leqslant r \leqslant g(x)\} \subset Y \times \mathbb{R}_+.$$

Specifically, the product space $Y \times \mathbb{R}_+$ is naturally equipped with the distance

$$d_{Y \times \mathbb{R}_+}\big((y, r), (y', r')\big) := d_Y(y, y') \vee |r - r'|, \tag{2.12}$$

and $\mathrm{Hyp}(g)$ is then a compact subspace[5] of the Polish space $(Y \times \mathbb{R}_+, d_{Y\times\mathbb{R}_+})$. Plainly, the hypograph $\mathrm{Hyp}(g)$ determines $g : K \to \mathbb{R}_+$. Namely, the domain K is the base of the hypograph, that is the image of $\mathrm{Hyp}(g)$ by the first projection $p_1 : Y \times \mathbb{R}_+ \to Y$, and

$$g(x) = \max\{r \geqslant 0 : (x, r) \in \mathrm{Hyp}(g)\}, \qquad \text{for every } x \in K.$$

In this work, we will interpret a decorated tree through its hypograph. Before continuing with the general study of hypographs and defining a suitable metric for comparing them, let us highlight the following consequence of Lemma 2.8 and establish a bound for the Hausdorff dimension of the hypograph of the decoration g.

Corollary 2.12. *Assume (2.4) and (2.6), and recall from (2.8) that $\partial_0 T$ denotes the subset of leaves of T on which the decoration g vanishes. Then, the Hausdorff dimension of the hypograph of g can be bounded by*

$$\dim_H(\mathrm{Hyp}(g)) \leqslant \dim_H(\partial_0 T) \vee 2,$$

where $\dim_H(\partial_0 T)$ stands for the Hausdorff dimension of $\partial_0 T$ equipped with the restriction of d_T.

Proof. Recall from Notation 2.9, that for every $n \geqslant 1$, the notation T^n stands for the subtree of T built by gluing segments up to generation n only. Both T and T^n are rooted at ρ and T is decorated with the usc function g, while T^n is decorated with g^n which verifies $g^n \leqslant g$ on T^n. The hypograph $\mathrm{Hyp}(g^n)$ can be constructed by gluing a countable family of hypographs of decorated segments, each having Hausdorff dimension smaller than 2, and therefore

$$\dim_H\Big(\bigcup_{n\geqslant 0} \mathrm{Hyp}(g^n)\Big) \leqslant 2.$$

Since the family $(\|f_u\|)_{u\in\mathbb{U}}$ is null, we infer that $\bigcup_{n\geqslant 0} \mathrm{Hyp}(g^n) = \bigcup_{n\geqslant 0} \mathrm{Hyp}(g_{T^n})$, where g_{T^n} stands for the restriction of g to T^n. Finally, recall also from the proof of Lemma 2.8 that $T \setminus \bigcup_{n\geqslant 0} T^n$ is a subset of $\partial_0 T$, so we can write

$$\mathrm{Hyp}(g) = \Bigg(\bigcup_{n\geqslant 0} \mathrm{Hyp}(g_{|T^n})\Bigg) \cup \Big(\partial_0 T \times \{0\}\Big),$$

and the desired bound follows. □

[5] This is one of the main reasons why we considered usc functions.

Our goal now is to define a metric to compare general hypographs. In this direction, consider two usc functions $g : K \to \mathbb{R}_+$ and $g' : K' \to \mathbb{R}_+$ with respective domains K and K' in $\mathcal{K}(Y)$. Their hypographs $\mathrm{Hyp}(g)$ and $\mathrm{Hyp}(g')$ are thus two elements of $\mathcal{K}(Y \times \mathbb{R}_+)$, and we can define

$$\mathrm{d}_{\mathrm{Hyp}}(g, g') \coloneqq \mathrm{d}_{\mathrm{Haus}}\left(\mathrm{Hyp}(g), \mathrm{Hyp}(g')\right),$$

where in the right-hand side, $\mathrm{d}_{\mathrm{Haus}}$ denotes the Hausdorff distance between two compact subsets in $Y \times \mathbb{R}_+$. In words, the hypograph distance $\mathrm{d}_{\mathrm{Hyp}}$ between two usc functions $g : K \to \mathbb{R}_+$ and $g' : K' \to \mathbb{R}_+$ is at most $\varepsilon > 0$ if and only if for every $x \in K$ we can find $x' \in K'$ such that $d_Y(x, x') \leqslant \varepsilon$ and $g(x) \leqslant g'(x') + \varepsilon$, and *vice versa* when the roles of g and g' are permuted. The hypograph convergence for sequences of usc functions can be characterized as a kind of pointwise convergence, see [13] for details.

Remark 2.13. In the case when $Y = \mathbb{R}$, it is natural to compare the distance between two functions in the sense of hypographs with other notions *à la* Skorohod; see [82, Chapter VI] for background. Specifically, take $a < z$ and $a' < z'$, and let $g : [a, z] \to \mathbb{R}_+$ and $g' : [a', z'] \to \mathbb{R}_+$ be two usc functions. It is then readily checked that for any increasing bijection $\beta : [a, z] \to [a', z']$, one has

$$\mathrm{d}_{\mathrm{Hyp}}(g, g') \leqslant \max\left\{\|g - g' \circ \beta\|, \|\beta - \mathrm{Id}_{[a,z]}\|\right\},$$

where we use the notation $\|\cdot\|$ for the supremum norm on the space of bounded functions on $[a, z]$ and $\mathrm{Id}_{[a,z]}$ for the identity function on $[a, z]$. As a consequence, convergence in the sense of Skorohod for sequences of rcll (right-continuous with left limits) functions defined on compact intervals entails convergence of the sequence of usc versions[6] in the sense of hypographs. Of course, the converse fails. For instance, consider $g_n : [0, 1] \to \mathbb{R}_+$ given by $g_n(x) = 1 + \cos(nx)$ for $n \geqslant 1$. Then the sequence $(g_n)_{n\geqslant 1}$ converges as $n \to \infty$ to the constant function 2 on $[0, 1]$ for the distance $\mathrm{d}_{\mathrm{Hyp}}$, but does not converge in the sense of Skorohod.

Let us now extend the main nomenclature and notations introduced for real trees to the setting of general compact metric spaces. In this direction, consider $K \in \mathcal{K}(Y)$, and note that this set is naturally endowed with the distance d_K induced by the restriction of d_Y to $K \times K$. We say that (K, d_K) is rooted if it is equipped with a distinguished point ρ in K, also referred to as the root. Similarly, the compact space (K, d_K) is said decorated (resp. measured) if it is equipped with an usc function $g : K \to \mathbb{R}_+$ (resp. with a finite Borel measure ν on K).We use short hand notations $\mathtt{K} \coloneqq (K, d_K, \rho, g)$ for a decorated compact space and $\mathbf{K} = (K, d_K, \rho, g, \nu)$ for a measured decorated compact space. Obviously, a decorated compact space can always be seen as a measured decorated compact space once equipped with the null measure, so that we directly develop the formalism in the more general case.

We write $\mathbb{H}_m(Y)$ for the space of measured decorated compact spaces on Y, that is formally

$$\mathbb{H}_m(Y) \coloneqq \bigsqcup_{K \in \mathcal{K}(Y)} \left\{(K, d_K, \rho, g, \nu) : \rho \in K, g : K \to \mathbb{R}_+ \text{ usc, and } \nu \in \mathcal{M}^f(K)\right\},$$

[6] Recall that if $g : [a, z] \to \mathbb{R}_+$ is a rcll function, its usc version is the function defined by $\check{g}(t) \coloneqq \max\{g(t-), g(t)\}$ for $t \in [a, z]$, with the convention $g(a-) = g(a)$.

where $\mathcal{M}^f(K)$ denotes the space of finite Borel measures on K. We endow $\mathbb{H}_m(Y)$ with a natural metric defined for any $\mathbf{K}, \mathbf{K}' \in \mathbb{H}_m(Y)$ in the obvious notation by

$$\mathrm{d}_{\mathbb{H}_m(Y)}(\mathbf{K}, \mathbf{K}') := d_Y\big(\rho, \rho'\big) \vee \mathrm{d}_{\mathrm{Hyp}}\big(g, g'\big) \vee \mathrm{d}_{\mathrm{Prok}}\big(\nu, \nu'\big),$$

where $\mathrm{d}_{\mathrm{Prok}}$ stands for the Prokhorov distance on $\mathcal{M}^f(Y)$.

Proposition 2.14. *The space $\mathbb{H}_m(Y)$ of measured decorated compact spaces on a Polish space (Y, d_Y) and equipped with the distance $\mathrm{d}_{\mathbb{H}_m(Y)}$ is also Polish.*

Proof. The heart of the argument is the observation, which is doubtless well-known, that the space of hypographs of usc functions with compact domains in Y is closed in $\mathcal{K}(Y \times \mathbb{R}_+)$. More precisely, we claim that if $(g_n : K_n \to \mathbb{R}_+)_{n \geqslant 1}$ is a sequence of usc functions with $K_n \in \mathcal{K}(Y)$, such that the sequence of hypographs $(\mathrm{Hyp}(g_n))_{n \geqslant 1}$ converges as $n \to \infty$ to some H for the Hausdorff distance in $\mathcal{K}(Y \times \mathbb{R}_+)$, then the sequence $(K_n)_{n \geqslant 1}$ converges to some K for the Hausdorff distance in $\mathcal{K}(Y)$. Furthermore, there exists an usc function $g : K \to \mathbb{R}_+$ such that $H = \mathrm{Hyp}(g)$, and hence the sequence of usc functions $(g_n)_{n \geqslant 1}$ converges to g as $n \to \infty$ for the hypograph distance $\mathrm{d}_{\mathrm{Hyp}}$.

Indeed, the projection on the first component $p_1 : Y \times \mathbb{R}_+ \to Y$ is a 1-Lipschitz map. Set $K = p_1(H)$ for the image of H by p_1, so K is a nonempty compact set. We now also regard p_1 as a map from $\mathcal{K}(Y \times \mathbb{R}_+)$ to $\mathcal{K}(Y)$; plainly this still is a 1-Lipschitz map, and therefore $K_n = p_1(\mathrm{Hyp}(g_n))$ converges to K for the Hausdorff distance in $\mathcal{K}(Y)$. Then take any $x \in K$ and $r \geqslant 0$ such that $(x, r) \in H$. There exists some sequence $(x_n, r_n)_{n \geqslant 1}$ with $(x_n, r_n) \in \mathrm{Hyp}(g_n)$ which converges to (x, r) in $Y \times \mathbb{R}_+$. Since $g_n(x_n) \geqslant r_n$, the segment $\{x_n\} \times [0, r_n]$ belongs to $\mathrm{Hyp}(g_n)$, and therefore $\{x\} \times [0, r] \subset H$. We set $g(x) = \sup\{r \geqslant 0 : \{x\} \times [0, r] \subset H\}$. Since H is compact, $(x, g(x)) \in H$, and we conclude that $H = \{(x, r) : x \in K \text{ and } 0 \leqslant r \leqslant g(x)\}$. Now using that H is closed, we infer that the function $g : K \to \mathbb{R}_+$ is usc, and $H = \mathrm{Hyp}(g)$.

Next, let E denote the space of rooted measured compact space on $(Y \times \mathbb{R}_+, d_{Y \times \mathbb{R}_+})$ endowed with the Hausdorff–Prokhorov distance, say $\mathrm{d}_{\mathrm{HP}(E)}$. Identifying the usc function g with its hypograph $\mathrm{Hyp}(g)$, the root ρ with $(\rho, 0) \in \mathrm{Hyp}(g)$ and similarly ν with a measure $\bar{\nu}$ on $Y \times \mathbb{R}_+$ supported by the base $K \times \{0\}$ of $\mathrm{Hyp}(g)$, yields a natural isometric embedding, say

$$\Phi : \big(\mathbb{H}_m(Y), \mathrm{d}_{\mathbb{H}_m(Y)}\big) \hookrightarrow \big(E, \mathrm{d}_{\mathrm{HP}(E)}\big).$$

Since it is well-known that $(E, \mathrm{d}_{\mathrm{HP}(E)})$ is a Polish space, all that is needed to establish Proposition 2.14 is to verify that the image $\Phi(\mathbb{H}_m(Y))$ of $\mathbb{H}_m(Y)$ under this embedding is closed in $(E, \mathrm{d}_{\mathrm{HP}(E)})$. So, consider a sequence $\big(\mathbf{K}_n\big)_{n \geqslant 1}$ in $\mathbb{H}_m(Y)$ such that the embedded sequence $(\Phi(\mathbf{K}_n))_{n \geqslant 1}$ converges in $(E, \mathrm{d}_{\mathrm{HP}(E)})$, say to $(H, d_H, \bar{\rho}, \bar{\nu})$, where $H \in \mathcal{K}(Y \times \mathbb{R}_+)$, d_H denotes the restriction of $d_{Y \times \mathbb{R}_+}$ to H, $\bar{\rho} \in Y \times \mathbb{R}_+$ and $\bar{\nu} \in \mathcal{M}^f(Y \times \mathbb{R}_+)$. We have just seen above that K_n converges to K for the Hausdorff distance on $\mathcal{K}(Y)$ and that $H = \mathrm{Hyp}(g)$ for some usc function $g : K \to \mathbb{R}_+$. Plainly, since $\bar{\rho}$ is the limit of $(\rho_n, 0)$ in $Y \times \mathbb{R}_+$ and $\rho_n \in K_n$ for all $n \geqslant 1$, we have $\bar{\rho} = (\rho, 0)$ for some $\rho \in K$. Similarly, by the Portemanteau theorem, the measure

$\bar{\nu}$ must be supported by $K \times \{0\}$, and can thus be identified as a finite measure ν on K. Last, d_H coincides with the restriction of $d_{K\times\mathbb{R}_+}$ to $H = \mathrm{Hyp}(g)$. Putting the pieces together, we get $(H, d_H, \bar{\rho}, \bar{\nu}) = \Phi(\mathbf{K})$ for some $\mathbf{K} = (K, d_K, \rho, g, \nu) \in \mathbb{H}_m(Y)$, as we wanted to check. $\square$

Roughly speaking, we are only interested in the general structure induced by a decorated compact space rather than by a specific realization. More precisely, two decorated compact spaces, say $\mathbf{K} = (K, d_K, \rho, g, \nu) \in \mathbb{H}_m(Y)$ and $\mathbf{K}' = (K', d_{K'}, \rho', g', \nu') \in \mathbb{H}_m(Y')$ have the same structure and then are viewed as equivalent (or isomorphic) if there exists a bijective isometry $\phi : (K, d_K) \to (K', d_{K'})$ with inverse denoted by ϕ^{-1}, such that $\rho' = \phi(\rho)$, $g' = g \circ \phi^{-1}$ and $\nu' = \nu \circ \phi^{-1}$ is the pushed forward of ν by ϕ. We then simply write $\mathbf{K} \approx \mathbf{K}'$. Note that the underlying Polish spaces (Y, d_Y) and $(Y', d_{Y'})$ play no role in this definition and we may simply take $(Y, d_Y) = (K, d_K)$ and $(Y', d_{Y'}) = (K', d_{K'})$ for definitiveness. Let us also make some comments concerning decorated trees and the general construction presented in Section 2.2. First, note that due to the 4-point criterion, it is clear that a decorated real tree can only be isomorphic to another decorated real tree. Moreover, we emphasize that the gluing construction described in Section 2.2 is not unique in the sense that different building blocks, $((f_u)_{u\in\mathbb{U}}, (t_u)_{u\in\mathbb{U}^*})$ can produce isomorphic trees. For instance, one may apply a bijective isometry (for the graph distance) $\varsigma : \mathbb{U} \to \mathbb{U}$ that fixes the root. More complex rearrangements are also possible; for example, during a birth event, a mother particle might switch identities with one of its daughters. Such modifications, called **bifurcations**, are studied in depth in the self-similar Markov setting in Chapter 6. Informally, each bifurcation provides a different way to decompose the associated decorated tree in building blocks.

Figure 2.3: Illustration of a bifurcation event where the identity of two particles is exchanged during a birth event. Obviously the underlying decorated tree is unchanged.

Our goal now is to define a notion of distance between equivalence classes. In this direction, we denote the set of all equivalence classes of measured decorated compact spaces by $\mathbb{H}_m$. Our goal now is to endow $\mathbb{H}_m$ with a natural distance and make it a Polish space. In this direction, recall that the Gromov–Hausdorff–Prokhorov distance between two rooted measured compact spaces (K, d_K, ρ, ν) and $(K', d_{K'}, \rho', \nu')$ is defined as

$$\begin{aligned}&\mathrm{d_{GHP}}\big((K, d_K, \rho, \nu), (K', d_{K'}, \rho', \nu')\big)\\ &\qquad := \inf_{Y,\phi,\psi} \mathrm{d}_{\mathrm{HP}(Y)}\Big(\big(\phi(K), d_{\phi(K)}, \phi(\rho), \nu \circ \phi^{-1}\big), \big(\psi(K'), d_{\psi(K')}, \psi(\rho'), \nu' \circ \psi^{-1}\big)\Big),\end{aligned}$$

where in the right-hand side, $\mathrm{d}_{\mathrm{HP}(Y)}$ stands for the Hausdorff–Prokhorov distance between rooted measured compact spaces in Y, the infimum is over all the metric spaces (Y, d_Y) and all the isometric embeddings $\phi : (K, d_K) \hookrightarrow (Y, d_Y)$ and $\psi : (K', d_{K'}) \hookrightarrow (Y, d_Y)$, and $\nu \circ \phi^{-1}$ is the pushforward measure of ν by ϕ. The quantity $\mathrm{d}_{\mathrm{GHP}}\big((K, d_K, \rho, \nu), (K', d_{K'}, \rho', \nu')\big)$ is null if and only if there exists an isomorphism between (K, d_K, ρ, ν) and $(K', d_{K'}, \rho', \nu')$, and $\mathrm{d}_{\mathrm{GHP}}$ defines a distance in the space of equivalence classes of rooted measured compact spaces, see [3].

This incites us to make the following definition. First, for any measured decorated compact space $\mathbf{K} = (\mathtt{K}, \nu) = (K, d_K, \rho, g, \nu)$ and any isometric embedding $\phi : (K, d_K) \hookrightarrow (Y, d_Y)$, we define

$$g \circ \phi^{-1} : \phi(K) \to \mathbb{R}_+ \quad , \quad g \circ \phi^{-1}(\phi(x)) \coloneqq g(x) \text{ for all } x \in K.$$

Plainly, $\phi(K)$ is a compact subset of Y and $g \circ \phi^{-1}$ is usc. We further write

$$\phi(\mathbf{K}) \coloneqq \big(\phi(\mathtt{K}), \nu \circ \phi^{-1}\big) = \big(\phi(K), d_{\phi(K)}, \phi(\rho), g \circ \phi^{-1}, \nu \circ \phi^{-1}\big).$$

For every $\mathbf{K}$ and $\mathbf{K}'$ measured decorated compact spaces, we then set

$$\mathrm{d}_{\mathbb{H}_m}\left(\mathbf{K}, \mathbf{K}'\right) \coloneqq \inf_{Y, \phi, \psi} \mathrm{d}_{\mathbb{H}_m(Y)}\big(\phi(\mathbf{K}), \psi(\mathbf{K}')\big), \tag{2.13}$$

where again the infimum is taken over all the Polish spaces (Y, d_Y) and all the isometric embeddings $\phi : (K, d_K) \hookrightarrow (Y, d_Y)$ and $\psi : (K', d_{K'}) \hookrightarrow (Y, d_Y)$. We point out that in the special case where g and g' are identically zero on their respective domains, the distances $\mathrm{d}_{\mathbb{H}_m}\big((K, d_K, \rho, 0, \nu), (K', d_{K'}, \rho', 0, \nu')\big)$ and $\mathrm{d}_{\mathrm{GHP}}((K, d_K, \rho, \nu), (K', d_{K'}, \rho', \nu'))$ coincide. Of course $\mathrm{d}_{\mathbb{H}_m}(\mathbf{K}, \mathbf{K}')$ is invariant by isomorphisms, and therefore can be viewed as a function $\mathbb{H}_m \times \mathbb{H}_m \to \mathbb{R}_+$ which we still denote by $\mathrm{d}_{\mathbb{H}_m}$ for simplicity. Our goal now is to establish the following result.

Theorem 2.15. *The map $\mathrm{d}_{\mathbb{H}_m} : \mathbb{H}_m \times \mathbb{H}_m \to \mathbb{R}_+$ defines a distance on $\mathbb{H}_m$ and $(\mathbb{H}_m, \mathrm{d}_{\mathbb{H}_m})$ is Polish.*

We prepare the proof of the theorem with a technical lemma. Roughly speaking, it states that given an arbitrary sequence $(\mathbf{K}_n)_{n \geqslant 1}$ of equivalence classes in $\mathbb{H}_m$, we can always find some Polish space (Z, d_Z) and representatives of the equivalence classes in $\mathbb{H}_m(Z)$, such that the distance in $\mathbb{H}_m(Z)$ between the representatives of two consecutive equivalence classes of the sequence is never significantly larger than the distance between the equivalence classes $(\mathbf{K}_n)_{n \geqslant 1}$ in $\mathbb{H}_m$. Here is the formal statement.

Lemma 2.16. *Let $(\varepsilon_n)_{n \geqslant 1}$ be a sequence in $\mathbb{R}_+$ and $(\mathbf{K}_n)_{n \geqslant 1} \coloneqq (K_n, d_{K_n}, \rho_n, g_n, \nu^n)_{n \geqslant 1}$ a sequence of measured decorated compact spaces with*

$$\mathrm{d}_{\mathbb{H}_m}(\mathbf{K}_n, \mathbf{K}_{n+1}) < \varepsilon_n, \qquad \textit{for all } n \geqslant 1.$$

Then there exist a Polish space (Z, d_Z) and isometric embeddings $\phi_1, \phi_2, \ldots$ respectively from $K_1, K_2, \ldots$ into Z such that

$$\mathrm{d}_{\mathbb{H}_m(Z)}\big(\phi_n(\mathbf{K}_n), \phi_{n+1}(\mathbf{K}_{n+1})\big) < \varepsilon_n, \qquad \textit{for all } n \geqslant 1.$$

Let us mention that similar results have already appeared in the literature for different variants of the Gromov–Hausdorff–Prokhorov distance. In this work, we adapt the proof of Lemma 5.7 in [74], which establishes the analog of our Lemma 2.16 for the Gromov–Prokhorov distance.

Proof. Without loss of generality, we may suppose that the compact spaces K_n are pairwise disjoint. First, by definition, we can find for every $n \geqslant 1$ a Polish space (Y_n, d_{Y_n}) and two isometric embeddings $\phi_n : K_n \hookrightarrow Y_n$ and $\psi_n : K_{n+1} \hookrightarrow Y_n$ such that:

$$\eta_n \coloneqq \mathrm{d}_{\mathbb{H}_m(Y_n)}\big(\phi_n(\mathbf{K}_n), \psi_n(\mathbf{K}_{n+1})\big) < \varepsilon_n.$$

Then, we introduce the disjoint union $Z \coloneqq \bigsqcup_{n\geqslant 1} K_n$, and we endow Z with the metric d_Z defined as the largest distance that coincides with d_{K_n} on $K_n \times K_n$ for each $n \geqslant 1$ and such that

$$d_Z(x, y) = d_{Y_n}\big(\phi_n(x), \psi_n(y)\big) + (\varepsilon_n - \eta_n)/2, \qquad \text{for } x \in K_n \text{ and } y \in K_{n+1},$$

where the term $(\varepsilon_n - \eta_n)/2$ ensures that $d_Z(x, y) > 0$ when $x \in K_n$ and $y \in K_{n+1}$. Specifically, for $x_j \in K_j$ and $x_{j+k} \in K_{j+k}$ with $1 \leqslant j, k$, we have

$$d_Z(x_j, x_{j+k}) \coloneqq \inf\Big\{ \sum_{\ell=j+1}^{j+k} d_Z(x_{\ell-1}, x_\ell) : x_\ell \in K_\ell \text{ for every } \ell = j+1, \dots, k-1 \Big\}.$$

Since the spaces K_n are separable, (Z, d_Z) is also separable. By a slight abuse of notation, we still write (Z, d_Z) for its completion, which is a Polish space. We claim that (Z, d_Z) and the sequence $(p_n)_{n\geqslant 1}$, where $p_n : K_n \hookrightarrow Z$ is a canonical embedding, satisfy the conclusion of the statement.

To begin with, note from the very definition of d_Z that, for every $n \geqslant 1$, the distance in Z between the roots of K_n and K_{n+1} satisfies

$$d_Z(\rho_n, \rho_{n+1}) = d_{Y_n}\big(\phi_n(\rho_n), \psi_n(\rho_{n+1})\big) + (\varepsilon_n - \eta_n)/2 \leqslant \eta_n + (\varepsilon_n - \eta_n)/2 < \varepsilon_n.$$

Then, recall that the Prokhorov distance between $\nu^n \circ \phi_n^{-1}$ and $\nu^{n+1} \circ \psi_n^{-1}$ is at most η_n. Take any $\eta > \eta_n$ and let A be an arbitrary Borel subset of Y_n. Writing A^η for the η-neighborhood of A in Y_n, we have

$$\nu^n \circ \phi_n^{-1}(A) \leqslant \nu^{n+1} \circ \psi_n^{-1}(A^\eta) + \eta.$$

Let B be an arbitrary Borel subset of Z, take $A = \phi_n(p_n^{-1}(B))$, and set $\eta' = \eta + (\varepsilon_n - \eta_n)/2$. Since ϕ_n and ψ_n are isometries, $\psi_n^{-1}(A^\eta)$ is contained into the pre-image by p_{n+1} of $B^{\eta'}$, the η'-neighborhood of B in Z. So the last displayed inequality shows that

$$\nu^n \circ p_n^{-1}(B) \leqslant \nu^{n+1} \circ p_{n+1}^{-1}(B^{\eta'}) + \eta.$$

The very same argument also shows that

$$\nu^{n+1} \circ p_{n+1}^{-1}(B) \leqslant \nu^n \circ p_n^{-1}(B^{\eta'}) + \eta.$$

Since η can be chosen arbitrarily close to η_n, this entails that

$$\mathrm{d}_{\mathrm{Prok}}(\nu^n \circ p_n^{-1}, \nu^{n+1} \circ p_{n+1}^{-1}) \leqslant \eta_n + (\varepsilon_n - \eta_n)/2 < \varepsilon_n.$$

Finally, we deal with the hypographs and check that

$$\mathrm{d}_{\mathrm{Hyp}}\Big(g_n \circ p_n^{-1}, g_{n+1} \circ p_{n+1}^{-1}\Big) < \varepsilon_n. \tag{2.14}$$

Indeed, for every $(x, r) \in \mathrm{Hyp}(g_n \circ p_n^{-1})$, we have

$$\begin{aligned}&\inf\Big\{d_Z(x,y) \vee |r-s| : \ (y,s) \in \mathrm{Hyp}(g_{n+1} \circ p_{n+1}^{-1})\Big\}\\ &\leqslant (\varepsilon_n - \eta_n)/2 + \inf\Big\{d_{Y_n}\big(\phi_n(x), \psi_n(y)\big) \vee |r-s| : \ (y,s) \in \mathrm{Hyp}(g_{n+1})\Big\}.\end{aligned}$$

The second term in the sum above is bounded by

$$\mathrm{d}_{\mathrm{Hyp}}\big(g_n \circ \phi_n^{-1}, g_{n+1} \circ \psi_n^{-1}\big) \leqslant \mathrm{d}_{\mathbb{H}_m(Z)}\Big(\phi_n(\mathtt{K}_n, \nu^n), \psi_n(\mathtt{K}_{n+1}, \nu^{n+1})\Big) = \eta_n,$$

and therefore

$$\sup\big\{d_{Z\times\mathbb{R}_+}\big((x,r), \mathrm{Hyp}(g_{n+1} \circ p_{n+1}^{-1})\big) : (x,r) \in \mathrm{Hyp}(g_n \circ p_n^{-1})\big\} < \varepsilon_n.$$

The same argument shows that

$$\sup\big\{d_{Z\times\mathbb{R}_+}\big((y,s), \mathrm{Hyp}(g_n \circ p_n^{-1})\big) : (y,s) \in \mathrm{Hyp}(g_{n+1} \circ p_{n+1}^{-1})\big\} < \varepsilon_n.$$

This establishes (2.14) and completes the proof of the lemma. □

We can now proceed with the proof of Theorem 2.15.

Proof Theorem 2.15. The proof relies heavily on Proposition 2.14 and Lemma 2.16. We shall use Lemma 2.16 to represent measured decorated compact spaces in the same well-chosen Polish space (Z, d^Z). It is convenient for this purpose to let Z systematically appear as an exponent in the notation, writing e.g. $\mathrm{d}^Z_{\mathrm{Hyp}}$ for the hypograph distance on $\mathbb{H}_m(Z)$, rather than using Z as an index or omitting it like in Lemma 2.16 and its proof.

We first establish that $\mathrm{d}_{\mathbb{H}_m}$ is a distance. Symmetry is clear; we now check the triangle inequality. Let $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ three measured decorated compact spaces and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\mathrm{d}_{\mathbb{H}_m}\big(\mathbf{K}_1, \mathbf{K}_2\big) < \varepsilon_1 \quad \text{and} \quad \mathrm{d}_{\mathbb{H}_m}\big(\mathbf{K}_2, \mathbf{K}_3\big) < \varepsilon_2.$$

By Lemma 2.16, there exist a Polish space Z and isometric embeddings ϕ_1, ϕ_2 and ϕ_3, respectively from K_1, K_2 and K_3 into Z, such that

$$\mathrm{d}_{\mathbb{H}_m(Z)}\big(\phi_n(\mathbf{K}_n), \phi_{n+1}(\mathbf{K}_{n+1})\big) < \varepsilon_n, \qquad \text{for } n = 1, 2.$$

By Proposition 2.15, $\mathrm{d}_{\mathbb{H}_m(Z)}$ is a distance on $\mathbb{H}_m(Z)$ and then the triangle inequality gives

$$\mathrm{d}_{\mathbb{H}_m}\big(\mathbf{K}_1, \mathbf{K}_3\big) \leqslant \mathrm{d}_{\mathbb{H}_m(Z)}\big(\phi_1(\mathbf{K}_1), \phi_3(\mathbf{K}_3)\big) < \varepsilon_1 + \varepsilon_2.$$

Passing to equivalent classes, we infer that $\mathrm{d}_{\mathbb{H}_m} : \mathbb{H}_m \times \mathbb{H}_m \to \mathbb{R}_+$ satisfies the triangle inequality.

We then check positivity. To this end we need to show that if $\mathbf{K}$ and $\mathbf{K}'$ are two measured decorated compact spaces such that $\mathrm{d}_{\mathbb{H}_m}(\mathbf{K}, \mathbf{K}') = 0$, then they have to be equivalent. Let us proceed. For every $n \geqslant 0$, set $\mathbf{K}_n = \mathbf{K}$ if n is odd and $\mathbf{K}_n = \mathbf{K}'$ if n is even. Again by Lemma 2.16, we can find a Polish space (Z, d^Z) and isometric embeddings $\phi_{2n-1} : K \hookrightarrow Z$ and $\phi_{2n} : K' \hookrightarrow Z$, for all $n \geqslant 1$, such that

$$\mathrm{d}_{\mathbb{H}_m(Z)}\big(\phi_n(\mathbf{K}_n), \phi_{n+1}(\mathbf{K}_{n+1})\big) < 2^{-n}.$$

It follows that the sequence $(\phi_n(\mathbf{K}_n))_{n\geqslant 1}$ is Cauchy, and by Proposition 2.14, converges in $\mathbb{H}_m(Z)$ to, say, $\mathbf{K}^Z$. Specifying this for odd integers, we get in the obvious notation,

$$\lim_{n\to\infty} d^Z(\phi_{2n-1}(\rho), \rho^Z) = 0 \quad , \quad \lim_{n\to\infty} \mathrm{d}^Z_{\mathrm{Hyp}}(g \circ \phi^{-1}_{2n-1}, g^Z) = 0 \quad , \quad \lim_{n\to\infty} \mathrm{d}^Z_{\mathrm{Prok}}(\nu \circ \phi^{-1}_{2n-1}, \nu^Z) = 0.$$

Recall now from the proof of Proposition 2.14 that the convergence of usc functions for the hypograph distance $\mathrm{d}^Z_{\mathrm{Hyp}}$ entails the convergence of the domains for the Hausdorff distance, so

$$\lim_{n\to\infty} \mathrm{d}^Z_{\mathrm{Haus}}(\phi_{2n-1}(K), K^Z) = 0.$$

This implies that for any $x \in K$, the sequence $(\phi_{2n-1}(x))_{n\geqslant 0}$ is relatively compact in Z. On the other hand, the sequence of isometries $(\phi_{2n-1})_{n\geqslant 0}$ is of course equicontinuous. These observations enable us to apply the Arzelà-Ascoli theorem, and we infer that there is a strictly increasing sequence of odd integers $(n_k)_{k\geqslant 1}$ such that $(\phi_{n_k})_{k\geqslant 1}$ converges uniformly to an isometric embedding $\phi : K \hookrightarrow Z$. It is now immediate to check that $K^Z = \phi(K)$, $\rho^Z = \phi(\rho)$, $g^Z = g \circ \phi^{-1}$, and $\nu^Z = \nu \circ \phi^{-1}$, so $\mathbf{K}$ and $\mathbf{K}^Z$ belong to the same equivalence class in $\mathbb{H}_m$. The same argument shows that $\mathbf{K}'$ and $\mathbf{K}^Z$ also belong to the same equivalence class in $\mathbb{H}_m$, and establish positivity.

Finally, we check that the space $(\mathbb{H}_m, \mathrm{d}_{\mathbb{H}_m})$ is Polish. Separability should be plain since the set – of isometry classes – of measured decorated rooted compact spaces with a finite cardinality and associated distances, measures and usc functions taking only rational values is dense in $(\mathbb{H}_m, \mathrm{d}_{\mathbb{H}_m})$. Completeness is also immediate from Lemma 2.16 and Proposition 2.14. Indeed, if $(\mathbf{K}_n)_{n\geqslant 1}$ is a sequence of measured decorated compact spaces such that:

$$\lim_{n\to\infty} \sup_{\ell\geqslant 1} \mathrm{d}_{\mathbb{H}_m}(\mathbf{K}_n, \mathbf{K}_{n+\ell}) = 0,$$

then Lemma 2.16 enables us to embed $(\mathbf{K}_n)_{n\geqslant 1}$ into a Cauchy sequence $(\mathbf{K}^Z_n)_{n\geqslant 1}$ in $(\mathbb{H}_m(Z), \mathrm{d}_{\mathbb{H}_m(Z)})$ for some Polish space (Z, d^Z). We know from Proposition 2.14 that the latter converges, say to some $\mathbf{K}^Z \in \mathbb{H}_m(Z)$. We now see from the definition (2.13) that the equivalent class of $\mathbf{K}_n$ converges to the equivalent class of $\mathbf{K}$ in $\mathbb{H}_m$ as $n \to \infty$. □

Finally, we turn back our attention to measured decorated real trees; recall Definition 2.1.

Corollary 2.17. *The set $\mathbb{T}_m$ of equivalence (up to isomorphisms) classes of decorated compact real trees is closed in $(\mathbb{H}_m, \mathrm{d}_{\mathbb{H}_m})$.*

Proof. Let $(\mathbf{T}_n)_{n\geqslant 1}$ be a sequence of decorated compact real trees converging to some $\mathbf{T}$ in $\mathbb{H}_m$. We already know that $\mathbf{T}$ must be a decorated compact space. We just need to check that the metric space associated to the latter is a real tree, which is immediate, using e.g. the four point condition. See also [70, Theorem 1]. □

In the remainder of this work, we write $\mathrm{d}_{\mathbb{T}_m}$ for the restriction of $\mathrm{d}_{\mathbb{H}_m}$ to $\mathbb{T}_m$ and equip $\mathbb{T}_m$ with the Borel sigma-field. Let us mention that the class of decorated trees, with a trivial measure, can be viewed as a closed subclass of $(\mathbb{T}_m, \mathrm{d}_{\mathbb{T}_m})$. This allow us to identify the set $\mathbb{T}$ of equivalence of (non-measured) decorated trees with the closed subset of $\mathbb{T}_m$ of equivalence classes equipped with the null measure. We denote by $\mathrm{d}_{\mathbb{T}}$ the induced distance so that $(\mathbb{T}, \mathrm{d}_{\mathbb{T}})$ is also a Polish space which we similarly equip with the Borel sigma-field. We will often abuse terminology and refer to a decorated real tree $\mathtt{T} = (T, d_T, \rho, g)$ or a measured decorated tree $\mathbf{T} = (T, d_T, \rho, g, \nu)$ instead of their equivalence class, implicitly identifying a decorated real tree and its equivalence class. When doing so, one must consider only notions that are invariant under isomorphisms, such as weighted length measures (if $\varpi \circ g \in L^1(\lambda_T)$). Other examples of notions well-defined in $\mathbb{T}_m$ include the functions that associate to a decorated real tree its diameter, the maximal value of the upper semi-continuous decoration, and its total mass. These functions are all continuous with respect to $\mathrm{d}_{\mathbb{T}_m}$ and are particularly well-defined on $\mathbb{T}_m$. We also stress that by definition the map from $(\mathbb{T}_m, \mathrm{d}_{\mathbb{T}_m})$ to $(\mathbb{T}, \mathrm{d}_{\mathbb{T}})$, defined by $(T, d_T, \rho, g, \nu) \mapsto (T, d_T, \rho, g)$, is continuous. It will also be convenient to write $\mathtt{0}$ (resp. $\mathbf{0}$) for the element of $\mathbb{T}$ (resp. $\mathbb{T}_m$) corresponding to a degenerate real tree reduced to a singleton with zero decoration (and zero measure).

In the sequel, we will sometimes have to consider equivalence classes of measured decorated real trees with marks. A minor difficulty however is that marking a decorated real tree is not unambiguously defined for equivalence classes, and as a remedy, we need to work with an extension of $\mathbb{T}_m$ and $\mathbb{T}$ for marked (measured) decorated real trees. This extension is straightforward and let us present the measured decorated case.

Let $\mathbf{T} = (T, d_T, \rho, g, \nu)$ be a fixed measured decorated tree and $(x_i)_{i\in I}$ a family of points in T, where I is a finite or countable set of indices. We then say that two pairs $(\mathbf{T}, (x_i)_{i\in I})$ and $(\mathbf{T}', (x_i')_{i\in I})$ are equivalent and then write $\big(\mathbf{T}, (x_i)_{i\in I}\big) \approx \big(\mathbf{T}, (x_i')_{i\in I}\big)$, if there exists an isometric bijection $\varphi : T \to T'$ which induces an isomorphism between $\mathbf{T}$ and $\mathbf{T}'$, such that furthermore $\varphi(x_i) = x_i'$ for all $i \in I$. We denote the set of such equivalence classes by $\mathbb{T}_m^{I\bullet}$, and simply $\mathbb{T}_m^{\bullet}$ when $I = \{1\}$. In order to extend the distance $\mathrm{d}_{\mathbb{T}_m}$ to the set $\mathbb{T}_m^{I\bullet}$ of equivalence classes of decorated real trees with marks, we fix some null family $(a_i)_{i\in I}$ of positive real numbers. First, when $(\mathbf{T}, (x_i)_{i\in I})$ and $(\mathbf{T}', (x_i')_{i\in I})$ are two marked decorated real spaces in some Polish space (Y, d_Y), we set first

$$\mathrm{d}_{\mathbb{H}_m^{I\bullet}(Y)}\big((\mathbf{T}, (x_i)_{i\in I}), (\mathbf{T}', (x_i')_{i\in I})\big) := \mathrm{d}_{\mathbb{H}_m(Y)}\big(\mathbf{T}, \mathbf{T}'\big) \vee \Big(\sup_{i\in I}\Big(\mathrm{d}_Y\big(x_i, x_i'\big) \wedge a_i\Big)\Big),$$

Then, in the general case, one defines

$$\begin{aligned}&\mathrm{d}_{\mathbb{T}_m^{I\bullet}}\big((\mathbf{T},(x_i)_{i\in I}),(\mathbf{T}',(x_i')_{i\in I})\big)\\&:=\inf_{\substack{\varphi:T\hookrightarrow Y\\ \varphi':T'\hookrightarrow Y}}\mathrm{d}_{\mathbb{H}_m(Y)}\big(\varphi(\mathbf{T}),\varphi'(\mathbf{T}')\big)\vee\Big(\sup_{i\in I}\Big(\mathrm{d}_Y\big(\varphi(x_i),\varphi'(x_i)\big)\wedge a_i\Big)\Big),\end{aligned}\tag{2.15}$$

where the infimum is over all the Polish spaces (Y, d_Y) and all the isometric embeddings $\varphi : T \hookrightarrow Y$ and $\varphi' : T' \hookrightarrow Y$.

By definition, $\mathrm{d}_{\mathbb{T}_m^{I\bullet}}$ induces a well defined function from $\mathbb{T}_m^{I\bullet} \times \mathbb{T}_m^{I\bullet}$ to $\mathbb{R}_+$, which we still denote by $\mathrm{d}_{\mathbb{T}_m^{I\bullet}}$ by a slight abuse of notation. It is straightforward to check that the space $(\mathbb{T}_m^{I\bullet}, \mathrm{d}_{\mathbb{T}_m^{I\bullet}})$ is Polish, and we stress that the resulting topology on $\mathbb{T}_m^{I\bullet}$ does not depend on the specific choice of $(a_i)_{i\in I}$. Specifically, Lemma 2.16 can be extended to this context using the exact same proof, and then the proof of Theorem 2.15 can easily be adapted; we leave the details to the reader. Just as in the unmarked case, we write $\mathbb{T}^{I\bullet}$ for the set of equivalence classes of marked, non-measured, decorated compact trees, and when no confusion is possible, we identify a marked (measured) decorated compact tree with its equivalence class in $\mathbb{T}_m^{I\bullet}$ or $\mathbb{T}^{I\bullet}$ and we use the notation $(\mathbf{T}, (x_i)_{i\in I})$ and $(\mathtt{T}, (x_i)_{i\in I})$. Later on, when we will deal with random decorated trees, we shall always implicitly work on the canonical space $\mathbb{T}, \mathbb{T}_m, \mathbb{T}^{I\bullet}$ or $\mathbb{T}_m^{I\bullet}$ equipped with their Borel sigma-fields, which will be endowed with different laws.

As an important example, we point out that the gluing operator defined in Section 2.1, which uses marks to specify the locations where gluing takes place on the base tree, can be made compatible with isomorphisms. Here is a formal statement.

Lemma 2.18. *Let $\mathtt{T}'$ be a decorated real tree with marks $(x_i)_{i\in I}$ and $(\mathtt{T}_i)_{i\in I}$ a family of decorated real trees such that the domains of $(\mathtt{T}_i)_{i\in I}$ are pairwise disjoint and also disjoint from the one of $\mathtt{T}'$. Assume that (2.2) and (2.3) hold.*

Let also $\check{\mathtt{T}}'$ be another decorated real tree with marks $(\check{x}_i)_{i\in I}$ and $(\check{\mathtt{T}}_i)_{i\in I}$ another family of decorated real trees such that the the domains of $(\check{\mathtt{T}}_i)_{i\in I}$ are pairwise disjoint and also disjoint from the one of $\check{\mathtt{T}}'$. Suppose that

$$\big(\mathtt{T}',(x_i)_{i\in I}\big)\approx\big(\check{\mathtt{T}}',(\check{x}_i)_{i\in I}\big)\quad\textit{and}\quad\mathtt{T}_i\approx\check{\mathtt{T}}_i\quad\textit{for all } i\in I.$$

Then (2.2) and (2.3) also hold for the second family of decorated real trees, and we have

$$\mathrm{Gluing}\Big(\big(\mathtt{T}',(x_i)_{i\in I}\big),\big(\mathtt{T}_i\big)_{i\in I}\Big)\approx\mathrm{Gluing}\Big(\big(\check{\mathtt{T}}',(\check{x}_i)_{i\in I}\big),\big(\check{\mathtt{T}}_i\big)_{i\in I}\Big).$$

Proof. With the obvious notation, let $\varphi' : T' \to \check{T}'$ and $\varphi_i : T_i \to \check{T}_i$, for $i \in I$, denote bijective isomorphisms underlying the assumptions of the Lemma. By gluing these isomorphisms at the marks $(x_i)_{i\in I}$ in an obvious way, we can construct a function φ from the domain of $\mathrm{Gluing}\big((\mathtt{T}',(x_i)_{i\in I}),(\mathtt{T}_i)_{i\in I}\big)$ to the one of $\mathrm{Gluing}\big((\check{\mathtt{T}}',(\check{x}_i)_{i\in I}),(\check{\mathtt{T}}_i)_{i\in I}\big)$. One readily checks that φ is in turn a bijective isomorphism for the glued decorated real trees. □

As a consequence, we can henceforth view the gluing operator as a map from a sub-domain of $\mathbb{T}^{I\bullet} \times (\mathbb{T})^I$ to $\mathbb{T}$; and as usual we keep the same notation for the latter as for the former. By convention, we can extend the definition when conditions (2.2) and (2.3) are not fulfilled by simply setting $\mathrm{Gluing}\big((\mathtt{T}', (x_i)_{i\in I}), (\mathtt{T}_i)_{i\in I}\big) = \mathtt{0}$.

2.5 Comments and bibliographical notes

The formalism presented here is chiefly inspired by the recursive construction of continuum random trees performed by Rembart and Winkel [124] in terms of so-called strings of beads and by the general gluing of metric spaces along points performed by Senizergues, see [130, Section 2]. Note in particular that gluing of real trees along points is a folklore operation in the literature on random real trees, see e.g. [4, Section 2.4], which can at least be traced back to Aldous [9] and the famous stick breaking construction of the Brownian Continuum Random Tree.

Let us conclude with a discussion, for readers familiar with "Gromov-type" topologies, on possible alternative topologies for decorated trees, and explain our choice. It is possible to define a complete separable topology on the set of compact metric spaces endowed with a continuous function taking values in a fixed Polish space and with a controlled regularity (e.g. a Lipschitz condition), see [12, Section 3] and especially Remark 3.2 there. However, since we aim to consider functions that are typically discontinuous on trees, finding an adaptation "à la Skorokhod" for trees decorated with rcll functions in the spirit of [86] or [120] seemed complicated. Another approach inspired by the Brownian and Lévy snake constructions of Duquesne & Le Gall [65], involves viewing the label of a point x of a "decorated" tree as a rcll path:

$$\zeta_x : [0, d_T(\rho, x)) \to \mathbb{R},$$

representing the entire history of the "decoration" from the root of T to x. By doing so, the labeling becomes Lipschitz over the real tree (for the appropriate topology of space of paths), allowing us to define a Gromov-type topology for such structures. The resulting "snake" topology[7] differs from the topology discussed in this chapter: snake convergence roughly corresponds to the convergence of the (measured) rooted tree in the Gromov-Hausdorff-Prokhorov sense, combined with a Skorokhod convergence of the decorations along the macroscopic branches, but it does not cover the decorations near the leaves. Conversely, our hypograph convergence uniformly controls the supremum of our function over any non trivial interval of the tree, but the topology is weaker "in the interior of branches", see Remark 2.13. We chose the hypograph-type topology because, besides the fact that functions are defined on leaves, it also works well with the gluing operation and is particularly suited for studying Markov properties and spinal decompositions in the context of decorated trees, see Chapters 5 and 6.

[7] Snake trajectories [1, 65, 126] encode more structure than just the tree and the labels. Roughly speaking, they also encode a canonical contour function of the underlying tree. The analogy with snake trajectories here is that we consider the entire ancestral path as a label. This is also reminiscent of the notion of historical processes in the theory of superprocesses [68].

Chapter 3

Branching processes with real types

Motivated by the evolution in continuous time of a population of individuals, Jagers [83] introduced general branching processes as Markov random fields indexed by the Ulam tree. Roughly speaking, each individual is labeled according to its ancestral lineage, and receives at birth a type in some abstract space which determines the statistics of a so-called life career. The lifetime, the reproduction process and further traits of an individual (which typically may evolve with the age of the individual) are all viewed as measurable functions of its life career. The branching property requests that conditionally on their types, the life careers of individuals at generation n are independent, and also independent of the life careers of individuals from the previous generations.

For the applications we have in mind, here types are positive real numbers[1]. Any individual may beget infinitely many children[2], but only finitely many with types greater than ε for every $\varepsilon > 0$. We apply results of the preceding chapter to construct, under fairly general assumptions, a random real tree that encodes the evolution of such a branching process, and such that lengths of branches correspond to time durations. We decorate the latter with a nonnegative usc function to represent some trait of every individual which may vary with the age. We further endow the resulting tree with different weighted length measures, and, assuming further the existence of a harmonic function on the space of types, we also define a natural Borel measure carried by the set of leaves of the genealogical tree.

The framework is made simpler when one further requests self-similarity and the Markov property in time. More precisely, it is well-known feature that general branching processes are Markovian when viewed as processes indexed by generations; however the Markov property in the time variable fails, except in the very special case when lifetimes have an exponential

[1]In Chapter 6, we shall also consider a bit more generally models with distinguished individuals, so that formally the type of an individual has then two components: a positive real number together with a distinction or an absence of distinction.

[2]This situation is usually excluded in the setting of general branching processes, as it would be awkward from a biological point of view. It is nonetheless relevant when populations and types have rather a geometric interpretation as we shall see later on.

distribution and reproduction processes are homogeneous Poisson processes. Nevertheless, the fact that we do not only consider the evolution of a branching population with types, but also endow individuals with a random decoration, will enable us to retrieve the Markov property in the time variable for a large family of self-similar models. Self-similar Markov branching processes are introduced in Section 3.3 and will be shown to arise as scaling limits of a variety of discrete Markov branching models in Part II. The construction relies on the so-called Lamperti transformation, which is classically applied to connect real Lévy processes to positive self-similar Markov processes, and we shall also provide some necessary background in this setting.

3.1 General branching processes as random decorated real trees

In this section, we shall apply general results from the preceding chapter and define the decorated real tree that depicts a general branching process endowed with some random decoration for individuals. In this direction, we will have to specify the distribution of the building blocks, and to start with, we recall more precisely how a general branching process with types in $(0, \infty)$ can be constructed.

We call **decoration-reproduction process** a pair (f, η) with $f : [0, z] \to \mathbb{R}_+$ a random rcll function on a random interval $[0, z]$, and $\eta = \eta(\mathrm{d}t, \mathrm{d}y)$ a point process on $(0, z] \times (0, \infty)$. Strictly speaking, we mean by this that η is a point process on $\mathbb{R}_+ \times (0, \infty)$ such that

$$\eta(\{0\} \times (0, \infty)) = 0 \quad \text{and} \quad \eta((z, \infty) \times (0, \infty)) = 0, \qquad \text{almost surely.} \tag{3.1}$$

See Figure 3.1 below. We refer to f, respectively to η, as the decoration process, respectively the reproduction process. We should think of z as the lifetime of an individual, of $f(t)$ as some trait of this individual at age t, and of η as a reproduction process, in the sense that each atom of η, say (t, y), is interpreted as a birth event of a child of type y occurring when the individual has reached age t (notice that an individual can produce multiple offspring at the same time). The requirement (3.1) means that the individual cannot beget at birth nor after its death which is a natural restriction (however it may happen that an individual produces offsprings at the exact time when it dies). Needless to say, decoration and reproduction are generally not independent.

We then call a **decoration-reproduction kernel** a family of probability laws $(P_x)_{x>0}$, where for every $x > 0$, the distribution P_x is the law of a random decoration-reproduction process (f, η) for an individual with type x. We shall implicitly assume that map $x \mapsto P_x$ is a measurable[3] function of the type x and we use the notation E_x for the mathematical expectation under P_x. We always assume that for every $\varepsilon > 0$, any individual has finitely many children with type greater than ε, almost surely for P_x for all $x > 0$. In other words, the types of the progeny of an individual always form a null family, even though the total progeny may be infinite.

We now describe formally the construction of a general branching process with a given decoration-reproduction kernel, using the Ulam tree $\mathbb{U}$ to encode the genealogy of individuals.

[3] This requirement will be automatically satisfied in the self-similar case.

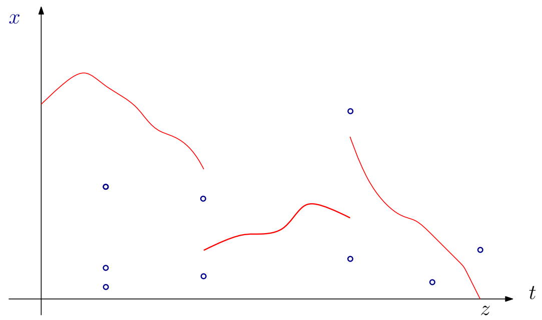

Figure 3.1: Decoration and reproduction point process (f, η). The centers of the small circles represent the locations of the atoms of η; observe that several atoms may share the same time-coordinate, and also that the time-coordinates of these atoms may or not be the time of a jump of the decoration. In the above case, there is a reproduction at the exact time of death of the individual.

Let us first explain how to assign to every individual a type, a reproduction process, and a random decoration. By assigning types to individuals, we mean a random process $(\chi(u))_{u\in\mathbb{U}}$ on the Ulam tree, so that $\chi(u)$ is the type of the individual labeled by u. In this direction, it is convenient to add 0 to the space of types; the type 0 will be assigned to fictitious individuals, that do not appear in the branching process, but nonetheless have to be represented by some vertex $u \in \mathbb{U}$ for definitiveness.

