Spatial Markov property in Brownian disks*

Jean-François Le Gall and Armand Riera

Université Paris-Saclay and Sorbonne Université

Abstract

We derive a new representation of the Brownian disk in terms of a forest of labeled trees, where labels correspond to distances from a subset of the boundary. We then use this representation to obtain a spatial Markov property showing that the complement of a hull centered at a boundary point of a Brownian disk is again a Brownian disk, with a random perimeter, and is independent of the hull conditionally on its perimeter. Our proofs rely in part on a study of the peeling process for triangulations with a boundary, which is of independent interest. The results of the present work will be applied to a continuous version of the peeling process for the Brownian half-plane in a companion paper.

1 Introduction

Brownian disks are basic models of random geometry that arise as scaling limits of random planar maps with a boundary, in the regime where the number of faces grows like the square of the boundary size. Brownian disks first appeared in the work of Bettinelli [6], who obtained the existence of subsequential limits of rescaled quadrangulations with a boundary in the Gromov-Hausdorff sense. The uniqueness of the limit, which is called the Brownian disk, was then obtained in the work [7] of Bettinelli and Miermont. In these scaling limits, [6] and [7] mainly deal with the case where both the boundary size and the volume are fixed, but it is also of interest to study the so-called free Brownian disk for which the boundary size (also called the perimeter) is fixed but the volume is random, cf. Section 1.5 in [7]. The free Brownian disk then appears as the limit of Boltzmann distributed random quadrangulations with a boundary, and in fact of much more general bipartite planar maps [7, Theorem 8]. In view of certain applications, it is desirable to consider the case of random planar maps with a simple boundary. Convergence to the free Brownian disk in that case was obtained for quadrangulations by Gwynne and Miller [13] and for triangulations by Albenque, Holden and Sun [2]. Both these papers prove convergence in a strong form of the Gromov-Hausdorff topology, which they call the GHPU convergence, which includes the convergence of the so-called boundary curves.

In the present work, we are primarily interested in the free Brownian disk, and we also consider the variant called the free pointed Brownian disk, where there is a distinguished point in the interior of the disk — if one "forgets" this distinguished point, the distribution of the free pointed Brownian disk becomes a size-biased version of the distribution of the free Brownian disk. The Bettinelli-Miermont construction applied to the free pointed Brownian disk (see Section 4.1 below) relies on a random forest made of a collection of labeled $\mathbb{R}$-trees, where labels correspond, up to a shift, to distances from the distinguished point of the Brownian disk. Different constructions, still based on labeled $\mathbb{R}$-trees, have been proposed in [18, 19] and shed light on various properties of (free) Brownian disks. In the construction of [18], labels correspond to distances from the boundary, and in [19] they represent distances from a distinguished point of the boundary. In the present work, we give yet another representation of the free Brownian disk (Theorem 17), where labels correspond to distances from a part of the boundary.

Let us briefly describe our new representation of the free Brownian disk. Let us fix $\xi > 0$, and let $C(\mathbb{R}_+, \mathbb{R}_+)$ denote the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$. Then let $\sum_{i\in I} \delta_{(t_i,e_i)}$ be a

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Poisson point measure on $[0,\xi] \times C(\mathbb{R}_+,\mathbb{R}_+)$ with intensity $2\,\mathrm{d}t\,\mathbf{n}(\mathrm{d}e)$, where $\mathbf{n}(\mathrm{d}e)$ denotes the Itô measure of positive excursions of linear Browian motion. It is well known that each excursion e_i codes a compact $\mathbb{R}$-tree, which is denoted by $\mathcal{T}_{e_i}$. We then assign a real label ℓ_a to every point a of $[0,\xi]$ and to every $a \in \mathcal{T}_{e_i}$, $i \in I$, in the following way. We consider a Brownian excursion $(\mathbf{e}_t)_{0\le t\le \xi}$ of duration ξ and we declare that the label of every $a \in [0,\xi]$ is $\ell_a := \sqrt{3}\,\mathbf{e}_a$. For every $i \in I$, we assign the label $\sqrt{3}\,\mathbf{e}_{t_i}$ to the root of $\mathcal{T}_{e_i}$, and then we require that labels evolve like linear Brownian motion along the segments of $\mathcal{T}_{e_i}$ (independently when i varies). In other words, $(\ell_a)_{a\in\mathcal{T}_{e_i}}$ is distributed as Brownian motion indexed by $\mathcal{T}_{e_i}$ started from $\sqrt{3}\,\mathbf{e}_{t_i}$ at the root. The preceding objects (the normalized Brownian excursion $(\mathbf{e}_t)_{0\le t\le\xi}$ and the labeled trees $\mathcal{T}_{e_i}$) are the basic ingredients of the Bettinelli-Miermont construction of free Brownian disks [6, 7] (see Section 4.1 below), but here we perform an additional step: we prune each tree $\mathcal{T}_{e_i}$ at levels where labels first hit the value 0, and write $\widetilde{\mathcal{T}}_{e_i}$ for the resulting pruned tree, which now carries nonnegative labels. We then consider the union

$$\mathfrak{T}^\star := [0,\xi] \cup \Big(\bigcup_{i\in I} \widetilde{\mathcal{T}}_{e_i}\Big),$$

where we identify the root of $\widetilde{\mathcal{T}}_{e_i}$ with the point t_i of $[0,\xi]$. We then proceed in a way very similar to known constructions of the Brownian sphere or the Brownian disk. Precisely, if a, b are two points of $\mathfrak{T}^\star$ with positive labels, we let $D^\circ_\star(a,b)$ be the sum of the labels of a and b minus twice the minimal label "between" a and b (see Section 4.2 for a more precise definition) if this minimal label is positive, and if not we set $D^\circ_\star(a,b) = +\infty$. We finally write $D_\star$ for the maximal pseudo-metric on $\mathfrak{T}^\star$ that is bounded above by $D^\circ_\star$. Theorem 17 then shows that the quotient space[1] $\mathbb{U} := \mathfrak{T}^\star/\{D_\star = 0\}$ equipped with the metric induced by $D_\star$ is a free pointed Brownian disk with (random) perimeter $\xi + \mathcal{Z}$, where $\mathcal{Z}$ is a random variable measuring, in some sense, the quantity of points with zero label in $\mathfrak{T}^\star$. Moreover, labels on $\mathbb{U}$ (which are inherited from the labels on $\mathfrak{T}^\star$) correspond to distances from the subset of the boundary that is the image of the set of points of $\mathfrak{T}^\star$ with zero label under the canonical projection from $\mathfrak{T}^\star$ onto $\mathbb{U}$. The complementary part of the boundary is the image of $[0,\xi]$ under the canonical projection.

An important motivation for Theorem 17 came from an application to the complement of hulls centered at a boundary point of a Brownian disk. Let $\mathbb{D}'$ be a free Brownian disk with perimeter ξ and boundary $\partial\mathbb{D}'$. One can define a "standard boundary curve" $(\Gamma(t))_{t\in[0,\xi]}$ that starts from a point uniformly distributed on $\partial\mathbb{D}'$ and runs along the boundary $\partial\mathbb{D}'$ at "uniform speed" (see Section 4.1). Then let α and β be distinct real numbers in $[0,\xi]$ and consider the two points a and b of the boundary defined by $a = \Gamma(\alpha)$ and $b = \Gamma(\beta)$. Fix $r > 0$ and write B_r for the ball of radius r centered at a in $\mathbb{D}'$. Conditionally on the event where the distance between a and b is greater than r, one may consider the connected component of $\mathbb{D}'\backslash B_r$ that contains b, and we denote this component by $\widehat{B}^\circ_r$. By definition, the hull of radius r centered at a, relative to b, is $B^\bullet_r := \mathbb{D}'\backslash\widehat{B}^\circ_r$. Then, $\widehat{B}^\circ_r$ (or rather its closure $\widehat{B}^\bullet_r$) equipped with the appropriate intrinsic distance, is again a free Brownian disk, now with a random perimeter (Theorem 23). Moreover, conditionally on its boundary size, this free Brownian disk is independent of the hull $B^\bullet_r$ also equipped with an intrinsic distance (Theorem 24). These results can be interpreted as a spatial Markov property of the free Brownian disk. Imagine that one starts exploring the free Brownian disk from the point a of the boundary. At the time where one has discovered the ball $B_r(a)$ and all connected components of $\mathbb{D}'\backslash B_r(a)$ not containing b, what remains to be explored is again a free Brownian disk (with a random perimeter) which conditionally on its boundary size is independent of what has already been discovered. This is also reminiscent of the peeling explorations of random planar maps, which have found a number of striking applications (see in particular [3, 10, 11]).

The preceding results take an even nicer form in the model called the Brownian half-plane [9, 12, 4], which will be studied in the companion paper [23]. The Brownian half-plane $\mathfrak{H}$ is a random non-compact metric space, which is homeomorphic to the usual half-plane $\mathbb{R}\times\mathbb{R}_+$, so that it makes sense to define its boundary $\partial\mathfrak{H}$. The Brownian half-plane comes with a distinguished point $\mathbf{x}$ on its boundary. For every $r > 0$, the hull of radius r centered at $\mathbf{x}$ is defined as the complement of the unbounded connected

[1] The notation $\mathfrak{T}^\star/\{D_\star = 0\}$ refers to the quotient space of $\mathfrak{T}^\star$ for the equivalence relation defined by setting $a \simeq_\star b$ if and only if $D_*(a,b) = 0$.

component of $\mathfrak{H}\backslash B_r(\mathfrak{H})$, where $B_r(\mathfrak{H})$ denotes the closed ball of radius r centered at $\mathbf{x}$. Let $B^\bullet_r(\mathfrak{H})$ denote this hull. Then, the closure of $\mathfrak{H}\backslash B^\bullet_r(\mathfrak{H})$ equipped with the intrinsic metric (and pointed at a boundary point which can be chosen in a deterministic way from the hull $B^\bullet_r(\mathfrak{H})$) is again a Brownian half-plane, which furthermore can be shown to be independent of the hull $B^\bullet_r(\mathfrak{H})$. This property is again a continuous analog of the peeling process of infinite half-planar planar maps (see in particular [3]). The proof, whose details will be given in [23], is based on a passage to the limit from Theorems 23 and 24. Similarly, a passage to the limit from Theorem 17 yields a simple new representation of the Brownian half-plane.

Let us comment on the proofs of the preceding results. The proof of Theorem 17 relies on discrete approximations. The underlying idea is to start from a free pointed Brownian disk $\mathbb{D}$, and to consider hulls centered at the distinguished point $\mathbf{x}_*$ (which is now a point of the interior of $\mathbb{D}$ and not of $\partial\mathbb{D}$ as above) relative to the boundary $\partial\mathbb{D}$. More precisely, we consider the hull H with a radius r_0 which is the distance between $\mathbf{x}_*$ and the boundary $\partial\mathbb{D}$, and we write U for the complement of H in $\mathbb{D}$. In other words, U is the connected component of the complement of the ball of radius r_0 centered at $\mathbf{x}_*$ that contains all the boundary $\partial\mathbb{D}$ but the single point $\mathbf{x}_0$ realizing the distance between $\mathbf{x}_*$ and $\partial\mathbb{D}$. The Bettinelli-Miermont construction of $\mathbb{D}$ (Section 4.1) allows one to get a representation of U in terms of a Brownian excursion $\mathbf{e}$ and a collection $(\widetilde{\mathcal{T}}_{e_i})_{i\in I}$ of labeled trees having exactly the distribution described above. On the other hand, one proves that the completion of U for the appropriate intrinsic metric is also a free Brownian disk whose boundary can be viewed as the union of $\partial\mathbb{D}$ and ∂H, provided that the unique point $\mathbf{x}_0$ of $\partial\mathbb{D}\cap\partial H$ is split into two points. This identification of the law of (the completion of) U is the difficult part of the proof of Theorem 17 in Section 5, and, for this, we first obtain an analogous discrete result: we observe that, for a Boltzmann distributed pointed triangulation with a simple boundary, the analog of the set U, which is conveniently defined via a particular version of the peeling process, is again a Boltzmann distributed triangulation with a random boundary size, and we use properties of the peeling process to investigate the asymptotics of this boundary size. This part of the argument relies on a study of the peeling process for a Boltzmann distributed pointed triangulation with a boundary (Section 3), which is of independent interest.

In order to derive the spatial Markov property of Theorems 23 and 24, we rely on Theorem 17 and on the representation of the Brownian disk in [19]. In this representation, the Brownian excursion $(\mathbf{e}_t)_{0\leq t\leq\xi}$ is replaced by a five-dimensional Bessel excursion $(\mathbf{b}_t)_{0\leq t\leq\xi}$, and the Poisson collection $(\mathcal{T}_{e_i})_{i\in I}$ of labeled trees is conditioned to have only positive labels. Furthermore labels now correspond to distances from a (uniformly distributed) point of the boundary. Thanks to this last property, the complement of the hull of radius r centered at the distinguished point of the boundary (and relative to another fixed point) can be coded by the "subexcursion" $\mathbf{b}^{(r)}$ of $\mathbf{b}$ above level r that straddles a given time of $[0,\xi]$, and by the labeled subtrees $\mathcal{T}_{e_i}$ for indices i such that t_i belongs to the time interval associated with $\mathbf{b}^{(r)}$, provided these subtrees are pruned at levels where labels first hit r. Under an appropriate conditioning, the pair consisting of $\mathbf{b}^{(r)}$ and the collection of pruned labeled trees (where labels are shifted by $-r$) has the same distribution as the pair $(\mathbf{e},(\widetilde{\mathcal{T}}_{e_i})_{i\in I})$ considered above, provided ξ is replaced by the quantity ξ' which is the duration of $\mathbf{b}^{(r)}$. This allows one to apply Theorem 17 in order to obtain that the hull complement is again a Brownian disk.

The paper is organized as follows. Section 2 gives several preliminaries. In particular, we recall the formalism of curve-decorated measure metric spaces, and the associated Gromov-Hausdorff-Prokhorov-uniform distance d_{GHPU}, which has been introduced in [12] and also used in [2]. Moreover, we recall basic facts about snake trajectories and the Brownian snake excursion measure, which provide a convenient setting to deal with our labeled trees. In Section 3, we discuss the peeling process of triangulations with a boundary, whose scaling limit is known to be the Brownian disk [2]. In Section 4, we introduce the space $\mathbb{U}$, and we explain how this space can be identified with (the completion of) the complement of a hull centered at the distinguished point in the free pointed Brownian disk. Section 5, which is the most technical part of the paper, is devoted to the proof of Theorem 17 identifying $\mathbb{U}$ as a Brownian disk. The general idea is to pass to the limit from the analogous discrete result for triangulations, but unfortunately this passage to the limit requires a number of technicalities. Finally, Section 6 presents the proof of Theorems 23 and 24.

Contents

2 Preliminaries

2.1 Convergence of metric spaces

In this work, we will consider different notions of convergence of a sequence of compact metric spaces, which we briefly present in this section. A bipointed compact metric space (E, d, x, x') is just a compact metric space (E, d) given with an ordered pair $(x, x') \in E \times E$ of distinguished points. We write $\mathbb{M}^{GH\bullet\bullet}$ for the set of all isometry classes of bipointed compact metric spaces (two pointed compact metric spaces (E_1, d_1, x_1, x_1') and (E_2, d_2, x_2, x_2') are isometry equivalent if there is an isometry Φ from E_1 onto E_2 such that $\Phi(x_1) = x_2$ and $\Phi(x_1') = x_2'$). We can equip $\mathbb{M}^{GH\bullet\bullet}$ with the bipointed Gromov-Hausdorff distance $\mathrm{d}_{GH\bullet\bullet}$, which is defined by setting

$$\mathrm{d}_{GH\bullet\bullet}((E_1, d_1, x_1, x_1'), (E_2, d_2, x_2, x_2')) \\ := \inf\Big\{ d_{\mathrm{H}}^E(\Phi_1(E_1), \Phi_2(E_2)) \vee d(\Phi_1(x_1), \Phi_2(x_2)) \vee d(\Phi_1(x_1'), \Phi_2(x_2'))\Big\},$$

where the infimum is over all isometric embeddings $\Phi_1 : E_1 \longrightarrow E$ and $\Phi_2 : E_2 \longrightarrow E$ of E_1 and E_2 into the same compact metric space (E, d), and d_{H}^E is the usual Hausdorff distance between compact subsets of E. Then, $(\mathbb{M}^{GH\bullet\bullet}, \mathrm{d}_{GH\bullet\bullet})$ is a Polish space. See in particular [8] (proofs in [8] are given in the non-pointed case, but are immediately adapted). We can also define $\mathrm{d}_{GH\bullet\bullet}$ in terms of correspondences. Recall that a correspondence between E_1 and E_2 is a subset $\mathcal{C}$ of $E_1 \times E_2$ such that the restrictions to

$\mathcal{C}$ of both canonical projections $E_1 \times E_2 \longrightarrow E_1$ and $E_1 \times E_2 \longrightarrow E_2$ are surjective. The distortion of $\mathcal{C}$ is then defined by

$$\mathrm{dis}(\mathcal{C}) := \sup\{|d_1(y_1, z_1) - d_2(y_2, z_2)| : (y_1, y_2) \in \mathcal{C}, (z_1, z_2) \in \mathcal{C}\},$$

and the $\mathrm{d}_{GH^{\bullet\bullet}}$ distance can be expressed as

$$\mathrm{d}_{GH^{\bullet\bullet}}((E_1, d_1, x_1, x_1'), (E_2, d_2, x_2, x_2')) = \frac{1}{2} \inf\{\mathrm{dis}(\mathcal{C})\}.$$

where the infimum is over all correspondences between E_1 and E_2 such that $(x_1, x_2) \in \mathcal{C}$ and $(x_1', x_2') \in \mathcal{C}$.

We will consider metric spaces equipped with additional structures. If (E, d) is a compact metric space, we let $C_0(\mathbb{R}, E)$ be the space of all continuous functions $\gamma : \mathbb{R} \longrightarrow E$ such that, for every $\varepsilon > 0$, there exists $T > 0$ such that $d(\gamma(t), \gamma(T)) < \varepsilon$ and $d(\gamma(-t), \gamma(-T)) < \varepsilon$ for every $t \geq T$. By convention, if $\gamma : [a, b] \longrightarrow E$ is only (continuous and) defined on an interval $[a, b]$, we view it as an element of $C_0(\mathbb{R}, E)$ by extending it so that it is constant on $(-\infty, a]$ and on $[b, \infty)$. A curve-decorated and pointed (compact) measure metric space is then a compact metric space (E, d) equipped with a finite Borel measure μ (sometimes called the volume measure), with a curve $\gamma \in C_0(\mathbb{R}, E)$, and with a distinguished point x. We write $\mathbb{M}^{GHPU^\bullet}$ for the set of all isometry classes of curve-decorated and pointed compact measure metric spaces (here (E, d, μ, γ, x) and $(E', d', \mu', \gamma', x')$ are isometry equivalent if there exists an isometry Φ from E onto E' such that $\Phi_*\mu = \mu'$, $\gamma' = \Phi \circ \gamma$, and $\Phi(x) = x'$). One can equip $\mathbb{M}^{GHPU^\bullet}$ with the so-called Gromov-Hausdorff-Prokhorov-uniform distance $\mathrm{d}_{GHPU^\bullet}$, which is defined by

$$\begin{aligned}&\mathrm{d}_{GHPU^\bullet}((E_1, d_1, \mu_1, \gamma_1, x_1), (E_2, d_2, \mu_2, \gamma_2, x_2))\\ &= \inf\Big\{d_{\mathrm{H}}^E(\Phi_1(E_1), \Phi_2(E_2)) \vee d_{\mathrm{P}}^E((\Phi_1)_*\mu_1, (\Phi_2)_*\mu_2) \vee \sup_{t \in \mathbb{R}} d(\Phi_1 \circ \gamma_1(t), \Phi_2 \circ \gamma_2(t)) \vee d(\Phi_1(x_1), \Phi_2(x_2))\Big\},\end{aligned}$$

where the infimum is over all isometric embeddings $\Phi_1 : E_1 \longrightarrow E$ and $\Phi_2 : E_2 \longrightarrow E$ of E_1 and E_2 into the same compact metric space (E, d), and d_{P}^E denotes the Prokhorov metric on the space of all finite measures on E. By a straightforward adaptation of the arguments of [12, Section 2.2], one verifies that $\mathrm{d}_{GHPU^\bullet}$ is a complete separable metric on $\mathbb{M}^{GHPU^\bullet}$.

Following [12], we will also use the space $\mathbb{M}^{GHPU}$ of all isometry classes of (non-pointed) curve-decorated compact measure metric spaces, which is equipped with distance d_{GHPU} defined exactly as $\mathrm{d}_{GHPU^\bullet}$ in the last display by just omitting the last term $d(\Phi_1(x_1), \Phi_2(x_2))$. Then $(\mathbb{M}^{GHPU}, \mathrm{d}_{GHPU})$ is again a Polish space [12].

Proposition 1. *Let $(E_n, d_n, \mu_n, \gamma_n, x_n)$, for $n \in \mathbb{N}$, and $(E_\infty, d_\infty, \mu_\infty, \gamma_\infty, x_\infty)$ be elements of $\mathbb{M}^{GHPU^\bullet}$. Suppose that $(E_n, d_n, \mu_n, \gamma_n, x_n)$ converges to $(E_\infty, d_\infty, \mu_\infty, \gamma_\infty, x_\infty)$ in $(\mathbb{M}^{GHPU^\bullet}, \mathrm{d}_{GHPU^\bullet})$, as $n \to \infty$. Then, we can find a compact metric space (E, d) and isometric embeddings $\Phi_n : E_n \longrightarrow E$ and $\Phi_\infty : E_\infty \longrightarrow E$ such that $\Phi_n(E_n) \longrightarrow \Phi_\infty(E_\infty)$ for the Hausdorff metric, $(\Phi_n)_*\mu_n \longrightarrow (\Phi_\infty)_*\mu_\infty$ for the Prokhorov metric, $\Phi_n \circ \gamma_n(t) \longrightarrow \Phi_\infty \circ \gamma_\infty(t)$ uniformly in t, and $\Phi_n(x_n) \longrightarrow \Phi_\infty(x_\infty)$, as $n \to \infty$.*

This is the exact analog of [12, Proposition 1.5], which deals with $\mathbb{M}^{GHPU}$ instead of $\mathbb{M}^{GHPU^\bullet}$. The proof is the same. In what follows, we will be interested in random metric spaces in $\mathbb{M}^{GHPU^\bullet}$, and particularly in the special case where the distinguished point is chosen "uniformly" according to the volume measure. The following lemma will be useful.

Lemma 2. *Let $(X^n, D^n, \Upsilon^n, \Gamma^n)$, for $n \in \mathbb{N} \cup \{\infty\}$, be random variables with values in $\mathbb{M}^{GHPU}$. Assume that $(X^n, D^n, \Upsilon^n, \Gamma^n)$ converges to $(X^\infty, D^\infty, \Upsilon^\infty, \Gamma^\infty)$ in distribution when $n \to \infty$. Also assume that $0 < \mathbb{E}[\Upsilon^n(X^n)] < \infty$ for every $n \in \mathbb{N} \cup \{\infty\}$, and that*

$$\mathbb{E}[\Upsilon^n(X^n)] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\Upsilon^\infty(X^\infty)]. \tag{1}$$

For every $n \in \mathbb{N} \cup \{\infty\}$, define a probability measure $\mathbf{\Theta}_n$ on $\mathbb{M}^{GHPU^\bullet}$ by setting, for every bounded continuous real function F on $\mathbb{M}^{GHPU^\bullet}$,

$$\int F \,\mathrm{d}\mathbf{\Theta}_n = \frac{1}{\mathbb{E}[\Upsilon^n(X^n)]} \mathbb{E}\Big[\int \Upsilon^n(\mathrm{d}x)\, F((X^n, D^n, \Upsilon^n, \Gamma^n, x))\Big].$$

Then $\mathbf{\Theta}_n$ converges weakly to $\mathbf{\Theta}_\infty$ as $n \to \infty$.

Proof. By the Skorokhod representation theorem, we may assume that $(X^n, D^n, \Upsilon^n, \Gamma^n)$ converges a.s. to $(X^\infty, D^\infty, \Upsilon^\infty, \Gamma^\infty)$. We then observe that, if F is bounded and continuous on $\mathbb{M}^{GHPU\bullet}$, the mapping

$$(E, d, \mu, \gamma) \mapsto \int \mu(\mathrm{d}x)\, F((E, d, \mu, \gamma, x))$$

is continuous on $\mathbb{M}^{GHPU}$ (we leave the proof to the reader). It follows that we have a.s.

$$\int \Upsilon^n(\mathrm{d}x)\, F((X^n, D^n, \Upsilon^n, \Gamma^n, x)) \underset{n\to\infty}{\longrightarrow} \int \Upsilon^\infty(\mathrm{d}x)\, F((X^\infty, D^\infty, \Upsilon^\infty, \Gamma^\infty, x)).$$

Using dominated convergence and our assumption (1), we obtain that $\int F\,\mathrm{d}\mathbf{\Theta}_n \longrightarrow \int F\,\mathrm{d}\mathbf{\Theta}_\infty$. □

2.2 Snake trajectories

To construct the models of random geometry that we consider, we will use the formalism of snake trajectories. A (one-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})} \geq 0$ is called the lifetime of w. We let $\mathfrak{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathfrak{W}}(\mathrm{w}, \mathrm{w}') := |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For $x \in \mathbb{R}$, we set $\mathfrak{W}_x := \{\mathrm{w} \in \mathfrak{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathfrak{W}_x$ with zero lifetime is identified with the point x of $\mathbb{R}$.

Definition 3. *Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathfrak{W}_x$ that satisfies the following two properties:*

(i) *We have $\omega_0 = x$ and the number $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$, called the duration of the snake trajectory ω, is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = x$ for every $s \geq 0$).*

(ii) (Snake property) *For every $0 \leq s \leq s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s\leq r\leq s'} \zeta_{(\omega_r)}]$.*

We will write $\mathcal{S}_x$ for the set of all snake trajectories with initial point x and $\mathcal{S} = \bigcup_{x\in\mathbb{R}} \mathcal{S}_x$ for the set of all snake trajectories. If $\omega \in \mathcal{S}$, we often write $W_s(\omega) := \omega_s$ and $\zeta_s(\omega) := \zeta_{(\omega_s)}$ for every $s \geq 0$. The set $\mathcal{S}$ is a Polish space for the distance

$$d_{\mathcal{S}}(\omega, \omega') := |\sigma(\omega) - \sigma(\omega')| + \sup_{s\geq 0} d_{\mathfrak{W}}(W_s(\omega), W_s(\omega')).$$

We stress that a snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$: See [1, Proposition 8].

Let $\omega \in \mathcal{S}$ be a snake trajectory and $\sigma = \sigma(\omega)$. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact $\mathbb{R}$-tree, which will be denoted by $\mathcal{T}_{(\omega)}$ and called the *genealogical tree* of the snake trajectory. This $\mathbb{R}$-tree is the quotient space $\mathcal{T}_{(\omega)} := [0, \sigma]/\sim$ of the interval $[0, \sigma]$ for the equivalence relation

$$s \sim s' \text{ if and only if } \zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s\wedge s'\leq r\leq s\vee s'} \zeta_r(\omega),$$

and $\mathcal{T}_{(\omega)}$ is equipped with the distance induced by

$$d_{(\omega)}(s, s') := \zeta_s(\omega) + \zeta_{s'}(\omega) - 2 \min_{s\wedge s'\leq r\leq s\vee s'} \zeta_r(\omega).$$

(notice that $d_{(\omega)}(s, s') = 0$ if and only if $s \sim s'$). We write $p_{(\omega)} : [0, \sigma] \longrightarrow \mathcal{T}_{(\omega)}$ for the canonical projection, and the mapping $[0, \sigma] \ni t \mapsto p_{(\omega)}(t)$ can be viewed as a cyclic exploration of $\mathcal{T}_{(\omega)}$. By convention, $\mathcal{T}_{(\omega)}$ is rooted at the point $\rho_{(\omega)} := p_{(\omega)}(0)$, and the volume measure on $\mathcal{T}_{(\omega)}$ is defined as the pushforward of Lebesgue measure on $[0, \sigma]$ under $p_{(\omega)}$. If $u, v \in \mathcal{T}_{(\omega)}$, $[\![u, v]\!]$ denotes the geodesic segment between u and v in $\mathcal{T}_{(\omega)}$. The segment $[\![\rho_{(\omega)}, u]\!]$ is called the ancestral line of u.

By property (ii) in the definition of a snake trajectory, the condition $p_{(\omega)}(s) = p_{(\omega)}(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ can be viewed as defined on the quotient space

$\mathcal{T}_{(\omega)}$. For $u \in \mathcal{T}_{(\omega)}$, we set $\ell_u(\omega) := \widehat{W}_s(\omega)$ whenever $s \in [0,\sigma]$ is such that $u = p_{(\omega)}(s)$ (by the previous observation, this does not depend on the choice of s). We interpret $\ell_u(\omega)$ as a "label" assigned to the "vertex" u of $\mathcal{T}_{(\omega)}$. Notice that the mapping $u \mapsto \ell_u(\omega)$ is continuous on $\mathcal{T}_{(\omega)}$, and that, for every $s \geq 0$, the path $W_s(\omega)$ records the labels $\ell_u(\omega)$ along the ancestral line $[\![\rho_{(\omega)}, p_{(\omega)}(s)]\!]$. We will use the notation $W_*(\omega) := \min\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$.

We now introduce an important operation on snake trajectories in $\mathcal{S}$. Let $x, y \in \mathbb{R}$ with $y < x$. For every $\mathrm{w} \in \mathfrak{W}_x$, set

$$\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$$

with the usual convention $\inf \varnothing = \infty$ (this convention will be in force throughout this work unless otherwise indicated). Then, if $\omega \in \mathcal{S}_x$, we set, for every $s \geq 0$,

$$\eta_s(\omega) := \inf\Big\{t \geq 0 : \int_0^t \mathrm{d}r\, \mathbf{1}_{\{\zeta_{(\omega_r)} \leq \tau_y(\omega_r)\}} > s\Big\}.$$

Note that the condition $\zeta_{(\omega_r)} \leq \tau_y(\omega_r)$ holds if and only if $\tau_y(\omega_r) = \infty$ or $\tau_y(\omega_r) = \zeta_{(\omega_r)}$. Then, setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geq 0$ defines an element ω' of $\mathcal{S}_x$, which will be denoted by $\mathrm{tr}_y(\omega)$ and called the truncation of ω at y (see [1, Proposition 10]). The effect of the time change $\eta_s(\omega)$ is to "eliminate" those paths ω_s that hit y and then survive for a positive amount of time. The genealogical tree $\mathcal{T}_{(\mathrm{tr}_y(\omega))}$ is canonically and isometrically identified to the closed set

$$\{v \in \mathcal{T}_{(\omega)} : \ell_u(\omega) > y \text{ for every } u \in [\![\rho_{(\omega)}, v]\!] \backslash \{v\}\},$$

and this identification preserves labels. In what follows, we will therefore view $\mathcal{T}_{(\mathrm{tr}_y(\omega))}$ as a subset of $\mathcal{T}_{(\omega)}$. Informally, $\mathcal{T}_{(\mathrm{tr}_y(\omega))}$ is obtained from $\mathcal{T}_{(\omega)}$ by pruning branches at the level where labels first take the value y.

We can then also define the excursions of ω away from a given level. Let $\omega \in \mathcal{S}_x$ and let $y < x$. Let (α_j, β_j), $j \in J$, be the connected components of the open set

$$\{s \in [0,\sigma] : \tau_y(\omega_s) < \zeta_{(\omega_s)}\},$$

and notice that we have $\omega_{\alpha_j} = \omega_{\beta_j}$, for every $j \in J$, by the snake property. For every $j \in J$ we define a snake trajectory $\omega^j \in \mathcal{S}_y$ by setting

$$\omega^j_s(t) := \omega_{(\alpha_j+s)\wedge\beta_j}(\zeta_{(\omega_{\alpha_j})} + t)\ ,\ \text{for } 0 \leq t \leq \zeta_{(\omega^j_s)} := \zeta_{(\omega_{(\alpha_j+s)\wedge\beta_j})} - \zeta_{(\omega_{\alpha_j})} \text{ and } s \geq 0.$$

We say that ω^j, $j \in J$, are the excursions of ω away from y. We note that, for every $j \in J$, the tree $\mathcal{T}_{(\omega^j)}$ is canonically identified to a subtree of $\mathcal{T}_{(\omega)}$ consisting of descendants of $p_{(\omega)}(\alpha_j) = p_{(\omega)}(\beta_j)$.

