Some explicit distributions for Brownian motion indexed by the Brownian tree*

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Abstract

We derive several explicit distributions of functionals of Brownian motion indexed by the Brownian tree. In particular, we give a direct proof of a result of Bousquet-Mélou and Janson identifying the distribution of the density at 0 of the integrated super-Brownian excursion.

1 Introduction

The main purpose of the present work is to derive certain explicit distributions for the random process which we call Brownian motion indexed by the Brownian tree, which has appeared in a variety of different contexts. As a key tool for the derivation of our main results we use the excursion theory developed in [1] for Brownian motion indexed by the Brownian tree. In many respects, this excursion theory is similar to the classical Itô theory, which applies in particular to linear Brownian motion and has proved a powerful tool for the calculation of exact distributions of Brownian functionals.

Let us briefly describe the objects of interest in this work. We define the Brownian tree $\mathcal{T}_\zeta$ as the random compact $\mathbb{R}$-tree coded by a Brownian excursion $\zeta = (\zeta_s)_{s\geq 0}$ distributed according to the (infinite) Itô measure of positive excursions of linear Brownian motion. If σ stands for the duration of the excursion ζ, this coding means that $\mathcal{T}_\zeta$ is the quotient space of $[0,\sigma]$ for the equivalence relation defined by $s \sim s'$ if and only if $\zeta_s = \zeta_{s'} = m_\zeta(s,s')$, where $m_\zeta(s,s') := \min\{\zeta_r : s\wedge s' \leq r \leq s \vee s'\}$, and this quotient space is equipped with the metric induced by $\mathrm{d}_\zeta(s,s') = \zeta_s + \zeta_{s'} - 2m_\zeta(s,s')$. The volume measure $\mathrm{vol}(\mathrm{d}a)$ on $\mathcal{T}_\zeta$ is defined as the push forward of Lebesgue measure on $[0,\sigma]$ under the canonical projection, and the root ρ of $\mathcal{T}_\zeta$ is the equivalence class of 0. We note that under the conditioning by $\sigma = 1$ (equivalently the total volume is equal to 1) the tree $\mathcal{T}_\zeta$ is Aldous' Brownian Continuum Random Tree (also called the CRT, see [2, 3]), up to an unimportant scaling factor 2.

Let us turn to Brownian motion indexed by $\mathcal{T}_\zeta$. Informally, given $\mathcal{T}_\zeta$, this is the centered Gaussian process $(V_a)_{a\in\mathcal{T}_\zeta}$ such that $V_\rho = 0$ and $\mathrm{Var}(V_a - V_b) = \mathrm{d}_\zeta(a,b)$ for every $a,b\in\mathcal{T}_\zeta$. This definition is a bit informal since we are dealing with a random process indexed by a *random* set. These difficulties can be overcome easily by using the Brownian snake approach. We let $(W_s)_{s\geq 0}$ be the Brownian snake (whose spatial motion is linear Brownian motion started at 0) driven by the Brownian excursion $(\zeta_s)_{s\geq 0}$. Then, for every $s \geq 0$, W_s is a finite path started at 0 and with lifetime ζ_s, and for every $a \in \mathcal{T}_\zeta$ we may define V_a as the terminal point $\widehat{W}_s$ of the path W_s, for any $s\in[0,\sigma]$ such that a is the equivalence class of s in $\mathcal{T}_\zeta$. The Brownian snake approach thus reduces the study of a tree-indexed Brownian motion to that a process indexed by the positive half-line, and we systematically use this approach in the next sections.

The total occupation measure $\Theta(\mathrm{d}x)$ of $(V_a)_{a\in\mathcal{T}_\zeta}$ is the push forward of $\mathrm{vol}(\mathrm{d}a)$ under the mapping $a\mapsto V_a$, or equivalently the push forward of Lebesgue measure on $[0,\sigma]$ under $s\mapsto \widehat{W}_s$. Under the special conditioning $\sigma = 1$, this random measure is known as ISE for Integrated Super-Brownian Excursion [4] (note that our normalization is different from the one in [4]).

At this point, we observe that both the Brownian tree (often under special conditionings) and Brownian motion indexed by the Brownian tree have appeared in different areas of probability theory.

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The Brownian snake is very closely related to the measure-valued process called super-Brownian motion and has proved an efficient tool to study this process (see [20] and the references therein). Super-Brownian motion and ISE arise in a number of limit theorems for discrete probability models, but also in the theory of interacting particle systems [10, 12, 16] and in a variety of models of statistical physics [15, 17, 18]. More recently, Brownian motion indexed by the Brownian tree has served as the essential building block in the construction of the universal model of random geometry called the Brownian map (see in particular [21, 23, 24, 26]). In this connection, we note that the distribution of certain functionals of Brownian motion indexed by the Brownian tree is investigated in the article [14], which was already motivated by asymptotics for random planar maps.

Let us now explain our main results more in detail. In agreement with the usual notation for the Brownian snake, we write $\mathbb{N}_0$ for the (infinite) measure under which $(\zeta_s)_{s\geq 0}$ and $(V_a)_{a\in\mathcal{T}_\zeta}$ are defined in the way we just explained — see Section 2 for more details. We are primarily interested in local times, which are the densities of the random measure $\Theta(\mathrm{d}x)$. It follows from the work of Bousquet-Mélou and Janson [7, 9] that $\Theta(\mathrm{d}x)$ has a continuous density $(\mathcal{L}^x)_{x\in\mathbb{R}}$ with respect to Lebesgue measure on $\mathbb{R}$, $\mathbb{N}_0$ a.e. (this fact could also be derived from the earlier work of Sugitani [30] dealing with super-Brownian motion, see in particular the introduction of [27]). We also consider the quantity $\sigma_+ = \Theta([0,\infty))$ (resp. $\sigma_- = \Theta((-\infty,0]))$ corresponding to the volume of the set of all points $a\in\mathcal{T}_\zeta$ such that $V_a\geq 0$ (resp. $V_a\leq 0$). One of our main technical results (Proposition 7) identifies the joint Laplace transform

$$\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu_1\sigma_+-\mu_2\sigma_-))\ , \quad \lambda,\mu_1,\mu_2>0,$$

as the solution of the equation $h_{\mu_1,\mu_2}(v)=\sqrt{6}\,\lambda$, where, for every $v\geq 0$,

$$h_{\mu_1,\mu_2}(v)=\sqrt{\sqrt{2\mu_1}+v}\,\Big(2v-\sqrt{2\mu_1}\Big)+\sqrt{\sqrt{2\mu_2}+v}\,\Big(2v-\sqrt{2\mu_2}\Big).$$

In the special case $\mu_1=\mu_2$, this equation can be solved explicitly and leads to the formula

$$\mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)\Big)=\begin{cases}\sqrt{2\mu}\,\cos\Big(\frac{2}{3}\,\arccos\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\leq 1,\\ \sqrt{2\mu}\,\cosh\Big(\frac{2}{3}\,\mathrm{arcosh}\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\geq 1.\end{cases} \tag{1}$$

We can extract the conditional distribution of $\mathcal{L}^0$ knowing σ from the preceding formula. In this way we obtain a short direct proof of a remarkable result of Bousquet-Mélou and Janson [9] stating that the local time $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot\mid\sigma=1)$ (equivalently the density of ISE at 0) is distributed as $(2^{3/4}/3)\,T^{-1/2}$, where T is a positive stable variable with index 2/3, whose Laplace transform is $\mathbb{E}[\exp(-\lambda T)]=\exp(-\lambda^{2/3})$ (Theorem 11). The original proof of Bousquet-Mélou and Janson relied on limit theorems for approximations of ISE by discrete labeled trees. Somewhat surprisingly, we are also able to obtain an analog of the latter result when instead of conditioning on $\sigma=1$ we condition on $\sigma_+=1$. Precisely, we get that the local time $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot\mid\sigma_+=1)$ is distributed as $(2^{9/4}/3)\,D\,T^{-1/2}$, where T is as previously and the random variable D is independent of T and has density $2x\,\mathbf{1}_{[0,1]}(x)$ (Theorem 12). Our proofs are computational and rely on explicit formulas for moments derived via the Lagrange inversion theorem. It would be interesting to have more probabilistic proofs and a better understanding of the reason why such simple distributions occur.

Because of the connections between the Brownian snake and super-Brownian motion, several of our results can be restated in terms of distributions of (one-dimensional) super-Brownian motion $(\mathbf{X}_t)_{t\geq 0}$ started from the Dirac measure δ_0. In particular, we get that the total local time at 0 (defined as the density at 0 of the measure $\int_0^\infty \mathrm{d}t\,\mathbf{X}_t$) is distributed as $3^{1/2}2^{-2/3}\,T$ where T is as previously a positive stable variable with index 2/3 (Corollary 6). This is by no means a difficult result (as pointed out to the authors by Edwin Perkins [28], the fact that the total local time is a stable variable with index 2/3 can also be derived by a scaling argument, see formula (2.13) in [27]), but it seems to have remained unnoticed by the specialists of super-Brownian motion. The fact that the same variable T occurs in the Bousquet-Mélou-Janson result suggests the existence of a direct connection between the two results, but we have been unable to find such a connection.