Given the type of the ancestor, say $\chi(\varnothing) = x > 0$, we pick a pair $(f_\varnothing, \eta_\varnothing)$ with law P_x as above. We enumerate the atoms $(t_1, y_1), (t_2, y_2), \ldots$ of $\eta_\varnothing$ using some deterministic rule, for instance the co-lexicographical order[4]. In the case when $\eta_\varnothing$ has only finitely many atoms, we complete with fictitious individuals of the form $(\dagger, 0)$ to get an infinite sequence. We then set $\chi(j) = y_j$ for every individual $j \in \mathbb{N}$ at the first generation. Once the types of the individuals (fictitious or not) at the first generation have been assigned, we iterate the construction using the branching property. That is, conditionally on $(f_\varnothing, \eta_\varnothing)$, we consider a sequence $(f_1, \eta_1), (f_2, \eta_2), \ldots$ of independent pairs distributed according to $P_{y_1}, P_{y_2}, \ldots$, where $y_j = \chi(j)$, and, for definitiveness, P_0 denotes the law of the trivial pair (f, η) corresponding to $z = 0$, $f(0) = 0$ and $\eta = 0$. The construction by iteration for the next generations should now be obvious, using independent decoration-reproduction processes for different individuals. We stress that the type $\chi(u)$ of an individual any generation $|u| \geqslant 1$ is determined by the reproduction process η_{u-} of its parent $u-$.

We write $\mathbb{P}_x$ for the probability law of the family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$ which results when the ancestor $\varnothing$ has the type $x > 0$; then we naturally write also $\mathbb{E}_x$ for the mathematical expectation under $\mathbb{P}_x$. Let us spell out in this setting what we shall refer to as the **branching property** in the sequel. For any initial type $x > 0$ and for every

[4] That is in non-increasing order of the types and ties are broken in non-increasing of the times.

$n \geqslant 1$, the families $(f_u, \eta_u)_{|u|<n}$ and $(f_v, \eta_v)_{|v|\geqslant n}$ are conditionally independent under the law $\mathbb{P}_x$ given the family $(\chi(w))_{|w|=n}$ of the types at generation n. Specifically, the conditional law of each subfamily $(f_{wv}, \eta_{wv})_{v\in\mathbb{U}}$ for a vertex w at generation n is $\mathbb{P}_{\chi(w)}$, and to different vertices at generation n correspond conditionally independent subfamilies. The proof of this branching property is immediate from the construction by a recursive argument.

Every vertex $u \in \mathbb{U}$ has now been assigned not only a type $\chi(u)$, but also a decoration $f_u : [0, z_u] \to \mathbb{R}_+$, and a reproduction process η_u. The latter encodes the age t_{uj} of the individual labeled by u at which its j-th child is born, and the type $y_{uj} = \chi(uj)$ of this child, for every $j \in \mathbb{N}$. We have therefore the building blocks needed for the gluing construction of Section 2.2, namely the families $(f_u)_{u\in\mathbb{U}}$ and $(t_u)_{u\in\mathbb{U}^*}$:

We stress that even though the decoration and reproduction processes above have been defined as random variables under $\mathbb{P}_x$, the setting still makes sense more generally for arbitrary (deterministic) family $(f_u, \eta_u)_{u\in\mathbb{U}}$ of decoration-reproduction pairs and we can always consider their associated building blocks $(f_u)_{u\in\mathbb{U}}$ and $(t_{ui})_{ui\in\mathbb{U}^*}$.

Definition 3.1 (Property $(\mathcal{P})$). *We say that such a family*

$$(f_u, \eta_u)_{u\in\mathbb{U}}$$

verifies Property $(\mathcal{P})$ if the associated building blocks $(f_u)_{u\in\mathbb{U}}$ and $(t_{ui})_{ui\in\mathbb{U}^*}$ satisfy the requirements (2.4) and (2.6).

When Property $(\mathcal{P})$ is satisfied, we write generically $\mathtt{T} = (T, d_T, \rho, g)$ for the decorated real tree which then stems from an application of Theorem 2.5. For the sake of notational simplicity, we still write $\mathtt{T}$ for the respective equivalence class up to isomorphisms (see Section 2.4), which enables us to view the latter as random variables with values in the Polish space $\mathbb{T}$ of equivalence classes of (non-measured) decorated compact trees; see Corollary 2.17 and the discussion below it. In this direction, we point out that the function which maps a family $(f_u, \eta_u)_{u\in\mathbb{U}}$ satisfying Property $(\mathcal{P})$ into a decorated real tree $\mathtt{T} = (T, d_T, \rho, g)$ is measurable when the set of such families indexed by $\mathbb{U}$ is equipped with the cylindrical sigma-algebra.[5] It may be tempting to think of $\mathtt{T}$ as the genealogical tree of the general branching process, and then the decoration g encodes some trait of individuals which may evolve with age. We stress however that if $\mathtt{T}'$ is another random decorated real tree which is a.s. isomorphic to $\mathtt{T}$, then one cannot fully recover the general branching process from $\mathtt{T}'$ as the precise genealogy of individuals may have been lost[6]. Nonetheless $\mathtt{T}'$ does keep track of evolution as time passes of the point process that records the values of traits of individuals in the population at any given time, which is sufficient for many applications. In general, we use the notation $\mathbb{P}$ to refer to distributions on families

[5]In short, measurability is straightforward when we restrict our attention to families having only finitely many non-fictitious elements, and the general setting follows by approximation, we leave details to scrupulous readers familiar with Gromov-Hausdorff-Prokhorov topologies.

[6]Recall the discussion around Figure 2.3.

of decoration-reproduction processes, and we use $\mathbb{Q}$ instead for distributions on the space of equivalence relations of decorated trees $\mathbb{T}$.

Our goal now is to introduce simple assumptions on the decoration-reproduction kernel $(P_x)_{x>0}$ to ensure that Property $(\mathcal{P})$ is verified $\mathbb{P}_x$-a.s., for every $x > 0$. These assumptions will become even more transparent in the case when the kernel $(P_x)_{x>0}$ is self-similar, as we will see at the end of the section.

In this direction, we consider first the total intensity of children of given types which an individual of type x begets, i.e. the measure $\imath_x$ on $(0, \infty)$ defined by

$$\imath_x(B) := E_x\left(\eta([0, z] \times B)\right), \qquad B \in \mathcal{B}((0, \infty)).$$

We shall henceforth suppose the existence of a function $\phi : [0, \infty) \to \mathbb{R}_+$ with $\phi(0) = 0$ and a positive constant $c_\imath < 1$, such that for all $x > 0$:

$$\int_{(0,\infty)} \phi(y)\imath_x(\mathrm{d}y) \leqslant c_\imath \phi(x). \tag{3.2}$$

The function ϕ will be referred to as a **strictly excessive function**. We shall also suppose that there exist $\gamma_0, \alpha > 0$ and finite constants c_z, c_f such that:

$$\left.\begin{aligned} E_x(z^{\gamma_0/\alpha}) &\leqslant c_z\phi(x), \\ E_x(\sup f^{\gamma_0}) &\leqslant c_f\phi(x), \end{aligned}\right\} \tag{3.3}$$

for all $x > 0$. The role of the first assumption (3.2) is enlightened by the following elementary result.

Lemma 3.2. *Assuming* (3.2)*, we have for every $x > 0$*

$$\mathbb{E}_x\left(\sum_{u\in\mathbb{U}} \phi(\chi(u))\right) \leqslant \phi(x)/(1 - c_\imath).$$

As a consequence, the family $(\phi(\chi(u)))_{u\in\mathbb{U}}$ is null, $\mathbb{P}_x$-a.s.

Proof. By the definition of the intensity measure $\imath_x$, there is the identity

$$\mathbb{E}_x\left(\sum_{j=1}^{\infty} \phi(\chi(j))\right) = \int_{(0,\infty)} \phi(y)\imath_x(\mathrm{d}y).$$

We deduce from the branching property and (3.2) that for $n \geqslant 1$ we have

$$\mathbb{E}_x\left(\sum_{|u|=n} \phi(\chi(u))\right) \leqslant c_\imath^n \phi(x), \tag{3.4}$$

and since $c_\imath < 1$, the first claim follows. In particular, the family $(\phi(\chi(u)))_{u\in\mathbb{U}}$ is summable and *a fortiori* null, $\mathbb{P}_x$-a.s. □

We are now able to state the main result of this section:

Theorem 3.3. *Suppose that* (3.3) *is fulfilled for some strictly excessive function* ϕ *(i.e.* ϕ *verifies* (3.2)*) and some exponents* $\alpha, \gamma_0 > 0$*. Then Property* $(\mathcal{P})$ *is satisfied by the family* $(f_u, \eta_u)_{u\in\mathbb{U}}$*,* $\mathbb{P}_x$*-a.s. for every* $x > 0$*.*

Proof. Fix $x > 0$ and recall the notation from Section 2.2, and notably that $z_u \geqslant 0$ is the length of the interval on which f_u is defined. As a warm-up, we observe from the branching property, (3.2), the first bound in (3.3), and Lemma 3.2, that

$$\mathbb{E}_x\left(\sum_{u\in\mathbb{U}} z_u^{\gamma_0/\alpha}\right) \leqslant c_z \mathbb{E}_x\left(\sum_{u\in\mathbb{U}} \phi(\chi(u))\right) \leqslant \frac{c_z}{1-c_i}\phi(x) < \infty. \tag{3.5}$$

In particular,

$$\sum_{u\in\mathbb{U}} z_u^{\gamma_0/\alpha} < \infty, \qquad \mathbb{P}_x\text{-a.s.},$$

and *a fortiori* $(z_u)_{u\in\mathbb{U}}$ is a null family, $\mathbb{P}_x$-a.s.

We check similarly that (2.7) (recall that this is a stronger requirement than (2.6)) holds $\mathbb{P}_x$-a.s. We distinguish two cases, depending on whether $\gamma_0 < \alpha$ or $\gamma_0 \geqslant \alpha$. In the first case where $\gamma_0 < \alpha$, we simply write

$$\left(\sum_{n=0}^{\infty} \sup_{|u|=n} z_u\right)^{\gamma_0/\alpha} \leqslant \sum_{n=0}^{\infty} \sup_{|u|=n} z_u^{\gamma_0/\alpha} \leqslant \sum_{u\in\mathbb{U}} z_u^{\gamma_0/\alpha}.$$

We then take the expectation and invoke (3.5) to see that (2.7) holds $\mathbb{P}_x$-a.s.

In the second case where $\gamma_0 \geqslant \alpha$, we write from Minkovski's inequality

$$\mathbb{E}_x\left(\left(\sum_{n=0}^{\infty} \sup_{|u|=n} z_u\right)^{\gamma_0/\alpha}\right)^{\alpha/\gamma_0} \leqslant \sum_{n=0}^{\infty} \mathbb{E}_x\left(\sup_{|u|=n} z_u^{\gamma_0/\alpha}\right)^{\alpha/\gamma_0}.$$

and then, from (3.3) and the branching property

$$\mathbb{E}_x\left(\sup_{|u|=n} z_u^{\gamma_0/\alpha}\right) \leqslant \sum_{|u|=n} \mathbb{E}_x(z_u^{\gamma_0/\alpha}) \leqslant c_z \sum_{|u|=n} \mathbb{E}_x\left(\phi(\chi(u))\right).$$

Thanks to (3.4), we can bound the right-hand side by $c_z c_i^n \phi(x)$ and infer

$$\mathbb{E}_x\left(\left(\sum_{n=0}^{\infty} \sup_{|u|=n} z_u\right)^{\gamma_0/\alpha}\right) \leqslant \left(\sum_{n=0}^{\infty} c_z^{\alpha/\gamma_0} c_i^{n\alpha/\gamma_0} \phi(x)^{\alpha/\gamma_0}\right)^{\gamma_0/\alpha} \leqslant c_z\phi(x)(1-c_i^{\alpha/\gamma_0})^{-\gamma_0/\alpha},$$

and again (2.7) holds $\mathbb{P}_x$-a.s.

We check likewise that the requirement (2.4) holds $\mathbb{P}_x$-a.s. Namely, we deduce from the branching property, the second bound in (3.3), and Lemma 3.2, that

$$\sum_{u\in\mathbb{U}} \mathbb{E}_x(\sup f_u^{\gamma_0}) \leqslant c_f \sum_{u\in\mathbb{U}} \mathbb{E}_x\left(\phi(\chi(u))\right) \leqslant \frac{c_z}{1-c_i}\phi(x) < \infty, \tag{3.6}$$

This ensures that $(\sup f_u)_{u\in\mathbb{U}}$ is a null family, $\mathbb{P}_x$-a.s. Since we have already observed that $(z_u)_{u\in\mathbb{U}}$ is a null family, $\mathbb{P}_x$-a.s., the same holds for $(\|f_u\|)_{u\in\mathbb{U}}$ as well. □

When the conditions of Theorem 3.3 are fulfilled, we can consider the random decorated real tree $\mathtt{T} = (T, d_T, \rho, g)$ which then stems from an application of Theorem 2.5. The above proof also shows:

Corollary 3.4. *Under the assumptions of Theorem 3.3, the random variables*

$$\mathrm{Height}(T)^{\gamma_0/\alpha} \quad \textit{and} \quad \max_T g^{\gamma_0}$$

belong to $L^1(\mathbb{P}_x)$.

The assumptions of Theorem 3.3 are more transparent when the decoration-reproduction kernel $(P_x)_{x>0}$ satisfies a scaling property. Fix some $\alpha > 0$. We start defining, for any $c > 0$ and a (rcll) function $f : [0, z] \to \mathbb{R}_+$, the rescaled function

$$f^{(c)} : [0, c^\alpha z] \to \mathbb{R}_+, \quad f^{(c)}(t) \coloneqq cf(c^{-\alpha}t),$$

and also, for any measure η on $[0, z] \times (0, \infty)$, the rescaled measure $\eta^{(c)}$ given by the push forward image of η by the map

$$[0, z] \times (0, \infty) \to [0, c^\alpha z] \times (0, \infty)\ ,\ (t, y) \mapsto (c^\alpha t, cy).$$

Definition 3.5. *We say that a decoration-reproduction kernel $(P_x)_{x>0}$ is **self-similar** with exponent $\alpha > 0$ if for every $x > 0$, the law P_x coincides with the distribution of the rescaled pair $(f^{(x)}, \eta^{(x)})$ under P_1. Then we also say that the general branching process is self-similar.*

The self-similarity of the decoration-reproduction kernel $(P_x)_{x>0}$ enables us to focus on the type $x = 1$. Indeed it is immediate from the self-similarity assumption and the construction of general branching processes that the distribution of the family of rescaled pairs

$$\left(f_u^{(x)}, \eta_u^{(x)}\right)_{u\in\mathbb{U}}$$

under $\mathbb{P}_1$ is the same as that of $(f_u, \eta_u)_{u\in\mathbb{U}}$ under $\mathbb{P}_x$. We write for simplicity $P \coloneqq P_1$ and $\imath \coloneqq \imath_1$ for the total intensity of children of given types beget by an individual of type 1.

We first point out that the self-similarity assumption has a simple and important consequence for the process $(\chi(u))_{u\in\mathbb{U}}$ that assigns types to individuals. For every $x > 0$, the distribution under P_x of the family of the logarithms of types $(\log \chi(j))_{j\geqslant 1}$ of the individuals at the first generation, is identical to the law under P of the same family shifted by $\log x$. Therefore, if we consider the point process on $\mathbb{R}$ induced by the logarithms of types at each generation,

$$\sum_{|u|=n} \delta_{\log \chi(u)}, \qquad n \geqslant 0, \tag{3.7}$$

(implicitly the fictitious individuals with type 0 are discarded in the sum), then we obtain a **branching random walk**; see e.g. [132]. This observation has simple consequences regarding the existence of the strictly excessive function ϕ of (3.2) (and later on of harmonic function) that we now explain.

We introduce first the function

$$\mathcal{M}(\gamma) := \int_{(0,\infty)} y^\gamma \imath(\mathrm{d}y), \quad \gamma \geqslant 0. \tag{3.8}$$

In words, $\gamma \mapsto \mathcal{M}(\gamma - 1)$ is the Mellin transform of the intensity measure $\imath$. Note also that the push forward of $\imath$ by the logarithm function yields the reproduction intensity of the branching random walk (3.7), that is the intensity measure of the point process at the first generation, $\sum_{j=1}^{\infty} \delta_{\log \chi(j)}$. Therefore $\mathcal{M}$ can also be viewed as the moment generating function of the latter. Except in the degenerate case where the reproduction process η is merely a Dirac point mass with a fixed type P-a.s., this function is strictly log-convex and takes its values in $\mathbb{R} \cup \{\infty\}$. By convexity $\log \mathcal{M}$ has at most two zeros. Notice that (3.2) is satisfied for $\phi(y) = y^{\gamma_0}$ as soon as $\mathcal{M}(\gamma_0) < 1$.

In the self-similar case, one can also easily bound from above the Hausdorff dimension of the set of leaves ∂T.

Lemma 3.6 (Upper-bound on the Hausdorff dimension of leaves)**.** *Suppose that the decoration-reproduction kernel $(P_x)_{x>0}$ is self-similar with exponent α. Assume that there exist $\omega < \gamma_0$ with $\mathcal{M}(\omega) = 1$ and $\mathcal{M}(\gamma_0) < 1$, and furthermore that the assumptions of Theorem 3.2 are satisfied with $\phi(y) = y^{\gamma_0}$. If we write $\mathtt{T} = (T, d_T, \rho, g)$ for the resulting random decorated tree, then we have for every $x > 0$ that*

$$\dim_H(\partial T) \leqslant \omega/\alpha, \quad \mathbb{P}_x\text{-a.s.},$$

where $\dim_H(\partial T)$ stands for the Hausdorff dimension of ∂T equipped with the restriction of d_T.

In particular, combining Lemma 3.6 with Lemma 2.8 and Corollary 2.12, we infer that $\dim_H(T) \leqslant \max(1, \omega/\alpha)$ and $\dim_H(\mathrm{Hyp}(g)) \leqslant \max(2, \omega/\alpha)$, $\mathbb{P}_x$-a.s., for every $x > 0$.

Proof. By Lemma 2.8 and self-similarity, it suffices to prove the bound on the Hausdorff dimension of ∂T under $\mathbb{P}_1$. In this direction, recall the notation T^n for the tree obtained by performing the gluing of the first n generation along Ulam's tree. We see T^n as a subset of T. In the proof of Lemma 2.8 we showed that T^n is a countable union of segments and as a consequence of the gluing construction $T^n \cap \partial T$ is included in the union of the extremities of those segments. We deduce that $T^n \cap \partial T$ is a countable set of points and thus $\cup_{n\geqslant 0} T^n \cap \partial T$ has Hausdorff dimension 0. It remains to show that the Hausdorff dimension of $\partial^* T = \partial T \setminus \cup_{n\geqslant 0} T^n$ is bounded above by ω/α. To this end, for every $u \in \mathbb{U}$, consider the subtree T_u obtained by performing the gluing in the subtree above u in Ulam's tree, seen as a subset of T, and write $\mathrm{Diam}(T_u)$ for its diameter in (T, d_T). Then, for every $n \geqslant 0$, the collection $\{T_u : u \in \mathbb{N}^n\}$ is a covering of $\partial^* T$ and note that, for every $u \in \mathbb{N}^n$, we have $\mathrm{Diam}(T_u) \leqslant 2 \cdot \mathrm{Height}(T_u)$. Moreover, for every $u \in \mathbb{N}^n$, the self-similarity of the decoration-reproduction process entails that

$$\mathbb{E}_1\left(\mathrm{Diam}(T_u)^{\gamma/\alpha}\right) \leqslant 2^{\gamma/\alpha} \cdot \mathbb{E}_1\left(\chi(u)^\gamma\right) \cdot \mathbb{E}_1\left(\mathrm{Height}(T)^{\gamma/\alpha}\right),$$

for every $\gamma \in (\omega, \gamma_0)$. Now remark that by Corollary 3.4, the quantity $\mathbb{E}_1\left(\mathrm{Height}(T)^{\gamma/\alpha}\right) \leqslant 1 + \mathbb{E}_1\left(\mathrm{Height}(T)^{\gamma_0/\alpha}\right)$ is finite and by log-convexity of $\mathcal{M}$ we also have $\mathcal{M}(\gamma) < 1$, for every $\gamma \in (\omega, \gamma_0)$. Hence, by the branching property, we infer that

$$\mathbb{E}_1\left(\sum_{u\in\mathbb{N}^n} \mathrm{Diam}(T_u)^{\gamma/\alpha}\right) = 2^{\gamma/\alpha} \cdot \mathcal{M}(\gamma)^n \cdot \mathbb{E}_1\left(\mathrm{Height}(T)^{\gamma/\alpha}\right) \xrightarrow[n\to\infty]{} 0, \qquad \text{for } \gamma \in (\omega, \gamma_0).$$

This proves that the Hausdorff dimension of $\partial^* T$ is bounded above by γ/α, for every $\gamma \in (\omega, \gamma_0)$, and so by ω/α. $\square$

3.2 Lévy, Itô, Lamperti, and self-similar Markov trees

We now proceed with the self-similar Markov case. Our objective is to construct a general branching process endowed with a decoration that satisfies both self-similarity and temporal Markov properties. For this purpose, we first discuss pairs (X, η), where X is a self-similar Markov process started from 1, which we interpret as a decoration-reproduction process in the sense of Section 3.1. We then define the kernel $(P_x)_{x>0}$ through a scaling transformation of the law $P = P_1$ of (X, η); in particular the decoration under P_x has the law of the self-similar Markov process X started from x. Note that the type of an individual now coincides with the initial value of its decoration; this was not necessarily so in the more general setting of Section 3.1. We stress that the decoration-reproduction kernel $(P_x)_{x>0}$ is *de facto* self-similar in the sense of Definition 3.5, and that Markovian aspects will be analyzed in greater details in Chapter 5. The framework that we develop here lies at the heart of the construction of self-similar Markov trees in the next Section 3.3, which constitute one of the primary objects of this work.

Introduce first the space $\mathcal{S}_1$ of non-increasing sequences $\mathbf{y} = (y_1, \ldots)$ in $[-\infty, \infty)$ with $\lim_{n\to\infty} y_n = -\infty$, and then set $\mathcal{S} := [-\infty, \infty) \times \mathcal{S}_1$. Agreeing that $\log 0 = -\infty$ and $\mathrm{e}^{-\infty} = 0$, we view $\log : [0, \infty) \to [-\infty, \infty)$ as a bijection with reciprocal given by the exponential function. Transforming an element $(y, \mathbf{y})$ of $\mathcal{S}$ by the exponential function yields a sequence $(\mathrm{e}^y, \mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \ldots)$ in $[0, \infty)$, whose first term e^y is distinguished and the next ones form a non-increasing sequence that converges to 0. Hence, applying the exponential function to each term of the sequence enables us to endow $\mathcal{S}$ with the distance induced by the supremum norm on the space of real sequences converging to 0. Then $\mathcal{S}$ equipped with this distance is a Polish space.

We can now introduce the following important terminology.

Definition 3.7. *Consider a measure $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$ on $\mathcal{S}$. Write $\Lambda_0 = \Lambda_0(\mathrm{d}y)$ and $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1(\mathrm{d}\mathbf{y})$ for its push-forward images by the first projection $(y, \mathbf{y}) \mapsto y$ from $\mathcal{S}$ to $[-\infty, \infty)$ and by the second projection $(y, \mathbf{y}) \mapsto \mathbf{y}$ from $\mathcal{S}$ to $\mathcal{S}_1$, respectively. We call $\mathbf{\Lambda}$ a **generalized Lévy measure** provided that*

$$\int_{\mathbb{R}} (1 \wedge y^2)\Lambda_0(\mathrm{d}y) < \infty \quad \textit{and} \quad \mathbf{\Lambda}_1\left(\left\{\mathbf{y} \in \mathcal{S}_1 : \mathrm{e}^{y_1} > \varepsilon\right\}\right) < \infty \qquad \textit{for all } \varepsilon > 0.$$

We then further call $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *a* ***characteristic quadruplet****, where* $\alpha > 0$ *is a* ***self-similarity exponent****,* $\sigma^2 \geqslant 0$ *a* ***Gaussian coefficient****,* $\mathrm{a} \in \mathbb{R}$ *a* ***drift coefficient****. We also refer to* $\mathrm{k} := \mathbf{\Lambda}(\{-\infty\} \times \mathcal{S}_1) < \infty$ *as the* ***killing rate****.*

We will next associate a decoration process to a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. We rely for this on the classical Lamperti construction of positive self-similar Markov processes from (possibly killed) real Lévy processes and refer the reader to [91] and Chapter 13 in [88] for a complete account. We introduce a standard Brownian motion B and a Poisson random measure $\mathbf{N} = \mathbf{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}\mathbf{y})$ on $[0, \infty) \times \mathcal{S}$ with intensity measure $\mathrm{d}t\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$. If the killing rate k is strictly positive, then we denote the time coordinate of the first atom of $\mathbf{N}$ belonging to $\{-\infty\} \times \mathcal{S}_1$ by $\zeta < \infty$, so that ζ is then an exponential variable with parameter the killing rate k. If $\mathrm{k} = 0$, then we agree that $\zeta = \infty$. We assume that B and $\mathbf{N}$ are independent.

Let us quickly recall the construction of real Lévy processes by the Lévy-Itô decomposition. Consider the first projection of $\mathbf{N}$ on $[0, \infty) \times \mathbb{R}$ and write $N_0 = N_0(\mathrm{d}t, \mathrm{d}y)$ for the resulting point process. By the mapping theorem for Poisson random measures, N_0 is a Poisson point process with intensity $\mathbf{1}_{y \in \mathbb{R}} \mathrm{d}t \Lambda_0(\mathrm{d}y)$. Additionally, we introduce the compensated Poisson measure

$$N_0^{(c)}(\mathrm{d}s, \mathrm{d}y) := N_0(\mathrm{d}s, \mathrm{d}y) - \mathrm{d}s\Lambda_0(\mathrm{d}y).$$

As a consequence of the conditions fulfilled by $\mathbf{\Lambda}$, the process ξ defined for $0 \leqslant t < \zeta$ by

$$\xi(t) := \sigma B(t) + \mathrm{a}t + \int_{[0,t] \times \mathbb{R}} N_0(\mathrm{d}s, \mathrm{d}y)\ y \mathbf{1}_{|y|>1} + \int_{[0,t] \times \mathbb{R}} N_0^{(c)}(\mathrm{d}s, \mathrm{d}y)\ y \mathbf{1}_{|y| \leqslant 1} \tag{3.9}$$

is a **Lévy process**. Furthermore, in the case $\mathrm{k} > 0$, it will be convenient to declare that $\xi(t) = -\infty$ for $t \geqslant \zeta$, so that $\exp(\gamma \xi(t)) = 0$ whenever $\gamma > 0$ and $t \geqslant \zeta$. In other words, k is the killing rate of the Lévy process ξ. For every $t \geqslant 0$, we have

$$E\big(\exp(\gamma \xi(t))\big) = E\big(\exp(\gamma \xi(t)), t < \zeta\big) = \exp\big(t\psi(\gamma)\big), \tag{3.10}$$

where ψ is known as the **Laplace exponent** of ξ and given by the **Lévy-Khintchine formula**

$$\psi(\gamma) := -\mathrm{k} + \frac{1}{2}\sigma^2\gamma^2 + \mathrm{a}\gamma + \int_{\mathbb{R}^*} \left(\mathrm{e}^{\gamma y} - 1 - \gamma y \mathbf{1}_{|y| \leqslant 1}\right) \Lambda_0(\mathrm{d}y). \tag{3.11}$$

Slightly more generally, one can start the Lévy process from any arbitrary $y \in \mathbb{R}$ by translating the entire process, i.e., by considering $y + \xi$.

We now present the **Lamperti transformation** which allows to construct a (positive) **self-similar Markov process**, for which the acronym pssMp is often used, from a real Lévy process ξ and a positive[7] exponent of self-similarity, $\alpha > 0$. We introduce first the exponential functional[8]

$$\epsilon(t) := \int_0^t \exp(\alpha \xi(s)) \mathrm{d}s \quad \text{for } 0 \leqslant t < \zeta, \quad z := \epsilon(\zeta-) = \int_0^\zeta \exp(\alpha \xi(s)) \mathrm{d}s, \tag{3.12}$$

[7]The case of a negative exponent can then be obtained by applying the simple inverse $x \mapsto 1/x$ transformation to a pssMp.

[8]We shall soon view the random interval $[0, z]$ as the domain of a rcll function; then the notation z has the same interpretation as in the preceding chapters. The reader should therefore not be worried about a possible confusion of notation.

and note that $\epsilon : [0, \zeta) \to [0, z)$ is an increasing bijection a.s. We then define τ as the reciprocal bijection, so that

$$\int_0^{\tau(t)} \exp(\alpha\xi(s))\mathrm{d}s = t, \quad \text{for any } 0 \leqslant t < z. \tag{3.13}$$

Lamperti [91] pointed out that the process

$$X(t) = \exp\big(\xi(\tau(t))\big), \qquad 0 \leqslant t < z, \tag{3.14}$$

obtained from the exponential of the Lévy process by time-substitution based on τ, is both Markovian and self-similar with scaling exponent α (in the literature, one often calls $1/\alpha$ the Hurst exponent). More precisely, X starts from 1, and for every $x > 0$, the rescaled process

$$xX(x^{-\alpha}t), \qquad \text{for } 0 \leqslant t < x^{\alpha}z, \tag{3.15}$$

is a version of X started from x (that is, the underlying Lévy process ξ starts from $\log x$). Conversely, any pssMp with a positive exponent α can be realized from some real Lévy process by this transformation. We stress that the lifetime z of X is finite a.s. if and only if either $\zeta < \infty$ a.s. or the Lévy process drifts to $-\infty$ (i.e. $\zeta = \infty$ a.s. and $\lim_{t\to\infty} \xi(t) = -\infty$ a.s.). More precisely, the boundary point 0 serves as a cemetery state for the self-similar Markov process X, it is reached by a jump when $\zeta < \infty$ (i.e. then $X(z-) > 0$ a.s.) and continuously when ξ drifts to $-\infty$ (i.e. then $X(z-) = 0$ a.s.). By convention we take $X(z) := 0$, when $z < \infty$, to see X as a rcll process on the segment $[0, z]$.

For later use, we also observe that for any $c > 0$, the Lévy process, say $\tilde{\xi}$, constructed from the scaled characteristics $(c^2\sigma^2, c\mathrm{a}, c\boldsymbol{\Lambda})$ has the same law as $(\xi_{ct})_{t\geqslant 0}$. Therefore, if we write $\tilde{X}$ for the pssMp induced by the Lamperti transformation applied to $\tilde{\xi}$, then there is the identity in distribution

$$\big(\tilde{X}_t : t \geqslant 0\big) = \big(X_{ct} : t \geqslant 0\big). \tag{3.16}$$

We now turn our attention to the reproduction process η. For convenience, we use the notation $[0, \zeta] = [0, \infty)$, if $\zeta = \infty$. The reproduction process η is going to be defined using the second projection of $\mathbf{N}$ on $[0, \zeta] \times \mathcal{S}_1$ that we denote by $\mathbf{N}_1 = \mathbf{N}_1(\mathrm{d}t, \mathrm{d}\mathbf{y})$. To this end, we expand each atom of the latter, say $(s, \mathbf{y})$, as a sequence $(s, y_\ell)_{\ell\geqslant 1}$ in $[0, \zeta] \times [-\infty, \infty)$ and, as a first step, introduce a point process on $[0, \zeta] \times \mathbb{R}_+$ by

$$\tilde{\eta} := \sum \mathbf{1}_{\{s\leqslant\zeta\}} \delta_{(s, \exp(\xi(s-)+y_\ell))}, \tag{3.17}$$

where the sum is taken over all the pairs (s, y_ℓ) obtained by developing the atoms $(s, \mathbf{y})$ of $\mathbf{N}_1$, possibly repeated according to their multiplicities, with $\mathbf{y} \neq \{-\infty, -\infty, \dots\}$. In words, $\tilde{\eta}$ has an atom at (s, x) for some $s \leqslant \zeta$ and $x > 0$ if and only if the Poisson random measure $\mathbf{N}_1$ has an atom at $(s, \mathbf{y})$, with $\mathbf{y} \neq \{-\infty, -\infty, \dots\}$, such that $\log x - \xi(s-)$ is a component of $\mathbf{y}$. As a second step, we perform the Lamperti transformation and consider the push-forward of the measure $\tilde{\eta}$ by the Lamperti time-change. Namely, recall that τ is defined by (3.13) as the inverse of the exponential functional ϵ (in particular $\epsilon(\zeta) = z$ is the lifetime of X), and set

$$\eta := \sum \mathbf{1}_{\{\epsilon(s) \leqslant z\}} \delta_{(\epsilon(s), \exp(\xi(s-)+y_\ell))}, \tag{3.18}$$

where the same convention for the summation as above applies. In particular several atoms may occur at the same time.

We now write $P = P_1$ for the law of the pair (X, η) constructed above and recall that $\alpha > 0$ denotes the self–similarity exponent. For every $x > 0$, we denote by P_x the image of P by the scaling transformation $(s, y) \mapsto (x^\alpha s, xy)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, which, by (3.15), transforms X into the version of the self-similar Markov process started from x, and the atoms $\delta_{(s,y)}$ of η into $\delta_{(x^\alpha s, xy)}$. We call $(P_x)_{x>0}$ the **self-similar Markov decoration-reproduction kernel** with characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$; the qualifier self-similar is taken in the sense of Definition 3.5. Note passing by that $\eta(\{0\} \times (0, \infty)) = 0$ and that, in absence of killing, i.e. when $\mathrm{k} = 0$, the reproduction process also satisfies $\eta(\{z\} \times (0, \infty)) = 0$, i.e. that no birth event can happen at the death of an individual.

Our goal now is to associate a decorated compact real tree to every characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ satisfying some additional requirements. In this direction, we start by computing the moment generating function of the branching random walk associated with the self-similar Markov decoration-reproduction kernel $(P_x)_{x>0}$ with characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. Recall that for simplicity, the index for the starting point x is omitted from the notation when $x = 1$, and following (3.8), we write

$$\mathcal{M}(\gamma) = E\left(\int_{[0,z]\times(0,\infty)} y^\gamma \eta(\mathrm{d}t, \mathrm{d}y)\right) = E\left(\sum_{j=1}^{\infty} \chi(j)^\gamma\right) = \int_{(0,\infty)} \imath(\mathrm{d}y) y^\gamma, \qquad \gamma \geqslant 0,$$

for the Mellin transform of the total intensity measure $\imath$ of children of given types. Then, recalling the Lévy-Khintchine formula (3.11) for the Laplace exponent ψ of ξ, we introduce the quantity

$$\begin{aligned}\kappa(\gamma) &:= \psi(\gamma) + \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \Big(\sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i}\Big) \\ &= \frac{1}{2}\sigma^2\gamma^2 + \mathrm{a}\gamma + \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \left(\mathrm{e}^{\gamma y} - 1 - \gamma y \mathbf{1}_{|y|\leqslant 1} + \sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i}\right),\end{aligned} \tag{3.19}$$

where we use again the convention $\mathrm{e}^{-\infty} = 0$, so that the possible killing rate k is incorporated in the integral with respect to $\mathbf{\Lambda}$. The function κ is called the **cumulant function** of $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. We stress that the cumulant does not depend on the exponent of self-similarity α, and that the functions ψ and κ are convex with $\psi \leqslant \kappa$. We will always assume that the cumulant is finite at least at some point $\gamma > 0$ (we will actually soon impose more, see the forthcoming Assumption 3.12). The cumulant function κ enables us to compute the Mellin transform $\mathcal{M}$.

Lemma 3.8. *The Mellin transform $\mathcal{M}$ in (3.8) is given by*

$$\mathcal{M}(\gamma) = 1 - \kappa(\gamma)/\psi(\gamma), \quad \textit{whenever } \psi(\gamma) < 0. \tag{3.20}$$

Proof. Recall that $\imath$ is the measure on $(0,\infty)$ that describes the total intensity of the children with given types that an individual with type 1 begets throughout its life. Note first that the connexion between η and $\tilde{\eta}$ via the Lamperti transformation yields, using the same convention for the summations as in (3.18),

$$\sum \mathbf{1}_{\{\epsilon(s)\leqslant z\}} \left(X(\epsilon(s)-)\exp(y_\ell)\right)^\gamma = \sum \mathbf{1}_{\{s\leqslant\zeta\}} \exp\left(\gamma(\xi(s-)+y_\ell)\right).$$

Next, the construction of $\tilde{\eta}$ in terms of the random measure $\mathbf{N}_1$ and the Lévy process ξ shows that:

$$\sum \mathbf{1}_{\{s\leqslant\zeta\}} \exp(\gamma(\xi(s-)+y_\ell)) = \int_{[0,\zeta]\times\mathcal{S}_1} \exp(\gamma\xi(s-)) \left(\sum_{i=1}^{\infty}\exp(\gamma y_i)\right) \mathbf{N}_1(\mathrm{d}s,\mathrm{d}\mathbf{y}).$$

Taking expectations using (3.10), we get by compensation whenever $\psi(\gamma)<0$ that

$$\begin{aligned}\int_{(0,\infty)} y^\gamma \imath(\mathrm{d}y) &= E\left(\int_0^\zeta \exp(\gamma\xi(s-))\mathrm{d}s\right)\left(\int_{\mathcal{S}_1}\sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i}\mathbf{\Lambda}_1(\mathrm{d}\mathbf{y})\right)\\ &= -\frac{1}{\psi(\gamma)}\left(\int_{\mathcal{S}_1}\sum_{i=1}^{\infty}\mathrm{e}^{\gamma y_i}\mathbf{\Lambda}_1(\mathrm{d}\mathbf{y})\right) = 1-\kappa(\gamma)/\psi(\gamma).\end{aligned}$$

□

We now have all the ingredients to properly define the self-similar Markov trees (for short, ssMt). We fix a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and we write $(P_x)_{x>0}$ for the associated self-similar Markov decoration-reproduction kernel. We also write $\mathbb{P}_x$ for the distribution of the family $(f_u,\eta_u)_{u\in\mathbb{U}}$ of decoration-reproduction processes of individuals in a general branching process governed by this kernel when the ancestral individual has type $x>0$, that is such that $(f_\varnothing,\eta_\varnothing)$ has the law P_x. Let κ denote the cumulant function associated by (3.19) to the characteristic quadruplet $(\sigma^2,\mathrm{a},\mathbf{\Lambda};\alpha)$. We make the following crucial assumption:

Assumption 3.9. *We say that a cumulant κ is* ***subcritical*** *if there exists $\gamma_0>0$ such that*

$$\kappa(\gamma_0)<0.$$

Subcriticality of the cumulant is the only assumption on the characteristic quadruplet needed to define self-similar Markov trees as random variables with values in the space $\mathbb{T}$ of equivalence classes up to isomorphisms of decorated real trees (see Section 2.4, and more specifically Corollary 2.17).

Proposition 3.10 (Construction of self-similar Markov trees)**.** *Let Assumption 3.9 be satisfied for some $\gamma_0>0$. Then the following assertions hold:*

(i) The function $\phi(x)=x^{\gamma_0}$ is strictly excessive in the sense of (3.2). Assumption 3.3 is verified, and as a consequence, so does Property $(\mathcal{P})$, $\mathbb{P}_x$-a.s. for all $x>0$.

(ii) The equivalence class (up to isomorphism) of the random decorated tree $\mathtt{T} = (T, d_T, \rho, g)$ *without measures constructed in Theorem 2.5 is called a **self-similar Markov tree** with characteristic quadruplet* $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$*. One has*

$$g(\rho) = \limsup_{\substack{r \to \rho \\ r \neq \rho}} g(r) = x, \qquad \mathbb{P}_x\text{-a.s. for all } x > 0.$$

In this framework, self-similarity means that for every $x > 0$*, the law of* $\mathtt{T}$ *under* $\mathbb{P}_x$ *is identical to that of the rescaled version* $(T, x^\alpha \cdot d_T, \rho, x \cdot g)$ *under* $\mathbb{P}_1$*, where the notation* $x^\alpha \cdot d_T$ *is for the distance on* T *given by* $x^\alpha \cdot d_T(y, y') = x^\alpha d_T(y, y')$*, and* $x \cdot g$ *denotes similarly the decoration with* $x \cdot g(y) = xg(y)$*.*

Proof. Let us check the assumptions of Theorem 3.3, for the exponents α, γ_0 and the function $\phi(y) = y^{\gamma_0}$. In this direction, recall that κ is convex, and since $\psi \leqslant \kappa$, we have $\psi(\gamma_0) < 0$. Hence, $\kappa(\gamma_0)/\psi(\gamma_0) \in (0,1)$ and we infer from Lemma 3.8 that

$$\int_{(0,\infty)} y^{\gamma_0} \imath(\mathrm{d}y) = \mathcal{M}(\gamma_0) < 1,$$

so (3.2) holds with $\phi(y) = y^{\gamma_0}$. By self-similarity, it suffices now to verify that $E_1\left(\sup X^{\gamma_0}\right) < \infty$ and $E_1(z^{\gamma_0/\alpha}) < \infty$, and by the Lamperti transformation, this boils down to establishing that

$$E_1\left(\sup_{t\geqslant 0} \exp\left(\gamma_0 \xi(t)\right)\right) < \infty \quad \text{ and } E_1\left(\left(\int_0^\zeta \exp(\alpha\xi(t))\mathrm{d}t\right)^{\gamma_0/\alpha}\right) < \infty.$$

These kinds of results are part of the folklore of the theory of Lévy processes. For example, see Lemma 3 of Rivero [128] for the second assertion when the killing rate is zero. Unfortunately, we have not been able to find a suitable reference that holds when killing is allowed. Since the display above follows from standard techniques, and because we believe it may be of independent interest, we provide details of the proof in the Appendix; see Lemma 7.1 there.

The assertion that the value of the decoration at the root is x, $\mathbb{P}_x$-a.s. follows readily from (2.4) and the fact that $f_\varnothing$ is rcll with $f_\varnothing(0) = x$, $\mathbb{P}_x$-a.s. Last, by construction, the random decorated tree $\mathtt{T}$ plainly inherits the self-similarity property from the kernel $(P_x)_{x>0}$; see Definition 3.5 and the discussion thereafter. □

Let us point right now at an important feature, which is closely related to the discussion around Figure 2.3. We have argued here that one can associate a self-similar Markov tree to any subcritical characteristic quadruplet, and the characteristic quadruplet then determines the distribution of this self-similar Markov tree. Nonetheless we stress that different characteristic quadruplets may yield self-similar Markov trees with the same distribution, just as in Section 2.2 where different families of building blocks could produce isomorphic decorated trees. In that case, one says that the characteristic quadruplets belong to the same equivalence class of bifurcators. Thus, in short, a characteristic quadruplet determines the law of a self-similar Markov tree, and in the converse direction, a self-similar Markov tree determines an equivalence class of bifurcators, where each such bifurcator corresponds to a unique characteristic quadruplet. This matter will be discussed in details in Chapter 6.

We also point out from (3.16) that for any $c > 0$, the self-similar Markov tree $(\tilde{T}, d_{\tilde{T}}, \tilde{\rho}, \tilde{g})$ with dilated characteristic quadruplet $(c^2\sigma^2, c\mathrm{a}, c\mathbf{\Lambda}; \alpha)$ has the same distribution as $\left(T, c^{-1} \cdot d_T, \rho, g\right)$ where $\mathtt{T} = (T, d_T, \rho, g)$ denotes the ssMt with characteristics $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$.

3.3 Weighted length and harmonic measures

Now that we have defined self-similar Markov trees, our next purpose is to endow the latter with certain natural measures. We are interested in measures compatible with self-similarity, and which are also consistent with the Markov property of the decoration. This leads us to consider power functions with adequate exponents as weight functions for weighted length measures. To define a natural measure on the leaves, we will require a stronger assumption, which ensures the existence of a simple harmonic function for the decoration-reproduction kernel. By analogy with the literature on branching random walk, we refer to this assumption as the first Cramér's condition[9], and its statement and implications will be addressed in the second part of this section. Finally, we point out that the harmonic measure on leaves can also be obtained as a limit of weighted length measures (Proposition 3.14). Throughout this section, we use the notation of Sections 3.1 and 3.2. In particular, we fix a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and we write $(P_x)_{x>0}$ for the associated self-similar Markov decoration-reproduction kernel which has been defined in the preceding section. We also write $\mathbb{P}_x$ for the distribution of the family $(f_u, \eta_u)_{u\in\mathbb{U}}$ of decoration-reproduction processes of individuals in a general branching process governed by this kernel when the ancestral individual has type $x > 0$, that is such that $(f_\varnothing, \eta_\varnothing)$ has the law P_x.

3.3.1 Weighted length measures

We start with weighted length measures which are simpler to construct. Recall Section 2.3 and focus on the case where ϖ is a power function.

Proposition 3.11 (Weighted length measures)**.** *Let Assumption 3.9 hold. Then for every $x > 0$, there is the identity*

$$\mathbb{E}_x\left(\sum_{u\in\mathbb{U}} \int_0^\infty f_u(t)^{\gamma_0-\alpha}\mathrm{d}t\right) = -\frac{x^{\gamma_0}}{\kappa(\gamma_0)}. \tag{3.21}$$

As a consequence, for every $\gamma \geqslant \gamma_0$, the measure $\varpi \circ g \cdot \lambda_T$ induced by the weight function[10] *$\varpi(x) = x^{\gamma-\alpha}$ is finite, $\mathbb{P}_x$-a.s. We denote it by λ^γ and then $(T, d_T, \rho, g, \lambda^\gamma)$ is a random measured decorated compact tree.*

[9]The second Cramér's condition will appear when defining conditional versions of ssMt.

[10]Our choice of using $\gamma - \alpha$ instead of γ as exponent is related to the Lamperti transformation. As we will see shortly, this will simplify notation thereafter.

Proof. Let us start establishing (3.21). Thanks to self-similarity, it is enough to treat the case $x = 1$. We have

$$\sum_{u\in\mathbb{U}} \mathbb{E}_1\left(\int_0^{z_u} f_u(t)^{\gamma_0-\alpha}\mathrm{d}t\right) = \mathbb{E}_1\left(\sum_{u\in\mathbb{U}} \chi(u)^{\gamma_0}\right) E_1\left(\int_0^{z} f(t)^{\gamma_0-\alpha}\mathrm{d}t\right).$$

By Assumption 3.9, $\psi(\gamma_0) < \kappa(\gamma_0) < 0$, and Lemma 3.8 gives $\mathcal{M}(\gamma_0) = 1 - \kappa(\gamma_0)/\psi(\gamma_0)$. Then by the very construction of the family $(f_u, \eta_u)_{u\in\mathbb{U}}$, we get

$$\mathbb{E}_1\left(\sum_{u\in\mathbb{N}^n} \chi(u)^{\gamma_0}\right) = \Big(1 - \kappa(\gamma_0)/\psi(\gamma_0)\Big)^n, \qquad \text{for } n \geqslant 0.$$

Since $1 - \kappa(\gamma_0)/\psi(\gamma_0) \in (0, 1)$, we deduce that

$$\mathbb{E}_1\left(\sum_{u\in\mathbb{U}} \chi(u)^{\gamma_0}\right) = \psi(\gamma_0)/\kappa(\gamma_0).$$

Next, we have by the Lamperti transformation,

$$\begin{aligned} E_1\left(\int_0^{z} f(t)^{\gamma_0-\alpha}\mathrm{d}t\right) &= E_1\left(\int_0^{z} \exp\left((\gamma_0-\alpha)\xi(\tau(t))\right)\mathrm{d}t\right) \\ &= E_1\left(\int_0^{\infty} \exp\left(\gamma_0\xi(s)\right)\mathrm{d}s\right) \\ &= \int_0^{\infty} \exp\left(\psi(\gamma_0)s\right)\mathrm{d}s \\ &= -\frac{1}{\psi(\gamma_0)}, \end{aligned}$$

where in the second equality we used that $\exp(\gamma\xi(s)) = 0$, for every $s \geqslant \zeta$, and in the ultimate equality, that $\psi(\gamma_0) \leqslant \kappa(\gamma_0) < 0$. This completes the proof of (3.21).

Let us now deduce the remaining claim. Recalling Section 2.3 and in particular (2.9), by Proposition 3.10 and self-similarity, it suffices to verify that for every $\gamma > \gamma_0$, we have

$$\sum_{u\in\mathbb{U}} \int_0^{z_u} f_u(t)^{\gamma-\alpha}\mathrm{d}t < \infty, \qquad \mathbb{P}_1\text{-a.s.}$$

In this direction, we notice that for every $\gamma_1 > \gamma_0$,

$$\sup_{\gamma\in[\gamma_0,\gamma_1]} \sum_{u\in\mathbb{U}} \int_0^{z_u} f_u(t)^{\gamma-\alpha}\mathrm{d}t \leqslant \sup\left\{1 + f_u(t)^{\gamma_1-\gamma_0} : \ u \in \mathbb{U}, t \in [0, z_u]\right\} \cdot \left(\sum_{u\in\mathbb{U}} \int_0^{z_u} f_u(t)^{\gamma_0-\alpha}\mathrm{d}t\right).$$

Recalling from Proposition 3.10 that $(\|f_u\|)_{u\in\mathbb{U}}$ is a null family; we infer from (3.21) that:

$$\sup_{\gamma\in[\gamma_0,\gamma_1]} \sum_{u\in\mathbb{U}} \int_0^{z_u} f_u(t)^{\gamma-\alpha}\mathrm{d}t < \infty, \quad \mathbb{P}_1\text{-a.s.}$$

This completes the proof of the proposition. □

In the setup of the above proposition, we have $\lambda^\gamma(T) < \infty$, $\mathbb{P}_x$-a.s. for every $x > 0$, and we even have

$$\mathbb{E}_x\big(\lambda^\gamma(T)\big) = -\frac{x^\gamma}{\kappa(\gamma)}, \qquad \text{as soon as } \kappa(\gamma) < 0. \tag{3.22}$$

We further stress that the self-similarity asserted in Proposition 3.10 then extends to these weighted length measures. Specifically, for every $x > 0$, the distribution of the equivalence class in $\mathbb{T}_m$ of $(T, d_T, \rho, g, \lambda^\gamma)$ under $\mathbb{P}_x$ is the same as that of $(T, x^\alpha \cdot d_T, \rho, x \cdot g, x^\gamma \cdot \lambda^\gamma)$ under $\mathbb{P}_1$. For this reason, we say that λ^γ is self-similar with exponent γ, which motivates *a posteriori* our choice for the parameter for the weight functions ϖ in Proposition 3.11.

3.3.2 First Cramér's condition and the harmonic measure on leaves

In this section, we define the natural measure on leaves for self-similar Markov trees. To do so, we shall rely on the constructed presented in Section 2.3 using an additional family $(m_u)_{u\in\mathbb{U}}$ verifying (2.10). Those m_u will be constructed using a harmonic function for the underlying branching random walk. Specifically, recall from Section 3.1 the notation $(f_u, \eta_u)_{u\in\mathbb{U}}$ for the general branching process index by Ulam's tree, where the decoration-reproduction kernel $(P_x)_{x>0}$ is self-similar and associated with the characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$. In this particular case, we have $f_u(0) = \chi(u)$, and, as noticed in (3.7), the process

$$\sum_{|u|=n} \delta_{\log \chi(u)}, \qquad n \geqslant 0$$

is a branching random-walk with moment generating function $\mathcal{M}$ given by Lemma 3.8. We deduce that if $\omega > 0$ is such that $\kappa(\omega) = 0$, then the function $h(x) = x^\omega$ is a **harmonic function** of types, namely that $h(0) = 0$ and

$$h(x) = \mathbb{E}_x\left(\sum_{j=1}^{\infty} h(\chi(j))\right) = \int_{(0,\infty)} h(y)\imath_x(\mathrm{d}y), \qquad \text{for all } x > 0. \tag{3.23}$$

It follows readily that by the branching property that the process

$$M_n \coloneqq \sum_{|u|=n} h(\chi(u)), \qquad n \geqslant 0, \tag{3.24}$$

is then a positive martingale, which is usually referred to as an additive martingale; it converges to its terminal value $\mathbb{P}_x$ almost surely. We have more generally for any vertex $u \in \mathbb{U}$

$$m_u := \lim_{n\to\infty} \sum_{|v|=n} h(\chi(uv)), \qquad \text{in } \mathbb{P}_x - a.s. \tag{3.25}$$

Verifying the spread of mass condition (2.10) is intimately connected to ensuring that the previous convergence also holds in mean. Whether or not the additive martingale (3.24) converges in $L^1(\mathbb{P})$ is well understood in the literature on branching random walks, see e.g. [33]. In our setup, this will be implied by the following assumption:

Assumption 3.12 (First Cramér's condition). *Suppose that there exist $\omega_- > 0$ and $p \in (1,2]$ such that*

$$\text{(i)} \quad \kappa(\omega_-) = 0,\ \kappa(p\omega_-) < 0, \qquad \textit{and} \qquad \text{(ii)} \quad \int_{\mathcal{S}_1} \mathbf{\Lambda}_1(\mathrm{d}\mathbf{y}) \left(\sum_{i=1}^{\infty} \mathrm{e}^{y_i \omega_-}\right)^p < \infty.$$

From now on, under Assumption 3.12, we set:

$$\omega_+ := \inf \left\{ t > \omega_- : \ \kappa(t) \geqslant 0 \right\}, \tag{3.26}$$

with the usual convention $\inf \varnothing = \infty$. Observe that, by convexity of the cumulant, we have $p\omega_- < \omega_+$. Furthermore, Assumption 3.12 ensures that both κ and ψ are negative over the interval (ω_-, ω_+). In particular, Assumption 3.9 holds for every $\gamma_0 \in (\omega_-, \omega_+)$ and therefore we can define the associated ssMt as well as the measures λ^γ, $\gamma > \omega_-$. Also, because of convexity, κ possesses a negative right-derivative at ω_- that we denote for simplicity by $\kappa'(\omega_-)$, see Figure 3.2 for an illustration. We are going to use the function $h(x) = x^{\omega_-}$ to define our measure on leaves.

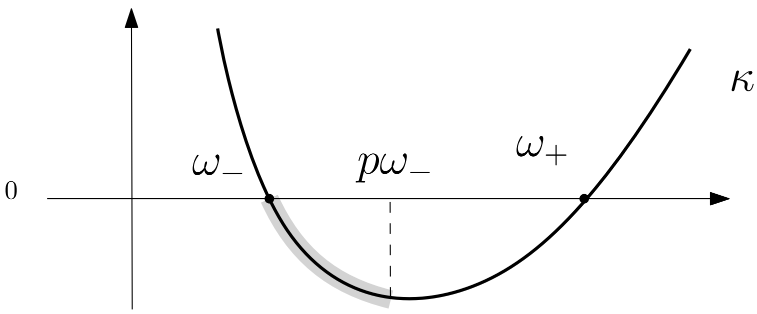

Figure 3.2: Illustration of the typical shape of the cumulant function κ. Assumption 3.12 requires that κ becomes strictly negative after its first zero (like in the gray region).