2.3 The Brownian snake excursion measure on snake trajectories

Let $x \in \mathbb{R}$. The Brownian snake excursion measure $\mathbb{N}_x$ is the σ-finite measure on $\mathcal{S}_x$ that satisfies the following two properties: Under $\mathbb{N}_x$,

(i) the distribution of the lifetime function $(\zeta_s)_{s\geq 0}$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x\Big(\sup_{s\geq 0} \zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon};$$

(ii) conditionally on $(\zeta_s)_{s\geq 0}$, the tip function $(\widehat{W}_s)_{s\geq 0}$ is a Gaussian process with mean x and covariance function

$$K(s,s') := \min_{s\wedge s' \leq r \leq s\vee s'} \zeta_r.$$

Informally, the lifetime process $(\zeta_s)_{s\geq 0}$ evolves under $\mathbb{N}_x$ like a Brownian excursion, and conditionally on $(\zeta_s)_{s\geq 0}$, each path W_s is a linear Brownian path started from x with lifetime ζ_s, which is "erased" from its tip when ζ_s decreases and is "extended" when ζ_s increases. The measure $\mathbb{N}_x$ can be interpreted as the excursion measure away from x for the Markov process in $\mathfrak{W}_x$ called the Brownian snake. We refer to [15] for a detailed study of the Brownian snake. For every $y < x$, we have

$$\mathbb{N}_x(W_* \le y) = \frac{3}{2(x-y)^2}. \tag{2}$$

See e.g. [15, Section VI.1] for a proof.

Exit measures. Let $x, y \in \mathbb{R}$, with $y < x$. Under the measure $\mathbb{N}_x$, one can make sense of a quantity that measures "how many" paths W_s hit y. One shows [18, Proposition 34] that the limit

$$L_t^y := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t \mathrm{d}s\, \mathbf{1}_{\{\tau_y(W_s)=\infty,\, \widehat{W}_s < y+\varepsilon\}} \tag{3}$$

exists uniformly in $t \ge 0$, $\mathbb{N}_x$ a.e., and defines a continuous nondecreasing function, which is obviously constant on $[\sigma, \infty)$. The process $(L_t^y)_{t\ge 0}$ is called the exit local time from (y, ∞), and the exit measure $\mathcal{Z}_y$ is defined by $\mathcal{Z}_y := L_\infty^y = L_\sigma^y$. Then, $\mathbb{N}_x$ a.e., the topological support of the measure $\mathrm{d}L_t^y$ is exactly the set $\{s \in [0,\sigma] : \tau_y(W_s) = \zeta_s\}$, and, in particular, $\mathcal{Z}_y > 0$ if and only if one of the paths W_s hits y. The definition of $\mathcal{Z}_y$ is a special case of the theory of exit measures (see [15, Chapter V] for this general theory). We will use the formula for the Laplace transform of $\mathcal{Z}_y$: For $\lambda > 0$,

$$\mathbb{N}_x\Big(1 - \exp(-\lambda \mathcal{Z}_y)\Big) = \Big((x-y)\sqrt{2/3} + \lambda^{-1/2}\Big)^{-2}. \tag{4}$$

See formula (6) in [11] for a brief justification.

It is useful to observe that $\mathcal{Z}_y$ can be defined in terms of the truncated snake $\mathrm{tr}_y(\omega)$. To this end, recall the time change $(\eta_s(\omega))_{s\ge 0}$ used to define $\mathrm{tr}_y(\omega)$ at the end of Section 2.2, and set $\widetilde{L}_t^y = L_{\eta_t}^y$ for every $t \ge 0$. Then $\widetilde{L}_\infty^y = L_\infty^y = \mathcal{Z}_y$, whereas formula (3) implies that

$$\widetilde{L}_t^y = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s(\mathrm{tr}_y(\omega)) < y+\varepsilon\}} \tag{5}$$

uniformly for $t \ge 0$, $\mathbb{N}_x$ a.e.

The special Markov property. We use the notation introduced in Section 2.2. More precisely, we write ω^j, $j \in J$, for the excursions of ω below y and (α_j, β_j), $j \in J$, for the associated time intervals. The special Markov property states that, conditionally on the truncation $\mathrm{tr}_y(\omega)$, the point measure:

$$\sum_{j\in J} \delta_{(L_{\alpha_j}^y, \omega^j)} \tag{6}$$

is Poisson with intensity $\mathbf{1}_{[0,\mathcal{Z}_y]}(t)\, \mathrm{d}t\, \mathbb{N}_y(\mathrm{d}\omega)$. We refer to the Appendix of [17] for a proof. By combining the special Markov property with the fact that the "law" of W_* under $\mathbb{N}_x$ has no atoms, one easily gets that, for every fixed $z \in (-\infty, x)$, z is $\mathbb{N}_x$ a.e. not a local minimum of the function $s \mapsto \widehat{W}_s$.

2.4 A technical lemma

In this section, we establish a lemma that will be useful in forthcoming proofs. This lemma is a direct consequence of arguments used in the proof of [18, Proposition 31], which was the key result needed for the extension of the distance to the boundary in the construction of the Brownian disk presented in [18]. We use the notation $\mathbb{N}_r^{[0]} := \mathbb{N}_r(\cdot \mid W_* \le 0)$ for every $r > 0$. Under $\mathbb{N}_r^{[0]}(\mathrm{d}\omega)$, we write $\widetilde{\omega} = \mathrm{tr}_0(\omega)$ to simplify notation. Recall that $\ell_a(\widetilde{\omega}) = \widehat{W}_s(\widetilde{\omega})$ if $a = p_{(\widetilde{\omega})}(s)$, and note that $\ell_a(\widetilde{\omega}) \ge 0$ for every $a \in \mathcal{T}_{(\widetilde{\omega})}$, $\mathbb{N}_r^{[0]}(\mathrm{d}\omega)$ a.s. We define the "boundary" of the tree $\mathcal{T}_{(\widetilde{\omega})}$ as the set $\partial\mathcal{T}_{(\widetilde{\omega})} := \{a \in \mathcal{T}_{(\widetilde{\omega})} : \ell_a(\widetilde{\omega}) = 0\}$. We set, for every $s, t \in [0, \sigma(\widetilde{\omega})]$,

$$\Delta^\circ_{(\widetilde{\omega})}(s,t) := \widehat{W}_s(\widetilde{\omega}) + \widehat{W}_t(\widetilde{\omega}) - 2 \min_{s\wedge t \le r \le s\vee t} \widehat{W}_r(\widetilde{\omega})$$

if the minimum in the last display is positive, and $\Delta^\circ_{(\widetilde{\omega})}(s,t) = \infty$ otherwise. We then set, for every $a, b \in \mathcal{T}_{(\widetilde{\omega})} \backslash \partial\mathcal{T}_{(\widetilde{\omega})}$,

$$\Delta^\circ_{(\widetilde{\omega})}(a,b) := \min\{\Delta^\circ_{(\widetilde{\omega})}(s,t) : s,t \in [0,\sigma(\widetilde{\omega})],\, p_{(\widetilde{\omega})}(s) = a, p_{(\widetilde{\omega})}(t) = b\},$$

and

$$\Delta_{(\tilde{\omega})}(a,b) := \inf\Big\{\sum_{i=1}^{p} \Delta^{\circ}_{(\tilde{\omega})}(a_{i-1},a_i)\Big\}$$

where the infimum is over all choices of the integer $p \geq 1$ and of $a_0 = a, a_1, \ldots, a_{p-1}, a_p = b$ in $\mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})}$. It is not hard to verify that the mapping $(a,b) \mapsto \Delta_{(\tilde{\omega})}(a,b)$ takes finite values and is continuous on $(\mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})}) \times (\mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})})$ — see the comments following Proposition 30 in [18].

Lemma 4. $\mathbb{N}^{[0]}_r$ *a.s., the mapping* $(a,b) \mapsto \Delta_{(\tilde{\omega})}(a,b)$ *has a unique continuous extension to* $\mathcal{T}_{(\tilde{\omega})} \times \mathcal{T}_{(\tilde{\omega})}$. *Moreover, there exists a finite constant* C *such that*

$$\mathbb{N}^{[0]}_r\Big(\sup_{a,b\in\mathcal{T}_{(\tilde{\omega})}} \Delta_{(\tilde{\omega})}(a,b)\Big) = C\,r.$$

Proof. By scaling, it is enough to consider the case $r = 1$. Let us start by proving the first assertion. Since $\Delta_{(\tilde{\omega})}$ satisfies the triangle inequality, it is enough to verify that, for any $a \in \partial\mathcal{T}_{(\tilde{\omega})}$, if $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})}$ that converges to a, we have $\Delta_{(\tilde{\omega})}(a_n,a_m) \longrightarrow 0$ as $n,m \to \infty$. To get this, write m_c for the minimal label along the ancestral line of c in $\mathcal{T}_{(\tilde{\omega})}$, for every $c \in \mathcal{T}_{(\tilde{\omega})}$, and, for every $\delta > 0$, let $\mathcal{C}^{\delta}_j$, $j \in \{1,\ldots,N_\delta\}$, be those connected components of the open set $\{c \in \mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})} : m_c < \delta\}$ whose closure intersects the "boundary" $\partial\mathcal{T}_{(\tilde{\omega})}$. By formula (53) in [18], we have

$$\sup_{1\leq j\leq N_\delta}\Big(\sup_{b,b'\in\mathcal{C}^{\delta}_j} \Delta_{(\tilde{\omega})}(b,b')\Big) \underset{\delta\to 0}{\longrightarrow} 0$$

(note that formula (53) in [18] deals with a function $\Delta(x,y)$ which is defined in a slightly different way than $\Delta_{(\tilde{\omega})}(x,y)$, but the arguments apply as well to $\Delta_{(\tilde{\omega})}(x,y)$). For every fixed $\delta > 0$, there is a unique index j such that a belongs to the closure of $\mathcal{C}^{\delta}_j$, and, for n large enough, a_n must belong to $\mathcal{C}^{\delta}_j$. The desired convergence of $\Delta_{(\tilde{\omega})}(a_n,a_m)$ to 0 then follows from the last display.

Let us turn to the second assertion. To simplify notation, we write ℓ_a instead of $\ell_a(\tilde{\omega})$ and ρ instead of $\rho_{(\tilde{\omega})}$. Our goal is to verify that

$$\mathbb{N}^{[0]}_1\Big(\sup_{a\in\mathcal{T}_{(\tilde{\omega})}} \Delta_{(\tilde{\omega})}(\rho,a)\Big) < \infty,$$

which will immediately give the second assertion (for $r = 1$) since $\Delta_{(\tilde{\omega})}$ satisfies the triangle inequality. We need to recall some ingredients of the proof of Proposition 31 in [18]. We first introduce the reduced tree of $\mathcal{T}_{(\tilde{\omega})}$, which consists of all points a of $\mathcal{T}_{(\tilde{\omega})}\backslash\partial\mathcal{T}_{(\tilde{\omega})}$ that have at least one descendant with label 0 (a belongs to the reduced tree if there exists $b \in \mathcal{T}_{(\tilde{\omega})}$ such that $\ell_b = 0$ and $a \in [[\rho,b]]$). Let $\mathcal{T}^{\triangledown}$ stand for this subtree. Then the tree $\mathcal{T}^{\triangledown}$ is a binary $\mathbb{R}$-tree, which can be constructed by induction as follows. One starts from a line segment connecting the root ρ to a first branching point $a_\varnothing$. To this branching point are attached two other line segments connecting $a_\varnothing$ to branching points a_1 and a_2, listed in the order prescribed by the exploration $t \mapsto p_{(\tilde{\omega})}(t)$ of $\mathcal{T}_{(\tilde{\omega})}$. To a_1 (respectively to a_2) are then attached two line segments connecting a_1 (resp. a_2) to branching points $a_{(1,1)}$ and $a_{(1,2)}$ (resp. $a_{(2,1)}$ and $a_{(2,2)}$) and so on. The reason for introducing this reduced tree is the bound

$$\sup_{a\in\mathcal{T}_{(\tilde{\omega})}} \Delta_{(\tilde{\omega})}(\rho,a) \leq 2 \sup_{(i_1,i_2,\ldots)\in\{1,2\}^{\mathbb{N}}}\left(\sum_{n=0}^{\infty} \ell_{a_{(i_1,\ldots,i_n)}}\right) + 4\sup_{a\in\mathcal{T}_{(\tilde{\omega})}} \ell_a, \tag{7}$$

which easily follows from the fact that $\Delta_{(\tilde{\omega})}(a_{(i_1,\ldots,i_{n-1})},a_{(i_1,\ldots,i_n)}) \leq \ell_{a_{(i_1,\ldots,i_{n-1})}} + \ell_{a_{(i_1,\ldots,i_n)}}$ (see the end of the proof of [18, Proposition 31] for more details). The second term in the right-hand side of (7) has finite expectation under $\mathbb{N}^{[0]}_1$ because, for every $x > 1$,

$$\mathbb{N}^{[0]}_1\Big(\sup_{a\in\mathcal{T}_{(\tilde{\omega})}} \ell_a > x\Big) \leq \frac{2}{3}\,\mathbb{N}_1\Big(\sup_{a\in\mathcal{T}_{(\tilde{\omega})}} \ell_a > x\Big) \leq (x-1)^{-2},$$

using (2). So it remains to verify that the first term in the right-hand side of (7) also has finite expectation under $\mathbb{N}_1^{[0]}$. To this end, we rely on the formula

$$\mathbb{N}_1^{[0]}\Big((\ell_{a_{(i_1,\dots,i_n)}})^{5/2}\Big) = \Big(\frac{24}{49}\Big)^{n+1},$$

which is obtained in the proof of [18, Proposition 31] as a consequence of the recursive structure of the tree $\mathcal{T}^{\nabla}$. We fix $\alpha \in (0,1)$ such that $2\alpha^{-5/2} < 49/24$. Then, for every $x > 1$,

$$\begin{aligned}
\mathbb{N}_1^{[0]}\Bigg(\sup_{(i_1,i_2,\dots)\in\{1,2\}^{\mathbb{N}}}\Bigg(\sum_{n=0}^{\infty}\ell_{a_{(i_1,\dots,i_n)}}\Bigg) > x\Bigg) &\leq \sum_{n=0}^{\infty}\mathbb{N}_1^{[0]}\Bigg(\Big(\sup_{(i_1,i_2,\dots,i_n)\in\{1,2\}^n}\ell_{a_{(i_1,\dots,i_n)}}\Big) > (1-\alpha)\alpha^n x\Bigg)\\
&\leq \sum_{n=0}^{\infty} 2^n \sup_{(i_1,i_2,\dots,i_n)\in\{1,2\}^n}\mathbb{N}_1^{[0]}\Big(\ell_{a_{(i_1,\dots,i_n)}} > (1-\alpha)\alpha^n x\Big)\\
&\leq \sum_{n=0}^{\infty} 2^n \times ((1-\alpha)\alpha^n x)^{-5/2} \times \Big(\frac{24}{49}\Big)^{n+1}\\
&= c\,x^{-5/2}
\end{aligned}$$

with some constant $c < \infty$. This completes the proof. □

3 Peeling of a triangulation with a boundary

Our goal in this section is to discuss certain properties of Boltzmann distributed pointed triangulations with a simple boundary. More precisely, we are interested in the discrete hull whose radius is the distance from the distinguished vertex to the boundary. Thanks to the results of [2], this study will allow us to derive similar properties for the free pointed Brownian disk. In our investigation of Boltzmann distributed triangulations, it will be convenient to use the peeling algorithm.

3.1 Peeling probabilities

For integers $L \geq 1$ and $k \geq 0$, we let $\mathbb{T}^1(L,k)$ be the set of all rooted planar triangulations of type I (i.e. loops and multiple edges are allowed) with a simple boundary of length L and k inner vertices. By convention, triangulations with a simple boundary are rooted on the boundary in such a way that the external face (of degree L) lies to the left of the root edge (see e.g. the introduction of [2] for a more detailed presentation of triangulations with a boundary). Then (see e.g. Theorem 1.1 in [5]), $\mathbb{T}^1(1,0) = \varnothing$ and, for $(L,k) \neq (1,0)$,

$$\#\mathbb{T}^1(L,k) = 4^{k-1}\,\frac{(2L+3k-5)!!}{k!\,(2L+k-1)!!}\,L\binom{2L}{L} \underset{k\to\infty}{\sim} C^{(1)}(L)(12\sqrt{3})^k\,k^{-5/2}, \tag{8}$$

where

$$C^{(1)}(L) := \frac{3^{L-2}}{4\sqrt{2\pi}}\,L\binom{2L}{L} \underset{L\to\infty}{\sim} \frac{1}{36\pi\sqrt{2}}\sqrt{L}\,12^L. \tag{9}$$

(When $L=2$ and $k=0$, formula (8) is valid with the convention $(-1)!!=1$, provided we consider the “trivial triangulation” as in [11].) Assuming that $L \geq 2$, we have

$$Z(L) := \sum_{k=0}^{\infty}(12\sqrt{3})^{-k}\,\#\mathbb{T}^1(L,k) = \frac{6^L(2L-5)!!}{8\sqrt{3}\,L!}. \tag{10}$$

(see e.g. [3, Section 2.2]). We set $\mathbb{T}^1(L) := \bigcup_{k\geq 0}\mathbb{T}^1(L,k)$. A random triangulation τ in $\mathbb{T}^1(L)$ is said to be Boltzmann distributed if $\mathbb{P}(\tau = \theta) = Z(L)^{-1}(12\sqrt{3})^{-k}$ for every $k \geq 0$ and $\theta \in \mathbb{T}^1(L,k)$. We will also consider rooted and pointed triangulations with a boundary, which in addition to the root edge have a distinguished vertex, which can be any inner vertex of the triangulation. We can then define Boltzmann distributed rooted and pointed planar triangulations in exactly the same way as we did in the non-pointed case (using the fact that $\sum_{k=0}^{\infty} k\,(12\sqrt{3})^{-k}\#\mathbb{T}^1(L,k) < \infty$, by (8)).

For integers $L \geq 1$, $p \geq 1$, and $k \geq 0$, let $\mathbb{T}^2(L,p,k)$ be the set of all planar triangulations with two simple boundaries of respective lengths L and p, and k inner vertices, that are rooted on both boundaries (with the same convention for the orientation of the root edges). Notice that we distinguish the first and the second boundary, and that the size of the first one is L. According to [14] (see also [5]),

$$\#\mathbb{T}^2(L,p,k) = \frac{4^k\,(2(L+p)+3k-2)!!}{k!\,(2(L+p)+k)!!}\,L\binom{2L}{L}\,p\binom{2p}{p}. \tag{11}$$

Set

$$Z'(L,p) := \sum_{k=0}^{\infty} (12\sqrt{3})^{-k}\,\#\mathbb{T}^2(L,p,k).$$

Using calculations in Krikun [14], one checks that

$$Z'(L,p) = \frac{1}{2}\,\frac{3^{L+p}}{L+p}\,L\binom{2L}{L}\,p\binom{2p}{p} = \frac{6^4\,\pi}{L+p} C^{(1)}(L) \times C^{(1)}(p). \tag{12}$$

In the appendix below, we explain how formula (12) can be deduced from [14].

From now on, we always assume that $L \geq 2$. Let $\mathbb{T}^2(L,p)$ stand for the union of all $\mathbb{T}^2(L,p,k)$ for $k \geq 0$. Consider a random triangulation τ of $\mathbb{T}^2(L,p)$ distributed according to Boltzmann weights. This means that, if θ is a given triangulation of $\mathbb{T}^2(L,p,k)$ for some $k \geq 0$,

$$\mathbb{P}(\tau = \theta) = Z'(L,p)^{-1}\,(12\sqrt{3})^{-k}.$$

Consider a given edge of the second boundary of τ. This edge, which will be called the revealed edge, is chosen in a deterministic manner given the root of the second boundary. Let Δ be the triangle incident to this edge, which is called the revealed triangle. Several configurations may occur (see Fig. 1 for an illustration).

1. The third vertex of Δ does not lie on any of the two boundaries. Then, if we “remove” Δ from τ, we get a triangulation of $\mathbb{T}^2(L,p+1,k)$ for some $k \geq 0$ — the root edge on the second boundary can be chosen again in a deterministic manner from the position of the second root edge in τ. Hence configuration 1 occurs if and only if τ is obtained by filling the space between the first boundary and the (new) second boundary by a triangulation of $\mathbb{T}^2(L,p+1,k)$ for some $k \geq 0$. For any fixed choice of the latter triangulation, the probability of the corresponding event is

$$Z'(L,p)^{-1}\,(12\sqrt{3})^{-k-1}.$$

Finally, the probability of configuration 1 is

$$Z'(L,p)^{-1}\,\sum_{k=0}^{\infty}(12\sqrt{3})^{-k-1}\,\#\mathbb{T}^2(L,p+1,k) = \frac{1}{12\sqrt{3}}\,\frac{Z'(L,p+1)}{Z'(L,p)},$$

and this quantity is also equal to

$$\frac{1}{12\sqrt{3}}\,\frac{L+p}{L+p+1}\,\frac{C^{(1)}(p+1)}{C^{(1)}(p)}. \tag{13}$$

2. The third vertex of Δ belongs to the second boundary, and the revealed triangle disconnects the first boundary from m edges of the second boundary, where $m \in \{0,1,\ldots,p-1\}$, and these edges may lie either to the right or to the left of the revealed edge. Consider the case where these edges lie to the right of the revealed edge (the other case is symmetric).

The complement of the revealed triangle in the initial triangulation has two connected components (when $m=1$, one of them may be the trivial triangulation). The one incident to the first boundary must be filled by a triangulation of $\mathbb{T}^2(L,p-m,k)$ for some $k \geq 0$, and the other one is filled by a triangulation of $\mathbb{T}^1(m+1,j)$ for some $j \geq 0$ ($j \geq 1$ if $m=0$). If these two triangulations are fixed, the probability of the resulting event is

$$Z'(L,p)^{-1}\,(12\sqrt{3})^{-k-j}.$$

Hence, the probability of the configuration is

$$Z'(L,p)^{-1} \sum_{k,j=0}^{\infty} (12\sqrt{3})^{-k-j} \, \#\mathbb{T}^2(L,p-m,k) \, \#\mathbb{T}^1(m+1,j) = Z(m+1) \, \frac{Z'(L,p-m)}{Z'(L,p)}.$$

The last quantity is also equal to

$$\frac{L+p}{L+p-m} \, Z(m+1) \, \frac{C^{(1)}(p-m)}{C^{(1)}(p)}. \tag{14}$$

3. The third vertex of the revealed triangle Δ belongs to the first boundary. To evaluate the probability of this event, we first notice that there are L possible choices for the third vertex. Then, given the revealed triangle, the initial triangulation τ is determined from a triangulation of $\mathbb{T}^1(L+p+1,k)$ for some $k \geq 0$, and if this triangulation is given, the probability is

$$Z'(L,p)^{-1} \, (12\sqrt{3})^{-k}.$$

The probability of configuration 3 is thus

$$L \times Z'(L,p)^{-1} \sum_{k=0}^{\infty} (12\sqrt{3})^{-k} \, \#\mathbb{T}^1(L+p+1,k) = L \, \frac{Z(L+p+1)}{Z'(L,p)}.$$

When L is large, we have

$$Z(L+p+1) \sim \frac{\sqrt{3}}{8\sqrt{\pi}} \, 12^{L+p} \, (L+p)^{-5/2},$$

uniformly in p (cf. Section 6.1 in [11]), whereas

$$Z'(L,p) \sim \frac{3^{L+p} \times 4^L}{2(L+p)} \sqrt{\frac{L}{\pi}} \, p \binom{2p}{p}.$$

It follows that the probability of configuration 3 behaves, when L and p are large, like

$$\frac{\sqrt{3\pi}}{4} \sqrt{\frac{L}{p}} \, (L+p)^{-3/2}.$$

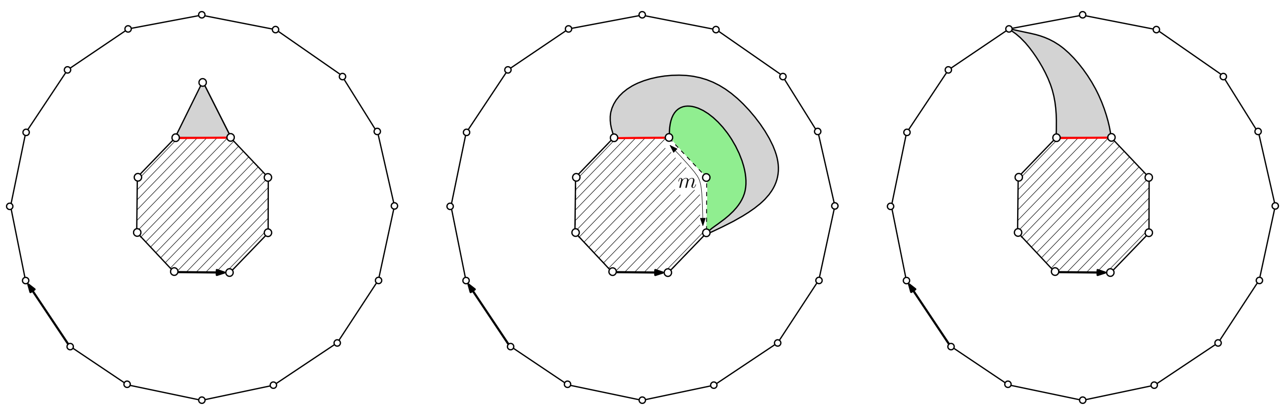

Figure 1: Illustration from left to right of configurations 1,2 and 3.

3.2 The peeling process

In this section, the integer $L \geq 2$ is fixed, and we also fix an integer $p_0 \geq 1$. As previously, we consider a Boltzmann triangulation τ^L in $\mathbb{T}^2(L,p_0)$. We define a peeling algorithm giving rise to a sequence $(\tau_n^L)_{0 \leq n < \zeta_L}$ of triangulations with two boundaries, where $\zeta_L \geq 1$ is a random integer. Precisely, we take $\tau_0^L = \tau^L$, and then we proceed inductively as follows.

At step n, assuming that $n < \zeta_L$, we choose an edge of the second boundary of τ_n^L and reveal the triangle incident to this edge in the way explained in the previous section. If configuration 1 occurs, we let τ_{n+1}^L be obtained by removing the revealed triangle in τ_n^L. If configuration 2 occurs, τ_{n+1}^L is obtained by removing both the revealed triangle and those triangles that are disconnected by the revealed triangle from the first boundary. Finally, if configuration 3 occurs, we take $\zeta_L := n+1$.

The preceding description is a little informal since we need to specify how the revealed edge is chosen at each step. To this end, define, for every $n < \zeta_L$, the planar map $\check{\tau}_n^L$ obtained by considering the second boundary of τ^L and all triangles of τ^L that do not appear in τ_n^L. We view $\check{\tau}_n^L$ as a rooted planar map whose root edge is the root of the second boundary of τ^L and call $\check{\tau}_n^L$ the revealed region at step n. We also define the revealed region at step ζ_L by adding to $\check{\tau}_{\zeta_L-1}^L$ the revealed triangle at step ζ_L (which necessarily has a vertex on the first boundary of $\mathcal{T}^L$). Then, at each step $n < \zeta_L$, the choice of the revealed edge is made as a deterministic function of $\check{\tau}_n^L$. Furthermore, τ_n^L is viewed as a triangulation with two boundaries, and the first root edge is the same as in τ^L, whereas the second root edge is chosen as a deterministic function of $\check{\tau}_n^L$.

For $n < \zeta_L$, write P_n^L for the size of the second boundary of τ_n^L and set $P_n^L = \dagger$ for $n \geq \zeta_L$, where $\dagger$ serves as a cemetery point. From the discussion of the previous section, it should be clear that conditionally on the event $\{n < \zeta_L, P_n^L = k\}$, τ_n^L is distributed as a Boltzmann triangulation of $\mathbb{T}^2(L,k)$. Furthermore, $(P_n^L)_{n\geq 0}$ is a Markov chain with values in $\mathbb{N} \cup \{\dagger\}$ with transition probabilities

$$\mathbb{P}(P_{n+1}^L = p+1 \mid P_n^L = p) = \frac{1}{12\sqrt{3}} \frac{Z'(L,p+1)}{Z'(L,p)} =: q_L(p,p+1)$$

and, for every $m \in \{0,1,\ldots,p-1\}$,

$$\mathbb{P}(P_{n+1}^L = p-m \mid P_n^L = p) = 2\,\frac{Z'(L,p-m)}{Z'(L,p)}\,Z(m+1) =: q_L(p,p-m),$$

and finally

$$\mathbb{P}(P_{n+1}^L = \dagger \mid P_n^L = p) = 1 - q_L(p,p+1) - \sum_{m=0}^{p-1} q_L(p,p-m) =: q_L(p,\dagger).$$

Now recall (13) and (14), and use the notation $q_\infty(p,j)$ for the transition probabilities of the peeling process of the type I UIPT, which is discussed in [11, Section 6.1]. According to [11], the nonzero values of $q_\infty(p,j)$ are determined as follows. For every integer $p \geq 1$, we have

$$q_\infty(p,p+1) = \frac{1}{12\sqrt{3}} \frac{C^{(1)}(p+1)}{C^{(1)}(p)},$$

and, for every $m \in \{0,1,\ldots,p-1\}$,

$$q_\infty(p,p-m) = 2\,Z(m+1)\,\frac{C^{(1)}(p-m)}{C^{(1)}(p)}.$$

Comparing the last two displays with (13) and (14), we see that we have

$$q_L(p,p+j) = \frac{L+p}{L+p+j}\,q_\infty(p,p+j)$$

for every $j \in \{1,0,-1,-2,\ldots,-p+1\}$. In other words, $(P_n^L)_{n\geq 0}$ is a h-transform of the peeling process of the UIPT, for the function $h = h_L$ defined by

$$h_L(j) := \frac{L}{L+j},$$

for $j = 1,2,\ldots$.

3.3 Asymptotics for the peeling process

We now want to derive asymptotics when $L \to \infty$ (the integer p_0 remains fixed). We set $Z_L = P^L_{\zeta_L - 1} + 1$ (thus $L + Z_L$ is interpreted as the boundary size of the triangulation that needs to be "pasted" to the revealed region at time ζ_L to recover τ^L). We also write $(P^\infty_n)_{n\geq 0}$ for the Markov chain with transition probabilities $q_\infty(p, j)$ started at p_0, which is known to be transient [11].

Proposition 5. *We have*

$$\frac{Z_L}{L} \xrightarrow[L\to\infty]{\text{(d)}} \mathcal{Z},$$

where $\mathcal{Z}$ has density $\frac{3}{2}(1+x)^{-5/2}$ on $\mathbb{R}_+$.