The present article is organized as follows. Section 2 gives a number of preliminaries concerning the Brownian snake. We have chosen to discuss the Brownian snake with a general spatial motion because it turns out to be useful to consider also the case where this spatial motion is the pair consisting of a linear Brownian motion and its local time at 0. In fact, Section 3 starts with a formula expressing the local time $\mathcal{L}^0$ in terms of certain exit measures of this two-dimensional Brownian snake (Proposition 3). This expression then leads to an easy calculation of the Laplace transform of $\mathcal{L}^0$, or more generally of $\mathcal{L}^x$ for any $x \in \mathbb{R}$, under $\mathbb{N}_0$ (Corollary 5). Section 4 gives the key Proposition 7 characterizing the joint Laplace transform of the triple $(\mathcal{L}^0, \sigma_+, \sigma_-)$ and establishes (1) as a consequence. Finally, Section 5 derives conditional distributions of the local time $\mathcal{L}^0$, and also discusses the interpretation of these distributions in continuous models of random geometry.

2 Preliminaries

2.1 The Brownian snake

In this section, we recall some basic facts about the Brownian snake with a general spatial motion. We let ξ stand for a Markov process with values in $\mathbb{R}^d$, which starts from $x \in \mathbb{R}^d$ under the probability measure $\mathbb{P}_x$. We assume that ξ has continuous sample paths, and moreover we require the following bound on the increments of ξ. There exist three positive constants C, $q > 2$ and $\chi > 0$ such that for every $t \in [0, 1]$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\Big[\sup_{0\le s\le t} |\xi_s - x|^q\Big] \le C\, t^{2+\chi}. \tag{2}$$

Under this moment assumption, we may define the Brownian snake with spatial motion ξ as a strong Markov process with values in the space of d-dimensional finite paths (see [20, Section IV.4]). In this work, we will only need the Brownian snake excursion measures, which we now introduce within the formalism of snake trajectories [1].

First recall that a (d-dimensional) finite path w is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}^d$, where the number $\zeta = \zeta_{(\mathrm{w})} \ge 0$ is called the lifetime of w. We let $\mathcal{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\ge 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For every $x \in \mathbb{R}^d$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x of $\mathbb{R}^d$.

Definition 1. *Let $x \in \mathbb{R}^d$. A snake trajectory with initial point x is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_x$ which satisfies the following two properties:*

(i) *We have $\omega_0 = x$ and the number $\sigma(\omega) := \sup\{s \ge 0 : \omega_s \ne x\}$, called the duration of the snake trajectory ω, is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = x$ for every $s \ge 0$).*

(ii) *For every $0 \le s \le s'$, we have $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s\le r\le s'} \zeta_{(\omega_r)}]$.*

If ω is a snake trajectory, we will write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$. We denote the set of all snake trajectories with initial point x by $\mathcal{S}_x$. The set $\mathcal{S}_x$ is equipped with the distance

$$d_{\mathcal{S}_x}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s\ge 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega'))$$

and the associated Borel σ-field.

Let $\mathbf{n}(\mathrm{d}e)$ denote the classical Itô measure of positive excursions of linear Brownian motion (see e.g. [29, Chapter XII]). Then $\mathbf{n}(\mathrm{d}e)$ is a σ-finite measure on the space of all continuous functions $s \mapsto e_s$ from $\mathbb{R}_+$ into $\mathbb{R}_+$, and without risk of confusion, we will write $\sigma(e) = \sup\{s \ge 0 : e_s \ne 0\}$. We consider the usual normalization of $\mathbf{n}(\mathrm{d}e)$, so that, for every $\varepsilon > 0$,

$$\mathbf{n}\Big(\sup\{e_s : s \ge 0\} > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

We have then also, for every $\lambda > 0$,

$$\mathbf{n}(1 - \exp(-\lambda\sigma(e))) = \sqrt{\lambda/2}, \tag{3}$$

and equivalently the distribution of $\sigma(e)$ under $\mathbf{n}(\mathrm{d}e)$ is $(2\sqrt{2\pi})^{-1}\, s^{-3/2}\,\mathrm{d}s$.

Definition 2. *For every $x \in \mathbb{R}^d$, the Brownian snake excursion measure $\mathbb{N}_x$ is the σ-finite measure on $\mathcal{S}_x$ characterized by the following two properties:*

(i) *The distribution of $(\zeta_s)_{s\geq 0}$ under $\mathbb{N}_x$ is $\mathbf{n}$;*

(ii) *Under $\mathbb{N}_x$ and conditionally on $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is a time-inhomogeneous Markov process whose transition kernels can be described as follows: For every $0 \leq s \leq s'$,*

- *$W_{s'}(t) = W_s(t)$ for all $0 \leq t \leq m_\zeta(s,s') := \min\{\zeta_r : s \leq r \leq s'\}$;*
- *conditionally on W_s, the random path $(W_{s'}(m_\zeta(s,s') + t), 0 \leq t \leq \zeta_{s'} - m_\zeta(s,s'))$ is distributed as the Markov process ξ started at $W_s(m_\zeta(s,s'))$.*

See again [20, Chapter IV] for more information about the measures $\mathbb{N}_x$. If F is a nonnegative function on $\mathcal{W}_x$, we have the first-moment formula

$$\mathbb{N}_x\Big(\int_0^\sigma F(W_s)\,\mathrm{d}s\Big) = \mathbb{E}_x\Big[\int_0^\infty F\Big((\xi_r)_{0\leq r\leq t}\Big)\,\mathrm{d}t\Big]. \tag{4}$$

We now turn to exit measures. Let O be an open set in $\mathbb{R}^d$ such that $x \in O$. For every $\mathrm{w} \in \mathcal{W}_x$, set

$$\tau_O(\mathrm{w}) = \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin O\}$$

with the usual convention $\inf \varnothing = +\infty$. Then $\mathbb{N}_x$ a.e. there exists a random finite measure $\mathcal{Z}_O$ supported on ∂O such that, for every bounded continuous function φ on ∂O, we have

$$\langle \mathcal{Z}_O, \varphi\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\tau_O(W_s)\leq\zeta_s\leq\tau_O(W_s)+\varepsilon\}}\,\varphi(W_s(\tau_O(W_s))). \tag{5}$$

See [20, Chapter V]. Then, for every nonnegative measurable function φ on $\mathbb{R}^d$,

$$\mathbb{N}_x(\langle \mathcal{Z}_O, \varphi\rangle) = \mathbb{E}_x[\varphi(\xi_{\tau_O})\,\mathbf{1}_{\{\tau_O<\infty\}}], \tag{6}$$

where in the right-hand side $\tau_O = \inf\{t \geq 0 : \xi_t \notin O\}$.

Let us now recall the special Markov property of the Brownian snake, referring to the appendix of [22] for the proof of a slightly more precise statement. To this end we consider again the open set O such that $x \in O$, and fix a snake trajectory $\omega \in \mathcal{W}_x$. We observe that the set $\{s \geq 0 : \tau_O(W_s) < \infty\}$ is open and thus can be written as a disjoint union of open intervals (a_i, b_i), $i \in I$ (the indexing set I may be empty if none of the paths W_s exits O). For every $i \in I$, we may define a new snake trajectory $\omega^{(i)}$ by setting for every $s \geq 0$,

$$\omega^{(i)}_s(t) := \omega_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)\ ,\ \text{for every } 0 \leq t \leq \zeta_{(\omega^{(i)}_s)} := \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}.$$

The snake trajectories $\omega^{(i)}$ represent the excursions of ω outside O (the word "outside" is somewhat misleading since these excursions typically come back into O though they start on ∂O). We also introduce a σ-field $\mathcal{E}_O$ corresponding informally to the information given by the paths W_s before they exit O (see [22] for a more precise definition), and note that $\mathcal{Z}_O$ is measurable withe respect to $\mathcal{E}_O$. Then the special Markov property states that, under $\mathbb{N}_x$ and conditionally on $\mathcal{E}_O$, the point measure

$$\sum_{i\in I} \delta_{\omega^{(i)}}$$

is a Poisson random measure with intensity $\int \mathcal{Z}_O(\mathrm{d}y)\,\mathbb{N}_y(\cdot)$.

2.2 Specific properties when ξ is linear Brownian motion

We finally mention a few more specific properties that hold in the special case where $d = 1$ and ξ is standard linear Brownian motion. In that case, we have the following scaling property. If $\lambda > 0$ and

$$W'_s(t) = \lambda\, W_{s/\lambda^4}(t/\lambda^2)\ , \quad \text{for every } 0 \le t \le \zeta'_s := \lambda^2 \zeta_{s/\lambda^4}, \tag{7}$$

then the distribution of $(W'_s)_{s\ge0}$ under $\mathbb{N}_x$ is $\lambda^2 \mathbb{N}_{\lambda x}$.