For this purpose, we need the following result:

Lemma 3.13. *Let Assumption 3.12 hold. Then the function $h(x) = x^{\omega_-}$ is harmonic in the sense* (3.23)*, and the process*

$$M_n(\omega_-) := \sum_{u \in \mathbb{N}^n} \chi(u)^{\omega_-}, \qquad n \geqslant 0,$$

is a martingale bounded in $L^p(\mathbb{P}_x)$ for every $x > 0$. As a consequence, the spread of mass condition (2.10) *is fulfilled for the family the $(m_u)_{u \in \mathbb{U}}$ defined by* (3.25).

Proof. Lemma 3.8 and Assumption 3.12 entail that $\mathcal{M}(\omega_-) = 1$, so $h(x) = x^{\omega_-}$ is a harmonic function, and we have already seen that $M_n(\omega_-)$, $n \geqslant 0$, is then an additive martingale of a branching random walk with moment generating function $\mathcal{M}$. Since $\mathcal{M}(p\omega_-) = 1 - \kappa(p\omega_-)/\psi(p\omega_-) < 1$, it is known that boundedness in L^p will follow provided that $\mathbb{E}(M_1(\omega_-)^p) < \infty$; see e.g. Theorem 3.1 of Alsmeyer and Kuhlbusch [10].

In this direction, we write

$$M_1(\omega_-) = \sum_{j=1}^{\infty} \chi(j)^{\omega_-} = \sum \mathbf{1}_{\{s \leqslant \zeta\}} \exp\left((\xi(s-) + y_\ell)\omega_-\right),$$

where the last sum is taken over all the pairs (s, y_ℓ) obtained by expanding the atoms $(s, \mathbf{y})$ of $\mathbf{N}_1$. This shows that $M_1(\omega_-)$ is a so-called Lévy-type perpetuity in the sense of Iksanov and Mallein [81], see Section 2 there. Specifying Theorem 3.1 of [81] in our setting, all that we need to verify are, first

$$E\left(\exp(\xi(1)p\omega_-)\right) = \exp(\psi(p\omega_-)) < 1,$$

and second,

$$\int_{\mathcal{S}_1} \mathbf{\Lambda}_1(\mathrm{d}\mathbf{y}) \left(\sum_{i=1}^{\infty} \mathrm{e}^{y_i\omega_-}\right)^p \mathbf{1}_{\{\sum_{i=1}^{\infty} \mathrm{e}^{y_i\omega_-} > 1\}} < \infty.$$[11]

The first requirement holds since $\psi(p\omega_-) < \kappa(p\omega_-) < 0$, and the second is part of Assumption 3.12. Let us now prove the spread of mass condition (2.10). Introduce for any $u \in \mathbb{U}$,

$$M_n(u) \coloneqq \sum_{|v|=n} h(\chi(uv)). \tag{3.27}$$

Note from the Markov property that conditionally on $\chi(u) = y$, the martingale $(M_n(u))_{n\geqslant 0}$ has the same law as $(M_n)_{n\geqslant 0}$ under $\mathbb{P}_y$. Plainly, we have on the one hand,

$$M_{n+1}(u) = \sum_{j=1}^{\infty} M_n(uj), \tag{3.28}$$

and on the other hand, there are the convergences a.s. under $\mathbb{P}_x$,

$$m_u = \lim_{n\to\infty} M_n(u) \quad \text{and} \quad m_{uj} = \lim_{n\to\infty} M_n(uj) \quad \text{for all } j \geqslant 1.$$

Hence Fatou's lemma yields

$$m_u \geqslant \sum_{j=1}^{\infty} m_{uj}, \qquad \mathbb{P}_x\text{-a.s.} \tag{3.29}$$

Moreover, we also know from the first point of the lemma that these martingales converge in $L^1(\mathbb{P}_x)$, which implies

$$\mathbb{E}_x(m_u) = \mathbb{E}_x(M_0(u)) = \mathbb{E}_x\left(h(\chi(u))\right) \quad \text{and} \quad \mathbb{E}_x(m_{uj}) = \mathbb{E}_x(M_0(uj)) = \mathbb{E}_x\left(h(\chi(uj))\right),$$

for all $j \geqslant 1$. Taking the expectation in (3.28) for $n = 0$, we deduce that both sides in (3.29) have the same mean, and are thus almost surely equal. $\square$

In the sequel, we shall refer to $(M_n(\omega_-))_{n\geqslant 0}$ as the **intrinsic martingale**. Under Assumption 3.12 and using Proposition 2.10, we can endow the self-similar Markov tree $\mathtt{T} = (T, d_T, \rho, g)$ with the measure induced by the $(m_u)_{u\in\mathbb{U}}$; recall that the latter is supported by the set ∂T of leaves of T. From now on, we will use the notation $\upmu$ for this measure and recall that it is referred to as the **harmonic measure**. Observe that the total mass $\upmu(T)$ of the harmonic measure coincides

[11] Theorem 3.1 in [81] assumes that the killing rate is 0. However, since the proof can be extended for a positive killing without modifications, we leave the details to the reader.

with the terminal value $m_\varnothing$ of the intrinsic martingale. Plainly, the self-similarity stated in Proposition 3.10 extends to μ as follows. For any $x > 0$, the distribution, the equivalence class in $\mathbb{T}_m$, of $(T, x^\alpha \cdot d_T, \rho, x \cdot g, x^{\omega_-} \cdot \mu)$, under $\mathbb{P}_1$, coincides with that of (T, d_T, ρ, g, μ), under $\mathbb{P}_x$. We say that μ is self-similar with exponent ω_-; note also that

$$\mathbb{E}_x(\mu(T)) = x^{\omega_-}, \qquad \text{for } x > 0. \tag{3.30}$$

We mention that when the second root ω_+ in (3.26) is finite, $M_n(\omega_+)$, $n \geqslant 0$ is also a martingale. Nonetheless it follows by general results on branching random walks, see [132], that its terminal value is always 0 a.s. Even though the martingale $M_n(\omega_+)$ has also interesting applications (see Part III), it cannot be used to construct a non-degenerate measure on self-similar Markov trees.

Proposition 3.11 also allows us to endow T with the weighted length measures λ^γ for any $\gamma > \omega_-$, as it was discussed at the end of Section 3.2. In this direction, one can infer from Lemma 3.13, that for every $\gamma' \leqslant \omega_-$, one has

$$\int_T g^{\gamma'-\alpha}(v)\ \lambda_T(\mathrm{d}v) = \sum_{u \in \mathbb{U}} \int_0^{z_u} f_u(t)^{\gamma'-\alpha}\ \mathrm{d}t = \infty, \quad \mathbb{P}_1\text{-a.s.}$$

In other words, the measures $c \cdot \lambda^\gamma$, for $\gamma > \omega_-$ and some $c > 0$, are the sole self-similar weighted length measures with a finite total mass. Additionally, since all the measures μ and λ^γ, $\gamma > \omega_-$, have distinct self-similarity exponents, a nontrivial linear combination cannot be self-similar. Consequently, up to an unimportant factor, μ and λ^γ, $\gamma > \omega_-$, are the only self-similar measures consistent with our framework. They constitute the family of measures that we will work with in the sequel, and will also naturally appear as scaling limits of discrete models in Part II.

3.3.3 The harmonic measure as limit of weighted length measures

The purpose of this section is to point out that the harmonic measure μ on leaves can also be obtained as a limit of re-scaled versions of weighted length measures λ^γ, as γ decreases towards ω_-. One important motivation for establishing such a result stems from the following observation. Weighted length measures are given intrinsically in terms of the decorated real tree $\mathtt{T} = (T, d_T, \rho, g)$ rather than in terms of a specific construction of $\mathtt{T}$, such as by gluing building blocks in Section 2.2. As a consequence, if $(T', d_{T'}, \rho', g') = \mathtt{T}'$ is another decorated real tree isomorphic to $\mathtt{T}$, then for any weight function ϖ with $\varpi \circ g \in L^1(\lambda_T)$, we have also in the notation of Definition 2.1 that

$$(T, d_T, \rho, g, \varpi \circ g \cdot \lambda_T) \approx (T', d_{T'}, \rho', g', \varpi \circ g' \cdot \lambda_{T'}).$$

At the opposite, the harmonic measure μ on a self-similar Markov tree has been defined in terms the family $(m_u)_{u \in \mathbb{U}}$ in (3.25), and it is not clear *a priori* whether μ can be given directly in terms of $\mathtt{T}$ only. Therefore, it is also unclear whether the compatibility with isomorphisms that we just stressed for weighted length measures remains valid for μ. Proposition 3.14 below implies that actually, this is indeed the case, and the harmonic measure on leaves is also an intrinsic quantity for self-similar Markov trees.

Proposition 3.14. *Let Assumption 3.12 hold. Then there exists a sequence $(\gamma_n)_{n\geqslant 1}$ with $\gamma_n > \omega_-$ and $\lim_{n\to\infty}\gamma_n = \omega_-$, such that $\mathbb{P}_1$-a.s.,*

$$\lim_{n\to\infty} -\kappa(\gamma_n)\cdot\lambda^{\gamma_n} = \mu,$$

in the sense of weak convergence for finite measures on T.

Recall that $\kappa(\gamma) < 0$ for $\gamma \in (\omega_-, \omega_+)$ so that $-\kappa(\gamma)\cdot\lambda^{\gamma}$ is a positive finite measure, and by (3.22), the factor $-\kappa(\gamma) > 0$ is simply chosen so that

$$\mathbb{E}_1\big(-\kappa(\gamma)\cdot\lambda^{\gamma}(T)\big) = \mathbb{E}_1(\mu(T)) = 1.$$

For technical reasons and to avoid extending the section unnecessarily, we restricted ourselves to convergence along a sequence $(\gamma_n)_{n\geqslant 1}$ in Proposition 3.14, which suffices for our purpose. We are nonetheless confident that the convergence should hold as $\gamma \downarrow \omega_-$.

The rest of this section is devoted to the proof of Proposition 3.14; we implicitly take Assumption 3.12 for granted. As a first step, we establish the convergence of the total mass.

Lemma 3.15. *There is the convergence*

$$\lim_{\gamma\downarrow\omega_-} \kappa(\gamma)\lambda^{\gamma}(T) = -\mu(T), \qquad \text{in } L^1(\mathbb{P}_1).$$

Proof. Recall from Lemma 3.8 that the Mellin transform of the reproduction intensity is given by $\mathcal{M}(\gamma) = 1 - \kappa(\gamma)/\psi(\gamma) \in (0,1]$ for every $\gamma \in [\omega_-, \omega_+]$. From the branching property,

$$M_n(\gamma) := \mathcal{M}(\gamma)^{-n} \sum_{u\in\mathbb{N}^n} \chi(u)^{\gamma} = \mathcal{M}(\gamma)^{-n} \sum_{u\in\mathbb{N}^n} f_u(0)^{\gamma}, \quad n \geqslant 0,$$

is then a nonnegative martingale, and we denote its terminal value by $M_\infty(\gamma)$. We are going to establish:

(i) $M_\infty(\gamma)$ converges in $L^1(\mathbb{P}_1)$ to $M_\infty(\omega_-)$ as $\gamma \downarrow \omega_-$,

(ii) $M_\infty(\gamma) + \kappa(\gamma)\cdot\lambda^{\gamma}(T)$ converges in $L^1(\mathbb{P}_1)$ to 0 as $\gamma \downarrow \omega_-$.

Combining these two claims implies the lemma, since, by definition $\mu(T) = M_\infty(\omega_-)$. We start by proving the first item which follows by standard branching random walk technics[12].

(i) Recall from Lemma 3.13 that $M_n(\omega_-) \in L^p(\mathbb{P}_1)$ for some $p > 1$ appearing in Assumption 3.12 and every fixed $n \geqslant 0$. We deduce readily from dominated convergence that

$$\lim_{\gamma\downarrow\omega_-} M_n(\gamma) = M_n(\omega_-), \qquad \text{in } L^1(\mathbb{P}_1). \tag{3.31}$$

[12]Similar results appeared in the theory of branching random walks, see notably [34]. Unfortunately, we have not been able to find a reference covering our specific situation.

The idea now is to control the difference $M_\infty(\gamma) - M_n(\gamma)$ for γ close enough of ω_-. In this direction, fix $q \in (1,p)$ and take $\gamma_1 \in (\omega_-, p\omega_-/q)$ such that

$$c := \sup_{\gamma \in [\omega_-, \gamma_1]} \mathcal{M}(q\gamma)/\mathcal{M}(\gamma)^q < 1.$$

This is indeed feasible since $\kappa(\omega_-) = 0$ and $\kappa(p\omega_-) < 0$.

For every $\gamma \in [\omega_-, \gamma_1]$, a direct computation gives:

$$M_1(\gamma)^q \leqslant \mathcal{M}(\gamma)^{-q} \left(\sup_{u \in \mathbb{N}} \chi(u)^{(\gamma-\omega_-)q} \right) M_1(\omega_-)^q \leqslant \mathcal{M}(\gamma)^{-q} \left(1 + M_1(\omega_-)^{p-q}\right) M_1(\omega_-)^q.$$

Since by Lemma 3.13, the variable $M_1(\omega_-)$ is in $L^p(\mathbb{P}_1)$ and $\inf_{[\omega_-,\gamma_1]} \mathcal{M}(\gamma) > 0$, we deduce from Hölder's inequality that

$$\sup_{\gamma \in [\omega_-, \gamma_1]} \mathbb{E}_1(M_1(\gamma)^q) := K < \infty.$$

We can now apply [34, Lemma 2(i)], combined with Jensen inequality, to infer that for every $n \geqslant 0$ and $\gamma \in [\omega_-, \gamma_1]$, we have

$$\mathbb{E}_1 \left(|M_\infty(\gamma) - M_n(\gamma)| \right) \leqslant 2^3 K^{1/q} c^{n/q}. \tag{3.32}$$

Using (3.31) and (3.32), we arrive at

$$\limsup_{\gamma \to \omega_-} \mathbb{E}_1 \left(|M_\infty(\gamma) - M_\infty(\omega_-)| \right) \leqslant 2^4 K^{1/q} c^{n/q}.$$

Finally, taking the limit as $n \to \infty$ in the right-hand side, we obtain (i).

(ii) From the very definition of the weighted length measure λ^γ and the canonical decomposition (2.9) of T into line segments indexed by $\mathbb{U}$, we get for every $\gamma \in (\omega_-, \gamma_1]$,

$$\kappa(\gamma)\lambda^\gamma(T) = \frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{u \in \mathbb{U}} \chi(u)^\gamma A_u(\gamma),$$

where for non-fictitious vertex u, we write

$$A_u(\gamma) = \psi(\gamma)\chi(u)^{-\gamma} \int_0^{z_u} f_u(t)^{\gamma-\alpha} \mathrm{d}t,$$

and by convention we take $A_u = 0$ if u is fictitious. Next from (3.32) that

$$\begin{aligned} \mathbb{E}_1 \left(\left| M_\infty(\gamma) - \frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{n \geqslant 0} \sum_{u \in \mathbb{N}^n} \chi(u)^\gamma \right| \right) &\leqslant \frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{n \geqslant 0} \mathcal{M}(\gamma)^n \mathbb{E}_1 \left(|M_n(\gamma) - M_\infty(\gamma)| \right) \\ &\leqslant \frac{2^4 K^{1/q} \kappa(\gamma)}{\psi(\gamma)(1 - \mathcal{M}(\gamma)c^{1/q})}, \end{aligned}$$

and the right-hand side converges to 0 as $\gamma \downarrow \omega_-$. Therefore, it suffices to show that

$$\lim_{\gamma \downarrow \omega_-} \mathbb{E}_1 \left(\left| \frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{n \geqslant 0} \sum_{u \in \mathbb{N}^n} \chi(u)^\gamma (A_u(\gamma) + 1) \right| \right) = 0. \tag{3.33}$$

In this direction, recalling that $q \in (1,2)$ and using the triangle and then the Jensen inequalities, we bound from above the expectation in (3.33) by

$$\frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{n \geqslant 0} \mathbb{E}_1 \left(\left(\sum_{u \in \mathbb{N}^n} \chi(u)^{\gamma} |A_u(\gamma) + 1| \right)^q \right)^{1/q}. \tag{3.34}$$

Now an application of the branching and the self-similarity properties, combined with the Lamperti transformation, shows that, for $n \geqslant 0$, under $\mathbb{P}_1$ and conditionally on the types at generation n, $(\chi(u))_{u \in \mathbb{N}^n}$, the variables $A_u(\gamma)$, for $u \in \mathbb{N}^n$ with $\chi(u) \neq 0$, are i.i.d with the same law as

$$\psi(\gamma) \int_0^{\zeta} \exp(\gamma \xi(t)) \mathrm{d}t, \text{ under } P_1.$$

We know from Tonelli's theorem that

$$E_1 \left(\int_0^{\zeta} \exp(\gamma \xi(t)) \mathrm{d}t \right) = \int_0^{\infty} E_1 \left(\exp(\gamma \xi(t), t < \zeta \right) \mathrm{d}t = -1/\psi(\gamma),$$

and we can apply the Marcinkiewicz-Zygmund inequality to infer that (3.34) is bounded above by

$$\begin{aligned}
&c(q) \frac{\kappa(\gamma)}{\psi(\gamma)} \sum_{n \geqslant 0} \mathbb{E}_1 \left(\sum_{u \in \mathbb{N}^n} \chi(u)^{\gamma q} \right)^{1/q} \cdot E_1 \left(\left| \psi(\gamma) \cdot \int_0^{\infty} \exp(\gamma \xi(t)) \mathrm{d}t + 1 \right|^q \right)^{1/q} \\
&= c(q) \frac{\kappa(\gamma)}{\psi(\gamma) \left(1 - \mathcal{M}(q\gamma)^{1/q}\right)} E_1 \left(\left| \psi(\gamma) \int_0^{\infty} \exp(\gamma \xi(t)) \mathrm{d}t + 1 \right|^q \right)^{1/q},
\end{aligned}$$

where $c(q) < \infty$ is some constant depending on q only.

On the one hand, $c(q)\kappa(\gamma) / \left(\psi(\gamma)\left(1 - \mathcal{M}(q\gamma)^{1/q}\right)\right)$ converges to 0 as $\gamma \downarrow \omega_-$. On the other hand, the bound $\mathrm{e}^{y\gamma} \leqslant \mathrm{e}^{y\omega_-} + \mathrm{e}^{y\gamma_1}$ for all $y \in \mathbb{R}$ and $\gamma \in [\omega_-, \gamma_1]$ yields

$$\begin{aligned}
&\sup_{\gamma \in [\omega_-, \gamma_1]} E_1 \left(\left(\int_0^{\infty} \exp(\gamma \xi(t)) \mathrm{d}t \right)^q \right) \\
&\leqslant 2^q E_1 \left(\left(\int \exp(\omega_- \xi(t)) \mathrm{d}t \right)^q \right) + 2^q E_1 \left(\left(\int \exp(\gamma_1 \xi(t)) \mathrm{d}t \right)^q \right),
\end{aligned}$$

and since $\psi(q\omega_-) < 0$ and $\psi(q\gamma_1) < 0$, the finiteness of the right-hand side above follows from standard properties of Lévy processes, see Lemma 7.1 in the Appendix. Putting the pieces together, we have checked (3.33), and the proof of (ii) is complete. $\square$

We can now establish Proposition 3.14.

Proof of Proposition 3.14. We work under $\mathbb{P}_1$ and equip the ssMt $\mathtt{T} = (T, d_T, \rho, g)$ with the measures $\upmu$ and $\tilde{\uplambda}^{\gamma} := -\kappa(\gamma) \cdot \uplambda^{\gamma}$ for $\gamma \in (\omega_-, \omega_+)$. We infer from Lemma 3.15 that there exists a sequence $(\gamma_n)_{n \geqslant 1}$ in $(\omega_-, p\omega_-)$ which converges to ω_- and such that $\tilde{\uplambda}^{\gamma_n}(T) \to \upmu(T)$ a.s. Our goal is to show that $\lim_{n \to \infty} \mathrm{d}_{\mathrm{Prok}}(\tilde{\uplambda}^{\gamma_n}, \upmu) = 0$ a.s. Since the convergence for the total masses is already know from Lemma 3.15, it suffices to establish that, for every $\delta > 0$ and every Borel set $A \subset T$, we have

$$\mu(A) \leqslant \tilde{\lambda}^{\gamma_n}(A^\delta) + \delta, \qquad \text{for all } n \text{ sufficiently large}, \tag{3.35}$$

where we use the standard notation A^δ to denote the δ-neighborhood of A in T.

In this direction, recall Notation 2.9, and in particular that for every $v \in \mathbb{U}$, $(T_v, d_{T_v}, \rho(v), g_v, \mu_v)$ stands for the subtree encoded by the sub-family $(f_{vu}, \eta_{vu})_{u\in\mathbb{U}}$. By the branching property, the conditional law of $(T_v, d_{T_v}, \rho(v), g_v, \mu_v)$, in $\mathbb{T}$, given the type $\chi(u) = y$ is that of the ssMt under $\mathbb{P}_y$. Therefore, by self-similarity, we get

$$\lim_{n\to\infty} \tilde{\lambda}^{\gamma_n}(T_v) = \mu_v(T_v) = \mu(T_v), \quad \text{for every } v \in \mathbb{U}, \qquad \text{a.s.} \tag{3.36}$$

Using again Notation 2.9, we can decompose the tree T at a generation $k \geqslant 1$ as

$$T = T^k \cup \left(\bigcup_{|v|=k} T_v \right).$$

We stress that for any two distinct vertices at the same generation, say $v \neq w$ with $|v| = |w| = k$, the subtrees T_v and T_w are either disjoint, or they share the same root and their intersection is then reduced to the latter. In both cases, we have $\mu(T_v \cap T_w) = 0$. Combining this observation with the fact from Corollary 3.4, that for any $k \geqslant 1$, the harmonic measure assigns no mass to T^k, we conclude that there is the identity

$$\mu(A) = \sum_{v\in\mathbb{N}^k} \mu(A \cap T_v).$$

We then recall (from Propositions 2.7 and 3.10, and Property $(\mathcal{P})$) that

$$\lim_{k\to\infty} \sup\{\mathrm{Height}(T_v) : v \in \mathbb{N}^k\} = 0;$$

we can therefore almost surely find a (random) integer k sufficiently large such that

$$\mathrm{Height}(T_v) < \delta/2, \qquad \text{for all } v \in \mathbb{N}^k. \tag{3.37}$$

Next, since $\mu(T) = \sum_{|v|=k} m_v$ and $m_v = \mu(T_v)$, we can select finitely many distinct vertices $v_1, \ldots, v_M$ at generation k, such that

$$\sum_{1\leqslant i\leqslant M} \mu\left(T_{v_i}\right) \geqslant \mu(T) - \delta/2.$$

As a consequence of (3.36), we can now find an integer $N \geqslant 1$, such that

$$\mu\left(T_{v_i}\right) \leqslant \tilde{\lambda}^{\gamma_n}\left(T_{v_i}\right) + \delta/(2M), \qquad \text{for every } 1 \leqslant i \leqslant M \text{ and } n \geqslant N.$$

Take any Borel subset A of T and combine the preceding observation. We get

$$\mu(A) \leqslant \sum_{1\leqslant i\leqslant M} \mu\left(A \cap T_{v_i}\right) + \delta/2 \leqslant \sum_{\substack{1\leqslant i\leqslant M \\ A\cap T_{v_i}\neq\varnothing}} \tilde{\lambda}^{\gamma_n}\left(T_{v_i}\right) + \delta.$$

Finally, (3.37) entails that for any vertex $v \in \mathbb{N}^k$ with $A \cap T_v \neq \varnothing$, T_v is included into the δ-neighborhood of A, and the previous display is therefore bounded from above by $\tilde{\lambda}^{\gamma_n}(A^\delta) + \delta$; here we used the facts that $\tilde{\lambda}^{\gamma_n}$ has no atoms and that the intersection of two different subtrees at the same generation is either empty or a singleton. This completes the proof of our claim. $\square$

3.4 Comments and bibliographical notes

Construction of ssMt. As mentioned in the previous chapter, the inspiration for the recursive random construction of self-similar Markov trees is the work of Rembart & Winkel. In [124] those authors already constructed the underlying tree structure (but without the decoration) of binary growth-fragmentation processes (a subclass of our ssMt) and gave a similar upper bound on the Hausdorff dimension as in Lemma 3.6. After the initial works on self-similar fragmentations [18, 78], the introduction of branching self-similar Markov processes can be traced back to [21] (in particular Lemma 3.8 is adapted from [21, Lemma 4]) in the context of (binary) Growth-Fragmentation and [26] in the context of branching Lévy processes. Most of the framework of Section 3.2 is adapted from the literature on branching Lévy processes [26]. A branching Lévy process can be seen as the continuous-time version of a branching random walk which describes a particle system on the real line in which particles move and reproduce independently in a Poissonian manner. Just as for Lévy processes, the law of a branching Lévy process is determined by its characteristic triplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda})$ where the decorated Lévy measure $\mathbf{\Lambda}$ describes the intensity of the Poisson point process of births and jumps. In a nutshell, the self-similar Markov branching trees can be interpreted as the random decorated trees obtained after performing a Lamperti transformation with exponent $\alpha > 0$ to the decorated trees coding for the genealogy of branching Lévy processes.

Critical case. Our construction of ssMt in Proposition 3.10 assumes sub-criticality of the cumulant, i.e. that $\kappa(\gamma_0) < 0$ for some $\gamma_0 > 0$. Indeed, when κ is strictly positive, the underlying branching random walk should witness local explosion, see [28], and it is hopeless to define a random compact tree from it. However, we left aside of this work the **critical case** when there exists $\omega > 0$ for which $\kappa(\omega) = 0$ and otherwise $\kappa \geqslant 0$. In this case, we do not believe that our Theorem 3.3 can apply, since there are cases when

$$\sum_{n \geqslant 0} \mathbb{E}\big(\sup_{|u|=n} \chi(u)\big) = \infty.$$

However, the application of Theorem 2.5 requires the more flexible assumption (2.6), namely $\lim_{k\to\infty} \sup\left\{ \sum_{n=k}^{\infty} z_{\bar{u}(n)} : \bar{u} \in \mathbb{N}^{\mathbb{N}} \right\} = 0$ and its associated branching random walk analog

$$\lim_{k\to\infty} \sup\left\{ \sum_{n=k}^{\infty} \chi(\bar{u}(n)) : \bar{u} \in \mathbb{N}^{\mathbb{N}} \right\} = 0.$$

In terms of *growth-fragmentation process* (see Section 4.3 for details), although the particle system evolving in continuous time makes sense as well in the critical case, we do not know in general whether its extinction time is finite a.s. or not (see [21, Corollary 3]) and in turn whether the construction of Proposition 3.10 yields to a compact decorated random real tree. A private communication of Elie Aidekon, Yueyun Hu and Zhan Shi reported progress in that direction and we believe that ssMt could even be defined in the critical case[13]. See Remark 4.4 and Example 4.12 for instances of critical cases.

On the intrinsic martingale. Our construction of the harmonic measure on ssMt is based on a natural additive martingale, which is often referred to as the intrinsic martingale in the branching random walk literature. Biggins [33] and Biggins & Kyprianou [36] have given explicit general criteria (extending the celebrated Kesten-Stigum theorem) which entail that the convergence also holds in mean of those martingales. In our case, Assumption 3.12 and Lemma 3.13 which ensures the convergence in mean builds on the recent works Alsmeyer & Kuhlbusch [10] and Iksanov & Mallein [81]. Although Lemma 3.13 proves a convergence in $L^p(\mathbb{P}_1)$ for some $p > 1$, Proposition 3.10 merely requires the uniform integrability. Even though we employ the condition of boundedness in $L^p(\mathbb{P}_1)$ to streamline some technical arguments, it might be possible to bypass it. Our results are likely to remain valid under less stringent assumptions ensuring the uniform integrability of the intrinsic martingale $(M_n(\omega_-))_{n\geqslant 1}$.

The law of M^-_∞, the limit of the intrinsic martingale is subject of intense study in the literature: Passing (3.28) to the limit using the convergence in mean, we deduce the following recursive distributional equation

$$M_\infty(\omega_-) \overset{(d)}{=} \sum_{u\in\mathbb{N}} \chi(u)^{\omega_-} \cdot M^u_\infty(\omega_-), \tag{3.38}$$

where on the right hand side the iid copies $M^u_\infty(\omega_-), u \in \mathbb{N}$ are independent of the vector $(\chi(u) : u \in \mathbb{N})$. The above fixed point equation, related to the equation "$X = AX + B$" is sometimes called a **smoothing transform** and has been studied in depth in recent years, see [59]. Under our assumptions, we infer in particular from results due to Biggins (see Section 2 in [33]), that this equation has a unique solution with given mean. In our Markovian setup, we can also write an "infinitesimal version" of the recursive distributional equation (3.38), see [25, bottom page 4], and deduce that the Laplace transform $w(\lambda) = \mathbb{E}\big(\mathrm{e}^{-\lambda M_\infty(\omega_-)}\big)$, $\lambda \geqslant 0$, satisfies the following integro-differential equation,

$$\begin{aligned} 0 = &- \mathrm{k}w(\lambda) + \frac{1}{2}\sigma^2\lambda^2 w''(\lambda) + a\,\lambda w'(\lambda) \\ &+ \int \nu(\mathrm{d}\mathbf{y}) \left(\prod_{i\geqslant 0} w(\lambda y_i^{\omega_-}) - w(\lambda) - y_0\lambda w'(\lambda)\mathbf{1}_{\mathrm{e}^{-1}\leqslant y_0 \leqslant \mathrm{e}} \right), \end{aligned} \tag{3.39}$$

where ν is the image measure of $\mathbf{\Lambda}$ by $x \mapsto \mathrm{e}^x$. Alas, despite the fact that the law of $M_\infty(\omega_-)$ satisfies (3.38) and (3.39), it seems to be difficult to identify its distribution in general. We

[13] Notice then that Assumption 3.12 cannot hold and natural harmonic measure μ must be constructed from the *derivative martingale*, [132, Section 3.4].

shall however be able to identify the law of $M_\infty(\omega_-)$ in some special cases in particular via a surprising connection with random planar maps, see Chapter 4.

Chapter 4

Examples

The purpose of this chapter is to illustrate the construction of self-similar Markov trees and to discuss some distinguished families of examples. Roughly speaking, many of those examples have the property that the decorations along branches are given by some versions of a stable Lévy process. We stress here again -see also the end of Section 3.2- that the law of a self-similar Markov tree is determined by a characteristic quadruplet, and that different characteristic quadruplets in the same equivalence class of bifurcators induce self-similar Markov trees with the same distribution. Choosing one characteristic quadruplet rather than another one within an equivalence class of bifurcators is often only a matter of preferences. The reader may wish to have first a glance at the forthcoming Section 6.3 that will provide a detailed account on bifurcators; notably Theorem 6.3 there describes these equivalence classes explicitly. Nonetheless, the examples treated in this chapter do not require any result from Section 6.3.

Before starting our list of examples and in order to give a purely analytic definition in terms of the characteristic quadruplet, we also need to make some comments on the concept of drift. The notion of drift coefficient for a Lévy process ξ can be defined canonically when the Lévy measure $\Lambda_0(\mathrm{d}y)$ integrates $1 \wedge |y|$, but depends otherwise of the arbitrary choice of the cutoff function in the Lévy-Khintchin formula (3.11). More precisely, if $\int(1 \wedge |y|)\Lambda_0(\mathrm{d}y) < \infty$, then no compensation is needed to make sense of the Poisson integral there and we can re-write the Lévy-Itô decomposition (3.9) in the simpler form

$$\xi(t) := \sigma B(t) + \mathrm{a}_{\mathrm{can}} t + \int_{[0,t]\times\mathbb{R}} N_0(\mathrm{d}s, \mathrm{d}y)\ y, \qquad \mathrm{a}_{\mathrm{can}} := \mathrm{a} - \int \Lambda_0(\mathrm{d}y) y \mathbf{1}_{|y|\leqslant 1}.$$

Note that $\mathrm{a}_{\mathrm{can}} = \mathrm{a}$ when all the jumps of ξ have size greater than 1 (i.e. when $\Lambda_0([-1,1]) = 0$), but otherwise these two coefficients are usually different. It is well-known that $\mathrm{a}_{\mathrm{can}}$ is then a much more relevant quantity for the Lévy process ξ than the rather artificial a, and we shall therefore call $\mathrm{a}_{\mathrm{can}}$ the **canonic drift coefficient**. Notice that in this case, the cumulant function takes the following simpler form

$$\kappa(\gamma) = \frac{1}{2}\sigma^2\gamma^2 + \mathrm{a}_{\mathrm{can}}\gamma + \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \left(\Big(\sum_{i=0}^{\infty} \mathrm{e}^{\gamma y_i}\Big) - 1 \right). \tag{4.1}$$

4.1 Finite branching activity

One often says that a Lévy process has a finite activity when almost surely, its sample paths have only a finite number of jumps along any time interval of finite duration; this is equivalent to the finiteness of the Lévy measure. In the setting of Section 3.2, recall that $\mathbf{\Lambda}_1$ denotes the image of the generalized Lévy measure $\mathbf{\Lambda}$ by the second projection from $\mathcal{S} = [-\infty, \infty) \times \mathcal{S}_1$ to $\mathcal{S}_1$, and that $\mathbf{\Lambda}_1$ bears a close relation to the reproduction process η; see (3.17) and (3.18). Recall also that in a reproduction event, the degenerate sequence $(-\infty, -\infty, \ldots) \in \mathcal{S}_1$ should be interpreted as empty. So strictly speaking, an atom of the Poisson random measure $\mathbf{N}$ of the form $(t, y, (-\infty, -\infty, \ldots))$ is not associated to any reproduction event, but only to a jump of the decoration at time t.

We say that a self-similar Markov tree with characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ has a **finite branching activity** when

$$\mathbf{\Lambda}_1\big(\mathcal{S}_1 \backslash \{(-\infty, -\infty, \ldots)\}\big) < \infty.$$

This is the case precisely when the first atom of the reproduction process η occurs at a (strictly) positive time a.s., and as a consequence, reproduction events for an individual can only (possibly) accumulate at the end of its lifetime. The tree has then essentially a discrete structure and its branching points do not pile up, except possibly at leaves; see Figures 4.1 and 4.2. Note also that since the killing rate $\mathrm{k} = \mathbf{\Lambda}(\{-\infty\} \times \mathcal{S}_1)$ is always finite, a self-similar Markov tree has a finite branching activity if and only if its generalized Lévy measure satisfies

$$\mathbf{\Lambda}\big(\{(y, \mathbf{y}) \in \mathcal{S} : y = -\infty \text{ or } \mathbf{y} \neq (-\infty, -\infty, \ldots)\}\big) < \infty. \tag{4.2}$$

Having or not a finite branching activity is an intrinsic property of a self-similar Markov tree, hence it does not depend on the choice of the characteristic quadruplet within a family of bifurcators. Moreover, if a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ has a finite branching activity, then we can always choose an equivalent bifurcator, which we denote again by $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ for the sake of simplicity, such that the generalized Lévy measure now fulfills

$$\mathbf{\Lambda}\big(\{(y, \mathbf{y}) \in \mathcal{S} : y \neq -\infty \text{ and } \mathbf{y} \neq (-\infty, -\infty, \ldots)\}\big) = 0. \tag{4.3}$$

This requirement means that reproduction events can only occur at the death of the parent, which makes the reproduction process especially simple to depict. In terms of general branching processes, considering such an equivalent bifurcator amounts to the following somewhat artificial transformation. We decide to kill each individual at the first instant when it begets, and declare that at its death, it gives birth to an additional child then viewed as its reincarnation and whose decoration is thus defined by shifting the decoration of the parent. Although this transformation affects some genealogical aspects of the branching process, it has no impact on the decorated real tree which is induced.

We assume throughout the rest of this section that the generalized Lévy measure $\mathbf{\Lambda}$ satisfies[1]

[1]Of course, (4.2) and (4.3) alone do not grant Assumption (3.9) and the latter is also needed to ensure the existence of the self-similar Markov tree.

(4.2) and (4.3). Then the self-similar Markov tree can be seen as the result of Lamperti's transformation applied to the decorated genealogical tree that records the evolution of a continuous time branching random walk in the sense of Biggins [34, Section 5]. More precisely, the latter depicts a population of individuals living in $\mathbb{R}$, starting at time 0 from a single ancestor located at the origin, and such that each individual lives for an exponentially distributed duration with a fixed parameter. During their lifetimes, individuals move according to independent copies of a Lévy process with characteristic triplet $(\sigma^2, \mathrm{a}, \Lambda_0)$ and are thus killed with rate

$$\mathrm{k} = \Lambda_0(\{-\infty\}) = \mathbf{\Lambda}\big(\{-\infty\} \times \mathcal{S}_1\big) = \mathbf{\Lambda}\big(\{(y, \mathbf{y}) \in \mathcal{S} : y = -\infty \text{ or } \mathbf{y} \neq (-\infty, -\infty, \ldots)\}\big),$$

where the second identity stems from (4.3). In particular, when a death event occurs, the individual which dies is chosen uniformly at random in the current population, independently of its location.

At the time of its death, the parent is replaced by its offspring. The distribution of the children positions relative to the parent is given by the normalized sub-probability measure $\mathrm{k}^{-1}\mathbf{\Lambda}_1$ on $\mathcal{S}_1 \backslash \{(-\infty, -\infty, \ldots)\}$, where the default of mass is the probability that an individual dies without begetting any child. The self-similar Markov tree is then obtained by interpreting the location of an individual in the continuous time branching random walk as a value of the (real) decoration on the genealogical tree, and then performing the Lamperti transformation on each line of descent. Last, we need to take the completion in order to deal with a compact structure.

We will now describe in more detail three examples in which the motions of individuals and their reproductions for the continuous time branching random walk are particularly simple. In the first example, individuals are static and the sole motion occurs at birth. In the second, the motion of individuals is merely a linear drift and children are born at the same location as their parents. In the third, the displacements of individuals are governed by independent Brownian motions with drift, and again children are born at the same location as their parents. For the sake of simplicity, we mostly focus on binary branching, meaning that an individual begets exactly two children when it dies.

Example 4.1 (Static, after Haas [77]). *Consider the characteristic quadruplet with* $\sigma^2 = 0$*,* $\mathrm{a}_{\mathrm{can}} = 0$*,* $\mathbf{\Lambda}_{\mathrm{half}} = \delta_{(-\infty,(-\log 2, -\log 2, -\infty, \ldots))}$ *and an arbitrary* $\alpha > 0$*. In the continuous time branching random walk, each individual lives for a standard exponential duration and does not move until it dies. At death, each parent given birth to two children, both located at distance* $\log 2$ *at the left of the parent.*

The self-similar Markov tree is then obtained by performing the Lamperti transformation, which is elementary. Its structure is very simple to described iteratively: it consists of a branch having a standard exponential length and decorated with the constant function 1*, at the extremity of which two branches of independent exponential lengths with mean* $2^{-\alpha}$*, each decorated with the constant function* $1/2$ *are grafted, and so on and so forth. Finally we take the closure; see Figure 4.1 for an illustration.*

The cumulant is simply

$$\kappa(\gamma) = 2^{1-\gamma} - 1, \qquad \text{for } \gamma > 0.$$

Assumption 3.12 holds with $\omega_- = 1$ and $\omega_+ = \infty$, and since the sum of the decoration of the two children always equals the decoration of the parent, the intrinsic martingale is constant and the total mass of the tree for the harmonic measure is merely 1. Turning our attention to weighted length measures, we observe from Proposition 3.11 that λ^γ is finite for any $\gamma > 1$. An application of the branching property shows that the total mass $\lambda^\gamma(T)$ satisfies the fixed-point equation in distribution

$$\lambda^\gamma(T) \overset{(d)}{=} \mathcal{E}(1) + 2^{-\gamma}(\lambda_1 + \lambda_2)$$

where λ_1 and λ_2 are two independent copies of $\lambda^\gamma(T)$ also independent of the exponential random variable $\mathcal{E}(1)$. ◇

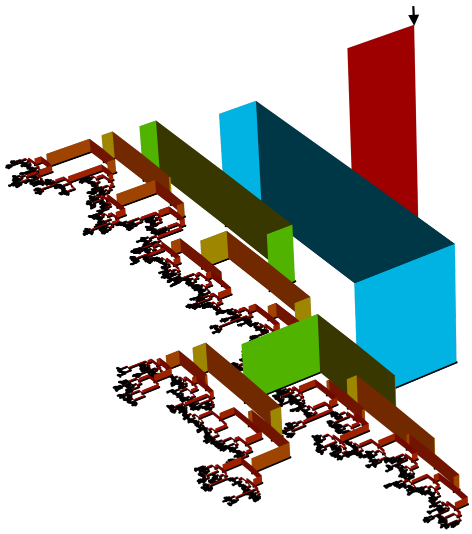

Figure 4.1: A simulation of the self-similar Markov tree in Example 4.1 for $\alpha = 0.4$. The tree T is embedded (non-isometrically) in the plane, and the decoration is represented in the vertical dimension. The root is at the bottom of the right hand side, marked by an arrow.

The second example is a simple variation of the Yule process, which has also a natural interpretation in terms of reduced stable trees, as it will be discussed afterwards.

Example 4.2 (Binary branching and drift)**.** *Consider here the characteristics $\sigma^2 = 0$, $\mathrm{a}_{\mathrm{can}} = -1$, $\boldsymbol{\Lambda}_{\mathrm{two}} = \delta_{(-\infty,(0,0,-\infty,\dots))}$ and $\alpha = 1$. In the continuous time branching random walk, individuals live in $\mathbb{R}_-$ and drift continuously with velocity -1; they die at rate $\mathrm{k} = 1$ and then are replaced by two children at the same location. This is an elementary spatial version of the Yule process (see [11, Chapter III.5]); in particular, the number of individuals at any given time $t \geqslant 0$ has* *the geometric distribution with parameter* e^{-t} *and all these individuals are located at* $-t$*. The sequence of times at which birth events occur is given by the partial sums* $\epsilon_1 + \cdots + \epsilon_n$ *for* $n \geqslant 1$*, where the variables* ϵ_i*'s are independent and each* ϵ_i *has the exponential distribution with parameter* i*. As a consequence, the variables* $\beta_i = \exp(-\epsilon_i)$ *are independent beta*$(i,1)$ *variables with distribution functions* $\mathbb{P}(\beta_i \leqslant r) = r^i$ *for* $r \in [0,1]$*. In the present case, the Lamperti transformation amounts to combining the time-change* $\tau(r) = -\log(1-r)$ *for* $r \in [0,\beta_1)$ *and the exponential map* $y \to \mathrm{e}^y$ *in space. In particular, the decoration process is simply given by* $X(r) = \exp(-\tau(r)) = 1-r$*, for* $r \in [0,\beta_1)$ *and* $X(\beta_1) = 0$*. Furthermore, the ranked sequence of the heights of the branching points in the self-similar Markov tree is* $1 - \beta_1 \cdots \beta_n$*, for* $n \geqslant 1$*.*

Equivalently, the self-similar Markov tree can then be constructed recursively as follows; see Figure 4.2 for an illustration. We start with the line segment with unit length. As a first step, we glue at height $1-\beta_1$ *a segment with length* β_1*. Next we choose one of the two points at height* $1-\beta_1\beta_2$ *uniformly at random and glue there a segment with length* $\beta_1\beta_2$*, and so on, and so forth. That it, at the* n*-th step, we pick on the tree constructed so far one of its* n *points at height* $1-\beta_1\cdots\beta_n$ *uniformly at random and glue there a segment with length* $\beta_1\cdots\beta_n$*. We end the construction by completing the tree and define the decoration as* 1 *minus the height function. The decoration* $g(x)$ *at vertex* x *is then the height of the fringe subtree that stems from* x*.*

We next turn our attention to the intrinsic martingale and the harmonic measure $\upmu$*. By* (3.19)*, we have*

$$\kappa(\gamma) = 1-\gamma, \qquad \textit{for } \gamma > 0,$$

so that Assumption 3.12 holds with $\omega_- = 1$ *and* $\omega_+ = +\infty$*. On the other hand, we see from* (3.18) *that reproduction process simply given by*

$$\eta = 2\delta_{(U,U)},$$

where $U = 1-\exp(-\epsilon_1)$ *has the uniform distribution on* $[0,1]$*. Recall that the harmonic mass* $\upmu(T)$ *is given by the terminal value of the intrinsic martingale, and we see from the branching property that the latter satisfies the distributional identity*

$$\upmu(T) \overset{(d)}{=} U(\mu_1 + \mu_2),$$

where in the right-hand side, the variables μ_1 *and* μ_2 *are independent copies of* $\upmu(T)$*, also independent of* U*. It is easy to deduce (for instance, by considering Laplace transforms) that* $\upmu(T)$ *has the standard exponential distribution, as we may expect from a well-known result on Yule processes (see, e.g. [11, Problem 2 on page 136]).*

We consider as well the weighted length $\uplambda^\gamma(T)$ *for* $\gamma > 1$*, and get similarly the identity in distribution*

$$\uplambda^\gamma(T) \overset{(d)}{=} \gamma^{-1}(1-U^\gamma) + U^\gamma(\lambda_1 + \lambda_2),$$

where in the right-hand side, the variables λ_1 *and* λ_2 *are independent copies of* $\uplambda^\gamma(T)$*, also independent of the uniform variable* U*.* ◇

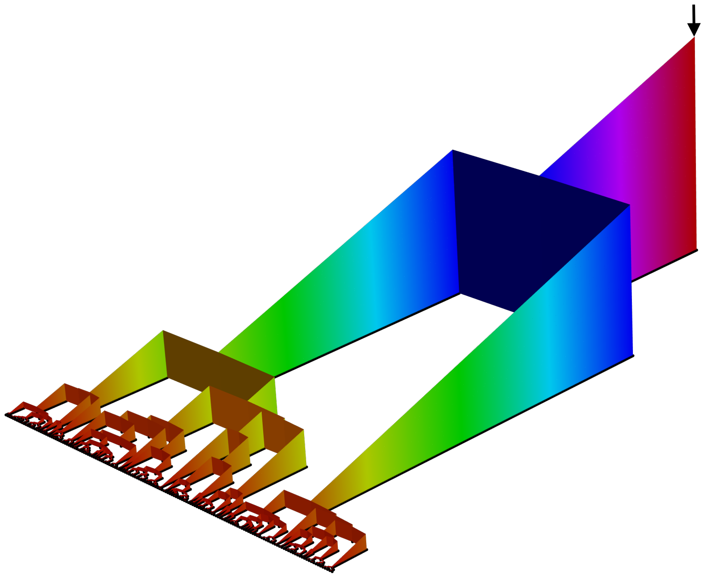

Figure 4.2: A simulation of reduced tree at height 1 in the Brownian CRT. The reduced tree is embedded in the plane and the decoration (corresponding to the height of subtrees) corresponds to the vertical dimension. The root is placed on the right and marked with an arrow.

The self-similar Markov tree in Example 4.2 is well-known in the literature and notably appeared in connection with the Brownian motion and the Brownian CRT. In this direction, let us first briefly recall basics about Itô's measure of Brownian excursions and the coding of real trees by excursions since this will serve in several examples below.

We consider the Itô measure $\mathbb{N}_{\mathrm{Ito}}$ of positive Brownian excursions, which can be constructed using Itô or William's decomposition, see [125, Chapter XII, Theorem 4.2 and 4.5]. Specifically, there exist a family of probability distributions $(\mathbb{N}_\ell)_{\ell>0}$ on excursions of fixed length ℓ, and a family $(\mathbb{N}^{(h)})_{h>0}$ on excursions of fixed height h, which both inherit self-similarity from Brownian motion, such that

$$\mathbb{N}_{\mathrm{Ito}} = \int_0^\infty \frac{\mathrm{d}h}{2h^2}\mathbb{N}^{(h)} = \int_0^\infty \frac{\mathrm{d}\ell}{2\sqrt{2\pi\ell^3}}\mathbb{N}_\ell. \tag{4.4}$$

Next, recall from [66] that any continuous excursion, say $\mathrm{e} : [0,\ell] \to \mathbb{R}_+$ with $\mathrm{e}(0) = \mathrm{e}(\ell) = 0$, encodes a rooted planar continuous tree, say $\mathcal{T}_\mathrm{e}$, via the contour function. The latter is a continuous surjection $c_\mathrm{e} : [0,\ell] \to \mathcal{T}_\mathrm{e}$, and it is then natural to endow $\mathcal{T}_\mathrm{e}$ with the push-forward image of the Lebesgue measure on $[0,\ell]$ by c_e, which we call here the contour measure[2] on $\mathcal{T}_\mathrm{e}$ and denote by γ_e. We shall see in Examples 4.5 and 4.6 that the random tree $\mathcal{T}_\mathrm{e}$ under $\mathbb{N}_1$ or $\mathbb{N}^{(1)}$ can be seen as self-similar Markov tree (see also Example 4.12 for a related example).

[2]The same tree can be encoded by different excursions, and each such excursion induces a contour measure.

At the heart of the connexions between Example 4.2 and Brownian excursions or CRT lies the observation that the sequence $(\beta_1 \cdots \beta_n)_{n\geqslant 1}$ of the random lengths used in the construction by gluing can be obtained as the sequence ranked in the decreasing order of the atoms of a Poisson point measure on $(0,1)$ with intensity $\ell^{-2}\mathrm{d}\ell$. By Brownian excursion theory, the latter has the same statistics as the ranked sequence of the depths of the family of the excursions below 1 for a Brownian motion started from 1 and killed when hitting 0. Now imagine that instead of ranking these lengths in decreasing order, we keep the natural order induced by time) of occurrences of the excursions of the Brownian motion (informally speaking, this corresponds to a uniform random shuffling of the ranked sequence), and add a unit length at the right-end to take into account the final excursion that brings the Brownian motion to 0. By gluing each length $\ell < 1$ at its bottom point to the first larger length at its right, we recover precisely the recursive construction devised in Example 4.2.

This construction can also be interpreted as follows. We work under the Itô measure of positive Brownian excursion conditioned to have height at least 1, which thanks to (4.4) can be expressed in the form

$$\int_1^\infty \frac{\mathrm{d}h}{h^2}\mathbb{N}^{(h)}.$$

Under this conditional probability measure, a sample path can be viewed as the contour process of a planar continuous tree with height at least 1. Imagine that we reduce this tree by removing first all the vertices at height greater than 1, and further by pruning all the remaining branches that do not reach height 1. Then this reduced tree has the same distribution as the self similar Markov tree in Example 4.2; see e.g. [53, Section 2.1].

This reduction process can be applied to the stable trees of the forthcoming Example 4.7. In this case we still have $\alpha = 1$, the drift is $\mathrm{a} = -1$, the Gaussian coefficient $\sigma^2 = 0$ and the generalized Lévy measure is given by

$$\boldsymbol{\Lambda} = \sum_{k\geqslant 1} \delta_{(-\infty,(\underbrace{0,\ldots,0}_{k \text{ times}},-\infty,\ldots))} \cdot \frac{\beta\Gamma(k-\beta)}{k!\Gamma(2-\beta)},$$

so that the associated cumulant function is

$$\kappa(\gamma) = \frac{1}{\beta-1} - \gamma.$$

In particular Assumption 3.12 still holds with $\omega_- = \frac{1}{\beta-1} \in [1,\infty)$ and $\omega_+ = +\infty$, and in this case, the total $\upmu$-mass has Laplace transform

$$\mathbb{E}\big(\mathrm{e}^{-\gamma M_\infty(\omega_-)}\big) = 1 - \left(1 + \frac{1}{\gamma^{\beta-1}}\right)^{-1/(\beta-1)},$$

see [109] and the references therein.

We end this subsection with a last example where now the motions of individuals are diffusion processes.

Example 4.3 (Branching Bessel processes). *Consider the characteristics $\sigma^2 = 1$, $\mathrm{a_{can}} \in \mathbb{R}$, $\boldsymbol{\Lambda}_{\mathrm{two}} = \delta_{(-\infty,(0,0,-\infty,\dots))}$, and self-similarity parameter $\alpha = 2$. The continuous time branching process is then the well-known binary branching Brownian motion with drift $\mathrm{a_{can}}$, and the cumulant function is*

$$\kappa(\gamma) = \frac{\gamma^2}{2} + \mathrm{a_{can}}\gamma + 1.$$

It satisfies Assumption 3.12 as soon as $\mathrm{a_{can}} < -\sqrt{2}$ and then $\omega_- = -\mathrm{a_{can}} - \sqrt{\mathrm{a_{can}}^2 - 2}$ and $\omega_+ = -\mathrm{a_{can}} + \sqrt{\mathrm{a_{can}}^2 - 2}$. By the Lamperti transformation, the self-similar Markov process X associated to the Brownian motion with drift $B_t + \mathrm{a_{can}}t$ is a Bessel process with dimension $d = 2\mathrm{a_{can}} + 2$. See Figure 4.3 for an illustration.

Similarly to Example 4.2, the spatial marginal of the reproduction measure is

$$\eta(\mathbb{R}_+ \times \mathrm{d}x) = 2\delta_{\exp(B_\epsilon + \mathrm{a_{can}}\epsilon)}(\mathrm{d}x),$$

where ϵ is an exponentially distributed random time independent of the Brownian motion B. The density of the variable $B_\epsilon + \mathrm{a_{can}}\epsilon$ is given by

$$\mathbb{P}(B_\epsilon + \mathrm{a_{can}}\epsilon \in \mathrm{d}y) = (2 + \mathrm{a_{can}}^2)^{-1/2} \exp(\mathrm{a_{can}}y - |y|\sqrt{2 + \mathrm{a_{can}}^2})\mathrm{d}y,$$

see [39, (1.0.5) on page 256]. The terminal value of the intrinsic martingale, that is the mass of the harmonic measure, then satisfies the distributional identity

$$M_\infty(\omega_-) \overset{(d)}{=} \exp(\omega_-(B_\epsilon + \mathrm{a_{can}}\epsilon))(M_\infty(\omega_-) + M'_\infty(\omega_-)),$$

where in the right-hand side, the variables $B_\epsilon + \mathrm{a_{can}}\epsilon$, $M_\infty(\omega_-)$ and M'_∞ are independent, and M'_∞ is a copy of $M_\infty(\omega_-)$. It is not known to us whether an explicit expression for the distribution of $M_\infty(\omega_-)$ can be derived from this. $\diamond$

Remark 4.4 (Critical case). If we take $\mathrm{a_{can}} = -\sqrt{2}$ in the above example, then the overall minimum of the cumulant function κ equals 0, which we called the *critical case* in the comments section of Chapter 3. In this example, we leave open the fact that the procedure of Chapter 3 actually produces a compact decorated random tree.