Proof. For every integer $j \geq 1$, we get, using the h-transform relation between the Markov chains $(P^L_n)_{n\geq 0}$ and $(P^\infty_n)_{n\geq 0}$,

$$\begin{aligned}
\mathbb{P}(Z_L = j+1) &= \sum_{n=0}^{\infty} \mathbb{P}(P^L_n = j, \zeta_L = n+1) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(P^L_n = j)\, q_L(j, \dagger) \\
&= \frac{1}{h_L(p_0)} \sum_{n=0}^{\infty} \mathbb{P}(P^\infty_n = j)\, h_L(j)\, q_L(j, \dagger) \\
&= \frac{1}{h_L(p_0)}\, U(p_0, j)\, h_L(j)\, q_L(j, \dagger),
\end{aligned} \tag{15}$$

where we have written $U(k, \ell)$ for the potential kernel of the Markov chain $(P^\infty_n)_{n\geq 0}$. We can explicitly compute $U(p_0, j)$ when $j \geq p_0$. To this end, set for every integer $k \in \{1, 0, -1, -2, \ldots\}$,

$$q_k := \lim_{p\to\infty} q_\infty(p, p+k) = \begin{cases} \dfrac{1}{\sqrt{3}} & \text{if } k = 1, \\ 2\, Z(k+1)\, 12^{-k} & \text{if } k \leq 0. \end{cases}$$

From [11], $(q_k)_{k\leq 1}$ defines a probability measure with mean zero on $\mathbb{Z}$. Let $(S_n)_{n\geq 0}$ denote the (recurrent) random walk with jump distribution $(q_k)_{k\leq 1}$, and set $T^S_0 = \inf\{n \geq 0 : S_n \leq 0\}$. Consider the killed random walk $(S^\bullet_n)_{n\geq 0}$ defined by $S^\bullet_n = S_n$ if $n < T^S_0$ and $S^\bullet_n = \dagger$ if $n \geq T^S_0$. According to [11], $(P^\infty_n)_{n\geq 0}$ is the h-transform of the Markov chain $(S^\bullet_n)_{n\geq 0}$, for the function

$$h_\bullet(p) := 12^{-p} C^{(1)}(p)$$

for every $p \geq 1$. For the random walk S started from 0, the expected number of visits of $j \geq 1$ before the first return to 0 is equal to 1, and is also equal to $1/\sqrt{3}$ times the expected number of visits of j for the Markov chain $S^\bullet$ started from 1. So, if $U^\bullet$ denotes the potential kernel of $S^\bullet$, we have $U^\bullet(1, j) = \sqrt{3}$ for every $j \geq 1$. The h-transform relation between the Markov chains $(P^\infty_n)_{n\geq 0}$ and $(S^\bullet_n)_{n\geq 0}$ then gives

$$U(1, j) = \frac{h_\bullet(j)}{h_\bullet(1)}\, \sqrt{3},$$

and since $(P^\infty_n)_{n\geq 0}$ is transient and its positive jumps are of size 1, it is immediate that $U(p, j) = U(1, j)$ whenever $p \leq j$. From (9), we have $h_\bullet(1) = 1/(72\sqrt{2\pi})$ and

$$h_\bullet(p) \underset{p\to\infty}{\sim} \frac{1}{36\pi\sqrt{2}}\, \sqrt{p}.$$

It follows that, for $j \geq p_0$,

$$U(p_0, j) = 72\sqrt{6\pi}\, h_\bullet(j) \underset{j\to\infty}{\sim} \frac{2\sqrt{3}}{\sqrt{\pi}}\, \sqrt{j}.$$

Now recall formula (15). From the end of Section 3.1, we know that $q_L(j,\dagger)$ behaves like

$$\frac{\sqrt{3\pi}}{4}\sqrt{\frac{L}{j}}\,(L+j)^{-3/2}$$

when both L and j are large. We thus get

$$\mathbb{P}(Z_L=j+1)\underset{L,j\to\infty}{\sim}\frac{2\sqrt{3}}{\sqrt{\pi}}\sqrt{j}\times\frac{L}{L+j}\times\frac{\sqrt{3\pi}}{4}\sqrt{\frac{L}{j}}\,(L+j)^{-3/2}=\frac{3}{2}\,L^{3/2}\,(L+j)^{-5/2}.$$

The result of the proposition follows. □

3.4 Convergence of rescaled triangulations

For every integer $L\geq 1$, let $\mathcal{T}'_L$ be a Boltzmann distributed rooted triangulation with a simple boundary of size L. We let $\mathrm{d_{gr}}$ stand for the graph distance on the vertex set $V(\mathcal{T}'_L)$. We write $\partial\mathcal{T}'_L$ for the set of all boundary vertices and we denote the set of all inner vertices by $V_i(\mathcal{T}'_L):=V(\mathcal{T}'_L)\backslash\partial\mathcal{T}'_L$. We also let ν'_L be the counting measure on $V_i(\mathcal{T}'_L)$ scaled by the factor $\frac{3}{4}L^{-2}$. We finally consider the "boundary path" $\Theta'_L=(\Theta'_L(k))_{0\leq k\leq L}$, which is obtained by letting $\Theta'_L(0)=\Theta'_L(L)$ be the root vertex of $\mathcal{T}'_L$ and then letting $\Theta'_L(1),\Theta'_L(2),\ldots,\Theta'_L(L-1)$ be the points of $\partial\mathcal{T}'_L$ enumerated in clockwise order from $\Theta'_L(0)$. We also set $\widehat{\Theta}'_L(t)=\Theta'_L(\lfloor Lt\rfloor)$ for $t\in[0,1]$. According to Theorem 1.1 of [2] we have

$$(V(\mathcal{T}'_L),\sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}},\nu'_L,\widehat{\Theta}'_L)\xrightarrow[L\to\infty]{(d)}(\mathbb{D}',D',\mathbf{V}',\Gamma'). \tag{16}$$

where the convergence holds in distribution in $(\mathbb{M}^{GHPU},\mathrm{d}_{GHPU})$, and $(\mathbb{D}',D',\mathbf{V}',\Gamma')$ is a curve-decorated free Brownian disk of perimeter 1 (the precise definition of this limiting space will be given below in Section 4). We note that $\mathbb{E}[\mathbf{V}'(\mathbb{D}')]=1$ (the density of $\mathbf{V}'(\mathbb{D}')$ is the function $r\mapsto(2\pi r^5)^{-1/2}\exp(-1/(2r))$, cf. [7, Section 1.5]).

The convergence (16) seems to be stated incorrectly since the paths $\widehat{\Theta}'_L$ are *not continuous* and thus $(V(\mathcal{T}'_L),\sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}},\nu'_L,\widehat{\Theta}'_L)$ does not belong to the space $\mathbb{M}^{GHPU}$. There is however a straightforward way of overcoming this difficulty, by replacing $V(\mathcal{T}'_L)$ with the union of all its edges, each edge being represented by a copy of the interval $[0,1]$, so that the boundary path can be made continuous and its range will be the union of the boundary edges — see [2] for more details.

We now want to argue that a result similar to (16) holds for rooted *and pointed* triangulations. So, for every integer $L\geq 1$, let $\mathcal{T}_L$ be a Boltzmann distributed rooted and pointed triangulation with a simple boundary of size L. We define $V_i(\mathcal{T}_L)$, ν_L and Θ_L in the same way as $V_i(\mathcal{T}'_L)$, ν'_L and Θ'_L were defined above, and we also write $v_*^{(L)}$ for the distinguished vertex of $\mathcal{T}_L$. Then, we have

$$(V(\mathcal{T}_L),\sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}},\nu_L,\widehat{\Theta}_L,v_*^{(L)})\xrightarrow[L\to\infty]{(d)}(\mathbb{D},D,\mathbf{V},\Gamma,\mathbf{x}_*), \tag{17}$$

where the convergence holds in distribution in $(\mathbb{M}^{GHPU\bullet},\mathrm{d}_{GHPU\bullet})$, and the limit $(\mathbb{D},D,\mathbf{V},\Gamma,\mathbf{x}_*)$ is now a curve-decorated free pointed Brownian disk of perimeter 1 (see Section 4 below).

Let us explain why (17) follows from (16). To simply notation, write $\mathbf{X}^L$, respectively $\mathbf{X}'^L$, for the space $(V(\mathcal{T}_L),\sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}},\nu_L,\widehat{\Theta}_L,v_*^{(L)})$, resp. for $(V(\mathcal{T}'_L),\sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}},\nu'_L,\widehat{\Theta}'_L)$. Also write $\langle\nu'_L,1\rangle$ for the total mass of ν'_L. Then, for every bounded continuous function F on $(\mathbb{M}^{GHPU\bullet},\mathrm{d}_{GHPU\bullet})$,

$$\mathbb{E}[F(\mathbf{X}^L)]=\frac{\mathbb{E}\Big[\sum_{x\in V_i(\mathcal{T}'_L)}F((\mathbf{X}'^L,x))\Big]}{\mathbb{E}[\#V_i(\mathcal{T}'_L)]}=\frac{\mathbb{E}\Big[\int\nu'_L(dx)F((\mathbf{X}'^L,x))\Big]}{\mathbb{E}[\langle\nu'_L,1\rangle]},$$

where $(\mathbf{X}'^L,x)$ obviously denotes the pointed space derived from $\mathbf{X}'_L$ by distinguishing the point x. We then claim that

$$\mathbb{E}[\langle\nu'_L,1\rangle]\underset{L\to\infty}{\longrightarrow}1=\mathbb{E}[\mathbf{V}'(\mathbb{D}')]. \tag{18}$$

Assuming that (18) holds, we can apply Lemma 2, which implies that $\mathbf{X}^L$ converges in distribution (in the space $(\mathbb{M}^{GHPU\bullet}, \mathrm{d}_{GHPU\bullet})$) to the random space $\mathbf{X}^\infty$ whose law is characterized by

$$\mathbb{E}[F(\mathbf{X}^\infty)] = \mathbb{E}\Big[\int \mathbf{V}'(\mathrm{d}x)\, F((\mathbb{D}', D', \mathbf{V}', \Gamma', x))\Big].$$

The last display exactly means that $\mathbf{X}^\infty$ is a (curve-decorated) free pointed Brownian disk of perimeter 1 — see e.g. the discussion in [19, Section 6]. It only remains to justify our claim (18). We already know (by (16)) that $\langle \nu'_L, 1\rangle$ converges in distribution to $\mathbf{V}'(\mathbb{D}')$, and therefore it suffices to verify that $\mathbb{E}[\langle \nu'_L, 1\rangle\, \mathbf{1}_{\{\langle \nu'_L, 1\rangle \geq a\}}]$ tends to 0 as $a \to +\infty$, uniformly in L. This can be checked from the explicit formulas (8),(9),(10) and we omit the details.

4 The limiting space

4.1 The Bettinelli-Miermont construction

In this section, we recall the Bettinelli-Miermont construction of the free Brownian disk [6, 7], which will play an important role in our proofs. We follow the presentation of Section 6 of [22], which is slightly different from [6, 7].

We fix $\xi > 0$, which will correspond to the boundary size of the Brownian disk. We consider a Brownian excursion $(\mathbf{e}_t)_{0\leq t\leq \xi}$ of duration ξ, and, conditionally on $(\mathbf{e}_t)_{0\leq t\leq \xi}$, a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{(t_i,\omega_i)}$ on $[0,\xi] \times \mathcal{S}$ with intensity

$$2\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\,\mathbf{e}_t}(\mathrm{d}\omega).$$

We let $\mathfrak{T}$ be the compact metric space obtained from the disjoint union

$$[0,\xi] \cup \Big(\bigcup_{i\in I} \mathcal{T}_{(\omega_i)}\Big) \tag{19}$$

by identifying the root $\rho_{(\omega_i)}$ of $\mathcal{T}_{(\omega_i)}$ with the point t_i of $[0,\xi]$, for every $i \in I$. The metric $\mathrm{d}_{\mathfrak{T}}$ on $\mathfrak{T}$ is defined as follows. First, the restriction of $\mathrm{d}_{\mathfrak{T}}$ to each tree $\mathcal{T}_{(\omega_i)}$ is the metric $d_{(\omega_i)}$. Then, if $u, v \in [0,\xi]$, we take $\mathrm{d}_{\mathfrak{T}}(u,v) := |v-u|$. If $u \in [0,\xi]$, and $v \in \mathcal{T}_{(\omega_i)}$ for some $i \in I$, we take $\mathrm{d}_{\mathfrak{T}}(u,v) := |u-t_i| + d_{(\omega_i)}(\rho_{(\omega_i)}, v)$. Finally if $u \in \mathcal{T}_{(\omega_i)}$ and $v \in \mathcal{T}_{(\omega_j)}$, with $j \neq i$, we let

$$\mathrm{d}_{\mathfrak{T}}(u,v) := d_{(\omega_i)}(u, \rho_{(\omega_i)}) + |t_i - t_j| + d_{(\omega_j)}(\rho_{(\omega_j)}, v).$$

The volume measure on $\mathfrak{T}$ is the sum of the volume measures on the trees $\mathcal{T}_{(\omega_i)}$, $i \in I$.

If $\Sigma := \sum_{i\in I} \sigma(\omega_i)$ is the total mass of the volume measure, we define a clockwise exploration $(\mathcal{E}_t)_{0\leq t\leq \Sigma}$ of $\mathfrak{T}$, informally by concatenating the mappings $p_{(\omega_i)} : [0, \sigma(\omega_i)] \longrightarrow \mathcal{T}_{(\omega_i)}$ in the order prescribed by the t_i's. To give a more precise definition, set

$$\beta_s := \sum_{i\in I} \mathbf{1}_{\{t_i \leq s\}}\, \sigma(\omega_i)\,,\ \beta_{s-} := \sum_{i\in I} \mathbf{1}_{\{t_i < s\}}\, \sigma(\omega_i)\,,$$

for every $s \in [0,\xi]$. Then, for every $t \in [0,\Sigma]$, we define $\mathcal{E}_t \in \mathfrak{T}$ as follows. We observe that there is a unique $s \in [0,\xi]$ such that $\beta_{s-} \leq t \leq \beta_s$, and:

- Either there is a (unique) $i \in I$ such that $s = t_i$, and we set $\mathcal{E}_t := p_{(\omega_i)}(t - \beta_{t_i-})$.
- Or there is no such i and we set $\mathcal{E}_t := s$.

Note that $\mathcal{E}_0 = 0$ and $\mathcal{E}_\Sigma = \xi$.

The clockwise exploration allows us to define "intervals" in $\mathfrak{T}$. Let us make the convention that, if $s, t \in [0,\Sigma]$ and $s > t$, the (real) interval $[s,t]$ is defined by $[s,t] := [s,\Sigma] \cup [0,t]$ (of course, if $s \leq t$, $[s,t]$ is the usual interval). Then, for every $u, v \in \mathfrak{T}$, such that $u \neq v$, there is a smallest interval $[s,t]$, with $s, t \in [0,\Sigma]$, such that $\mathcal{E}_s = u$ and $\mathcal{E}_t = v$, and we define

$$[|u,v|] := \{\mathcal{E}_r : r \in [s,t]\}.$$

Observe that in general $[|u,v|] \neq [|v,u|]$. We also take $[|u,u|] = \{u\}$. Note that we use the notation $[|u,v|]$ rather than $[u,v]$ to avoid confusion with intervals of the real line.

We then assign labels $(\ell_a)_{a\in\mathfrak{T}}$ to the points of $\mathfrak{T}$. If $a = s \in [0,\xi]$, we take $\ell_a := \sqrt{3}\,\mathbf{e}_s$, and if $a \in \mathcal{T}_{(\omega_i)}$ for some $i \in I$, we simply let ℓ_a be the label of a in $\mathcal{T}_{(\omega_i)}$. The function $a \mapsto \ell_a$ is continuous on $\mathfrak{T}$. The following simple fact will be important for us: For every $\varepsilon > 0$, formula (2) and the property $\int_0^\varepsilon \mathrm{d}t/(\mathbf{e}_t)^2 = \infty$ imply that some of the trees $\mathcal{T}_{(\omega_i)}$ such that $t_i < \varepsilon$ carry negative labels.

For every $a, b \in \mathfrak{T}$, we set

$$D^\circ(a,b) := \ell_a + \ell_b - 2\max\Big(\min_{c\in[|a,b|]} \ell_c, \min_{c\in[|b,a|]} \ell_c\Big).$$

Notice that $D^\circ(0,\xi) = 0$ (because $\ell_0 = \ell_\xi = 0$ and the "interval" $[|\xi,0|]$ is the pair $\{0,\xi\}$). We define a pseudo-metric on $\mathfrak{T}$ by setting, for every $a, b \in \mathfrak{T}$,

$$D(a,b) := \inf_{a_0=a,a_1,\dots,a_{p-1},a_p=b} \sum_{i=1}^{p} D^\circ(a_{i-1},a_i)$$

where the infimum is over all choices of the integer $p \geq 1$ and of the points $a_1,\dots,a_{p-1}$ in $\mathfrak{T}$. One can prove [6, Theorem 13] that $D(a,b) = 0$ if and only if $D^\circ(a,b) = 0$ (the "if" part is trivial). We set $\mathbb{D} := \mathfrak{T}/\{D = 0\}$, where the notation $\mathfrak{T}/\{D = 0\}$ refers to the quotient space of $\mathfrak{T}$ for the equivalence relation defined by setting $a \simeq b$ if and only if $D(a,b) = 0$, and this quotient space is equipped with the metric induced by D — similar notation will be used several times in what follows. It is immediate that $D(a,b) \geq |\ell_a - \ell_b|$, and therefore $D(a,b) = 0$ implies that $\ell_a = \ell_b$, so that we can make sense of labels on $\mathbb{D}$, for which we keep the same notation ℓ_x. We write Π for the canonical projection from $\mathfrak{T}$ onto $\mathbb{D}$. The volume measure $\mathbf{V}$ on $\mathbb{D}$ is the pushforward of the volume measure on $\mathfrak{T}$ under Π. The metric space $(\mathbb{D}, D)$ is a.s. homeomorphic to the closed unit disk of the plane [6], and, in any such homeomorphism, the unit circle corresponds to the "boundary" $\partial\mathbb{D} := \Pi([0,\xi])$ (which is therefore the set of all points of $\mathbb{D}$ that have no neighborhood homeomorphic to the open unit disk).

There is a unique point $a_* \in \mathfrak{T}$ such that $\ell_{a_*} = \min\{\ell_a : a \in \mathfrak{T}\} < 0$ and we set $\mathbf{x}_* = \Pi(a_*)$. For every $x \in \mathbb{D}$, we have $D(\mathbf{x}_*, x) = \ell_x - \ell_{\mathbf{x}_*}$. In particular, the distance from $\mathbf{x}_*$ to $\partial\mathbb{D}$ is $-\ell_{\mathbf{x}_*}$, and $\Pi(0) = \Pi(\xi)$ is the unique point of $\partial\mathbb{D}$ at minimal distance from x_*.

The free pointed Brownian disk with perimeter ξ may then be defined as the random measure metric space $(\mathbb{D}, D, \mathbf{V})$ with the distinguished point $\mathbf{x}_*$ but, for our purposes, it will be convenient to view $\mathbb{D}$ as a curve-decorated space. We first observe that the mapping $[0,\xi] \ni t \mapsto \Pi(t)$ is a simple loop (recall that $\Pi(0) = \Pi(\xi)$) whose range is $\partial\mathbb{D}$. More precisely, the loop $[0,\xi] \ni t \mapsto \Pi(t)$ is a *standard boundary curve* in the following sense. We first recall from [19, Theorem 9] that the measures $\varepsilon^{-2}\mathbf{1}_{\{D(x,\partial\mathbb{D})\leq\varepsilon\}}\mathbf{V}(\mathrm{d}x)$ converge weakly (a.s.) to a measure on the boundary $\partial\mathbb{D}$, which we denote by $\mu_{\partial\mathbb{D}}$ and whose total mass is the perimeter ξ of $\mathbb{D}$ (the measure $\mu_{\partial\mathbb{D}}$ is known as the uniform measure on the boundary of $\mathbb{D}$). We then say that $f : [0,\xi] \to \partial\mathbb{D}$ is a standard boundary curve of $\mathbb{D}$ if f is a simple loop whose range is $\partial\mathbb{D}$, and if the pushforward of Lebesgue measure on $[0,\xi]$ under f is $\mu_{\partial\mathbb{D}}$. The latter property is equivalent to

$$t = \lim_{\varepsilon\to 0} \varepsilon^{-2} \int \mathbf{V}(\mathrm{d}x)\mathbf{1}_{\{D(x,f([0,t]))\leq\varepsilon\}}, \qquad \text{for every } t \in [0,\xi]. \tag{20}$$

It then follows from [19, Theorem 9] that the loop $[0,\xi] \ni t \mapsto \Pi(t)$ is a standard boundary curve, and the same holds for the time-reversed loop $\check{\Pi}(t) := \Pi(\xi - t)$. Moreover, for every $x \in \partial\mathbb{D}$, there are exactly two standard boundary curves with starting point x, which are obtained by changing the origin of the loops $\Pi(t)$ and $\check{\Pi}(t)$.

We observe that the starting point $\Pi(0)$ of the loop $t \mapsto \Pi(t)$ is not a "typical" point of $\partial\mathbb{D}$, since it is the point of $\partial\mathbb{D}$ at minimal distance from $\mathbf{x}_*$. So we will consider the loop $t \mapsto \Pi(t)$ "re-rooted at a uniform boundary point". To this end, let $\mathfrak{U}$ be uniformly distributed over $[0,\xi]$ and independent of the random quantities involved in the definition of $\mathbb{D}$. We set $\Gamma(t) = \Pi(\mathfrak{U} + t)$ for $t \in [0, \xi - \mathfrak{U}]$, and $\Gamma(t) = \Pi(\mathfrak{U} + t - \xi)$ for $t \in (\xi - \mathfrak{U}, \xi]$. Then Γ is again a standard boundary curve (now rooted at a uniform boundary point).

The *curve-decorated free pointed Brownian disk with perimeter* ξ that appears (for $\xi = 1$) in formula (17) is the random space $(\mathbb{D}, D, \mathbf{V}, \Gamma, \mathbf{x}_*)$, which is a random variable taking values in $\mathbb{M}^{GHPU\bullet}$. The curve-decorated free Brownian disk with perimeter ξ (appearing in (16)) is then the random space $(\mathbb{D}', D', \mathbf{V}', \Gamma')$ in $\mathbb{M}^{GHPU}$ whose distribution is characterized by

$$\mathbb{E}[F((\mathbb{D}', D', \mathbf{V}', \Gamma'))] = \xi^2\, \mathbb{E}\Big[\frac{F((\mathbb{D}, D, \mathbf{V}, \Gamma))}{\mathbf{V}(\mathbb{D})}\Big]. \tag{21}$$

See the discussion at the beginning of [19, Section 6]. It will sometimes be convenient to "forget" the curve Γ' and to keep track only of its initial point: The pointed measure metric space $(\mathbb{D}', D', \mathbf{V}', \Gamma'(0))$ is the free Brownian disk of perimeter ξ *pointed at a uniform boundary point*, which is discussed in [19, Section 6] (informally, given the unpointed space $(\mathbb{D}', D', \mathbf{V}')$, the distinguished point $\Gamma'(0)$ is chosen according to the uniform measure on $\partial\mathbb{D}'$).

Let us finally discuss simple geodesics in $\mathbb{D}$. Let $x = \Pi(a)$ be a point of $\mathbb{D}$, and $r \in [0, \Sigma]$ such that $a = \mathcal{E}_r$. For every $t \leq \ell_x - \ell_* = D(\mathbf{x}_*, x)$, set

$$\varphi_r(t) := \begin{cases} \inf\{s \in [r, \Sigma] : \ell_{\mathcal{E}_s} = \ell_x - t\} & \text{if } \{s \in [r, \Sigma] : \ell_{\mathcal{E}_s} = \ell_x - t\} \neq \varnothing, \\ \inf\{s \in [0, r] : \ell_{\mathcal{E}_s} = \ell_x - t\} & \text{otherwise.} \end{cases}$$

Then $\big(\Pi(\mathcal{E}_{\varphi_r(t)})\big)_{0\leq t\leq D(\mathbf{x}_*, x)}$ is a geodesic from x to $\mathbf{x}_*$, which is called a simple geodesic. It is easy to verify that, if $x = \Pi(a)$ and $y = \Pi(b)$ are two points of $\mathbb{D}$, there are two simple geodesics starting from x and from y respectively that coalesce at a point whose label is

$$\max\Big(\min_{c\in[|a,b|]} \ell_c, \min_{c\in[|b,a|]} \ell_c\Big).$$

Consequently, the quantity $D^\circ(a, b)$ is the length of a path from $\Pi(a)$ to $\Pi(b)$ obtained by concatenating two simple geodesics up to the point where they merge.

4.2 Construction of the limiting space

We will now slightly modify the preceding construction of the Brownian disk to get another random metric space, which later will be identified to a particular subset of $\mathbb{D}$ equipped with an intrinsic metric. We start from the same Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{(t_i,\omega_i)}$ as in the previous section, but, for every $i \in I$, we now consider the truncation $\widetilde{\omega}_i := \mathrm{tr}_0(\omega_i)$ of ω_i at level 0, and we write $\mathrm{tr}_0(\mathcal{N}) = \sum_{i\in I} \delta_{(t_i,\widetilde{\omega}_i)}$. Recall that the genealogical tree $\mathcal{T}_{(\widetilde{\omega}_i)}$ is identified to the subtree obtained from $\mathcal{T}_{(\omega_i)}$ by pruning the branches when labels first hit 0. We then consider the geodesic space $\mathfrak{T}^\star$ which is obtained from the disjoint union

$$[0, \xi] \cup \Big(\bigcup_{i\in I} \mathcal{T}_{(\widetilde{\omega}_i)}\Big)$$

by identifying the root of $\mathcal{T}_{(\widetilde{\omega}_i)}$ with the point t_i of $[0, \xi]$, for every $i \in I$. Then $\mathfrak{T}^\star$ is identified to a closed subset of $\mathfrak{T}$, so that labels ℓ_a are defined for every $a \in \mathfrak{T}^\star$ and we can equip $\mathfrak{T}^\star$ with the restriction of the distance on $\mathfrak{T}$. We also define the volume measure on $\mathfrak{T}^\star$ as the sum of the volume measures on the trees $\mathcal{T}_{(\widetilde{\omega}_i)}$, $i \in I$.

We note that labels ℓ_a are nonnegative for every $a \in \mathfrak{T}^\star$, because we have replaced ω_i by $\mathrm{tr}_0(\omega_i)$. We set $\partial\mathfrak{T}^\star := \{a \in \mathfrak{T}^\star : \ell_a = 0\}$ and note that $0, \xi \in \partial\mathfrak{T}^\star$.

We can introduce a clockwise exploration process $(\mathcal{E}^\star_s)_{0\leq s\leq\Sigma^\star}$ of $\mathfrak{T}^\star$ in exactly the same way as we did for $\mathfrak{T}$, and use this exploration process to define the "interval" $[|a, b|]_\star$ for every $a, b \in \mathfrak{T}^\star$ (we have in fact $[|a, b|]_\star = [|a, b|] \cap \mathfrak{T}^\star$). We then set, for every $a, b \in \mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$,

$$D^\circ_\star(a, b) := \ell_a + \ell_b - 2\max\Big(\min_{c\in[|a,b|]_\star} \ell_c, \min_{c\in[|b,a|]_\star} \ell_c\Big) \tag{22}$$

if the maximum in the right-hand side is positive, and $D^\circ_\star(a, b) := \infty$ otherwise. Finally, in exactly the same way as we defined D from D°, we set, for every $a, b \in \mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$,

$$D_\star(a, b) := \inf_{a_0=a, a_1, \ldots, a_{p-1}, a_p=b} \sum_{i=1}^{p} D^\circ_\star(a_{i-1}, a_i) \tag{23}$$

where the infimum is over all choices of the integer $p \geq 1$ and of the points $a_1, \ldots, a_{p-1}$ in $\mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star$. It is not hard to verify that $D_\star(a,b) < \infty$ (see Proposition 30 (i) in [18] for a very similar argument). The mapping $(a,b) \mapsto D_\star(a,b)$ is continuous on $(\mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star) \times (\mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star)$, and we have again $D_\star(a,b) \geq |\ell_a - \ell_b|$. We also notice that $D^\circ_\star(a,b) = D^\circ(a,b)$ whenever $D^\circ_\star(a,b) < \infty$ (if labels are positive on $[|a,b|]_\star$, this implies that $[|a,b|]_\star = [|a,b|]$). Consequently, $D_\star(a,b) \geq D(a,b)$ for every $a,b \in \mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star$.

Proposition 6. *The function $(a,b) \mapsto D_\star(a,b)$ has a continuous extension to $\mathfrak{T}^\star \times \mathfrak{T}^\star$, which satisfies the triangle inequality and the bound $D_\star(a,b) \geq |\ell_a - \ell_b|$ for every $a,b \in \mathfrak{T}^\star$. Furthermore, the property $D_\star(a,b) = 0$ holds if and only if either a and b both belong to $\mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star$ and $D^\circ_\star(a,b) = 0$, or a and b both belong to $\partial \mathfrak{T}^\star$ and we have $\{a,b\} = \{\mathcal{E}^\star_s, \mathcal{E}^\star_t\}$, with $0 \leq s \leq t \leq \Sigma^\star$, and $\ell_{\mathcal{E}^\star_r} > 0$ for every $r \in (s,t)$.*

Proof. Let us start by proving the first assertion. Since $D_\star$ satisfies the triangle inequality, it is enough to verify that, for any $a \in \partial \mathfrak{T}^\star$, if $(a_n)_{n \in \mathbb{N}}$ is a sequence in $\mathfrak{T}^\star \backslash \partial \mathfrak{T}^\star$ that converges to a, we have $D_\star(a_n, a_m) \longrightarrow 0$ as $n, m \to \infty$. If $a \in \partial \mathfrak{T}^\star \backslash \{0, \xi\}$, this is a straightforward consequence of Lemma 4. Indeed, let $i \in I$ such that $a \in \mathcal{T}_{(\tilde{\omega}_i)}$. Then, we have also $a_n \in \mathcal{T}_{(\tilde{\omega}_i)}$ when n is large. However, when a_n and a_m both belong to $\mathcal{T}_{(\tilde{\omega}_i)}$, we immediately get from our definitions that $D_\star(a_n, a_m) \leq \Delta_{(\tilde{\omega}_i)}(a_n, a_m)$ with the notation introduced before Lemma 4. The convergence of $D_\star(a_n, a_m)$ to 0 now follows as a consequence of the first assertion of Lemma 4.

Let us turn to the case where $a = 0$ or $a = \xi$. For definiteness, take $a = 0$ (the other case is similar). We can list all indices $i \in I$ such that $t_i \in [0, \xi/2]$ and $W_*(\omega_i) \leq 0$ in a sequence $i_1, i_2, \ldots$ such that $\xi/2 > t_{i_1} > t_{i_2} > \cdots$. We set $h_j = \sqrt{3}\,\mathbf{e}_{t_{i_j}}$ for every $j \geq 1$. Then, conditionally on $(\mathbf{e}_t)_{0 \leq t \leq \xi}$ and on the sequence $(t_{i_1}, t_{i_2}, \ldots)$, the snake trajectories $\omega_{i_1}, \omega_{i_2}, \ldots$ are independent and the conditional distribution of ω_{i_j} is $\mathbb{N}^{[0]}_{h_j}$. As already mentioned, we have for every $a,b \in \mathcal{T}_{(\tilde{\omega}_{i_j})} \backslash \partial \mathcal{T}_{(\tilde{\omega}_{i_j})}$,

$$D_\star(a,b) \leq \Delta_{(\tilde{\omega}_{i_j})}(a,b), \tag{24}$$

For every $j \geq 1$, set

$$H_j := \sup \{ \Delta_{(\tilde{\omega}_{i_j})}(a,b) : a,b \in \mathcal{T}_{(\tilde{\omega}_{i_j})} \backslash \partial \mathcal{T}_{(\tilde{\omega}_{i_j})} \}.$$

Thanks to Lemma 4 and a scaling argument, we have $H_j = h_j H'_j$ where the random variables H'_j are independent of $(\mathbf{e}, t_{i_1}, t_{i_2}, \ldots)$ and have the same distribution with finite expectation. Next observe that

$$\mathbb{E}\Big[\sum_{j=1}^{\infty} h_j \,\Big|\, \mathbf{e} \Big] = 2 \int_0^{\xi/2} \mathrm{d}t\, \sqrt{3}\,\mathbf{e}_t\, \mathbb{N}_{\sqrt{3}\mathbf{e}_t}(W_* \leq 0) = 3\sqrt{3} \int_0^{\xi/2} \frac{\mathrm{d}t}{\mathbf{e}_t} < \infty \quad \text{a.s.}$$

and thus

$$\sum_{j=1}^{\infty} h_j < \infty \quad \text{a.s.}$$

It also follows that

$$\sum_{j=1}^{\infty} H_j < \infty \quad \text{a.s.}$$

since

$$\mathbb{E}\Big[\sum_{j=1}^{\infty} H_j \,\Big|\, \mathbf{e}, t_{i_1}, t_{i_2}, \ldots \Big] = \sum_{j=1}^{\infty} h_j\, \mathbb{E}[H'_j] = C \sum_{j=1}^{\infty} h_j.$$

where the constant C is as in Lemma 4. For every $j \geq 1$, write $\rho_j = t_{i_j}$ for the root of $\mathcal{T}_{(\tilde{\omega}_{i_j})}$. Fix an integer $k \geq 1$, and let $a \in [|0, \rho_k|]_\star$ such that $\ell_a > 0$. Then there is an index $j > k$ such that either $a \in \mathcal{T}_{(\tilde{\omega}_{i_j})}$ or $a \in [|\rho_j, \rho_{j-1}|]_\star$. We observe that in both cases we have

$$D_\star(a, \rho_j) \leq H_j + \ell_a. \tag{25}$$

If $a \in \mathcal{T}_{(\tilde{\omega}_{i_j})}$, this is immediate from the bound (24) and the definition of H_j. If $a \in [|\rho_j, \rho_{j-1}|]_\star$, we choose $\varepsilon > 0$ smaller than the minimal label in $[|\rho_j, \rho_{j-1}|]_\star$ and we write $a_{(\varepsilon)}$ for the last vertex (in the clockwise exploration of $\mathfrak{T}^\star$) of $\mathcal{T}_{\tilde{\omega}_{i_j}}$ with label ε. Then we have $D_\star(a, \rho_j) \leq D_\star(a_{(\varepsilon)}, \rho_j) + D_\star(a, a_{(\varepsilon)})$, and on one hand $D_\star(a_{(\varepsilon)},\rho_j)\leq H_j$, on the other hand, $D_\star(a,a_{(\varepsilon)})\leq \ell_a-\varepsilon$ (because labels "between" $a_{(\varepsilon)}$ and a remain greater than ε).