Suppose that the open set O is the interval $(-\infty, y)$ with $y > x$, or the interval (y, ∞) with $y < x$. In both cases, the exit measure $\mathcal{Z}_O$ can be written as $\mathcal{Z}_y\, \delta_y$, where $\mathcal{Z}_y \ge 0$ and δ_y denotes the Dirac measure at y, and we have, for every $\lambda > 0$,

$$\mathbb{N}_x\Big(1 - \exp(-\lambda \mathcal{Z}_y)\Big) = \Big(\lambda^{-1/2} + |y - x|\sqrt{2/3}\Big)^{-2}. \tag{8}$$

See formula (6) in [13].

Let $\mathcal{R} := \{\widehat{W}_s : s \ge 0\} = \{W_s(t) : s \ge 0, 0 \le t \le \zeta_s\}$ denote the range of the Brownian snake. Then, for every $y \in \mathbb{R}$, $y \ne x$,

$$\mathbb{N}_x(y \in \mathcal{R}) = \frac{3}{2(y-x)^2} = \mathbb{N}_x(\mathcal{Z}_y > 0). \tag{9}$$

See [20, Section VI.1] for the first equality, and note that the second one follows from (8).

Finally, it follows from the results of [9] that there exists $\mathbb{N}_0$ a.e. a continuous function $(\mathcal{L}^y)_{y\in\mathbb{R}}$, which is supported on $\mathcal{R}$, such that, for every nonnegative measurable function φ on $\mathbb{R}$,

$$\int_0^\sigma \mathrm{d}s\, \varphi(\widehat{W}_s) = \int_{\mathbb{R}} \mathrm{d}y\, \varphi(y)\, \mathcal{L}^y.$$

We call $\mathcal{L}^y$ the Brownian snake local time at y. Note that [9] deals with the case of ISE, that is, with the conditional measure $\mathbb{N}_0(\cdot \mid \sigma = 1)$, but then a scaling argument gives the desired result under $\mathbb{N}_0$. Next suppose that, for a given $\lambda > 0$, W' is defined from W as in (7). Then, with an obvious notation, we have $\sigma' = \lambda^4 \sigma$ and $\mathcal{L}'^x = \lambda^3 \mathcal{L}^{x/\lambda}$ for every $x \in \mathbb{R}$, $\mathbb{N}_0$ a.e. As a consequence, for every $s > 0$, the distribution of $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = s)$ is equal to the distribution of $s^{3/4}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = 1)$.

The scaling property also implies the existence of a constant C such that, for every $s > 0$ and $x \in \mathbb{R}$, we have $\mathbb{N}_0(\mathcal{L}^x \mid \sigma = s) \le C\, s^{3/4}$ (the case $s = 1$ follows from [9, Corollary 11.3], or from a simple argument using Fatou's lemma and the approximation of $\mathcal{L}^x$ by $(2\varepsilon)^{-1} \int_0^\sigma \mathrm{d}r\, \mathbf{1}_{\{|\widehat{W}_r - x| < \varepsilon\}}$).

3 The local time at 0

In this section and the next ones, we consider the Brownian snake excursion measure $\mathbb{N}_0$ in the case where $\xi = B$ is linear Brownian motion. For every $a \in \mathbb{R}$ and $t \ge 0$, we use the notation $L^a_t(B)$ for the local time of the Brownian motion B at level a and at time t.

For every fixed $s \ge 0$, the path W_s is distributed under $\mathbb{N}_0$ and conditionally on ζ_s as a linear Brownian path started at 0 with lifetime ζ_s, and we can define its local time process at 0,

$$L^0_t(W_s) = \lim_{\varepsilon\to0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[0,\varepsilon]}(W_s(r))\, \mathrm{d}r\ , \quad 0 \le t \le \zeta_s\ ,\ \text{a.s.}$$

We may view $L^0(W_s) = (L^0_t(W_s))_{0\le t\le\zeta_s}$ as a random element of $\mathcal{W}_0$ with lifetime ζ_s. Simple moment estimates show that we can choose a continuous modification of $(L^0(W_s))_{s\ge0}$ (as a random process with values in $\mathcal{W}_0$). Moreover, the distribution under $\mathbb{N}_0$ of the two-dimensional process $(W_s, L^0(W_s))_{s\ge0}$ is the Brownian snake excursion measure (from the point $(0,0)$ of $\mathbb{R}^2$) for the Markov process $\xi'_t = (B_t, L^0_t(B))$ (note that ξ' may be viewed as a Markov process with values in $\mathbb{R}^2$, which satisfies (2)).

The point of the preceding discussion is that, under $\mathbb{N}_0$, we can define exit measures for the process $(W_s, L^0(W_s))_{s\ge0}$ from open subsets O of $\mathbb{R}^2$ containing $(0,0)$, in the way explained in Section 2 (e.g. from the approximation formula (5)). For every $r > 0$, we consider the exit measure from the open set $O = \mathbb{R} \times (-\infty, r)$ and denote its mass by $\mathscr{X}_r$ (as an application of the first-moment formula (6), this exit measure is a random multiple of the Dirac measure at $(0,r)$). By convention we also take $\mathscr{X}_0 = 0$.

On the other hand, as explained in Section 8 of [1], we can use a famous theorem of Lévy [29, Theorem VI.2.3] to give a different presentation of the process $(|W_s|, L^0(W_s))$. To this end, for every $s \geq 0$, write

$$W_s^\bullet(t) := W_s(t) - \min\{W_s(r) : 0 \leq r \leq t\} \,, L_s^\bullet(t) = -\min\{W_s(r) : 0 \leq r \leq t\} \,, \quad \text{for } 0 \leq t \leq \zeta_s.$$

Then the distribution of the pair $(W_s^\bullet, L_s^\bullet)_{s\geq 0}$ under $\mathbb{N}_0$ is equal to the distribution of $(|W_s|, L^0(W_s))_{s\geq 0}$ under the same measure.

Using the preceding identity in distribution of two-dimensional snake trajectories, and the approximation (5) of exit measures, we get that the process $(\mathscr{X}_r)_{r>0}$ has the same distribution under $\mathbb{N}_0$ as the process $(\mathcal{Z}_{-r})_{r>0}$, where we recall that, for every $x \in \mathbb{R}\backslash\{0\}$, $\mathcal{Z}_x$ denotes the (total mass of the) exit measure of $(W_s)_{s\geq 0}$ from the open interval (x,∞) if $x<0$, or $(-\infty,x)$ if $x>0$ — of course, by symmetry, $(\mathcal{Z}_{-r})_{r>0}$ has the same distribution as $(\mathcal{Z}_r)_{r>0}$. In particular $\mathbb{N}_0(\mathscr{X}_r > 0) = \mathbb{N}_0(\mathcal{Z}_r > 0) < \infty$ by (9). The discussion in [25, Section 2.4] now shows that the process $(\mathscr{X}_r)_{r>0}$ has a càdlàg modification under $\mathbb{N}_0$, which we consider from now on. Furthermore the distribution of this càdlàg modification under $\mathbb{N}_0$ can be interpreted as the excursion measure of the continuous-state branching process with branching mechanism $\phi(u) = \sqrt{8/3}\,u^{3/2}$ (the ϕ-CSBP in short, see [20, Chapter II] for a brief presentation of continuous-state branching processes). This means that, if $\alpha > 0$, and $\sum_{i\in I}\delta_{\omega_i}$ is a Poisson point measure with intensity $\alpha\,\mathbb{N}_0$, the process Y defined by $Y_0 = \alpha$ and

$$Y_r = \sum_{i\in I} \mathscr{X}_r(\omega_i)$$

for every $r > 0$, is a ϕ-CSBP started from α. Note that the right-hand side of the last display is a finite sum since $\mathbb{N}_0(\mathscr{X}_r > 0) < \infty$.

Recall our notation $\mathcal{L}^0$ for the Brownian snake local time at 0.

Proposition 3. *We have*

$$\mathcal{L}^0 = \int_0^\infty \mathrm{d}r\, \mathscr{X}_r \,, \quad \mathbb{N}_0 \ a.e.$$

This proposition is obviously related to the identity (37) in [25, Proposition 25], which is however concerned with the local time $\mathcal{L}^x$ at a level $x > 0$. Unfortunately, the case $x = 0$ seems to require a different argument.