4.2 Non-increasing decorations and fragmentations

We say that a decoration on rooted real tree is non-increasing when its restriction to any segment from the root defines a non-increasing function of the height. We then call a self-similar Markov tree **non-increasing** if its decoration is non-increasing, almost-surely. In that case, the self-similar Markov process X that describes the decoration for a typical individual must plainly have non-increasing sample paths, that is, equivalently, the underlying Lévy process ξ that gives X after the Lamperti transformation must be the negative of a subordinator (possibly with killing). However, this requirement is clearly not sufficient; one needs to impose further that

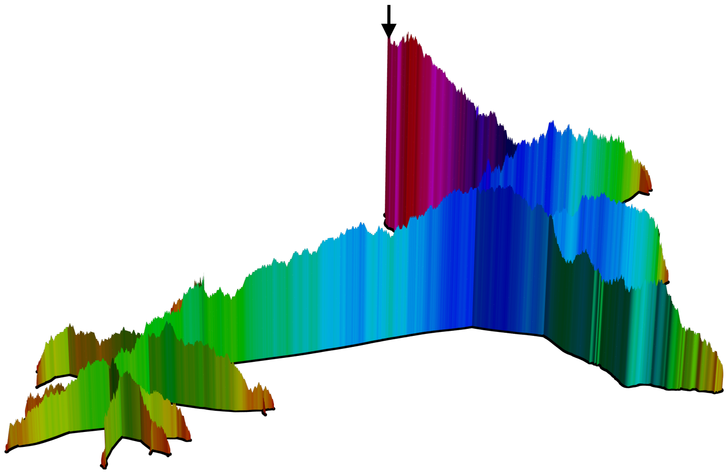

Figure 4.3: A simulation of the Branching Bessel process with $\mathrm{a_{can}} = -3$. As usual, the decoration (the values of the Bessel processes) is displayed in the vertical coordinate. The root of the decorated tree is marked by an arrow.

the reproduction process η has no atom, say at (t, x), with $X(t-) < x$. The latter translates in terms of the generalized Lévy measure as

$$\boldsymbol{\Lambda}_1\big(\{\mathbf{y} = (y_1, \dots) \in \mathcal{S}_1 : y_1 > 0\}\big) = 0. \tag{4.5}$$

In the converse direction, it is immediately seen that if (4.5) holds and if the underlying Lévy process ξ is the negative of a subordinator, then the self-similar Markov tree is non-increasing. For instance, Examples 4.1 and 4.2 are non-increasing self-similar Markov trees (with a finite branching activity), but not Example 4.3. In particular, a (possibly killed) real Lévy process is a subordinator – that is, it has non-decreasing sample paths until it eventually dies almost surely – if and only if its Gaussian coefficient is $\sigma^2 = 0$, its Lévy measure $\Lambda_0(\mathrm{d}y)$ gives zero mass to $(-\infty, 0)$ and integrates the function $1 \wedge |y|$, and finally, its canonic drift coefficient is nonnegative, $\mathrm{a_{can}} \geqslant 0$.

Putting the pieces together, we arrive at the following analytic definition in terms of the characteristic quadruplet[3]. A self-similar Markov tree is non-increasing if and only if its Gaussian coefficient is zero, $\sigma^2 = 0$, its generalized Lévy measure verifies

$$\boldsymbol{\Lambda}\big(\{(y, \mathbf{y}) \in \mathcal{S} : y > 0 \text{ or } y_1 > 0\}\big) = 0,$$

and

$$\int_{(-\infty,0)} (1 \wedge |y|)\Lambda_0(\mathrm{d}y) < \infty, \tag{4.6}$$

[3]Whether or not a self-similar Markov tree is non-increasing is obviously an intrinsic property, which does not depend of the choice of the characteristic quadruplet within a family of bifurcators.

and finally, its canonic drift coefficient is non-positive,

$$\mathrm{a}_{\mathrm{can}} = \mathrm{a} - \int_{[-1,0)} y\Lambda_0(\mathrm{d}y) \leqslant 0.$$

We stress however that at this stage, the cumulant function κ in (3.19) could potentially be infinite everywhere, and *a fortiori* Assumption (3.9) may fail in general.

We now present an example of a non-increasing self-similar Markov tree with an infinite branching activity which is related to Example 4.2. Recall the notation regarding Itô excursion measure and contours of planar trees that was introduced after Example 4.2.

Example 4.5 (Heights of Brownian sub-trees). *Let us denote the Brownian CRT of height* 1 *by* $\mathcal{T}^{(1)}$*, i.e. the tree* $\mathcal{T}_{\mathrm{e}^{(1)}}$ *where* $\mathrm{e}^{(1)}$ *follows the law* $\mathbb{N}^{(1)}$ *in* (4.4)*. We further endow* $\mathcal{T}^{(1)}$ *with the deterministic decoration which assigns to each vertex* $v \in \mathcal{T}^{(1)}$ *the height of the fringe-subtree* $\mathcal{T}_v^{(1)}$ *rooted at* v *as defined in Section 2.1; see Figure 4.4 for an illustration. We shall now argue that this decorated random real tree is a self-similar Markov tree and determine its characteristics.*

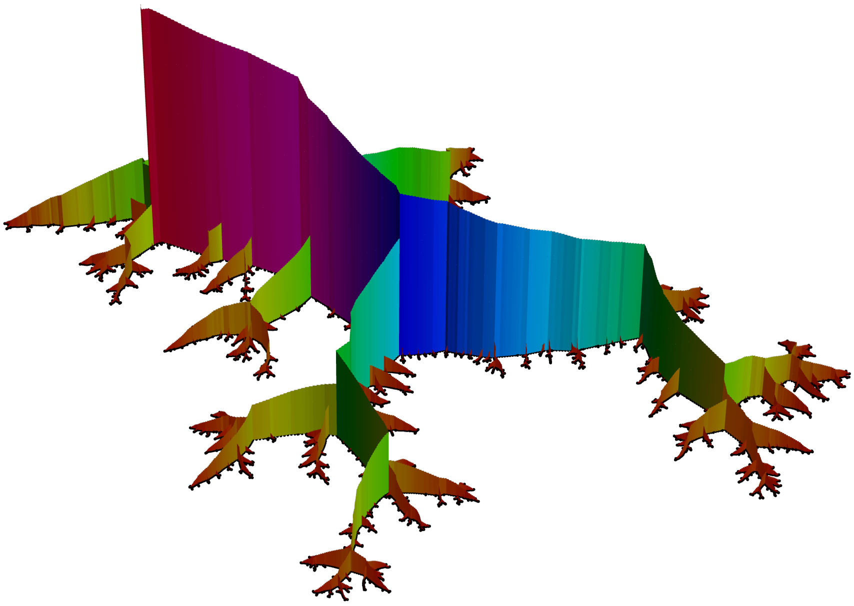

Figure 4.4: A simulation of a Brownian CRT normalized by the height. The tree is embedded non-isometrically in $\mathbb{R}^2$; the decoration function represents the height of fringe subtrees and is depicted in the vertical coordinate.

As a first step, recall from a well-known result of David Williams, that the Brownian excursion with height 1 *can be constructed by gluing back to back the trajectories of two independent copies of a* 3*-dimensional Bessel process started from* 0 *and killed when hitting* 1*, say* $R = (R(s))_{0 \leqslant s \leqslant z}$*. See [125, Theorem 4.5 on page 499]. We observe from Brownian excursion theory and classical relations between the Brownian motion and the* 3*-dimensional Bessel process* *also due to Williams, that if we set $\sigma_u := \sup\{s \geqslant 0 : R(s) \leqslant u\}$ for the last passage time of R below some level $u \in (0,1)$, then the following assertion holds. The point process of sub-excursions of R above its future infimum, whose atoms are given by*

$$\left(u, (R_{\sigma_{u-}+s} - u : 0 \leqslant s \leqslant \sigma_u - \sigma_{u-})\right)$$

for those u such that $\sigma_u > \sigma_{u-}$, is a Poisson random measure with intensity

$$\mathbf{1}_{h(\mathrm{e})<1-u}\mathrm{d}u\, 2\mathbb{N}_{\mathrm{Ito}}(\mathrm{de}).$$

Here e stands for a generic excursion, $h(\mathrm{e}) = \max \mathrm{e}$ for the height of e, and the restriction $h(\mathrm{e}) < 1-u$ stems from the fact that the maximum of R during the excursion interval $[\sigma_{u-}, \sigma_u]$ must be less than 1. See for instance [121], and also [2] for a more general version in the setting of Lévy's CRT. If we think of R as the left-contour of $\mathcal{T}^{(1)}$ until the highest vertex is reached, then an atom of this point measure, say at (u, e), should be interpreted as follows. The vertex at height u on the branch from the root to the highest vertex is a branchpoint of $\mathcal{T}^{(1)}$, and a contour function of the left subtree that stems from this vertex is induced by the excursion e.

Now from (4.4) we have $2\mathbb{N}_{\mathrm{Ito}}(h(\mathrm{e}) \in \mathrm{d}x) = x^{-2}\mathrm{d}x$. Since by Williams' decomposition, the full contour process of $\mathcal{T}^{(1)}$ is obtained from two independent copies of R, one now readily sees from the Brownian scaling property that the process that records the heights of the collection of the subtrees of $\mathcal{T}^{(1)}$ above height $t \in [0,1]$ is a general branching process in the sense of Section 3.1, whose decoration-reproduction kernel can be described as follows. For every $x > 0$, under P_x, the decoration process is simply given by $X(t) = x - t$ for $0 \leqslant t < x$, and the reproduction process $\eta(\mathrm{d}u, \mathrm{d}y)$ is a Poisson random measure on the triangle $\{(u, y) : 0 < y < x - u < x\}$ with intensity $2y^{-2}\mathrm{d}u\mathrm{d}y$ (the factor 2 stems from the fact that two independent copies of R are needed to construct the contour function of $\mathcal{T}^{(1)}$). One immediately checks that the family of laws $(P_x)_{x>0}$ is self-similar with exponent $\alpha = 1$ in the sense of Definition 3.5, and we shall now argue that it is actually one of the self-similar Markov decoration-reproduction kernels devised in Section 3.2.

In this direction, set $\sigma^2 = 0$, $\mathrm{a}_{\mathrm{can}} = -1$ and $\alpha = 1$, and introduce the generalized Lévy measure $\mathbf{\Lambda}_{\mathrm{Height}}$ on $\mathcal{S}$ given by

$$\int_{\mathcal{S}} F(y_0, (y_1, y_2, \dots))\mathbf{\Lambda}_{\mathrm{Height}}(\mathrm{d}y, \mathrm{d}\mathbf{y}) = 2\int_{-\infty}^{0} F(0, (y, -\infty, \dots))\mathrm{e}^{-y}\mathrm{d}y,$$

where F denotes a generic nonnegative functional on $\mathcal{S}$. In particular, notice that we have $\mathrm{a}_{\mathrm{can}} = \mathrm{a} = -1$. Just as in Section 3.2, consider a Poisson random measure $\mathbf{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}\mathbf{y})$ on $[0,\infty) \times \mathcal{S}$ with intensity $\mathrm{d}t\mathbf{\Lambda}_{\mathrm{Height}}(\mathrm{d}y, \mathrm{d}\mathbf{y})$. In this setting, the Lévy process is merely linear, $\xi(t) = -t$ for $t \geqslant 0$, and the exponential functional $\epsilon(t) = 1 - \mathrm{e}^{-t}$ has inverse $\tau(s) = -\log(1-s)$ for $0 \leqslant s < 1$. After the Lamperti transformation, the positive self-similar Markov process is hence $X(t) = 1 - t$ for $0 \leqslant t < 1$ as we wished. Next observe that the push-forward image of the measure $\mathrm{d}t\mathrm{e}^{-y}\mathrm{d}y$ on $\mathbb{R}_+ \times (-\infty, 0)$ by the map $(t, y) \mapsto (u, x) = (1 - \mathrm{e}^{-t}, \mathrm{e}^{-t+y})$ is the measure $x^{-2}\mathrm{d}u\mathrm{d}x$ on the triangle $\{(u, x) : 0 < x < 1 - u < 1\}$. It follows from the mapping *theorem for Poisson point processes that the reproduction process defined by* (3.18) *has the same distribution as the reproduction process* η *above under* P_1. *Putting the pieces together, we can now conclude that the decorated tree* $(\mathcal{T}^{(1)}, v \to \mathrm{Height}(\mathcal{T}_v^{(1)}), 0)$ *is a self-similar Markov tree with characteristic quadruplet* $(0, -1, \mathbf{\Lambda}_{\mathrm{Height}}; 1)$.

As a consequence, we compute the cumulant by (4.1), *which is simply*

$$\kappa_{\mathrm{Height}}(\gamma) = -\gamma + 2\int_{-\infty}^{0} \mathrm{e}^{(\gamma-1)y}\mathrm{d}y = -\gamma + 2/(\gamma - 1), \qquad \gamma > 1.$$

It follows that $\omega_- = 2$ *and* $\omega_+ = \infty$ *and Assumption 3.12 holds. We will now identify the harmonic measure* μ *in terms of the contour measure* $\gamma_{\mathrm{e}^{(1)}}$ *on* $\mathcal{T}^{(1)}$, *which has been defined above as the push-forward image of the Lebesgue measure by the contour function induced by* $\mathrm{e}^{(1)}$. *We claim that there is the identity*

$$\mu = \frac{3}{2}\gamma_{\mathrm{e}^{(1)}}. \tag{4.7}$$

To establish this assertion, recall that the expectation of the lifetime z *of the killed Bessel process* R *equals* $1/3$. *Therefore, the total mass of contour measure has expectation* $\mathbb{E}(\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}^{(1)})) = 2/3$; *moreover we have also* $\mathrm{Var}(\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}^{(1)})) < \infty$. *Now recall that* $\chi(v)$ *denotes the type of the individual labelled by* $v \in \mathbb{U}$ *in the general branching process, so that in the present setting,* $\chi(v)$ *is the height of some sub-excursion of* $\mathrm{e}^{(1)}$. *We then deduce from the branching property and self-similarity that*

$$\begin{aligned}
\mathbb{E}\left(\left|\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}^{(1)}) - \frac{2}{3}M_n(\omega_-)\right|^2\right) &= \mathbb{E}\left(\sum_{|v|=n}\left|\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}_v^{(1)}) - \frac{2}{3}\chi(v)^2\right|^2\right) \\
&\leqslant c \cdot \mathbb{E}\left(\sum_{|v|=n}\chi(v)^4\right) \\
&\leqslant c \cdot (1 + \kappa(4)/4)^n,
\end{aligned}$$

where the last line stems from Lemma 3.8. Since $\kappa(4) < 0$, *we deduce that*

$$\mu(\mathcal{T}^{(1)}) = \lim_{n\to\infty} M_n(\omega_-) = \frac{3}{2}\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}^{(1)}).$$

Again by the branching property, we have more generally that

$$\mu(\mathcal{T}_v^{(1)}) = \frac{3}{2}\gamma_{\mathrm{e}^{(1)}}(\mathcal{T}_v^{(1)}), \qquad v \in \mathbb{U},$$

and (4.7) *follows.* ◇

We next discuss an important family of non-increasing self-similar Markov trees that have received much attention in the literature; see for instance the survey [18] and references therein. A **self-similar fragmentation** process can be thought of as a model for an inert length that splits as time passes into smaller and smaller pieces called fragments. One assumes the branching property and self-similarity, in the sense that different fragments evolve independently the ones from the others and according to the same dynamics up to a proper rescaling of time and size. Intuitively speaking, the inertia of the length falling apart yields a natural genealogical structure of the family of fragments: we view a fragment present at time t as a forebear of another fragment present at time $t' > t$ if the latter was part of the former at time t. Hence a fragmentation can be depicted by a continuous genealogical tree where branching points represent the sudden dislocations of a fragment, endowed with a decoration which records sizes.

Roughly speaking, self-similarity entails that the rate at which a fragment with size $x > 0$ breaks into a (possibly infinite) sequence of sizes $xs_1, xs_2, \ldots$ is given by $x^{-\alpha}\boldsymbol{\Xi}(\mathrm{d}\mathbf{s})$, where $\mathbf{s} = (s_1, s_2, \ldots)$ is a non-increasing sequence in $[0,1)$ and $\boldsymbol{\Xi}$ is known as the **dislocation measure**. Beware that the convention for the self-similarity exponent α that we follow in this text has the opposite sign of the one used in the literature on self-similar fragmentations, see e.g. [77, 78]. So for $\alpha > 0$, small fragments break up at a higher pace than larger ones, hence faster and faster as time passes. An important consequence is that the entire length becomes fully shattered after a finite time; in other words, there are no more fragments with positive length present in the system after a while.

The assumption of inertia of the length falling apart translates into the requirement that the sum of the sizes of the smaller fragments after a dislocation event never exceeds that of the fragment before the dislocation, and can even be strictly smaller in the case where some dust (fragments of infinitesimal sizes) is produced. So more precisely, $\boldsymbol{\Xi}$ is a measure on the space of non-increasing sequences $\mathbf{s} = (s_i)_{i \geqslant 1}$ in $[0,1)$ with $\sum_{i \geqslant 1} s_i \leqslant 1$, which furthermore fulfills the integral condition

$$\int (1 - s_1)\boldsymbol{\Xi}(\mathrm{d}\mathbf{s}) < \infty. \tag{4.8}$$

Note that this requirement allows the dislocation measure $\boldsymbol{\Xi}$ to be infinite, that is, the fragmentation to have an infinite activity. If (4.8) failed, then the intensity of dislocations would then be too strong and any length would be instantaneously reduced to dust.

Self-similar fragmentations naturally yield non-increasing self-similar Markov trees. More precisely, the dislocation measure $\boldsymbol{\Xi}$ of a fragmentation is related to the generalized Lévy measure $\boldsymbol{\Lambda}$ as follows. The former is the push-forward image of the latter by the function which maps $(y_0, \mathbf{y}) \in \mathcal{S}$ to $(s_i)_{i \geqslant 1}$, the version ranked in the non-increasing order of sequence $(\mathrm{e}^{y_0}, \mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \ldots)$. Note that (4.8) follows from (4.6) and that

$$\boldsymbol{\Lambda}\left(\left\{(y_0, (y_1, \ldots)) \in \mathcal{S} : \sum_{j=0}^{\infty} \mathrm{e}^{y_j} > 1\right\}\right) = \boldsymbol{\Xi}\left(\left\{\mathbf{s} : \sum_{i=1}^{\infty} s_i > 1\right\}\right) = 0;$$

one then says that $\boldsymbol{\Lambda}$ is **dissipative**. The last parameter of the model is the so-called **erosion coefficient**, a non-negative real number which accounts for rate at which fragments continuously shrink and that is simply identified as the negative of the canonic drift coefficient $\mathrm{a}_{\mathrm{can}}$. Putting the pieces together, a self-similar Markov tree encodes a self-similar fragmentation if and only if it is non-increasing and its generalized Lévy measure is dissipative. Note from (4.1) that the cumulant can also be expressed in terms of the dislocation measure as

$$\kappa(\gamma) = \mathrm{a}_{\mathrm{can}}\gamma + \int \boldsymbol{\Xi}(\mathrm{d}\mathbf{s}) \left(\sum_{i \geqslant 1} s_i^{\gamma} - 1 \right).$$

By dissipativity, κ is a non-increasing function with $\kappa(1) \leqslant 0$, and in particular, Assumption (3.9) holds for $\gamma > 1$ as soon as $\boldsymbol{\Xi}$ is not degenerated, i.e. $\boldsymbol{\Xi} \neq \delta_{(1,0,\dots)}$.

The cumulant κ has a simple probabilistic interpretation in terms of the so-called tagged fragment. The latter is the process that, as time passes, records the size of the fragment that currently contains a point that has been initially tagged uniformly at random in the length. It is easily seen from the branching property that this tagged fragment has the Markov property, and it is also naturally self-similar. The Lévy process that underlies the latter via the Lamperti transformation is non-increasing, hence the negative of a subordinator. In this framework, the Laplace exponent of this subordinator is given by the function $\gamma \mapsto -\kappa(\gamma + 1)$; see e.g. [18, Corollary 3.1]. This observation is in close relation to the forthcoming Lemma 6.4.

A fragmentation is called pure when there is no erosion, that is when the canonic drift coefficient is $\mathrm{a}_{\mathrm{can}} = 0$, and then **conservative** when further total sizes are preserved at dislocation events, that is when the dislocation measure satisfies

$$\boldsymbol{\Xi}\left(\left\{ \mathbf{s} = (s_i)_{i \geqslant 1} : \sum_{i=1}^{\infty} s_i < 1 \right\} \right) = 0,$$

or equivalently in terms of the generalized Lévy measure

$$\boldsymbol{\Lambda}\left(\left\{ (y_0, (y_1, \dots)) : \sum_{j=0}^{\infty} \mathrm{e}^{y_j} \neq 1 \right\} \right) = 0. \tag{4.9}$$

Example 4.1 is an elementary case of a conservative fragmentation.

For any conservative self-similar fragmentation, we have $\kappa(1) = 0$, so Assumption 3.12 holds with $\omega_- = 1$ and $\omega_+ = +\infty$. Much more precisely, conservativeness readily entails in terms of the reproduction process η that there is the identity

$$\int_{[0,\infty) \times (0,\infty)} x \eta(\mathrm{d}t, \mathrm{d}x) = 1 \qquad \text{a.s.},$$

and therefore the intrinsic martingale is trivial, $M_n(\omega_-) \equiv 1$. As a consequence, the decoration of a self-similar conservative fragmentation simply assigns to any vertex of the tree which is not a branching point, the $\upmu$-mass of the subtree that stems from this vertex.

Genealogical trees of self-similar fragmentations have been constructed first by Haas & Miermont [78] in the conservative setting. They were notably able, under mild hypotheses, to compute Hausdorff dimensions as well as the maximal Hölder exponents of the height functions. Specifically, they showed that the Hausdorff dimension of the set of leaves is $1/\alpha$. We will extend this result to our general framework of ssMt in Section 6.4, and specifically in Proposition 6.14 (actually, [78] has a mild assumption that is not needed Proposition 6.14). Then this was extended to more general self-similar fragmentations where the conservativeness condition is dropped by Stephenson [134]. We also mention that in the case $\mathrm{a}_{\mathrm{can}} = 0$ without erosion, the total length measure $\lambda^\gamma(T)$ has been introduced and studied in [19].

We already dealt in Example 4.5 (see also the discussion after Example 4.2) with a variation of the most distinguished member of the family of self-similar fragmentation trees, namely the ubiquitous Brownian Continuum Random Tree [8]. Here is a precise discussion.

Example 4.6 (Brownian CRT and its fragmentation)**.** *The real tree $\mathcal{T}_1$ constructed from a standard Brownian excursion of length 1, say $(\mathrm{e}_1(s))_{0\leqslant s\leqslant 1}$ with the law $\mathbb{N}_1$ defined in (4.4), is known as the **Brownian CRT**[4]. We denote by γ_{e_1} its contour measure and endow $\mathcal{T}_1$ with the decoration which assigns to each vertex $v \in \mathcal{T}_1$ the contour-mass $\gamma_{\mathrm{e}_1}(\mathcal{T}_{1,v})$ of the fringe-subtree above point v (this is indeed a usc decoration), see Figure 4.5.*

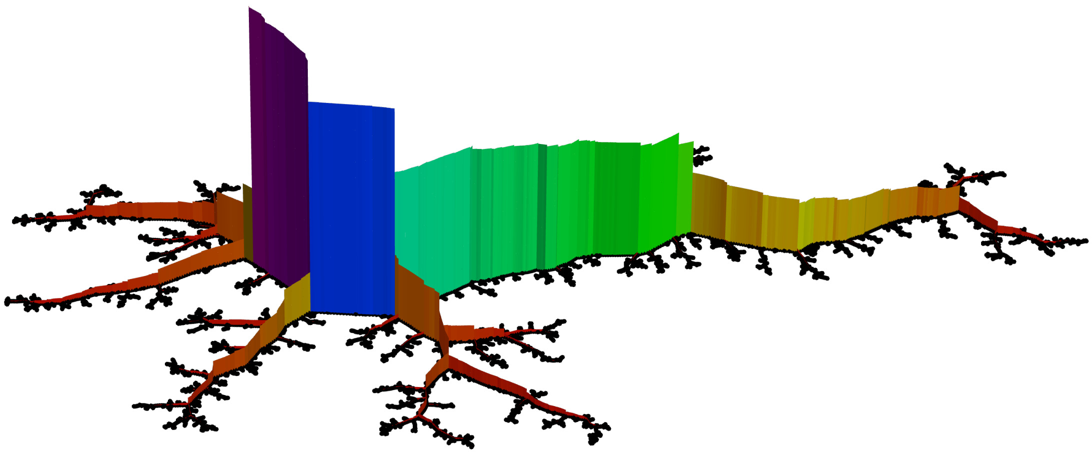

Figure 4.5: A simulation of a Brownian CRT. The tree is embedded (non-isometrically in $\mathbb{R}^2$) and the decoration function representing the μ-mass above each point is depicted in the vertical coordinate.

Just as in Example 4.5, this decorated random real tree is a self-similar Markov tree. By construction, it encodes a self-similar fragmentation whose state at time $t \geqslant 0$ is given by the ranked sequence of decorations on the sphere $\{v \in \mathcal{T}_1 : d(\rho, v) = t\}$. Equivalently in terms of the excursion e_1, for every $t \geqslant 0$, the random open set $\{s \in [0,1] : \mathrm{e}_1(s) > t\}$ can be decomposed into a (possibly empty) sequence of open intervals; the process in the variable t that records the sequence of the lengths of these intervals ranked in the decreasing order, is the Brownian fragmentation. The characteristics are found using [17, pages 338-340] and [135].

The Brownian fragmentation tree is self-similar with index $\alpha = 1/2$, no erosion, and binary

[4]Beware that, as in [94] and many works in that field, there is a difference by a factor 2 between our definition of the Brownian CRT and that in [8].

conservative dislocation measure Ξ_{Bro} *given by*

$$\int F(s_1, s_2, \dots)\ \Xi_{\mathrm{Bro}}(\mathrm{d}\mathbf{s}) := \sqrt{\frac{2}{\pi}} \int_{1/2}^{1} F(x, 1-x, 0, 0, \ldots) \frac{\mathrm{d}x}{(x(1-x))^{3/2}}, \tag{4.10}$$

where F *stands for a generic nonnegative functional of the sequence of fragments. The cumulant function can be evaluated using* (4.1)*:*

$$\kappa_{\mathrm{Bro}}(\gamma) = \sqrt{\frac{2}{\pi}} \int_0^1 (x^\gamma + (1-x)^\gamma - 1) \frac{\mathrm{d}x}{(x(1-x))^{3/2}} = -2\sqrt{2}\frac{\Gamma(\gamma - 1/2)}{\Gamma(\gamma-1)}, \quad \gamma > 1. \tag{4.11}$$

As for all conservative fragmentations, we have $\omega_- = 1$ *and* $\omega_+ = \infty$*, Assumption 3.12 holds and the total* $\upmu$*-mass is trivially* x *under* $\mathbb{P}_x$*. In particular, we have the almost sure equality* $\upmu = \gamma_{\mathrm{e}_1}$ *so that the decoration in* $\mathcal{T}_1$ *can be recovered from its harmonic measure and vice-versa.*

A possible choice for the generalized Lévy measure in the family of bifurcators is then obtained by following the largest fragment at each dislocation. This amounts to defining first the measure $\Lambda_{\mathrm{Bro,max}}$ *on* $\mathbb{R}_-$ *as the push-forward image of the dislocation measure* Ξ_{Bro} *by the map* $\mathbf{s} \mapsto \log x$ *with* $\mathbf{s} = (x, 1-x, 0, \dots)$ *for* $x \in [1/2, 1)$*. Concretely, this gives*

$$\begin{aligned}\int_{\mathbb{R}_-} f(y) \Lambda_{\mathrm{Bro,max}}(\mathrm{d}y) &= \sqrt{\frac{2}{\pi}} \int_{1/2}^{1} f(\log x) \frac{\mathrm{d}x}{(x(1-x))^{3/2}} \\ &= \sqrt{\frac{2}{\pi}} \int_{-\log 2}^{0} f(y) \frac{\mathrm{d}y}{\sqrt{\mathrm{e}^y (1-\mathrm{e}^y)^3}}.\end{aligned}$$

Then the generalized Lévy measure $\boldsymbol{\Lambda}_{\mathrm{Bro,max}}$ *is simply obtained by*

$$\int_{\mathbb{R}\times\mathcal{S}_1} F(y_0, \mathbf{y}) \boldsymbol{\Lambda}_{\mathrm{Bro,max}}(\mathrm{d}y, \mathrm{d}\mathbf{y}) = \int_{\mathbb{R}_-} F(y, (\log(1-\mathrm{e}^y), -\infty, \dots)) \Lambda_{\mathrm{Bro,max}}(\mathrm{d}y),$$

where now F *stands for a generic nonnegative functional on* $\mathcal{S}$*.*

Notice also that using the fact that the Brownian CRT is coded by the Brownian excursion [94, 16], the quantity $\uplambda^{3/2}(T) = \int_T \mathrm{d}\uplambda\ g$ *can be interpreted as the area under a standard Brownian excursion, known as the Airy law and whose moments are explicit, see [84] for details. Other length measures appeared recently in the literature [71, 60, 5].*

This Brownian fragmentation has been studied in depth in the literature and we did not try here to survey all its known properties, see e.g. [38] for a recent application to the study of increasing subsequences in the Brownian separable permuton. Finally, it is interesting to recall that Aldous [9] has a construction of the Brownian CRT by recursive gluing of line segments that is somewhat similar to the construction in Example 4.1. ◇

One says that a conservative self-similar fragmentation is **binary** if, just like as in Example 4.6, exactly two fragments are produced at each dislocation event. Specifically, the dislocation measure must satisfy

$$\boldsymbol{\Xi}\left(\{\mathbf{s} = (s_i)_{i\geqslant 1} : s_3 > 0\}\right) = 0,$$

that is equivalently in terms of the generalized Lévy measure to asking

$$\mathbf{\Lambda}\left(\{(y_0,(y_1,...)) : y_3 \neq -\infty\}\right) = 0. \tag{4.12}$$

We already observed in Example 4.6 that in the binary conservative case, the generalized Lévy measure $\mathbf{\Lambda}$ is entirely determined by its first marginal Λ_0 and the identity

$$\int_{\mathbb{R}\times\mathcal{S}_1} F(y_0,\mathbf{y})\mathbf{\Lambda}(\mathrm{d}y,\mathrm{d}\mathbf{y}) = \int_{\mathbb{R}_-} F(y,(\log(1-\mathrm{e}^y),-\infty,\ldots))\Lambda_0(\mathrm{d}y). \tag{4.13}$$

Moreover, the reproduction process η is determined by the decoration process X since we have

$$\eta = \sum_{t>0} \delta_{(t,-\Delta X(t))}. \tag{4.14}$$

Pitman and Winkel [122] and [123, Section 3] devised first a recursive construction of binary conservative self-similar fragmentation trees using a so-called bead splitting processes that generalizes Aldous' line-breaking construction of the Brownian CRT.

The stable CRT, introduced by Le Gall – Le Jan & Duquesne [101, 66], form a one-parameter family indexed by $\beta \in (1,2]$ of random continuous trees, where the boundary case $\beta = 2$ is the Brownian CRT. They appear as scaling limits of critical Galton–Watson trees whose offspring distribution belong to the domain of attraction of a spectrally positive stable law with index β. They also belong to the family of self-similar fragmentation trees of Haas and Miermont [78], and satisfy the striking property that their distribution is invariant under re-rooting at a random μ-point [80]. A notable difference in the case $\beta < 2$ is that branchpoints have infinite degrees almost surely, whereas branching are always binary for the Brownian CRT.

Example 4.7 (Stable fragmentation trees)**.** *Just as in Example 4.6, the stable trees can be interpreted as self-similar Markov trees. More precisely, for $\beta \in (1,2)$, if $\mathrm{h}_\beta : [0,1] \to \mathbb{R}_+$ is the stable excursion height process introduced in [101], we can consider the random real tree $\mathcal{T}_{\mathrm{h}_\beta}$ decorated by the $\gamma_{\mathrm{h}_\beta}$-mass of its fringe-subtrees. It then follows from the work of Miermont [113] that this is a self-similar Markov tree with self-similar exponent $\alpha = 1 - 1/\beta$, canonical drift $\mathrm{a}_{\mathrm{can}} = 0$ and no Brownian component. The generalized Lévy measure $\mathbf{\Lambda}_{\beta-\mathrm{stable}}$ is conservative but non-binary, and can be related to the Poisson-Dirichlet measure with parameter $(1/\beta,-1)$. More precisely, it can be given by*

$$\int F\left(\mathrm{e}^{y_0},(\mathrm{e}^{y_1},\ldots)\right)\,\mathbf{\Lambda}_{\beta-\mathrm{stable}}(\mathrm{d}y_0,\mathrm{d}\mathbf{y}) \underset{[113, Thm1]}{:=} \frac{\beta^2\Gamma(2-1/\beta)}{\Gamma(2-\beta)}\cdot\mathbb{E}\left[S_1 F\left(\frac{\Delta S_{t_i}}{S_1} : i \geqslant 1\right)\right],$$

where ΔS_{t_i} are the jumps of a stable $1/\beta$ subordinator $(S_t)_{0\leqslant t\leqslant 1}$ started from 0 on the unit time interval, ranked in decreasing order.

As for all conservative fragmentations we have $\omega_- = 1$ and $\omega_+ = +\infty$, Assumption 3.12 holds with trivial total μ mass, and as in the previous example, $\mu = \gamma_{\mathrm{e}_{h_\beta}}$ and the decoration $g(x)$ is the μ-measure of the subtree above point $x \in T$. The length measures on T have also been considered in [71, 60, 5].

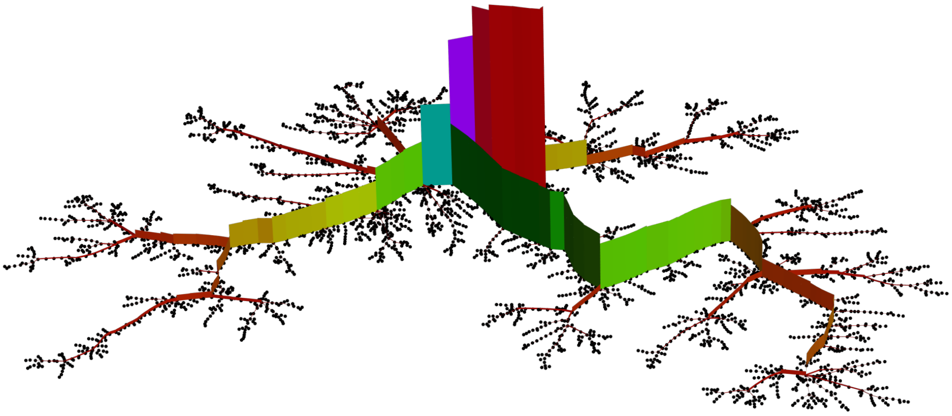

Figure 4.6: A simulation of a 3/2-stable fragmentation tree embedded in the plane with the decoration in the third coordinate.

The exact form of the cumulant function was computed in Section 3.4 in [113], which established that

$$\kappa_{\beta-\text{stable}}(\gamma) \quad = \quad -\beta \cdot \frac{\Gamma(\gamma - 1/\beta)}{\Gamma(\gamma - 1)}.$$

In this normalization, the β-stable tree encodes the genealogy of the CSBP with branching mechanism $z \mapsto z^\beta$, and when $\beta = 2$ the associated random tree is $\sqrt{2}$ times the Brownian CRT (hence the disappearance of a factor of $\sqrt{2}$ in (4.11), see [51] for careful normalizations). Last, we also refer to [73] and [124] for constructions of stable CRTs by recursive random gluing of segments, in the same vein as in Aldous [9] and Example 4.1. ◇

We refer to [79, 77] for other examples of conservative fragmentation trees that we do not describe in these pages. Let us now present a few natural generalized fragmentation trees that are not covered by the Haas–Miermont framework. One natural way to obtain them is to consider dissipative fragmentations obtained by trimming a (conservative) fragmentation using a local rule to keep only certain particles. Specifically, in the continuous world, a **trimming** rule is a function

$$\mathrm{Trim} : \begin{cases} \mathcal{S} \times [0,1] \to \mathcal{S} \\ \Big((y_0, (y_1, y_2, ...)), \omega \Big) \mapsto (\tilde{y}_0, (\tilde{y}_1, \tilde{y}_2, ...)), \end{cases}$$

which associates to a point $(y_0, (y_1, y_2, ...)) \in \mathcal{S}$ and an additional source of randomness $\omega \in [0,1]$ a random variable $(\tilde{y}_0, (\tilde{y}_1, \tilde{y}_2, ...))$ in $\mathcal{S}$ where $\tilde{y}_0$ is either y_0 or $-\infty$ (we interpret the latter case as a deletion) together with a -possibly finite[5]- subsequence $(\tilde{y}_1, \tilde{y}_2, ...)$ excerpted from $(y_1, y_2, ...)$. Given a trimming rule, we write

$$\mathrm{Trim}(\mathbf{\Lambda}) := \mathrm{Trim}_{\#}(\mathbf{\Lambda} \otimes \mathrm{Leb}[0,1]),$$

[5]In that case, the finite sequence is completed by infinitely many's $-\infty$ for the sake of definitiveness.

for the image of a generalized Lévy measure by this rule. We shall furthermore suppose that the killing rate does not explode after trimming, i.e. that

$$\mathrm{k}_{\mathrm{trim}} = \mathrm{Trim}(\mathbf{\Lambda})\left(\{-\infty\} \times \mathcal{S}_1\right) < \infty;$$

plainly $\mathrm{Trim}(\mathbf{\Lambda})$ is in turn a generalized Lévy measure. Remark that the cumulant function associated to $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ is clearly upper bounded by that of $(\sigma^2, \mathrm{a}, \mathrm{Trim}(\mathbf{\Lambda}); \alpha)$ so that by Proposition 3.10, we can construct simultaneously on the same probability space (T, d_T, ρ_T, g) and $(T_{\mathrm{trim}}, d_{T_{\mathrm{trim}}}, \rho_{T_{\mathrm{trim}}}, g_{\mathrm{trim}})$ with characteristics $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and $(\sigma^2, \mathrm{a}, \mathrm{Trim}(\mathbf{\Lambda}); \alpha)$ so that T_{trim} is a subtree of T with the restriction of the decoration (the trees do not carry any measure). Let us give a concrete example where explicit computations are doable:

Example 4.8 (k-sampling in the Brownian case)**.** *Consider the Brownian fragmentation of Example 4.6 which we trim as follows: Fix $k \geqslant 2$ and for $(s_0, s_1) \in \mathcal{S}$, write $x_0 = \mathrm{e}^{s_0}$ and $x_1 = \mathrm{e}^{s_1}$. Next, using an independent source of randomness, sample k i.i.d. Bernoulli variables $\epsilon_1, ..., \epsilon_k$ with $\mathbb{P}(\xi = i) = x_i$ for $i = 0, 1$. Then delete x_i if and only if none of the Bernoulli variables ϵ_j for $j = 1, \ldots, k$ takes the value i. This yields a non-conservative binary self-similar fragmentation with characteristics $(\sigma^2 = 0, \mathrm{a}_{\mathrm{can}} = 0, \mathbf{\Lambda}_{\mathrm{Bro},k}; \alpha = 1/2)$, where the generalized Lévy measure is given by*

$$\begin{aligned}
&\int_{\mathcal{S}} \mathbf{\Lambda}_{\mathrm{Bro},k}(\mathrm{d}y_0, \mathrm{d}(y_i)_{i\geqslant 1}) F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, ...)\big) \\
&= \sqrt{\frac{2}{\pi}} \int_{1/2}^{1} \frac{\mathrm{d}x}{(x(1-x))^{3/2}} \left(\begin{array}{rl} & F\big(x, (1-x, 0, ...)\big)(1 - x^k - (1-x)^k) \\ + & F\big(x, (0, 0...)\big) x^k \\ + & F\big(0, (1-x, 0, ...)\big)(1-x)^k \end{array} \right).
\end{aligned}$$

In this example, we have:

- *For $k = 2$,*
  $$\kappa_{\mathrm{Bro},2}(\gamma) = \sqrt{2} \cdot \frac{(1-2\gamma)\Gamma(\gamma + 1/2)}{\Gamma(\gamma + 1)},$$
  so that Assumption 3.12 holds with $\omega_- = 1/2$ and $\omega_+ = +\infty$. We believe that the total harmonic measure $\upmu(T)$ should be distributed as a multiple of a Rayleigh law with density $x\mathrm{e}^{-x^2/2}$ on $\mathbb{R}_+$.

- *for $k = 3$,*
  $$\kappa_{\mathrm{Bro},3}(\gamma) = \frac{(3 - 2\gamma(1+2\gamma))\Gamma(\gamma + 1/2)}{\sqrt{2}\Gamma(2+\gamma)},$$
  so that Assumption 3.12 holds with $\omega_- = \frac{\sqrt{13}-1}{4}$ and $\omega_+ = +\infty$. The underlying ssMt is thus a rather natural subtree of the Brownian CRT with Hausdorff dimension $\frac{\sqrt{13}-1}{2}$.

$\diamond$

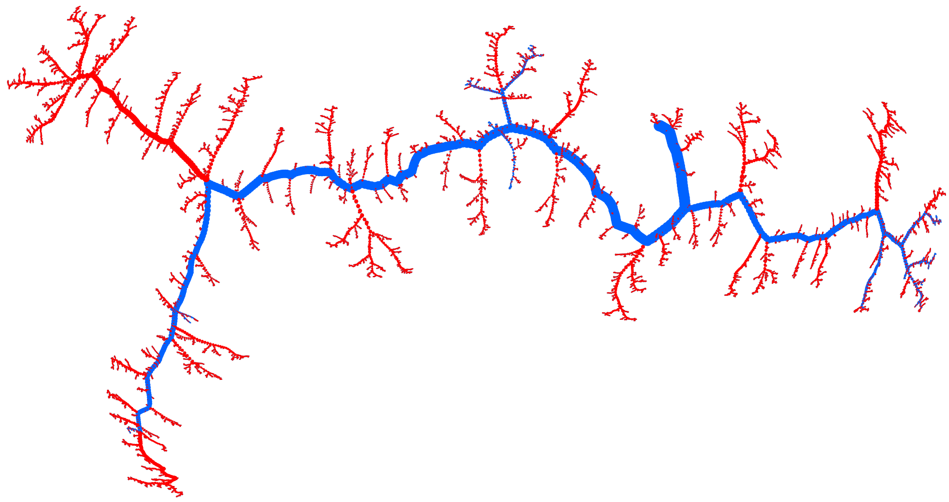

Figure 4.7: An illustration of the Trimmed Brownian tree for $k = 2$ (in blue) inside the standard Brownian tree of Example 4.6. The thickness indicates the labels.

4.3 Conservative binary growth-fragmentations

Markovian growth-fragmentations processes have been introduced in [21] as stochastic models describing the evolution of a system of living cells, where at any given time cells are simply determined by their sizes. Imagine that as time passes, cells may grow or shrink continuously and are further involved in birth events at which a daughter cell split from the mother cell (this is called mitosis in biology). More precisely, let $X(t)$ be the size of a typical cell at age t; we suppose that the process $X = (X(t))_{t\geqslant 0}$ is Markovian, has only negative jumps, and reaches 0 continuously after some finite time which we view as the death of the cell. Suppose further that each jump of X is a birth event, such that if $\Delta X(s) = -x < 0$, then s is the birthtime of a daughter cell with size x which then evolves independently and according to the same dynamics, i.e. giving birth in turn to great-daughters, and so on and so forth. This description fits the setting of general branching processes of Section 3.1, where the reproduction process η is simply the point process of the negative of the jumps of the decoration, see (4.14). We point out that the case when cells are actually inert (i.e. may split but otherwise do not grow nor shrink) corresponds to bead splitting processes introduced by Pitman and Winkel [122, 123].

Suppose further now that the Markov process X is self-similar with exponent $\alpha > 0$, and note from (4.14) that the reproduction process η inherits self-similarity from X. Recall also that X should have no positive jumps (to match the assumption that cells grow continuously and hence have only negative jumps at mitosis events) and die when reaching continuously a size 0. We are then in the conservative (4.9) and binary (4.12) case. The Lévy process $\xi = (\xi(t))_{t\geqslant 0}$ that underlies X in the Lamperti transformation is thus spectrally negative (i.e. its Lévy measure Λ_0 is carried on $(-\infty, 0)$), drifts to $-\infty$, and has also killing coefficient $\mathrm{k} = 0$. Therefore, if we write $(\sigma^2, \mathrm{a}, \Lambda_0)$ for its characteristic Lévy triplet, then Λ_0 is a measure on $(-\infty, 0)$ with

$$\int_{(-\infty,0)} (1 \wedge y^2)\Lambda_0(\mathrm{d}y) < \infty \quad \text{and} \quad \mathrm{a} + \int_{(-\infty,-1)} y\Lambda_0(\mathrm{d}y) < 0.$$

We stress that the integrability condition (4.6) may fail, and that the second requirement above is the necessary and sufficient condition for ξ to drift to $-\infty$. The generalized Lévy measure is then given by (4.13) and hence the cumulant by

$$\kappa(\gamma) = \frac{1}{2}\sigma^2\gamma^2 + \mathrm{a}\gamma + \int_{(-\infty,0)} (\mathrm{e}^{\gamma y} + (1-\mathrm{e}^y)^\gamma - 1 - \gamma y \mathbf{1}_{y \geqslant -1})\Lambda_0(\mathrm{d}y).$$

It can be shown that if $\kappa(\gamma) \leqslant 0$ for some $\gamma > 0$, then almost-surely, the family of the sizes of cells is null at any time $t \geqslant 0$, whereas if $\kappa(\gamma) \in (0, \infty]$ for all $\gamma > 0$, then almost surely, there exist times $t > 0$ at which there are infinitely many cells with size, say, larger than 1. See [21] and [28]. We stress that on top of the restriction above (binary, conservative, no killing), the main conceptual difference between the growth-fragmentation point of view and our approach in these pages is that a growth-fragmentation process is a process $(\mathbf{X}(t) : t \geqslant 0)$ with values in the space of point measures on $\mathbb{R}_+$ (or sequences of non-negative reals), whereas we focus here on the construction of random decorated trees. Specifically, when $\kappa(\gamma) < 0$ for some $\gamma > 0$, that is in the subcritical case, the self-similar Markov tree with characteristics $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ where $\mathbf{\Lambda}$ is given by (4.13) can be thought of as the decorated genealogical tree underlying the self-similar growth-fragmentation process. This correspondence is similar to the one made between superprocesses and their encoding by random trees and snake trajectories [65]. It is still open to decide whether this correspondence also holds in the critical case, see the end of Chapter 3. Rembart and Winkel [124, Section 4.2] were the first to describe a recursive construction of binary conservative self-similar growth-fragmentation trees that inspired our general construction by gluing in Section 2.4.

We now describe an interesting family of self-similar Markov trees for which cumulant functions will always be expressed as ratio of Gamma functions (and simple trigonometric functions). We start with the most important example:

Example 4.9 (Brownian growth-fragmentation tree)**.** *The Brownian growth-fragmentation tree is a self-similar Markov tree related to a remarkable growth-fragmentation that has appeared as the scaling limit of cactus trees inside random triangulations [23] or directly within the free Brownian disk [104]. It bears some obvious similarities with the Brownian fragmentation tree of Example 4.6.*

The Brownian growth-fragmentation is self-similar with exponent $\alpha = 1/2$, and its cumulant function is given by

$$\kappa_{\mathrm{BroGF}}(\gamma) = \frac{\Gamma(\gamma - 3/2)}{\Gamma(\gamma - 3)}, \quad \textit{for } \gamma > 3/2.$$

Its Gaussian coefficient is $\sigma^2 = 0$, its generalized Lévy measure $\mathbf{\Lambda}_{\mathrm{BroGF}}$ is binary conservative and given by

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \ldots)\big)\ \mathbf{\Lambda}_{\mathrm{BroGF}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \quad := \quad \frac{3}{4\sqrt{\pi}} \int_{1/2}^{1} F\big(x, (1-x, 0, 0, \ldots)\big) \frac{\mathrm{d}x}{(x(1-x))^{5/2}}. \tag{4.15}$$

Figure 4.8: A simulation of the Brownian growth-fragmentation tree. The process is binary and conservative: at each splitting event, the total mass is conserved and split between two children.

Since $\mathbf{\Lambda}_{\mathrm{BroGF}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y})$ *does not integrate* $1 \wedge |y_0|$, *the drift coefficient is not canonic; for the cut-off function as in* (3.11) *we have*

$$\mathrm{a}_{\mathrm{BroGF}} = -\frac{2}{\sqrt{\pi}} + \frac{3}{4\sqrt{\pi}} \int_{1/2}^{1} \frac{\log x + 1 - x}{(x(1-x))^{5/2}}\, \mathrm{d}x = \frac{4(7-3\pi)}{3\sqrt{\pi}},$$

see [23, Eq (32)] or [22, Proof of Proposition 5.2] for a different drift using a different cut-off function.

In particular, we have $\omega_- = 2$ *and* $\omega_+ = 3$, *and Assumption 3.12 holds. The distribution of the total mass* $\mu(T)$ *is known from [22, Corollary 6.7], it satisfies the striking property that its size-biased transform is a* $1/2$*-stable law. Let us develop in more details the connection with Brownian geometry since this is one of central motivation for this whole work. The Brownian sphere is a random compact metric space almost surely homeomorphic to the* 2*-sphere, and which has famously proved to be the scaling limit of various classes of random planar maps equipped with the graph distance [114, 99], or shown to be the random metric space obtained by exponentiating a planar Gaussian Free Field with the proper parameter* $\sqrt{8/3}$, *see [116, 118, 75]. The Brownian sphere has a variant, having the topology of the disk, called the* ***Brownian disk****, see [32]. In particular, the boundary of the Brownian disk may be defined as the set of all points that have no neighborhood homeomorphic to the open unit disk. Let us denote a* ***free Brownian disk*** *with boundary size x (it has a random volume) by $\mathbb{D}_x$ and its boundary by $\partial\mathbb{D}_x$. For every point $u \in \mathbb{D}_x$ we write $H(u)$ for the height of u, defined as its distance to $\partial\mathbb{D}_x$, and for every $r > 0$, we*

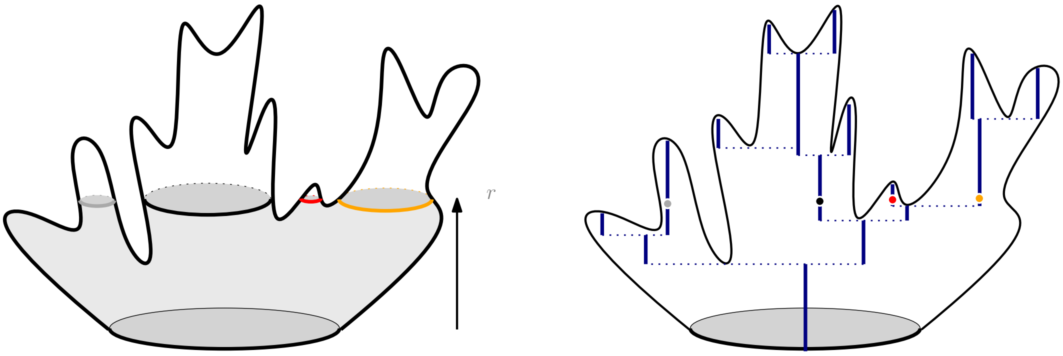

Figure 4.9: Illustration of the Cactus tree of a Brownian disk. The ball of radius r (measured from $\partial\mathbb{D}_x$) is depicted in light gray and it has several boundary components. Each of these components has a "size" which enables us to decorated the cactus tree (on the right).

consider the "ball" $\mathrm{B}_r = \{u \in \mathbb{D}_x : H(x) \leqslant r\}$. Its boundary $\partial\mathrm{B}_r$ is made of several components homeomorphic to circles. Each of these boundary components $\mathcal{C}$ is a fractal curve (of dimension 2) but it is possible to give a meaning to its size $|\mathcal{C}|$ by approximation (a.s. simultaneously for every $r \geqslant 0$ and for each boundary component), see [104, Theorem 3]. In particular, the boundary $\partial\mathbb{D}_x$ has size x a.s. Furthermore, those boundary components have a natural tree genealogy as r varies. Specifically, consider the pseudo-metric d_{Cac} defined by

$$d_{\mathrm{Cac}}(u,v) = H(u) + H(v) - 2\sup_{\gamma:u\to v}\left(\min_{0\leqslant t\leqslant 1} H(\gamma(t))\right),$$

*where the supremum is over all continuous curves $\gamma : [0,1] \to \mathbb{D}_x$ such that $\gamma(0) = u$ and $\gamma(1) = v$. Then, we introduce $\mathrm{Cac}(\mathbb{D}_x)$ the quotient space of $\mathbb{D}_x$ for the equivalence relation defined by setting $a \sim_{\mathrm{Cac}} b$ if and only if $d_{\mathrm{Cac}}(a,b) = 0$. The quotient $\mathrm{Cac}(\mathbb{D}_x)$, equipped with the metric induced by d_{Cac}, is a compact real tree called **the cactus** of $\mathbb{D}_x$ seen from $\partial\mathbb{D}_x$. We root $\mathrm{Cac}(\mathbb{D}_x)$ at $\rho_{\partial\mathbb{D}_x}$ the equivalence class of $\partial\mathbb{D}_x$, see [55, Section 2.2]. It is easy to see that the equivalence classes for $\sim_{\mathrm{Cac}}$ are precisely the boundary cycles of $\partial\mathrm{B}_r$ for all $r \geqslant 0$, so that the size function $|\mathcal{C}|$ of boundary components is a well defined decoration rcll on branches on the cactus tree and we denote its usc modification by $g_{|\cdot|}$. Then, it follows from [104] (see in particular Theorem 3 there) that the law of the equivalence class in $\mathbb{T}$ of the decorated tree*

$$\left(\mathrm{Cac}(\mathbb{D}_x), d_{\mathrm{Cac}}, \rho_{\partial\mathbb{D}_x}, g_{|\cdot|}\right)$$

is that of $\mathtt{T} = (T, d_T, \rho, g)$ under $\mathbb{P}_x$ for the characteristic quadruplet $(0, \frac{2\sqrt{2}}{\sqrt{3}} \cdot \mathrm{a}_{\mathrm{BroGF}}, \frac{2\sqrt{2}}{\sqrt{3}} \cdot \boldsymbol{\Lambda}_{\mathrm{BroGF}}; \frac{1}{2})$. Actually, [104] only deals with the growth-fragmentation point of view, but the results there are established using cell processes so that the previous display is a consequence of *the arguments therein. Let us sketch the reasoning: It is proved in [104, Proposition 16] that the self-similar process obtained by following the locally largest exploration is precisely the pssMp X constructed from $(0, \frac{2\sqrt{2}}{\sqrt{3}} \cdot \mathrm{a}_{\mathrm{BroGF}}, \frac{2\sqrt{2}}{\sqrt{3}} \cdot \boldsymbol{\Lambda}_{\mathrm{BroGF}}; \frac{1}{2})$ in Section 3.2. In particular, in the growth-fragmentation case (binary conservative case), the decoration-reproduction η is recovered from X. Finally, conditionally on this exploration, by [104, Proposition 18], the decorated subtrees branching are conditionally independent given their initial decorations. Using these two ingredients, one can couple the construction of Section 3.1 with the iterative locally largest exploration of a Brownian disk so that they coincide.* $\diamond$

Related to the above example, one can consider the self-similar Markov tree with the same first three characteristics $(\sigma^2, \mathrm{a}_{\mathrm{BroGF}}, \boldsymbol{\Lambda}_{\mathrm{BroGF}})$ but with self-similarity parameter $3/2$ (instead of $1/2$). This modification has the effect of performing a length change along branches *à la* Lamperti. This tree actually appears as the scaling of the so-called **peeling** trees associated to random planar maps with small faces, see [22, Section 6]; we expect it to be further the scaling limit of several other discrete models such as critical fully parked trees [49], fighting-fish [62] or even the peeling trees of plane Weil-Petersson surfaces with n punctures. See Part II for more details.