In the case $a=\rho_{j-1}$, (25) gives $D_\star(\rho_{j-1},\rho_j)\leq H_j+h_{j-1}$. Using the triangle inequality, it follows that, for every $a,b\in[|0,\rho_k|]_\star$ with $\ell_a\wedge\ell_b>0$,

$$D_\star(a,b)\leq\sum_{j=k}^{\infty}(H_j+h_j)+2\sup\{\ell_c:c\in[|0,\rho_k|]_\star\}.$$

The right-hand side can be made arbitrarily small by choosing k large. This proves that $D_\star(a,b)$ tends to 0 when $a,b\to 0$ in $\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$. This completes the proof of the first assertion of the proposition.

We note that the continuous extension of $(a,b)\mapsto D_\star(a,b)$ also satisfies the triangle inequality and the bound $D_\star(a,b)\geq|\ell_a-\ell_b|$.

Let us turn to the second assertion. Trivially the property $D^\circ_\star(a,b)=0$ for $a,b\in\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$ implies $D_\star(a,b)=0$. Then suppose that $a,b\in\partial\mathfrak{T}^\star$, and $a=\mathcal{E}^\star_s$, $b=\mathcal{E}^\star_t$, with $0\leq s<t\leq\Sigma^\star$, and $\ell_{\mathcal{E}^\star_r}>0$ for every $r\in(s,t)$. Take $u=(s+t)/2$ and for every $\varepsilon\in(0,\ell_{\mathcal{E}^\star_u})$, define

$$s_\varepsilon:=\sup\{r\in[s,u]:\ell_{\mathcal{E}^\star_r}=\varepsilon\}\,,\quad t_\varepsilon:=\inf\{r\in[u,t]:\ell_{\mathcal{E}^\star_r}=\varepsilon\}.$$

Then, $D_\star(\mathcal{E}^\star_{s_\varepsilon},\mathcal{E}^\star_{t_\varepsilon})=0$ and by letting $\varepsilon\to0$ we get $D_\star(a,b)=0$.

Conversely, if $D_\star(a,b)=0$, with $\ell_a=\ell_b>0$, this implies a fortiori that $D(a,b)=0$ and, from results recalled in Section 4.1, this can only occur if $D^\circ(a,b)=0$. In particular labels are greater than or equal to ℓ_a on $[|a,b|]$ (or on $[|b,a|]$) and it readily follows that we have also $D^\circ_\star(a,b)=0$.

Finally, suppose that $D_\star(a,b)=0$ for distinct points $a,b\in\partial\mathfrak{T}^\star$. Let us write $a=\mathcal{E}^\star_s$, $b=\mathcal{E}^\star_t$, with $0\leq s<t\leq\Sigma^\star$, and exclude the case $\{s,t\}=\{0,\Sigma^\star\}$. Note that $[|a,b|]_\star=\{\mathcal{E}^\star_r:s\leq r\leq t\}$. Argue by contradiction and suppose that ℓ_c vanishes for some $c\in[|a,b|]_\star\backslash\{a,b\}$. Then there must also exist $c'\in[|a,b|]$ such that $\ell_{c'}<0$ (because 0 cannot be a local minimum for the function $s\mapsto\widehat{W}_s(\omega_i)$, for any $i\in I$, see the end of Section 2.3). Since it is also clear that the minimum of labels on $[|b,a|]$ is negative, it follows that $D^\circ(a,b)>0$, which in turn implies that $D(a,b)>0$, and a fortiori $D_\star(a,b)>0$ (the bound $D_\star(a,b)\geq D(a,b)$ remains valid for any $a,b\in\mathfrak{T}^\star$ by continuity). This contradicts our initial assumption $D_\star(a,b)=0$.

The preceding argument does not work for $a=0$ and $b=\xi$ because $D^\circ(0,\xi)=0$, but we can argue as follows. Suppose that $D_\star(0,\xi)=0$. For every integer $n>2/\xi$, we choose $r_n\in(0,1/n)$ small enough so that $D_\star(0,r_n)<1/n$ and $D_\star(\xi-r_n,\xi)<1/n$. Then $D_\star(r_n,\xi-r_n)<2/n$ by the triangle inequality. It follows that we can find $a^{(n)}_0=r_n,a^{(n)}_1,\ldots,a^{(n)}_{p_n-1},a^{(n)}_{p_n}=\xi-r_n$ in $\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$ such that $\sum_{j=1}^{p_n}D^\circ_\star(a^{(n)}_{j-1},a^{(n)}_j)<2/n$. Next fix any $i_0\in I$ such that labels on $\mathcal{T}_{(\tilde\omega_{i_0})}$ vanish. For n large enough so that $r_n<t_{i_0}<\xi-r_n$, at least one of the points $a^{(n)}_j$, $1\leq j\leq p_{n-1}$, say $a^{(n)}_{j_n}$, must belong to $\mathcal{T}_{(\tilde\omega_{i_0})}$ (otherwise we would have $D^\circ_\star(a^{(n)}_{j-1},a^{(n)}_j)=\infty$ for some j). By extracting a convergent sequence from the sequence $(a^{(n)}_{j_n})$, we get a point $a^{(\infty)}$ of $\mathcal{T}_{(\tilde\omega_{i_0})}$ such that $D_\star(0,a^{(\infty)})=0$. By the cases treated previously, this is impossible, and this contradiction completes the proof. $\square$

We then consider the quotient space $\mathbb{U}:=\mathfrak{T}^\star/\{D_\star=0\}$, and the canonical projection $\Pi_\star:\mathfrak{T}^\star\longrightarrow\mathbb{U}$. In contrast with the construction of $\mathbb{D}$, we observe that $\Pi_\star(0)\neq\Pi_\star(\xi)$. The function $(a,b)\mapsto D_\star(a,b)$ induces a metric on $\mathbb{U}$, which we still denote by $D_\star$, and the metric space $(\mathbb{U},D_\star)$ is equipped with the pushforward of the volume measure on $\mathfrak{T}^\star$ under $\Pi_\star$. This measure will be denoted by $\mathbf{V}_\star$. We also write $\partial_0\mathbb{U}=\Pi_\star([0,\xi])$ and $\partial_1\mathbb{U}=\Pi_\star(\partial\mathfrak{T}^\star)$. We note that labels ℓ_x make sense for $x\in\mathbb{U}$ (because $D_\star(a,b)=0$ implies $\ell_a=\ell_b$). Furthermore, we have $D_\star(x,\partial_1\mathbb{U})=\ell_x$ for every $x\in\mathbb{U}$ (here and below, we use the notation $D_\star(x,A):=\inf\{D_\star(x,y):y\in A\}$). Indeed, the bound $D_\star(x,\partial_1\mathbb{U})\geq\ell_x$ is immediate since $D_\star(x,y)\geq|\ell_x-\ell_y|$ for every $y\in\mathbb{U}$. Conversely, if $x=\Pi_\star(\mathcal{E}^\star_r)$ and $s=\inf\{t\geq r:\ell_{\mathcal{E}^\star_t}=0\}$, it is easy to verify that $D_\star(x,\mathcal{E}^\star_s)=\ell_x$.

Our goal is to verify that the random measure metric space $(\mathbb{U},D_\star,\mathbf{V}_\star)$ equipped with an appropriate standard boundary curve is a curve-decorated free Brownian disk with a random perimeter. The boundary of this Brownian disk will be $\partial_0\mathbb{U}\cup\partial_1\mathbb{U}$. To this end, we will first identify $(\mathbb{U},D_\star,\mathbf{V}_\star)$ with another space constructed directly from the Brownian disk $\mathbb{D}$.

4.3 Identification of $\mathbb{U}$

We consider the free pointed Brownian disk $\mathbb{D}$ with perimeter ξ constructed in Section 4.1. Recall the notation Π for the canonical projection from $\mathfrak{T}$ onto $\mathbb{D}$, and $\mathbf{x}_*$ for the distinguished point of $\mathbb{D}$. To simplify notation, we will also write $\mathbf{x}_0 = \Pi(0) = \Pi(\xi)$ for the unique point of $\partial\mathbb{D}$ at minimal distance from $\mathbf{x}_*$, and $r_0 = D(\mathbf{x}_*, \mathbf{x}_0) = -\ell_{\mathbf{x}_*}$. Let $\mathcal{B}_{\mathbb{D}}(\mathbf{x}_*, r_0)$ stand for the closed ball of radius r_0 centered at $\mathbf{x}_*$ in $\mathbb{D}$. The hull H of radius r_0 centered at $\mathbf{x}_*$ is the closed subset of $\mathbb{D}$ defined by saying that $\mathbb{D}\backslash H$ is the connected component of $\mathbb{D}\backslash\mathcal{B}_{\mathbb{D}}(\mathbf{x}_*, r_0)$ that contains $\partial\mathbb{D}\backslash\{\mathbf{x}_0\}$. We also let U be the closure of $\mathbb{D}\backslash H$, and write $\mathrm{Int}(U) = \mathbb{D}\backslash H$ for the topological interior of U and $\partial U = U\backslash\mathrm{Int}(U) = \partial H$ for its topological boundary. Notice that $\mathrm{Int}(U)$ contains $\partial\mathbb{D}\backslash\{\mathbf{x}_0\}$. We also observe that $D(\mathbf{x}_*, x) = r_0$ for every $x \in \partial H$. Since $\mathbb{D}$ is a geodesic space, it follows that, for every $x \in \mathrm{Int}(U)$, $D(x, \partial H) = D(x, \mathbf{x}_*) - r_0 = \ell_x$.

Let d_∞ stand for the intrinsic distance on $\mathrm{Int}(U)$. For every $x, y \in \mathrm{Int}(U)$, $d_\infty(x, y)$ is the infimum of lengths of paths connecting x to y that stay in $\mathrm{Int}(U)$ — here lengths are of course evaluated with respect to the metric D on $\mathbb{D}$. The fact that $\mathbb{D}$ is a geodesic space easily implies that $d_\infty(x, y) < \infty$ for every $x, y \in \mathrm{Int}(U)$. We let (U', d_∞) stand for the completion of the metric space $(\mathrm{Int}(U), d_\infty)$. We write $\mathbf{V}_U$ for the restriction of the volume measure $\mathbf{V}$ to $\mathrm{Int}(U)$, which may also be viewed as a measure on U' since $\mathrm{Int}(U)$ is an open subset of U'. Finally, we recall the abuse of notation that consists in viewing $\mathfrak{T}^\star$ as a subset of $\mathfrak{T}$.

Proposition 7. *The measure metric spaces $(U', d_\infty, \mathbf{V}_U)$ and $(\mathbb{U}, D_\star, \mathbf{V}_\star)$ are almost surely equal. More precisely, the following property holds a.s.: there is a one-to-one mapping Ψ from $\mathrm{Int}(U)$ into $\mathbb{U}$ such that Ψ maps $\Pi(a)$ to $\Pi_\star(a)$, for every $a \in \mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$, and the mapping Ψ extends to a measure-preserving isometry from $(U', d_\infty, \mathbf{V}_U)$ onto $(\mathbb{U}, D_\star, \mathbf{V}_\star)$.*

Proof. Let us first identify the hull H in terms of our construction of $\mathbb{D}$. To this end, for every $x \in \mathbb{D}$, set $m_x = \ell_x$ if $x \in [0, \xi]$, and, if $x = \Pi(a)$ where $a \in \mathcal{T}_{(\omega_i)}$ for some $i \in I$, let m_x be the minimal label along the ancestor line of a in $\mathcal{T}_{(\omega_i)}$. Notice that this definition does not depend on the choice of a such that $x = \Pi(a)$. Then an easy extension of the so-called cactus bound (cf. Proposition 3.1 in [16]) shows that any continuous curve from x to $\partial\mathbb{D}$ has to come within distance $r_0 + m_x$ from $\mathbf{x}_*$. On the other hand, if $x = \Pi(a)$ with $a \in \mathcal{T}_{(\omega_i)}$, the image of the ancestral line of a under Π provides a path from x to $\partial\mathbb{D}\backslash\{\mathbf{x}_0\}$ whose minimal distance from $\mathbf{x}_*$ is $r_0 + m_x$. It follows that $x \in \mathbb{D}\backslash H$ if and only if $m_x > 0$, and consequently

$$\mathrm{Int}(U) = \mathbb{D}\backslash H = \Pi(\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star), \quad H = \Pi(\mathfrak{T}\backslash\mathfrak{T}^\star) \cup \Pi(\partial\mathfrak{T}^\star), \quad \partial H = \partial U = \Pi(\partial\mathfrak{T}_\star)\,.$$

The next step is to prove that, if $x = \Pi(a) \in \mathrm{Int}(U)$ and $y = \Pi(b) \in \mathrm{Int}(U)$, where $a, b \in \mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$, we have

$$d_\infty(x, y) = D_\star(a, b). \tag{26}$$

The upper bound $d_\infty(x, y) \leq D_\star(a, b)$ is easy because each quantity of the form

$$\sum_{i=1}^{p} D^\circ_\star(a_{i-1}, a_i),$$

where $a_0 = a$, $a_p = b$ and $a_1, \ldots, a_{p-1} \in \mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$ are such that $D^\circ_\star(a_{i-1}, a_i) < \infty$ for $1 \leq i \leq p$, is the length of a curve from x to y that stays in $\mathrm{Int}(U)$ (use the simple geodesics defined at the end of Section 4.1 and note that the condition $D^\circ_\star(a_{i-1}, a_i) < \infty$ precisely prevents the curve from hitting the set of points with zero label).

Let us now justify the reverse bound $d_\infty(x, y) \geq D_\star(a, b)$. It is enough to verify that, if $\gamma = (\gamma(t))_{0\leq t\leq 1}$ is a curve connecting x to y and staying in $\mathrm{Int}(U)$, the length of γ is bounded below by $D_\star(a, b)$. Since γ stays in $\mathrm{Int}(U)$, a compactness argument shows that $\mathrm{dist}_D(\gamma, \partial U) > 0$ (here we write $\mathrm{dist}_D(\gamma, \partial U)$ for the minimal D-distance between a point of the curve γ and ∂U). So we can fix an integer $n \geq 1$ such that, for every $i \in \{1, \ldots, n\}$, we have

$$D\Big(\gamma(\frac{i-1}{n}), \gamma(\frac{i}{n})\Big) < \frac{1}{2}\mathrm{dist}_D(\gamma, \partial U). \tag{27}$$

Since the length of γ is bounded below by

$$\sum_{i=1}^{n} D\Big(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n})\Big) \tag{28}$$

it is enough to verify that the latter sum is bounded below by $D_\star(a,b)$. Fix $\varepsilon\in(0,\frac{1}{4}\mathrm{dist}_D(\gamma,\partial U))$. By the definition of the metric D, for every $i\in\{1,\dots,n\}$, we can find $a^i_0,a^i_1,\dots,a^i_{p_i}\in\mathfrak{T}$ with $\Pi(a^i_0)=\gamma(\frac{i-1}{n})$ and $\Pi(a^i_{p_i})=\gamma(\frac{i}{n})$ such that

$$D\Big(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n})\Big)\geq\sum_{j=1}^{p_i} D^\circ(a^i_{j-1},a^i_j)-\frac{\varepsilon}{n}. \tag{29}$$

This implies in particular that, for every $k\in\{1,\dots,p_i\}$,

$$D(a^i_0,a^i_k)\leq\sum_{j=1}^{k} D^\circ(a^i_{j-1},a^i_j)\leq D\Big(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n})\Big)+\frac{\varepsilon}{n}<\mathrm{dist}_D(\gamma,\partial U).$$

Since $\Pi(a^i_0)$ belongs to the range of γ, it follows that $\Pi(a^i_k)\in\mathrm{Int}(U)$. Thus all points a^i_k must belong to $\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$ (recall that $\Pi(\mathfrak{T}\backslash\mathfrak{T}^\star)\cup\Pi(\partial\mathfrak{T}^\star)=H$). We now claim that

$$D^\circ(a^i_{j-1},a^i_j)=D^\circ_\star(a^i_{j-1},a^i_j), \tag{30}$$

for every $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,p_i\}$. If the claim is proved, we obtain that

$$\sum_{i=1}^{n} D\Big(\gamma(\frac{i-1}{n}),\gamma(\frac{i}{n})\Big)\geq\sum_{i=1}^{n}\sum_{j=1}^{p_i} D^\circ_\star(a^i_{j-1},a^i_j)-\varepsilon\geq D_\star(a,b)-\varepsilon,$$

by the very definition of $D_\star(a,b)$. Since ε was arbitrary, this shows that the sum (28) is bounded below by $D_\star(a,b)$, as desired.

Let us prove our claim (30). From our definitions, it is enough to verify that

$$\max\Big(\min_{c\in[|a^i_{j-1},a^i_j|]}\ell_c,\min_{c\in[|a^i_j,a^i_{j-1}|]}\ell_c\Big)>0 \tag{31}$$

(note that, if for instance $\min_{c\in[|a^i_{j-1},a^i_j|]}\ell_c>0$, this implies that $[|a^i_{j-1},a^i_j|]=[|a^i_{j-1},a^i_j|]_\star$). Fix $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,p_i\}$. Using (27), (29) and the bound $D^\circ(u,v)\geq|\ell_u-\ell_v|$, we have

$$\begin{aligned}
\mathrm{dist}_D(\gamma,\partial U)&\geq\sum_{k=1}^{p_i} D^\circ(a^i_{k-1},a^i_k)\\
&\geq|\ell_{\gamma(\frac{i-1}{n})}-\ell_{a^i_{j-1}}|+\ell_{a^i_{j-1}}+\ell_{a^i_j}-2\max\Big(\min_{c\in[|a^i_{j-1},a^i_j|]}\ell_c,\min_{c\in[|a^i_j,a^i_{j-1}|]}\ell_c\Big)+|\ell_{\gamma(\frac{i}{n})}-\ell_{a^i_j}|\\
&\geq\ell_{\gamma(\frac{i-1}{n})}+\ell_{\gamma(\frac{i}{n})}-2\max\Big(\min_{c\in[|a^i_{j-1},a^i_j|]}\ell_c,\min_{c\in[|a^i_j,a^i_{j-1}|]}\ell_c\Big)\\
&\geq2\Bigg(\mathrm{dist}_D(\gamma,\partial U)-\max\Big(\min_{c\in[|a^i_{j-1},a^i_j|]}\ell_c,\min_{c\in[|a^i_j,a^i_{j-1}|]}\ell_c\Big)\Bigg),
\end{aligned}$$

thanks to the equality $D(x,\partial U)=\ell_x$ for $x\in\mathrm{Int}(U)$. Clearly this implies that (31) holds, completing the proof of (26).

Recall that $\mathrm{Int}(U)=\Pi(\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star)$, and also recall the notation $\partial_1\mathbb{U}=\Pi_\star(\partial\mathfrak{T}^\star)$. By Proposition 6, two points a and b of $\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star$ are identified in the quotient space defining $\mathbb{U}$ if and only if they are identified in the quotient space $\mathbb{D}$. This shows that $\mathbb{U}\backslash\partial_1\mathbb{U}=\Pi_\star(\mathfrak{T}^\star\backslash\partial\mathfrak{T}^\star)$ is identified as a set to $\mathrm{Int}(U)$, and (26) entails that this identification is an isometry when $\mathrm{Int}(U)$ is equipped with d_∞ and $\mathbb{U}\backslash\partial_1\mathbb{U}$ is equipped with $D_\star$. Since $\mathbb{U}\backslash\partial_1\mathbb{U}$ is dense in the compact set $\mathbb{U}$, it follows that the completion U' can also be identified isometrically to $\mathbb{U}$. Furthermore, it is clear that the volume measure $\mathbf{V}_U$ corresponds to $\mathbf{V}_\star$ in this identification. This completes the proof of the proposition. □

In the identification of U' with $\mathbb{U}$, the "boundary" $\partial U' := U' \backslash \mathrm{Int}(U)$ is mapped bijectively onto $\partial_1 \mathbb{U}$. Note that every point of $\partial U' = \partial_1 \mathbb{U}$ has to correspond to a point of ∂U. This correspondence is one-to-one except that the two points $\Pi_\star(0)$ and $\Pi_\star(\xi)$ of $\partial_1 \mathbb{U}$ correspond to the same point $\mathbf{x}_0$ of ∂U.

We now record two technical properties that will be useful in the proofs of the next section.

Proposition 8. *The following properties hold $\mathbb{P}$ a.s.*
(i) *For every $\delta > 0$, let $U_{(\delta)}$ be the set of all $x \in \mathbb{D}$ such that there is a continuous path from x to $\partial \mathbb{D}$ that stays at distance at least $r_0 - \delta$ from $\mathbf{x}_*$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $U_{(\delta)} \subset \{x \in \mathbb{D} : D(x, U) < \varepsilon\}$.*
(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x \in \mathbb{D}$ and $D(x, H) \geq \varepsilon$, there is a continuous path from x to $\partial \mathbb{D}$ that stays at distance at least $r_0 + \delta$ from $\mathbf{x}_*$.*

Proof. (i) Recall the notation m_x introduced in the proof of Proposition 7 for $x \in \mathbb{D}$. As it was explained in this proof, any continuous path from x to $\partial \mathbb{D}$ has to come within distance $r_0 + m_x$ from $\mathbf{x}_*$. So the proof boils down to verifying that, given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\{x \in \mathbb{D} : m_x \geq -\delta\} \subset \{x \in \mathbb{D} : D(x, U) < \varepsilon\}.$$

Note that the mapping $x \mapsto m_x$ is continuous. Moreover, the property $m_x = 0$ may hold only if $x \in \partial U$. To see this, first note that, a.s. for every $x \in \mathbb{D}$ of the form $x = \Pi(a)$ with $a \in \mathcal{T}_{(\omega_i)}$, the property $m_x = 0$ implies that $\ell_x = 0$ and that labels on $[\![\rho_{(\omega_i)}, a]\!] \backslash \{a\}$ are positive. Indeed, if this were not the case, this would mean that the Brownian path describing the labels along the ancestral line $[\![\rho_{(\omega_i)}, a]\!]$ has a local minimum equal to 0 at a point distinct from a, which does not occur for any $a \in \mathfrak{T}$, a.s. It follows that $a \in \partial \mathfrak{T}_\star$ and $x \in \partial U$.

The desired result then follows from a compactness argument, since the intersection of compact sets

$$\bigcap_{\delta > 0} \Big(\{x \in \mathbb{D} : m_x \geq -\delta\} \backslash \{x \in \mathbb{D} : D(x, U) < \varepsilon\} \Big)$$

is empty by the preceding considerations and the fact that $m_x > 0$ implies $x \in \mathrm{Int}(U)$.
(ii) For every $x \in \mathbb{D} \backslash H$, let $\varphi(x)$ be the supremum of all $\delta > 0$ such that there is a continuous path from x to $\partial \mathbb{D}$ that stays at distance at least $r_0 + \delta$ from $\mathbf{x}_*$. Then $\varphi(x) > 0$ for every $x \in \mathbb{D} \backslash H$ and the mapping $x \mapsto \varphi(x)$ is continuous. It follows that $\inf\{\varphi(x) : x \in \mathbb{D}, D(x, H) \geq \varepsilon\} > 0$, giving the desired result. □

5 Passage to the limit

5.1 Preliminaries

For every integer $L \geq 1$, let $\mathcal{T}_L$ be a Boltzmann distributed rooted and pointed type I triangulation with a simple boundary of size $n_L \geq 1$. We assume throughout this section that $n_L / L \longrightarrow \xi > 0$ as $L \to \infty$. We use a similar notation as in Section 3.4. In particular, ν_L is the counting measure on the set $V_i(\mathcal{T}_L)$ scaled by the multiplicative factor $\frac{3}{4} L^{-2}$, $\Theta_L = (\Theta_L(0), \Theta_L(1), \ldots, \Theta_L(n_L))$ is the boundary path of $\mathcal{T}_L$, and $\widehat{\Theta}_L(t) = \Theta_L(\lfloor Lt \rfloor)$ for $0 \leq t \leq n_L / L$, and $v_*^{(L)}$ is the distinguished vertex of $\mathcal{T}_L$. From (a trivial extension of) (17), we have

$$(V(\mathcal{T}_L), \sqrt{3/2}\, L^{-1/2} \mathrm{d_{gr}}, \nu_L, \widehat{\Theta}_L, v_*^{(L)}) \xrightarrow[L \to \infty]{(d)} (\mathbb{D}, D, \mathbf{V}, \Gamma, \mathbf{x}_*), \tag{32}$$

where the convergence holds in distribution in $(\mathbb{M}^{GHPU\bullet}, \mathrm{d}_{GHPU\bullet})$, and the limit $(\mathbb{D}, D, \mathbf{V}, \Gamma, \mathbf{x}_*)$ is the curve-decorated free pointed Brownian disk with perimeter ξ as defined at the end of Section 4.1. Recall that the range of Γ is the boundary $\partial \mathbb{D}$.

Assuming that $\mathrm{d_{gr}}(v_*^{(L)}, \partial \mathcal{T}_L) \geq 2$, we can make sense of the hull of radius 1 centered at $v_*^{(L)}$, which we denote by $H_1(v_*^{(L)})$: We first define the ball $B_1(v_*^{(L)})$ as the union of all faces of $\mathcal{T}_L$ that are incident to $v_*^{(L)}$, and the hull $H_1(v_*^{(L)})$ is obtained by adding to the ball $B_1(v_*^{(L)})$ all faces of $\mathcal{T}_L$ that are disconnected from $\partial\mathcal{T}_L$ by the ball. This hull can be viewed as a triangulation with a boundary, such that every boundary vertex is at distance 1 from $v_*^{(L)}$.

Note that the probability of the event $\{\mathrm{d_{gr}}(v_*^{(L)}, \partial\mathcal{T}_L) \geq 2\}$ tends to 1 as $L \to \infty$. From now on, we will argue conditionally on this event. Then the complement of the hull $H_1(v_*^{(L)})$ in $\mathcal{T}_L$ is a triangulation $\mathcal{T}'_L$ with two boundaries. The first boundary is $\partial\mathcal{T}_L$ and the second one is the boundary of the hull $H_1(v_*^{(L)})$ — we may choose the root on the second boundary uniformly at random, independently of $\mathcal{T}_L$. Furthermore, conditionally on the event that the boundary size of the hull $H_1(v_*^{(L)})$ is equal to $p \geq 1$, the triangulation $\mathcal{T}'_L$ is Boltzmann distributed on $\mathbb{T}^2(n_L, p)$, so that we can apply the results of Section 3. In view of this application, we also observe that the boundary size of $H_1(v_*^{(L)})$ (that is, the size of the second boundary of $\mathcal{T}'_L$) remains bounded in probability when $L \to \infty$. This follows from an easy counting argument since this boundary size is bounded above by the degree of $v_*^{(L)}$ in $\mathcal{T}_L$.

We can then run the peeling algorithm of $\mathcal{T}'_L$ as described in Section 3. With a slight abuse of terminology, we will consider that the revealed region at step $n \leq \zeta_L$, as defined in Section 3.2, also includes the hull $H_1(v_*^{(L)})$, and thus can be viewed as a triangulation with a (simple) boundary of size P_n^L. We consider the "peeling by layers" algorithm, which involves particular choices of the revealed edge at each step (see e.g. [11] for more details). The only property of this algorithm that we will use is the fact that, for every step $n = 0, 1, \ldots, \zeta_L$, there exists an integer $r_n \geq 1$ such that every vertex of the boundary of the revealed version at step n is at distance r_n or $r_n + 1$ from $v_*^{(L)}$. We let $r_0^{(L)}$ be $\sqrt{3/2}\, L^{-1/2}$ times the maximal graph distance between $v_*^{(L)}$ and a point of the boundary of the revealed region at time ζ_L. So a path from any point of the revealed region at time ζ_L to the boundary $\partial\mathcal{T}_L$ must come within distance $\sqrt{2/3}\, L^{1/2}\, r_0^{(L)}$ from the distinguished point $v_*^{(L)}$. On the other hand, by the properties of the peeling by layers algorithm, we have $\mathrm{d_{gr}}(v_*^{(L)}, \partial\mathcal{T}_L) = \sqrt{2/3}\, L^{1/2}\, r_0^{(L)}$.

The collection of all faces of $\mathcal{T}_L$ that do not belong to the revealed region at time ζ_L will be called the unrevealed region. It will be convenient to write $\mathcal{V}_L$ for the set of all vertices of the unrevealed region, and $\mathcal{W}_L$ for the set of all vertices of the revealed region (at time ζ_L). The unrevealed region is rooted at the root of $\mathcal{T}_L$ and then corresponds to a triangulation with a simple boundary where two boundary vertices, say α'_L and α''_L, have been glued to give the unique vertex α_L of $\partial\mathcal{T}_L$ that belongs to the revealed region at step ζ_L (see Figure 2). If these two vertices are "unglued", we get a triangulation $\mathcal{U}_L$ with a simple boundary $\partial\mathcal{U}_L$ of size $n_L + Z_L$, where we know from Proposition 5 that Z_L/n_L converges in distribution to a random variable denoted by $\mathcal{Z}$. Moreover, if e stands for the unique oriented edge of $\partial\mathcal{T}_L$ whose tail is α_L, and such that the external face of $\mathcal{T}_L$ lies to the left of e, we may root $\mathcal{U}_L$ at the edge corresponding to e, whose origin is α'_L (see again Figure 2). From the discussion in Section 3.1, one easily checks that, conditionally on $\{Z_L = k\}$, the triangulation $\mathcal{U}_L$ is Boltzmann distributed in $\mathbb{T}^1(n_L + k)$.

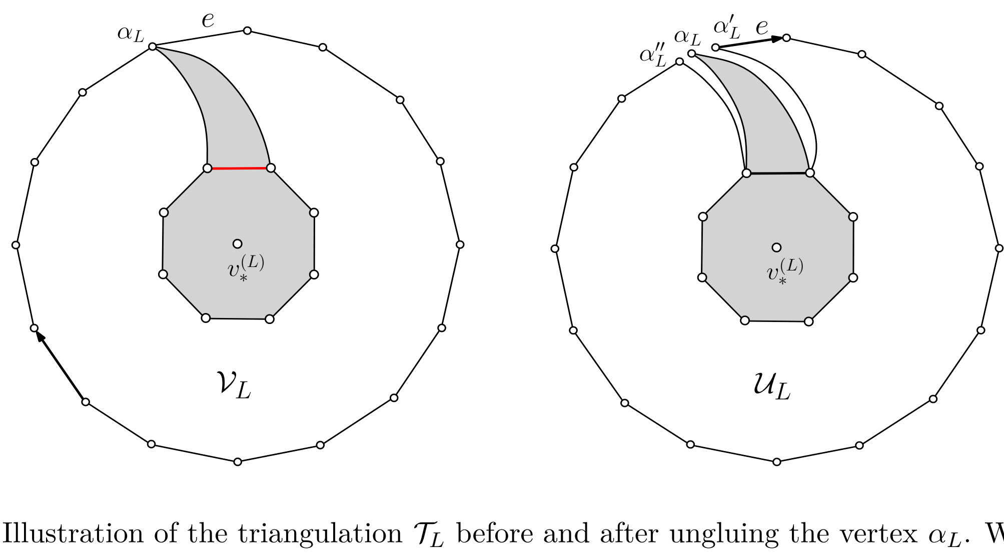

Figure 2: Illustration of the triangulation $\mathcal{T}_L$ before and after ungluing the vertex α_L. We represent the revealed and unrevealed region in green and in white respectively.

Since, conditionally on Z_L, the map $\mathcal{U}_L$ is a Boltzmann distributed rooted triangulation with perimeter $n_L + Z_L$, and $L^{-1} Z_L$ converges in distribution to $\mathcal{Y} := \xi \mathcal{Z}$, we can also apply the convergence (16) to the triangulations $\mathcal{U}_L$. Write $(\Theta'_L(j))_{0\leq j\leq n_L+Z_L}$ for the boundary path of $\mathcal{U}_L$, and $\widehat{\Theta}'_L(t) = \Theta'_L(\lfloor Lt \rfloor)$ for $0 \leq t \leq (n_L + Z_L)/L$. Also let d_L stand for the graph distance on $V(\mathcal{U}_L)$ scaled by the factor $\sqrt{3/2}L^{-1/2}$, and let θ_L be the counting measure on $V_i(\mathcal{U}_L)$ scaled by $\frac{3}{4}L^{-2}$. Then

$$(V(\mathcal{U}_L), d_L, \theta_L, \widehat{\Theta}'_L) \xrightarrow[L\to\infty]{(d)} \Big(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\mathbf{V}}, \widetilde{\Gamma}\Big), \tag{33}$$

where $(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\mathbf{V}}, \widetilde{\Gamma})$ is a curve-decorated free (non-pointed) Brownian disk with perimeter $\xi + \mathcal{Y}$, and the convergence in distribution holds in $(\mathbb{M}^{GHPU}, \mathrm{d}_{GHPU})$.