Proof. It will be convenient to write $\widehat{L}(W_s) = L^0_{\zeta_s}(W_s)$ and

$$L^* = \max\{\widehat{L}(W_s) : 0 \leq s \leq \sigma\}.$$

For every $\varepsilon > 0$, set

$$\mathcal{L}^{0,\varepsilon} := \varepsilon^{-1}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon\}}.$$

Then $\mathcal{L}^{0,\varepsilon} \longrightarrow \mathcal{L}^0$ as $\varepsilon \to 0$, $\mathbb{N}_0$ a.e. We also introduce, for every fixed $\delta > 0$,

$$\mathcal{L}^{0,\varepsilon,(\delta)} := \varepsilon^{-1}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, \widehat{L}(W_s)>\delta\}}.$$

We observe that, for every $\varepsilon, \delta > 0$, we can use the first-moment formula (4) to compute

$$\mathbb{N}_0(\mathcal{L}^{0,\varepsilon} - \mathcal{L}^{0,\varepsilon,(\delta)}) = \mathbb{N}_0\Big(\varepsilon^{-1}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, \widehat{L}(W_s)\leq\delta\}}\Big) = \varepsilon^{-1}\mathbb{E}_0\Big[\int_0^\infty \mathrm{d}t\, \mathbf{1}_{\{0<B_t<\varepsilon, L^0_t(B)\leq\delta\}}\Big] = \delta, \quad (10)$$

where the last equality follows from a standard Ray-Knight theorem for Brownian local times [29, Theorem IX.2.3]. We then need the following lemma.

Lemma 4. *For every $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0} \mathcal{L}^{0,\varepsilon,(\delta)} = \int_\delta^\infty \mathrm{d}r\, \mathscr{X}_r \;,$$

in probability under $\mathbb{N}_0(\cdot \mid L^* \geq \delta)$.

Let us postpone the proof of this lemma and complete that of Proposition 3. Write $\widetilde{\mathcal{L}}^0 = \int_0^\infty \mathrm{d}r\, \mathscr{X}_r$ and $\widetilde{\mathcal{L}}^{0,(\delta)} = \int_\delta^\infty \mathrm{d}r\, \mathscr{X}_r$ to simplify notation, and for $a > 0$ set $\mathbb{N}_0^{(a)} = \mathbb{N}_0(\cdot \mid L^* \geq a)$. Then, for every $\alpha > 0$,

$$\begin{aligned}\mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \widetilde{\mathcal{L}}^0| > \alpha) \leq{}& \mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \mathcal{L}^{0,\varepsilon}| > \alpha/4) + \mathbb{N}_0^{(a)}(|\mathcal{L}^{0,\varepsilon} - \mathcal{L}^{0,\varepsilon,(\delta)}| > \alpha/4) \\ &+ \mathbb{N}_0^{(a)}(|\mathcal{L}^{0,\varepsilon,(\delta)} - \widetilde{\mathcal{L}}^{0,(\delta)}| > \alpha/4) + \mathbb{N}_0^{(a)}(|\widetilde{\mathcal{L}}^{0,(\delta)} - \widetilde{\mathcal{L}}^0| > \alpha/4). \qquad (11)\end{aligned}$$

Let $\gamma > 0$. We can fix $\delta > 0$ small enough so that, for every $\varepsilon > 0$, the second and the fourth term in the right-hand side of (11) are smaller than $\gamma/4$ (we use (10) for the second term). Then, if $\varepsilon > 0$ is small enough, the first and the third term are also smaller than $\gamma/4$ (using Lemma 4 for the third term). We conclude that $\mathbb{N}_0^{(a)}(|\mathcal{L}^0 - \widetilde{\mathcal{L}}^0| > \alpha) \leq \gamma$ and since α and γ were arbitrary this gives the desired result $\widetilde{\mathcal{L}}^0 = \mathcal{L}^0$. □

Proof of Lemma 4. We keep the notation $\widetilde{\mathcal{L}}^{0,(\delta)}$ introduced in the previous proof. We first observe that

$$\widetilde{\mathcal{L}}^{0,(\delta)} = \lim_{\varepsilon\to 0} \varepsilon \sum_{k=0}^{\infty} \mathscr{X}_{\delta+k\varepsilon} \;, \quad \mathbb{N}_0 \text{ a.e.} \qquad (12)$$

and on the other hand,

$$\mathcal{L}^{0,\varepsilon,(\delta)} = \varepsilon^{-1} \sum_{k=0}^{\infty} \mathcal{H}_k^{\varepsilon,(\delta)}, \qquad (13)$$

where

$$\mathcal{H}_k^{\varepsilon,(\delta)} = \int_0^\sigma \mathrm{d}s\; \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon,\; \delta+k\varepsilon<\widehat{L}(W_s)\leq\delta+(k+1)\varepsilon\}}.$$

The idea of the proof is to bound $\mathbb{N}_0(|\varepsilon\mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-1}\mathcal{H}_k^{\varepsilon,(\delta)}|)$, for every fixed $k \geq 0$. To this end, we apply the special Markov property to the Brownian snake with spatial motion $(B_t, L_t^0(B))$ and the open set $O = \mathbb{R} \times (-\infty, \delta + k\varepsilon)$, noting that the event $\{L^* \geq \delta\}$ is then $\mathcal{E}_O$-measurable. It follows that, under $\mathbb{N}_0(\cdot \mid L^* \geq \delta)$ and conditionally on $\mathscr{X}_{\delta+k\varepsilon} = a$, the quantity $\mathcal{H}_k^{\varepsilon,(\delta)}$ is distributed as

$$\int \mathcal{N}(\mathrm{d}\omega)\, \mathcal{U}_\varepsilon(\omega)$$

where $\mathcal{N}(\mathrm{d}\omega)$ is a Poisson point measure with intensity $a\,\mathbb{N}_0$, and the random variable $\mathcal{U}_\varepsilon$ is defined under $\mathbb{N}_0$ by

$$\mathcal{U}_\varepsilon = \int_0^\sigma \mathrm{d}s\; \mathbf{1}_{\{0<\widehat{W}_s<\varepsilon, 0<\widehat{L}(W_s)\leq\varepsilon\}}.$$

Hence, conditionally on $\mathscr{X}_{\delta+k\varepsilon} = a$, $\mathcal{H}_k^{\varepsilon,(\delta)}$ has the distribution of U_a^ε, where $(U_t^\varepsilon)_{t\geq 0}$ is the subordinator whose Lévy measure is the distribution of $\mathcal{U}_\varepsilon$ under $\mathbb{N}_0$. Note that $\mathbb{E}[U_1^\varepsilon] = \mathbb{N}_0(\mathcal{U}_\varepsilon) = \varepsilon^2$ by (10).

By a scaling argument, we get that $(U_t^\varepsilon)_{t\geq 0}$ has the same distribution as $(\varepsilon^4\, U^1_{\varepsilon^{-2}t})_{t\geq 0}$. Next the law of large numbers shows that

$$\lim_{t\to\infty} \sup_{s\leq t} \mathbb{E}\Big[\frac{|U_s^1 - s|}{t}\Big] = 0. \qquad (14)$$

Fix $A>0$ and consider the event $E_A := \{L^* \leq A\} \cap \{\sup\{\mathscr{X}_r : r \geq 0\} \leq A\}$. Notice that on this event we have $\mathscr{X}_{\delta+k\varepsilon} = 0$ and $\mathcal{H}_k^{\varepsilon,(\delta)} = 0$ as soon as $\delta + k\varepsilon > A$. It follows that

$$\begin{aligned}
&\mathbb{N}_0\Big(\mathbf{1}_{E_A}\, |\varepsilon \sum_{k=0}^{\infty} \mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-1} \sum_{k=0}^{\infty} \mathcal{H}_k^{\varepsilon,(\delta)}| \,\Big|\, L^* \geq \delta\Big) \\
&\leq \varepsilon(\lfloor A/\varepsilon \rfloor + 1) \sup_{0 \leq k \leq \lfloor A/\varepsilon \rfloor} \mathbb{N}_0\Big(\mathbf{1}_{\{\mathscr{X}_{\delta+k\varepsilon} \leq A\}}\, |\mathscr{X}_{\delta+k\varepsilon} - \varepsilon^{-2}\mathcal{H}_k^{\varepsilon,(\delta)}| \,\Big|\, L^* \geq \delta\Big) \\
&\leq \varepsilon(\lfloor A/\varepsilon \rfloor + 1) \sup_{0 \leq a \leq A} \mathbb{E}[|\varepsilon^{-2} U_a^{\varepsilon} - a|] \\
&= \varepsilon(\lfloor A/\varepsilon \rfloor + 1) \sup_{0 \leq s \leq A/\varepsilon^2} \mathbb{E}[\varepsilon^2 |U_s^1 - s|],
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$, by (14). The statement of the lemma follows, recalling (12) and (13). □

Corollary 5. *For every $\lambda > 0$,*

$$\mathbb{N}_0(1 - e^{-\lambda \mathcal{L}_0}) = \frac{3^{1/3}}{2}\, \lambda^{2/3}. \tag{15}$$

The distribution of $\mathcal{L}^0$ under $\mathbb{N}_0$ has density

$$h(\ell) = \frac{3^{-2/3}}{\Gamma(1/3)}\, \ell^{-5/3}$$

with respect to Lebesgue measure on $(0,\infty)$.