At bit more generally, one can allow the trajectories of cells to have positive jumps, keeping up with the convention that only negative jumps are interpreted as mitosis events. Then [22, Theorem 5.1] presents the following generalization of the preceding example:

Example 4.10 (The (β, ϱ)-stable family)**.** *There exists a family parametrized by $(\mathfrak{a} \in (0,1], \mathfrak{b} \in (0,1/2])$ of self-similar Markov trees with cumulant functions given by*

$$\kappa_{\mathfrak{a},\mathfrak{b}}(\gamma) = -\frac{\Gamma(1+2\mathfrak{a}+2\mathfrak{b}-\gamma)\Gamma(\gamma-\mathfrak{a}-\mathfrak{b})}{\Gamma(1+\mathfrak{a}+2\mathfrak{b}-\gamma)\Gamma(\gamma-\mathfrak{a}-2\mathfrak{b})}, \quad \gamma \in (\mathfrak{a}+\mathfrak{b}, 2\mathfrak{a}+2\mathfrak{b}+1). \tag{4.16}$$

Specifically, the generalized Lévy measure $\boldsymbol{\Lambda}_{\mathfrak{a},\mathfrak{b}}$ is prescribed by three parts

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \cdots)\big)\ \boldsymbol{\Lambda}_{\mathfrak{a},\mathfrak{b}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \quad =$$

$$\begin{aligned}
& \frac{\Gamma(\beta+1)\sin(\pi\mathfrak{b})}{\pi} \int_{1/2}^{1} \frac{\mathrm{d}x}{(x(1-x))^{\beta+1}} F\big(x, (1-x, 0, \ldots)\big) && (\textit{conservative binary splitting}) \\
+ \quad & \frac{\Gamma(\beta+1)\sin(\pi\mathfrak{a})}{\pi} \int_{0}^{\infty} \frac{\mathrm{d}x}{(x(1+x))^{\beta+1}} F\big(1+x, (0, 0, \ldots)\big) && (\textit{unique positive jump, growth}) \\
+ \quad & \cos(\pi\mathfrak{b}) \frac{2\Gamma(2(\mathfrak{a}+\mathfrak{b}))}{\Gamma(\mathfrak{a}+\mathfrak{b})} \cdot F\big(0, (0, 0, \ldots)\big) && (\textit{killing}),
\end{aligned} \tag{4.17}$$

for a generic positive function $F : \mathcal{S} \to \mathbb{R}_+$. The Gaussian part is degenerate, $\sigma^2 = 0$, and the drift coefficients are fine tuned so that the (4.16) holds. In particular, they satisfy $\omega_- = \mathfrak{a} + 2\mathfrak{b}, \omega_+ = \mathfrak{a} + 2\mathfrak{b} + 1$ and Assumption 3.12 holds. Those expressions may seem ad-hoc for *the moment, but we will see in Section 6.5 that they appear in relation to so-called Lamperti-stable processes, [89, Section 4.3], and are naturally associated with β-stable Lévy process with positivity parameter ϱ for*

$$\beta = \mathtt{a} + \mathtt{b} \quad \text{and} \quad \varrho = \frac{\mathtt{a}}{\mathtt{a}+\mathtt{b}}, \tag{4.18}$$

see Section 6.5 for details. We also denote the killing rate $\mathtt{k} = \cos(\pi \mathtt{b})\frac{2\Gamma(2(\mathtt{a}+\mathtt{b}))}{\Gamma(\mathtt{a}+\mathtt{b})}$ *to lighten notation.*

Let us give a more explicit description of the characteristics in two cases (in red and blue on Figure 4.10):

No killing. *We take* $\mathtt{b} = 1/2$ *in (4.18) so that* $\mathtt{k} = 0$, $1/2 < \beta = \mathtt{a}+\mathtt{b} \leqslant 3/2$ *and* $\beta(1-\varrho) = 1/2$. *This gives the self-similar Markov tree with exponent* $\alpha = \beta$, *no Gaussian component, generalized Lévy measure given by*

$$\begin{aligned} &\frac{\pi}{\Gamma(\beta+1)} \int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \cdots)\big)\ \mathbf{\Lambda}_{\mathtt{a},\frac{1}{2}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \\ &= \int_{1/2}^{1} \frac{\mathrm{d}x}{(x(1-x))^{\beta+1}} F\big(x, (1-x, 0, \ldots)\big) \\ &\quad + \cos((\beta+1)\pi) \cdot \int_0^\infty \frac{\mathrm{d}x}{(x(1+x))^{\beta+1}} F\big(1+x, (0,0,\ldots)\big), \end{aligned} \tag{4.19}$$

and where the drift coefficient a *is prescribed so that the cumulant function equals*

$$\kappa_{\mathtt{a},\frac{1}{2}}(\gamma) = \frac{\cos(\pi(\gamma-\beta))}{\sin(\pi(\gamma-2\beta))}\frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-2\beta)}, \qquad \text{for } \beta < \gamma < 2\beta+1, \tag{4.20}$$

in particular we have $\omega_- = \beta + \frac{1}{2}$ *and* $\omega_+ = \beta + \frac{3}{2}$, *see [22, Section 5]. Assumption 3.12 holds and the limiting mass measure has the law of a positive* $1/(\beta+1/2)$*-stable random variable biased by* $x \mapsto 1/x$, *see [42, Proposition 4]. Those decorated trees appear as the scaling limit of the peeling trees in critical discrete stable planar maps, see [22, Section 6] and Part II for details. They also appear (implicitly) in Liouville Quantum Gravity in [119]. We can add to this family the limiting point* $\mathtt{b} = 1/2$, $\mathtt{a} = 0$ *corresponding to Example 4.6.*

Conservative. *We take* $\mathtt{a} = 1$ *in (4.18) so that* $1 < \beta = \mathtt{a}+\mathtt{b} \leqslant 3/2$ *and* $\varrho = \frac{1}{\beta}$. *This gives the self-similar Markov tree with index* $\alpha = \beta$, *Gaussian coefficient* $\sigma^2 = 0$, *and binary generalized Lévy measure given by*

$$\begin{aligned} &\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \cdots)\big) \mathbf{\Lambda}_{1,\mathtt{b}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \\ &= \frac{\Gamma(\beta+1)\sin(\pi(\beta-1))}{\pi} \left(\int_{1/2}^{1} \frac{\mathrm{d}x}{(x(1-x))^{\beta+1}} F\big(x, (1-x, 0, \ldots)\big) \right) \\ &+ \frac{2\Gamma(2\beta)\cos(\pi(\beta-1))}{\Gamma(\beta)} \cdot F\big(0, (0,0,\ldots)\big), \end{aligned} \tag{4.21}$$

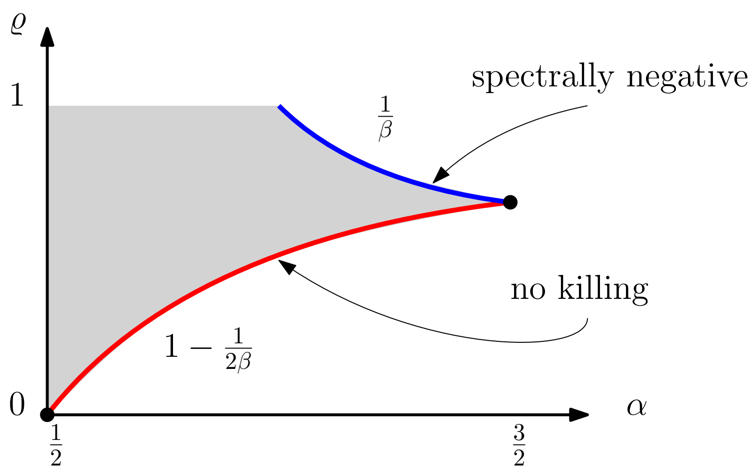

Figure 4.10: The two-parameters family of "stable" binary conservative self-similar Markov trees. See Section 6.5 for the explanation of the relation with β-stable processes with positivity parameter ϱ. The upper blue boundary case corresponds to stable Lévy process with no positive jumps (but the generalized Lévy measure $\mathbf{\Lambda}$ has a positive killing). This case is encountered in the scaling limit of fully parked trees, see Part II. The lower red boundary corresponds to the case with no killing. This case is encountered in critical stable random planar maps, see Part II. The case $\beta = 3/2, \varrho = \frac{2}{3}$ where those two lines merge is the spectrally negative $3/2$-stable case discussed in Example 4.9, but with self-similarity parameter $\alpha = \frac{3}{2}$. The case $\beta = 1/2$ and $\varrho = 0$ corresponds to the pure fragmentation of the Brownian CRT, see Example 4.6.

and where the drift $a \in \mathbb{R}$ is prescribed so that the cumulant function is equal to

$$\kappa_{1,\mathfrak{b}}(\gamma) = -\frac{\Gamma(1+2\beta-\gamma)\Gamma(\gamma-\beta)\sin(\pi(2\beta-\gamma))}{\pi}.$$

Assumption 3.12 holds, however the distribution of the total harmonic mass remains elusive. We expect that these self-similar Markov trees provide the scaling limits of the so-called dilute critical fully-parked tree studied in [47]. See also Section 6.5 for a related family of Examples with generalized Lévy measures similar to (4.21) but where $\beta \in (0, 1/2]$. $\diamond$

4.4 An overlay on the stable family and a critical example

Our final example is a family of binary non-conservative self-similar Markov trees, so that a trimmed version of which gives the family with no killing in Example 4.10. Those ssMt should appear in connection with $O(n)$-decorated random planar maps, see Part II for details. There is a critical case in this family which is naturally associated with the Brownian CRT (Example 4.6) in a rather surprising way.

Example 4.11 (An overlay on the stable family). *Recall the case $\mathtt{b} = 1/2$ in Example 4.10. In particular, with the notation used there we have $\alpha = \beta = \mathtt{a}+\mathtt{b}$. We consider now the "augmented" self-similar Markov tree obtained by adding; in case of positive jumps $x \mapsto x + y$, a new particle of mass y. Formally this is done by replacing the generalized Lévy measure defined in (4.19) by $\mathbf{\Lambda}^+_{\mathtt{a},\frac{1}{2}}$, where*

$$\begin{aligned} &\frac{\pi}{\Gamma(\beta+1)} \int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \cdots)\big) \ \mathbf{\Lambda}^+_{\mathtt{a},\frac{1}{2}}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) \\ &= \int_{1/2}^1 \frac{\mathrm{d}x}{(x(1-x))^{\beta+1}} F\big(x, (1-x, 0, \ldots)\big) \\ &+ \cos((\beta+1)\pi) \cdot \int_0^\infty \frac{\mathrm{d}x}{(x(1+x))^{\beta+1}} F\big(1+x, (x, 0, \ldots)\big), \end{aligned} \tag{4.22}$$

(notice the small difference with (4.19): we replaced $F(1-x,(0,...))$ there by $F(1-x,(x,0,...))$). The cumulant function is easily updated and becomes

$$\begin{aligned} &\underbrace{\kappa(\gamma)}_{\text{in (4.20)}} + \frac{\Gamma(\beta+1)}{\pi} \cos((\beta-1)\pi) \int_0^\infty (x-1)^\gamma \frac{\mathrm{d}x}{(x(x-1))^{\beta+1}} \\ = \ & -\frac{\Gamma(\gamma-\beta)\mathrm{Sec}(\frac{\pi}{2}(\gamma-2\beta))\sin(\pi\gamma/2)}{\Gamma(\gamma-2\beta)}, \qquad \text{for } \beta < \gamma < 2\beta+1, \end{aligned}$$

for which now the two roots are $\{2\beta, 2\}$ so that $\omega_- = \min\{2\beta, 2\}$ and $\omega_+ = \max\{2\beta, 2\}$. Assumption 3.9 holds and so the self-similar Markov tree $\mathtt{T} = (T, d_T, \rho, g)$ exists as soon as $\beta \neq 1$. Under this condition, Assumption 3.12 also holds and the law of harmonic mass $\upmu(T)$ under $\mathbb{P}_1$ has been identified in [48, Theorem 4.3] and related to an inverse-Gamma distribution. Watson [136] extended the model where the exploration of positive jumps appears with probability $r \in [0,1]$ and found explicit cumulant functions as well. ◇

In the example above, when $\beta = 1$ the cumulant function has a double root and stays non negative so that we cannot directly apply Proposition 3.10 to construct a self-similar Markov tree. However, the work of Aidekon & Da Silva [6] identified directly the underlying ssMt as a variant of the Brownian CRT of Example 4.6.

Example 4.12 (Half-plane excursions after Aidekon & Da Silva [6]). *We consider a two-dimensional excursion (X, e) where e is a Brownian excursion of length ℓ and X is a Brownian bridge of length ℓ going from 0 to 1 under the normalized excursion measure $\mathbb{N}^{[1]}_{\text{ads}}$ for plane Brownian motion. This law can be expressed using Itô's positive Brownian excursion measure (4.4) as*

$$\mathbb{N}^{[1]}_{\text{ads}}\big(\mathrm{d}(X, \mathrm{e})\big) = \int_{\mathbb{R}_+} \mathrm{d}\ell \ \frac{\mathrm{e}^{-\frac{1}{2\ell}}}{2\ell^2} \ P^{0\to 1}_\ell(\mathrm{d}X) \otimes \mathbb{N}_\ell(\mathrm{de}),$$

where $P^{0\to1}_\ell$ is the law of the Brownian bridge of length ℓ going from 0 to 1; see [6, Proposition 2.9]. Given the pair (X, e) distributed according to $\mathbb{N}^{[1]}_{\text{ads}}$, we can first construct the rooted tree

$(\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}}, \rho)$ *coded by the excursion* $\mathrm{e} : [0,\ell] \to \mathbb{R}_+$ *as presented in Section 4.1 using [66]. The random real tree* $\mathcal{T}_{\mathrm{e}}$ *is nothing but a mixture of Brownian CRT (Example 4.6) whose size is distributed according to the* $1/2$*-stable law* ℓ*. We then endow it with the following decoration using the process* X*: for any point* $u \in \mathcal{T}_{\mathrm{e}}$*, in the coding of* $\mathcal{T}_{\mathrm{e}}$ *from* e*, let us denote by* s_u, t_u *respectively the minimal and maximal pre-images*[6] *of* u *in* $[0,\ell]$*. The interval* $[s_u, t_u]$ *is then a subexcursion of* e *in the sense that* $\mathrm{e}(r) \geqslant \mathrm{e}(s_u) = \mathrm{e}(t_u)$ *for all* $r \in (s_u, t_u)$ *and* $t_u - s_u$ *correspond to the size of the fringe subtree above* u*. The point* $u \in \mathcal{T}_{\mathrm{e}}$ *is labeled by the* X*-displacement over the time interval* $[s_u, t_u]$*, i.e.*

$$\tilde{g}(u) := |X(t_u) - X(s_u)|.$$

It is not hard to see that $\tilde{g}$ *is then a rcll function over the branches of* $\mathcal{T}_{\mathrm{e}}$ *and we thus consider*

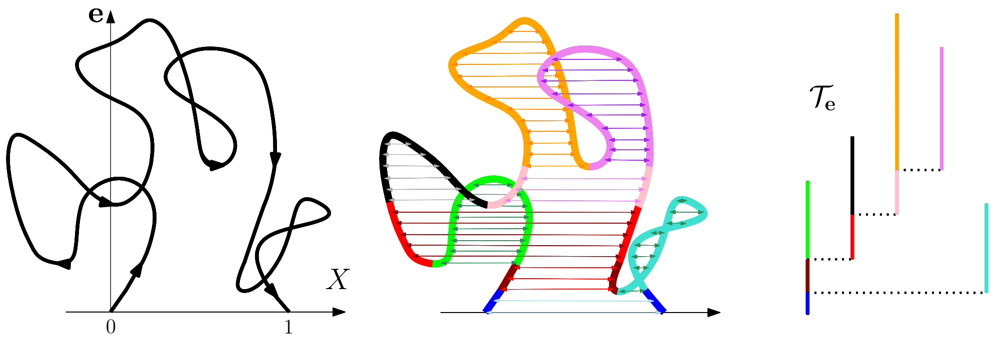

Figure 4.11: Illustration of the construction of a decorated tree from an excursion in the half-plane. The vertical coordinate encodes the tree structure, whereas the horizontal displacement encodes the decoration.

the usc modification g *of* $\tilde{g}$ *as the decoration on* $\mathcal{T}_{\mathrm{e}}$ *to fit the framework of Chapter 2. Remark in particular that we have* $g(\rho) = 1$ *under the law* $\mathbb{N}^{[1]}_{\mathrm{ads}}$*. It follows from the arguments of [6] that the law of the random decorated tree*

$$(\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}}, \rho, g) \qquad \textit{under } \mathbb{N}^{[1]}_{\mathrm{ads}},$$

is that of the ssMt $\mathtt{T}$ *under* $\mathbb{P}_1$ *with the characteristic quadruplet* $(0, \mathrm{a}_{\mathrm{ads}}, \mathbf{\Lambda}_{\mathrm{ads}}; 1)$ *where*

$$\int F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \dots)\big) \mathbf{\Lambda}_{\mathrm{ads}}(\mathrm{d}y_0, \mathrm{d}(y_i)_{i\geqslant 1})$$
$$= \frac{2}{\pi}\left(\int_{1/2}^{1} \frac{\mathrm{d}x}{\big(x(1-x)\big)^2} F\big(x, (1-x, 0, \dots)\big) + \int_{1}^{\infty} \frac{\mathrm{d}x}{\big(x(1+x)\big)^2} F\big(1+x, (x, 0, \dots)\big)\right),$$

$$\textit{and} \qquad \mathrm{a}_{\mathrm{ads}} = -\frac{4}{\pi} + \frac{2}{\pi}\int_{-\log 2}^{\infty} \mathrm{d}y\, \big(\mathbf{1}_{|y|\leqslant 1} - (\mathrm{e}^y - 1)\big) \frac{\mathrm{e}^{-y}}{(\mathrm{e}^y - 1)^2}.$$

[6] Recall that almost surely, any point $u \in \mathcal{T}_{\mathrm{e}}$ has at most three pre-images by the projection $\pi : [0,\ell] \to \mathcal{T}_e$. The points having three pre-images correspond to the branch points of $\mathcal{T}_{\mathrm{e}}$ and those having at least two pre-images correspond to the skeleton of $\mathcal{T}_{\mathrm{e}}$.

That is, the decorated Lévy measure is twice that of (4.22) *and the drift coefficient is set so that we have*

$$\kappa(\gamma) = 2(\gamma - 2)\tan\left(\frac{\pi\gamma}{2}\right).$$

Some explanations are in order. First, in the work [6], the authors only consider "positive excursions", i.e. they trimmed the tree at points $u \in \mathcal{T}_{\mathrm{e}}$ *where* $X(t_u) - X(s_u)$ *becomes negative. If we do so, we instead recover the ssMt corresponding to* (4.19) *with* $\beta = 2$*. However, we prefer to keep positive and negative horizontal excursions (their proof adapts easily) so that the underlying tree is exactly* $\mathcal{T}_{\mathrm{e}}$*. Second, as in [104], the authors state their results in terms of growth-fragmentation processes, but we can argue similarly as in Example 4.9 using [6, Theorem 3.3 and Theorem 3.6] and prove that we can couple the construction of Chapter 3 with that of [6] so that the underlying branching process with types and decoration-reproduction is actually the same. We deduce that in this case the gluing can be performed and yields a compact decorated tree without appealing to Proposition 3.10 nor to Property* $(\mathcal{P})$*. Indeed, Assumption 3.9 is* ***not*** *fulfilled and so one cannot use Proposition 3.10 to justify a priori that* $\mathtt{T}$ *is well-defined. Also, Assumption 3.12 is not fulfilled either and actually the "natural measure on the leaves of* $\mathtt{T}$*" which coincides with the contour measure* $\frac{1}{2} \cdot \gamma_{\mathrm{e}}$ *should be constructed via the derivative martingale, see [6, Theorem 5.3].*

This example shows that the construction of self-similar Markov trees can, at least in some examples, still be performed in the critical case $\min \kappa = 0$ *and gives credits to the discussion in the comments section of Chapter 3. Furthermore, it sheds yet another point of view on the usual Brownian CRT by showing that a version of it can be constructed as critical self-similar Markov tree.*

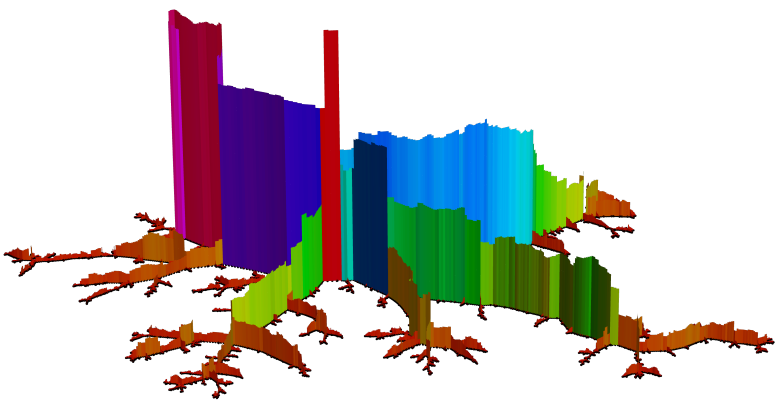

Figure 4.12: The decorated random tree associated with half-planar Brownian excursion.

◇

Chapter 5

Markov properties

In this chapter, we discuss several Markov properties of a self-similar Markov tree with a given characteristic quadruplet, hence justifying *a posteriori* the terminology. We state them as properties satisfied by the laws of the equivalence class in $\mathbb{T}$ of the decorated trees $\mathtt{T} = (T, d_T, \rho, g)$. Heuristically, these properties claim that certain natural families of disjoint decorated subtrees are conditionally independent given the values of the decoration at their respective roots, and that after a proper rescaling, they are distributed as the original self-similar Markov tree. The chapter is divided as follows. First in Section 5.1, we introduce the notion of local decomposition, which provides a general and rigorous formulation of the Markov property. Then in Sections 5.2 and 5.3, we apply this framework to the case of subtrees dangling from spines, balls or hulls. The Markov properties will play a pivotal role in our study of self-similar Markov trees, see Chapter 6, and are also crucial for establishing invariance principles in Part II.

5.1 Local decompositions

Our goal here is to introduce a general framework that enables us to state rigorously Markov properties of self-similar Markov trees. Throughout this section, we fix a probability space on which various random variables will be defined. Heuristically, a local decomposition of a random decorated tree $\mathtt{T}$ is a way to reconstruct it from an initial random decorated real tree $\mathtt{T}'$ with marks and then by gluing on the latter a family of random decorated real trees satisfying some branching property. This notion is similar in a random framework to the deterministic one we used in Section 2.1. See also the end of Section 2.4, and notably Lemma 2.18 therein, for the version of the gluing operator in the setting of equivalence classes modulo isomorphisms of decorated real trees.

Let us now provide a formal definition. Recall from Section 2.4 that $\mathbb{T}$ is the Polish space of all isomorphism classes of rooted decorated compact trees equipped with the distance $\mathrm{d}_{\mathbb{T}}$. Fix $(\mathbb{Q}_x)_{x \geqslant 0}$ a (measurable) kernel of probability measures on $\mathbb{T}$ and I some countable set of indices. In the same probability space, we let $(\mathtt{T}', (r_i)_{i \in I})$ be a random decorated real tree endowed with a family of marks, that is a random variable in $\mathbb{T}^{I\bullet}$, $(\ell_i)_{i \in I}$ a family of random variables in $\mathbb{R}_+$, and finally $(\tau_i)_{i\in I}$ a family of random decorated real trees in $\mathbb{T}$. Recall the definition of the gluing operation as a map from $\mathbb{T}^{I\bullet} \times (\mathbb{T})^I$ to $\mathbb{T}$, which was given after Lemma 2.18 in Section 2.4.

Definition 5.1. *We say that* $\Big((\mathtt{T}', (r_i)_{i\in I}), (\ell_i)_{i\in I}, (\tau_i)_{i\in I}\Big)$ *is a* $(\mathbb{Q}_x)$*-**local decomposition** of* $\mathtt{T}$ *if it satisfies the following:*

(LD1) We have

$$\mathrm{Gluing}\Big((\mathtt{T}', (r_i)_{i\in I}), (\tau_i)_{i\in I}\Big) = \mathtt{T}, \qquad \textit{in } \mathbb{T}, \ \textit{a.s.}$$

(LD2) Conditionally on $\big((\mathtt{T}', (r_i)_{i\in I}), (\ell_i)_{i\in I}\big)$*, the random decorated real trees* $(\tau_i)_{i\in I}$ *are independent and the conditional law of* τ_i *is* $\mathbb{Q}_{\ell_i}$ *for every* $i \in I$*.*

We stress that the first and third components of the triplet involved in a local decomposition are random variables in spaces of equivalent classes $\mathbb{T}^{I\bullet}$ and $\mathbb{T}$. Since by Lemma 2.18, the equivalence class of the glued tree in (LD1) does not depend on the choice for representatives, the above definition does makes sense. In this situation, we refer to $\mathtt{T}'$ as the base of the local decomposition, and to the τ_i's as subtrees dangling from $\mathtt{T}'$. Recall also that in the construction by gluing, only the decorated real trees τ_i that are not degenerate play a role. We also stress that local decompositions are only of interest if conditions (2.2) and (2.3) are fulfilled a.s.; otherwise, we will simply have $\mathtt{T} = \mathtt{0}$.

We interpret a $(\mathbb{Q}_x)_{x\geqslant 0}$-local decomposition as a type of Markov property. Conditioned on the "present" $(\ell_i)_{i\in I}$, the "past" $(\mathtt{T}', (r_i)_{i\in I})$ and the "future" $(\tau_i)_{i\in I}$ are independent. Furthermore, the conditional distribution of the future is determined by the probability kernel $(\mathbb{Q}_x)_{x\geqslant 0}$. This property can actually be seen as an extension of the branching property discussed in Section 3.1, within the context of random decorated trees. It can be related to the so-called **strong Markov branching property** of general branching processes in [83], as we shall see notably in the proof of the forthcoming Proposition 5.4. Let us illustrate the notions that we just introduced with a basic example involving general branching processes.

Example 5.2 (Local decomposition of general branching processes along the spine)**.** *Consider a decoration-reproduction kernel* $(P_x)_{x>0}$ *as defined in Section 3.1. We write as usual* $\mathbb{P}_x$ *for the law of the family of decoration-reproduction processes* $(f_u, \eta_u)_{u\in\mathbb{U}}$ *which is induced when the ancestor has type* x*. Assume that Property* $(\mathcal{P})$ *holds* $\mathbb{P}_x$*-a.s. for all* $x > 0$*. We take for* $\mathcal{T} \in \mathbb{T}$ *the random decorated tree defined in Theorem 2.5 and denote its law under* $\mathbb{P}_x$ *by* $\mathbb{Q}_x$*. We also write* $\mathbb{Q}_0$ *for the Dirac point mass at the degenerate decorated real tree* $\mathcal{O}$ *in* $\mathbb{T}$*.*

Let us work under $\mathbb{P}_y$ *for some* $y > 0$*. Recall from Notation 2.9, that* $T^0 = \{\rho(\varnothing, t) : t \in [0, z_\varnothing]\}$ *stands for the base subtree of* T *induced by the ancestral individual of the general branching process. We interpret* T^0 *as the spine of* T*. We also write* d_{T^0} *for the restriction of* d_T *to* T^0 *and* g^0 *for the associated usc-decoration – which corresponds to the usc version of* $f_\varnothing$*. Next, recall that the atoms of the reproduction process* $\eta_\varnothing$ *of the ancestor have been enumerated, say* $(r_1, \ell_1), (r_2, \ell_2), \ldots,$ *using some deterministic rule, for instance the co-lexicographic order, and, if needed, are completed with fictitious pairs* $(\dagger, 0)$ *to get an infinite sequence. For any* $j \geqslant 1$

such that $r_j \neq \dagger$, the sub-family of decoration-reproduction processes $(f_{ju}, \eta_{ju})_{u \in \mathbb{U}}$ also satisfies the Property $(\mathcal{P})$ almost surely and we write $\mathtt{T}_j$ for the random decorated tree that it induces. When $r_j = \dagger$, we simply decide that $\mathtt{T}_j$ is the degenerate decorated real tree.

On the one hand, we see from the gluing construction that

$$\mathrm{Gluing}\Big(\big((T^0, d_{T^0}, \rho, g^0), (\mathsf{\rho}(\varnothing, r_i))_{i\in\mathbb{N}}\big), (\mathtt{T}_i)_{i\in\mathbb{N}}\Big) = \mathtt{T},$$

which is the requirement (LD1). On the other hand, the branching property of general branching processes discussed in Section 3.1 entails that conditionally on $(f_\varnothing, \eta_\varnothing)$, or equivalently conditionally on $\big((T^0, d_{T^0}, \rho, g^0), (\mathsf{\rho}(\varnothing, r_i))_{i\in\mathbb{N}}\big)$ and $(\ell_i)_{i\in\mathbb{N}}$, the sub-families of decoration-reproduction processes $(f_{ju}, \eta_{ju})_{u\in\mathbb{U}}$ for $j \geqslant 1$ are independent and the conditional law of $(f_{ju}, \eta_{ju})_{u\in\mathbb{U}}$ is $\mathbb{P}_{\ell_j}$. Therefore the random decorated real trees $(\mathtt{T}_i)_{i\in\mathbb{N}}$ are also conditionally independent and the conditional law of $\mathtt{T}_i$ is $\mathbb{Q}_{\ell_i}$ for every $i \in \mathbb{N}$, which is the requirement (LD2). We conclude that

$$\bigg(\Big(\big(T^0, d_{T^0}, \rho, g^0\big), \big(\mathsf{\rho}(\varnothing, r_i)\big)_{i\in\mathbb{N}}\Big), \big(\ell_i\big)_{i\in\mathbb{N}}, \big(\mathtt{T}_i\big)_{i\in\mathbb{N}}\bigg)$$

is a $(\mathbb{Q}_x)_{x\geqslant 0}$ local decomposition of $\mathtt{T}$; see Figure 5.1 for an illustration. $\diamond$

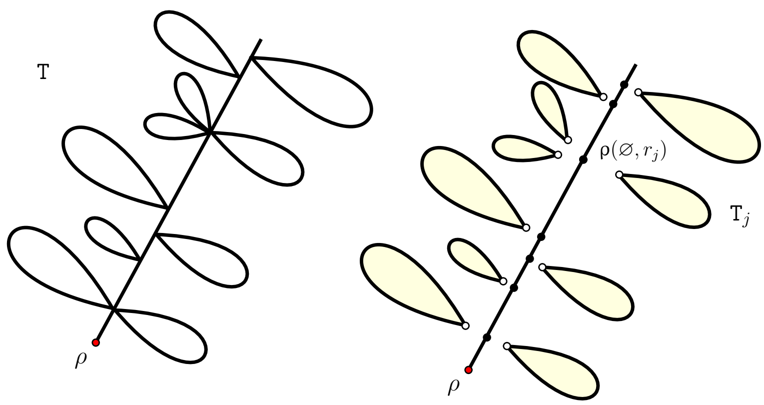

Figure 5.1: Illustration of the local decomposition of branching processes along the spine T^0: conditionally on the reproduction measure, the dangling subtrees $\mathtt{T}_i$ are independent and of law $\mathbb{Q}_{\ell_i}$, $i \in \mathbb{N}$.

Let us now focus on our case of interest, namely when $\mathtt{T} = (T, d_T, \rho, g)$ is built as in Proposition 3.10 from decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$ induced by a characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ satisfying Assumption 3.9. Recall the notation $\mathbb{P}_x$ for the law of $(f_u, \eta_u)_{u\in\mathbb{U}}$ when starting from initial decoration $x > 0$ and that $\mathbb{Q}_x$ is the law of the equivalence class of $\mathtt{T} = (T, d_T, \rho, g)$ in $\mathbb{T}$. Note the subtlety here: under $\mathbb{P}_x$ the variable $\mathtt{T}$ is an actual and explicit decorated tree built from gluing decorated segments obtained from $(f_u, \eta_u)_{u\in\mathbb{U}}$ as in Section 2.2, whereas under $\mathbb{Q}_x$ the variable $\mathtt{T}$ is rather an equivalence class of such trees. Recall also that $\mathbb{Q}_0$ stands for the law of the degenerate decorated tree $\mathtt{0}$ in $\mathbb{T}$.

We shall work under $\mathbb{P} = \mathbb{P}_1$ and establish a decomposition of the actual decorated tree $\mathtt{T} = (T, d_T, \rho, g)$ that will provide a local decomposition in the sense of Definition 5.1 once seen as variables in $\mathbb{T}$. In this direction, we need first to introduce a standard procedure to decorate subtrees of T. Specifically, we say that $\tau \subset T$ is a subtree if it is non-empty, connected and closed. We can endow τ with the distance d_τ induced by d_T and root it at the unique point $\rho_\tau \in \tau$ such that $\inf_{y \in \tau} d_T(\rho, y) = d_T(\rho, \rho_\tau)$. Then, $(\tau, d_\tau, \rho_\tau)$ is a compact rooted real tree and we further say that τ is a **base subtree** if furthermore $\rho \in \tau$. We equip τ with the decoration g_τ that coincides with g on $\tau \backslash \{\rho_\tau\}$ and is given at the root by

$$g_\tau(\rho_\tau) := \limsup_{\substack{y \in \tau \backslash \{\rho_\tau\} \\ y \to \rho_\tau}} g(y), \tag{5.1}$$

with the convention that $g_\tau(\rho_\tau) = 0$, if $\tau = \{\rho_\tau\}$. We refer to $g_\tau(\rho_\tau)$ as the **germ of the decoration** on τ. It is important to note that this quantity has been defined not only to ensure upper semi-continuity of the decoration, but also so that it can be evaluated by peeping only infinitesimally at the boundary point of a connected component $\tau \backslash \{\rho_\tau\}$, revealing the latter as little as possible. In particular, since $f_u(0) = \chi(u)$ in the self-similar setup, if τ is a base subtree then we must have $g_\tau(\rho_\tau) = g(\rho)$. In the rest of the section we will use the standard notation $\mathtt{\tau} := (\tau, d_\tau, \rho_\tau, g_\tau)$, and for definiteness we extend the definition when τ is empty by taking $\mathtt{\tau} := (\{\rho\}, 0, \rho, 0)$ which is isomorphic to $\mathtt{0}$. We refer to $\mathtt{\tau}$ as the standard decoration of τ.

We can now explain the road map to encode local decompositions under $\mathbb{P}$. First, assume that we have defined a base subtree $T' \subset T$ by means of some algorithm or geometric definition (using the $(f_u, \eta_u)_{u \in \mathbb{U}}$). We then write $(\tau_i^*)_{i \in I}$ for the family of connected components of $T \backslash T'$, agreeing for definitiveness that some τ_i^* may be empty, notably when the number of components is finite. We stress that the precise choice of the indexing set I is irrelevant at the stage, but will have some importance latter. Next, we write τ_i for the closure of τ_i^*, in particular we have $\tau_i = \tau_i^* \cup \{\rho_{\tau_i}\}$ when τ_i^* is non empty and $\tau_i = \varnothing$ otherwise. If, we write r_i and ℓ_i for the root and initial usc-decoration of $\mathtt{\tau}_j$, that is $(r_i, \ell_i) := (\rho_{\tau_i}, g_{\tau_i}(\rho_{\tau_i}))$ if τ_i is non empty and $(r_i, \ell_i) := (\rho, 0)$ otherwise, then the requirement (LD1) for a local decomposition is clearly fulfilled, i.e.

$$\mathrm{Gluing}\Big((\mathtt{T}', (r_i)_{i \in I}), (\mathtt{\tau}_i)_{i \in I}\Big) = \mathtt{T}, \quad \text{ in } \mathbb{T}, \text{ a.s..}$$

So the remaining crucial issue is to verify (LD2), and this requires in particular a proper a choice of the indexation of the connected components[1], which in our cases will use the explicit construction from $(f_u, \eta_u)_{u \in \mathbb{U}}$. Under $\mathbb{P}$, we will say in short that $T' \subset T$ **induces a local decomposition** of the self-similar Markov tree $\mathtt{T}$ whenever it exists such an indexation so that $\Big((\mathtt{T}', (r_i)_{i \in I}), (\ell_i)_{i \in I}, (\mathtt{\tau}_i)_{i \in I}\Big)$ is a $(\mathbb{Q}_x)_{x>0}$ local decomposition of $\mathtt{T}$. Ours proofs will consists in constructing such an indexation and checking the independence property.

[1]For instance, the algorithm that ranks components in the decreasing order of their heights would not serve our purpose.

5.2 Genealogical Markov property

In this section we establish local decompositions induced by base subtrees that are constructed using the genealogy of Ulam's tree. Recall that we work under $\mathbb{P}$ and that we write $\mathtt{T} = (T, d_T, \rho, g)$ for the decorated tree built from the family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$. Recall also from Notation 2.9 that for every $n \geqslant 0$, we write T^n for the base subtree of T induced by individuals of the general branching process up to generation n only. The following local decomposition can be seen as an easy extension of Example 5.2.

Proposition 5.3. *For every $n \geqslant 0$, the subtree T^n induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$.*

Proof. We will establish the claim by induction. We start considering the case $n = 0$ which has essentially been already discussed in Example 5.2, and we use the notation therein. The slight difference is that we decorate here the spine segment T^0 with the restriction of g to T^0, whereas in Example 5.2, we rather used the usc-modification of $f_\varnothing$. However, these two functions may only differ at marked points of the segment where gluing is performed. More precisely, for every $i \in \mathbb{N}$, we have $g(r_i) = f_\varnothing(r_i) \vee \ell_i$. We deduce that (LD2) still holds when we decorate T^0 with the restriction of g instead of $f_\varnothing$, and hence T^0 induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$.

For $n = 1$, we decompose in turn each (non-degenerate) connected component along its own spine, using $I = \mathbb{N}^2$ to index the family of subtrees at the second generation. And so on, and so forth, generation after generation. For the sake of clarity, let us spell the elements of the local decomposition out. At generation n, we use $I = \mathbb{N}^{n+1}$ as set of indices. For $u \in \mathbb{N}^{n+1}$, $r_u = \mathsf{p}(u)$ is the equivalence class of the root of the segment labelled by u in the construction by gluing, and $\ell_u = \chi(u)$ is the type of the individual u, that is the germ of the decoration $g_u(\rho_u)$ for the subtree $\tau_u = T_u$, again with the Notation 2.9. Condition (LD2) follows directly from the branching property. $\square$

In the remaining of the section, we focus on the model of pruned tree below a fixed level, which can be though of as a variation of T^n where we only keep the finitely many individuals with types larger than ε. We shall prove after, see Corollary 5.5, that the associated decorated subtree $\mathtt{T}^{[\varepsilon]}$ is an approximation of $\mathtt{T}$ as $\varepsilon \to 0$; and this will be notably useful in Part II, when we shall consider scaling limits of Galton-Watson branching processes with integer types.

To define properly the pruning transformation, we fix some threshold $\varepsilon \in (0, 1)$ and introduce the set of vertices

$$F(\varepsilon) := \left\{u \in \mathbb{U} : f_u(0) < \varepsilon \text{ and } f_v(0) \geqslant \varepsilon \text{ for all } v \prec u\right\}, \tag{5.2}$$

where the notation $v \prec u$ means that v is a strict prefix of u in $\mathbb{U}$. For every $u \in F(\varepsilon)$, we change the entire descent of u and make it fictitious by setting $(f_v^{[\varepsilon]}, \eta_v^{[\varepsilon]}) = (0, 0)$ for all $v \succeq u$ (u prefix of v). The decoration-reproduction of the ascendants of individuals of $F(\varepsilon)$ are unchanged, i.e. $(f_v^{[\varepsilon]}, \eta_v^{[\varepsilon]}) = (f_v, \eta_v)$ for all $v \prec u$ with $u \in F(\varepsilon)$. We know from Lemma 3.2 that the family

$$A(\varepsilon) := \left\{v \in \mathbb{U} : v \prec u \text{ for some } u \in F(\varepsilon)\right\}$$

is finite $\mathbb{P}$-a.s. The tree pruned at level $\varepsilon > 0$, denoted by $T^{[\varepsilon]}$, is the real tree obtained by gluing as in Section 3.1 the line segments $[0, z_v]$ for $v \in A(\varepsilon)$ only, i.e.

$$T^{[\varepsilon]} := \big\{\mathsf{\rho}(u,t) :\ u \in A(\varepsilon) \text{ and } t \in [0, z_u]\big\},$$

where we recall that $\mathsf{\rho}(u,t)$ denotes the – equivalent class of the – point on the segment S_u at distance t from the root $\rho(u)$. In particular, $T^{[\varepsilon]}$ is a base subtree of T built from a finite number of line segments.

Proposition 5.4. *For every $\varepsilon \in (0,1)$, the subtree $T^{[\varepsilon]}$ induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$.*

We stress that the heart of the proof is that the set of vertices $F(\varepsilon)$ in (5.2) is an optional line in the sense of Jagers [83] to which the strong Markov branching property applies. Other choices of optional lines would thus yield analogous local decompositions.

Proof. First, for every $v \in F(\varepsilon)$ non-fictitious, we consider the subtree

$$\tau_v = \big\{\mathsf{\rho}(vu,t) :\ u \in \mathbb{U} \text{ and } t \in [0, z_{vu}]\big\},$$

and we write $\mathtt{\tau}_v$ for the induced decorated tree. By construction, the connected components of $T \backslash T^{[\varepsilon]}$ are precisely the τ_v, for $v \in F(\varepsilon)$ non-fictitious. Furthermore, recalling Notation 2.9, we see that $\mathtt{\tau}_v$ is isomorphic to the decorated tree $\mathtt{T}_v$ associated with the family $(f_{vu}, \eta_{vu})_{u\in\mathbb{U}}$. We also let $r_v = \mathsf{\rho}(v)$ and $\ell_v = \chi(v) = f_v(0)$. For convenience, we extend the above construction to all the Ulam tree by taking $\mathtt{\tau}_v = (\{\rho\}, 0, \rho, 0)$, $r_v = \rho$ and $\ell_v = f_v(0)$, for $v \notin F(\varepsilon)$ or v fictitious.

We have to verify (LD2) in this setting, which can be derived from Proposition 5.3 by constructing $T^{[\varepsilon]}$ recursively branch by branch. Nonetheless, since the same argument can be used to establish many other local decompositions, it may be more instructive to rather use the strong Markov branching property of general branching processes, which has been developed in great generality in [83].

We first observe that the set of vertices $F(\varepsilon)$ is a random line, that is $F(\varepsilon)$ does not contain two vertices u and v with $u \prec v$. It is furthermore optional, in the following sense. For any subset of vertices $V \subset \mathbb{U}$, the event $\{F(\varepsilon) \preceq V\}$ that every $v \in V$ has some prefix in $F(\varepsilon)$, is measurable with respect to the sigma-algebra $\mathcal{F}_V$ generated by $((f_w, \eta_w) : w \not\succeq v \text{ for all } v \in V)$. In other words, the event $\{F(\varepsilon) \preceq V\}$ does not depend on the decoration-reproduction processes indexed by vertices with a prefix in V.

Then define the pre-$F(\varepsilon)$-algebra $\mathcal{F}_{F(\varepsilon)}$ of events A such that $A \cap \{F(\varepsilon) \preceq V\} \in \mathcal{F}_V$ for all $V \subset \mathbb{U}$. The strong Markov branching property [83, Theorem 4.14] of general branching processes at optional stopping lines can now be stated in our framework as follows. Consider for every $u \in \mathbb{U}$ a measurable functional $\varphi_u : \mathbb{T} \to [0,1]$; then there is the identity

$$\mathbb{E}\Big(\prod_{u\in F(\varepsilon)} \varphi_u(\mathtt{T}_u) \ \Big|\ \mathcal{F}_{F(\varepsilon)} \Big) = \prod_{u\in F(\varepsilon)} \mathbb{E}_{\chi(u)}\big(\varphi_u(\mathtt{T})\big).$$

Let us now impose furthermore $\varphi_u(0) = 1$ for a while, so that we can rewrite the preceding in the form

$$\mathbb{E}\Big(\prod_{u\in\mathbb{U}} \varphi_u(\tau_u) \,\Big|\mathcal{F}_{F(\varepsilon)}\Big) = \prod_{u\in\mathbb{U}} \mathbb{E}_{\ell_u}\big(\varphi_u(\mathtt{T})\big).$$

It is readily seen that this identity remains valid if we drop the requirement $\varphi_u(0) = 1$. Indeed, it suffices to replace φ_v by $\mathbf{1}_0$ for any given vertex $v \in \mathbb{U}$, use linearity, and repeat the operation for every vertex in $\mathbb{U}$. Since $(\mathtt{T}^{[\varepsilon]}, (r_v)_{v\in\mathbb{U}})$ and $(\ell_v)_{v\in\mathbb{U}}$, as variable in $\mathbb{T}^{\mathbb{U}\bullet}$ and $\mathbb{R}_+^{\mathbb{U}}$, are measurable with respect to the sigma-algebra $\mathcal{F}_{F(\varepsilon)}$, this shows (LD2) and hence completes the proof of the proposition. □

We conclude this section by establishing that the subtree pruned at level ε approximates the self-similar Markov tree as $\varepsilon \to 0+$. We shall furthermore endow $T^{[\varepsilon]}$ with approximations of the weighted length and harmonic measures constructed in Section 3.3. More precisely, for any $\gamma \geqslant \gamma_0$, the tree $T^{[\varepsilon]}$ can be equipped with the restriction $\mathbf{1}_{T^{[\varepsilon]}} \cdot \lambda^\gamma$ of the length measure $g^{\gamma-\alpha}\lambda_T$ constructed in Proposition 3.11. However, when Assumption 3.12 holds, the harmonic measure μ assigns zero mass to T^n for any $n \geqslant 0$, and therefore also to the pruned subtree $T^{[\varepsilon]}$ for any $\varepsilon > 0$. In particular the restriction of μ to $T^{[\varepsilon]}$ is always null and does not converge to μ as $\varepsilon \to 0+$. For this reason, we shall then rather equip $T^{[\varepsilon]}$ with another measure, namely we let $\mu^{[\varepsilon]}$ be the image of the harmonic measure μ by the canonical projection of T on $T^{[\varepsilon]}$. Specifically, we introduce

$$\mu^{[\varepsilon]} := \sum_{u\in F(\varepsilon)} \mu(T_u) \cdot \delta_{\rho(u)}, \tag{5.3}$$

where the optional line $F(\varepsilon)$ has been defined in (5.2), and, as usual, $\rho(u)$ denotes the point in $T^{[\varepsilon]}$ at which the segment S_u is glued. Then we have the following property.

Corollary 5.5 (Convergence of cutoff approximations)**.** *In the notation above, the following assertions hold $\mathbb{P}$-a.s.*

(i) Suppose Assumption 3.9. Then for any $\gamma \geqslant \gamma_0$, we have

$$\lim_{\varepsilon\to 0+} \big(T^{[\varepsilon]}, d_{T^{[\varepsilon]}}, \rho, g_{T^{[\varepsilon]}}, \mathbf{1}_{T^{[\varepsilon]}} \cdot \lambda^\gamma\big) = \big(T, d_T, \rho, g, \lambda^\gamma\big), \qquad \text{in } \mathbb{T}_m.$$

(ii) Suppose Assumption 3.12. Then we have

$$\lim_{\varepsilon\to 0+} \big(T^{[\varepsilon]}, d_{T^{[\varepsilon]}}, \rho, g_{T^{[\varepsilon]}}, \mu^{[\varepsilon]}\big) = \big(T, d_T, \rho, g, \mu\big), \qquad \text{in } \mathbb{T}_m.$$

Proof. We will establish the statements with convergence in probability instead of almost surely, as then the sharper claims follow readily by an argument of monotonicity. To start with, recall from Proposition 3.10 and Lemma 3.2 that

$$\mathbb{E}\Big(\sum_{u\in\mathbb{U}} \chi(u)^{\gamma_0}\Big) < \infty.$$

As a consequence, we have

$$\lim_{\varepsilon\to 0+} \mathbb{E}\Big(\sum_{u\in\mathbb{U}} \mathbf{1}_{\{\chi(u)<\varepsilon\}} \chi(u)^{\gamma_0} \Big) = 0,$$

and *a fortiori*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \chi(u)^{\gamma_0} \Big) = 0. \tag{5.4}$$

Next, we deduce from the local decomposition Proposition 5.4, Corollary 3.4 and the scaling property that there is some finite constant $c > 0$ such that for every $\varepsilon > 0$, we have

$$\mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \mathrm{Height}(T_u)^{\gamma_0/\alpha} \Big) \leqslant c \cdot \mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \chi(u)^{\gamma_0} \Big)$$

and

$$\mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \max_{T_u} g_u^{\gamma_0} \Big) \leqslant c \cdot \mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \chi(u)^{\gamma_0} \Big).$$

We deduce that

$$\lim_{\varepsilon\to 0+} \mathbb{E}\Big(\sum_{u\in F(\varepsilon)} \Big(\mathrm{Height}(T_u)^{\gamma_0/\alpha} + \max_{T_u} g_u^{\gamma_0} \Big) \Big) = 0.$$

Since the decorated real tree $\mathtt{T}$ can be recovered by gluing on $\mathtt{T}^{[\varepsilon]}$ the subtrees T_u for $u \in F(\varepsilon)$, there is the bound

$$\mathrm{d}_{\mathbb{H}(\mathtt{T})}(\mathtt{T}, \mathtt{T}^{[\varepsilon]}) \leqslant \sup_{u\in F(\varepsilon)} \Big(\mathrm{Height}(T_u) \vee \max_{T_u} g_u \Big),$$

and we conclude that

$$\lim_{\varepsilon\to 0+} \mathrm{d}_{\mathbb{H}(\mathtt{T})}(\mathtt{T}, \mathtt{T}^{[\varepsilon]}) = 0, \qquad \text{in probability.}$$

Note that this immediately yields the claim (ii) (again with convergence in probability in place of almost surely), since from the definition (5.3), the Prokhorov distance between $\mu^{[\varepsilon]}$ and μ is bounded from above by the Hausdorff distance between $T^{[\varepsilon]}$ and T.

We are thus left to deal with the weighted length measure in (i). In this direction, remark that for every $\gamma \geqslant \gamma_0$ we have

$$\mathrm{d}_{\mathrm{Prok}}(\lambda^{\gamma}, \mathbf{1}_{\mathtt{T}^{[\varepsilon]}} \cdot \lambda^{\gamma}) \leqslant \lambda^{\gamma}(T \backslash T^{[\varepsilon]}) \leqslant \big(\max_{T} g^{\gamma-\gamma_0} \big) \cdot \lambda^{\gamma_0}(T \backslash T^{[\varepsilon]}).$$

Consequently, by Corollary 3.4, to conclude it suffices to establish that $\lambda^{\gamma_0}(T\backslash T^{[\varepsilon]})$ converges to 0 in probability, as $\varepsilon \downarrow 0$. To this end note that, from Proposition 5.4 and the scaling property, that for every $\varepsilon > 0$

$$\mathbb{E}\big(\sum_{u\in F(\varepsilon)} \lambda^{\gamma}(T_u) \big) \leqslant \mathbb{E}\big(\lambda^{\gamma_0}(T)\big) \cdot \mathbb{E}\big(\sum_{u\in F(\varepsilon)} \chi(u)^{\gamma_0} \big) = -\frac{1}{\kappa(\gamma_0)} \cdot \mathbb{E}\big(\sum_{u\in F(\varepsilon)} \chi(u)^{\gamma_0} \big),$$

where to obtain the last equality we used (3.22). The desired results follows now by (5.4). $\square$

Remark 5.6 (An intrinsic cutoff)**.** The reader may compare the subtree $\mathtt{T}^{[\varepsilon]}$ with the subtree $\mathtt{T}^{(\varepsilon)}$ obtained heuristically from $\mathtt{T}$ by starting from the root and cutting each branch as soon as the decoration drops below level ε. The inclusion $\mathtt{T}^{(\varepsilon)} \subset \mathtt{T}^{[\varepsilon]}$ should be plain. The nice feature of $\mathtt{T}^{(\varepsilon)}$ is that its equivalence class in $\mathbb{T}$ only depends on that of $\mathtt{T}$, that is, it is an intrinsic geometric subtree, as opposed to $\mathtt{T}^{[\varepsilon]}$ which uses the construction from $(f_u, \eta_u)_{u\in\mathbb{U}}$. However, it will be technical simpler to deal with $\mathtt{T}^{[\varepsilon]}$ rather than $\mathtt{T}^{(\varepsilon)}$.