5.2 Paths avoiding part of the boundary

The boundary $\partial\mathcal{U}_L$ of $\mathcal{U}_L$ coincides with the set $\{\Theta'_L(j) : 0 \leq j \leq n_L + Z_L\}$. We will consider the subset $\partial_1\mathcal{U}_L$ of this boundary corresponding to the boundary of the revealed region at time ζ_L. Precisely, we set $\partial_1\mathcal{U}_L := \{\Theta'_L(j) : n_L \leq j \leq n_L + Z_L\}$ and $\partial_0\mathcal{U}_L := \{\Theta'_L(j) : 0 \leq j \leq n_L\}$. We also define $\partial_1\widetilde{\mathbb{D}} := \{\widetilde{\Gamma}(t) : \xi \leq t \leq \xi + \mathcal{Y}\}$.

By the Skorokhod representation theorem, we may and will assume in this section that the convergence (33) holds a.s., and moreover $Z_L/L \longrightarrow \mathcal{Y}$ a.s.

Lemma 9. *Let $\delta > 0$ and $\eta > 0$. Then a.s. there exists $\varepsilon_0 > 0$ such that, for every large enough L, for every $x, y \in V(\mathcal{U}_L)$ with $d_L(x, \partial_1\mathcal{U}_L) \geq \delta$ and $d_L(y, \partial_1\mathcal{U}_L) \geq \delta$, there is a path from x to y that stays at d_L-distance at least ε_0 from $\partial_1\mathcal{U}_L$, and whose d_L-length is bounded above by $d_L(x, y) + \eta$.*

Proof. Let us fix ω in the underlying probability space such that the (a.s.) convergence (33) holds and $Z_L/L \longrightarrow \mathcal{Y}$. By [12, Proposition 1.5] (or Proposition 1), we may assume that all metric spaces $(V(\mathcal{U}_L), d_L)$ and $(\widetilde{\mathbb{D}}, \widetilde{D})$ are embedded isometrically simultaneously in the same compact metric space (E, Δ), in such a way that we have $V(\mathcal{U}_L) \longrightarrow \widetilde{\mathbb{D}}$ for the Hausdorff distance and $(\widehat{\Theta}'_L(t\wedge(n_L+Z_L)/L))_{t\geq 0}$ converges uniformly to $(\widetilde{\Gamma}(t \wedge (\xi + \mathcal{Y})))_{t\geq 0}$. In particular, $\partial_1\mathcal{U}_L \longrightarrow \partial_1\widetilde{\mathbb{D}}$ in the sense of the Hausdorff distance between compact subsets of (E, Δ).

We may assume that $\eta < \delta$. We argue by contradiction. If the statement of the lemma does not hold, we can find a sequence $\varepsilon_k \downarrow 0$ and a sequence $L_k \uparrow \infty$ such that, for every k, there exist $x_k, y_k \in V(\mathcal{U}_{L_k})$ with $d_{L_k}(x_k, \partial_1\mathcal{U}_{L_k}) \geq \delta$ and $d_{L_k}(y_k, \partial_1\mathcal{U}_{L_k}) \geq \delta$, such that any path from x_k to y_k that stays at distance at least ε_k from $\partial_1\mathcal{U}_{L_k}$ has d_{L_k}-length at least $d_{L_k}(x_k, y_k) + \eta$.

By compactness, we may assume that $x_k \longrightarrow x_\infty$ and $y_k \longrightarrow x_\infty$, where $x_\infty, y_\infty \in \widetilde{\mathbb{D}}$, and $\widetilde{D}(x_\infty, \partial_1\widetilde{\mathbb{D}}) \geq \delta$, $\widetilde{D}(y_\infty, \partial_1\widetilde{\mathbb{D}}) \geq \delta$. We take k large enough so that $\Delta(x_k, x_\infty) < \eta/16$ and $\Delta(y_k, y_\infty) < \eta/16$ and then we have also $d_{L_k}(x_k, y_k) = \Delta(x_k, y_k) \geq \Delta(x_\infty, y_\infty) - \eta/8 = \widetilde{D}(x_\infty, y_\infty) - \eta/8$. Next, by Lemma 19 in [7], we can find a point $x'_\infty \in \widetilde{\mathbb{D}}$ with $\widetilde{D}(x_\infty, x'_\infty) < \eta/16$ and a point $y'_\infty \in \widetilde{\mathbb{D}}$ with $\widetilde{D}(y_\infty, y'_\infty) < \eta/16$, such that there is a $\widetilde{D}$-geodesic from x'_∞ to y'_∞ that does not hit $\partial\widetilde{\mathbb{D}}$, and thus stays at distance at least $\alpha > 0$ from $\partial\widetilde{\mathbb{D}}$, for some $\alpha > 0$. We can assume that $\alpha < \eta/8$, and we write $(\gamma(t))_{0\leq t\leq \tilde{D}(x'_\infty, y'_\infty)}$ for the latter geodesic. To simplify notation, we set $d_* = \widetilde{D}(x'_\infty, y'_\infty)$ and we note that $d_{L_k}(x_k, y_k) \geq \widetilde{D}(x_\infty, y_\infty) - \eta/8 \geq d_* - \eta/2$.

Then pick an integer $N \geq 4$ large enough so that $\frac{d_*}{N} < \frac{\alpha}{4}$. Taking k even larger if necessary, we can choose, for every $0 \leq i \leq N$, a point $x_i^{(k)}$ in $V(\mathcal{U}_{L_k})$ such that $\Delta(\gamma(\frac{id_*}{N}), x_i^{(k)}) < \alpha/(2N)$. Hence, for every $0 \leq i \leq N - 1$, we have

$$\Delta(x_i^{(k)}, x_{i+1}^{(k)}) \leq \Delta(x_i^{(k)}, \gamma(\frac{id_*}{N})) + \Delta(\gamma(\frac{id_*}{N}), \gamma(\frac{(i+1)d_*}{N})) + \Delta(\gamma(\frac{(i+1)d_*}{N}), x_{i+1}^{(k)}) < \frac{d_*}{N} + \frac{\alpha}{N} \leq \frac{\alpha}{2}.$$

Moreover,

$$\Delta(x_k, x_0^{(k)}) \leq \Delta(x_k, x_\infty) + \Delta(x_\infty, x'_\infty) + \Delta(x'_\infty, x_0^{(k)}) < \frac{\eta}{16} + \frac{\eta}{16} + \frac{\alpha}{2N} \leq \frac{3\eta}{16},$$

and similarly $\Delta(y_k, x_N^{(k)}) < \frac{3\eta}{16}$. If we now concatenate a geodesic from x_k to $x_0^{(k)}$ in $\mathcal{U}_{L_k}$ with geodesics from $x_i^{(k)}$ to $x_{i+1}^{(k)}$, for $0 \leq i \leq N - 1$, and finally with a geodesic from $x_N^{(k)}$ to y_k, we get a path from x_k to y_k with length smaller that $d_* + \alpha + \frac{3\eta}{8} < d_{L_k}(x_k, y_k) + \eta$. Furthermore, recalling that the Δ-Hausdorff distance between $\partial_1\mathcal{U}_L$ and $\partial_1\widetilde{\mathbb{D}}$ tends to 0 when $L \to \infty$, and using the bounds $\widetilde{D}(x_\infty, \partial_1\widetilde{\mathbb{D}}) \geq \delta > \eta$, $\widetilde{D}(y_\infty, \partial_1\widetilde{\mathbb{D}}) \geq \delta > \eta$, $\Delta(x_k, x_0^{(k)}) < \frac{3\eta}{16}$, $\Delta(y_k, x_N^{(k)}) < \frac{3\eta}{16}$, and the fact that the geodesic γ stays at distance at least α from $\partial\widetilde{\mathbb{D}}$, one easily verifies that the path from x_k to y_k that we constructed stays at distance at least $\alpha/4$ from $\partial_1\mathcal{U}_{L_k}$ when k is large. Taking k such that $\varepsilon_k < \alpha/4$, we get a contradiction with our initial assumption. This contradiction completes the proof. □

For $\delta > 0$, we set

$$\mathcal{V}_{L,\delta} := \{x \in V(\mathcal{U}_L) : d_L(x, \partial_1\mathcal{U}_L) \geq \delta\}.$$

From the a.s. convergence (33), the latter set is not empty for δ small. We will view $\mathcal{V}_{L,\delta}$ as a bipointed metric space with distinguished points

$$z_{L,\delta} := \Theta'_L(\inf\{j \in \{0, 1, \dots, n_L\} : d_L(\Theta'_L(j), \partial_1\mathcal{U}_L) \geq \delta\}),$$
$$z'_{L,\delta} := \Theta'_L(\sup\{j \in \{0, 1, \dots, n_L\} : d_L(\Theta'_L(j), \partial_1\mathcal{U}_L) \geq \delta\}).$$

Again this definition makes sense if δ is small enough, which we assume in what follows. We also view $(V(\mathcal{U}_L), d_L)$ as a bipointed metric space whose distinguished points are $\Theta'_L(0)$ and $\Theta'_L(n_L)$. Finally, we recall the notation $\mathrm{d}_{GH\bullet\bullet}$ for the bipointed Gromov-Hausdorff distance (Section 2.1).

Lemma 10. *Almost surely, we can find a sequence of values of δ tending to 0 along which we have*

$$\lim_{\delta \to 0} \left(\limsup_{L \to \infty} \mathrm{d}_{GH\bullet\bullet}\Big((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)) \Big) \right) = 0.$$

Proof. We may assume that all metric spaces $(V(\mathcal{U}_L), d_L)$ and $(\widetilde{\mathbb{D}}, \widetilde{D})$ are embedded isometrically in (E, Δ) in the way explained at the beginning of the proof of Lemma 9. Set

$$\widetilde{\mathbb{D}}_\delta := \{x \in \widetilde{\mathbb{D}} : \widetilde{D}(x, \partial_1\widetilde{\mathbb{D}}) \geq \delta\}.$$

We view $(\widetilde{\mathbb{D}}_\delta, \widetilde{D})$ as a bipointed metric space whose distinguished points are

$$\widetilde{z}_\delta := \widetilde{\Gamma}\big(\inf\{t \in [0, \xi] : \widetilde{D}(\widetilde{\Gamma}(t), \partial_1\widetilde{\mathbb{D}}) \geq \delta\}\big),\ \widetilde{z}'_\delta := \widetilde{\Gamma}\big(\sup\{t \in [0, \xi] : \widetilde{D}(\widetilde{\Gamma}(t), \partial_1\widetilde{\mathbb{D}}) \geq \delta\}\big)$$

(this makes sense for $\delta > 0$ small). We claim that

$$\mathcal{V}_{L,\delta} \underset{L \to \infty}{\longrightarrow} \widetilde{\mathbb{D}}_\delta, \tag{34}$$

for the Δ-Hausdorff distance, provided that $\delta > 0$ is not a local maximum of the function $x \mapsto \widetilde{D}(x, \partial_1\widetilde{\mathbb{D}})$ on $\mathbb{D}$ (there are only countably many such local maxima).

Indeed, suppose that, for some $\alpha > 0$, there exists a sequence $L_k \uparrow \infty$ and, for every k, a point $x_k \in \widetilde{\mathbb{D}}_\delta$ such that $\Delta(x_k, \mathcal{V}_{L_k,\delta}) > \alpha$. By compactness, we may assume that $x_k \longrightarrow x_\infty \in \widetilde{\mathbb{D}}_\delta$. From the condition we imposed on δ, we can find $x'_\infty \in \widetilde{\mathbb{D}}$ such that $\widetilde{D}(x_\infty, x'_\infty) < \alpha/3$ and $\widetilde{D}(x'_\infty, \partial_1\widetilde{\mathbb{D}}) > \delta$. Since $V(\mathcal{U}_{L_k})$ converges to $\widetilde{\mathbb{D}}$ in the sense of the Δ-Hausdorff distance, we can find a point $x'_k \in V(\mathcal{U}_{L_k})$, for every k, in such a way that $\Delta(x'_k, x'_\infty) \longrightarrow 0$ as $k \to \infty$. This last property and the convergence of $\partial_1\mathcal{U}_{L_k}$ to $\partial_1\widetilde{\mathbb{D}}$ ensure that, for k large enough, we have $\Delta(x'_k, \partial_1\mathcal{U}_{L_k}) > \delta$ and thus $x'_k \in \mathcal{V}_{L_k,\delta}$. Finally, writing $d_{L_k}(x_k, x'_k) \leq \Delta(x_k, x_\infty) + \Delta(x_\infty, x'_\infty) + \Delta(x'_\infty, x'_k)$, we see that we have also $d_{L_k}(x_k, x'_k) < \alpha$ when k is large, which contradicts $\Delta(x_k, \mathcal{V}_{L_k,\delta}) > \alpha$. This contradiction shows that, for every $\varepsilon > 0$, the set $\widetilde{\mathbb{D}}_\delta$ is contained in $\{x \in E : \Delta(x, \mathcal{V}_{L,\delta}) < \varepsilon\}$ when L is large enough.

A similar (easier) argument shows that, for every $\varepsilon > 0$, the set $\mathcal{V}_{L,\delta}$ is contained in $\{x \in E : \Delta(x, \widetilde{\mathbb{D}}_\delta) < \varepsilon\}$ when L is large. This completes the proof of our claim (34).

We then observe that we have also, on the event where $\widetilde{z}_\delta$ and $\widetilde{z}'_\delta$ are defined,

$$z_{L,\delta} \underset{L \to \infty}{\longrightarrow} \widetilde{z}_\delta\,, \quad z'_{L,\delta} \underset{L \to \infty}{\longrightarrow} \widetilde{z}'_\delta\,, \tag{35}$$

provided that δ is not a local maximum of the function $[0, \xi + \mathcal{Y}] \ni t \mapsto \widetilde{D}(\widetilde{\Gamma}(t), \partial_1\widetilde{\mathbb{D}})$. In fact, under the last assumption, using the uniform convergence of $\widehat{\Theta}'_L$ to $\widetilde{\Gamma}$ and the fact that $\partial_1\mathcal{U}_L$ converges to

$\partial_1\widetilde{\mathbb{D}}$ for the Δ-Hausdorff distance, it is easy to verify that $\inf\{t \in [0, n_L/L] : d_L(\widehat{\Theta}'_L(t), \partial_1\mathcal{U}_L) \geq \delta\}$ converges to $\inf\{t \in [0,\xi] : \widetilde{D}(\widetilde{\Gamma}(t), \partial_1\widetilde{\mathbb{D}}) \geq \delta\}$ as $L \to \infty$, which gives the first convergence in (35). A similar argument applies to the second convergence.

From (34) and (35), we conclude that $\mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (\widetilde{\mathbb{D}}_\delta, \widetilde{D}, \widetilde{z}_\delta, \widetilde{z}'_\delta))$ converges to 0 as $L \to \infty$. Finally writing

$$\begin{aligned}
&\mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L))) \\
&\quad \leq \mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (\widetilde{\mathbb{D}}_\delta, \widetilde{D}, \widetilde{z}_\delta, \widetilde{z}'_\delta)) \\
&\quad + \mathrm{d}_{GH\bullet\bullet}((\widetilde{\mathbb{D}}_\delta, \widetilde{D}, \widetilde{z}_\delta, \widetilde{z}'_\delta), (\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))) \\
&\quad + \mathrm{d}_{GH\bullet\bullet}((\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi)), (V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L))),
\end{aligned}$$

we obtain (for δ belonging to a suitable sequence converging to 0)

$$\limsup_{L\to\infty} \mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}), (V(\mathcal{U}_L), d_L, \Theta'_L(0))) \leq \mathrm{d}_{GH\bullet\bullet}((\widetilde{\mathbb{D}}_\delta, \widetilde{D}, \widetilde{z}_\delta), (\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0))).$$

This completes the proof since the right-hand-side tends to 0 as $\delta \to 0$. □

5.3 Comparing the length of discrete and continuous paths

We again use Skorokhod's representation theorem, which allows us to assume that (32) holds almost surely. We then fix ω (in the underlying probability space) such that (32) holds. We may and will also assume that ω has been chosen so that the properties of Proposition 8 hold. By Proposition 1, we may embed all measure metric spaces $(V(\mathcal{T}_L), \sqrt{3/2}\, L^{-1/2}\mathrm{d_{gr}})$ and $(\mathbb{D}, D)$ isometrically in the same compact metric space (E, Δ), in such a way that

$$V(\mathcal{T}_L) \underset{L\to\infty}{\longrightarrow} \mathbb{D}$$

for the Hausdorff distance Δ_{Haus}, the distinguished point $v_*^{(L)}$ of $V(\mathcal{T}_L)$ converges to the distinguished point $\mathbf{x}_*$ of $\mathbb{D}$, the boundary paths $\widehat{\Theta}_L$ converge uniformly to Γ (and in particular $\partial\mathcal{T}_L$ converges to $\partial\mathbb{D}$ for the Hausdorff distance), and we also have the weak convergence

$$\nu_L \underset{L\to\infty}{\longrightarrow} \mathbf{V}.$$

Notice that we have then $r_0^{(L)} \longrightarrow r_0$, where we recall that $r_0 = D(x_*, \partial\mathbb{D})$ and $r_0^{(L)}$ is the rescaled graph distance between $v_*^{(L)}$ and $\partial\mathcal{T}_L$.

We state a simple lemma that follows from the preceding convergences. The notation $\mathrm{Length}(\gamma)$ stands for the length of a continuous path γ in $(\mathbb{D}, D)$.

Lemma 11. *Let ω be fixed as explained above, and let $A > 0$. For every $\varepsilon > 0$, we can find L_0 such that the following properties hold for every $L \geq L_0$.*

(i) *Let $(\gamma(t))_{0\leq t\leq 1}$ be a continuous path in $\mathbb{D}$. Then we can find a discrete path $\gamma_L = (u_0, u_1, \ldots, u_p)$ in $\mathcal{T}_L$ such that $\Delta(\gamma(0), u_0) < \varepsilon$, $\Delta(\gamma(1), u_p) < \varepsilon$ and $\Delta(u_i, \{\gamma(t) : 0 \leq t \leq 1\}) < \varepsilon$ for every $i \in \{1, \ldots, p-1\}$. If $\mathrm{Length}(\gamma) \leq A$, we can construct $(u_0, u_1, \ldots, u_p)$ so that the rescaled $\mathrm{d_{gr}}$-length of γ_L is bounded above by $\mathrm{Length}(\gamma) + \varepsilon$. In all cases, if $\gamma(1) \in \partial\mathbb{D}$, we can take $u_p \in \partial\mathcal{T}_L$.*

(ii) *If $(u_0, u_1, \ldots, u_p)$ is a discrete path in $\mathcal{T}_L$, we can find a continuous path $\gamma = (\gamma(t))_{0\leq t\leq 1}$ in $\mathbb{D}$ such that $\Delta(\gamma(0), u_0) < \varepsilon$, $\Delta(\gamma(1), u_p) < \varepsilon$ and, for every $t \in [0,1]$, $\Delta(\gamma(t), \{u_0, u_1, \ldots, u_p\}) < \varepsilon$. If the rescaled $\mathrm{d_{gr}}$-length of $(u_0, \ldots, u_p)$ is smaller than A, we can construct γ so that the length of γ is bounded above by the rescaled $\mathrm{d_{gr}}$-length of $(u_0, \ldots, u_p)$ plus ε. In all cases, if $u_p \in \partial\mathcal{T}_L$, we can take $\gamma(1) \in \partial\mathbb{D}$.*

Proof. (i) Let $N \geq 2$ be an integer such that $N(\varepsilon/4) \geq A$. We choose L_0 large enough so that $\Delta_{\mathrm{Haus}}(V(\mathcal{T}_L), \mathbb{D}) < \frac{\varepsilon}{4N}$ and $\Delta_{\mathrm{Haus}}(\partial\mathcal{T}_L, \partial\mathbb{D}) < \frac{\varepsilon}{4N}$ for every $L \geq L_0$. Then, let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $D(\gamma(t_{j-1}), \gamma(t_j)) \leq \varepsilon/4$ for every $1 \leq j \leq k$. For every $0 \leq j \leq k$, we can find

$v_j \in V(\mathcal{T}_L)$ with $\Delta(v_j, \gamma(t_j)) < \varepsilon/(4N)$ (and in the case $\gamma(1) \in \partial\mathbb{D}$ we can take $v_k \in \partial\mathcal{T}_L$). We note that, for $1 \leq j \leq N$,

$$\Delta(v_{j-1}, v_j) \leq \Delta(v_{j-1}, \gamma(t_{j-1})) + \Delta(\gamma(t_{j-1}), \gamma(t_j)) + \Delta(\gamma(t_j), v_j) \leq \Delta(\gamma(t_{j-1}), \gamma(t_j)) + \frac{\varepsilon}{2N} \leq \frac{\varepsilon}{2}.$$

We construct the path $(u_0, \ldots, u_p)$ as the concatenation of (graph distance) geodesics between v_{j-1} and v_j, for $1 \leq j \leq k$. Then, for $i \in \{0, 1, \ldots, p\}$, the vertex u_i belongs to the geodesic between v_{j-1} and v_j, for some j, and $\Delta(u_i, \gamma(t_j)) \leq \Delta(u_i, v_j) + \Delta(v_j, \gamma(t_j)) \leq \Delta(v_{j-1}, v_j) + \varepsilon/4 < \varepsilon$.

If $\mathrm{Length}(\gamma) \leq A$, we observe that we can then take $k = N$ in the preceding considerations (thanks to the condition $N(\varepsilon/4) \geq A$). It follows that the rescaled $\mathrm{d_{gr}}$-length of the path $(u_0, \ldots, u_p)$ is equal to

$$\sum_{j=1}^{N} \Delta(v_{j-1}, v_j) \leq \sum_{j=1}^{N} \Delta(\gamma(t_{j-1}), \gamma(t_j)) + N\,\frac{\varepsilon}{2N} \leq \mathrm{Length}(\gamma) + \frac{\varepsilon}{2}.$$

The proof of (ii) is similar and omitted. □

5.4 The key technical result

Recall the convergences in distribution (32) and (33). By a tightness argument, we may find a sequence $L_n \uparrow \infty$ along which these two convergences hold jointly. From now on, we restrict our attention to values of L belonging to this sequence. Then, by an application of the Skorokhod representation theorem, we may assume that both convergences (32) and (33) hold a.s.

Proposition 12. *Under the preceding assumptions, the bipointed metric space* $(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))$ *coincides a.s. with the space* $(\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi))$.

The proof of Proposition 12 occupies the remaining part of this section, and will rely on several lemmas. From now on until the end of this section, we fix ω such that both convergences (32) and (33) hold. We also embed all measure metric spaces $(V(\mathcal{T}_L), \sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}})$ and $(\mathbb{D}, D)$ isometrically in the same compact metric space (E, Δ) in the way explained at the beginning of Section 5.3. In the following, the word “distance” refers to the distance Δ on the space E in which the measure metric spaces $(V(\mathcal{T}_L), \sqrt{3/2}\,L^{-1/2}\mathrm{d_{gr}})$ and $(\mathbb{D}, D)$ are embedded isometrically.

Recall that $\mathcal{W}_L$ denotes the set of all vertices of the revealed region at time ζ_L, and that $\mathcal{V}_L$ is the set of all vertices of the unrevealed region at the same time. We note that $V(\mathcal{T}_L) = \mathcal{V}_L \cup \mathcal{W}_L$, and that $\mathcal{V}_L \cap \mathcal{W}_L = \partial\mathcal{W}_L$ is the boundary of the revealed region. We also observe that $\partial\mathcal{W}_L \cap \partial\mathcal{T}_L$ consists of the single point α_L, and every point of $\mathcal{V}_L$ except α_L corresponds to a single point of $V(\mathcal{U}_L)$. We already noticed that any path from $\mathcal{W}_L$ to $\partial\mathcal{T}_L$ must visit a vertex whose distance from $v_*^{(L)}$ is bounded above by $r_0^{(L)}$.

Lemma 13. *For ω fixed as above, we have both*

$$\mathcal{V}_L \underset{L\to\infty}{\longrightarrow} U \tag{36}$$

and

$$\mathcal{W}_L \underset{L\to\infty}{\longrightarrow} H \tag{37}$$

for the Hausdorff distance in (E, Δ).

Proof. For any (nonempty) subset A of E, we write $A_\varepsilon := \{x \in E : \Delta(x, A) < \varepsilon\}$. Let us first discuss the convergence (36). Fix $\varepsilon > 0$. We start by verifying that $U \subset (\mathcal{V}_L)_\varepsilon$ for all large enough L. To this end, first suppose that $x \in \mathrm{Int}(U)$. Then $\Delta(x, H) > 0$ and we can find a continuous path from x to $\partial\mathbb{D}$ that does not intersect H, and therefore this path stays at distance at least $r_0 + \delta$ from the distinguished point $\mathbf{x}_*$, for some $\delta > 0$ depending on x. For L large enough, Lemma 11 (i) then allows us to find $v_{(L)} \in V(\mathcal{T}_L)$ such that $\Delta(x, v_{(L)}) < \varepsilon$, and a path from $v_{(L)}$ to $\partial\mathcal{T}_L$ in $\mathcal{T}_L$ that stays at distance at least $r_0 + \delta/2 > r_0^{(L)}$ from the distinguished point $v_*^{(L)}$. Then $v_{(L)}$ must belong to $\mathcal{V}_L$, and we conclude that $x \in (\mathcal{V}_L)_\varepsilon$ for L large enough. This also holds when $x \in \partial U$, because we can then approximate x by a point of $\mathrm{Int}(U)$ close to x and use the preceding conclusion with ε replaced by $\varepsilon/2$. We can now use a compactness argument to verify that $U \subset (\mathcal{V}_L)_\varepsilon$ for all sufficiently large L. Indeed, if this does not hold, we can find a sequence $L_k \uparrow \infty$ and, for every k, a point $x_k \in U$ that does not belong to $(\mathcal{V}_{L_k})_\varepsilon$. Up to extracting a subsequence we can assume that $x_k \longrightarrow x_\infty \in U$, and the fact that $x_\infty \in (\mathcal{V}_L)_{\varepsilon/2}$ for L sufficiently large forces $x_k \in (\mathcal{V}_{L_k})_\varepsilon$ for k large, which is a contradiction.

To complete the proof of (36), we also need to verify that $\mathcal{V}_L \subset U_\varepsilon$ for L large. If $v \in \mathcal{V}_L$, we can find a path from v to $\partial\mathcal{T}_L$ that stays at distance at least $r_0^{(L)} - \sqrt{2/3}\,L^{-1/2}$ from $v_*^{(L)}$. Assuming that L is large enough, Lemma 11 (ii) allows us to find $x \in \mathbb{D}$ with $\Delta(x,v) < \varepsilon/2$, such that there exists a path from x to $\partial\mathbb{D}$ that stays at distance at least $r_0 - \delta$ from $\mathbf{x}_*$, where $\delta > 0$ can be chosen arbitrarily small (note that the choice of L "large enough" depends on ε and δ, but can be made uniformly in v). From the property stated in Proposition 8 (i), if δ has been chosen small enough, this implies that $x \in U_{\varepsilon/2}$ and thus $v \in U_\varepsilon$.

Let us turn to (37). We need to verify that, for every $\varepsilon > 0$, we have for L large enough,

$$H \subset (\mathcal{W}_L)_\varepsilon \tag{38}$$

and

$$\mathcal{W}_L \subset H_\varepsilon. \tag{39}$$

Consider the second inclusion (39). It is enough to verify that, for L sufficiently large, if $v \in V(\mathcal{T}_L)$ and $v \notin H_\varepsilon$, then necessarily $v \in V(\mathcal{T}_L)\backslash\mathcal{W}_L$. By the property stated in Proposition 8 (ii), we can find $\delta = \delta(\varepsilon) > 0$ such that, if $x \in \mathbb{D}\backslash H_{\varepsilon/2}$, there is a path from x to $\partial\mathbb{D}$ in $\mathbb{D}$ that stays at distance greater than $r_0 + \delta(\varepsilon)$ from $\mathbf{x}_*$ (and thus at distance greater than $\delta(\varepsilon)$ from H). Then, if $v \in V(\mathcal{T}_L)$ and $v \notin H_\varepsilon$, we can find (for L large, independently of the choice of v) $x \in \mathbb{D}$ such that $\Delta(v,x) < \alpha$, where $\alpha = (\varepsilon/2) \wedge (\delta(\varepsilon)/4)$. Since $\alpha \leq \varepsilon/2$, it is immediate that $x \notin H_{\varepsilon/2}$, hence we can find a path from x to $\partial\mathbb{D}$ that stays at distance greater than $r_0 + \delta(\varepsilon)$ from $\mathbf{x}_*$. Using Lemma 11 (i), we then obtain that, for L large (independently of our choice of v), there is a path from v to $\partial\mathcal{T}_L$ in $V(\mathcal{T}_L)$ that stays at distance greater than $r_0^{(L)} + \delta(\varepsilon)/2$ from $v_*^{(L)}$. This implies that $v \in V(\mathcal{T}_L)\backslash\mathcal{W}_L$ as required.

Let us finally discuss (38). Fix $\alpha \in (0, (\varepsilon/3) \wedge r_0)$, and let $x \in H$ such that $\Delta(x,U) \geq \alpha$. Since we know by the first part of the proof that $\mathcal{V}_L$ converges to U in the Hausdorff metric, we have $\mathcal{V}_L \subset U_{\alpha/2}$ for L large, and this implies that $\Delta(x,\mathcal{V}_L) \geq \alpha/2$. Assuming that L is even larger, we can find $v_{(L)} \in \mathcal{T}_L$ such that $\Delta(x, v_{(L)}) < \alpha/4$, and then $\Delta(v_{(L)}, \mathcal{V}_L) \geq \alpha/4$, which implies that $v_{(L)} \in V(\mathcal{T}_L)\backslash\mathcal{V}_L$. We conclude that, for L large,

$$\{x \in H : \Delta(x,U) \geq \alpha\} \subset (V(\mathcal{T}_L)\backslash\mathcal{V}_L)_{\alpha/4} \subset (\mathcal{W}_L)_{\alpha/4}.$$

On the other hand, if $x \in H$ is such that $\Delta(x,U) < \alpha$, then necessarily $\Delta(\mathbf{x}_*, x) \leq r_0 + \alpha$ (points of $\partial H = \partial U$ are at distance r_0 from $\mathbf{x}_*$) and we can find on a geodesic from x to $\mathbf{x}_*$ a point x' such that $\Delta(\mathbf{x}_*, x') = r_0 - \alpha$ and $\Delta(x,x') \leq 2\alpha$. Clearly $\Delta(x',U) \geq \alpha$, and x' belongs to the set in the left-hand side of the last display. We conclude that $x \in (\mathcal{W}_L)_{9\alpha/4}$. By combining the two cases, and using $\alpha < \varepsilon/3$, we obtain that $H \subset (\mathcal{W}_L)_\varepsilon$, which completes the proof of (38) and Lemma 13. □

Recall from Section 4.3 the definition of the metric space (U', d_∞) and of its "boundary" $\partial U' = U'\backslash\mathrm{Int}(U)$. For $\delta > 0$, we set

$$U_{(\delta)} := \{x \in U' : d_\infty(x, \partial U') \geq \delta\}$$

(this set is not empty if δ is small enough). We state an analog of Lemma 9, which is easier to establish.

Lemma 14. *Let $\delta > 0$ and $\eta > 0$. Then there exists $\varepsilon_0 > 0$ such that, for every $x, y \in U_{(\delta)}$, there is a path from x to y in $\mathrm{Int}(U)$ that stays at d_∞-distance at least ε_0 from $\partial U'$, and whose d_∞-length is bounded above by $d_\infty(x,y) + \eta$.*

Proof. If the conclusion of the lemma fails, then we can find a sequence $\varepsilon_k \downarrow 0$, and, for every k, two points x_k and y_k in $U_{(\delta)}$ such that any path connecting x_k to y_k and staying at d_∞-distance at least ε_k from $\partial U'$ has length at least $d_\infty(x_k, y_k) + \eta$. By compactness, we may assume that $x_k \longrightarrow x_\infty$ and $y_k \longrightarrow y_\infty$, with $x_\infty, y_\infty \in U_{(\delta)}$. By the definition of the intrinsic distance d_∞, there is a curve $(\gamma(t))_{0 \leq t \leq 1}$ connecting x_∞ to y_∞ and staying in $\mathrm{Int}(U)$ whose length is smaller than $d_\infty(x,y) + \eta/2$.

We can find $\alpha > 0$ such that $d_\infty(\gamma(t), \partial U') > \alpha$ for every $t \in [0,1]$. By concatenating the path γ with D-geodesics from x_k to x_∞ and from y_∞ to y_k, we get, for all large k, a path from x_k to y_k which stays at distance at least $\alpha/2$ from $\partial U'$ and whose length is smaller than $d_\infty(x,y) + \eta$. This gives a contradiction when $\varepsilon_k < \alpha/2$. □

Recall that Proposition 7 yields a canonical identification of the measure metric spaces $(U', d_\infty, \mathbf{V}_U)$ and $(\mathbb{U}, D_\star, \mathbf{V}_\star)$, so that we may view $U_{(\delta)}$ as a closed subset of $\mathbb{U}$. We will point the metric space $(U_{(\delta)}, d_\infty)$ at the distinguished points

$$z_\delta := \Pi_\star(\inf\{t \in [0,\xi] : \ell_{\Pi_\star(t)} \geq \delta\}),\ z'_\delta := \Pi_\star(\sup\{t \in [0,\xi] : \ell_{\Pi_\star(t)} \geq \delta\}),$$

which are the "first" and "last" points of $\partial_0\mathbb{U}$ at distance δ from $\partial_1\mathbb{U}$. The definition of z_δ and z'_δ makes sense for δ small enough, which we assume from now on.