Proof. By Proposition 3 and the interpretation of the distribution of $(\mathscr{X}_r)_{r>0}$ under $\mathbb{N}_0$, we have

$$\mathbb{N}_0(1 - e^{-\lambda \mathcal{L}_0}) = -\log \mathbb{E}\Big[\exp\Big(-\lambda \int_0^{\infty} \mathrm{d}r\, X_r\Big)\Big],$$

where $(X_r)_{r\geq 0}$ denotes a ϕ-CSBP started from 1, and we recall that $\phi(u) = \sqrt{8/3}\, u^{3/2}$. The classical Lamperti transformation [11, 19] shows that $\int_0^{\infty} \mathrm{d}r\, X_r$ has the same distribution as $T_0 := \inf\{t \geq 0 : Y_t = 0\}$, where $(Y_t)_{t\geq 0}$ denotes a stable Lévy process with no negative jumps started from 1, whose distribution is characterized by the Laplace transform $\mathbb{E}[\exp(-\lambda(Y_t - 1))] = \exp(t\,\phi(\lambda))$. It is then classical (see e.g. [6, Chapter VII]) that

$$\mathbb{E}[e^{-\lambda T_0}] = e^{-\phi^{-1}(\lambda)},$$

where $\phi^{-1}(\lambda) = (3/8)^{1/3}\, \lambda^{2/3}$ is the inverse function of ϕ. This completes the proof of the first assertion. The density of $\mathcal{L}^0$ is then obtained by inverting the Laplace transform. □

In the next corollary, we consider a one-dimensional super-Brownian motion $(\mathbf{X}_t)_{t\geq 0}$ with quadratic branching mechanism $\psi(u) = 2u^2$ (the choice of the constant 2 is only for convenience, since a scaling argument will give a similar result with a general quadratic branching mechanism). Then it is well known that we can define the associated (total) local times as the unique (random) continuous function $(\mathbf{L}^a)_{a\in\mathbb{R}}$ such that

$$\int_0^{\infty} \mathrm{d}t\, \langle \mathbf{X}_t, f\rangle = \int_{\mathbb{R}} \mathrm{d}a\, f(a)\, \mathbf{L}^a,$$

for every Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}_+$. See in particular Sugitani [30].

Corollary 6. *Suppose that $\mathbf{X}_0 = \alpha\, \delta_0$ for some $\alpha > 0$. Then, for every $a \in \mathbb{R}$ and $\lambda > 0$,*

$$\mathbb{E}[e^{-\lambda \mathbf{L}^a}] = \exp\left(-\alpha\, \frac{3^{1/3}}{2} \Big(\lambda^{-1/3} + 3^{-1/3}\, |a|\Big)^{-2}\right). \tag{16}$$

In particular,

$$\mathbb{E}[e^{-\lambda \mathbf{L}^0}] = \exp\left(-\alpha \frac{3^{1/3}}{2}\, \lambda^{2/3}\right), \tag{17}$$

so that $\mathbf{L}^0$ is a positive stable variable with index $2/3$.

Proof. We rely on the Brownian snake construction of super-Brownian motion (see in particular [20, Chapter 4]). We may assume that $(\mathbf{X}_t)_{t\geq 0}$ is constructed in such a way that there exists a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{\omega_i}$ with intensity $\alpha\mathbb{N}_0$, such that, for every Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}_+$,

$$\int_{\mathbb{R}} \mathrm{d}a\, f(a)\, \mathbf{L}^a = \int_0^\infty \mathrm{d}t\, \langle \mathbf{X}_t, f\rangle = \sum_{i\in I} \int_0^{\sigma(\omega_i)} \mathrm{d}s\, f(\widehat{W}_s(\omega_i)) = \sum_{i\in I} \int_{\mathbb{R}} \mathrm{d}a\, f(a)\, \mathcal{L}^a(\omega_i).$$

It follows that we have

$$\mathbf{L}^a = \sum_{i\in I} \mathcal{L}^a(\omega_i) \tag{18}$$

for Lebesgue a.e. $a \in \mathbb{R}$. The left-hand side is continuous in a, and the right-hand side is continuous on $\mathbb{R}\backslash\{0\}$ since, for every $\delta > 0$, there are only finitely many $i \in I$ such that $\mathcal{L}^a(\omega_i)$ is nonzero for some a with $|a| > \delta$. So (18) holds for every $a \in \mathbb{R}\backslash\{0\}$. In fact it is easy to see that (18) also holds for $a = 0$. First note that, by Fatou's lemma, $\mathbf{L}^0 \geq \sum_{i\in I} \mathcal{L}^0(\omega_i)$, so that it suffices to check that

$$\mathbb{E}[e^{-\mathbf{L}^0}] = \mathbb{E}\Big[\exp\Big(-\sum_{i\in I} \mathcal{L}^0(\omega^i)\Big)\Big].$$

The left-hand side is the limit when $a \to 0$ of $\mathbb{E}[e^{-\mathbf{L}^a}] = \exp(-\mathbb{N}_0(1 - e^{-\mathcal{L}^a}))$ and the right-hand side is equal to $\exp(-\mathbb{N}_0(1 - e^{-\mathcal{L}^0}))$. So we only need to verify that $\mathbb{N}_0(1 - e^{-\mathcal{L}^a})$ tends to $\mathbb{N}_0(1 - e^{-\mathcal{L}^0})$ as $a \to 0$, which is easy by conditioning on σ and then using the bound $\mathbb{N}_0(1 - e^{-\mathcal{L}^a} \,|\, \sigma = s) \leq C(s^{3/4} \wedge 1)$ to justify dominated convergence.

Formula (17) follows from the case $a = 0$ of (18) as an immediate application of (15) and the exponential formula for Poisson measures. As for formula (16), it is enough to verify that

$$\mathbb{N}_0(1 - e^{-\lambda\mathcal{L}^a}) = \frac{3^{1/3}}{2}\Big(\lambda^{-1/3} + 3^{-1/3}\,|a|\Big)^{-2}. \tag{19}$$

Fix $a > 0$ for definiteness, and recall our notation $\mathcal{Z}_a$ for the total mass of the exit measure from $(-\infty, a)$. Write $(\omega'_j)_{j\in J}$ for the excursions of the Brownian snake outside $(-\infty, a)$. By the special Markov property, under $\mathbb{N}_0$ and conditionally on $\mathcal{Z}_a$, the point measure $\sum_{j\in J} \delta_{\omega'_j}$ is Poisson with intensity $\mathcal{Z}_a\, \mathbb{N}_a$. Moreover, the first part of the proof shows that we have $\mathbf{L}^a = \sum_{j\in J} \mathcal{L}^a(\omega'_j)$, $\mathbb{N}_0$ a.e., and therefore

$$\mathbb{N}_0(1 - e^{-\lambda\mathcal{L}^a}) = \mathbb{N}_0\Big(1 - \exp\Big(-\mathcal{Z}_a\mathbb{N}_0(1 - \exp(-\lambda\mathcal{L}^0))\Big)\Big).$$

Then (19) follows from (15) and (8). □

Remark. An alternative way to derive the previous two corollaries would be to use the known connections between super-Brownian motion or the Brownian snake and partial differential equations. See formula (1.13) in [27], and note that, as a function of a, the right-hand side of (19) solves the differential equation $\frac{1}{2}u'' = 2u^2 - \lambda\delta_0$ in the sense of distributions. On the other hand, our method provides a better probabilistic understanding of the results and the derivation of (15) in particular relies on Proposition 3 which is of independent interest and will play a significant role in the proofs of the next section.

4 The joint distribution of the local time and the time spent above and below 0

Our next goal is to discuss the joint distribution of $(\mathcal{L}^0, \sigma_+, \sigma_-)$ under $\mathbb{N}_0$, where we write

$$\sigma_+ := \int_0^\sigma \mathbf{1}_{\{\widehat{W}_s > 0\}}\, \mathrm{d}s\,, \quad \sigma_- := \int_0^\sigma \mathbf{1}_{\{\widehat{W}_s < 0\}}\, \mathrm{d}s.$$

Proposition 7. *Let $\lambda, \mu_1, \mu_2 \geq 0$, and consider the function $h_{\mu_1,\mu_2} : [0,\infty) \longrightarrow \mathbb{R}$ defined by*

$$h_{\mu_1,\mu_2}(v) = \sqrt{\sqrt{2\mu_1} + v}\,\Big(2v - \sqrt{2\mu_1}\Big) + \sqrt{\sqrt{2\mu_2} + v}\,\Big(2v - \sqrt{2\mu_2}\Big).$$

Then the quantity

$$v(\lambda,\mu_1,\mu_2) := \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu_1\sigma_+-\mu_2\sigma_-))$$

is the unique solution of the equation $h_{\mu_1,\mu_2}(v)=\sqrt{6}\,\lambda$.