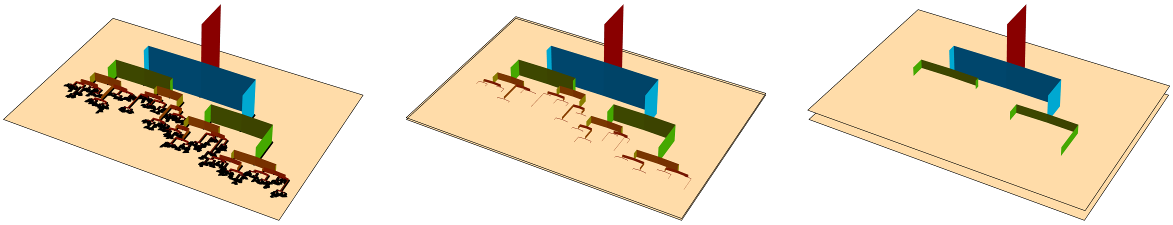

Figure 5.2: Illustration of the decorated subtrees $T^{(\varepsilon)}$ pruned at increasing levels from left to right. The subtrees $T^{[\varepsilon]}$ would actually be larger since they contain all branches whose initial decoration is larger than ε.

5.3 Temporal Markov property

Roughly speaking, the local decompositions described in Propositions 5.3 and 5.4 only rely on the genealogical branching property of self-similar Markov trees, and could have been stated as well for general branching processes; see Example 5.2 and also the proof of Proposition 5.4. In this section, we turn our attention to another type of local decompositions for self-similar Markov trees, which now rather stems from the following temporal Markov property of decoration-reproduction processes. Recall that $(P_x)_{x\geqslant 0}$ denotes the self-similar decoration-reproduction kernel with characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$. For every $t \geqslant 0$, we use the notation $(f, \eta)\circ\theta_t$ for the pair of shifted processes at time t, that is $(f, \eta) \circ \theta_t = (0, 0)$ if $t \geqslant z$, and otherwise,

$$f \circ \theta_t : [0, z-t] \to \mathbb{R}_+, \quad f \circ \theta_t(s) = f(t+s) \quad \text{for } 0 \leqslant s \leqslant z-t,$$

and

$$\int_{[0,\infty)\times\mathbb{R}_+} \varphi(s, x)\eta \circ \theta_t(\mathrm{d}s, \mathrm{d}x) = \int_{(t,\infty)\times\mathbb{R}_+} \varphi(s-t, x)\eta(\mathrm{d}s, \mathrm{d}x),$$

for any measurable function $\varphi : [0, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$.

Lemma 5.7 (Markov property of the decoration-reproduction process)**.** *Write $(\mathcal{G}_t)_{t\geqslant 0}$ for the natural filtration of the decoration-reproduction process (f, η), i.e. $\mathcal{G}_t$ is the (completed) sigma-algebra generated by $\mathbf{1}_{[0,t]} \cdot f$ and $\mathbf{1}_{[0,t]\times\mathbb{R}_+} \cdot \eta$. For every $x \geqslant 0$ and every $(\mathcal{G}_t)$-stopping time R, the conditional distribution under P_x of the shifted decoration-reproduction process $(f, \eta) \circ \theta_R$ given $\mathcal{G}_R$ is $P_{f(R)}$.*

Proof. The claim is an easy extension of the strong Markov property of positive self-similar Markov processes; we merely sketch below the argument and refer to [88, Chapter 13] for more details. Recall the setting of Section 3.2, and in particular that $\mathbf{N} = \mathbf{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}\mathbf{y})$ is a Poisson random measure on $[0,\infty) \times \mathcal{S}$ with intensity measure $\mathrm{d}t\boldsymbol{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$, and ξ the Lévy process with Lévy-Itô decomposition (3.9) and possibly killed at the exponentially distributed time ζ. Writing $(\mathcal{H}_t)_{t\geqslant 0}$ for the natural filtration of $(\xi, \mathbf{N})$, we have that for every $t \geqslant 0$ and conditionally on $t < \zeta$, the pair $(\xi^{(t)}, \mathbf{N}^{(t)})$ given by

$$\xi^{(t)}(s) = \xi(s+t) - \xi(t), \qquad \text{for } s \geqslant 0,$$

and

$$\int_{[0,\infty)\times\mathcal{S}} \varphi(s, y, \mathbf{y})\mathbf{N}^{(t)}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}\mathbf{y}) = \int_{(t,\infty)\times\mathcal{S}} \varphi(s-t, y, \mathbf{y})\mathbf{N}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}\mathbf{y}),$$

is independent of $\mathcal{H}_t$ and has the same law as $(\xi, \mathbf{N})$. This observation extends to the situation where the fixed time t is replaced, first by a simple $(\mathcal{H}_t)$-stopping time, and then, by approximation, by any a.s. finite $(\mathcal{H}_t)$-stopping time. The statement then follows readily by inspection of the Lamperti transformation described in Section 3.2. □

We now derive from Lemma 5.7 the first two temporal local decompositions that are both induced by a stopping time $R \leqslant z_\varnothing$ in the natural filtration of the decoration-reproduction process of the ancestor $(f_\varnothing, \eta_\varnothing)$. We consider the spine truncated at distance R from the root, that is the segment $T^0_R := \{\mathsf{p}(\varnothing, r) : \ r \in [0, R]\} \subset T^0$. We consider also the hull generated by the truncated spine,

$$B^\bullet_R(T) := \{y \in T : d_T(\rho, p_\varnothing(y)) \leqslant R\},$$

where $p_\varnothing : T \to T^0$ denotes the projection on the spine and ρ stands for the root of T. So roughly speaking, the hull $B^\bullet_R(T)$ is the subtree that results from killing the ancestor immediately after time R in the population model. The closure of its complement $\check{B}^\bullet_R(T)$ is connected and hence also a real tree on the event $R < z_\varnothing$, and empty on the complementary event. Notice that when $R < z_\varnothing$, we have

$$\check{B}^\bullet_R(T) = \{y \in T : d_T(0, p_\varnothing(y)) > R\} \cup \{\mathsf{p}(\varnothing, R)\}.$$

Proposition 5.8. *For every stopping time $R \leqslant z_\varnothing$ in the natural filtration of the ancestral decoration-reproduction process $(f_\varnothing, \eta_\varnothing)$, both the truncated spine T^0_R and the hull $B^\bullet_R(T)$ induce a local decomposition of $\mathtt{T}$ under $\mathbb{P}$.*

Proof. The claim readily follows from a variation of the $(\mathbb{Q}_x)_{x\geqslant 0}$ local decomposition along the spine, i.e. Proposition 5.3 when $n = 0$. Recall the notation there and in Example 5.2, and notably that the atoms of the ancestral reproduction process $\eta_\varnothing$ are enumerated in the co-lexicographic order, $(r_1, \ell_1), (r_2, \ell_2), \ldots$, and that for every $j \geqslant 1$, we denoted by $\mathtt{T}_j$ the random decorated tree induced by the sub-family $(f_{ju}, \eta_{ju})_{u\in\mathbb{U}}$. We have seen in the beginning of the proof of Proposition 5.3 that

$$\big((\mathtt{T}^0, (\mathsf{p}(\varnothing, r_i))_{i\in\mathbb{N}}), (\ell_i)_{i\in\mathbb{N}}, (\mathtt{T}_i)_{i\in\mathbb{N}}\big) \ \text{ is a } (\mathbb{Q}_x)_{x\geqslant 0} \text{ local decomposition of } \mathtt{T}. \tag{5.5}$$

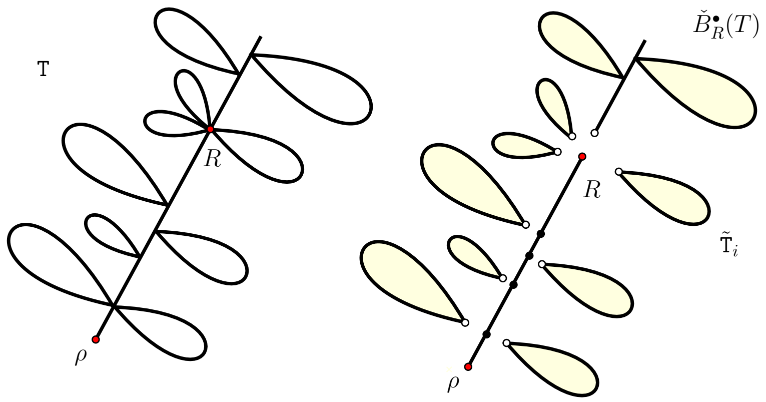

Figure 5.3: Illustration of the Markov property for the truncated spine: conditionally on the reproduction measure, all dangling subtrees are independent and of law $\mathbb{Q}_{\ell_i}$ where ℓ_i are the germ decorations.

We now re-order those elements depending on their positions relative to the stopping time R as follows, taking $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ as new set of indices. We construct recursively $1 \leqslant R(1) < R(2) < \dots$ such that for any $j \geqslant 1$, $R(j) = k$ where k is the rank of the j-th atom (r_k, ℓ_k) such that $r_k > R$ if any, and $R(j) = \infty$ otherwise. We define similarly $1 \leqslant R(-1) < R(-2) < \dots$ such that for any $j \geqslant 1$, $R(-j) = k$ where k is the rank of the j-th atom (r_k, ℓ_k) such that $r_k \leqslant R$ if any, and $R(-j) = \infty$ otherwise. We then set for any $j \in \mathbb{Z}^*$, $\tilde{r}_j = r_{R(j)}$, $\tilde{\ell}_j = \ell_{R(j)}$ and $\widetilde{\mathtt{T}}_j = \mathtt{T}_{R(j)}$ when $R(j) < \infty$, and $\tilde{r}_j = 0$, $\tilde{\ell}_j = 0$ and $\widetilde{\mathtt{T}}_j = \mathtt{0}$ otherwise. It is immediate to check from (5.5), the fact that R is measurable with respect to $((\mathtt{T}^0, (\rho(\varnothing, r_i))_{i\in\mathbb{N}}), (\ell_i)_{i\in\mathbb{N}})$, and the fact that the original ordering can be recovered from the new one, that

$$\big((\mathtt{T}^0, (\rho(\varnothing, \tilde{r}_i))_{i\in\mathbb{Z}^*}), (\tilde{\ell}_i)_{i\in\mathbb{Z}^*}, (\widetilde{\mathtt{T}}_i)_{i\in\mathbb{Z}^*}\big) \text{ is again a } (\mathbb{Q}_x)_{x\geqslant 0} \text{ local decomposition of } \mathtt{T}. \tag{5.6}$$

We then split the spine at R. First, write $\widetilde{\mathtt{T}}_0$ for the decorated version of $\check{B}^{\bullet}_R(T)$ and note that it can be obtained by taking the set $\{\rho(\varnothing, r) : r \in [R, z_\varnothing]\}$ with its associated decoration, and then gluing on it the family $(\widetilde{\mathtt{T}}_i)_{i\geqslant 1}$ at the position prescribed by the points $(\rho(\varnothing, \tilde{r}_i))_{i\geqslant 1}$. Second, write $\mathtt{T}^R_0$ for the decorated truncated spine and observe that the variables $(\mathtt{T}^0_R, (\rho(\varnothing, \tilde{r}_i))_{i\leqslant -1})$ and $(\tilde{\ell}_i)_{i\leqslant -1}$ are measurable in the sigma-algebra $\mathcal{G}_R$ generated by the ancestral decoration-reproduction process stopped at time R. By comparison with the local decomposition along the spine in Proposition 5.3 and the temporal Markov property in Lemma 5.7, we deduce that the conditional law of $\widetilde{\mathtt{T}}_0$ given

$$\Big(\big(\mathtt{T}^0_R, \big(\rho(\varnothing, \tilde{r}_i)\big)_{i\leqslant -1}\big), (\tilde{\ell}_i)_{i\leqslant -1}, (\widetilde{\mathtt{T}}_i)_{i\leqslant -1}\Big)$$

is $\mathbb{Q}_{f_\varnothing(R)}$. Thus writing $\tilde{r}_0 = R$ and $\tilde{\ell}_0 = f_\varnothing(R)$, we conclude from (5.6) that

$$\Big(\big(\mathtt{T}^0_R, \big(\rho(\varnothing, \tilde{r}_i)\big)_{i\leqslant 0}\big), (\tilde{\ell}_i)_{i\leqslant 0}, (\widetilde{\mathtt{T}}_i)_{i\leqslant 0}\Big)$$

is a $(\mathbb{Q}_x)$-local decomposition of $\mathtt{T}$; we have thus proved the claim for the truncated spine.

Finally the claim for the hull follows again from (5.6) and the observation that the decorated version of $B^{\bullet}_R(T)$ can be obtained by gluing the sequence of decorated trees $(\widetilde{\mathtt{T}}_i)_{i\leqslant -1}$ on the truncated spine $\mathtt{T}^0_R$ at the locations $(\mathsf{\rho}(\varnothing,\tilde{r}_i))_{i\leqslant -1}$. $\square$

Similarly as in the previous section, one can iterate the local decomposition of the truncated spine to infer new Markov properties. We illustrate this procedure with another important example of base subtrees, namely the **closed ball** of radius a,

$$B_a(T) := \big\{v \in T:\ d_T(\rho, v) \leqslant a\big\}.$$

Proposition 5.9. *For every $a > 0$, the ball $B_a(T)$ induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$.*

Proof. Recall that T^n denotes the base subtree of T induced by individuals of the general branching process up to generation n only, and set $B_a(T^n) := B_a(T)\cap T^n$. For $n = 0$, the subtree $B_a(T^0)$ is merely the spine truncated at the fixed height a, and we know from Proposition 5.8 that $B_a(T^0)$ induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$. Just as in Proposition 5.3, we can then recursively decompose $\mathtt{T}$ along the truncated segments at generations $1, 2, \dots$, and get that for any $n \geqslant 0$, the base tree $B_a(T^n)$ induces a local decomposition of $\mathtt{T}$ under $\mathbb{P}$. In this framework, the connected components of $T\backslash B_a(T^n)$ are indexed by the set of vertices $u \in \mathbb{U}$.

More specifically, for every $u \in \mathbb{U}$ with $|u| \leqslant n$, note that there is at most one point $\mathsf{\rho}(u, t)$ in $\{\mathsf{\rho}(u, s) :\ s \in [0, z_u]\}$ at distance a from ρ. When there is such a point, we set $r^n_u := \mathsf{\rho}(u, t)$, $\ell^n_u := f_u(t)$ and we write τ^n_u for the subtree formed by the set of points $v \in T$ such that $\mathsf{\rho}(u, t) \in [\![\rho, v]\!]$. Remark that we have $t = a - d_T(\rho, \mathsf{\rho}(u))$ and τ^n_u can be seen as the closure of the complement of the hull of radius $a - d_T(\rho, \mathsf{\rho}(u))$ of T_u. If there no such point or if $|u| > n$, we let $r^n_u := \rho$, $\ell^n_u := 0$ and $\tau^n_u := 0$. If we write $\mathsf{\tau}^n_u$ for the decorated version of τ^n_u and $\mathtt{B}_a(T^n)$ for the decorated version of $B_a(T^n)$, by recursively applying Proposition 5.3 we infer that, under $\mathbb{P}$ the family:

$$\Big(\big(\mathtt{B}_a(T^n), (r^n_u)_{u\in\mathbb{U}}\big), \big(\ell^n_u\big)_{u\in\mathbb{U}}, \big(\mathsf{\tau}^n_u\big)_{u\in\mathbb{U}}\Big)$$

is a $(\mathbb{Q}_x)_{x\geqslant 0}$ local decomposition of $\mathtt{T}$. We stress that the labeling is consistent as we let generations increase, that is, for any vertex u with $|u| \leqslant n$, the mark r^n_u, the variable ℓ^n_u and the subtree $\mathsf{\tau}^n_u$ for the decomposition of $T\backslash B_a(T^n)$ are identical to those obtained for the decomposition of $T\backslash B_a(T^m)$, for any $m > n$. With the obvious notation, for $u \in \mathbb{U}$, let us write r_u, ℓ_u and $\mathsf{\tau}_u$ for the corresponding terminal value. Now, using that $\big(\|f_u\| : u \in \mathbb{U}\big)$ is a null family, the fact that $\sum_{u\in\mathbb{U}} z_u < \infty$ a.s. and the definition of $(\mathbb{T}^{\mathbb{U}\bullet}, \mathrm{d}_{\mathbb{T}^{\mathbb{U}\bullet}})$ given in (2.15), it is straightforward to infer that $\big(\mathtt{B}_a(T^n), (r^n_u)_{u\in\mathbb{U}}\big)$ converges to $\big(\mathtt{B}_a(T), (r_u)_{u\in\mathbb{U}}\big)$, where $\mathtt{B}_a(T)$ stands for the standard decorated version of $B_a(T)$. Therefore, taking the limit when $n \to \infty$, we deduce that

$$\Big(\big(\mathtt{B}_a(T), (r_u)_{u\in\mathbb{U}}\big), \big(\ell_u\big)_{u\in\mathbb{U}}, \big(\mathsf{\tau}_u\big)_{u\in\mathbb{U}}\Big)$$

is a $(\mathbb{Q}_x)_{x\geqslant 0}$ local decomposition of $\mathtt{T}$. This completes the proof of the proposition. $\square$

We now conclude this chapter by stressing a connection with measured-valued branching processes; we refer to [108] for background and precise definition. Consider for every $a \geqslant 0$, the random integer-valued measure on $(0, \infty)$

$$Z_a := \sum_{i \in I} \delta_{\ell_i(a)},$$

where $\ell_i(a)$ denotes the germ decoration of the connected component τ_i^* of $T \backslash B_a(T)$ and we implicitly discard empty components with $\ell_i(a) = 0$ in the sum. Plainly, Z_a does not depend on the choice for indexing these connected components. We then deduce from Proposition 5.9 that for ever $a \geqslant 0$, conditionally on Z_a, the shifted process $(Z_{a+b})_{b \geqslant 0}$ is independent of the process $(Z_b)_{0 \leqslant b \leqslant a}$. Moreover, the distribution of the shifted process given Z_a is that of $\sum_{i \in I} Y^i$, where the Y^i's are independent measured-valued processes and each Y^i has the same law as Z under $\mathbb{Q}_{\ell_i(a)}$. In short, $(Z_a)_{a \geqslant 0}$ is measured-valued branching process. For instance, $(Z_a)_{a \geqslant 0}$ may be a self-similar fragmentation process as discussed in Section 4.2, or a growth-fragmentation process as in Section 4.3.

Comments and bibliographical notes

The Markov property is of course a most important concept in probability theory. Born with the theory of stochastic processes, it has been developed for more sophisticated objets such as branching processes [83] (the stopping lines used many times in this chapter), superprocesses [67] (the so-called special Markov property) and random snakes, see [95, 65] and [126]. The Gaussian Free Field (GFF, in abbreviation) is essentially characterized by its domain Markov property [14] and random sets coupled with the GFF that satisfy a strong Markov property are called **local sets**, see [115] for their introduction and [138] for a comprehensive survey. In particular, our formulation of the Markov property (Definition 5.1) is inspired from GFF local sets and we use the terminology *local decomposition* to emphasize this connection. See also [105, 106] for a theory of a general Markov property in Brownian geometry. We also recall from the introduction that we expect that our self-similar Markov trees (including the critical case discussed in Section 3.4) are essentially all random decorated trees with a positive decoration on the skeleton satisfying a Markov and self-similar property.

Chapter 6

Spinal decompositions and bifurcators

The main purpose of this chapter is to investigate so-called spinal decompositions for self-similar Markov trees governed by a characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ satisfying Assumption 3.9 which is fixed thorough this chapter. We first take the point of view of general branching processes, and then provide an explicit description in the self-similar Markovian setting. We use either a weighted length measure or the harmonic measure to mark a point $\rho^\bullet$ at random on the tree, and describe the joint distribution of the decoration-reproduction along the segment $[\![\rho, \rho^\bullet]\!]$ and the family of subtrees that stem from this segment, that is the collection of – the closure of – the connected components of the complement $T\backslash[\![\rho, \rho^\bullet]\!]$. Although the decorated trees dangling from $[\![\rho, \rho^\bullet]\!]$ are, conditionally on their initial decorations, independent ssMt; the decoration-reproduction process along $[\![\rho, \rho^\bullet]\!]$ is a different self-similar Markov decoration-reproduction whose characteristics $(\sigma^2, \mathrm{a}_\gamma, \boldsymbol{\Lambda}_\gamma; \alpha)$ are explicitly determined in terms of the initial ones. As a first application, we address an important issue already mentioned in Sections 2.4 and 3.2, namely the fact that different characteristic quadruplets may yield self-similar Markov trees with the same distribution, and provide an explicit characterization of such bifurcators in Section 6.3. Another application to the determination of the Hausdorff dimension of self-similar Markov trees that fulfill the first Cramer's condition is given in Section 6.4.

6.1 Spine decompositions in the setting of general branching processes

The notion of spinal decomposition is one of the most useful and powerful tools in the study of branching structures. The purpose of this section is to present its basic aspect, first from the point of view of general branching processes, and then more specifically in the situation where the decoration-reproduction kernel is Markovian and self-similar.

Let $(P_x)_{x>0}$ denote a decoration-reproduction kernel; using the notation introduced in Section 3.1, we write $\mathbb{P}_x$ for the probability law of the associated family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$, which results when the ancestor has type $x > 0$. We assume the existence of a harmonic function $h : (0, \infty) \to (0, \infty)$ in the sense of (3.23). Recall that $\chi(u)$ stands for the type of the individual labelled by the vertex u of the Ulam tree $\mathbb{U}$, and that the process

$M_n = \sum_{|u|=n} h(\chi(u))$ for $n \geqslant 0$ in (3.24) is a nonnegative martingale. We stress that at this stage, we do not require Property $(\mathcal{P})$ to hold, nor the additive martingale (3.24) to converge in $L^1(\mathbb{P})$. The harmonic function h serves here to distinguish an infinite lineage at random, and we will use the symbol $\star$ as an exponent to refer to distinction. That is, a single individual $u^\star(n) \in \mathbb{N}^n$ is distinguished at each generation $n \geqslant 0$ such that $u^\star(n) \prec u^\star(n+1)$, i.e. the individual distinguished at generation n is always the parent of the individual distinguished at generation $n+1$, and $(u^\star(n))_{n\geqslant 0}$ constitutes the distinguished lineage. Specifically, we introduce a probability measure $\bar{\mathbb{P}}^h_x$ that describes the joint law of the family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$ together with $(u^\star(n))_{n\geqslant 0}$. For every $n \geqslant 0$, every $v \in \mathbb{N}^n$ and every nonnegative functional Φ of the decoration-reproduction processes up to generation n, we define

$$\bar{\mathbb{E}}^h_x\big(\Phi((f_u,\eta_u)_{|u|\leqslant n})\mathbf{1}_{u^\star(n)=v}\big) := h(x)^{-1}\mathbb{E}_x\Big(\Phi((f_u,\eta_u)_{|u|\leqslant n})h(\chi(v))\Big). \tag{6.1}$$

The martingale property ensures the coherence of this definition. It is an easily checked and well-known fact in the field of branching processes (see for instance [132, Chapter 4] or [35]) that the probability measures $\bar{\mathbb{P}}^h_x$ also describe a generalized branching process, where individuals not only have a type $x > 0$, but are furthermore either distinguished or not. More precisely, non-distinguished individuals beget only non-distinguished children and the statistics of their decoration-reproduction processes are given by the original kernel $(P_x)_{x>0}$. Distinguished individuals always give birth to a single distinguished child, possibly together with further non-distinguished children, the distinguished child being picked at random in the progeny with probability proportional to the value assigned by h to the type. The decoration-reproduction process of a distinguished individual follows a biased version of the original kernel, denoted by $(\bar{P}^h_x)_{x>0}$. Specifically, writing $t^\star$ for the age of the parent when its distinguished child is born, and $x^\star$ for the type of the latter, then for every nonnegative functional Ψ, there is the identity

$$\bar{E}^h_x\big(\Psi(f,\eta,t^\star,x^\star)\big) = h(x)^{-1}E_x\Big(\int_{[0,\infty)\times(0,\infty)}\eta(\mathrm{d}t,\mathrm{d}y)\Psi(f,\eta,t,y)h(y)\Big). \tag{6.2}$$

Let us now provide a more explicit description of the biased law $\bar{P}^h_x$ in the case when the original decoration-reproduction process is self-similar Markovian with characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ as in Section 3.2. We exclude implicitly the degenerate case $\eta = 0$ when there is no reproduction at all, and suppose further that the cumulant function κ in (3.19) is well-defined and vanishes at some $\omega > 0$. The function $h(x) = x^\omega$ is harmonic; see Lemma 3.13 and observe that the sole harmonicity of h only requires Lemma 3.8 and not the more stringent Assumption 3.12. In this direction, we shall now write $\bar{P}^\omega$ instead of $\bar{P}^h$ for the sake of clarity. Recall from Section 3.2 that in the self-similar Markov setting, the decoration process is denoted by X rather than by f, so (6.2) reads

$$\bar{E}^\omega_x\big(\Psi(X,\eta,t^\star,x^\star)\big) = x^{-\omega}E_x\Big(\int_{[0,\infty)\times(0,\infty)}\eta(\mathrm{d}t,\mathrm{d}y)\Psi(X,\eta,t,y)y^\omega\Big). \tag{6.3}$$

We now argue that a similar feature occurs also for $\gamma > 0$ when $\kappa(\gamma) < 0$. As a motivation, we define first a probability measure $\widetilde{\mathbb{P}}^\gamma_x$, with associated mathematical expectation $\widetilde{\mathbb{E}}^\gamma_x$, describing the joint law of the family of decoration-reproduction processes together with a marked vertex $u^\bullet$ and a marked time $t^\bullet$, such that for every $v \in \mathbb{U}$, every measurable $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and every nonnegative functional Φ,

$$\widetilde{\mathbb{E}}_x^\gamma\Big(\Phi((f_u,\eta_u)_{u\in\mathbb{U}})\varphi(t^\bullet)\mathbf{1}_{u^\bullet=v}\Big) := -\kappa(\gamma)x^{-\gamma}\mathbb{E}_x\Big(\Phi((f_u,\eta_u)_{u\in\mathbb{U}})\int_0^{z_v}\varphi(t)f_v(t)^{\gamma-\alpha}\mathrm{d}t\Big), \tag{6.4}$$

where the assertion that $\widetilde{\mathbb{P}}_x^\gamma$ is probability is seen from Proposition 3.11. In words, the distribution of the family of decoration-reproduction processes $(f_u,\eta_u)_{u\in\mathbb{U}}$ under $\widetilde{\mathbb{P}}_x^\gamma$ is $x^{-\gamma}\,\boldsymbol{\nu}(T)\cdot\mathbb{P}_x$, further conditionally on this family, the marked vertex $u^\bullet$ is picked at random in $\mathbb{U}$ with probability proportional to $\int_0^{z_v} f_v(t)^{\gamma-\alpha}\mathrm{d}t$, and finally, the conditional law of the marked time $t^\bullet$ given $(f_u,\eta_u)_{u\in\mathbb{U}}$ and $u^\bullet$ is proportional to $\mathbf{1}_{t\in[0,z_{u^\bullet})}f_{u^\bullet}(t)^{\gamma-\alpha}\mathrm{d}t$.

We then relate the law $\widetilde{\mathbb{P}}_x^\gamma$ to that of another generalized branching process where individuals have a positive type and are either distinguished or not. Again non-distinguished individuals beget only non-distinguished children and the statistics of their decoration-reproduction processes are given by the original kernel $(P_x)_{x>0}$. In turn distinguished individuals, for which an exponent $\star$ is used in the notation, give birth to at most one distinguished child and further non-distinguished children. We stress that now, a distinguished individual may have no distinguished child (then, of course, the distinguished lineage dies out), in which case we rather distinguish an age. Specifically, when the distinguished parent has a distinguished child, we write $x^\star > 0$ for the type of the latter and $t^\star$ for the age of the parent at the distinguished birth event. Otherwise, i.e. when the distinguished parent has no distinguished child, we set $x^\star = 0$ and write $t^\star$ for the distinguished age. With this notation at hand, the decoration-reproduction kernel $(\bar{P}_x^\gamma)_{x>0}$ for distinguished individuals is defined in terms of the original kernel $(P_x)_{x>0}$ by the following variation of (6.2). We set for every nonnegative functional Ψ,

$$\begin{aligned}&\bar{E}_x^\gamma\big(\Psi(X,\eta,t^\star,x^\star)\big)\\ &= x^{-\gamma}E_x\left(\int_{[0,z)\times(0,\infty)}\Psi(X,\eta,t,y)y^\gamma\eta(\mathrm{d}t,\mathrm{d}y) - \kappa(\gamma)\int_0^z\Psi(X,\eta,t,0)X(t)^{\gamma-\alpha}\mathrm{d}t\right),\end{aligned} \tag{6.5}$$

where as usual $\bar{E}_x^\gamma$ stands for the mathematical expectation with respect to $\bar{P}_x^\gamma$. We write $\bar{\mathbb{P}}_x^\gamma$ for the law of the generalized branching process with distinguished individuals defined above , when the ancestor has type $x > 0$ and is distinguished. More precisely, $\bar{\mathbb{P}}_x^\gamma$ is the joint distribution of the family of decoration-reproduction processes $(f_u,\eta_u)_{u\in\mathbb{U}}$ together with the (finite) sequence of distinguished individuals and distinguished ages. We further write $u^\bullet$ for the ultimate distinguished individual and $t^\bullet$ for its distinguished age.

Lemma 6.1. *Suppose $\kappa(\gamma) < 0$. Then the law of $((f_u,\eta_u)_{u\in\mathbb{U}}, t^\bullet, u^\bullet)$ under $\bar{\mathbb{P}}_x^\gamma$ is $\widetilde{\mathbb{P}}_x^\gamma$.*

Proof. To start with, we observe that for any $n \geqslant 1$, any vertex $v \in \mathbb{N}^n$ at generation n, and every functional $F \geqslant 0$, there is the identity

$$\bar{\mathbb{E}}_x^\gamma\big(F\left((f_u,\eta_u)_{|u|\leqslant n-1}\right)\mathbf{1}_{v\text{ is distinguished}}\big) = \widetilde{\mathbb{E}}_x^\gamma\left(F\left((f_u,\eta_u)_{|u|\leqslant n-1}\right)\mathbf{1}_{u^\bullet(n)=v}\right), \tag{6.6}$$

where as usual $u^\bullet(n)$ denotes the forebear of $u^\bullet$ at generation n if $|u^\bullet| \geqslant n$ and otherwise the indicator in the left-hand side is interpreted as 0. Indeed, this identity is immediately checked for $n = 1$ from (6.4) and Proposition 3.11. The general case $n \geqslant 1$ then follows by induction, applying the branching property under $\bar{\mathbb{P}}_x^\gamma$ and under $\mathbb{P}_x$.

Next, consider a measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a nonnegative functional Φ. By the branching property under $\mathbb{P}_x$, there is a functional $\Psi(f, \eta)$ of the decoration-reproduction process such that for any $x > 0$,

$$\mathbb{E}_x\big(\Phi((f_u, \eta_u)_{u\in\mathbb{U}}) \mid f_\varnothing, \eta_\varnothing\big) = \Psi(f_\varnothing, \eta_\varnothing).$$

Using the branching property under $\bar{\mathbb{P}}_x^\gamma$ and (6.5), we get by conditioning on the decoration-reproduction of the ancestor that

$$\begin{aligned}\bar{\mathbb{E}}_x^\gamma\left(\Phi((f_u, \eta_u)_{u\in\mathbb{U}})\varphi(t^\bullet)\mathbf{1}_{u^\bullet=\varnothing}\right) &= -\kappa(\gamma)x^{-\gamma}\mathbb{E}_x\left(\Psi(f_\varnothing, \eta_\varnothing)\int_0^{z_\varnothing}\varphi(t)f_\varnothing(t)^{\gamma-\alpha}\mathrm{d}t\right)\\ &= \widetilde{\mathbb{E}}_x^\gamma\left(\Phi((f_u, \eta_u)_{u\in\mathbb{U}})\varphi(t^\bullet)\mathbf{1}_{u^\bullet=\varnothing}\right),\end{aligned} \tag{6.7}$$

where the second equality stems from (6.4). The statement now readily follows from the combination of (6.5) and (6.6) with the branching property. □

Our goal now is to characterize the law of $(X, \eta, t^\star, x^\star)$ under $(\bar{P}_x^\gamma)_{x\geqslant 0}$ in both regimes. In this direction, we fix $\gamma > 0$ such that $\kappa(\gamma) \leqslant 0$, and we decompose the decoration-reproduction process into three parts. The first corresponds to the time interval $[0, t^\star)$, the second to the time interval $(t^\star, \infty)$ up to a natural shift and rescaling, and the third focusses at the exact time $t^\star$ when the distinguished child is born. We now spell out these parts explicitly. We write $A = A(X, \eta, t^\star)$ for the pair resulting from the restriction of (X, η) to the time-interval $[0, t^\star)$, that is

$$A = \Big(\mathbf{1}_{[0,t^\star)}X, \mathbf{1}_{[0,t^\star)\times(0,\infty)} \cdot \eta\Big).$$

We write $B = B(X, \eta, t^\star)$ for the pair given by the decoration-reproduction shifted at time $t^\star$ and rescaled, namely the image of $\big(\mathbf{1}_{[t^\star,\infty)}X, \mathbf{1}_{(t^\star,\infty)\times(0,\infty)} \cdot \eta\big)$ by the map

$$(s, y) \mapsto \Big(\frac{s - t^\star}{X(t^\star)^\alpha}, \frac{y}{X(t^\star)}\Big), \qquad s \geqslant t^\star, y \geqslant 0.$$

Finally, we consider the (possibly finite) sequence $(x_j)_{j\geqslant 1}$ of the types of children (distinguished or not) which are born at time $t^\star$, as usual ranked in the non-increasing order. In other words, the restriction of the reproduction process η to the fiber $\{t^\star\} \times (0, \infty)$ is

$$\mathbf{1}_{\{t^\star\}\times(0,\infty)} \cdot \eta = \sum_{j\geqslant 1}\delta_{(t^\star, x_j)},$$

and $x^\star$ is one of the terms of the sequence $(x_j)_{j\geqslant 1}$. We set

$$C = C(X, \eta, t^\star, x^\star) = \Big(\frac{X(t^\star)}{X(t^\star-)}, \frac{x^\star}{X(t^\star-)}, \Big(\frac{x_j}{X(t^\star-)}\Big)_{j\geqslant 1}\Big).$$

Plainly, the variable $X(t^\star-)$ is measurable with respect to A, so $X(t^\star)$ can be recovered from A and C. Therefore A, B, and C entirely determine the decoration-reproduction process of a distinguished individual. We conclude this section by an explicit description of their joint law. In this direction, recall that the function ψ is the Laplace exponent given by the Lévy-Khintchine formula (3.11), and from (3.19) that

$$\psi(\gamma) - \kappa(\gamma) = -\int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \left(\sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i} \right) \in (-\infty, 0).$$

Proposition 6.2. *Let $\gamma > 0$ such that $\kappa(\gamma) \leqslant 0$. For every $x > 0$, the random variables A, B and C are independent under $\bar{P}_x^\gamma$. Moreover we have:*

(i) A is a self-similar Markov decoration-reproduction process with characteristic quadruplet $(\sigma^2, \bar{\mathrm{a}}_\gamma, \bar{\mathbf{\Lambda}}_\gamma; \alpha)$ and type x, where

$$\bar{\mathbf{\Lambda}}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}) := \mathrm{e}^{\gamma y} \cdot \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) - \psi(\gamma)\delta_{(-\infty,(-\infty,\ldots))}(\mathrm{d}y, \mathrm{d}\mathbf{y}),$$

and

$$\bar{\mathrm{a}}_\gamma := \mathrm{a} + \sigma^2\gamma + \int_{\mathcal{S}} y(\mathrm{e}^{\gamma y} - 1)\mathbf{1}_{|y|\leqslant 1}\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

(ii) B has the law P_1 of the initial self-similar Markov decoration-reproduction process with characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$.

(iii) The law of C is determined by

$$\bar{E}_x^\gamma\left(F(C)\right) = |\psi(\gamma)|^{-1} \left(-\kappa(\gamma)F(1, 0, (0, \ldots)) + \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{i\geqslant 1} \mathrm{e}^{\gamma y_i} F(\mathrm{e}^{\gamma y}, \mathrm{e}^{\gamma y_i}, (\mathrm{e}^{\gamma y_j})_{j\geqslant 1}) \right),$$

for every nonnegative functional F.

Proof. By self-similarity, there is no loss of generality in assuming that the type of the initial distinguished individual is $x = 1$. As usual, we then drop the indices 1 in the notation for probabilities and expectations. Roughly speaking, the cornerstone of the argument consists in applying the well-known Mecke equation for Poisson random measures and analyzing its consequences. For this, we need first to reformulate (6.3) in terms of the Lévy process ξ and the Poisson random measure $\mathbf{N}$, which essentially amounts to undoing the Lamperti transformation.

The decoration-reproduction process (X, η) under P is constructed by applying the Lamperti transformation to the Lévy process ξ and the point process η (see (3.14) and (3.18)), and since η is defined in terms of the Lévy process ξ and the Poisson random measure $\mathbf{N}$ by (3.17), we have

$$\begin{aligned}\bar{E}^\gamma\left(\Phi(\xi, \mathbf{N}, \epsilon(t^\star), x^\star)\right) =& E\Big(\int_{[0,\infty)\times\mathbb{R}\times\mathcal{S}_1} \mathbf{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{j\geqslant 1} \Phi\left(\xi, \mathbf{N}, t, \mathrm{e}^{\xi(t-)+y_j}\right) \mathrm{e}^{\gamma(\xi(t-)+y_j)}\Big) \\ &- \kappa(\gamma) E\Big(\int_{[0,\infty)} \mathrm{d}t\ \Phi\big(\xi, \mathbf{N}, t, 0\big)\mathrm{e}^{\gamma\xi(t)}\Big),\end{aligned}$$

where Φ denotes another generic nonnegative functional and ϵ is given by (3.12).

We can now apply the Mecke equation (see for instance [92, Section 4.1]), which involves adding a Dirac mass at $(t, y, \mathbf{y})$, under $\mathbf{1}_{t>0}dt\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$, to the Poisson random measure $\mathbf{N}$. Note from the Lévy-Itô decomposition (3.9) that this addition also changes ξ into $\xi + y\mathbf{1}_{[t,\infty)}$. We rewrite the first expectation of the right-hand side above as

$$\int_0^\infty \mathrm{d}t \int_{\mathbb{R}\times\mathcal{S}_1} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) E\Big(\sum_{j\geqslant 1} \Phi\left(\xi + y\mathbf{1}_{[t,\infty)}, \mathbf{N} + \delta_{(t,y,\mathbf{y})}, t, \mathrm{e}^{\xi(t-)+y_j}\right) \mathrm{e}^{\gamma(\xi(t-)+y_j)}\Big)$$
$$= \int_{\mathbb{R}\times\mathcal{S}_1} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{j\geqslant 1} \mathrm{e}^{\gamma y_j} \int_0^\infty E\Big(\mathrm{e}^{\gamma\xi(t-)}\Phi\Big(\xi + y\mathbf{1}_{[t,\infty)}, \mathbf{N} + \delta_{(t,y,\mathbf{y})}, t, \mathrm{e}^{\xi(t-)+y_j}\Big)\Big)\mathrm{d}t.$$

Recall that variables A, B, and C in the statement essentially correspond to splitting the decoration-reproduction process of a distinguished individual at time $t^\star$. Aiming similarly at undoing the Lamperti transformation for the parts before and after $t^\star$, this leads us to decompose $(\xi, \mathbf{N})$ under $\bar{P}^\gamma$ into three parts corresponding to times strictly before, strictly after, and exactly at $\epsilon(t^\star)$. Specifically, we write A' for the restriction of $(\xi, \mathbf{N})$ to $[0, \epsilon(t^\star))$, and B' for the pair

$$\Big(\mathbf{1}_{(\epsilon(t^\star),\infty)}(\xi - \xi(\epsilon(t^\star)), \mathbf{1}_{(\epsilon(t^\star),\infty)\times\mathcal{S}} \cdot \mathbf{N}\Big)$$

further shifted in time by $\epsilon(t^\star)$. Last, we consider the atom of $\mathbf{N}$ at time $\epsilon(t^\star)$, say $(\epsilon(t^\star), w, (w_i)_{i\geqslant 1})$. We denote by $w^\star$ the distinguished element of the sequence $(w_i)_{i\geqslant 1}$ and set $C' = (w, w^\star, (w_i)_{i\geqslant 1})$. In particular, w coincides with the jump $\Delta\xi(\epsilon(t^\star))$ of ξ at time $\epsilon(t^\star)$ and $w^\star = \log x^\star - \xi(\epsilon(t^\star)-)$. Then A and B are recovered by applying the Lamperti transformation to A' and to B' respectively, and C by exponentiating the elements of C'.

Specializing the previous discussion to functionals Φ of the form

$$\Phi(\xi, \mathbf{N}, \epsilon(t^\star), x^\star) = \Phi_1(A')\Phi_2(B')\Phi_3(C'),$$

and using that for Lebesgue almost every $t > 0$ we have $\xi(t-) = \xi(t)$, we obtain that the quantity $\bar{E}^\gamma\left(\Phi_1(A')\Phi_2(B')\Phi_3(C')\right)$ equals

$$\Big(\int_{\mathbb{R}\times\mathcal{S}_1} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{j\geqslant 1} \mathrm{e}^{\gamma y_j}\Phi_3\big(y, y_j, \mathbf{y}\big) - \kappa(\gamma)\Phi_3\big(0, -\infty, (-\infty, ...)\big)\Big)$$
$$\cdot \int_0^\infty E\left(\mathrm{e}^{\gamma\xi(t)}\Phi_1(\mathbf{1}_{[0,t)}\xi, \mathbf{1}_{[0,t)\times\mathcal{S}} \cdot \mathbf{N}))\Phi_2(\xi^{(t)}, \mathbf{N}^{(t)})\right) \mathrm{d}t,$$

where $\xi^{(t)}(s) := \xi(t+s) - \xi(t)$, for $s \geqslant 0$, and $\mathbf{N}^{(t)}(\mathrm{d}s, \mathrm{d}y', \mathrm{d}\mathbf{y}') := \mathbf{1}_{\{s>0\}\times\mathcal{S}} \cdot \mathbf{N}(t + \mathrm{d}s, \mathrm{d}y', \mathrm{d}\mathbf{y}')$ are the shifted versions of ξ and $\mathbf{N}$. It is now elementary to check that for any $t \geqslant 0$, the expectation in the right-hand side can be expressed as the product

$$E\Big(\mathrm{e}^{\gamma\xi(t)}\Phi_1(\mathbf{1}_{[0,t)}\xi, \mathbf{1}_{[0,t)\times\mathcal{S}} \cdot \mathbf{N}))\Big) \cdot E\big(\Phi_2(\xi, \mathbf{N})\big);$$

see e.g. the proof of Proposition 5.8 which is closely related.

This proves the independence claim in the statement, as well as (ii) and (iii), and we are left the computation of

$$\int_0^\infty E\left(\mathrm{e}^{\gamma\xi(t)}\Phi_1(\mathbf{1}_{[0,t)}\xi, \mathbf{1}_{[0,t)\times\mathcal{S}}\cdot\mathbf{N})\right)\mathrm{d}t.$$

For this, recall that $\psi(\gamma)\in(-\infty,0)$ is the value taken at γ by the Laplace exponent ψ of the Lévy process ξ, and rewrite the preceding quantity as

$$\int_0^\infty \mathrm{e}^{\psi(\gamma)t}E\left(\mathrm{e}^{\gamma\xi(t)-t\psi(\gamma)}\Phi_1(\mathbf{1}_{[0,t)}\xi, \mathbf{1}_{[0,t)\times\mathcal{S}}\cdot\mathbf{N})\right)\mathrm{d}t.$$

The process $\exp(\gamma\xi(t)-\psi(\gamma)t)$ is a martingale under P, which can be use as a density to define a locally equivalent probability measure. This procedure is known as an Esscher transform. In the current setting, if we write $(P'_x)_{x>0}$ for the self-similar Markov decoration-reproduction kernel with characteristic quadruplet $(\sigma^2, \bar{\mathrm{a}}_\gamma, \bar{\mathbf{\Lambda}}_\gamma;\alpha)$ defined in the statement, then an elementary calculation as in the proof of [88, Theorem 3.9] yields

$$|\psi(\gamma)|\int_0^\infty \mathrm{e}^{\psi(\gamma)t}E\left(\mathrm{e}^{\gamma\xi(t)-t\psi(\gamma)}\Phi_1(\mathbf{1}_{[0,t)}\xi, \mathbf{1}_{[0,t)\times\mathcal{S}}\cdot\mathbf{N})\right)\mathrm{d}t = E'\left(\Phi_1(\mathbf{1}_{[0,\zeta)}\xi, \mathbf{1}_{[0,\zeta)\times\mathcal{S}}\cdot\mathbf{N})\right),$$

where we recall that ζ stands for the lifetime of the Lévy process, which is exponentially distributed with parameter $|\psi(\gamma)|$ under P', and E' corresponds to the mathematical expectation with respect to P'. This completes the proof. $\square$

6.2 Size-biased spine decompositions for self-similar Markov trees

In the setting of self-similar Markov trees, a spinal decomposition can be viewed as a remarkable local decomposition in the sense of Definition 5.1 for some size-biased version of a self-similar Markov tree under $\mathbb{P}_x$. Slightly more precisely, the base subtree that induces this decomposition is merely the segment $[\![\rho,\rho^\bullet]\!]$ from the root ρ to a marked point $\rho^\bullet$ which is picked at random according to some natural probability measure on the tree. We may work either with a weighted length measure or the harmonic measure on T, and to avoid many repetitions, we will use the same notation for both cases. In this setting, a spinal decomposition further describes explicitly the joint law of the decoration on the segment $[\![\rho,\rho^\bullet]\!]$ and of the point measure induced by the germs of the decorations of the subtrees dandling from $[\![\rho,\rho^\bullet]\!]$.

We consider a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda};\alpha)$ satisfying Assumption 3.9. Either under this sole requirement, we fix $\gamma>0$ such that $\kappa(\gamma)<0$ and write $\nu := -\kappa(\gamma)\lambda^\gamma$, or under the stronger Assumption 3.12, we take $\gamma=\omega_-$, so $\kappa(\gamma)=0$, and write $\nu:=\mu$ for the harmonic measure. Recall from (3.22) and (3.30) that the total mass $\nu(T)$ has expectation

$$\mathbb{E}_x(\nu(T)) = x^\gamma, \qquad \text{for } x>0. \tag{6.8}$$

The characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda};\alpha)$ induces self-similar laws $(\mathbb{P}_x)_{x>0}$ for families of reproduction processes $(f_u,\eta_u)_{u\in\mathbb{U}}$, and in turn the latter yield distributions $(\mathbb{Q}_x)_{x>0}$ on the space $\mathbb{T}$ of (equivalence classes up to isomorphisms of) measured decorated real trees $\mathbf{T}=$

$(T, d_T, \rho, g, \mathsf{v})$. We next introduce for every $x > 0$ a probability measure $\widetilde{\mathbb{Q}}_x$ on the space $\mathbb{T}^\bullet$ of (equivalence classes up to isomorphisms of) non-measured decorated real trees with a single marked point, $\mathtt{T}^\bullet = (\mathtt{T}, \rho^\bullet)$, such that for any positive measurable function $F : \mathbb{T}^\bullet \to \mathbb{R}_+$,

$$\widetilde{\mathbb{Q}}_x^\gamma\Big(F\big(\mathtt{T}^\bullet\big)\Big) := x^{-\gamma}\mathbb{Q}_x\left(\int_T F\big(\mathtt{T}, r\big)\mathsf{v}(\mathrm{d}r)\right), \tag{6.9}$$

where for the sake of simplicity, we use the same notation for the probability measures $\widetilde{\mathbb{Q}}_x$ and $\mathbb{Q}_x$ as for their corresponding mathematical expectations. In words, $\widetilde{\mathbb{Q}}_x^\gamma$ is obtained from $\mathbb{Q}_x$ by first biasing the latter with the total mass $x^{-\gamma}\mathsf{v}(T)$, and then marking a point $\rho^\bullet$ at random according to the normalized law $\mathsf{v}(\mathrm{d}r)/\mathsf{v}(T)$ on T. The assertion that $\widetilde{\mathbb{Q}}_x^\gamma$ is a probability measure on $\mathbb{T}^\bullet$ is seen from (6.8).

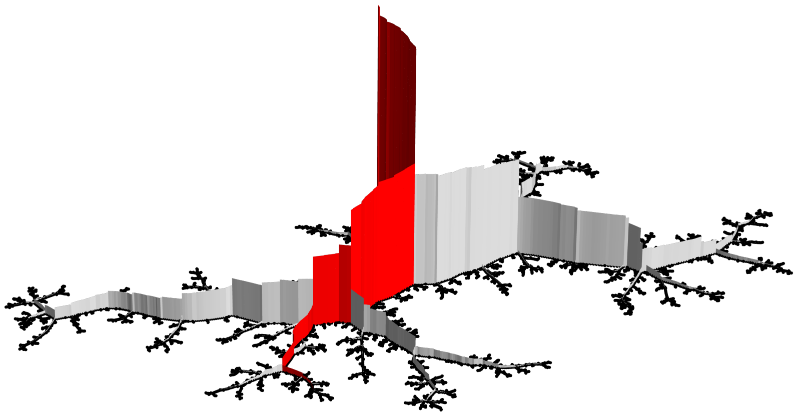

Figure 6.1: Illustration of the spinal decomposition where a branch (in red above) has been distinguished and along which the decoration-reproduction process evolves according to tilted characteristics.

It will often be convenient to work with a specific realization of the marked decorated tree $\mathtt{T}^\bullet = (\mathtt{T}, \rho^\bullet)$ under $\widetilde{\mathbb{Q}}_x^\gamma$ in terms of a certain general branching process. In this direction, let us first make the connexion with the preceding section in the case when Assumption 3.12 holds. When $\kappa(\gamma) < 0$ this connection is transparent. Specifically, if we set $\varrho^\bullet := \varrho(u^\bullet, t^\bullet)$, then by Proposition 3.10 and Lemma 6.1 , under $\bar{\mathbb{P}}_x^\gamma$, the family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$ satisfies Property $(\mathcal{P})$, and the construction of decorated trees by gluing building blocks immediately yields that

$$\text{the distribution of } (\mathtt{T}, \varrho^\bullet) \text{ under } \bar{\mathbb{P}}_x^\gamma \text{ is } \widetilde{\mathbb{Q}}_x^\gamma. \tag{6.10}$$

Let us now explain why the analog result also holds for the harmonic measure. More precisely, recall that the probability measure $\bar{\mathbb{P}}_x^{\omega}$ stands for the joint law of the family of decoration-reproduction processes $(f_u, \eta_u)_{u\in\mathbb{U}}$ together with the distinguished lineage $(u^\star(n))_{n\geqslant 0}$.

Proposition 6.3. *Let Assumption 3.12 be satisfied and take $\gamma = \omega_- = \omega$. Then the following assertions hold under $\bar{\mathbb{P}}_x^{\omega_-}$ for any $x > 0$:*

(i) The law of $(f_u, \eta_u)_{u\in\mathbb{U}}$ is $x^{-\omega_-}\nu(T)\cdot\mathbb{P}_x$ and satisfy Property $(\mathcal{P})$. We write as usual $\mathtt{T}$ for the resulting decorated tree.

(ii) The sequence $(\varrho(u^\star(n)))_{n\geqslant 0}$ of the locations on T induced by the births of the distinguished individuals $u^\star(n)$ converges a.s. We write $\varrho^\bullet$ for its limit.

(iii) The law of the marked decorated tree $(\mathtt{T}, \varrho^\bullet)$ is $\widetilde{\mathbb{Q}}_x^{\omega_-}$.

Proof. (i) By definition, for any $n \geqslant 0$, the distribution of $(f_u, \eta_u)_{|u|\leqslant n}$ under $\bar{\mathbb{P}}_x^{\omega_-}$ is the same as under $x^{-\omega_-}M_{n+1}(\omega_-)\cdot\mathbb{P}_x$, where $(M_n(\omega_-))_{n\geqslant 0}$ is the intrinsic martingale. The remaining assertions are then immediate from Proposition 3.10.

(ii) The sequence $(\varrho(u^\star(n)))_{n\geqslant 0}$ is monotone increasing, in the sense that $\varrho(u^\star(n))$ is te parent of $\varrho(u^\star(n+1))$ for all $n \geqslant 0$. Since T is compact, this sequence converges in T.