Lemma 15. *We have*

$$\mathrm{d}_{GH\bullet\bullet}((U_{(\delta)}, d_\infty, z_\delta, z'_\delta), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi))) \underset{\delta\to 0}{\longrightarrow} 0.$$

This is immediate since $U_{(\delta)}$ is identified to the closed subset $\{x \in \mathbb{U} : D_\star(x, \partial_1\mathbb{U}) \geq \delta\}$ which converges to $\mathbb{U}$ in the sense of the Hausdorff distance, and we have also $z_\delta \longrightarrow \Pi_\star(0)$ and $z'_\delta \longrightarrow \Pi_\star(\xi)$ as $\delta \to 0$.

Lemma 16. *For all but countably many values of $\delta > 0$, we have*

$$\mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (U_{(\delta)}, d_\infty, z_\delta, z'_\delta)) \underset{L\to\infty}{\longrightarrow} 0.$$

Assume for the moment that we have proved Lemma 16. Then, writing

$$\begin{aligned}
&\mathrm{d}_{GH\bullet\bullet}((V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi)))\\
&\quad\leq \mathrm{d}_{GH\bullet\bullet}((V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)), (\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}))\\
&\quad+ \mathrm{d}_{GH\bullet\bullet}((\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta}), (U_{(\delta)}, d_\infty, z_\delta, z'_\delta))\\
&\quad+ \mathrm{d}_{GH\bullet\bullet}((U_{(\delta)}, d_\infty, z_\delta, z'_\delta), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi))),
\end{aligned}$$

we get, except possibly for countably many values of δ,

$$\begin{aligned}
&\limsup_{L\to\infty} \mathrm{d}_{GH\bullet\bullet}((V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi)))\\
&\quad\leq \mathrm{d}_{GH\bullet\bullet}((U_{(\delta)}, d_\infty, z_\delta, z'_\delta), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi)))\\
&\quad+ \limsup_{L\to\infty} \mathrm{d}_{GH\bullet\bullet}((V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)), (\mathcal{V}_{L,\delta}, d_L, z_{L,\delta}, z'_{L,\delta})).
\end{aligned}$$

By Lemmas 10 and 15, the right-hand-side can be made arbitrarily small by taking δ small. Hence, we have proved that

$$\mathrm{d}_{GH\bullet\bullet}((V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L)), (\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi))) \underset{L\to\infty}{\longrightarrow} 0.$$

Now notice that (33) (which we have assumed to hold pointwise for the fixed value of ω) implies the $\mathrm{d}_{GH\bullet\bullet}$-convergence of the pointed spaces $(V(\mathcal{U}_L), d_L, \Theta'_L(0), \Theta'_L(n_L))$ to $(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))$. The statement of Proposition 12 follows by comparing with the last display.

Proof of Lemma 16. Although $\mathcal{V}_{L,\delta}$ was defined as a subset of $V(\mathcal{U}_L)$, we may and will identify $\mathcal{V}_{L,\delta}$ with the corresponding subset of $\mathcal{V}_L$ (recall that points of $\mathcal{V}_L$ are in one-to-one correspondence with the points of $V(\mathcal{U}_L)$, with the exception of α_L that corresponds to two points of $V(\mathcal{U}_L)$), and we have also $\mathcal{V}_{L,\delta} = \{x \in \mathcal{V}_L : d_L(x, \mathcal{W}_L) \geq \delta\}$. We first observe that

$$\mathcal{V}_{L,\delta} \underset{L\to\infty}{\longrightarrow} U_{(\delta)}, \tag{40}$$

in the sense of the Hausdorff distance $\Delta_{\rm Haus}$, except possibly for countably many values of δ. The proof of (40) is essentially the same as the proof of (34) above, using now the convergence of $\mathcal{W}_L$ to H in Lemma 13. The reader may be puzzled by the fact that the limit of $\mathcal{V}_{L,\delta}$ in (40) is different than the limit in (34): The point is that $\mathcal{V}_{L,\delta}$ was viewed as a subset of $V(\mathcal{U}_L)$ in (34), and we were using (in the proof of Lemma 10) a specific embedding of the metric spaces $V(\mathcal{U}_L)$ whereas in this section we are using another specific embedding of the spaces $V(\mathcal{T}_L)$.

We then also observe that, for all but countably many values of δ, we must have

$$z_{L,\delta} \underset{L\to\infty}{\longrightarrow} z_\delta\,, \quad z'_{L,\delta} \underset{L\to\infty}{\longrightarrow} z'_\delta. \tag{41}$$

Indeed, in the cyclic exploration of $\partial\mathcal{U}_L$ induced by the path Θ_L, $z_{L,\delta}$ is the first visited point after α'_L such that $d_L(z_{L,\delta},\mathcal{W}_L)\geq\delta$ and similarly, in the cyclic exploration of $\partial\mathbb{D}$ induced by Γ, z_δ is the first point after $\mathbf{x}_0$ such that $D(z_\delta,H)\geq\delta$. Moreover, it is easy to verify that α_L converges as $L\to\infty$ to the point $\mathbf{x}_0$ (which is the unique point of $\partial\mathbb{D}$ at minimal distance from the distinguished point of $\mathbb{D}$). Taking the preceding facts into account, one can use the convergence of $\mathcal{W}_L$ to H and the uniform convergence of $\widehat{\Theta}_L$ to Γ to derive the convergence of $z_{L,\delta}$ toward z_δ. A similar argument applies to the convergence of $z'_{L,\delta}$ toward z'_δ.

We then define a correspondence between $(\mathcal{V}_{L,\delta},d_L)$ and $(U_{(\delta)},d_\infty)$ by setting

$$\mathcal{C}_L=\{(x,x')\in\mathcal{V}_{L,\delta}\times U_{(\delta)}:\Delta(x,x')\leq\kappa_L\}$$

where

$$\kappa_L=\Delta_{\rm Haus}(\mathcal{V}_{L,\delta},U_{(\delta)})+\Delta(z_{L,\delta},z_\delta)+\Delta(z'_{L,\delta},z'_\delta).$$

Note that $\kappa_L\longrightarrow 0$ as $L\to\infty$. By construction, the pairs $(z_{L,\delta},z_\delta)$ and $(z'_{L,\delta},z'_\delta)$ belongs to $\mathcal{C}_L$. From the characterization of $\mathrm{d}_{GH\bullet\bullet}$ in terms of correspondences (Section 2.1), it therefore suffices to verify that the distortion of $\mathcal{C}_L$ tends to 0 as $L\to\infty$.

We first verify that

$$\sup_{(x,x')\in\mathcal{C}_L,(y,y')\in\mathcal{C}_L}\Big(d_L(x,y)-d_\infty(x',y')\Big)\underset{L\to\infty}{\longrightarrow}0. \tag{42}$$

To this end, fix $\eta\in(0,1)$, and choose ε_0 as in Lemma 14. We can assume that $\varepsilon_0\leq\eta\wedge\frac{\delta}{2}$. Then choose L_0 as in Lemma 11, with $\varepsilon=\frac{\varepsilon_0}{2}\wedge\frac{\eta}{8}$ and $A=\mathrm{diam}(U',d_\infty)+1$, where $\mathrm{diam}(U',d_\infty)$ denotes the diameter of the metric space (U',d_∞). Taking L_0 larger if necessary, we can assume that $\kappa_L\leq\frac{\varepsilon_0}{4}$ and $\Delta_{\rm Haus}(\mathcal{W}_L,H)\leq\frac{\varepsilon_0}{2}$, for every $L\geq L_0$. Then, let $L\geq L_0$ and let $(x,x')\in\mathcal{C}_L$ and $(y,y')\in\mathcal{C}_L$. By Lemma 14, we can find a path $(\gamma(t))_{t\in[0,1]}$ from x' to y' in $\mathrm{Int}(U)$, which stays at distance at least ε_0 from H and whose length is smaller than $d_\infty(x',y')+\eta$. By Lemma 11 (i), we can then find a discrete path $\gamma_L=(u_0,u_1,\ldots,u_p)$ such that $\Delta(\gamma(0),u_0)<\varepsilon$, $\Delta(\gamma(1),u_p)<\varepsilon$ and $\Delta(u_i,\{\gamma(t):0\leq t\leq 1\})<\varepsilon$ for every $i\in\{1,\ldots,p-1\}$, and the rescaled $\mathrm{d_{gr}}$-length of γ_L is bounded above by $\mathrm{Length}(\gamma)+\varepsilon$. Since $\Delta_{\rm Haus}(\mathcal{W}_L,H)\leq\frac{\varepsilon_0}{2}$ and γ stays at distance at least ε_0 from H, it follows that the path γ_L does not visit $\mathcal{W}_L$, and therefore $d_L(u_0,u_p)$ is bounded above by the rescaled length of γ_L which is bounded by $d_\infty(x',y')+\eta+\varepsilon$. Since $\Delta(x,x')\leq\kappa_L\leq\frac{\varepsilon_0}{4}$, we have $\Delta(u_0,x)\leq\varepsilon+\frac{\varepsilon_0}{4}<\varepsilon_0\leq\frac{\delta}{2}$, and (since $x\in\mathcal{V}_{L,\delta}$) it follows that a geodesic from u_0 to x in $\mathcal{T}_L$ does not intersect $\mathcal{W}_L$, and therefore $d_L(x,u_0)=\Delta(u_0,x)<\varepsilon_0$, and similarly $d_L(y,u_p)<\varepsilon_0$. Finally we get $d_L(x,y)\leq d_L(u_0,u_p)+2\varepsilon_0\leq d_\infty(x',y')+3\eta+\varepsilon$. This shows that the supremum in (42) is bounded above by 4η when $L\geq L_0$, which completes the proof of (42).

It remains to verify that

$$\sup_{(x,x')\in\mathcal{C}_L,(y,y')\in\mathcal{C}_L}\Big(d_\infty(x',y')-d_L(x,y)\Big)\underset{L\to\infty}{\longrightarrow}0. \tag{43}$$

This is very similar to the proof of (42). Fix $\eta>0$. Then, we can use Lemma 9 to find $\varepsilon_0>0$ such that the following holds for all $L\geq L_1$, for some integer $L_1\geq 1$. For any $x,y\in\mathcal{V}_{L,\delta}$, there is a discrete path $\gamma_L=(u_0,\ldots,u_p)$ from x to y in $\mathcal{V}_L$, which stays at rescaled graph distance at least ε_0 from $\mathcal{W}_L$ and whose rescaled length is bounded above by $d_L(x,y)+\eta$. Let L_0 be as in Lemma 11, with A such that $\mathrm{diam}(V(\mathcal{U}_L),d_L)\le A$ for every L (the convergence (33) allows us to find A with this property) and $\varepsilon=\frac{\varepsilon_0}{4}\wedge\frac{\eta}{4}\wedge\frac{\delta}{4}$. Taking L_0 larger if necessary, we can assume that $L_0\ge L_1$ and $\Delta_{\mathrm{Haus}}(\mathcal{W}_L,H)<\varepsilon$ for every $L\ge L_0$, and moreover $\kappa_L<\varepsilon$ for every $L\ge L_0$. Then, if $L\ge L_0$ and $(x,x')\in\mathcal{C}_L$ and $(y,y')\in\mathcal{C}_L$, and if the discrete path $\gamma_L=(u_0,\dots,u_p)$ from x to y is chosen as explained above, Lemma 11 (ii) allows us to find a continuous path $(\gamma(t))_{t\in[0,1]}$ such that $\Delta(\gamma(0),u_0)<\varepsilon$, $\Delta(\gamma(1),u_p)<\varepsilon$ and, for every $t\in[0,1]$, $\Delta(\gamma(t),\{u_0,u_1,\dots,u_p\})<\varepsilon$, and moreover the length of γ is bounded above by the rescaled length of γ_L plus ε. Recalling that $\Delta_{\mathrm{Haus}}(\mathcal{W}_L,H)<\varepsilon$ and that γ_L stays at rescaled graph distance at least ε_0 from $\mathcal{W}_L$, we see that γ does not visit H because this would contradict the property $\Delta(\gamma(t),\{u_0,u_1,\dots,u_p\})<\varepsilon$ for every $t\in[0,1]$. It follows that $d_\infty(\gamma(0),\gamma(1))\le d_L(x,y)+\eta+\varepsilon$. However, we have $\Delta(\gamma(0),x')\le\Delta(\gamma(0),u_0)+\Delta(x,x')\le 2\varepsilon$, and similarly $\Delta(\gamma(1),y')\le 2\varepsilon$, and it follows that a geodesic (in $\mathbb{D}$) from x' to $\gamma(0)$, or from y' to $\gamma(1)$, does not hit H. We finally conclude that $d_\infty(x',y')\le d_\infty(\gamma(0),\gamma(1))+4\varepsilon\le d_L(x,y)+3\eta$. Thus, for $L\ge L_0$, the supremum in (43) is bounded above by 3η. This completes the proof of Lemma 16. □

5.5 The main theorem

Recall the notation $\partial_0\mathbb{U}=\Pi_\star([0,\xi])$ and $\partial_1\mathbb{U}=\Pi_\star(\partial\mathfrak{T}^\star)$ introduced at the end of Section 4.2. We also set

$$\mathcal{Z}_{\mathbb{U}}:=\sum_{i\in I}\mathcal{Z}_0(\omega_i),$$

where we recall the point measures $\mathcal{N}=\sum_{i\in I}\delta_{(t_i,\omega_i)}$ and $\widetilde{\mathcal{N}}=\sum_{i\in I}\delta_{(t_i,\widetilde{\omega}_i)}$ used in Section 4 to define $\mathbb{D}$ and $\mathbb{U}$ respectively. A first-moment argument shows that $\mathcal{Z}_{\mathbb{U}}<\infty$ a.s. Specifically, using formula (4) one gets that $\mathbb{N}_y(\mathcal{Z}_0)=1$ for every $y>0$, and consequently:

$$\mathbb{E}[\mathcal{Z}_{\mathbb{U}}]=2\,\mathbb{E}\Big[\int_0^\xi \mathrm{d}t\ \mathbb{N}_{\sqrt{3}\mathbf{e}_t}(\mathcal{Z}_0)\Big]=2\,\xi. \tag{44}$$

We next introduce a path $\Gamma^\star$ parametrizing $\partial_0\mathbb{U}\cup\partial_1\mathbb{U}$. For every $i\in I$, we let $(\widetilde{L}^0_t(\omega_i))_{t\ge0}$ be the (time changed) exit local time of ω_i from $(0,\infty)$, as given by formula (5) with $y=0$. In the time scale of the clockwise exploration $(\mathcal{E}^\star_s)_{0\le s\le\Sigma^\star}$ of $\mathfrak{T}^\star$, each snake trajectory $\widetilde{\omega}_i$ corresponds to the interval $[r_i,r_i+\sigma(\widetilde{\omega}_i)]$, where

$$r_i:=\sum_{j\in I:t_j<t_i}\sigma(\widetilde{\omega}_i).$$

We then set, for every $s\in[0,\Sigma^\star]$,

$$\widetilde{L}^\star_s:=\sum_{i\in I}\widetilde{L}^0_{(s-r_i)^+}(\omega_i),$$

so that $\widetilde{L}^\star_s$ represents the total exit local time accumulated at 0 by the clockwise exploration of $\mathfrak{T}^\star$ up to time s. We set $\kappa_\star(t):=\inf\{s\ge0:\widetilde{L}^\star_s>t\}$ for every $t\in[0,\mathcal{Z}_{\mathbb{U}})$, and $\kappa_\star(\mathcal{Z}_{\mathbb{U}}):=\Sigma^\star$.

Then, for every $t\in[0,\xi]$, we set $\Gamma^\star(t):=\Pi_\star(t)$, and we also define $\Gamma^\star(t)$ for $\xi\le t\le\xi+\mathcal{Z}_{\mathbb{U}}$ by setting

$$\Gamma^\star(\xi+\mathcal{Z}_{\mathbb{U}}-s):=\Pi_\star(\mathcal{E}^\star_{\kappa_\star(s)}),\quad\text{for every } s\in[0,\mathcal{Z}_{\mathbb{U}}]. \tag{45}$$

The two definitions of $\Gamma^\star(\xi)$ are consistent since $\mathcal{E}^\star_{\kappa_\star(\mathcal{Z}_{\mathbb{U}})}=\mathcal{E}^\star_{\Sigma^\star}=\xi$. We also note that $\Gamma^\star(0)=\Gamma^\star(\xi+\mathcal{Z}_{\mathbb{U}})=\Pi_\star(0)$. Furthermore, we have $\{\Gamma^\star(t):0\le t\le\xi\}=\partial_0\mathbb{U}$, and $\{\Gamma^\star(t):\xi\le t\le\xi+\mathcal{Z}_{\mathbb{U}}\}\}=\Pi_\star(\partial\mathfrak{T}^\star)=\partial_1\mathbb{U}$, by the support property of the exit local time. So the range of $(\Gamma^\star(s))_{0\le s\le\mathcal{Z}_{\mathbb{U}}+\xi}$ is $\partial_0\mathbb{U}\cup\partial_1\mathbb{U}$.

We next observe that $\Gamma^\star$ is continuous, and is injective on the interval $[0,\xi+\mathcal{Z}_{\mathbb{U}})$, The injectivity property is a direct consequence of Proposition 6, and we omit the details. On the other hand, we already know that the function $[0,\xi]\ni t\mapsto\Pi_\star(t)$ is continuous (from the continuity properties of $(a,b)\mapsto D_\star(a,b)$, cf. Section 4.2). So it remains to show that $(\Gamma^\star(\xi+\mathcal{Z}_{\mathbb{U}}-s))_{0\le s\le\mathcal{Z}_{\mathbb{U}}}$ is continuous. Let us briefly justify this point. Since $[0,\Sigma^\star]\ni t\mapsto\Pi_\star(\mathcal{E}^\star_t)$ is continuous and $t\mapsto\kappa_\star(t)$ is right-continuous, we only need to show that if $\kappa_\star(t-)<\kappa_\star(t)$ then $\Pi_\star(\mathcal{E}^\star_{\kappa_\star(t-)})=\Pi_\star(\mathcal{E}^\star_{\kappa_\star(t)})$. To this end, notice that if $\kappa_\star(t-)<\kappa_\star(t)$, the support property of the exit local time implies that all points of the form $\mathcal{E}^\star_u$ with $u\in(\kappa_\star(t-),\kappa_\star(t))$ have a positive label. Proposition 6 then ensures that $\Pi_\star(\mathcal{E}^\star_{\kappa_\star(t-)})=\Pi_\star(\mathcal{E}^\star_{\kappa_\star(t)})$, as desired.

Theorem 17. *Let* $(\mathbb{U}, D_\star, \mathbf{V}_\star)$ *be defined as in Section 4, and let* $\Gamma^\star$ *be as above. Then* $(\mathbb{U}, D_\star, \mathbf{V}_\star, \Gamma^\star)$ *is a curve-decorated free Brownian disk with a random perimeter* $\xi + \mathcal{Z}_\mathbb{U}$ *distributed according to the measure* $\frac{3}{2}\,\xi^{3/2}\,z^{-5/2}\,\mathbf{1}_{\{z>\xi\}}\,\mathrm{d}z$.

Proof. By Proposition 12, we may suppose that there exists a curve-decorated free Brownian disk $(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\mathbf{V}}, \widetilde{\Gamma})$ with perimeter $\xi + \mathcal{Y}$ such that the random bipointed metric spaces $(\mathbb{U}, D_\star, \Pi_\star(0), \Pi_\star(\xi))$ and $(\widetilde{\mathbb{D}}, \widetilde{D}, \widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))$ are a.s. equal. We now want to argue that this equality carries over to the volume measures and to the decorating curves, meaning that we have also $\mathbf{V}_\star = \widetilde{\mathbf{V}}$ and $\Gamma^\star = \widetilde{\Gamma}$ (the equality $\Gamma^\star = \widetilde{\Gamma}$ will imply in particular that $\mathcal{Z}_\mathbb{U} = \mathcal{Y}$, which has density $\frac{3}{2}\,\xi^{3/2}\,(\xi + x)^{-5/2}$ by Proposition 5).

The equality $(\mathbb{U}, D_\star) = (\widetilde{\mathbb{D}}, \widetilde{D})$ allows us to define the boundary $\partial\mathbb{U}$ (as the set of all points of $\mathbb{U}$ that have no neighborhood homeomorphic to the unit disk), and we also know that $\partial\mathbb{U}$ is the range of a simple loop. It is easy to verify that any point x of $\mathbb{U}\backslash(\partial_0\mathbb{U} \cup \partial_1\mathbb{U})$ has a neighborhood homeomorphic to the unit disk (just note that the "interior" $\mathbb{U}\backslash(\partial_0\mathbb{U} \cup \partial_1\mathbb{U})$ is identified to a subset of $\mathbb{D}$ in such a way that, in a sufficiently small neighborhood of x, the distance $D_\star$ coincides with the distance D). It follows that the $\partial\mathbb{U}$ is contained in $\partial_0\mathbb{U} \cup \partial_1\mathbb{U}$. Now recall that the range of the simple loop $(\Gamma^\star(s))_{0\le s\le \mathcal{Z}_\mathbb{U}+\xi}$ is precisely $\partial_0\mathbb{U} \cup \partial_1\mathbb{U}$. It follows that the boundary of $\mathbb{U}$ is contained in a simple loop, which is only possible if $\partial\mathbb{U}$ is the whole loop. We have thus $\partial\mathbb{U} = \partial_0\mathbb{U} \cup \partial_1\mathbb{U}$.

Let us then verify that $\mathbf{V}_\star = \widetilde{\mathbf{V}}$. It follows from the main result of [21] that $\widetilde{\mathbf{V}}$ may be defined by

$$\widetilde{\mathbf{V}}(A) = \mathbf{c}\, m_h(A)$$

for every Borel subset A of $\widetilde{\mathbb{D}}$, where $\mathbf{c} > 0$ is a constant, and m_h stands for the Hausdorff measure with gauge function $h(r) = r^4 \log\log(1/r)$. To be precise, the results of [21] apply to the Brownian sphere and not to the Brownian disk. However, we may use the connections between the Brownian sphere and the Brownian disk, and in particular Theorem 8 in [20] showing that the complement of a hull in the Brownian sphere is a Brownian disk (this complement needs to be equipped with the intrinsic distance, but this makes no difference for Hausdorff measures as long as we consider sets that do not intersect the boundary, and on the other hand we know that the boundary has zero volume measure, and zero h-Hausdorff measure by [6]).

So to prove that $\mathbf{V}_\star = \widetilde{\mathbf{V}}$, it is enough to verify that we have also $\mathbf{V}_\star(A) = \mathbf{c}\, m_h(A)$ for every Borel subset A of $\mathbb{U}$. We may restrict our attention to subsets of $\mathbb{U}\backslash\partial_1\mathbb{U}$ since we know that $m_h(\partial\mathbb{U}) = 0$ and $\mathbf{V}_\star(\partial_1\mathbb{U}) = 0$. Then we can use the fact that $\mathbb{U}\backslash\partial_1\mathbb{U}$ is identified to the open subset $\mathrm{Int}(U)$, and the previously mentioned result for the Brownian disk $\mathbb{D}$ to obtain that the equality $\mathbf{V}_\star(A) = \mathbf{c}\, m_h(A)$ holds for every Borel subset of $\mathbb{U}\backslash\partial_1\mathbb{U}$ — here again the fact that we deal with the intrinsic distance on $\mathrm{Int}(U)$ instead of the distance of $\mathbb{D}$ makes no difference for Hausdorff measures. This completes the proof of the equality $\mathbf{V}_\star = \widetilde{\mathbf{V}}$.

To complete the proof of the theorem, we rely on the next lemma.

Lemma 18. *Almost surely,* $\Gamma^\star$ *is a standard boundary curve of* $(\mathbb{U}, D_\star, \mathbf{V}_\star)$.

Assuming the result of the lemma, the proof of the theorem is easily completed. Indeed, since $\Gamma^\star$ and $\widetilde{\Gamma}$ are both standard boundary curves, the property $(\Gamma^\star(0), \Gamma^\star(\xi)) = (\widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))$ can only hold if $\Gamma^\star = \widetilde{\Gamma}$, which was the desired result. □

Remark. The reason for dealing with bipointed spaces throughout this section is the fact that we need the equality $(\Gamma^\star(0), \Gamma^\star(\xi)) = (\widetilde{\Gamma}(0), \widetilde{\Gamma}(\xi))$ to identify $\Gamma^\star$. The equality $\Gamma^\star(0) = \widetilde{\Gamma}(0)$ alone would not be sufficient for this identification.

Proof of Lemma 18. Let $\mu_{\partial\mathbb{U}}$ be the boundary measure of the Brownian disk $\mathbb{U}$. We already know that $(\Gamma^\star(t))_{0\le t\le \xi+\mathcal{Z}_\mathbb{U}}$ is a simple loop, and we will verify that, almost surely for any continuous function Φ on $\partial\mathbb{U}$, we have

$$\int_0^{\xi+\mathcal{Z}_\mathbb{U}} \mathrm{d}t\, \Phi(\Gamma^\star(t)) = \int \mu_{\partial\mathbb{U}}(\mathrm{d}x)\, \Phi(x). \tag{46}$$

This will imply that the perimeter of $\mathbb{U}$ is $\mathcal{Z}_\mathbb{U} + \xi$ (take $\Phi = 1$), and then, by the very definition, that $\Gamma^\star$ is a standard boundary curve of U.

Let us prove (46). We first observe that the restriction of $\mu_{\partial\mathbb{U}}$ to $\partial_0\mathbb{U}$ is the pushforward of Lebesgue measure on $[0,\xi]$ under $\Pi_\star$. This is easy from the identification of Proposition 7, the fact that the boundary measure $\mu_{\partial\mathbb{D}}$ of $\partial\mathbb{D}$ is the pushforward of Lebesgue measure on $[0,\xi]$ under Π, and the approximations of the boundary measure by the volume measure restricted to a tubular neighborhood of the boundary. It follows that

$$\int_0^\xi \mathrm{d}t\,\Phi(\Gamma^\star(t)) = \int_0^\xi \mathrm{d}s\,\Phi(\Pi_\star(s)) = \int_{\partial_0\mathbb{U}} \mu_{\partial\mathbb{U}}(\mathrm{d}x)\,\Phi(x), \tag{47}$$

and in particular $\mu_{\partial\mathbb{U}}(\partial_0\mathbb{U}) = \xi$. We then claim that we have also

$$\int_\xi^{\xi+\mathcal{Z}_\mathbb{U}} \mathrm{d}t\,\Phi(\Gamma^\star(t)) = \int_{\partial_1\mathbb{U}} \mu_{\partial\mathbb{U}}(\mathrm{d}x)\,\Phi(x), \tag{48}$$

which will complete the proof of (46). To derive (48), introduce, for every $\varepsilon > 0$, the measure ν_ε on $\mathbb{U}$ defined by

$$\langle \nu_\varepsilon, \Phi\rangle = \varepsilon^{-2}\int \mathbf{V}_\star(\mathrm{d}x)\,\mathbf{1}_{\{D_\star(x,\partial_1\mathbb{U})<\varepsilon\}}\Phi(x).$$

It follows from the approximations of the boundary measure of $\mathbb{U}$ that

$$\lim_{\varepsilon\to 0}\langle \nu_\varepsilon, \Phi\rangle = \int_{\partial_1\mathbb{U}} \mu_{\partial\mathbb{U}}(\mathrm{d}x)\,\Phi(x).$$

On the other hand, we can also verify that $\langle \nu_\varepsilon, \Phi\rangle$ converges to the left-hand side of (48). Consider first the case $\Phi = 1$. Then, we know that $\langle \nu_\varepsilon, 1\rangle$ converges to $\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U}) = \mu_{\partial\mathbb{U}}(\partial\mathbb{U}) - \xi$, which has density $\frac{3}{2}\,\xi^{3/2}\,(\xi+z)^{-5/2}$ by Theorem 17. In particular, $\mathbb{E}[\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U})] = 2\xi$. On the other hand, recall the notation used in the proof of Theorem 17 and in particular the fact that each $\widetilde{\omega}^i$, $i\in I$, corresponds to the interval $[r_i, r_i+\sigma(\widetilde{\omega}_i)]$ in the time scale of the clockwise exploration $\mathcal{E}^\star$. We also note that we have $D_\star(x,\partial_1\mathbb{U}) = \ell_x = \widehat{W}_{s-r_i}(\widetilde{\omega}_i)$ when $x = \Pi_\star(\mathcal{E}^\star_s)$ with $s\in[r_i, r_i+\sigma(\widetilde{\omega}_i)]$. Using Fatou's lemma, we have

$$\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U}) = \lim_{\varepsilon\to 0}\langle \nu_\varepsilon, 1\rangle \geq \sum_{i\in I}\Big(\liminf_{\varepsilon\to 0}\varepsilon^{-2}\int_0^{\sigma(\widetilde{\omega}_i)}\mathrm{d}t\,\mathbf{1}_{\{\widehat{W}_t(\widetilde{\omega}_i)<\varepsilon\}}\Big) = \sum_{i\in I}\mathcal{Z}_0(\omega_i) = \mathcal{Z}_\mathbb{U},$$

and we know from (44) that $\mathbb{E}[\mathcal{Z}_\mathbb{U}] = 2\xi = \mathbb{E}[\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U})]$. It follows that $\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U}) = \mathcal{Z}_\mathbb{U}$ a.s.

Let us consider now a general continuous function Φ on $\mathbb{U}$. Without loss of generality we can assume that $0\leq\Phi\leq 1$. By Fatou's lemma and the definition of $V_\star$, we have

$$\int_{\partial_1\mathbb{U}} \mu_{\partial\mathbb{U}}(\mathrm{d}x)\,\Phi(x) = \lim_{\varepsilon\to 0}\langle \nu_\varepsilon, \Phi\rangle \geq \sum_{i\in I}\Big(\liminf_{\varepsilon\to 0}\varepsilon^{-2}\int_0^{\sigma(\widetilde{\omega}_i)}\mathrm{d}t\,\mathbf{1}_{\{\widehat{W}_t(\widetilde{\omega}_i)<\varepsilon\}}\Phi(\Pi_\star(\mathcal{E}^\star_{r_i+t}))\Big).$$

Moreover, (5) entails that, for every $i\in I$, the measures $\varepsilon^{-2}\mathbf{1}_{\{\widehat{W}_t(\widetilde{\omega}_i)<\varepsilon\}}\mathbf{1}_{[0,\sigma(\widetilde{\omega}_i)]}(t)\mathrm{d}t$ converge weakly to $\mathrm{d}\widetilde{L}^0_t(\omega^i)$ as $\varepsilon\to 0$. Using the definition of $L^\star_t$, it follows that

$$\int_{\partial_1\mathbb{U}} \mu_{\partial\mathbb{U}}(\mathrm{d}x)\,\Phi(x) = \lim_{\varepsilon\to 0}\langle \nu_\varepsilon, \Phi\rangle \geq \int_0^{\Sigma^\star}\mathrm{d}\widetilde{L}^\star_t\;\Phi(\Pi_\star(\mathcal{E}^\star_t)) = \int_\xi^{\xi+\mathcal{Z}_\mathbb{U}}\mathrm{d}s\;\Phi(\Gamma^\star(s)),$$

where the last equality holds by the definition of $\Gamma^\star(s)$ when $\xi\leq s\leq\xi+\mathcal{Z}_\mathbb{U}$. If we combine the bound of the last display with the same bound when Φ is replaced by $1-\Phi$ (using $\mu_{\partial\mathbb{U}}(\partial_1\mathbb{U}) = \mathcal{Z}_\mathbb{U}$), we arrive at the desired equality (48). This completes the proof. □

Another way of verifying that the boundary size of $\mathbb{U}$ is $\xi+\mathcal{Z}_\mathbb{U}$ would have been to prove the convergence in distribution of $L^{-1}Z_L$ to $\mathcal{Z}_\mathbb{U}$ (independently of Proposition 5) and to check that this convergence holds jointly with the convergence of $(V(\mathcal{U}_L), d_L)$ to $(U', d_\infty) = (\mathbb{U}, D_\star)$.