Proof. First note that the quantities $v(\lambda,\mu_1,\mu_2)$ are finite, since $v(\lambda,\mu,\mu)\leq \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0))+\mathbb{N}_0(1-\exp(-\mu\sigma))<\infty$ by (3) and (15). Then, suppose that, under the probability measure $\mathbb{P}$, we are given a sequence $(\eta_i)_{i\geq 0}$ of independent Bernoulli variables with parameter 1/2, and a sequence $(U_i)_{i\geq 0}$ of i.i.d. nonnegative random variables with density $(2\pi u^5)^{-1/2}\exp(-1/2u)$ for $u>0$. We note that, for every $\beta>0$, we have

$$\mathbb{E}[\exp(-\beta U_1)] = (1+\sqrt{2\beta})\,\exp(-\sqrt{2\beta}). \tag{20}$$

The reason for introducing these two sequences is the following fact. If $(t_i)_{i\geq 0}$ is a measurable enumeration of the jump times of the process $(\mathscr{X}_t)_{t\geq 0}$ (under $\mathbb{N}_0$), the conditional distribution of the pair (σ_+,σ_-) under $\mathbb{N}_0$ and knowing $(\mathscr{X}_t)_{t\geq 0}$ is the law of

$$\Big(\sum_{i=0}^{\infty}\eta_i\,U_i\,(\Delta\mathscr{X}_{t_i})^2,\sum_{i=0}^{\infty}(1-\eta_i)\,U_i\,(\Delta\mathscr{X}_{t_i})^2\Big).$$

This fact is a consequence of the excursion theory developed in [1] (in particular Theorem 4 and Proposition 31 of [1]). In this theory, excursions away from 0 are in one-to-one correspondence with the jumps of $(\mathscr{X}_t)_{t\geq 0}$, so that in the preceding display η_i gives the sign of the associated excursion ($\eta_i=1$ for a positive excursion and $\eta_i=0$ for a negative one), and $U_i\,(\Delta\mathscr{X}_{t_i})^2$ corresponds to the duration of this excursion. We refer to [1] for more details.

Using also Proposition 3 and (20), it follows that

$$\mathbb{N}_0\Big(\exp(-\lambda\mathcal{L}^0-\mu_1\sigma_+-\mu_2\sigma_-)\,\Big|\,(\mathscr{X}_t)_{t\geq 0}\Big)=\exp\Big(-\lambda\int_0^\infty \mathrm{d}t\,\mathscr{X}_t\Big)\prod_{i=0}^{\infty}F(\mu_1,\mu_2,(\Delta\mathscr{X}_{t_i})^2),$$

where we have set, for every $x>0$,

$$F(\mu_1,\mu_2,x):=\frac{1}{2}\Big((1+\sqrt{2\mu_1x})\,\exp(-\sqrt{2\mu_1x})+(1+\sqrt{2\mu_2x})\,\exp(-\sqrt{2\mu_2x})\Big).$$

Hence, with the notation of the theorem, we have

$$v(\lambda,\mu_1,\mu_2)=\mathbb{N}_0\Big(1-\exp\Big(-\lambda\int_0^\infty \mathrm{d}t\,\mathscr{X}_t\Big)\prod_{i=0}^{\infty}F(\mu_1,\mu_2,(\Delta\mathscr{X}_{t_i})^2)\Big).$$

We now recall that the distribution of $(\mathscr{X}_t)_{t\geq 0}$ is the excursion measure of the ϕ-CSBP in order to rewrite this equality in a slightly different form. Suppose that $\sum_{k\in K}\delta_{\omega_k}$ is a Poisson point measure with intensity $\mathbb{N}_0$. The process $(X_t)_{t\geq 0}$ defined by $X_0=1$ and $X_t=\sum_{k\in K}\mathscr{X}_t(\omega_k)$ if $t>0$ is then a ϕ-CSBP started at 1. Furthermore, the exponential formula for Poisson measures and the last display immediately give

$$\mathbb{E}\Big[\exp\Big(-\lambda\int_0^\infty \mathrm{d}t\,X_t\Big)\prod_{j=0}^{\infty}F(\mu_1,\mu_2,(\Delta X_{s_j})^2)\Big]=\exp(-v(\lambda,\mu_1,\mu_2)) \tag{21}$$

where we have written $(s_j)_{j\geq 0}$ for a measurable enumeration of the jumps of X.

Let $t\geq 0$. Using the Markov property of X at time t, the left-hand side of (21) is also equal to

$$\mathbb{E}\Big[\Big(\exp\Big(-\lambda\int_0^t \mathrm{d}s\,X_s\Big)\prod_{j:s_j\leq t}F(\mu_1,\mu_2,(\Delta X_{s_j})^2)\Big)\exp(-v(\lambda,\mu_1,\mu_2)X_t)\Big]. \tag{22}$$

To simplify notation, we write $v=v(\lambda,\mu_1,\mu_2)$ in the following calculations, which are very similar to the proof of Proposition 4.8 in [13]. We also set, for every $s\geq 0$,

$$V_s:=\exp\Big(-\lambda\int_0^s \mathrm{d}u\,X_u\Big)\prod_{j:s_j\leq s}F(\mu_1,\mu_2,(\Delta X_{s_j})^2).$$

From the form of the generator of the ϕ-CSBP, we have

$$e^{-vX_t} = e^{-v} + M_t + \phi(v) \int_0^t X_s \, e^{-vX_s} \, \mathrm{d}s,$$

where $(M_s)_{s\geq 0}$ is a martingale, which is bounded on every compact time interval. By using the integration by parts formula as in [13, formula (28)], we get

$$e^{-vX_t} V_t = e^{-v} + \int_0^t V_{s-} \, \mathrm{d}M_s + \phi(v) \int_0^t V_s X_s \, e^{-vX_s} \, \mathrm{d}s + \int_0^t e^{-vX_s} \, \mathrm{d}V_s.$$

From (21) and (22), we have $\mathbb{E}[e^{-vX_t} V_t] = e^{-v}$. Hence, taking expectations in the last display, we obtain

$$\phi(v) \, \mathbb{E}\Big[\int_0^t V_s X_s \, e^{-vX_s} \, \mathrm{d}s \Big] = -\mathbb{E}\Big[\int_0^t e^{-vX_s} \, \mathrm{d}V_s \Big].$$

Next we observe that

$$\int_0^t e^{-vX_s} \, \mathrm{d}V_s = -\lambda \int_0^t V_s X_s \, e^{-vX_s} \, \mathrm{d}s + \sum_{j:s_j\leq t} e^{-vX_{s_j}} \, V_{s_j-} \, (F(\mu_1, \mu_2, (\Delta X_{s_j})^2) - 1),$$

and so we get

$$(\phi(v) - \lambda) \, \mathbb{E}\Big[\int_0^t V_s X_s \, e^{-vX_s} \, \mathrm{d}s \Big] = -\mathbb{E}\Big[\sum_{j:s_j\leq t} e^{-vX_{s_j}} \, V_{s_j-} \, (F(\mu_1, \mu_2, (\Delta X_{s_j})^2) - 1) \Big].$$

We multiply both sides of this identity by $1/t$ and let $t \downarrow 0$. We have first

$$\lim_{t\downarrow 0} \frac{1}{t} \, \mathbb{E}\Big[\int_0^t V_s X_s \, e^{-vX_s} \, \mathrm{d}s \Big] = e^{-v}.$$

On the other hand, as a consequence of the classical Lamperti representation of continuous-state branching processes [19, 11], we know that the dual predictable projection of the random measure

$$\sum_{i=0}^{\infty} \delta_{(s_j, \Delta X_{s_j})}(\mathrm{d}s, \mathrm{d}x)$$

is the measure $X_s \, \mathrm{d}s \, \kappa(\mathrm{d}x)$, where $\kappa(\mathrm{d}x) = \sqrt{3/2\pi} \, x^{-5/2} \, \mathbf{1}_{\{x>0\}} \mathrm{d}x$ is the Lévy measure of the Lévy process appearing in the Lamperti representation of X. This implies that

$$\mathbb{E}\Big[\sum_{j:s_j\leq t} e^{-vX_{s_j}} \, V_{s_j-} \, (F(\mu_1, \mu_2, (\Delta X_{s_j})^2) - 1) \Big] = \mathbb{E}\Big[\int_0^t \mathrm{d}s \, e^{-vX_s} \, V_s X_s \int \kappa(\mathrm{d}x) \, e^{-vx} (F(\mu_1, \mu_2, x^2) - 1) \Big].$$

Consequently,

$$\lim_{t\downarrow 0} \frac{1}{t} \, \mathbb{E}\Big[\sum_{j:s_j\leq t} e^{-vX_{s_j}} \, V_{s_j-} \, (F(\mu_1, \mu_2, (\Delta X_{s_j})^2) - 1) \Big] = e^{-v} \int \kappa(\mathrm{d}x) \, e^{-vx} (F(\mu_1, \mu_2, x^2) - 1).$$

Finally, we have obtained

$$\phi(v) - \lambda = -\int \kappa(\mathrm{d}x) \, e^{-vx} (F(\mu_1, \mu_2, x^2) - 1).$$