(iii) Again it is seen from the very construction that for every $n \geqslant 0$, the conditional distribution under $\bar{\mathbb{P}}_x^{\omega_-}$ of $\varrho(u^\star(n))$ given $\mathtt{T}$ is $\mu^n(\mathrm{d}v)/\mu(T)$, where μ^n stands for the projection of the harmonic measure on T^n (recall that the latter denotes the subtree induced by the individuals up to the generation n only). Since we know from Proposition 2.10 that μ^n converges towards μ as $n \to \infty$ in the sense of Prokhorov, $\bar{\mathbb{P}}_x^{\omega_-}$-a.s. given $\mathtt{T}$, we conclude that the conditional law under $\bar{\mathbb{P}}_x^{\omega_-}$ of $\varrho^\bullet$ given $\mathtt{T}$ is indeed $\mu(\mathrm{d}v)/\mu(T) = \nu(\mathrm{d}v)/\nu(T)$. $\square$

Now that we have defined the ν-marked version of a ssMt and explained how it can be constructed using an explicit family of decoration-reproduction processes, let us turn our attention to the spinal decomposition, which involves decomposing the latter along the marked segment. Informally, our goal is to explicitly describe the joint law of the decoration on the segment $[\![\rho, \rho^\bullet]\!]$, the standard decorated versions of the subtrees dandling from $[\![\rho, \rho^\bullet]\!]$, and the point measure induced by the germs of the decorations of these dandling subtrees. Let us explain the road map that we will follow. First, we introduce the candidate for the law of the decoration-reproduction process encoding the decoration and germs on the segment $[\![\rho, \rho^\bullet]\!]$ by means of a new characteristic quadruplet. As in Section 2.2, this new decoration-reproduction process allows us to define a decorated segment, and by analogy with Section 5.2 we refer to it as the marked spine. Next, we demonstrate in Proposition 6.5 that we can construct a decorated tree by gluing self-similar Markov trees, with the original characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$, onto this marked spine. Finally, we show that the resulting decorated tree, marked at the endpoint of the marked spine, is distributed according to $\widetilde{\mathbb{Q}}_x^{\gamma}$, see Theorem 6.6. The proof will rely on the characterization given in Proposition 6.2.

Let us proceed by introducing our candidate for the decoration-reproduction process along the marked spine. In this direction, we introduce a measure $\mathbf{\Lambda}_\gamma$ on $\mathcal{S}$ which is derived from the generalized Lévy measure $\mathbf{\Lambda}$ as follows. Fix $i \geqslant 1$ and consider for any pair $(y, \mathbf{y})$ in $\mathcal{S} = \mathbb{R} \times \mathcal{S}_1$ the pair $(y, \mathbf{y})^{\backsim i}$ in $\mathcal{S}$ that results by swapping[1] y and y_i. Specifically, the first element of $(y, \mathbf{y})^{\backsim i}$ is given by the i-th term y_i of the sequence $\mathbf{y} = (y_j)_{j \geqslant 1}$, and its second element is obtained from the sequence $(y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots)$ (i.e. we replace the i-th term y_i in $\mathbf{y}$ by y) after re-ordering terms in the non-increasing order. Then $\mathbf{\Lambda}^{\backsim i}$ is simply the push-forward measure of $\mathbf{\Lambda}$ by the map $(y, \mathbf{y}) \mapsto (y, \mathbf{y})^{\backsim i}$. We next define the measure $\mathbf{\Lambda}_\gamma^{\backsim}$ on $\mathcal{S}$ by

$$\mathbf{\Lambda}_\gamma^{\backsim}(\mathrm{d}y, \mathrm{d}\mathbf{y}) := \mathrm{e}^{\gamma y} \cdot \left(\sum_{i \geqslant 1} \mathbf{\Lambda}^{\backsim i}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \right),$$

and observe from the finiteness of $\kappa(\gamma)$ and (3.19) that

$$\mathbf{\Lambda}_\gamma^{\backsim}(\mathcal{S}) = \int_{\mathcal{S}} \left(\sum_{i \geqslant 1} \mathrm{e}^{\gamma y_i} \right) \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) = \kappa(\gamma) - \psi(\gamma) < \infty. \tag{6.11}$$

Finally, we define

$$\mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}) := \mathrm{e}^{\gamma y} \cdot \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) + \mathbf{\Lambda}_\gamma^{\backsim}(\mathrm{d}y, \mathrm{d}\mathbf{y}) - \kappa(\gamma)\delta_{(-\infty,(0,-\infty,\cdots))}(\mathrm{d}y, \mathrm{d}\mathbf{y}). \tag{6.12}$$

We stress from the finiteness of $\psi(\gamma)$ that $\int_{\mathcal{S}} (1 \wedge y^2)\mathrm{e}^{\gamma y}\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) < \infty$, and therefore $\mathbf{\Lambda}_\gamma$ is a generalized Lévy measure, since

$$\int_{\mathcal{S}} (1 \wedge y^2)\mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}) < \infty.$$

Of course (6.12) is reminiscent of the definition of $\bar{\mathbf{\Lambda}}_\gamma$ given in Proposition 6.2. One of our motivations for introducing $\mathbf{\Lambda}_\gamma$ stems from the following observation.

Lemma 6.4. *The function*

$$\psi_\gamma(q) := \kappa(\gamma + q), \qquad q \geqslant 0 \tag{6.13}$$

can then be expressed in the Lévy-Khintchine form

$$\psi_\gamma(q) = \frac{1}{2}\sigma^2 q^2 + \mathrm{a}_\gamma q + \int_{\mathcal{S}} \left(\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{|y| \leqslant 1}\right) \mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}),$$

where the drift coefficient a_γ *is given by*

$$\mathrm{a}_\gamma := \mathrm{a} + \sigma^2\gamma + \int_{\mathcal{S}} \left(y(\mathrm{e}^{\gamma y} - 1)\mathbf{1}_{|y| \leqslant 1} + \sum_{i=1}^{\infty} y_i \mathrm{e}^{\gamma y_i}\mathbf{1}_{|y_i| \leqslant 1} \right) \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}), \tag{6.14}$$

[1] Observe that we often use the symbol $\sim$ to indicate a tilting transformation of probability measures; this should not be confused with $\backsim$ which rather refers to swapping two elements.

Proof. The claim should be viewed as a variation of a well-known result related to the Esscher transform, see e.g. [88, Theorem 3.9]. It states that if ψ is the Laplace exponent of a real Lévy process with characteristic triplet $(\sigma^2, \mathrm{a}, \Lambda)$ (recall our convention that the killing rate $\mathrm{k} = \Lambda(\{-\infty\})$ is specified by the mass of the Lévy measure at $-\infty$), and if $\psi(\gamma) \leqslant 0$, then the shifted function $\psi(\gamma + \cdot)$ can be expressed in the Lévy-Khintchine form (3.11) for the same Gaussian coefficient σ^2, the tilted drift coefficient $\mathrm{a}+\sigma^2\gamma+\int(\mathrm{e}^{\gamma y}-1)y\mathbf{1}_{|y|\leqslant 1}\Lambda(\mathrm{d}y)$, and the tilted Lévy measure $\mathrm{e}^{\gamma y}\Lambda(\mathrm{d}y) - \psi(\gamma)\delta_{-\infty}$.

Let us now proceed with the proof of the lemma. Since $\mathbf{\Lambda}_\gamma$ is a generalized Lévy measure with killing rate $-\kappa(\gamma) \geqslant 0$, we have

$$\begin{aligned}&\int_{\mathcal{S}} \left(\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{|y|\leqslant 1}\right) \mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y})\\ &= \int_{\mathcal{S}} \left(\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{|y|\leqslant 1}\right) \mathrm{e}^{\gamma y}\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) + \int_{\mathcal{S}} \Big(\sum_{i=1}^{\infty} \mathrm{e}^{\gamma y_i}\left(\mathrm{e}^{qy_i} - 1 - qy_i\mathbf{1}_{|y_i|\leqslant 1}\right)\Big)\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) + \kappa(\gamma).\end{aligned}$$

Now, by the Esscher transformation, the first term in the sum on the right-hand side can be expressed as

$$\begin{aligned}&\int_{\mathcal{S}} \left(\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{|y|\leqslant 1}\right) \mathrm{e}^{\gamma y}\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})\\ &= \psi(\gamma+q) - \psi(\gamma) - \frac{1}{2}\sigma^2 q^2 - q\left(\mathrm{a} + \sigma^2\gamma + \int_{\mathcal{S}} (\mathrm{e}^{\gamma y} - 1)y\mathbf{1}_{|y|\leqslant 1}\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})\right).\end{aligned}$$

Recall also from (6.11) that $\kappa(\gamma) = \psi(\gamma) + \mathbf{\Lambda}_\gamma^{\smile}(\mathcal{S})$. Therefore, if we set

$$\mathrm{a}_\gamma := \mathrm{a} + \sigma^2\gamma + \int_{\mathcal{S}} \Big(y(\mathrm{e}^{\gamma y} - 1)\mathbf{1}_{|y|\leqslant 1} + \sum_{i=1}^{\infty} y_i \mathrm{e}^{\gamma y_i}\mathbf{1}_{|y_i|\leqslant 1}\Big)\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}),$$

and use (3.19), then we arrive at the identity

$$\kappa(\gamma+q) = \frac{1}{2}\sigma^2 q^2 + \mathrm{a}_\gamma q + \int_{\mathcal{S}} \left(\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{|y|\leqslant 1}\right) \mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}),$$

and our claim is checked. □

Lemma 6.4 allows to apply the construction devised in Section 3.2 by keeping the same Gaussian coefficient σ^2 and the same exponent of self-similarity α, but replacing the Poisson random measure $\mathbf{N}$ on $[0,\infty)\times\mathcal{S}$ with intensity $\mathrm{d}t\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$ there by a Poisson random measure $\mathbf{N}_\gamma$ with intensity $\mathrm{d}t\mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y})$ and the drift coefficient a by a_γ given in (6.14). We write $(P_x^\gamma)_{x>0}$ for the self-similar kernel of decoration-reproduction laws induced there by the characteristic quadruplet $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$. In particular, the decoration process under this kernel is a self-similar Markov process X_γ with exponent α that is associated by the Lamperti transformation to a Lévy process ξ_γ with Laplace exponent ψ_γ. We also write η_γ for the reproduction process. We stress that in the case where $\kappa(\gamma) < 0$, the decoration is strictly positif immediately before the deathtime, $X_\gamma(z-) > 0$, and the reproduction process η_γ has an atom at $(z, X_\gamma(z-))$ (and there are no further atoms at time z).

Our goal now is to glue self-similar Markov trees, with characteristics $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$, onto the decorated segment induced by (X_γ, η_γ). To this end, we rely on the following technical result.

Proposition 6.5. *Fix $x > 0$. Under P_x^γ, consider $(t_1, y_1), (t_2, y_2), \dots$ the atoms of η_γ in co-lexicographical order. Let $(\mathtt{T}_j = (T_j, d_{T_j}, \rho_j, g_j))_{j\geqslant 1}$ denote a sequence of independent ssMt with characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and such that each $\mathtt{T}_j$ has the law $\mathbb{P}_{y_j}$. Then, almost surely, the two families*

$$\big(\mathrm{Height}(T_j)\big)_{j\geqslant 1} \ \textit{and}\ \big(\max_{T_j} g_j\big)_{j\geqslant 1}$$

are null.

Before proving the proposition, let us discuss some of its implications. First, with the notation above, it allows us to use Lemma 2.3 and consider the non-degenerated decorated tree

$$\mathrm{Gluing}\Big(\big([0,z], d, 0, X_\gamma\big), (y_j)_{j\geqslant 1}, \big(\mathtt{T}_i\big)_{j\geqslant 1}\Big),$$

where z stands for the lifetime of X_γ and d for the usual distance on segments. We then let $\widehat{\mathbb{Q}}_x^\gamma$ be the resulting law on the space $\mathbb{T}^\bullet$ of decorated real trees with a single marked point, where the marked point is induced by the right-extremity z of the ancestral segment. Heuristically speaking, under $\widehat{\mathbb{Q}}_x^\gamma$, the evolution along the spine is governed by the characteristic $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$, while the other branches are governed by $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. We can now provide a formal statement for the spinal decomposition. Recall the definition (6.9) of the law $\widetilde{\mathbb{Q}}_x^\gamma$.

Theorem 6.6 (Spinal decomposition)**.** *Either let Assumption 3.9 hold and pick any $\gamma > 0$ such that $\kappa(\gamma) < 0$, or let the stronger Assumption 3.12 hold and take $\gamma = \omega_-$. For every $x > 0$, the probability measures $\widehat{\mathbb{Q}}_x^\gamma$ and $\widetilde{\mathbb{Q}}_x^\gamma$ on $\mathbb{T}^\bullet$ are identical.*

The proofs of Proposition 6.5 and Theorem 6.6 rely on two further technical lemmas that connect the distribution of the decoration-reproduction process (X_γ, η_γ) under P_1^γ to (X, η) under P_1. These two lemmas will allow us to prove Proposition 6.5 and Theorem 6.6 simultaneously by comparing these results with the characterization of $\bar{P}_1^\gamma$ given in Proposition 6.2. To start with, (6.12) suggests to decompose $\mathbf{N}_\gamma$ under P_1^γ as the sum of two independent Poisson point processes, $\mathbf{N}'$ and $\mathbf{N}''$, with respective intensities

$$\mathrm{e}^{\gamma y} \cdot \mathrm{d}t \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \quad \text{and} \quad \mathrm{d}t \left(\mathbf{\Lambda}_\gamma^{\frown} - \kappa(\gamma)\delta_{(-\infty,(0,-\infty,\cdots))}\right)(\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

Since $\mathbf{\Lambda}_\gamma^{\frown}$ is a finite measure, the set of times at which $\mathbf{N}''$ has an atom is discrete P_1^γ-a.s., and we write r'' for the first one,

$$r'' := \sup\big\{r \geqslant 0 : \mathbf{N}''([0,r] \times \mathcal{S}) = 0\big\}.$$

In particular, by (6.11), r'' is exponentially distributed with parameter $\mathbf{\Lambda}_\gamma^{\frown}(\mathcal{S}) - \kappa(\gamma) = -\psi(\gamma)$.

We next write $\epsilon_\gamma(t) = \int_0^t \exp(\alpha\xi_\gamma(s))\mathrm{d}s$ for the exponential functional (3.12) of the Lévy process ξ_γ that appears in the Lamperti transformation. So $t'' := \epsilon_\gamma(r'')$ is the first time at which the reproduction process η_γ has atoms originating from $\mathbf{N}''$. Write furthermore $(t'', x_j'')_{j\geqslant 1}$ for the sequence of atoms of η_γ at time t'', in non-increasing order of $(x_j)_{j\geqslant 1}$.

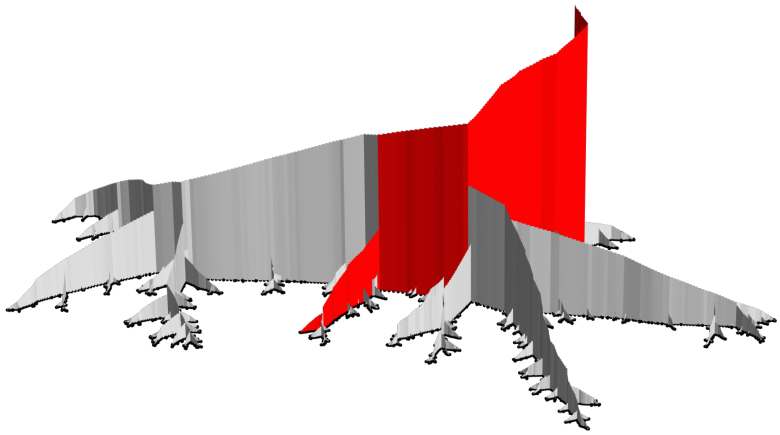

Figure 6.2: Illustration of Theorem 6.6.

Lemma 6.7. *Under P_1^γ, and in the notation above, the pair*

$$\left(X_\gamma(t'')/X_\gamma(t''-), \left(x''_j/X_\gamma(t''-)\right)_{j\geqslant 1}\right),$$

is independent of the restriction of the decoration-reproduction process to the time-interval $[0,t'')$, $\left(\mathbf{1}_{[0,t'')}X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)}\cdot\eta_\gamma\right)$, and the distribution of the former is the push-forward of the law on $\mathcal{S}$

$$|\psi(\gamma)|^{-1}\left(\mathbf{\Lambda}_\gamma^{\smile} - \kappa(\gamma)\delta_{(-\infty,(0,-\infty,\cdots))}\right)$$

by the exponential map $(y,(y_j)_{j\geqslant 1}) \mapsto (\mathrm{e}^y, (\mathrm{e}^{y_j})_{j\geqslant 1})$. Moreover, for every functional $F \geqslant 0$, there is the identity

$$E_1^\gamma\left(F\left(\mathbf{1}_{[0,t'')}X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)}\cdot\eta_\gamma\right)\right) = |\psi(\gamma)|\int_0^\infty \mathrm{d}t\, E_1\left(X(t)^{\gamma-\alpha}F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right)\right).$$

Proof. Let $(r'',(y'',\mathbf{y}''))$ denote the first atom of $\mathbf{N}''$. By basic properties of Poisson point measures, $(y'',\mathbf{y}'')$ is a variable in $\mathcal{S}$ distributed according to the normalized intensity

$$|\psi(\gamma)|^{-1}\left(\mathbf{\Lambda}_\gamma^{\smile} - \kappa(\gamma)\delta_{(-\infty,(0,-\infty,\cdots))}\right).$$

Moreover r'', $(y'',\mathbf{y}'')$, and $\mathbf{N}'$ are independent.

Next, let ξ' denote the process derived from ξ_γ by suppressing the jumps of the latter coming from $\mathbf{N}''$. Then ξ' is a Lévy process whose Lévy-Itô decomposition (3.9) uses the same Brownian component as ξ_γ, the Poisson random measure $\mathbf{N}'$ instead of $\mathbf{N}_\gamma$, and finally the drift coefficient

$$\mathrm{a}' := \mathrm{a}_\gamma - \int_{\mathcal{S}} y\mathbf{1}_{|y|\leqslant 1}\mathbf{\Lambda}_\gamma^{\smile}(\mathrm{d}y,\mathrm{d}\mathbf{y}) = \mathrm{a} + \sigma^2\gamma + \int_{\mathcal{S}} y(\mathrm{e}^{\gamma y}-1)\mathbf{1}_{|y|\leqslant 1}\mathbf{\Lambda}(\mathrm{d}y,\mathrm{d}\mathbf{y}).$$

We stress that the latter quantity has been tuned up to take into account the compensation in Poissonian integrals. By the Lévy-Khintchine formula, the Laplace exponent of ξ' is $q \mapsto \psi(\gamma+q)-\psi(\gamma)$. Obviously, r'', $(y'', \mathbf{y}'')$ and $(\xi', \mathbf{N}')$ are independent.

Since ξ_γ and ξ' coincide on the time-interval $[0, r'')$, we have

$$\epsilon_\gamma(t) = \epsilon'(t) := \int_0^t \exp(\alpha \xi'(s)) \mathrm{d}s \qquad \text{for all } t \leqslant r'', \tag{6.15}$$

and by the Lamperti transformation, $(y'', \mathbf{y}'')$ and $\left(\mathbf{1}_{[0,t'')} X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)} \cdot \eta_\gamma\right)$ are independent. By construction, there are the identities

$$X_\gamma(\epsilon_\gamma(r''))/X_\gamma(\epsilon_\gamma(r'')-) = \exp(y'') \quad \text{and} \quad x''_j/X_\gamma(\epsilon_\gamma(r'')-) = \exp(y''_j),$$

and the first two claims of the statement follow.

We turn our attention to the third claim. We deduce by the discussion above that for every functional $F \geqslant 0$, there is the identity

$$\begin{aligned} &E_1^\gamma\Big(F\left(\mathbf{1}_{[0,t'')} X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)} \cdot \eta_\gamma\right)\Big) \\ &= |\psi(\gamma)| \int_0^\infty \mathrm{d}t\, E_1^\gamma\left(\mathrm{e}^{\psi(\gamma)t} F\left(\mathbf{1}_{[0,\epsilon'(t))} X', \mathbf{1}_{[0,\epsilon'(t))\times(0,\infty)} \cdot \eta'\right)\right), \end{aligned}$$

where (X', η') denotes the decoration-reproduction process derived from $(\xi', \mathbf{N}')$ by the Lamperti transformation. On the other hand, again from a version of the Esscher transformation ([88, Theorem 3.9]), we know that the process $(\exp(\gamma\xi(t) - t\psi(\gamma)))_{t\geqslant 0}$ is a martingale under P_1, and for every $t > 0$, the distribution of the pair $\left(\mathbf{1}_{[0,t]}\xi, \mathbf{1}_{[0,t]\times\mathcal{S}} \cdot \mathbf{N}\right)$ under the tilted law $\exp(\gamma\xi(t) - t\psi(\gamma)) \cdot P_1$ is the same as that of the pair $\left(\mathbf{1}_{[0,t]}\xi', \mathbf{1}_{[0,t]\times\mathcal{S}} \cdot \mathbf{N}'\right)$. A final application of the Lamperti transformation yields

$$\begin{aligned} &\int_0^\infty \mathrm{d}t\, E_1^\gamma\left(\mathrm{e}^{\psi(\gamma)t} F\left(\mathbf{1}_{[0,\epsilon'(t))} X', \mathbf{1}_{[0,\epsilon'(t))\times(0,\infty)} \cdot \eta'\right)\right) \\ &= \int_0^\infty \mathrm{d}s\, E_1\Big(X(\epsilon_\gamma(s))^\gamma F\left(\mathbf{1}_{[0,\epsilon_\gamma(s))} X, \mathbf{1}_{[0,\epsilon_\gamma(s))\times(0,\infty)} \cdot \eta\right)\Big), \end{aligned}$$

and combining Tonelli's theorem with a time change using (6.15) we get that the previous displays equals

$$\int_0^\infty \mathrm{d}s\, E_1\left(X(s)^{\gamma-\alpha} F\left(\mathbf{1}_{[0,s)} X, \mathbf{1}_{[0,s)\times(0,\infty)} \cdot \eta\right)\right).$$

which completes the proof. □

The second technical lemma shows that the distributions in Lemma 6.7 also naturally appear under a tilted version of P_1; we need to introduce some notation in this direction. Consider a realization of the decoration-reproduction process (X, η) under the law P_1, and assume that (t, x) is an atom of η, that is a child with type x is born at time t. Let $\mathbf{x}(t) = (x_j)_{j\geqslant 1}$ be the sequence of the types of the children born at that time, repeated according to their multiplicities and ranked in the non-increasing order. In particular, x is one of the terms of the sequence $\mathbf{x}$, say $x = x_i$. We then write $(X(t), \mathbf{x})^{x\backsim}$ for the pair whose first element is x and second element is derived from the sequence $(x_1, \dots, x_{i-1}, X(t), x_{i+1}, \dots)$ (i.e. we replace a term x in $\mathbf{x}$ by $X(t)$) after re-ordering in the non-increasing order.

Lemma 6.8. *For every nonnegative functionals F, G, there is the identity*

$$\begin{aligned}
&E_1^\gamma \left(F\left(\mathbf{1}_{[0,t'')}X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)}\cdot\eta_\gamma\right) G(X_\gamma(t''), (x''_j)_{j\geqslant 1})\right) \\
&= E_1\left(\int_{[0,z)\times(0,\infty)} \eta(\mathrm{d}t, \mathrm{d}x) x^\gamma F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) G\big((X(t), \mathbf{x}(t))^{x\backsim}\big)\right) \\
&- \kappa(\gamma)\cdot E_1\Big(\int_0^z \mathrm{d}s\ X(t)^{\gamma-\alpha} F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) G\big(0, (X(t), 0, \dots)\big)\Big).
\end{aligned}$$

Proof. The notion of compensators of optional processes and random measures (see [82, Sections I.3 and II.1]) lies at the heart of the proof. Roughly speaking it plays a role similar to that of the Mecke equation in Proposition 6.2. To start with, recall from Lemma 6.7 that

$$\begin{aligned}
&E_1^\gamma \left(F\left(\mathbf{1}_{[0,t'')}X_\gamma, \mathbf{1}_{[0,t'')\times(0,\infty)}\cdot\eta_\gamma\right) G(X_\gamma(t''), (x''_j)_{j\geqslant 1})\right) \\
&= E_1\left(\int_0^z \mathrm{d}t\, X(t)^{\gamma-\alpha} F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) \int_{\mathcal{S}} \mathbf{\Lambda}_\gamma^\backsim(\mathrm{d}y, \mathrm{d}\mathbf{y}) G(X(t)\mathrm{e}^y, (X(t)\mathrm{e}^{y_j})_{j\geqslant 1})\right) \\
&- \kappa(\gamma) E_1\left(\int_0^\infty \mathrm{d}t\, X(t)^{\gamma-\alpha} F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) G(0, (X(t), 0, \dots))\right).
\end{aligned}$$

Comparing this identity with that of the statement, it suffices to identify the first terms of the differences in the respective right-hand sides.

We work under P_1 and may assume without loss of generality that the functionals F and G are bounded. In the natural filtration of (X, η), we consider the optional increasing process

$$A(t) := \int_{[0,t]\times(0,\infty)} \eta(\mathrm{d}s, \mathrm{d}x) x^\gamma G((X(s), \mathbf{x}(s))^{x\backsim}).$$

We claim that its compensator $A^{(\mathrm{p})}$ is the predictable increasing process given by

$$A^{(\mathrm{p})}(t) = \int_0^t \mathrm{d}s X(s)^{\gamma-\alpha} \int_{\mathcal{S}} \mathbf{\Lambda}_\gamma^\backsim(\mathrm{d}y, \mathrm{d}\mathbf{y}) G(X(s)\mathrm{e}^y, (X(s)\mathrm{e}^{y_j})_{j\geqslant 1}), \tag{6.16}$$

in the sense that $A(t) - A^{(\mathrm{p})}(t)$ is a martingale. Since the process $F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right)$ is predictable and bounded, it follows that

$$\begin{aligned}
&E_1\Big(\int_{[0,z)\times(0,\infty)} F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) \mathrm{d}A(t)\Big) \\
&= E_1\Big(\int_{[0,z)\times(0,\infty)} F\left(\mathbf{1}_{[0,t)}X, \mathbf{1}_{[0,t)\times(0,\infty)}\cdot\eta\right) \mathrm{d}A^{(\mathrm{p})}(t)\Big),
\end{aligned}$$

which proves of the statement.

We still have to check (6.16), for which we need to return to the construction of the reproduction process η in terms of the Poisson point measure $\mathbf{N}$ and Lamperti's transformation in Section 3.2. The expression for the optional increasing process A leads us to introduce the optional process (now in the natural filtration of $(\xi, \mathbf{N})$)

$$D(t) := \int_{[0,t]\times\mathcal{S}} \mathbf{N}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{i\geqslant 1} \exp(\gamma(\xi(s-)+y_i))H(\xi(s-), (y, \mathbf{y})^{\backsim i}),$$

for the functional

$$H(r, (y, \mathbf{y})) := G(\mathrm{e}^{r+y}, (\mathrm{e}^{r+y_i})_{i\geqslant 1}).$$

Observe that if we write $x = \mathrm{e}^{r+y}$ and $\mathbf{x} = (\mathrm{e}^{r+y_i})_{i\geqslant 1}$, then

$$H(r, (y, \mathbf{y})^{\backsim i}) = G((x, \mathbf{x})^{x_i \backsim}).$$

Finally, by Poissonian calculus, the compensator $D^{(\mathrm{p})}$ of D is

$$\begin{aligned} D^{(\mathrm{p})}(t) &= \int_0^t \mathrm{d}s \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{i\geqslant 1} \exp(\gamma(\xi(s-)+y_i))H(\xi(s-), (y, \mathbf{y})^{\backsim i}) \\ &= \int_0^t \mathrm{d}s \exp(\gamma\xi(s-)) \int_{\mathcal{S}} \mathbf{\Lambda}_\gamma^{\backsim}(\mathrm{d}y, \mathrm{d}\mathbf{y}) H(\xi(s-), (y, \mathbf{y})), \end{aligned}$$

and (6.16) follows from an application of the Lamperti time-substitution. $\square$

We can now tackle the proofs of Proposition 6.5 and Theorem 6.6.

Proofs of Proposition 6.5 and Theorem 6.6. Depending on whether $\gamma = \omega_-$ or $\kappa(\gamma) < 0$, we use the realization of $\widetilde{\mathbb{Q}}_x^\gamma$ in terms of a general branching process with a distinguished lineage, provided in Proposition 6.3 and the discussion above it. We henceforth work under the associated law $\bar{\mathbb{P}}_x^\gamma$, and by scaling, we may assume without loss of generality that $x = 1$. This enables us to explore the segment $[\![\rho, \rho^\bullet]\!]$ by following the trajectory of distinguished individuals, switching from parent to child at each distinguished birth event, in the sense that we then stop following the distinguished parent and rather follow its distinguished child. This allows us to analyze the decoration-reproduction process along the spine $[\![\rho, \rho^\bullet]\!]$, say $(f^\bullet, \eta^\bullet)$, which is defined rigorously in terms of the distinguished lineage as follows. In the case $\kappa(\gamma) < 0$, we write $n^\bullet := |u^\bullet|$ for the generation of the ultimate distinguished individual, so $n^\bullet + 1$ has the geometric law with success probability $\kappa(\gamma)/\psi(\gamma)$, as it can readily be checked from (6.4) and using the first lines of the proof of Proposition 3.11. In the case $\gamma = \omega_-$, we set $n^\bullet = \infty$. Next for every generation $0 \leqslant n < n^\bullet$, we write $t^\star(n)$ for the distinguished age at which the distinguished individual $u^\star(n)$ begets its distinguished child, and further set $t^\star(n^\bullet) = t^\bullet$ in the notation of Lemma 6.1 when $\kappa(\gamma) < 0$. In this framework, the segment $[\![\rho, \rho^\bullet]\!]$ is realized by concatenating one after the other the segments corresponding to distinguished individuals and truncated at their distinguished ages. We set $s^\star(n) := \sum_{k=0}^n t^\star(k)$, in particular $s^\star(-1) := 0$ and $s^\star(n^\bullet) := z^\bullet$ is the length of $[\![\rho, \rho^\bullet]\!]$. The (rcll) decoration $f^\bullet : [0, z^\bullet) \to \mathbb{R}_+$ is simply obtained by concatenating the truncated decorations of distinguished individuals,

$$f^\bullet(t) := f_{u^\star(n)}\big(t - s^\star(n-1)\big) \quad \text{for any } s^\star(n-1) \leqslant t < s^\star(n).$$

In turn, the reproduction process $\eta^\bullet$ only partly results from the concatenation of the truncated reproduction processes of distinguished individuals; a special attention must be given to the birth-times of distinguished children. Discarding the latter at first, we define the point process η° by

$$\int_{[0,z^\bullet)\times(0,\infty)} \eta^\circ(\mathrm{d}t,\mathrm{d}x)\varphi(t,x) := \sum_{k=0}^{n^\bullet} \int_{[0,t^\star(k))\times(0,\infty)} \eta_{u^\star(k)}(\mathrm{d}t,\mathrm{d}x)\varphi(t+s^\star(k-1),x),$$

where $\varphi:\mathbb{R}_+\times(0,\infty)\to\mathbb{R}_+$ stands for a generic measurable function. Next, for any $0\leqslant k<n^\bullet$, the distinguished individual $u^\star(k)$ begets children at its distinguished age $t^\star(k)$; for simplicity just write $(x_j)_{j\geqslant 1}$ for the sequence of the types of those children, repeated according to their multiplicities and, say, listed in the non-increasing order. The type of the distinguished child is an element of this sequence, and we then write $(x_j^\backsim)_{j\geqslant 1}$ for the sequence obtained from $(x_j)_{j\geqslant 1}$ after replacing the type of the distinguished child by $f_{u^\star(k)}(t^\star(k))$, the value of the decoration immediately after this birth event. This transformation reflects the fact that at distinguished birth events, we cease to follow the distinguished parent and rather switch to its distinguished child. We then write

$$\eta_k^{\star\backsim} := \sum_{j\geqslant 1} \delta_{(s^\star(k),x_j^\backsim)}.$$

Finally, in the case $\kappa(\gamma)<0$, the ultimate distinguished individual $u^\bullet$ at generation $n^\bullet$ does not beget any distinguished child, and we set

$$\eta_{n^\bullet}^{\star\backsim} := \delta_{(z^\bullet,f_{u^\bullet}(t^\bullet))}.$$

We now have all the ingredients to define the reproduction process on $[\![\rho,\rho^\bullet]\!]$ by

$$\eta^\bullet := \eta^\circ + \sum_{k=0}^{n^\bullet} \eta_k^{\star\backsim}.$$

We claim that

(i) The distribution of $(f^\bullet,\eta^\bullet)$ is P_1^γ.

(ii) If we write $\eta^\bullet = \sum_{i\geqslant 1}\delta_{(t_i,x_i)}$ where indices are chosen in co-lexicographical order, then conditionally on $(f^\bullet,\eta^\bullet)$, the associated standard decorated subtrees $(\mathtt{T}_i)_{i\geqslant 1}$ dangling from $[\![\rho,\rho^\bullet]\!]$ in $\mathtt{T}^\bullet=(\mathtt{T},\rho^\bullet)$ are independent and, for every $i\geqslant 1$, the law of $\mathtt{T}_i$ is $\mathbb{P}_{x_i}$.

Before proving the claim let us explain why Proposition 6.5 and Theorem 6.6 follows directly from it. First, by Proposition 3.10 and Lemma 6.1, combined with the definition of $\bar{\mathbb{P}}_1^\gamma$, we infer that under $\bar{\mathbb{P}}_1^\gamma$ the family $(f_u,\eta_u)_{u\in\mathbb{U}}$ satisfies Property $(\mathcal{P})$. It follows that the families

$$\big(\mathrm{Height}(T_i)\big)_{i\geqslant 1} \text{ and } \big(\max_{T_i} g_i\big)_{i\geqslant 1}$$

are both null. Therefore Point (i) implies Proposition 6.5. Moreover, by definition of the dangling subtrees and Point (ii), we must have:

$$\mathtt{T} = \mathrm{Gluing}\Big(\big([0,z^\bullet],d,0,f^\bullet\big),(x_i)_{i\geqslant 1},\big(\mathtt{T}_i\big)_{j\geqslant 1}\Big),$$

where as usual d stands for the usual distance on segments. This entails Theorem 6.6, since by Point (i) the right-side hand of the previous display is distributed according to $\widehat{\mathbb{Q}}_x^\gamma$. Let us now conclude by proving the claim. We focus on the case $\kappa(\gamma)<0$, as the case $\gamma=\omega_-$ follows from similar (actually, slightly simpler) arguments and we leave the extension the reader – it can also be directly deduced from the case $\kappa(\gamma)<0$ by taking the limit $\gamma\downarrow\omega_-$ and using Proposition 3.14. In order to verify (i), recall the setting of Lemma 6.7. Imagine that under the law P_1^γ, we tag the times at which the reproduction process η_γ has atoms originating from the Poisson random measure $\mathbf{N}''$. In particular the first tagged time is t'', and the ultimate one is related via the Lamperti time-change to the first time when $\mathbf{N}''$ has an atom on the fiber $[0,\infty)\times\{(-\infty,(0,-\infty,\dots))\}$. It follows that the total number of tagged times has the geometric distribution with success parameter $-\kappa(\gamma)/\left(\boldsymbol{\Lambda}_\gamma^\frown(\mathcal{S})-\kappa(\gamma)\right)=\kappa(\gamma)/\psi(\gamma)$, that is the same distribution as the number $n^\bullet+1$ of distinguished individuals under $\bar{\mathbb{P}}_1^\gamma$. This is of course not a mere coincidence, and we shall actually see that we can couple $\mathbb{P}_1^\gamma$ and $\bar{\mathbb{P}}_1^\gamma$ such that the tagged times for η_γ correspond to the times $t^\star(k)$ for $k=0,\dots,n^\bullet$ at which a new distinguished individual is born under $\bar{\mathbb{P}}_1^\gamma$.

Indeed, consider the restriction of the decoration-reproduction process under $\bar{\mathbb{P}}_1^\gamma$ to the closed time interval $[0,t^\star(0)]$,

$$\left(\mathbf{1}_{[0,t^\star(0)]}f^\bullet,\mathbf{1}_{[0,t^\star(0)]\times\mathcal{S}}\cdot\eta^\bullet\right).$$

By the construction of $(f^\bullet,\eta^\bullet)$ and the very definition of the general branching process with law $\bar{\mathbb{P}}_1^\gamma$, the former has the same law as

$$\left(\mathbf{1}_{[0,t^\star)}f+\mathbf{1}_{\{t^\star\}}x^\star,\mathbf{1}_{[0,t^\star)\times\mathcal{S}}\cdot\eta+\sum_{j\geqslant1}\delta_{(t^\star,x_j^\frown)}\right)$$

under $\bar{P}_1^\gamma$, where the sequence $(x_j^\frown)_{j\geqslant1}$ is reduced to the single term $f(t^\star)$ if $x^\star=0$, and otherwise is obtained by replacing $x^\star$ in the sequence of the types of the children born at time $t^\star$ by $f(t^\star)$.

The comparison of Proposition 6.2 with Lemma 6.8 now confirms that the law under $\bar{\mathbb{P}}_1^\gamma$ of the restriction $\left(\mathbf{1}_{[0,t^\star(0)]}f^\bullet,\mathbf{1}_{[0,t^\star(0)]\times\mathcal{S}}\cdot\eta^\bullet\right)$ is indeed the same as that of the restriction $\left(\mathbf{1}_{[0,t'']}f,\mathbf{1}_{[0,t'']\times\mathcal{S}}\cdot\eta\right)$ under P_1^γ. The verification of (i) can then be completed by an application of the strong Markov property, conditionally on $f^\bullet(t^\star(0))$ under $\bar{\mathbb{P}}_1^\gamma$, respectively conditionally on $f(t'')$ under P_1^γ, iteratively as long as these quantities remain non-zero.

It remains to check Point (ii) about the dangling subtrees. Since the non-distinguished individuals under $\bar{\mathbb{P}}_1^\gamma$ have the same evolution as under $\mathbb{P}_1$, we only need to consider the subtrees induced by distinguished individuals strictly after their distinguished age $t^\star$. Recall then from Proposition 6.2(ii) that the decoration-reproduction shifted at time t and properly rescaled, denote by B, has law P_1. Since by definition, a distinguished parent does not have any further distinguished children strictly after time $t^\star$, we conclude, by an application of the branching property and Lemma 5.7, that the subtree induced by the k-th distinguished individual strictly after time $t^\star(k)$ (i.e. strictly after the distinguished individual has reached the age $t^\star(k)$) and properly rescaled has indeed the law $\mathbb{Q}_1$, independently of all the others dangling subtrees. This completes the verification of (ii).

$\square$

6.3 Bifurcators

In this section, we use the spinal decomposition to determine all the characteristic quadruplets satisfying Assumption 3.9 which yield the same law on (unmarked, non-measured) decorated trees. To this end, we introduce the map

$$\mathrm{ord} : \mathcal{S} \to \mathcal{S}_1, \quad \mathrm{ord}\big(y_0, (y_i)_{i\geqslant 1}\big) = (y_i^{\downarrow})_{i\geqslant 1},$$

where the right-hand side denotes the sequence obtained by ranking the collection of $(y_i)_{i\geqslant 0}$ in non-increasing order. Borrowing the terminology from [123, 131], we say that two characteristic quadruplets $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ and $(\sigma^2_{\prec}, \mathrm{a}_{\prec}, \boldsymbol{\Lambda}_{\prec}; \alpha_{\prec})$ are **bifurcators** of one another and we write

$$(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha) \approx (\sigma^2_{\prec}, \mathrm{a}_{\prec}, \boldsymbol{\Lambda}_{\prec}; \alpha_{\prec})$$

if and only if

$$\sigma^2 = \sigma^2_{\prec} \quad , \quad \boldsymbol{\Lambda} \circ \mathrm{ord}^{-1} = \boldsymbol{\Lambda}_{\prec} \circ \mathrm{ord}^{-1} \quad , \quad \alpha = \alpha_{\prec}, \tag{6.17}$$

and

$$\mathrm{a} - \mathrm{a}_{\prec} = \lim_{\varepsilon\to 0+} \Big(\int_{\varepsilon<|y|\leqslant 1} \boldsymbol{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})\; y - \int_{\varepsilon<|y|\leqslant 1} \boldsymbol{\Lambda}_{\prec}(\mathrm{d}y, \mathrm{d}\mathbf{y})\; y \Big). \tag{6.18}$$

Remark 6.9. If (6.17) holds for two characteristic quadruplets that both fulfill Assumption 3.9, then we will see in the proof of Theorem 6.11 below that $\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_{\prec}$ is always a finite signed measure on $\mathcal{S}$. As a consequence, the condition (6.18) can then be re-expressed in the simpler form

$$\mathrm{a}_{\prec} = \mathrm{a} + \int_{|y|\leqslant 1} y \,(\boldsymbol{\Lambda}_{\prec} - \boldsymbol{\Lambda})\,(\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

Plainly $\approx$ is an equivalence relation; the notation is also meant to suggest that when both quadruplet fulfill Assumption 3.9, the self-similar Markov trees, say $\mathtt{T}$ with characteristic quadruplet $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha)$ and $\mathtt{T}_{\prec}$ with characteristic quadruplet $(\sigma^2_{\prec}, \mathrm{a}_{\prec}, \boldsymbol{\Lambda}_{\prec}; \alpha_{\prec})$, should be thought of as isomorphic; see Chapter 2. We immediately see from (3.19) that if $(\sigma^2, \mathrm{a}, \boldsymbol{\Lambda}; \alpha) \approx (\sigma^2_{\prec}, \mathrm{a}_{\prec}, \boldsymbol{\Lambda}_{\prec}; \alpha_{\prec})$, then the cumulant function $\kappa_{\prec}$ associated to $(\sigma^2_{\prec}, \mathrm{a}_{\prec}, \boldsymbol{\Lambda}_{\prec})$ is identical to κ. Therefore, if $\kappa(\gamma) < 0$ for some $\gamma > 0$, then $\kappa_{\prec}(\gamma) < 0$ as well, and Assumption 3.9 holds for both characteristic quadruplets.

Example 6.10. *An important example of bifurcator is obtained by taking $\boldsymbol{\Lambda}_{\prec} = \boldsymbol{\Lambda}_*$ to be the push-forward of $\boldsymbol{\Lambda}$ by the transformation on $\mathcal{S}$ that swaps y and y_1 if $y_1 > y$ and leaves $(y, \mathbf{y})$ unchanged otherwise, and then choosing a_* so that (6.18) is verified. See the proof of Theorem 6.11 below for details. The bifurcator $(\sigma^2, \mathrm{a}_*, \boldsymbol{\Lambda}_*; \alpha)$ is called the **locally largest** bifurcator since, in the genealogical interpretation, all children have a type smaller than the value of the decoration of their parent immediately after the birth event. This allows us to distinguish a canonical element in every equivalence class of bifurcators.*

We stress that it is implicitly assumed in the definition of bifurcators that both $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_{\prec}$ are generalized Lévy measures, and in particular that their images by the first projection $\mathcal{S} \to \mathbb{R}$, $(y, \mathbf{y}) \mapsto y$, are standard Lévy measures. For instance, if $\mathbf{\Lambda}^{\frown}$ denotes the measure on $\mathcal{S}$ obtained from $\mathbf{\Lambda}$ by swapping the first coordinate y and the first term y_1 of the sequence $\mathbf{y}$, no matter whether $y_1 > y$ or not, then plainly $\mathbf{\Lambda} \circ \mathrm{ord}^{-1} = \mathbf{\Lambda}^{\frown} \circ \mathrm{ord}^{-1}$. However generically $\mathbf{\Lambda}^{\frown}((-\infty, -1] \times \mathcal{S}_1) = \infty$ and therefore $\mathbf{\Lambda}^{\frown}$ is **not** a generalized Lévy measure.

In the sequel, we fix $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and $(\sigma^2_{\prec}, \mathrm{a}_{\prec}\mathbf{\Lambda}_{\prec}; \alpha_{\prec})$ two characteristic quadruplets and use the obvious notation P_x and $P^{\prec}_x$ for the laws of the decoration-reproduction process of an individual with type $x > 0$, $\mathbb{P}_x$ and $\mathbb{P}^{\prec}_x$ for the law of the family of decoration-reproduction processes indexed by the Ulam tree, and finally $\mathbb{Q}_x$ and $\mathbb{Q}^{\prec}_x$ for the laws of the self-similar Markov trees endowed with the zero measure – when Assumption 3.9 is verified.

Theorem 6.11 (Bifurcators)**.** *Assume that the two characteristic quadruplets,* $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *and* $(\sigma^2_{\prec}, \mathrm{a}_{\prec}, \mathbf{\Lambda}_{\prec}; \alpha_{\prec})$*, satisfy Assumption 3.9. Then,* $\mathbb{Q}_x = \mathbb{Q}^{\prec}_x$*, for all* $x > 0$*, if and only if* $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *and* $(\sigma^2_{\prec}, \mathrm{a}_{\prec}, \mathbf{\Lambda}_{\prec}; \alpha_{\prec})$ *are bifurcators of one another.*

Before establishing Theorem 6.11, we shall first illustrate the statement by establishing a special case, which will turn out later to be a key step of the proof. In this direction, let $\mathbf{\Lambda}'$ and $\mathbf{\Lambda}''$ two measures on $\mathcal{S}$ such that $\mathbf{\Lambda} = \mathbf{\Lambda}' + \mathbf{\Lambda}''$, and satisfying $\mathbf{\Lambda}'(\{-\infty\} \times \mathcal{S}_1) = 0$ and $\mathbf{\Lambda}''(\mathcal{S}) < \infty$. We then write $\mathbf{\Lambda}'''$ for the push-forward of $\mathbf{\Lambda}''$ by the transformation on $\mathcal{S}$

$$(y, \mathbf{y}) \mapsto (y, \mathbf{y})^{\dagger} := (-\infty, \mathrm{ord}(y, \mathbf{y})).$$

We set $\mathbf{\Lambda}^{\dagger} := \mathbf{\Lambda}' + \mathbf{\Lambda}'''$ and

$$\mathrm{a}^{\dagger} := \mathrm{a} - \int_{\mathcal{S}} y \mathbf{1}_{|y| \leqslant 1} \mathbf{\Lambda}''(\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

Clearly, $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha)$ is a characteristic quadruplet and we have $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha) \approx (\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$, and the next statement is thus a version of Theorem 6.11 in this setting. With transparent notations, we write $\mathbb{Q}^{\dagger}_x$ for the laws of the self-similar Markov trees associated with the characteristic quadruplet $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha)$, when the latter is well defined.

Lemma 6.12. *If* $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ *satisfies Assumption 3.9, then so does* $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha)$ *and the law of the self-similar Markov tree with characteristic quadruplet* $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha)$ *coincides with* $(\mathbb{Q}_x)_{x>0}$.

Proof. The strategy of the proof is similar to that of Theorem 6.6. Let us proceed. First, notice that by scaling it suffices to treat the case $x = 1$, and remark that $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha)$ directly satisfies Assumption 3.9 since $(\sigma^2, \mathrm{a}^{\dagger}, \mathbf{\Lambda}^{\dagger}; \alpha) \approx (\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. We shall use a coupling argument, and for this recall the construction of self-similar Markov decoration-reproduction processes in Section 3.2. Let B be a standard Brownian motion, and $\mathbf{N}'$ and $\mathbf{N}''$, two Poisson point measures on $[0, \infty) \times \mathcal{S}$ with respective intensities $\mathrm{d}t\mathbf{\Lambda}'(\mathrm{d}y, \mathrm{d}\mathbf{y})$ and $\mathrm{d}t\mathbf{\Lambda}''(\mathrm{d}y, \mathrm{d}\mathbf{y})$. We assume that B, $\mathbf{N}'$, and $\mathbf{N}''$ are independent. Let $\mathbf{N}'''$ denote the image of $\mathbf{N}''$ by the map $(t, (y, \mathbf{y})) \mapsto (t, (y, \mathbf{y})^{\dagger})$. Then $\mathbf{N} = \mathbf{N}' + \mathbf{N}''$ is a Poisson point measure with intensity $\mathrm{d}t\mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y})$, whereas $\mathbf{N}^{\dagger} = \mathbf{N}' + \mathbf{N}'''$ is a Poisson point measure with intensity $\mathrm{d}t\mathbf{\Lambda}^{\dagger}(\mathrm{d}y, \mathrm{d}\mathbf{y})$, and each of these point measures is independent of B. We then construct the Lévy processes ξ and $\xi^\dagger$, and the decoration-reproduction processes (X,η) and $(X^\dagger,\eta^\dagger)$ with law P and $P^\dagger$, by using the same Brownian motion B, the point measures $\mathbf{N}$ and $\mathbf{N}^\dagger$, and the drift coefficients a and $\mathrm{a}^\dagger$, respectively.

Since $0<\mathbf{\Lambda}''(\mathcal{S})<\infty$, we may consider the first time τ_1 at which $\mathbf{N}''$ has an atom. The Lévy processes ξ and $\xi^\dagger$ coincide on the time interval $[0,\tau_1)$, and so do the point processes $\mathbf{N}$ and $\mathbf{N}^\dagger$. Performing the Lamperti transformation with the exponential functional ϵ in (3.12), the restriction of the decoration processes X and $X^\dagger$ to the time interval $[0,\epsilon(\tau_1))$ coincide. Since $\mathbf{\Lambda}'(\{-\infty\}\times\mathcal{S}_1)$, the measure $\mathbf{N}'$ has no atoms on $\mathbb{R}_+\times\{-\infty\}\times\mathcal{S}_1$ and

$$\tau_1=\inf\left\{t\geqslant 0:\mathbf{N}^\dagger([0,t]\times\{-\infty\}\times\mathcal{S}_1)>0\right\}.$$

Hence $\epsilon(\tau_1)=z^\dagger$ is the lifetime of $X^\dagger$, and is smaller than the lifetime z of X. Moreover, from the very definition of the transformation $(y,\mathbf{y})\mapsto(y,\mathbf{y})^\dagger$, we can express the entire reproduction process $\eta^\dagger$ as

$$\eta^\dagger=\mathbf{1}_{[0,\epsilon(\tau_1)]\times(0,\infty)}\cdot\eta+\mathbf{1}_{X(\epsilon(\tau_1))>0}\cdot\delta_{(\epsilon(\tau_1),X(\epsilon(\tau_1)))}.$$

The interpretation in terms of the evolution of populations is that $(X^\dagger,\eta^\dagger)$ results from (X,η) by killing at time $z^\dagger$, the reproduction event for η occurring at time $z^\dagger$ being still taken into account in $\eta^\dagger$, and adding an extra child with type $X(z^\dagger)$ at that time (which can be of type 0 if X is also killed at $z^\dagger$). Let us tag this extra child to distinguish it from the progeny of the †-parent that stems from η. It should now be intuitively clear from the Markov property for (X,η) (see Lemma 5.7) that such a killing combined with the addition of an extra child at the killing time having precisely the type given by the decoration of the parent when it is killed is essentially a neutral operation for the population model (even though it clearly impacts the genealogical structure). More precisely, when $X(z^\dagger)>0$, we write τ_2 for the second time at which $\mathbf{N}''$ has an atom. We can the use for the decoration-reproduction process of the tagged child the restriction of (X,η) to the time-interval $(\tau_1,\tau_2]$ and shifted by $-\tau_1$ backward in time. The concatenation of the decoration-reproduction processes of the †-parent and its tagged child is then given by the restriction of (X,η) to the time-interval $[0,\tau_2]$. Concatenating iteratively along the tagged lineage, we then recover the entire process (X,η), and hence in this respect, the claimed identity $\mathbb{Q}_1=\mathbb{Q}_1^\dagger$ thus should not come as a surprise – here we use the transparent notation $\mathbb{Q}_1^\dagger$ for the law of the self-similar Markov tree with initial decoration 1 associated with $(\sigma^2,\mathrm{a}^\dagger,\mathbf{\Lambda}^\dagger;\alpha)$.

More formally, recalling the gluing construction of self-similar Markov trees, the analysis above enables us to construct recursively two decorated trees $\mathtt{T}=(T,d_T,\rho,g)$ and $\mathtt{T}^\dagger=(T^\dagger,d_{T^\dagger},\rho^\dagger,g^\dagger)$, such that the distributions induced on $\mathbb{T}$ are respectively $\mathbb{Q}_1$ and $\mathbb{Q}_1^\dagger$, and $T^\dagger$ is a subtree of T, $\rho^\dagger=\rho$, $d_{T^\dagger}$ corresponds to the restriction distance of d_T to $T^\dagger$, and $g^\dagger(v)\leqslant g(v)$, for every $v\in T^\dagger$. To conclude the proof, we need to establish that $T=T^\dagger$ and $g=g^\dagger$. To this end, fix $\gamma>0$ such that $\kappa(\gamma)<0$. Recalling that the functions used in the gluing construction of Section 2.2 are the usc-modification of rcll functions that can vanish only at the end of their lifetimes, we infer that to obtain the desired result, it suffices to show that:

$$\int_T g(\upsilon)^{\gamma-\alpha}\, \lambda_T(\mathrm{d}\upsilon) = \int_{T^\dagger} \big(g(\upsilon)^\dagger\big)^{\gamma-\alpha}\, \lambda_{T^\dagger}(\mathrm{d}\upsilon).$$

We already know that the left-hand term is greater than or equal to the right-hand side. Moreover, by Proposition 3.11 and since $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and $(\sigma^2, \mathrm{a}^\dagger, \mathbf{\Lambda}^\dagger; \alpha)$ have the same cumulant function, both quantities have the same expectation and therefore they must be equal. This completes the proof of the lemma. $\square$

We now prove Theorem 6.11.