Remark. It follows from Theorem 17 that the density of the distribution of $\mathcal{Z}_\mathbb{U}$ is $\frac{3}{2}\,\xi^{3/2}\,(\xi+z)^{-5/2}$. This can be verified by the following direct calculation, where we take $\xi = 1$ for simplicity. For $\lambda > 0$, we have

$$\mathbb{E}[e^{-\lambda\mathcal{Z}_\mathbb{U}}\mid \mathbf{e}] = \exp\Big(-2\int_0^1\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\mathbf{e}_t}(1-\exp(-\lambda\mathcal{Z}_0))\Big) = \exp\Big(-\int_0^1\mathrm{d}t\,(\mathbf{e}_t+(2\lambda)^{-1/2})^{-2}\Big),$$

using formula (4). Then, for every $\alpha > 0$,

$$\mathbb{E}\Big[\exp\Big(-\int_0^1 \mathrm{d}t\,(\mathbf{e}_t+\alpha)^{-2}\Big)\Big] = 1-\sqrt{\frac{\pi}{2}}\,\alpha^{-3}\,\chi_1(\frac{1}{2\alpha^2}),$$

where $\chi_1(x) = \frac{1}{\sqrt{\pi}}x^{-1/2} - e^x\,\mathrm{erfc}(\sqrt{x})$ as in [22]. This formula is the special case $F = 1$ of Lemma 19 below. It follows that

$$\mathbb{E}[e^{-\lambda \mathcal{Z}_{\mathbb{U}}}] = 1 - 2\sqrt{\pi}\,\lambda^{3/2}\,\chi_1(\lambda).$$

To invert the Laplace transform, start from the classical formula

$$\int_0^\infty e^{-\lambda x}\,\frac{\mathrm{d}x}{\sqrt{\pi(x+1)}} = \frac{1}{\sqrt{\lambda}}\,e^{\lambda}\mathrm{erfc}(\sqrt{\lambda}).$$

Two integrations by parts then give

$$\frac{3}{2}\int_0^\infty (x+1)^{-5/2}\,e^{-\lambda x}\,\mathrm{d}x = 1-2\lambda+2\sqrt{\pi}\lambda^{3/2}\,e^{\lambda}\,\mathrm{erfc}(\sqrt{\lambda}) = 1-2\sqrt{\pi}\,\lambda^{3/2}\,\chi_1(\lambda) = \mathbb{E}[e^{-\lambda\mathcal{Z}_{\mathbb{U}}}],$$

and we conclude that the density of $\mathcal{Z}_{\mathbb{U}}$ is $\frac{3}{2}(x+1)^{-5/2}$.

6 Peeling the Brownian disk

Our goal in this section is to discuss a peeling exploration for Brownian disks. In particular, we will see that the complement of a hull centered at a boundary point in a Brownian disk is again a Brownian disk. Our study relies on the representation derived in Theorem 17.

6.1 Preliminary distributional identities

We first need to introduce some notation. We fix $\xi > 0$ and $r > 0$. As previously, we write $(\mathbf{e}_t)_{0\le t\le\xi}$ for a positive Brownian excursion of duration ξ. We then consider a five-dimensional Bessel process $(X_t)_{t\ge 0}$ started from r. Since r is a regular point for this Markov process, we can define an (infinite) excursion measure away from r, and we can also make sense of the law of the excursion of duration ξ above level r for the process X. We let $(\overline{\mathbf{e}}^{(r)}_t)_{0\le t\le\xi}$ be distributed according to this law, and set $\mathbf{e}^{(r)}_t := \overline{\mathbf{e}}^{(r)}_t - r$ for $0 \le t \le \xi$, so that $\mathbf{e}^{(r)}$ starts and ends at 0. We let $C([0,\xi],\mathbb{R})$ stand for the set of all continuous functions from $[0,\xi]$ into $\mathbb{R}$, which is equipped with the sup norm.

Lemma 19. *For every $u \ge 0$, set*

$$\Phi(u) := 1-2u+2\sqrt{\pi}u^{3/2}e^u\mathrm{erfc}(\sqrt{u}) = 1-2\sqrt{\pi}u^{3/2}\chi_1(u).$$

Then, for every bounded continuous function $F : C([0,\xi],\mathbb{R}) \longrightarrow \mathbb{R}$, we have

$$\mathbb{E}\Big[F\Big((\mathbf{e}^{(r)}_t)_{0\le t\le\xi}\Big)\Big] = \Phi(\frac{\xi}{2r^2})^{-1}\,\mathbb{E}\Big[F\Big((\mathbf{e}_t)_{0\le t\le\xi}\Big)\,\exp\Big(-\int_0^\xi \frac{\mathrm{d}s}{(\mathbf{e}_s+r)^2}\Big)\Big].$$

Proof. For every $\varepsilon > 0$, let $(\mathbf{f}^{(\varepsilon,r)}_t)_{0\le t\le\xi}$ be distributed as a five-dimensional Bessel process started from $r+\varepsilon$ and conditioned to hit r exactly at time ξ. See Proposition 3 in [19] for a precise definition, noting that this proposition deals with a Bessel process of dimension -1 instead of a five-dimensional Bessel process, but this replacement gives the same conditioned process because of the h-process relation linking the Bessel processes of dimension 5 and of dimension -1 (cf. formula (4) in [19]). By standard arguments, $(\mathbf{f}^{(\varepsilon,r)}_t)_{0\le t\le\xi}$ converges in distribution to $(\overline{\mathbf{e}}^{(r)}_t)_{0\le t\le\xi}$ as $\varepsilon \to 0$, and therefore

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[F\Big((\mathbf{f}^{(\varepsilon,r)}_t - r)_{0\le t\le\xi}\Big)\Big] = \mathbb{E}\Big[F\Big((\mathbf{e}^{(r)}_t)_{0\le t\le\xi}\Big)\Big]. \tag{49}$$

On the other hand, Lemma 5 in [19] shows that

$$\mathbb{E}\Big[F\Big((\mathbf{f}^{(\varepsilon,r)}_t - r)_{0\le t\le\xi}\Big)\Big] = \frac{r+\varepsilon}{r}\,\frac{q_\xi(\varepsilon)}{\rho_\xi(\varepsilon,r)}\,\mathbb{E}\Big[F\Big((\mathbf{g}^{(\varepsilon)}_t)_{0\le t\le\xi}\Big)\,\exp\Big(-\int_0^\xi \frac{\mathrm{d}s}{(\mathbf{g}^{(\varepsilon)}_s+r)^2}\Big)\Big], \tag{50}$$

where $(\mathbf{g}^{(\varepsilon)}_s)_{0\leq s\leq \xi}$ is distributed as a linear Brownian motion started from ε and conditioned to hit 0 exactly at time ξ, and the functions $q_\xi(\varepsilon)$ and $\rho_\xi(\varepsilon,r)$ are given by

$$q_\xi(\varepsilon):=\frac{\varepsilon}{\sqrt{2\pi\xi^3}}\,\exp\Big(-\frac{\varepsilon^2}{2\xi}\Big)$$

and

$$\rho_\xi(\varepsilon,r):=\varepsilon\,e^{-\varepsilon^2/(2\xi)}\Bigg(\frac{1}{2r^3}\mathrm{erfc}\Big(\frac{\sqrt{\xi}}{r\sqrt{2}}+\frac{\varepsilon}{\sqrt{2\xi}}\Big)\exp\Big(\Big(\frac{\sqrt{\xi}}{r\sqrt{2}}+\frac{\varepsilon}{\sqrt{2\xi}}\Big)^2\Big)-\frac{1}{r^2\sqrt{2\pi\xi}}+\frac{r+\varepsilon}{r\sqrt{2\pi\xi^3}}\Bigg).$$

Since

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[F\Big((\mathbf{g}^{(\varepsilon)}_t)_{0\leq t\leq\xi}\Big)\,\exp\Big(-\int_0^\xi\frac{\mathrm{d}s}{(\mathbf{g}^{(\varepsilon)}_s+r)^2}\Big)\Big]=\mathbb{E}\Big[F\Big((\mathbf{e}_t)_{0\leq t\leq\xi}\Big)\,\exp\Big(-\int_0^\xi\frac{\mathrm{d}s}{(\mathbf{e}_s+r)^2}\Big)\Big],$$

the desired result will follow from (49) and (50) if we can verify that

$$\lim_{\varepsilon\to 0}\frac{\rho_\xi(\varepsilon,r)}{q_\xi(\varepsilon)}=\Phi(\frac{\xi}{2r^2}).$$

This is immediately checked from the formulas for $q_\xi(\varepsilon)$ and $\rho_\xi(\varepsilon,r)$. □

In what follows, we will consider $\mathbf{e}^{(r/\sqrt{3})}$ rather than $\mathbf{e}^{(r)}$, and in order to simplify notation we set $r'=r/\sqrt{3}$.

As in Section 4, we assume that, conditionally on $\mathbf{e}$, $\mathcal{N}=\sum_{i\in I}\delta_{(t_i,\omega_i)}$ is a measure on $[0,\xi]\times\mathcal{S}$ with intensity

$$2\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\mathbf{e}_t}(\mathrm{d}\omega).$$

We write $\mathcal{Z}=\int\mathcal{Z}_0(\omega)\mathcal{N}(\mathrm{d}t\,\mathrm{d}\omega)=\sum_{i\in I}\mathcal{Z}_0(\omega_i)$ for the total exit measure at 0 of the atoms ω_i ($\mathcal{Z}=\mathcal{Z}_{\mathbb{U}}$ in the notation of Section 5) and $\mathrm{tr}_0(\mathcal{N})=\sum_{i\in I}\delta_{(t_i,\mathrm{tr}_0(\omega_i))}$. Furthermore, conditionally given $\mathbf{e}^{(r')}$, we let $\widetilde{\mathcal{N}}$ be distributed as a Poisson measure on $[0,\xi]\times\mathcal{S}$ with intensity

$$2\,\mathbf{1}_{\{W_*(\omega)>-r\}}\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\mathbf{e}^{(r')}_t}(\mathrm{d}\omega). \tag{51}$$

We introduce the same notation $\widetilde{\mathcal{Z}}=\int\mathcal{Z}_0(\omega)\widetilde{\mathcal{N}}(\mathrm{d}t\,\mathrm{d}\omega)$ and $\mathrm{tr}_0(\widetilde{\mathcal{N}})$ for the analogs of $\mathcal{Z}$ and $\mathrm{tr}_0(\mathcal{N})$ when $\mathcal{N}$ is replaced by $\widetilde{\mathcal{N}}$.

Lemma 20. *For any nonnegative measurable functions F and G,*

$$\mathbb{E}\Big[F(\mathbf{e}^{(r')})\,G(\mathrm{tr}_0(\widetilde{\mathcal{N}}))\Big]=\Phi(\frac{3\xi}{2r^2})^{-1}\,\mathbb{E}\Big[F(\mathbf{e})\,G(\mathrm{tr}_0(\mathcal{N}))\,\exp\Big(-\frac{3\mathcal{Z}}{2r^2}\Big)\Big].$$

Proof. Suppose that, conditionally on $\mathbf{e}^{(r')}$, $\mathcal{N}'$ is distributed as a Poisson measure on $[0,\xi]\times\mathcal{S}$ with intensity

$$2\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\mathbf{e}^{(r')}_t}(\mathrm{d}\omega).$$

Let $\min\mathcal{N}'$ stand for the minimal value of $W_*(\omega)$ for all atoms (t,ω) of $\mathcal{N}'$. Then, conditionally on $\mathbf{e}^{(r')}$, $\widetilde{\mathcal{N}}$ is distributed as $\mathcal{N}'$ conditioned to have $\min\mathcal{N}'>-r$. By formula (2),

$$\mathbb{P}\Big(\min\mathcal{N}'>-r\,\Big|\,\mathbf{e}^{(r')}\Big)=\exp\Big(-2\int_0^\xi\frac{3}{2\times(\sqrt{3}\mathbf{e}^{(r')}_t+r)^2}\,\mathrm{d}t\Big)=\exp\Big(-\int_0^\xi\frac{\mathrm{d}t}{(\mathbf{e}^{(r')}_t+r')^2}\Big).$$

Hence, using the same notation as explained before the lemma to define $\mathrm{tr}_0(\mathcal{N}')$,

$$\mathbb{E}\Big[F(\mathbf{e}^{(r')})\,G(\mathrm{tr}_0(\widetilde{\mathcal{N}}))\Big]=\mathbb{E}\Big[F(\mathbf{e}^{(r')})\,G(\mathrm{tr}_0(\mathcal{N}'))\,\mathbf{1}_{\{\min\mathcal{N}'>-r\}}\,\exp\Big(\int_0^\xi\frac{\mathrm{d}t}{(\mathbf{e}^{(r')}_t+r')^2}\Big)\Big].$$

By the special Markov property and (2), we have

$$\mathbb{P}\Big(\min \mathcal{N}' > -r \,\Big|\, \mathbf{e}^{(r')}, \mathrm{tr}_0(\mathcal{N}')\Big) = \exp\Big(-\frac{3\mathcal{Z}'}{2r^2}\Big),$$

where $\mathcal{Z}'$ is the total exit measure at 0 of the atoms of $\mathcal{N}'$. We thus arrive at the formula

$$\mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\widetilde{\mathcal{N}}))\Big] = \mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\mathcal{N}'))\, \exp\Big(-\frac{3\mathcal{Z}'}{2r^2} + \int_0^\xi \frac{\mathrm{d}t}{(\mathbf{e}^{(r')}_t + r')^2}\Big)\Big].$$

At this stage, we use Lemma 19 to observe that the law of the pair $(\mathbf{e}^{(r')}, \mathcal{N}')$ has a density with respect to the law of $(\mathbf{e}, \mathcal{N})$ which is given by

$$\Phi(\frac{3\xi}{2r^2})^{-1}\, \exp\Big(-\int_0^\xi \frac{\mathrm{d}t}{(e(t)+r')^2}\Big),$$

where $(e(t))_{0\leq s\leq \xi}$ stands for the generic element of $C([0,\xi],\mathbb{R})$. Thanks to this observation, we get

$$\mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\mathcal{N}'))\, \exp\Big(-\frac{3\mathcal{Z}'}{2r^2} + \int_0^\xi \frac{\mathrm{d}t}{(\mathbf{e}^{(r')}_t + r')^2}\Big)\Big] = \Phi(\frac{3\xi}{2r^2})^{-1}\, \mathbb{E}\Big[F(\mathbf{e})\, G(\mathrm{tr}_0(\mathcal{N}))\, \exp\Big(-\frac{3\mathcal{Z}}{2r^2}\Big)\Big],$$

and this completes the proof. □

Lemma 20 applied with $F = 1$ and $G(\mathrm{tr}_0(\widetilde{\mathcal{N}})) = \varphi(\widetilde{\mathcal{Z}})$, for any test function φ, implies that

$$\mathbb{E}[\varphi(\widetilde{\mathcal{Z}})] = \Phi(\frac{3\xi}{2r^2})^{-1}\, \mathbb{E}\Big[\varphi(\mathcal{Z})\, \exp\Big(-\frac{3\mathcal{Z}}{2r^2}\Big)\Big]. \tag{52}$$

Since we know that the density of $\mathcal{Z}$ is the function $z \mapsto \frac{3}{2}\xi^{3/2}(\xi+z)^{-5/2}$, we get that the density of $\widetilde{\mathcal{Z}}$ is

$$z \mapsto \frac{3}{2}\, \Phi(\frac{3\xi}{2r^2})^{-1}\, \xi^{3/2}(\xi+z)^{-5/2}\, \exp(-\frac{3z}{2r^2}). \tag{53}$$

Proposition 21. *For any nonnegative measurable functions F and G, we have for Lebesgue almost every $z > 0$,*

$$\mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\widetilde{\mathcal{N}})) \,\Big|\, \widetilde{\mathcal{Z}} = z\Big] = \mathbb{E}\Big[F(\mathbf{e})\, G(\mathrm{tr}_0(\mathcal{N})) \,\Big|\, \mathcal{Z} = z\Big].$$

In other words, the conditional distributions of $(\mathbf{e}^{(r')}, \widetilde{\mathcal{N}})$ given $\widetilde{\mathcal{Z}}$ and of $(\mathbf{e}, \mathcal{N})$ given $\mathcal{Z}$ are the same.

Proof. This easily follows from Lemma 20. Set

$$\gamma_1(z) := \mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\widetilde{\mathcal{N}})) \,\Big|\, \widetilde{\mathcal{Z}} = z\Big]\,,\; \gamma_2(z) := \mathbb{E}\Big[F(\mathbf{e})\, G(\mathrm{tr}_0(\mathcal{N})) \,\Big|\, \mathcal{Z} = z\Big],$$

noting that both γ_1 and γ_2 are defined up to a set of zero Lebesgue measure of values of $z > 0$. By Lemma 20 and formula (52), for any nonnegative measurable function φ on $\mathbb{R}_+$,

$$\begin{aligned}
\mathbb{E}\Big[\gamma_1(\widetilde{\mathcal{Z}})\, \varphi(\widetilde{\mathcal{Z}})\Big] &= \mathbb{E}\Big[F(\mathbf{e}^{(r')})\, G(\mathrm{tr}_0(\widetilde{\mathcal{N}}))\, \varphi(\widetilde{\mathcal{Z}})\Big] \\
&= \Phi(\frac{3\xi}{2r^2})^{-1}\, \mathbb{E}\Big[F(\mathbf{e})\, G(\mathrm{tr}_0(\mathcal{N}))\, \varphi(\mathcal{Z})\, \exp\Big(-\frac{3\mathcal{Z}}{2r^2}\Big)\Big] \\
&= \Phi(\frac{3\xi}{2r^2})^{-1}\, \mathbb{E}\Big[\gamma_2(\mathcal{Z})\, \varphi(\mathcal{Z})\, \exp\Big(-\frac{3\mathcal{Z}}{2r^2}\Big)\Big] \\
&= \mathbb{E}[\gamma_2(\widetilde{\mathcal{Z}})\, \varphi(\widetilde{\mathcal{Z}})].
\end{aligned}$$

Since φ was arbitrary it follows that $\gamma_1(z) = \gamma_2(z)$, $\mathrm{d}z$ a.e. □

6.2 The complement of a hull in a Brownian disk

We now recall the construction of the Brownian disk "viewed from a boundary point" which is given in [19]. As in [19], we deal with a Brownian disk of perimeter 1, but the construction and the results of this section can easily be extended to an arbitrary perimeter $\xi > 0$ via scaling arguments. We start from a pair $(\mathbf{b}, \mathcal{M})$, where $\mathbf{b} = (\mathbf{b}_t)_{0\leq t\leq 1}$ is a five-dimensional Bessel bridge from 0 to 0 over the time interval $[0, 1]$ and, conditionally on $\mathbf{b}$, $\mathcal{M}(\mathrm{d}t\mathrm{d}\omega)$ is a Poisson measure on $[0, 1] \times \mathcal{S}$ with intensity

$$2\,\mathbf{1}_{\{W_*(\omega)>0\}}\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\,\mathbf{b}_t}(\mathrm{d}\omega).$$

We write

$$\mathcal{M} = \sum_{j\in J} \delta_{(t'_j,\omega'_j)},$$

and $\Sigma' := \sum_{j\in J} \sigma(\omega'_j)$. From $\mathcal{M}$, we can define a compact measure metric space $\mathfrak{T}'$ exactly in the same way as $\mathfrak{T}$ was defined in (19). We also introduce an associated clockwise exploration $(\mathcal{E}'_s)_{s\in[0,\Sigma']}$, and intervals $[|a, b|]'$ in $\mathfrak{T}'$ are defined as previously from the clockwise exploration. We specify labels $(\ell'_u)_{u\in\mathfrak{T}'}$ by setting $\ell'_t := \sqrt{3}\,\mathbf{b}_t$ for $t \in [0, 1]$, and $\ell'_u := \ell_u(\omega'_j)$ for $u \in \mathcal{T}_{(\omega'_j)}$, $j \in J$. A fundamental difference is the fact that $\ell'_u \geq 0$ for every $u \in \mathfrak{T}'$ (because by construction $W_*(\omega'_j) > 0$ for every $j \in J$). Furthermore 0 and 1 are the only elements of $\mathfrak{T}'$ with zero label.

For every $a, b \in \mathfrak{T}'$ we set

$$D'^{\circ}(a, b) := \ell'_a + \ell'_b - 2 \max\Big(\min_{c\in[|a,b|]'} \ell'_c,\ \min_{c\in[|b,a|]'} \ell'_c\Big), \tag{54}$$

and then

$$D'(a, b) := \inf_{a_0=a,a_1,\dots,a_{p-1},a_p=b} \sum_{i=1}^{p} D'^{\circ}(a_{i-1}, a_i) \tag{55}$$

where the infimum is over all choices of the integer $p \geq 1$ and of the points $a_1, \dots, a_{p-1}$ in $\mathfrak{T}'$. We notice that $D'(0, 1) = D'^{\circ}(0, 1) = 0$ and that, for every $a, b \in \mathfrak{T}'$,

$$D'(a, b) = 0 \quad \text{if and only if } D'^{\circ}(a, b) = 0. \tag{56}$$

This follows from the analogous result [6, Theorem 13] in the Bettinelli-Miermont construction of Section 4.1 via an absolute continuity argument.

Finally, we set $\mathbb{D}' = \mathfrak{T}'/\{D' = 0\}$. We observe that $D'(a, b) = 0$ implies $\ell'_a = \ell'_b$ so that we can make sense of labels on $\mathbb{D}'$, for which we keep the same notation ℓ'_x, $x \in \mathbb{D}'$. We write Π' for the canonical projection from $\mathfrak{T}'$ onto $\mathbb{D}'$, and $\mathbf{V}'$ for the pushforward of the volume measure on $\mathfrak{T}'$ under Π'.

Theorem 22. [19, Theorem 15] *The quotient space $(\mathbb{D}', D', \mathbf{V}')$ equipped with the distinguished point $\Pi'(0)$, is a free Brownian disk with perimeter 1 pointed at a uniform boundary point.*

See the end of Section 4.1, or [19, Section 6] for the definition of the free Brownian disk pointed at a uniform boundary point. The boundary $\partial\mathbb{D}'$ coincides with $\Pi'([0, 1])$ (as in formula (19), $[0, 1]$ is viewed as a subset of $\mathfrak{T}'$). In a way similar to the formula $D(\mathbf{x}_*, u) = \ell_x - \ell_{\mathbf{x}_*}$ in the Bettinelli-Miermont construction, labels ℓ'_x exactly correspond to distances from the distinguished point $\Pi'(0)$, which lies on $\partial\mathbb{D}'$.

We fix $\alpha \in (0, 1)$ and set $\mathbf{x}_1 := \Pi'(\alpha)$, which is a point of $\partial\mathbb{D}'$ distinct from $\Pi'(0)$. Note that $D'(\Pi'(0), \mathbf{x}_1) = \sqrt{3}\,\mathbf{b}_\alpha$. We also fix $r > 0$ and write B_r for the closed ball of radius r centered at $\Pi'(0)$ in $\mathbb{D}'$. On the event where $D'(\Pi'(0), \mathbf{x}_1) > r$, we let $\widehat{B}^{\circ,\mathbf{x}_1}_r$ denote the connected component of $\mathbb{D}'\backslash B_r$ that contains $\mathbf{x}_1$, and define the hull $B^{\bullet,\mathbf{x}_1}_r := \mathbb{D}'\backslash\widehat{B}^{\circ,\mathbf{x}_1}_r$. Notice that $D'(\Pi'(0), x) = r$ for every x belonging to the topological boundary of $B^{\bullet,\mathbf{x}_1}_r$. We also let $\widehat{B}^{\bullet,\mathbf{x}_1}_r$ be the closure of $\widehat{B}^{\circ,\mathbf{x}_1}_r$.

Let us argue on the event where $D'(\Pi'(0), \mathbf{x}_1) = \sqrt{3}\,\mathbf{b}_\alpha > r$. On this event, we set $T_- := \sup\{t \in [0, \alpha] : \mathbf{b}_t = r/\sqrt{3}\}$ and $T_+ := \inf\{t \in [\alpha, 1] : \mathbf{b}_t = r/\sqrt{3}\}$, so that (T_-, T_+) is the excursion interval of $\mathbf{b}$ above level $r/\sqrt{3}$ that straddles α. We also set $P_0 = T_+ - T_-$, and

$$P_1 = \sum_{j\in J: T_-<t'_j<T_+} \mathcal{Z}_r(\omega'_j).$$

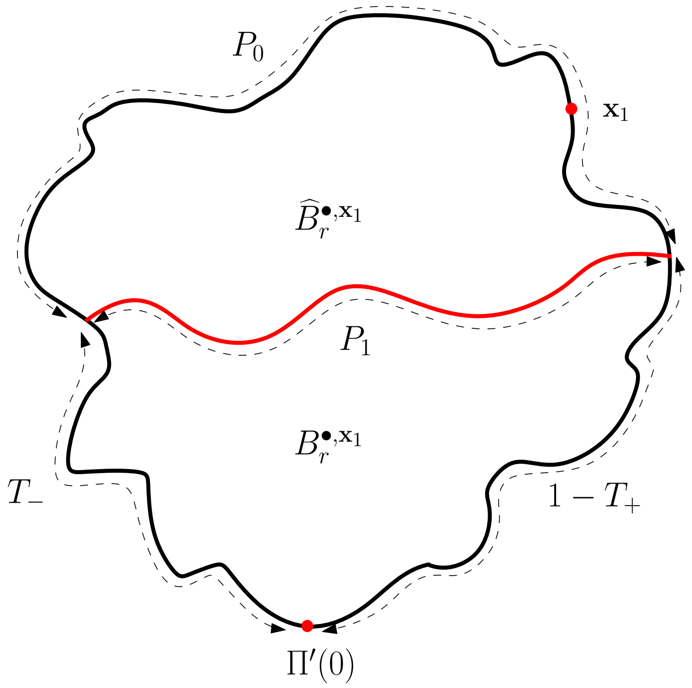

Figure 3: Illustration of the Brownian disk $\mathbb{D}'$ and the subsets $B_r^{\bullet,\mathbf{x}_1}$ and $\widehat{B}_r^{\bullet,\mathbf{x}_1}$. The intersection $B_r^{\bullet,\mathbf{x}_1} \cap \widehat{B}_r^{\bullet,\mathbf{x}_1}$ is represented in red and we interpret P_1 as its length. The variables T_-, P_0 and $1-T_+$ can be thought of as the lengths of the associated subsets of the boundary of $\mathbb{D}'$.

Theorem 23. *Almost surely under the conditional probability $\mathbb{P}(\cdot \mid D'(\Pi'(0), \mathbf{x}_1) > r)$, the intrinsic metric on $\widehat{B}_r^{\circ,\mathbf{x}_1}$ has a unique continuous extension to $\widehat{B}_r^{\bullet,\mathbf{x}_1}$, which is a metric on $\widehat{B}_r^{\bullet,\mathbf{x}_1}$. Furthermore, conditionally on the pair (P_0, P_1), the resulting metric space equipped with the volume measure $\widehat{\mathbf{V}}'_r$ which is the restriction of $\mathbf{V}'$ to $\widehat{B}_r^{\bullet,\mathbf{x}_1}$, and with the distinguished point $\Pi'(T_-)$, is a free Brownian disk with perimeter $P_0 + P_1$ pointed at a uniform boundary point.*

Proof. Throughout the proof, we argue under the conditional probability $\mathbb{P}(\cdot \mid D'(\Pi'(0), \mathbf{x}_1) > r)$. We set $\mathbf{b}_t^\diamond = \mathbf{b}_{T_-+t} - r/\sqrt{3}$ for every $t \in [0, P_0]$, and

$$\mathcal{M}^\diamond := \sum_{j \in J : T_- < t'_j < T_+} \delta_{(t'_j - T_-,\, \omega'_j - r)},$$

with the abuse of notation consisting in writing $\omega - r$ for the snake trajectory ω shifted by $-r$. We have then $P_1 = \int \mathcal{Z}_0(\omega)\, \mathcal{M}^\diamond(\mathrm{d}t\mathrm{d}\omega)$. Let $\mathfrak{T}^\diamond$ be the metric space constructed from $\mathrm{tr}_0(\mathcal{M}^\diamond)$ in the same way as $\mathfrak{T}^\star$ was constructed from $\mathrm{tr}_0(\mathcal{N})$ at the beginning of Section 4.2. More precisely, $\mathfrak{T}^\diamond$ is obtained from the disjoint union

$$[0, P_0] \cup \Bigg(\bigcup_{j \in J : T_- < t'_j < T_+} \mathcal{T}_{(\mathrm{tr}_0(\omega'_j - r))} \Bigg)$$

by identifying the root of $\mathcal{T}_{(\mathrm{tr}_0(\omega'_j - r))}$ with the point $t'_j - T_-$ of $[0, P_0]$, for every $j \in J$ such that $T_- < t'_j < T_+$. We also assign labels $\ell_a^\diamond$ to the points of $\mathfrak{T}^\diamond$ in the same manner as in Section 4.2, so that $\ell_a^\diamond = \sqrt{3}\, \mathbf{b}_a^\diamond$ if $a \in [0, P_0]$, and points of $\mathcal{T}_{(\mathrm{tr}_0(\omega'_j - r))}$ keep their labels. We set $\partial \mathfrak{T}^\diamond := \{a \in \mathfrak{T}^\diamond : \ell_a^\diamond = 0\}$ and

$$\Sigma^\diamond := \int \sigma(\mathrm{tr}_0(\omega))\, \mathcal{M}^\diamond(\mathrm{d}t\mathrm{d}\omega) = \sum_{j \in J : T_- < t'_j < T_+} \sigma(\mathrm{tr}_r(\omega'_j)).$$

As in Section 4.2, we can introduce the clockwise exploration $(\mathcal{E}_t^\diamond)_{0 \le t \le \Sigma^\diamond}$ of $\mathfrak{T}^\diamond$, which allows us to define intervals in $\mathfrak{T}^\diamond$. Then, for every $a, b \in \mathfrak{T}^\diamond \backslash \partial \mathfrak{T}^\diamond$, we define the functions $D_\diamond^\circ(a, b)$ and $D_\diamond(a, b)$ by the analogs of formulas (22) and (23).

Let $\xi \in (0,1)$. We observe that, conditionally on $P_0 = \xi$, $(\mathbf{b}_{T_-+t})_{0\le t\le \xi}$ is distributed as a five-dimensional Bessel process excursion of length ξ above level $r/\sqrt{3}$ and thus has the same distribution as the process $(\overline{\mathbf{e}}^{(r')}_t)_{0\le t\le\xi}$ introduced at the beginning of Section 6.1. Hence, conditionally on $P_0 = \xi$, $(\mathbf{b}^\diamond_t)_{0\le t\le\xi}$ has the same distribution as $(\mathbf{e}^{(r')}_t)_{0\le t\le\xi}$. By construction, conditionally on $P_0 = \xi$ and on $\mathbf{b}^\diamond$, $\mathcal{M}^\diamond$ is Poisson on $[0,\xi]\times\mathcal{S}$ with intensity

$$2\,\mathbf{1}_{\{W_*(\omega)>-r\}}\,\mathrm{d}t\,\mathbb{N}_{\sqrt{3}\,\mathbf{b}^\diamond_t}(\mathrm{d}\omega).$$

Comparing with formula (51) for the intensity of the Poisson measure $\widetilde{\mathcal{N}}$, and using both identities $\widetilde{\mathcal{Z}} = \int \mathcal{Z}_0(\omega)\widetilde{\mathcal{N}}(\mathrm{d}t\mathrm{d}\omega)$ and $P_1 = \int \mathcal{Z}_0(\omega)\,\mathcal{M}^\diamond(\mathrm{d}t\mathrm{d}\omega)$, we get that the conditional distribution of the pair $(\mathbf{b}^\diamond, \mathrm{tr}_0(\mathcal{M}^\diamond))$ knowing $(P_0,P_1) = (\xi, z)$ coincides with the conditional distribution of $(\mathbf{e}^{(r')}, \mathrm{tr}_0(\widetilde{\mathcal{N}}))$ given $\widetilde{\mathcal{Z}} = z$. By Proposition 21, this is also the conditional distribution of $(\mathbf{e}, \mathrm{tr}_0(\mathcal{N}))$ given $\mathcal{Z} = z$.

As a consequence of the latter identity in distribution, we can use Proposition 6 to get that the function $(a,b)\mapsto D_\diamond(a,b)$ has almost surely a continuous extension to $\mathfrak{T}^\diamond\times\mathfrak{T}^\diamond$. We then consider the quotient metric space $\mathbb{D}_\diamond := \mathfrak{T}^\diamond/\{D_\diamond = 0\}$, and the canonical projection $\Pi_\diamond : \mathfrak{T}^\diamond \longrightarrow \mathbb{D}_\diamond$. As usual, the metric space $(\mathbb{D}_\diamond, D_\diamond)$ is equipped with the pushforward of the volume measure on $\mathfrak{T}^\diamond$ under $\Pi_\diamond$, which is denoted by $\mathbf{V}_\diamond$. Moreover, Theorem 17 implies that, conditionally on the pair (P_0,P_1), the space $(\mathbb{D}_\diamond, D_\diamond, \mathbf{V}_\diamond)$ with the distinguished point $\Pi_\diamond(0)$ is a free Brownian disk with perimeter P_0+P_1 pointed at a uniform boundary point.