Using the equality $\phi(v) = \int \kappa(\mathrm{d}x)(e^{-vx} - 1 + vx)$, straightforward calculations left to the reader show that

$$\phi(v) + \int \kappa(\mathrm{d}x) \, e^{-vx} (F(\mu_1, \mu_2, x^2) - 1) = \frac{1}{\sqrt{6}} \, h_{\mu_1,\mu_2}(v),$$

where h_{μ_1,μ_2} is as in the statement. This proves that $v = v(\lambda, \mu_1, \mu_2)$ solves $h_{\mu_1,\mu_2}(v) = \sqrt{6} \, \lambda$. Uniqueness is clear since the function h_{μ_1,μ_2} is monotone increasing over $[0, \infty)$. □

Corollary 8. *For every $\lambda \geq 0$ and $\mu > 0$, we have*

$$\mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)\Big) = \begin{cases} \sqrt{2\mu}\,\cos\Big(\frac{2}{3}\,\arccos\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \leq 1, \\ \sqrt{2\mu}\,\cosh\Big(\frac{2}{3}\,\mathrm{arcosh}\Big(\frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}}\Big)\Big) & \text{if } \frac{\sqrt{3}\,\lambda}{2(2\mu)^{3/4}} \geq 1. \end{cases}$$

Proof. Set $w(\lambda,\mu) = \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))$. Note that $w(\lambda,\mu) \geq \mathbb{N}_0(1-\exp(-\mu\sigma)) = \sqrt{\mu/2}$ by (3). It follows from Proposition 7 applied with $\mu_1=\mu_2=\mu$ that $w(\lambda,\mu)$ is the unique solution of the equation

$$4w^3 - 6\mu w + (2\mu)^{3/2} = \frac{3}{2}\lambda^2$$

in $[\sqrt{\mu/2},\infty)$ (note that the left-hand side is a monotone increasing function of w on $[\sqrt{\mu/2},\infty)$). Set $\widetilde{w}(\lambda,\mu) = w(\lambda,\mu)/\sqrt{2\mu}$ and $a = \sqrt{3}\,\lambda/(2(2\mu)^{3/4})$. We immediately get that $\widetilde{w}(\lambda,\mu)$ is the unique solution of

$$4\widetilde{w}^3 - 3\widetilde{w} + 1 = 2a^2$$

in $[1/2,\infty)$. A simple calculation now shows that

$$\widetilde{w} = \begin{cases} \cos(\frac{2}{3}\,\arccos(a)) & \text{if } a \leq 1, \\ \cosh(\frac{2}{3}\,\mathrm{arcosh}(a)) & \text{if } a \geq 1, \end{cases}$$

solves the preceding equation. This completes the proof. □

We can also derive an explicit formula for $\mathbb{N}_x(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))$, for every $x\in\mathbb{R}$, from Corollary 8. Fix $x>0$ for definiteness and argue under the measure $\mathbb{N}_x$. Write $T_0(\mathrm{w}) = \inf\{t\in[0,\zeta_{(\mathrm{w})}] : \mathrm{w}(t)=0\}$ for any finite path w and define

$$\mathcal{Y}_0 = \int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{T_0(W_s)=\infty\}}.$$

Also let $(\omega_i)_{i\in I}$ be the excursions outside $(0,\infty)$ defined as in Section 2. Then, we have $\mathbb{N}_x$ a.e.

$$\sigma = \mathcal{Y}_0 + \sum_{i\in I}\sigma(\omega_i)\;,\quad \mathcal{L}^0 = \sum_{i\in I}\mathcal{L}^0(\omega_i)$$

where the second equality follows from the proof of Corollary 6. Using the special Markov property (with the fact that $\mathcal{Y}_0$ is $\mathcal{E}_{(0,\infty)}$-measurable), we get

$$\mathbb{N}_x\Big(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)\Big) = \mathbb{N}_x\Big(1-\exp\Big(-\mu\mathcal{Y}_0 - \mathcal{Z}_0\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))\Big)\Big). \tag{23}$$

On the other hand, Lemma 4.5 in [13] shows that, for every $\mu,\theta>0$ such that $\theta\geq\sqrt{\mu/2}$,

$$\mathbb{N}_x(1-\exp(-\mu\mathcal{Y}_0-\theta\mathcal{Z}_0)) = \sqrt{\frac{\mu}{2}}\left(3\left(\coth\left((2\mu)^{1/4}x + \coth^{-1}\sqrt{\frac{2}{3}+\frac{1}{3}\sqrt{\frac{2}{\mu}}\,\theta}\right)\right)^2 - 2\right) \tag{24}$$

with the convention that the right-hand side equals $\sqrt{\mu/2}$ if $\theta=\sqrt{\mu/2}$.

Taking $\theta = \mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma)) \geq \sqrt{\mu/2}$ in (24), using the formula of Corollary 8, then yields a (complicated but explicit) expression for $\mathbb{N}_x(1-\exp(-\lambda\mathcal{L}^0-\mu\sigma))$.

Corollary 9. *For every $\mu_1,\mu_2\geq 0$, we have*

$$\mathbb{N}_0(1-\exp(-\mu_1\sigma_+-\mu_2\sigma_-)) = \frac{\sqrt{2}}{3}\,\frac{\mu_1^{3/2}-\mu_2^{3/2}}{\mu_1-\mu_2},$$

with the convention that the right-hand side equals $\sqrt{\mu_1/2}$ if $\mu_1=\mu_2$. The distribution of the pair (σ_+,σ_-) under $\mathbb{N}_0$ has density

$$g(s_1,s_2) := \frac{1}{2\sqrt{2\pi}}\,(s_1+s_2)^{-5/2}$$

with respect to Lebesgue measure on $(0,\infty)^2$. In particular, the distribution of σ_+ (or of σ_-) under $\mathbb{N}_0$ has density $(3\sqrt{2\pi})^{-1}\,s^{-3/2}$ on $(0,\infty)$.

The form of the density $g(s_1, s_2)$ shows that the conditional distribution of σ_+ knowing that $\sigma = s$ is uniform over $[0, s]$. This is a well-known fact, which can be derived from the invariance of the CRT under uniform re-rooting (see e.g. [4, Section 3.2]).

Proof. The formula for $\mathbb{N}_0(1 - \exp(-\mu_1\sigma_+ - \mu_2\sigma_-))$ is obtained by solving the equation $h_{\mu_1,\mu_2}(v) = 0$. We can then verify that the function g satisfies

$$\int_0^\infty\int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,g(s_1,s_2)\,(1 - e^{-\mu_1 s_1 - \mu_2 s_2}) = \frac{\sqrt{2}}{3}\,\frac{\mu_1^{3/2} - \mu_2^{3/2}}{\mu_1 - \mu_2},$$

which gives the second assertion. □

We finally give an application to super-Brownian motion in the spirit of Corollary 6.

Corollary 10. *Let* $\mathbf{X}$ *be a one-dimensional super-Brownian motion with branching mechanism* $\psi(u) = 2u^2$*, such that* $\mathbf{X}_0 = \alpha\delta_0$*. Set*

$$\mathbf{R}_+ = \int_0^\infty \mathrm{d}t\,\langle \mathbf{X}_t, \mathbf{1}_{[0,\infty)}\rangle\ , \quad \mathbf{R}_- = \int_0^\infty \mathrm{d}t\,\langle \mathbf{X}_t, \mathbf{1}_{(-\infty,0]}\rangle.$$

Then, for every $\mu_1, \mu_2 > 0$*,*

$$\mathbb{E}[\exp(-\mu_1\mathbf{R}_+ - \mu_2\mathbf{R}_-)] = \exp\Big(-\frac{\alpha\sqrt{2}}{3}\,\frac{\mu_1^{3/2} - \mu_2^{3/2}}{\mu_1 - \mu_2}\Big).$$

Given Corollary 9, the proof of Corollary 10 is an immediate application of the Brownian snake construction of super-Brownian motion along the lines of the proof of Corollary 6.

5 Conditional distributions of the local time at 0

We will now use the preceding results to recover the conditional distribution of $\mathcal{L}^0$ given σ, which was first obtained by Bousquet-Mélou and Janson [9] with a very different method.

Theorem 11. *Let* $s > 0$*. Under the probability measure* $\mathbb{N}_0(\cdot \mid \sigma = s)$*, the local time* $\mathcal{L}^0$ *is distributed as* $(2^{3/4}/3)\,s^{3/4}\,T^{-1/2}$*, where* T *is a positive stable variable with index* $2/3$*, whose Laplace transform is* $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{2/3})$*.*

Remark. In Corollary 3.4 of [9], the constant $2^{3/4}/3$ is replaced by $2^{1/4}/3$. This is due to a different normalization: In [9] (as in [4]) the random function coding the genealogy of ISE is twice the Brownian excursion, and it follows that our random variable $\mathcal{L}^0$ is distributed under $\mathbb{N}_0(\cdot\,|\,\sigma = 1)$ as $\sqrt{2}$ times the quantity $f_{\mathrm{ISE}}(0)$ considered in [9].