Proof of Theorem 6.11. We first establish the sufficiency part. Consider a characteristic quadruplet $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ that fulfills Assumption 3.9 and fix any $\gamma > 0$ such that $\kappa(\gamma) < 0$. As we suggested in Example 6.10, it is natural to consider the locally largest bifurcator. In order to introduce the latter rigorously, we consider the partition $\mathcal{S} = \mathcal{S}_< \sqcup \mathcal{S}_\geqslant$, where $\mathcal{S}_< := \{(y, \mathbf{y}) \in \mathcal{S} : y < y_1\}$ and $\mathcal{S}_\geqslant := \mathcal{S} \backslash \mathcal{S}_<$, and observe, from the fact that $\Lambda_0(\mathrm{d}y) = \mathbf{\Lambda}(\mathrm{d}y, \mathcal{S}_1)$ is a Lévy measure and the finiteness of $\kappa(\gamma)$, that

$$\begin{aligned}
\mathbf{\Lambda}(\mathcal{S}_<) &\leqslant \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \mathbf{1}_{y \leqslant -1} + \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \mathbf{1}_{y_1 \geqslant -1} \\
&\leqslant \Lambda_0([-\infty, -1]) + \mathrm{e}^\gamma \int_{\mathcal{S}} \mathbf{\Lambda}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \sum_{i \geqslant 1} \mathrm{e}^{\gamma y_i} < \infty.
\end{aligned}$$

Write $\mathbf{\Lambda}_*$ for the push-forward of the generalized Lévy measure $\mathbf{\Lambda}$ by the transformation $(y, \mathbf{y}) \mapsto \mathrm{ord}(y, \mathbf{y})$, so $\mathbf{\Lambda}_* - \mathbf{\Lambda}$ is a finite signed measure on $\mathcal{S}$. We then set

$$\mathrm{a}_* := \mathrm{a} + \int_{|y| \leqslant 1} y\, (\mathbf{\Lambda}_* - \mathbf{\Lambda})\, (\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

By construction, $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and $(\sigma^2, \mathrm{a}_*, \mathbf{\Lambda}_*; \alpha)$ are bifurcators one another. Now remark that $\mathbf{\Lambda}_* - \mathbf{1}_{\mathcal{S}_\geqslant} \mathbf{\Lambda}$ is a finite measure, and let us write $\mathbf{\Lambda}_*'''$ for the push-forward measure of $\mathbf{\Lambda}_* - \mathbf{1}_{\mathcal{S}_\geqslant} \mathbf{\Lambda}$ by $(y, \mathbf{y}) \mapsto (y, \mathbf{y})^\dagger$. We set $\mathbf{\Lambda}_*^\dagger = \mathbf{1}_{\mathcal{S}_\geqslant} \mathbf{\Lambda} + \mathbf{\Lambda}_*'''$ and

$$\mathrm{a}_*^\dagger := \mathrm{a}_* + \int_{|y| \leqslant 1} y\, \big(\mathbf{\Lambda}_*^\dagger - \mathbf{\Lambda}_*\big)\, (\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

We define similarly $\mathbf{\Lambda}^\dagger$ and $\mathrm{a}^\dagger$ replacing $\mathbf{\Lambda}$ by $\mathbf{\Lambda}^\dagger$, $\mathbf{\Lambda}_* - \mathbf{1}_{\mathcal{S}_\geqslant} \mathbf{\Lambda}$ by $\mathbf{1}_{\mathcal{S}_<} \mathbf{\Lambda} = \mathbf{\Lambda} - \mathbf{1}_{\mathcal{S}_\geqslant} \mathbf{\Lambda}$ and a_* by a in the construction above. By definiton, we have the identity

$$(\sigma^2, \mathrm{a}^\dagger, \mathbf{\Lambda}^\dagger; \alpha) = (\sigma^2, \mathrm{a}_*^\dagger, \mathbf{\Lambda}_*^\dagger; \alpha),$$

and we deduce from Lemma 6.12 that

$$(\mathbb{Q}_x)_{x>0} = (\mathbb{Q}_x^*)_{x>0},$$

where as usual $\mathbb{Q}_x^*$ stands for the law of the self-similar Markov tree with initial decoration x associated with $(\sigma^2, \mathrm{a}_*, \mathbf{\Lambda}_*; \alpha)$. Finally consider another characteristic quadruplet $(\sigma^2, \mathrm{a}_\prec, \mathbf{\Lambda}_\prec; \alpha) \approx$

$(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$. Then it is readily checked that the locally largest bifurcator for $(\sigma^2, \mathrm{a}_\prec, \mathbf{\Lambda}_\prec; \alpha)$ is again $(\sigma^2, \mathrm{a}_*, \mathbf{\Lambda}_*; \alpha)$, from which we conclude that $(\mathbb{Q}_x)_{x>0} = (\mathbb{Q}_x^\prec)_{x>0}$.

We turn our attention to the necessary condition. Suppose that the laws $\mathbb{Q}_x$ and $\mathbb{Q}_x^\prec$ coincide for all $x > 0$. Comparing for instance the distribution of the height of T under $\mathbb{Q}_x$, $\mathbb{Q}_x^\prec$, $\mathbb{Q}_1$ and $\mathbb{Q}_1^\prec$, we immediately see that the exponents of self-similarity α and $\alpha_\prec$ must be the same. We can now focus on the case $x = 1$ and then drop the index x from the notation, i.e. we write as often $\mathbb{Q} = \mathbb{Q}_1$ and $\mathbb{Q}_\prec = \mathbb{Q}_1^\prec$.

Next, consider the random variable $\int_T g(v)^{\gamma-\alpha} \lambda_T(\mathrm{d}v)$, where we recall that λ_T stands for the length (or Lebesgue) measure on T, and we write κ and $\kappa_\prec$ for the cumulant function associated respectively with $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ and $(\sigma^2, \mathrm{a}_\prec, \mathbf{\Lambda}_\prec; \alpha)$. According to Proposition 3.11, the variable $\int_T g(v)^{\gamma-\alpha} \lambda_T(\mathrm{d}v)$ has expectation $-1/\kappa(\gamma) < \infty$ under $\mathbb{Q}$ and $-1/\kappa_\prec(\gamma) < \infty$ under $\mathbb{Q}_\prec$. This forces $\kappa_\prec(\gamma) = \kappa(\gamma) < 0$. So we can equip the decorated real tree $\mathtt{T}$ with the (finite) weighted length measure $\nu = \lambda^\gamma$ under $\mathbb{Q}$ as well as under $\mathbb{Q}_\prec$. The identity $\mathbb{Q} = \mathbb{Q}_\prec$ then extends to the framework of decorated real trees with a single marked point of Section 6.2. More precisely, we define the law $\widetilde{\mathbb{Q}}^\gamma$ (respectively, $\widetilde{\mathbb{Q}}_\prec^\gamma$) on $\mathbb{T}^\bullet$ as in (6.9), that is by first biasing $\mathbb{Q}$ (respectively, $\mathbb{Q}_\prec$) with the variable $\kappa(\gamma)\nu(T)$ and then picking a point $\rho^\bullet$ in T at random according to the normalized probability measure $\nu(\mathrm{d}v)/\nu(T)$. Obviously, we have again $\widetilde{\mathbb{Q}}^\gamma = \widetilde{\mathbb{Q}}_\prec^\gamma$.

Recall from the preceding section that in this setting, $f^\bullet$ denotes the decoration on the marked segment $[\![\rho, \rho^\bullet]\!]$ and $\eta^\bullet$ the point process on $[\![\rho, \rho^\bullet]\!] \times (0, \infty)$ that records the germs of the decorations of the subtrees dandling from $[\![\rho, \rho^\bullet]\!]$, see the proof of Theorem 6.6 for a formal definition. We know from the spinal decomposition, i.e. Theorem 6.6, that the law of $(f^\bullet, \eta^\bullet)$ under $\widetilde{\mathbb{Q}}^\gamma$ is P^γ, that is that of the self-similar Markov decoration-reproduction process with tilted characteristic quadruplet $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$, where the drift a_γ is given by (6.14) and the generalized Lévy measure $\mathbf{\Lambda}_\gamma$ by (6.12). Using an obvious notation under $\widetilde{\mathbb{Q}}_\prec^\gamma$, we arrive at the identity $P^\gamma = P_\prec^\gamma$ where the right-hand side is the law of the self-similar Markov decoration-reproduction process with tilted characteristic quadruplet $(\sigma_\prec^2, \mathrm{a}_\gamma^\prec, \mathbf{\Lambda}_\gamma^\prec; \alpha)$.

Just in the same way as the distribution of a Lévy process determines its characteristic triplet, one readily sees by undoing the Lamperti transformation that the law of a self-similar Markov decoration-reproduction process with a given exponent of self-similarity α entirely determines its characteristic quadruplet. We infer from above that there is the identity

$$(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma) = (\sigma_\prec^2, \mathrm{a}_\gamma^\prec, \mathbf{\Lambda}_\gamma^\prec).$$

To conclude the proof of the necessary part, it now suffices to observe from (6.12) that the push-forward of the generalized Lévy measure $\mathbf{\Lambda}$ by the function $\mathrm{ord} : \mathcal{S} \to \mathcal{S}$ that ranks all the terms of $(y, \mathbf{y})$ in the non-increasing order can be expressed in terms of $\mathbf{\Lambda}_\gamma$ and $\kappa(\gamma)$ via the identity

$$\left(\mathrm{e}^{\gamma y} + \sum_{i \geqslant 1} \mathrm{e}^{\gamma y_i} \right) \cdot \left(\mathbf{\Lambda} \circ \mathrm{ord}^{-1} \right) (\mathrm{d}y, \mathrm{d}\mathbf{y}) = \left(\left(\mathbf{\Lambda}_\gamma + \kappa(\gamma) \delta_{\{-\infty\} \times \{0, -\infty, \cdots\}} \right) \circ \mathrm{ord}^{-1} \right) (\mathrm{d}y, \mathrm{d}\mathbf{y}).$$

Recalling that $\kappa(\gamma) = \kappa_{\prec}(\gamma)$, we now see that $\mathbf{\Lambda} \circ \mathrm{ord}^{-1} = \mathbf{\Lambda}_{\prec} \circ \mathrm{ord}^{-1}$. Finally, we check (6.18) using again $\kappa(\gamma) = \kappa_{\prec}(\gamma)$, (3.19), and the Lévy-Khintchine formula (3.11). ☐

We saw in the above proof that the tilted characteristics $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ can be recovered from the laws $(\mathbb{Q}_x)_{x>0}$. Those tilted characteristics are in fact more intrinsic that the initial ones since they generically uniquely characterize the law of the ssMt. We say that a generalized Lévy measure $\mathbf{\Lambda}$ is **asymmetric** if there exists a point $\{y_0, (y_1, y_2, ...)\} \in \mathcal{S}$ in the support of $\mathbf{\Lambda}$ so that $y_0 > y_1 > -\infty$. Then we have

Corollary 6.13. *Suppose that $\mathbf{\Lambda}$ is asymmetric. The law $(\mathbb{Q}_x)_{x>0}$ is uniquely characterized by the data $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$.*

Remark that in the binary conservative case, the tilted characteristics are completely described by the sole Lévy–Khintchine exponent ψ_γ of the pssMp decoration X_γ along the tagged branch since the decoration-reproduction process η_γ is given in terms of X_γ by (4.14). See the end of the next section for applications.

Proof. We saw in the previous proof that $(\mathbb{Q}_x)_{x>0}$ is characterized by the data of $(\kappa(\gamma), \gamma)$ together with the tilted characteristic quadruplet $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$. Using (6.13), we deduce that the function κ can itself be recovered from $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ together with the parameter γ. To complete the proof it suffices to show that $\gamma > 0$ can be recovered from the generalized Lévy measure $\mathbf{\Lambda}_\gamma$ only. To prove this, let us denote by $\mathbf{\Lambda}_*$ the generalized Lévy measure of the locally largest bifurcator so that we have

$$\mathbf{\Lambda}_\gamma(\mathrm{d}y, \mathrm{d}\mathbf{y}) = \mathrm{e}^{\gamma y} \mathbf{\Lambda}_*(\mathrm{d}y, \mathrm{d}\mathbf{y}) + \mathrm{e}^{\gamma y} \cdot \left(\sum_{i \geqslant 1} \mathbf{\Lambda}_*^{\smile i}(\mathrm{d}y, \mathrm{d}\mathbf{y}) \right).$$

Since $\mathbf{\Lambda}_*$ is asymmetric (this is equivalent to asking that a bifurcator $\mathbf{\Lambda}$ or even $\mathbf{\Lambda}_\gamma$ is asymmetric), then there exists a point $(y_0, y_1, y_2, ...) \in \mathcal{S}_1$ with $y_0 > y_1 = y_2 = \cdots = y_k > y_{k+1} \geqslant .... \geqslant -\infty$ in the support of $\mathbf{\Lambda}_*$. Let us now consider the measure $\mathbf{\Lambda}_\gamma$ restricted to the vicinity of the points $(y_0, (y_1, y_2, ...))$ and $(y_1, (y_0, y_2, ...))$ in $\mathcal{S}$. Denote them respectively by $\mathbf{\Lambda}_\gamma^0$ and $\mathbf{\Lambda}_\gamma^1$. Using the previous display, we deduce that $\mathbf{\Lambda}_\gamma^1 \circ \mathrm{Ord}^{-1}$ and $\mathbf{\Lambda}_\gamma^0$ are absolutely continuous with respect to each-other in the vicitinity of $(y_0, (y_1, y_2, ...))$ with Radon–Nikodym derivative equal to

$$\frac{\mathrm{d}\big(\mathbf{\Lambda}_\gamma^1 \circ \mathrm{Ord}^{-1}\big)}{\mathrm{d}\mathbf{\Lambda}_\gamma^0}(y_0, (y_1, y_2, ...)) = \frac{c \cdot \mathrm{e}^{\gamma y_1}}{\mathrm{e}^{\gamma y_0}},$$

where $c > 0$ is an explicit constant. Since $y_1 \neq y_0$ are known, this formula thus enables us to recover γ from the knowledge of $\mathbf{\Lambda}_\gamma$. ☐

In the symmetric case, the previous corollary may not hold. Consider for example the two finite generalized Lévy measures

$$\begin{aligned} \mathbf{\Lambda} &= \delta_{(-\log 3, (-\log 3, -\log 3, -\infty, ...))} + 3 \cdot \delta_{(\log 3, (-\infty, -\infty, ...))} + 7 \cdot \delta_{(-\infty, (-\infty, -\infty, ...))}, \\ \tilde{\mathbf{\Lambda}} &= 3 \cdot \delta_{(-\log 3, (-\log 3, -\log 3, -\infty, ...))} + \delta_{(\log 3, (-\infty, -\infty, ...))} + 7 \cdot \delta_{(-\infty, (-\infty, -\infty, ...))}. \end{aligned}$$

Then a straightforward calculation shows that the ($\gamma = 1$)-tilted characteristics of $(0, 0, \mathbf{\Lambda}; \alpha)$ coincide with those of the ($\tilde{\gamma} = 2$)-tilted characteristics of $(0, 0, \tilde{\mathbf{\Lambda}}; \alpha)$.

6.4 Hausdorff dimensions

In this section, we use the spinal decomposition of Theorem 6.6 to determine the Hausdorff dimension of some random sets that appear naturally for self-similar Markov trees satisfying the first Cramer's condition, completing Lemma 3.6 in this context.

Proposition 6.14. *Fix $(\sigma^2, \mathrm{a}, \mathbf{\Lambda}; \alpha)$ satisfying Assumption 3.12 for some $\omega_- > 0$. Then $\mathbb{P}_1$-a.s., the Hausdorff dimensions of $\partial_0 T$, T and $\mathrm{Hyp}(g)$ are ω_-/α, $1 \vee (\omega_-/\alpha)$ and $2 \vee (\omega_-/\alpha)$ respectively.*

Proof. Thanks to Lemma 3.6, we only need to establish that $\dim_H(\partial_0 T) \geqslant \omega_-/\alpha$, $\mathbb{P}_1$-a.s. In this direction, we claim that it suffices to establish that

$$\limsup_{r \to 0} \frac{\upmu\big(B_r(\rho^\bullet)\big)}{r^{\frac{\omega_-}{\alpha} - \delta}} = 0, \quad \widehat{\mathbb{Q}}_1^{\omega_-}\text{-a.s.}, \tag{6.19}$$

where we write $\rho^\bullet$ for the point corresponding in $\mathtt{T}$ to the extremity $z_\varnothing$ of the ancestral individual $(f_\varnothing, \eta_\varnothing)$, and $B_r(\rho^\bullet)$ for the closed ball of radius r centered at $\rho^\bullet$. Indeed, recall from Proposition 2.11 that the harmonic measure is supported on $\partial_0 T$, $\mathbb{P}_1$- a.s. Since by (6.9) and Theorem 6.6, under $\widehat{\mathbb{Q}}_1^{\omega_-}$, the marked point $\rho^\bullet$ is distributed according to $\upmu(\mathrm{d}v)/\upmu(T)$, it follows from standard density theorems for Hausdorff measures that $\dim_H(\partial_0 T) \geqslant \omega_-/\alpha$, $\mathbb{P}_1$-a.s.

To prove the claim (6.19) recall that, under $\widehat{\mathbb{Q}}_1^{\omega_-}$, the decoration-reproduction process $(f_\varnothing, \eta_\varnothing)$ of the spine is distributed according to the biased decoration-reproduction kernels $(P_x^{\omega_-})_{x>0}$. Recall also from Section 5.3 the notation $\check{B}_a^\bullet(T)$ for the closure of the complement of the hull of radius a when $d_T(\rho, \rho^\bullet) > a$. By definition, $\check{B}_a^\bullet(T)$ contains the open ball $B_r(\rho^\bullet)$ for the radius $r = d_T(\rho, \rho^\bullet) - a$. Finally, fix $\delta > 0$ and for every $\varepsilon > 0$, set $\vartheta_\varepsilon := \inf\{t \geqslant 0 : f_\varnothing(t) < \varepsilon\}$. We shall show that for every $\delta_1, \delta_2 > 0$ such that

$$\omega_- - \delta_1 > (\alpha + \delta_2)(\omega_-/\alpha - \delta), \tag{6.20}$$

we have $\widehat{\mathbb{Q}}_1^{\omega_-}$-a.s. that for every $k \in \mathbb{N}$ large enough,

$$\upmu\big(\check{B}^\bullet_{\vartheta_{2^{-k}}}(T)\big) \leqslant 2^{-k(\omega_- - \delta_1)} \quad \text{and} \quad 2^{-k(\alpha+\delta_2)} \leqslant d_T(\rho, \rho^\bullet) - \vartheta_{2^{-k}}. \tag{6.21}$$

The desired result (6.19) then follows, since we deduce from the inclusions

$$B_{2^{-k(\alpha+\delta_2)}}(\rho^\bullet) \subset B_{d_T(\rho,\rho^\bullet) - \vartheta_{2^{-k}}}(\rho^\bullet) \subset \check{B}^\bullet_{\vartheta_{2^{-k}}}(T)$$

that

$$\upmu\big(B_{2^{-k(\alpha+\delta_2)}}(\rho^\bullet)\big) \leqslant 2^{-k(\omega_- - \delta_1)}, \quad \text{for } k \text{ sufficiently large,}$$

and we conclude from (6.20).

Let us proceed with the proof of (6.21). First remark that by the Markov property of $(f_\varnothing, \eta_\varnothing)$ of Lemma 5.7 and the branching property, conditionally on $f_\varnothing(\vartheta_\varepsilon)$, the variable $\upmu(\check{B}^\bullet_{\vartheta_{2^{-k}}}(T))$ is distributed as $\upmu(T)$ under $\widehat{\mathbb{Q}}^{\omega_-}_{f_\varnothing(\vartheta_\varepsilon)}$. Therefore, by the scaling property combined with the fact that $f_\varnothing(\vartheta_\varepsilon) \leqslant \varepsilon$, we get

$$\widehat{\mathbb{Q}}^{\omega_-}_1\Big(\upmu(\check{B}^\bullet_{\vartheta_{2^{-k}}}(T)) \geqslant \varepsilon^{\omega_- - \delta_1}\Big) \leqslant \widehat{\mathbb{Q}}^{\omega_-}_1\Big(\upmu(T) \geqslant \varepsilon^{-\delta_1}\Big).$$

Since $\mathbb{E}_1(\upmu(T)^p) < \infty$ for some $p > 1$ appearing in Assumption 3.12, the variable $\upmu(T)^{p-1}$ has a finite mean under $\widehat{\mathbb{Q}}^{\omega_-}_1$. By the Markov inequality, the right-hand side of the last display is $O(\varepsilon^{\delta_1(p-1)})$. The first inequality in (6.21) follows by taking $\varepsilon = 2^{-k}$ and applying the Borel–Cantelli lemma. On the other hand, recall that under $\widehat{\mathbb{Q}}^{\omega_-}_1$, the decoration process $f_\varnothing = X_{\omega_-}$ along the spine is obtained by performing the Lamperti transformation to a Lévy process ξ_{ω_-}, and from Lemma 6.4 that the latter has Laplace exponent $q \mapsto \kappa(\omega_- + q)$. From basic results on Lévy processes, we have $\xi_{\omega_-}(t) \sim t \cdot \widehat{\mathbb{E}}^{\omega_-}_1(\xi_{\omega_-}(1))$ almost surely as $t \to \infty$, we infer from the Lamperti transformation that a.s. we eventually have

$$2^{-k(\alpha+\delta_2)} \leqslant d_T(\rho, \rho^\bullet) - \vartheta_{2^{-k}}.$$

This completes the verification of (6.21) and hence the proof. □

6.5 Back to Examples

We now revisit the Examples of Chapter 4 and explicit their spinal decomposition. We also recall some background about stable Lévy processes conditioned to die continuously at zero, since they appear in the ω_--spinal decomposition of many natural examples. Interestingly, we will see that the only spectrally negative β-stable Lévy processes conditioned to die continuously at zero that may appear in binary conservative ssMt are present in Examples 4.6 and 4.10, and satisfy

$$\beta \in (0, 1/2] \cup (1, 3/2].$$

Recall from (6.12) and (6.14) the definition of the tilted characteristics $(\sigma, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ in the spinal decomposition (Theorem 6.6) for $\kappa(\gamma) < 0$ or $\gamma = \omega_-$ together with Assumption 3.12. Recall also from (6.13) that the Lévy–Khintchine exponent ψ_γ of the Lévy process underlying the pssMp X_γ of the decoration along the distinguished tagged branch under $\widetilde{\mathbb{Q}}^\gamma$ is given by

$$\psi_\gamma(q) = \kappa(\gamma + q).$$

Similarly to what we did in the opening of Chapter 4, in the case when the image $\Lambda_{\gamma,0}$ of $\mathbf{\Lambda}_\gamma$ by the application $(y, \mathbf{y}) \mapsto y$ integrates $1 \wedge |y|$ we define the canonical drift coefficient

$$\mathrm{a}^{\mathrm{can}}_\gamma := \mathrm{a}_\gamma - \int \Lambda_{\gamma,0}(\mathrm{d}y)\; y \mathbf{1}_{|y| \leqslant 1}.$$

It is easy to see that when $\kappa(\gamma) \leqslant 0$ then $\Lambda_{\gamma,0}$ integrates $1 \wedge |y|$ if and only if Λ_0 does and in this case we have

$$\mathrm{a}_\gamma^{\mathrm{can}} = \mathrm{a}_{\mathrm{can}} + \sigma^2 \gamma. \tag{6.22}$$

It is rather straightforward to compute the tilted characteristics $(\sigma^2, \mathrm{a}_\gamma, \mathbf{\Lambda}_\gamma; \alpha)$ in the finite branching activity case, see Examples 4.1, 4.2 and 4.3. We shall do so only in the case $\gamma = \omega_-$ to lighten the prose and because there is no killing involved in this case.

- In Example 4.1 we have $\omega_- = 1$ and Assumption 3.12 holds. After performing the spinal decomposition with $\gamma = \omega_-$ we still have $\mathrm{a}_\gamma^{\mathrm{can}} = 0$, $\sigma^2 = 0$ and the tilted generalized Lévy measure is merely equal to the initial one, more precisely $\mathbf{\Lambda}_{\mathrm{half},\omega_-} = \mathbf{\Lambda}_{\mathrm{half}} \circ \mathrm{Ord}^{-1}$. The tagged branch thus evolves as a standard branch in this model.

- In Example 4.2 we have $\omega_- = 1$ and Assumption 3.12 holds. After performing the spinal decomposition with $\gamma = \omega_-$ we still have $\mathrm{a}_\gamma^{\mathrm{can}} = -1$, $\sigma^2 = 0$ and the tilted generalized Lévy measure is twice the original one, more precisely $\mathbf{\Lambda}_{\mathrm{two},\omega_-} = 2 \times \mathbf{\Lambda}_{\mathrm{two}} \circ \mathrm{Ord}^{-1}$. Along the tagged branch, the intensity of splittings is twice that of a standard branch. Undoing the Lamperti transformation we recover the famous spine decomposition of Yule trees, see e.g. [53, Proposition 5].

- In Example 4.3 we have $\omega_- = -\mathrm{a}_{\mathrm{can}} - \sqrt{{\mathrm{a}_{\mathrm{can}}}^2 - 2}$ and Assumption 3.12 holds as long as $\mathrm{a}_{\mathrm{can}} < -\sqrt{2}$. After performing the spinal decomposition with $\gamma = \omega_-$ we still have $\sigma^2 = 1$, the tilted generalized Lévy measure is again $\mathbf{\Lambda}_{\mathrm{two},\omega_-} = 2 \times \mathbf{\Lambda}_{\mathrm{two}} \circ \mathrm{Ord}^{-1}$ and the canonical drift gets changed to $\mathrm{a}_\gamma^{\mathrm{can}} = -\sqrt{{\mathrm{a}_{\mathrm{can}}}^2 - 2}$ using (6.22). In particular, the intensity of splittings along the tagged branch is multiplied by two as in the previous case, and the decoration X_{ω_-} evolves as a Bessel process with dimension $2\mathrm{a}_\gamma^{\mathrm{can}} + 2$.

Let us now move to examples with an infinite branching activity, starting with Example 4.5. We have $\omega_- = 2$ and Assumption 3.12 holds. After performing the spinal decomposition with $\gamma = \omega_-$, we still have $\mathrm{a}_\gamma^{\mathrm{can}} = -1$, $\sigma^2 = 0$ and the tilted generalized Lévy measure is the sum of the original one $\mathbf{\Lambda}$ and a finite measure $\mathbf{\Lambda}'$ given by

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \mathrm{e}^{y_2}, \ldots)\big) \mathbf{\Lambda}'(\mathrm{d}\mathbf{y}) = 2 \int_0^1 \mathrm{d}u \; F(u, (1, 0, 0, \ldots)).$$

In probabilistic terms, this means that the decoration X_{ω_-} along the tagged branch is a pssMp which evolves as follows: starting from a value x, it decreases until $x \cdot (1 - \sqrt{U})$ as a standard branch, and then jumps to $x \cdot (1 - \sqrt{U}) \cdot V$ where U, V are independent and uniform on $[0, 1]$. Afterwards, the evolution iterates the same dynamic.

The next, and perhaps one of the most important, example is given by the Brownian CRT with mass 1, see Example 4.6. It is an example of conservative fragmentation so we have $\omega_- = 1$

and Assumption 3.12 holds. The spinal decomposition with $\gamma = \omega_-$ has characteristics $\sigma^2 = 0$, canonical drift $\mathrm{a}_{\mathrm{can}} = 0$ and the tilted generalized Lévy measure is

$$\int_{\mathcal{S}} F\big(\mathrm{e}^{y_0}, (\mathrm{e}^{y_1}, \dots)\big)\ \mathbf{\Lambda}_{\mathrm{Bro},\omega_-}(\mathrm{d}y_0, \mathrm{d}\mathbf{y}) = \sqrt{\frac{2}{\pi}} \int_0^1 F(x, 1-x, 0, 0, \dots) \cdot x \cdot \frac{\mathrm{d}x}{(x(1-x))^{3/2}}.$$

In particular, the Lévy–Khintchine exponent of the underlying Lévy process of the pssMp decoration X_{ω_-} along the tagged branch is

$$\psi_{\omega_-}(z) = \kappa_{\mathrm{Bro}}(z + \omega_-) = -2\sqrt{2} \cdot \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z)}.$$

We deduce from [135, Proposition 1] (see [89, Chapter 5] for more general results that we shall use below) that the decoration along the tagged branch X_{ω_-} has the law of the opposite of a stable subordinator of index 1/2 starting from 1, conditioned to visit 0, and finally killed when hitting 0. See below for details. Notice that thanks to the conservative and binary properties, the decoration-reproduction η_{ω_-} is recovered from X_{ω_-} using (4.14). Interestingly, the reproduction process bears a close relation to the Poisson–Dirichlet distribution with parameters $(1/2, 1/2)$; see [135, Theorem 1]. A similar phenomenon occurs in the spinal decomposition of the stable trees of Example 4.7. Again, since the fragmentation is conservative we have $\omega_- = 1$ and Assumption 3.12 holds. As above, the decoration along the tagged branch has the same law as the opposite of a stable subordinator of index $1 - 1/\beta$ starting from 1, conditioned to visit 0, and finally killed when hitting 0; see [135, Proposition 1] and also [113, Proposition 1]. See also below for details. Since this case is not binary, the decoration-reproduction process is more involved, as above it bears close relation with the Poisson–Dirichlet distribution with parameter $(1 - \frac{1}{\beta}, 1 - \frac{1}{\beta})$, see [113, 135] for details.

Finally, let us consider the family of Examples 4.10. Recall that for $\mathtt{a} \in (0, 1]$ and $\mathtt{b} \in (0, 1/2]$ the characteristics $(\sigma^2 = 0, \mathrm{a}_{\mathtt{a},\mathtt{b}}, \mathbf{\Lambda}_{\mathtt{a},\mathtt{b}}; \alpha)$ yield a ssMt for which $\omega_- = \mathtt{a} + 2\mathtt{b}$ and $\omega_+ = \mathtt{a} + 2\mathtt{b} + 1$ and for which Assumption 3.12 always holds. We introduce the notation

$$\beta = \mathtt{a} + \mathtt{b}, \qquad \text{and} \qquad \varrho = \frac{\mathtt{a}}{\mathtt{a} + \mathtt{b}}.$$

Then using (4.16) and Lemma 6.4 we see that the Lévy–Khintchine exponent of the pssMp decoration along the ω_--tagged branch is given by

$$\psi_{\omega_-}(z) = -\Gamma(1 + \mathtt{a} - z)\Gamma(\mathtt{b} + z)\frac{\sin(\pi z)}{\pi}.$$

Using [89, Theorem 5.15] we deduce that in the ω_-- spine decomposition of those ssMt, the decoration X_{ω_-} along the tagged branch evolves as a stable Lévy process with parameters (β, ϱ) conditioned to die continuously at 0. Since the generalized Lévy measure is conservative (4.9) and binary (4.12), the decoration-reproduction process is recovered from X_{ω_-} using (4.14). In the case of the overlay Example 4.11, the decoration X_{ω_-} in the ω_--spinal decomposition is a so-called ricocheted stable process, introduced by Budd [41] and recently studied in [90, 136].

The critical case Example 4.12 is left aside since Assumption 3.12 does not hold, see Section 3.4 for a discussion.

We saw in Examples 4.6, 4.7 and 4.10 the appearance of conditioned stable processes as the pssMp decoration along the tagged branch of a ssMt. We will actually show that not all such processes can appear in spine decomposition of binary conservative ssMt. Let us first give some background on stable Lévy processes, their conditioned versions and their relations with hypergeometric Lévy processes to unify the results. Recall that a Lévy process $(\xi_t)_{t\geqslant 0}$ is **stable** with index $\alpha \in (0,2]$ if it satisfies the scaling relation $c^{-1}\cdot(\xi(c^{\alpha}t))_{t\geqslant 0} = (\xi(t))_{t\geqslant 0}$ in law. Up to dilation, they can be classified by their index index of similarly $\alpha \in (0,2]$ together with the positivity parameter $\varrho \geqslant 0$ given by $\varrho = \mathbb{P}(\xi(t) \geqslant 0)$. Specifically, if

$$(\alpha,\varrho) \in \left\{\alpha \in (0,1), \varrho \in [0,1]\right\} \cup \left\{\alpha = 1, \varrho = 1/2\right\} \cup \left\{\alpha \in (1,2), \varrho \in [1-\frac{1}{\alpha},\frac{1}{\alpha}]\right\},$$

their Lévy measure is given by

$$\Pi(\mathrm{d}x) = \frac{\mathrm{d}x}{|x|^{\alpha+1}}\left(\Gamma(1+\alpha)\frac{\sin(\pi\alpha\varrho)}{\pi}\mathbf{1}_{x>0} + \Gamma(1+\alpha)\frac{\sin(\pi\alpha(1-\varrho))}{\pi}\mathbf{1}_{x<0}\right).$$

The case $\alpha = 2$ is the case of Brownian motion (no jumps). We refer to [15, Chapter VIII] or [89, Chapter 4] for details. A stable Lévy process naturally gives rise to a pssMp without performing the Lamperti transformation: if ξ is an α-stable Lévy process starting from 1, then the **censored** process $\xi^{\dagger}$ defined by

$$\xi^{\dagger}(t) := \xi(t)\mathbf{1}_{t\leqslant T_{\mathbb{R}_-}}, \qquad \text{with } T_{\mathbb{R}_-} := \inf\left\{t \geqslant 0 : \xi(t) < 0\right\},$$

is a pssMp. There are other ways to build pssMp from ξ using h-transformations, see [43]. More precisely, recall that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive harmonic function for $\xi^{\dagger}$ if for each $x > 0$ we have

$$h(x) = \mathbb{E}_x\left(h(\xi^{\dagger}(t))\mathbf{1}_{t<T_{\mathbb{R}_-}}\right).$$

Given such a harmonic function, it is possible to define a new process ξ^h by the formula

$$\mathbb{P}_x(\xi^h \in A) = \frac{1}{h(x)}\mathbb{P}_x\left(\xi^{\dagger} \in A \cdot h(\xi^{\dagger}(t))\mathbf{1}_{t<T_{\mathbb{R}_-}}\right),$$

where A is measurable with respect to $\mathcal{F}_t$ (a priori it is unclear whether the obtained process is conservative or not, but it will be the case in what follows). It turns out that for stable Lévy processes, any positive harmonic function for $\xi^{\dagger}$ is a linear combination of the two functions

$$h^{\uparrow}(x) = x^{\alpha(1-\varrho)}, \quad \text{and} \quad h^{\downarrow}(x) = x^{\alpha(1-\varrho)-1},$$

see [133]. The two processes $\xi^{\uparrow}$ and $\xi^{\downarrow}$ obtained this way are respectively called the (α,ϱ)-stable Lévy processes conditioned to survive, resp. to die continuously at 0, since they can be alternatively obtained by a limiting conditioning procedure associated to their names. See [89, Chapter 5] or [45] for details. Then the three processes $\xi^\dagger, \xi^\downarrow, \xi^\uparrow$ are positive self-similar Markov processes, for which it is possible to compute the characteristics of the underlying Lévy process in the Lamperti transformation [43] (beware, those are not stable processes anymore). To present them in a unified way, it is convenient to introduce the formalism of hypergeometric pssMp and Lévy processes. Recall that a pssMp is called hypergeometric if the underlying Lévy-Khintchine exponent is of the form

$$\psi(z) = -\frac{\Gamma(1-\varsigma+\upsilon-z)}{\Gamma(1-\zeta-z)} \cdot \frac{\Gamma(\hat{\varsigma}+\hat{\upsilon}+z)}{\Gamma(\hat{\varsigma}+z)}, \tag{6.23}$$

where $(\varsigma, \upsilon, \hat{\varsigma}, \hat{\upsilon})$ belongs to the admissible set of parameters $\{\varsigma \leqslant 1,\ \upsilon \in [0,1),\ \hat{\varsigma} \geqslant 0,\ \hat{\upsilon} \in (0,1)\}$.[2] We refer and use the same notation as in [87] where we replaced β by ς and γ by υ to avoid conflict in the notation. In particular, it is proved there that the pssMp $\xi^\dagger, \xi^\downarrow$ and $\xi^\uparrow$ are hypergeometric with parameters

	ς	υ	$\hat{\varsigma}$	$\hat{\upsilon}$
$\xi^\dagger$	$1-\alpha(1-\varrho)$	$\alpha\varrho$	$1-\alpha(1-\varrho)$	$\alpha(1-\varrho)$
$\xi^\uparrow$	1	$\alpha\varrho$	1	$\alpha(1-\varrho)$
$\xi^\downarrow$	0	$\alpha\varrho$	0	$\alpha(1-\varrho)$

Let us now examine which of the pssMp $\xi^\downarrow, \xi^\dagger$ can arise as the decoration along a tagged branch in a conservative and binary ssMt. We shall actually consider the case where the generalized Lévy measure $\mathbf{\Lambda}$ is binary, and **almost conservative** meaning that the only possible loss of mass during splitting event is due to a killing:

$$\mathbf{\Lambda}\Big(\Big\{(y_0,(y_1,...)) : \sum_{j=0}^{\infty} \mathrm{e}^{y_j} \neq 1\Big\}\Big) = \mathbf{\Lambda}\big(\{(-\infty,(-\infty,-\infty...)\}\big), \tag{6.24}$$

note that in this case the decoration-reproduction process η is still recovered from the pssMp evolution as in (4.14) except that the final jump does not yield to an atom in the decoration-reproduction. Since we are in the spectrally negative case, we shall treat separately the subordinator case $\alpha \in (0,1)$ and the case $\alpha \in (1,2)$. In the case $\alpha \in (0,1)$, consider the opposite of a α-stable subordinator started from 1 either killed $\xi^\dagger$ when reaching $\mathbb{R}_-$ or conditioned $\xi^\downarrow$ to die continuously at 0. Then we have $\varrho = 0$. The Lévy-Khintchine exponent in the Lamperti representation is explicit and given by

$$\psi_\alpha^\dagger(z) = -\frac{\Gamma(1+z)}{\Gamma(1-\alpha+z)}, \quad \text{and} \quad \psi_\alpha^\downarrow(z) = -\frac{\Gamma(\alpha+z)}{\Gamma(z)}. \tag{6.25}$$

By [89, Theorem 4.6 (ii)], their Lévy measures are, after a push-forward by $x \mapsto \mathrm{e}^x$, respectively given by

$$\pi_\alpha^\dagger(\mathrm{d}x) = \frac{-1}{\Gamma(-\alpha)} \cdot \frac{\mathrm{d}x}{(1-x)^{1+\alpha}} \mathbf{1}_{x\in[0,1]}, \quad \text{and} \quad \pi_\alpha^\downarrow(\mathrm{d}x) = \frac{-1}{\Gamma(-\alpha)} \cdot \frac{\mathrm{d}x}{x^{1-\alpha}(1-x)^{1+\alpha}} \mathbf{1}_{x\in[0,1]}.$$

[2] To be precise, the borderline case $\upsilon = 0$ is often excluded in the definition. Nonetheless, it can be added to the family of hypergeometric processes thanks to Proposition 4.1 and Theorem 4.4 in [89].

Moreover, we infer from (6.25) that their killing rate are $\frac{1}{\Gamma(1-\alpha)}$ and 0 respectively. In order to obtain $\xi^{\dagger}$ and $\xi^{\downarrow}$ as tilted versions of a characteristic quadruplet $(\mathrm{a}_*, \sigma_*^2, \mathbf{\Lambda}_*; \alpha_*)$, we must have $\mathrm{a}_*^{\mathrm{can}} = \sigma_* = 0$ and $\alpha_* = \alpha$; where we use the transparent notation $\mathrm{a}_*^{\mathrm{can}}$ for the associated canonical drift. Furthermore, without loss of generality we may assume that $\mathbf{\Lambda}_*$ is a locally largest generalized Lévy measure, and by (6.26) and the almost conservative assumption it must be of the form:

$$\int_{\mathcal{S}} F(\mathrm{e}^{y_0},(\mathrm{e}^{y_1},\mathrm{e}^{y_2},...)\mathbf{\Lambda}_*(\mathrm{d}y_0,\mathrm{d}\mathbf{y})$$
$$:= \int_{1/2}^{1} \frac{-1}{\Gamma(-\alpha)} \cdot \frac{\mathrm{d}x}{(x(1-x))^{1+\alpha}} F\big(x,(1-x,0,0,...)\big) + \mathrm{k}_* \cdot F\big(0,(0,\dots)\big), \tag{6.26}$$

for some constant $\mathrm{k}_* \geqslant 0$ corresponding to the killing rate. Recalling (6.11), we see that we must tilt the latter by $\gamma = \alpha + 1$ in the case of $\xi^{\dagger}$ and $\gamma = 2\alpha$ in the case of $\xi^{\downarrow}$. Finally using the (6.13) combined with the almost conservative assumption, we infer that we have to take $\mathrm{k}_* = -\psi_\alpha^{\dagger}(-\alpha)$ for $\xi^{\dagger}$ and $\mathrm{k}_* = -\psi_\alpha^{\downarrow}(1-2\alpha)$ for $\xi^{\downarrow}$. Performing the calculation, it turns out that

$$\psi_\alpha^{\dagger}(-\alpha) = \psi_\alpha^{\downarrow}(1-2\alpha) = \frac{-\Gamma(-\alpha)}{2\Gamma(-2\alpha)} = \frac{-4^{\alpha}\sqrt{\pi}}{\Gamma(1/2-\alpha)},$$

and the above display is non-positive if and only if $\alpha \in (0,1/2]$. So, to summarize, the processes $\xi^{\dagger}$ and $\xi^{\downarrow}$ can be obtained as the decoration processes along the tagged branch in a ssMt only when $\alpha \in (0,1/2]$, and in this case we can even use the same ssMt. Also, note that the associated generalized Lévy measure has killing in all cases except for $\alpha = 1/2$.

The case when $\alpha \in (1,2)$ is very similar except that some care is needed to deal correctly with compensation. The spectrally negative case correspond to $\varrho = \frac{1}{\alpha}$. The Lévy–Khintchine exponents are now given by

$$\psi_\alpha^{\dagger}(z) = \frac{1}{\pi}\left(\Gamma(\alpha-z)\Gamma(1+z)\sin(\pi(\alpha-z))\right) \quad \text{and} \quad \psi_\alpha^{\downarrow}(z) = \frac{1}{\pi}\left(-\Gamma(2-z)\Gamma(\alpha-1+z)\sin(\pi z)\right),$$

and the underlying Lévy measures are, after a push-forward by $x \mapsto \mathrm{e}^x$, similarly given by

$$\pi_\alpha^{\dagger}(\mathrm{d}x) = \frac{-\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}\frac{\mathrm{d}x}{(1-x)^{1+\alpha}} \quad \text{and} \quad \pi_\alpha^{\downarrow}(\mathrm{d}x) = \frac{-\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}\frac{\mathrm{d}x}{x^{2-\alpha}(1-x)^{1+\alpha}}.$$

As above, those measures are necessarily obtained by starting from a locally largest of the form

$$\int_{\mathcal{S}} F(\mathrm{e}^{y_0},(\mathrm{e}^{y_1},\mathrm{e}^{y_2},...)\mathbf{\Lambda}_*(\mathrm{d}y_0,\mathrm{d}\mathbf{y})$$
$$:= \int_{1/2}^{1} \frac{-\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}\frac{\mathrm{d}x}{(x(1-x))^{1+\alpha}} F(x,(1-x,0,0,...)) + \mathrm{k}_* \cdot F\big(0,(0,\dots)\big),$$

after tilting by $\gamma = \alpha+1$ in the case of $\xi^{\dagger}$ and by $\gamma = 2\alpha - 1$ in the case of $\xi^{\downarrow}$. The drift is then adjusted in the characteristics of the ssMt to match the one obtained in $\psi_\alpha^{\dagger}$ or $\psi_\alpha^{\downarrow}$. As above, the only condition to check is that the killing rate is non-negative. Using a formal calculation software, it is possible to obtain a closed formula for the cumulant function in terms of the killing rate, and we get the condition $\mathrm{k}_* = \frac{-4^\alpha\sqrt{\pi}}{\Gamma(1/2-\alpha)}$, in both regimes. The latter is non-negative if and only if $\alpha \in (1, 3/2]$. Notice in particular, the perhaps striking fact that the only stable processes conditioned to die continuously at 0 that can appear as the decoration X_γ in a binary conservative ssMt without killing are the 1/2 and 3/2-stable spectrally negative cases.

Comments and bibliographical notes

The notion of spinal decomposition is one of the most useful and powerful tools in the study of branching structures. It can be traced back at least to Kahane & Peyrière [85], while the first geometric formulation on trees is due to Chauvin & Rouault [46]. It has been famously popularized in the 90's by Lyons, Pemantle and Peres [111, 110] and has found numerous applications, notably in branching random walk theory, see [132]. In a context close to ours, these tools have been applied also for growth-fragmentation in [22] and branching Lévy processes [25].

The notion of bifurcators has already been introduced by Pitman & Winkel in the setting of fragmentations trees by [123] and by Shi [131] for growth-fragmentations processes. Section 6.3 build upon these works. See [22, Section 5] for a similar use of the tilted characteristics in the growth-fragmentation case.

The **profile** of a random measured (decorated) tree $\mathbf{T} = (T, d_T, \rho, g, \nu)$ is the push forward of ν by the distance to the origin $x \in T \mapsto d_T(\rho, x)$. After size-biaising by the total mass, it is related to the law of $d_T(\rho, \rho^\bullet)$ defined in (6.9). The profile has been studied in details in the fragmentation [76] and the growth-fragmentation [72] cases. In particular, it is proved there that in the case of the harmonic measure, the profile has a continuous density if and only if the self-similarity exponent α is strictly larger than ω_-. In the case of the Brownian CRT of Example 4.6, the profile is famously linked to Ray-Knight theorems and a similar phenomenon, though more complicated, has been established in the case of the Brownian growth-fragmentation tree of Example 4.9, see [100, 107, 44]. We wonder whether a similar Markov property of local times holds in a greater generality.

Chapter 7

Appendix

This appendix contains certain results which may be of broader interest.

Lemma 7.1. *Let ξ be a Lévy process (possibly with killing) and denote its law by P. We write ψ for its Laplace exponent as defined in Section 3.2; see (3.10). We further assume that there exists $\gamma > 0$ such that $\psi(\gamma) < 0$. Then, we have*

$$E\Big(\sup_{t\geqslant 0}\exp\big(\gamma\xi(t)\big)\Big) < \infty \quad \text{and} \quad E\Big(\big(\int_0^\zeta \exp(\alpha\xi(t))\mathrm{d}t\big)^{\gamma/\alpha}\Big) < \infty, \tag{7.1}$$

for every $\alpha > 0$, where ζ stands for the lifetime of ξ.

Proof. We work under the assumption of the lemma for some $\gamma > 0$ and we set $I = \int_0^\infty \exp(\gamma\xi(t))\mathrm{d}t$, with the usual convention $\xi(t) = -\infty$, for every $t \geqslant \zeta$. Then remark that by Fubini:

$$E\Big(\int_0^\infty \exp(\gamma\xi(t))\mathrm{d}t\Big) = \int_0^\infty E\big(\exp(\gamma\xi(t))\big)\mathrm{d}t = \int_0^\infty \exp(t\psi(\gamma))\mathrm{d}t = -\frac{1}{\psi(\gamma)} < \infty.$$

The idea now is to use the variable I to control the expectations appearing in (7.1). In this direction, we fix $b > 0$ such that $P_1(I \geqslant b) > 0$, and, for every $r > 0$, we introduce the stopping time $T_r := \inf\{s \geqslant 0 : \ \xi(s) \geqslant \log(r)/\gamma\}$. An application of the Markov property then yields

$$\begin{aligned} P\big(I \geqslant b\cdot r\big) &\geqslant P\Big(\big\{T_r < \infty\big\} \cap \Big\{\int_{T_r}^\infty \exp\big(\gamma(\xi(t)-\xi(T_r))\big)\mathrm{d}t \geqslant b\cdot r\exp\big(-\gamma\xi(T_r)\big)\Big\}\Big) \\ &\geqslant P\big(T_r < \infty\big)\cdot P\big(I \geqslant b\big), \end{aligned}$$

where in the second line we used that $\xi(T_r) \geqslant \log(r)/\gamma$ on the event $\{T_r < \infty\}$. Henceforth, we have

$$P\Big(\sup_{t\geqslant 0}\exp(\gamma\xi(t)) \geqslant r\Big) \leqslant \frac{P\big(I \geqslant br\big)}{P(I \geqslant b)}, \qquad \text{for } r > 0.$$

Now, an application of Fubini gives:

$$E\Big(\sup_{t\geqslant 0}\exp\big(\gamma\xi(t)\big)\Big) = \int_0^\infty P\big(\sup_{t\geqslant 0}\exp(\gamma\xi) \geqslant r\big)\mathrm{d}r \leqslant \int_0^\infty \frac{P\big(I \geqslant br\big)}{P(I \geqslant b)}\mathrm{d}r = \frac{E_1(I)}{b\cdot P_1(I \geqslant b)} < \infty. \tag{7.2}$$

Let us focus on the variable $\int_0^\infty \exp(\alpha\xi(t))\mathrm{d}t$, which belongs to the family of exponential functionals of Lévy processes. This family of variables has already been studied in depth [30], and analog results to the second inequality in (7.1) can be found in the literature. For instance, see Rivero [128, Lemma 3] for the case when the killing component is null, and [127, Lemma 2] for the case when $\alpha > \gamma$. Unfortunately, we have not been able to find a reference that holds under our assumptions; therefore, we adapt, with minor modifications, the latter proofs. In this direction, notice that

$$E\Big(\Big(\int_0^s \exp(\alpha\xi(t))\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}}\Big) = \frac{\gamma}{\alpha}E\Big(\int_0^s \exp\big(\gamma\xi(u)\big)\mathbf{1}_{u<\zeta}\Big(\int_u^\infty \exp\big(\alpha(\xi(t)-\xi(u))\big)\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}-1}\mathrm{d}u\Big),$$

for every $s \geqslant 0$. Moreover, the Markov property entails that

$$E\Big(\Big(\int_0^s \exp(\alpha\xi(t))\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}}\Big) = \frac{\gamma}{\alpha}E\Big(\int_0^s \exp\big(\gamma\xi(u)\big)\mathrm{d}u\Big)\cdot E\Big(\Big(\int_0^\infty \Big(\exp\big(\alpha\xi(t)\big)\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}-1}\Big).$$

Finally, since $\int_0^s \exp(\alpha\xi(t))\mathrm{d}t \leqslant s\cdot\sup_{t\in[0,s]}\exp(\xi(t))$, we derive from (7.2) that the previous display is finite for every $s \geqslant 0$. As a consequence, we get $E\big(\big(\int_0^\infty \exp\big(\alpha\xi(t)\big)\mathrm{d}t\big)^{\frac{\gamma}{\alpha}-1}\big) < \infty$, and, by monotone convergence taking the limit when $s \to \infty$, we obtain that:

$$E\Big(\Big(\int_0^\infty \exp(\alpha\xi(t))\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}}\Big) = \frac{\gamma}{\alpha}E(I)\cdot E\Big(\Big(\int_0^\infty \Big(\exp\big(\alpha\xi(t)\big)\mathrm{d}t\Big)^{\frac{\gamma}{\alpha}-1}\Big) < \infty,$$

which completes the proof of the lemma. □

Lemma 7.2 (A law of semi-large numbers)**.** *Let $(X_i : i \geqslant 1)$ be a uniformly integrable family of independent real random variables of mean 1. For any $\delta \in (0,1)$, there exists some constant $C_\delta > 0$ such that for any sequence of non-negative numbers $(x_i : i \geqslant 1)$ of sum $\mathbf{s} = \sum_i x_i < \infty$, the random sum $S = \sum_i X_i \cdot x_i$ satisfies*

$$\mathbb{P}\left(|S-\mathbf{s}| \geqslant \delta\mathbf{s}\right) \leqslant \delta + \frac{C_\delta}{\mathbf{s}}\sup_{i\geqslant 1} x_i.$$

Proof. Fix $\delta \in (0,1)$, and for every $i \geqslant 1$ set $Y_i := X_i - 1$. In particular, $(Y_i : i \geqslant 1)$ is a uniformly integrable family of independent centered random variables and thus we can find $M_\delta > 1$ such that $\mathbb{E}(|Y_i|\mathbf{1}_{|Y_i|>M_\delta}) \leqslant \delta^2/2$, for $i \geqslant 1$. Next, using the triangle inequality, we write:

$$\mathbb{P}\left(|S-\mathbf{s}| \geqslant \delta\mathbf{s}\right) = \mathbb{P}\Big(\Big|\sum_{i\geqslant 1} Y_i x_i\Big| \geqslant \delta\mathbf{s}\Big) \leqslant \mathbb{P}\Big(\Big|\sum_{i\geqslant 1} Y_i\mathbf{1}_{|Y_i|>M_\delta}x_i\Big| \geqslant \frac{\delta\mathbf{s}}{2}\Big) + \mathbb{P}\Big(\Big|\sum_{i\geqslant 1} Y_i\mathbf{1}_{|Y_i|\leqslant M_\delta}x_i\Big| \geqslant \frac{\delta\mathbf{s}}{2}\Big).$$

Let us now bound each term separately. On one hand, an application of Markov inequality gives:

$$\mathbb{P}\Big(\Big|\sum_{i\geqslant 1} Y_i\mathbf{1}_{|Y_i|>M_\delta}x_i\Big| \geqslant \frac{\delta\mathbf{s}}{2}\Big) \leqslant \frac{2}{\delta\mathbf{s}}\sum_{i\geqslant 1}\mathbb{E}(|Y_i|\mathbf{1}_{|Y_i|>M_\delta})x_i \leqslant \delta.$$

On the other hand, using that $\mathbb{E}(\sum_{i\geqslant 1} Y_i x_i) = 0$, it is plain that the truncated sum $\mathrm{TS} := \sum_{i\geqslant 1} Y_i\mathbf{1}_{|Y_i|<M_\delta}x_i$ has mean in $[-\delta^2\mathbf{s}/2, \delta^2\mathbf{s}/2]$ and by independence its variance is bounded

above by $4M_\delta^2 \sum_{i \geqslant 1} x_i^2$. Therefore, the triangle inequality followed by a classical application of Chebytchev inequality entails that

$$\mathbb{P}\Big(|\mathrm{TS}| \geqslant \frac{\delta \mathbf{s}}{2}\Big) \leqslant \frac{16 M_\delta^2 \sum_{i \geqslant 1} x_i^2}{(\delta - \delta^2)^2 \mathbf{s}^2} \leqslant \frac{16 M_\delta^2}{(\delta - \delta^2)^2 \mathbf{s}} \sup_{i \geqslant 1} x_i.$$

This completes the proof of the lemma. $\square$

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