In order to complete the proof of Theorem 23, we now need to explain that the space $\widehat{B}^{\bullet,x_1}_r$ equipped with its (extended) intrinsic metric and with the restriction of the volume measure $\mathbf{V}'$ is identified isometrically to $(\mathbb{D}_\diamond, D_\diamond, \mathbf{V}_\diamond)$, and this identification maps $\Pi'(T_-)$ to $\Pi_\diamond(0)$. To this end, we first introduce the subset of $\mathfrak{T}'$ defined by

$$\mathcal{G}^{\circ,\mathbf{x}_1}_r := (T_-,T_+)\cup\Bigg(\bigcup_{j\in J: T_-<t'_j<T_+}\{a\in\mathcal{T}_{(\omega'_j)} : m'_a > r\}\Bigg),$$

where, if $a\in\mathcal{T}_{(\omega'_j)}$, m'_a denotes the minimal label along the ancestral line $[\![\rho_{(\omega'_j)}, a]\!]$, and we (of course) make the same identifications as in the definition of $\mathfrak{T}'$. Let us verify that $\widehat{B}^{\circ,\mathbf{x}_1}_r = \Pi'(\mathcal{G}^{0,\mathbf{x}_1}_r)$. We know that a point x of $\mathbb{D}'$ belongs to $\widehat{B}^{\circ,\mathbf{x}_1}_r$ if and only if there is a continuous path from x to $\mathbf{x}_1$ that does not intersect the ball B_r. From this observation and the cactus bound already used in the proof of Proposition 7, it is not hard to verify that x belongs to $\widehat{B}^{\circ,\mathbf{x}_1}_r$ if and only if $x = \Pi'(a)$ where either $a\in(T_-,T_+)$ or a belongs to one of the trees $\mathcal{T}_{(\omega'_j)}$, with $T_- < t'_j < T_+$, and labels along the ancestral line of a stay greater than r. This leads to the desired identity $\widehat{B}^{\circ,\mathbf{x}_1}_r = \Pi'(\mathcal{G}^{0,\mathbf{x}_1}_r)$.

Next, set

$$\partial\mathcal{G}^{\circ,\mathbf{x}_1}_r := \{T_-,T_+\}\cup\Bigg(\bigcup_{j\in J:T_-<t'_j<T_+}\{a\in\mathcal{T}_{(\omega'_j)} : \ell'_a = r \text{ and } \ell'_b > r \text{ for every } b\in[\![\rho_{(\omega'_j)}, a]\!]\backslash\{a\}\}\Bigg).$$

One easily verifies that the topological boundary of $\widehat{B}^{\circ,\mathbf{x}_1}_r$ is $\Pi'(\partial\mathcal{G}^{\circ,\mathbf{x}_1}_r)$. Consequently, $\widehat{B}^{\bullet,\mathbf{x}_1}_r = \Pi'(\mathcal{G}^{\bullet,\mathbf{x}_1}_r)$ where $\mathcal{G}^{\bullet,\mathbf{x}_1}_r = \mathcal{G}^{\circ,\mathbf{x}_1}_r\cup\partial\mathcal{G}^{\circ,\mathbf{x}_1}_r$. Note that we have

$$\mathcal{G}^{\bullet,\mathbf{x}_1}_r = [T_-,T_+]\cup\Bigg(\bigcup_{j\in J:T_-<t'_j<T_+}\mathcal{T}_{(\mathrm{tr}_r(\omega'_j))}\Bigg),$$

where as usual we identify $\mathcal{T}_{(\mathrm{tr}_r(\omega'_j))}$ with a subset of $\mathcal{T}_{(\omega'_j)}$.

We can then identify $\mathcal{G}^{\bullet,\mathbf{x}_1}_r$ with $\mathfrak{T}^\diamond$ in the following manner: $a\in[T_-,T_+]$ is identified with $a - T_-$, and a point $a\in\mathcal{T}_{(\mathrm{tr}_r(\omega'_j))}$ (where j is such that $T_- < t'_j < T_+$) is identified to the corresponding point of $\mathcal{T}_{(\mathrm{tr}_0(\omega'_j-r))}$. Moreover, using (56) and Proposition 6, one checks that, for every $a,b\in\mathcal{G}^{\bullet,\mathbf{x}_1}_r$, the property $D'(a,b) = 0$ holds if and only if the points $\widetilde{a}$ and $\widetilde{b}$ of $\mathfrak{T}^\diamond$ corresponding to a and b satisfy $D_\diamond(\widetilde{a},\widetilde{b}) = 0$. This leads to the desired identification of the sets $\widehat{B}^{\bullet,\mathbf{x}_1}_r$ and $\mathbb{D}_\diamond$. Then one verifies that the intrinsic metric on $\widehat{B}^{\circ,\mathbf{x}_1}_r$ coincides (modulo the preceding identification) with the restriction of the metric $D_\diamond$ to $\mathbb{D}_\diamond \backslash \partial \mathbb{D}_\diamond$. This relies on arguments very similar to the proof of Proposition 7, and we omit the details. Finally, it is immediate that the restriction of $\mathbf{V}'$ to $\widehat{B}_r^{\bullet,\mathbf{x}_1}$ corresponds to the volume measure on $\mathbb{D}_\diamond$. This completes the proof. $\square$

We will write $\widehat{D}'_{\mathcal{I}}$ for the metric on $\widehat{B}_r^{\bullet,\mathbf{x}_1}$ constructed in Theorem 23 as the extension of the intrinsic metric on $\widehat{B}_r^{\circ,\mathbf{x}_1}$. In view of future applications, it will also be convenient to introduce the standard boundary curve of the Brownian disk $\widehat{B}_r^{\bullet,\mathbf{x}_1}$ that is defined as follows. Recall the construction of the boundary curve $(\Gamma^\star(t))_{0\leq t\leq \xi+\mathcal{Z}_{\mathbb{U}}}$ from the pair $(\mathbf{e}, \mathrm{tr}_0(\mathcal{N}))$ as explained before Theorem 17. Since the conditional distribution of the pair $(\mathbf{b}^\diamond, \mathrm{tr}_0(\mathcal{M}^\diamond))$ knowing $(P_0, P_1) = (\xi, z)$ coincides with the conditional distribution of $(\mathbf{e}, \mathrm{tr}_0(\mathcal{N}))$ given $\mathcal{Z} = z$, we can use the same construction to get a standard boundary curve of the Brownian disk $\mathbb{D}_\diamond$, hence (via the identification of the preceding proof) a standard boundary curve $(\widehat{\Gamma}'_r(t))_{0\leq t\leq P_0+P_1}$ of the Brownian disk $\widehat{B}_r^{\bullet,\mathbf{x}_1}$. More precisely, $\widehat{\Gamma}'_r(t) = \Pi'(T_- + t)$ for $0 \leq t \leq P_0$, and the values of $\widehat{\Gamma}'_r(t)$ for $P_0 \leq t \leq P_0 + P_1$ are defined by the analog of formula (45) (cf. formula (61) below). We have then $\{\widehat{\Gamma}'_r(t) : 0 \leq t \leq P_0\} = \widehat{B}_r^{\bullet,\mathbf{x}_1} \cap \partial \mathbb{D}'$, and the set $\{\widehat{\Gamma}'_r(t) : P_0 \leq t \leq P_0 + P_1\}$ is the topological boundary of $\widehat{B}_r^{\bullet,\mathbf{x}_1}$ or, equivalently, of the hull $B_r^{\bullet,\mathbf{x}_1}$.

6.3 A spatial Markov property

Our goal in this section is to prove that the free Brownian disk $(\widehat{B}_r^{\bullet,\mathbf{x}_1}, \widehat{D}'_r, \widehat{\mathbf{V}}'_r, \Pi'(T_-))$ in Theorem 23 is independent of the hull $B_r^{\bullet,\mathbf{x}_1}$ conditionally on the pair (P_0, P_1). To make this assertion precise, we need to explain how the hull $B_r^{\bullet,\mathbf{x}_1}$ is viewed as a random measure metric space. We argue on the event $\{D'(0, \mathbf{x}_1) > r\}$ and we keep the notation of the previous section. We introduce the subset $\mathcal{K}_r$ of $\mathfrak{T}'$ defined by

$$\mathcal{K}_r = [0, T_-] \cup [T_+, 1] \cup \Big(\bigcup_{j\in J:\, t'_j \in [0,T_-]\cup[T_+,1]} \mathcal{T}_{(\omega'_j)} \Big) \cup \Big(\bigcup_{j\in J:\, T_- < t'_j < T_+} \{a \in \mathcal{T}_{(\omega'_j)} : m'_a \leq r\} \Big).$$

where we recall that m'_a stands for the minimal label of $a \in \mathcal{T}_{(\omega'_j)}$ along its ancestral line. Note that labels on $\mathfrak{T}' \backslash \mathcal{K}_r$ are greater than r. We have $\mathcal{K}_r = \mathfrak{T}' \backslash \mathcal{G}_r^{\circ,\mathbf{x}_1}$, and therefore $\Pi'(\mathcal{K}_r) = B_r^{\bullet,\mathbf{x}_1}$ as a consequence of the equality $\widehat{B}_r^{\circ,\mathbf{x}_1} = \Pi'(\mathcal{G}_r^{0,\mathbf{x}_1})$. In view of forthcoming applications, we also mention the following simple fact. Let $a, b \in \mathcal{K}_r$. Then, in formula (54) defining $D'^\circ(a, b)$, we may replace the intervals $[|a, b|]'$ and $[|b, a|]'$ by $[|a, b|]' \cap \mathcal{K}_r$ and $[|b, a|]' \cap \mathcal{K}_r$ respectively: the point is that, if the interval $[|a, b|]'$ contains a point $c \notin \mathcal{K}_r$, then, necessarily, it contains another point c' (belonging to $\mathcal{K}_r$) whose label is r and is thus smaller than the label of c. Informally, the definition of $D'^\circ(a, b)$, when $a, b \in \mathcal{K}_r$ only depends on the labels on $\mathcal{K}_r$, despite the fact that the interval $[|a, b|]'$ may not be contained in $\mathcal{K}_r$.

For every $a, b \in \mathcal{K}_r$, we set

$$D'_r(a, b) := \inf_{\substack{a_0,a_1,\ldots,a_p \in \mathcal{K}_r \\ a_0=a,\, a_p=b}} \sum_{i=1}^{p} D'^\circ(a_{i-1}, a_i), \tag{57}$$

where the infimum is over all choices of the integer $p \geq 1$ and of the finite sequence $a_0, a_1, \ldots, a_p$ in $\mathcal{K}_r$ such that $a_0 = a$ and $a_p = b$. This is similar to the definition (55) of $D'(a, b)$, but we restrict the infimum to "intermediate" points $a_1, \ldots, a_{p-1}$ that belong to $\mathcal{K}_r$. Clearly, we have $D'(a, b) \leq D'_r(a, b) \leq D'^\circ(a, b)$ for every $a, b \in \mathcal{K}_r$. Since the condition $D'(a, b) = 0$ can only hold if $D'^\circ(a, b) = 0$, we get that, for every $a, b \in \mathcal{K}_r$, we have $D'_r(a, b) = 0$ if and only if $D'(a, b) = 0$. Hence D'_r induces a metric on $\Pi'(\mathcal{K}_r) = B_r^{\bullet,\mathbf{x}_1}$ and we keep the notation D'_r for this metric. Using simple geodesics and the definition (55) of D' as an infimum, one verifies that the restriction of D'_r to the interior of $B_r^{\bullet,\mathbf{x}_1}$ coincides with the intrinsic distance induced by D'. This follows by an adaptation of the proof of Proposition 7 and we omit the details since this is not really needed in what follows. Additionally, we have $D'_r(\Pi'(0), x) = D'(\Pi'(0), x)$ for every $x \in B_r^{\bullet,\mathbf{x}_1}$ (note that a D'-geodesic from x to $\Pi'(0)$ cannot exit the hull $B_r^{\bullet,\mathbf{x}_1}$). In particular, $D'_r(\Pi'(0), \widehat{\Gamma}'_r(t)) = D'(\Pi'(0), \widehat{\Gamma}'_r(t)) = r$ for every $t \in [P_0, P_0 + P_1]$.

We equip the metric space $(B_r^{\bullet,\mathbf{x}_1}, D'_r)$ with the restriction of the volume measure $\mathbf{V}'$, which we denote by $\mathbf{V}'_r$. It is also convenient to introduce a boundary curve of $B_r^{\bullet,\mathbf{x}_1}$, which we define as follows. We set, for every $t \in [0, P_1 + 1 - P_0]$,

$$\Gamma'_r(t) := \begin{cases} \Pi'(t) & \text{if } t \in [0, T_-], \\ \widehat{\Gamma}'_r(P_0 + P_1 - (t - T_-)) & \text{if } t \in (T_-, T_- + P_1), \\ \Pi'(t - (T_- + P_1) + T_+) & \text{if } t \in [T_- + P_1, P_1 + 1 - P_0]. \end{cases}$$

Note that Γ'_r is a simple loop taking values in $B_r^{\bullet,\mathbf{x}_1}$, and $\Gamma'_r(0) = \Gamma'_r(P_1 + 1 - P_0) = \Pi'(0)$. In fact, using Jordan's theorem, it is not hard to to verify that $B_r^{\bullet,\mathbf{x}_1}$ has the topology of the closed unit disk, which makes it possible to consider the "boundary" of $B_r^{\bullet,\mathbf{x}_1}$, and this boundary (which is not the topological boundary) is precisely the range of Γ'_r. We will consider the hull $(B_r^{\bullet,\mathbf{x}_1}, D'_r, \mathbf{V}'_r)$ as equipped with the curve Γ'_r: we view the 4-tuple

$$(B_r^{\bullet,\mathbf{x}_1}, D'_r, \mathbf{V}'_r, \Gamma'_r)$$

as a random variable taking values in the space $\mathbb{M}^{GHPU}$. Then it is not hard to verify that the quantities T_-, T_+, P_0, P_1 are measurable functions of the latter random space. In fact, recalling that $D'_r(\Pi'(0), \Pi'(t)) = D'(\Pi'(0), \Pi'(t)) = \sqrt{3}\,\mathbf{b}_t$ for $t \in [0, T_-] \cup [T_+, 1]$, one sees that T_- is the first time $t \geq 0$ such that there exists $\varepsilon > 0$ verifying

$$D'_r(\Pi'(0), \Gamma'_r(t + s)) = r, \quad \forall s \in [0, \varepsilon],$$

and an analogous representation holds for T_+. Furthermore $P_1 = \inf\{t \geq 0 : D'_r(0, \Gamma'_r(T_- + t)) \neq r\}$.

Theorem 24. *Under the conditional probability* $\mathbb{P}(\cdot \mid D'(0, \mathbf{x}_1) > r)$*, the space* $(\widehat{B}_r^{\bullet,\mathbf{x}_1}, \widehat{D}'_r, \widehat{\mathbf{V}}'_r, \widehat{\Gamma}'_r)$ *is independent of* $(B_r^{\bullet,\mathbf{x}_1}, D'_r, \mathbf{V}'_r, \Gamma'_r)$ *conditionally on the pair* (P_0, P_1)*.*

Proof. The general strategy of the proof is to show that the space $(B_r^{\bullet,\mathbf{x}_1}, D'_r, \mathbf{V}'_r, \Gamma'_r)$ can be constructed from random quantities that are independent of $(\widehat{B}_r^{\bullet,\mathbf{x}_1}, \widehat{D}'_r, \widehat{\mathbf{V}}'_r, \widehat{\Gamma}'_r)$ conditionally on (P_0, P_1). To this end, we will describe the hull $B_r^{\bullet,\mathbf{x}_1}$ in terms of the labeled tree $\mathfrak{T}'$. Recall the notation $(\mathcal{E}'_s)_{s \in [0, \Sigma']}$ for the clockwise exploration of $\mathfrak{T}'$. For every $j \in J$ such that $T_- < t'_j < T_+$, write $[u_j, u_j + \sigma(\omega'_j)]$ for the interval corresponding to ω'_j in the time scale of the clockwise exploration $\mathcal{E}'$ (meaning that $\mathcal{E}'_s \in \mathcal{T}_{(\omega'_j)}$ if and only if $s \in [u_j, u_j + \sigma(\omega'_j)]$), and recall the notation $(L^r_t(\omega'_j))_{t \geq 0}$ for the exit local time of ω'_j from (r, ∞). We set, for every $t \geq 0$,

$$L'_t := \sum_{j \in J : T_- < t'_j < T_+} L^r_{(t - u_j)^+}(\omega'_j), \tag{58}$$

Then, for every index j such that $T_- < t'_j < T_+$, we denote the excursions of ω'_j outside (r, ∞) by $(\omega^{\#}_{j,k})_{k \in I_j}$ where I_j is an appropriate indexing set (if the index j is such that $W_*(\omega'_j) \geq r$, I_j is the empty set). In the time scale of $\mathcal{E}'$, the excursion $\omega^{\#}_{j,k}$ corresponds to an interval $(\alpha_{j,k}, \beta_{j,k})$ and we set $t^{\#}_{j,k} = L'_{\alpha_{j,k}} = L'_{\beta_{j,k}}$. Finally, we introduce the point measure

$$\mathcal{N}_{\#}(\mathrm{d}t\mathrm{d}\omega) := \sum_{j \in J : T_- < t'_j < T_+} \sum_{k \in I_j} \delta_{(t^{\#}_{j,k}, \omega^{\#}_{j,k})}(\mathrm{d}t\mathrm{d}\omega).$$

We will show that the hull $B_r^{\bullet,\mathbf{x}_1}$ is a function of the triple

$$\mathbb{T}_{\#} := \Big(\mathcal{N}_{\#}(\mathrm{d}t\mathrm{d}\omega); \Big((\mathbf{b}_t)_{0 \leq t \leq T_-}, \sum_{j \in J : 0 \leq t'_j \leq T_-} \delta_{(t'_j, \omega'_j)}\Big); \Big((\mathbf{b}_t)_{T_+ \leq t \leq 1}, \sum_{j \in J : T_+ \leq t'_j \leq 1} \delta_{(t'_j, \omega'_j)}\Big)\Big). \tag{59}$$

To this end, we consider the interval $[0, 1 - P_0 + P_1]$, which we view as the union of the three intervals $[0, T_-]$, $[T_-, T_- + P_1]$ and $[T_- + P_1, T_- + P_1 + 1 - T_+]$. We define $\mathfrak{T}^{\#}$ as the union

$$[0, 1 - P_0 + P_1] \cup \Big(\bigcup_{j \in J : t'_j \in [0, T_-] \cup [T_+, 1]} \mathcal{T}_{(\omega'_j)}\Big) \cup \Big(\bigcup_{j \in J : T_- < t'_j < T_+} \bigcup_{k \in I_j} \mathcal{T}_{(\omega^{\#}_{j,k})}\Big),$$

where, for $j \in J$,

- if $0 \le t'_j \le T_-$, the root of $\mathcal{T}_{(\omega'_j)}$ is identified to $t'_j \in [0, T_-]$;
- if $T_- < t'_j < T_+$, then, for every $k \in I_j$, the root of $\mathcal{T}_{(\omega^{\#}_{j,k})}$ is identified to $T_- + t^{\#}_{j,k} \in [T_-, T_- + P_1]$;
- if $T_+ \le t'_j \le 1$, the root of $\mathcal{T}_{(\omega'_j)}$ is identified to $T_- + P_1 + (t'_j - T_+) \in [T_- + P_1, 1 - P_0 + P_1]$.

As in the Bettinelli-Miermont construction of Section 4.1, we view $\mathfrak{T}^{\#}$ as a measure metric space and we write $\Sigma^{\#}$ for the total mass of its volume measure. We can also assign labels to the points of $\mathfrak{T}^{\#}$: points of the trees $\mathcal{T}_{(\omega'_j)}$ and $\mathcal{T}_{(\omega^{\#}_{j,k})}$ obviously keep their labels, the label of $s \in [0, T_-]$, resp. of $s \in [T_- + P_1, T_- + P_1 + 1 - T_+]$, is $\sqrt{3}\,\mathbf{b}_s$, resp. $\sqrt{3}\,\mathbf{b}_{T_+ + s - (T_- + P_1)}$, and finally the label of each $s \in [T_-, T_- + P_1]$ is r. We define the exploration function $(\mathcal{E}^{\#}_t)_{0 \le t \le \Sigma^{\#}}$ in a way similar to Section 4.1: informally, we concatenate the exploration functions of the trees $\mathcal{T}_{(\omega'_j)}$ and $\mathcal{T}_{(\omega^{\#}_{j,k})}$ in the order prescribed by their roots viewed as elements of $[0, 1 - P_0 + P_1]$. This exploration function allows us to define intervals on $\mathfrak{T}^{\#}$, and then to introduce the functions $D^{\circ}_{\#}(a,b)$ and $D_{\#}(a,b)$ for $a, b \in \mathfrak{T}^{\#}$, exactly as we did in Section 4.1 to define $D^{\circ}(a,b)$ and $D(a,b)$. Similarly, we consider the quotient space $\mathbb{D}_{\#} := \mathfrak{T}^{\#}/\{D_{\#} = 0\}$ and we write $\Pi_{\#}$ for the canonical projection. As usual, we write $\mathbf{V}_{\#}$ for the pushforward of the volume measure on $\mathfrak{T}^{\#}$ under $\Pi_{\#}$. Finally, we set $\Gamma_{\#}(t) := \Pi_{\#}(t)$, for $t \in [0, 1 - P_0 + P_1]$.

We then claim that we have the almost sure equality

$$(B^{\bullet,\mathbf{x}_1}_r, D'_r, \mathbf{V}'_r, \Gamma'_r) = (\mathbb{D}_{\#}, D_{\#}, \mathbf{V}_{\#}, \Gamma_{\#}). \tag{60}$$

Let us explain why Theorem 24 follows from (60). On one hand, we know that the space $(\mathbb{D}_{\#}, D_{\#}, \mathbf{V}_{\#}, \Gamma_{\#})$ is obtained as a function of the triple $\mathbb{T}_{\#}$ in (59). On the other hand, the proof of Theorem 23 (see also the remark after this proof) shows that the space $(\widehat{B}^{\bullet,\mathbf{x}_1}_r, \widehat{D}'_r, \widehat{\mathbf{V}}'_r, \widehat{\Gamma}'_r)$ is a function of the pair $(\mathbf{b}^{\diamond}, \mathrm{tr}_0(\mathcal{M}^{\diamond}))$ introduced in this proof. So the statement of Theorem 24 reduces to checking that the pair $(\mathbf{b}^{\diamond}, \mathrm{tr}_0(\mathcal{M}^{\diamond}))$ is independent of $\mathbb{T}_{\#}$ conditionally on (P_0, P_1). To this end, notice that the excursion $\mathbf{b}^{\diamond}$ is independent of $((\mathbf{b}_t)_{0 \le t \le T_-}, (\mathbf{b}_t)_{T_+ \le t \le 1})$ conditionally on P_0. It follows that, conditionally on P_0, the pair $(\mathbf{b}^{\diamond}, \mathcal{M}^{\diamond})$ is independent of the pair

$$\Big((\mathbf{b}_t)_{0 \le t \le T_-}, \sum_{j \in J: 0 \le t'_j \le T_-} \delta_{(t'_j, \omega'_j)}\Big), \quad \Big((\mathbf{b}_t)_{T_+ \le t \le 1}, \sum_{j \in J: T_+ \le t'_j \le 1} \delta_{(t'_j, \omega'_j)}\Big).$$

Furthermore, an application of the special Markov property (6) entails that, conditionally on (P_0, P_1), the point measure $\mathcal{N}_{\#}(\mathrm{d}t\mathrm{d}\omega)$ is Poisson with intensity $\mathbf{1}_{[0,P_1]}(t)\,\mathrm{d}t\,\mathbb{N}_r(\mathrm{d}\omega \cap \{W_*(\omega) > 0\})$, and is independent of the pair $(\mathbf{b}^{\diamond}, \mathrm{tr}_0(\mathcal{M}^{\diamond}))$. It follows that conditionally on (P_0, P_1), the variable $(\mathbf{b}^{\diamond}, \mathrm{tr}_0(\mathcal{M}^{\diamond}))$ is independent of $\mathbb{T}_{\#}$, which was the desired result.

It only remains to justify our claim (60). This relies on arguments similar to the proof of Theorem 31 in [22], which is a statement analogous to Theorem 24 for the hull centered at the distinguished point of the Brownian plane. For this reason, we will skip some details. Recall the definition (58) of the process $(L'_t)_{t \ge 0}$, and, for every $s \in [0, P_1)$, set

$$\kappa'(t) := \inf\{s \ge 0 : L'_s > t\}.$$

By convention we let $\kappa'(P_1) = \kappa'(P_1-)$ be the left limit of $t \mapsto \kappa'(t)$ at P_1. We have then $\mathcal{E}'_{\kappa'(0)} = T_-$ and $\mathcal{E}'_{\kappa'(P_1)} = T_+$. From the construction of the boundary curve $\widehat{\Gamma}'_r$, we have

$$\Gamma'_r(T_- + s) = \widehat{\Gamma}'_r(P_0 + P_1 - s) = \Pi'(\mathcal{E}'_{\kappa'(s)}) \quad \text{for every } s \in [0, P_1]. \tag{61}$$

We then define a mapping $\Phi : \mathcal{K}_r \longrightarrow \mathfrak{T}^{\#}$ by the following prescriptions. Let $a \in \mathcal{K}_r$:

(i) If $a \in [0, T_-]$, $\Phi(a)$ is the "same" point of $\mathfrak{T}^{\#}$.

(ii) If $a \in [T_+, 1]$, $\Phi(a)$ is the point $a + P_1 - P_0$ of $\mathfrak{T}^{\#}$.

(iii) If $a \in \mathcal{T}_{(\omega'_j)}$, for some $j \in J$ with $t'_j \in [0, T_-] \cup [T_+, 1]$, $\Phi(a)$ is the "same" point in $\mathfrak{T}^\#$.

(iv) If $a \in \mathcal{T}_{(\omega^\#_{j,k})}$, for some $j \in J$ such that $t'_j \in (T_-, T_+]$ and some $k \in I_j$, $\Phi(a)$ is the "same" point in $\mathfrak{T}^\#$ — we use the fact that $\mathcal{T}_{(\omega^\#_{j,k})}$ can be viewed as a subset of the tree $\mathcal{T}_{(\omega'_j)}$.

(v) If a is of the form $\mathcal{E}'_{\kappa'(s)}$ with $s \in [0, P_1]$, or of the form $\mathcal{E}'_{\kappa'(s-)}$ with $s \in (0, P_1]$, $\Phi(a)$ is the point $T_- + s \in \mathfrak{T}^\#$.

One verifies that these prescriptions are consistent with the identifications made when defining $\mathcal{K}_r$ and $\mathfrak{T}^\#$. In particular, for $j \in J$ such that $t'_j \in (T_-, T_+)$, and $k \in I_j$, the root $\rho_{(\omega^\#_{j,k})}$ of $\mathcal{T}_{(\omega^\#_{j,k})}$ (viewed as an element of $\mathcal{K}_r$) is easily seen to coincide with $\mathcal{E}'_{\kappa'(t^\#_{j,k})}$ and thus (by property (v)) is mapped to $T_- + t^\#_{j,k}$, which we know to be identified to $\rho_{(\omega^\#_{j,k})}$ in $\mathfrak{T}^\#$. Moreover, (i) — (v) define $\Phi(a)$ for *every* $a \in \mathcal{K}_r$. The point is that, if $a \in \mathcal{K}_r$ belongs to a tree $\mathcal{T}_{(\omega'_j)}$, for some j such that $T_- < t'_j < T_+$, and if a does not belong to any of the subtrees $\mathcal{T}^\#_{j,k}$ with $k \in I_j$, then necessarily $\ell'_a = m'_a = r$, and the support property of the exit local time ensures that we have $a = \mathcal{E}'_{\kappa'(s)}$ or $a = \mathcal{E}'_{\kappa'(s-)}$ for some $s \in [0, P_1]$.

We note that Φ preserves labels and is surjective. However, Φ is not injective because $\Phi(\mathcal{E}'_{\kappa'(s)}) = \Phi(\mathcal{E}'_{\kappa'(s-)})$ for $s \in (0, P_1]$. Nonetheless, it follows from our definitions and the support property of the exit local time that $\Pi'(\mathcal{E}'_{\kappa'(s)}) = \Pi'(\mathcal{E}'_{\kappa'(s-)})$ for every $s \in (0, P_1]$. We also note that, if $a, b \in \mathcal{K}_r$, the image under Φ of $[|a, b|]' \cap \mathcal{K}_r$ is the "interval" from $\Phi(a)$ to $\Phi(b)$ in $\mathfrak{T}^\#$. Using this observation and the simple fact stated before formula (57), it is straightforward to verify that, for every $a, b \in \mathcal{K}_r$, we have $D'^\circ(a, b) = D^\circ_\#(\Phi(a), \Phi(b))$, and then $D'_r(a, b) = D_\#(\Phi(a), \Phi(b))$. It follows that Φ induces an isometry from $B^{\bullet,\mathbf{x}_1}_r = \Pi'(\mathcal{K}_r)$ onto $\mathbb{D}_\#$, and this isometry, for which we keep the same notation Φ, is easily seen to preserve the volume measures. Finally, from (61) and properties (i),(ii),(v), we immediately get that $\Phi(\Gamma'_r(s)) = \Pi_\#(s) = \Gamma_\#(s)$ for every $s \in [0, 1 - P_0 + P_1]$. This completes the proof of our claim (60) and of Theorem 24. □

Theorems 23 and 24 should be interpreted as giving a way to define a peeling exploration of the Brownian disk $\mathbb{D}'$. Starting from the point $\Pi'(0)$ uniformly distributed on the boundary, one may start by "peeling" the hull $B^{\bullet,\mathbf{x}_1}_r$ of small radius $r > 0$ centered at this point (relative to another fixed point $\mathbf{x}_1$ of the boundary). Then the remaining part $\widehat{B}^{\bullet,x_1}_r$ of the initial Brownian disk is a Brownian disk with a different perimeter (Theorem 23) and, if we choose a point on its boundary as a function of the part that has been removed, this point will again be distributed uniformly on the boundary of the new Brownian disk (as a consequence of the independence property in Theorem 24). In the notation of this section, we may choose the next point to be "peeled" as $\widehat{\Gamma}'_r(U)$ where U is a measurable function of $(B^{\bullet,\mathbf{x}_1}_r, D'_r, \mathbf{V}'_r, \Gamma'_r)$.

Appendix

In this appendix, we briefly explain how formula (12) is derived from [14]. We consider the generating function

$$G(y, z) = \sum_{k=0}^{\infty} \sum_{L=1}^{\infty} \sum_{p=1}^{\infty} \#\mathbb{T}^2(L, p, k)\, (12\sqrt{3})^{-k}\, y^L\, z^p.$$

A direct application of formula (27) in [14] gives

$$G(y, z) = \int_0^3 \frac{2sy}{(1 - 4sy)^{3/2}} \frac{2sz}{(1 - 4sz)^{3/2}} \frac{\mathrm{d}s}{2s}.$$

Note that the parameter k corresponding to the number of inner vertices is replaced by a parameter counting the number of edges of the triangulation in [14], but of course this makes no difference thanks to Euler's formula.

On the other hand, we can also consider the generating function

$$\widetilde{G}(y,z) = \sum_{L=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{2}\,\frac{3^{L+p}}{L+p}\; L\binom{2L}{L}\, p\binom{2p}{p}\, y^L\, z^p.$$

If we set $F_{y,z}(t) = \widetilde{G}(\frac{ty}{3},\frac{tz}{3})$, we have

$$F_{y,z}(t) = \frac{1}{2}\sum_{L=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{L+p}\; L\binom{2L}{L}\, p\binom{2p}{p}\, t^{L+p}\, y^L\, z^p$$

and

$$tF'_{y,z}(t) = \frac{1}{2}\sum_{L=1}^{\infty}\sum_{p=1}^{\infty} L\binom{2L}{L}\, p\binom{2p}{p}\, t^{L+p}\, y^L\, z^p = \frac{1}{2}\,\varphi(ty)\,\varphi(tz),$$

where

$$\varphi(x) = \sum_{n=1}^{\infty} n\binom{2n}{n} x^n = \frac{2x}{(1-4x)^{3/2}}.$$

We conclude that

$$\widetilde{G}(y,z) = F_{y,z}(3) = \int_0^3 \varphi(ty)\,\varphi(tz)\,\frac{\mathrm{d}t}{2t} = G(y,z)$$

as desired.

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