The occurence of a stable variable with index 2/3 in Theorem 11 is of course reminiscent of Corollary 6 above. It would be very interesting to establish a direct connection between this corollary and Theorem 11.

Proof. From the scaling properties of the end of Section 2, it is enough to treat the case $s = 1$. Recall the notation $v(\lambda, \mu_1, \mu_2)$ in Proposition 7. For every $\lambda \geq 0$, set

$$F(\lambda) := 2\,\mathbb{N}_0\Big(e^{-\sigma/2}(1 - e^{-\lambda\mathcal{L}^0})\Big) = 2\,v(\lambda, \frac{1}{2}, \frac{1}{2}) - 1,$$

where the second equality holds because $\mathbb{N}_0(1 - \exp(-\sigma/2)) = 1/2$. The function F is continuous and vanishes at 0. As a straightforward consequence of Proposition 7, we have for every $\lambda \geq 0$,

$$F(\lambda) = \lambda\sqrt{\frac{3}{3 + F(\lambda)}}. \tag{25}$$

In particular, the right derivative of F at 0 is 1, and consequently $\mathbb{N}_0(\mathcal{L}^0\,\exp(-\sigma/2)) = 1/2$. The fact that $\mathbb{N}_0(\mathcal{L}^0\,\exp(-\sigma/2))$ is finite allows us to make sense of $F(\lambda)$ for every $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$, and the restriction of F to $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ is analytic.

Set $\psi(z) = \sqrt{3/(3+z)}$ so that ψ is analytic on a neighborhood of 0 in $\mathbb{C}$. Since $\psi(0) \neq 0$, we can find an analytic function G defined on a neighborhood of 0 such that $z\psi(G(z)) = G(z)$ for $|z|$ small enough. By (25), we must have $F(z) = G(z)$ for $\mathrm{Re}(z) > 0$ and $|z|$ small, and this means that F can be extended to an analytic function on a neighborhood of 0. By the Lagrange inversion theorem, we have then, for every integer $n \geq 1$,

$$[z^n]F(z) = \frac{1}{n}[z^{n-1}]\psi(z)^n = \frac{3^{n/2}}{n!}\,\frac{\mathrm{d}^{n-1}(3+z)^{-n/2}}{\mathrm{d}z^{n-1}}\Big|_{z=0} = \frac{(-1)^{n-1}3^{1-n}}{n!}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})},$$

using the standard notation $[z^n]F(z)$ for the coefficient of z^n in the series expansion of $F(z)$ near 0. On the other hand, the fact that the function $z \mapsto F(z)$ is analytic in a neighborhood of 0 implies that all moments $\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2})$, $n \geq 1$, are finite and given by

$$\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2}) = \frac{1}{2}\,(-1)^{n-1}n! \times [z^n]F(z) = \frac{1}{2}\,3^{1-n}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}. \tag{26}$$

To complete the proof, we use a scaling argument. We recall that the distribution of $\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = s)$ coincides with the distribution of $s^{3/4}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot \mid \sigma = 1)$. It follows that

$$\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2}) = \int_0^\infty \frac{\mathrm{d}s}{2\sqrt{2\pi s^3}}\,e^{-s/2}\mathbb{N}_0\Big(s^{3n/4}(\mathcal{L}^0)^n\,\Big|\,\sigma=1\Big) = \frac{2^{\frac{3n}{4}-2}}{\sqrt{\pi}}\,\Gamma(\frac{3n}{4}-\frac{1}{2}) \times \mathbb{N}_0\Big((\mathcal{L}^0)^n\,\Big|\,\sigma=1\Big).$$

By combining the last two displays and using the duplication formula for the Gamma function, we arrive at

$$\mathbb{N}_0\Big((\mathcal{L}^0)^n\,\Big|\,\sigma=1\Big) = \frac{\sqrt{\pi}\,3^{1-n}}{2^{\frac{3n}{4}-1}}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})\Gamma(\frac{3n}{4}-\frac{1}{2})} = \frac{2^{\frac{3n}{4}}}{3^n}\,\frac{\Gamma(\frac{3n}{4}+1)}{\Gamma(\frac{n}{2}+1)} = \Big(\frac{2^{3/4}}{3}\Big)^n\,\mathbb{E}[T^{-n/2}],$$

where T is as in the theorem (to check the last equality, write $T^{-n/2} = (\Gamma(n/2))^{-1}\int_0^\infty \mathrm{d}s\, s^{n/2-1}e^{-sT}$). The growth of the moments of the distribution of $T^{-1/2}$ ensures that this distribution is characterized by its moments, which completes the proof. □

Remark. Rather than using the Lagrange inversion theorem, we could have derived formula (26) for the moments $\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma/2})$ from a series expansion of the quantity $\mathbb{N}_0(1-\exp(-\lambda\mathcal{L}^0-\sigma/2))$ as given in Corollary 8. This would still have required some calculations. We preferred to use the previous method because it also serves as a prototype for the proof of the (more delicate) Theorem 12 below.

Proposition 7 can also be used to derive the conditional distribution of $\mathcal{L}^0$ given σ_+. Perhaps surprisingly, this distribution turns out again to be remarkably simple.

Theorem 12. *Let $s > 0$. Under the probability measure $\mathbb{N}_0(\cdot \mid \sigma_+ = s)$, the local time $\mathcal{L}^0$ is distributed as $(2^{9/4}/3)\,s^{3/4}\,D\,T^{-1/2}$, where the random variables D and T are independent, T is a positive stable variable with index $2/3$, whose Laplace transform is $\mathbb{E}[\exp(-\lambda T)] = \exp(-\lambda^{2/3})$, and D has density $2x\,\mathbf{1}_{[0,1]}(x)$ with respect to Lebesgue measure on $\mathbb{R}_+$.*

Proof. It is enough to treat the case $s = 1$. For every $\lambda \geq 0$, set

$$F_+(\lambda) := \mathbb{N}_0\Big(1-\exp(-\lambda\mathcal{L}^0-\frac{1}{2}\sigma_+)\Big) = v(\lambda,\frac{1}{2},0)$$

with the notation of Proposition 7. As for Theorem 11, the strategy of the proof is to compute the coefficient $[\lambda^n]F_+(\lambda)$ in two different ways. Unfortunately, the details of the argument are more involved than in the proof of Theorem 11.

By Proposition 7, we have

$$(2F_+(\lambda)-1)\sqrt{F_+(\lambda)+1} + 2\,F_+(\lambda)^{3/2} = \sqrt{6}\,\lambda.$$

We cannot apply directly the Lagrange inversion theorem, but the idea will be to find a rational parametrization of the preceding equation (see e.g. [8, Section 3]). It follows from the last display that we have $P(F_+(\lambda),\lambda)=0$, where

$$P(y,z)=96\,y^3z^2-36\,z^4-36\,yz^2+12\,z^2-9\,y^2+6\,y-1,\qquad y,z\in\mathbb{C}.$$

We now introduce[1] the rational functions

$$Q(z)=-\frac{1}{124416}\,z^3+\frac{1}{48}\,z,\qquad R(z)=\frac{1}{3456}\,z^2-\frac{1}{2}+\frac{216}{z^2},$$

which satisfy $P(R(z),Q(z))=0$ for every $z\in\mathbb{C}\backslash\{0\}$. We have $Q^{-1}(0)=\{-36\sqrt{2},0,36\sqrt{2}\}$, and the derivative Q' does not vanish on $Q^{-1}(0)$. It follows that we can find $r_0>0$ and three analytic functions $\gamma_1,\gamma_2,\gamma_3$ defined on the disk $\mathbb{D}_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$ and with disjoint ranges, such that $\gamma_1(0)=-36\sqrt{2}$, $\gamma_2(0)=0$, $\gamma_3(0)=36\sqrt{2}$ and for every $z\in\mathbb{D}_{r_0}$, $Q^{-1}(z)=\{\gamma_1(z),\gamma_2(z),\gamma_3(z)\}$. Note that $R(\gamma_1(0))=1/3=R(\gamma_3(0))$ and $R'(\gamma_1(0))=-\sqrt{2}/54=-R'(\gamma_3(0))$. Also the fact that $Q(\gamma_i(z))=z$ readily implies that $\gamma_1'(0)=\gamma_2'(0)=-24$.

Since $P(R(z),Q(z))=0$ for every $z\in\mathbb{C}\backslash\{0\}$, we get that $P(R(\gamma_i(z)),z)=0$ for every $i\in\{1,2,3\}$ and $z\in\mathbb{D}_{r_0}\backslash\{0\}$. We claim that $F_+(\lambda)=R(\gamma_1(\lambda))$ for $\lambda>0$ small enough. To see this, observe that for $z\neq 0$ and $|z|$ small enough, then the quantities $R(\gamma_i(z))$, $i\in\{1,2,3\}$, are distinct. Indeed, since $|R(y)|\longrightarrow\infty$ as $|y|\to 0$ it is clear that $R(\gamma_2(z))$ is distinct from $R(\gamma_1(z))$ and $R(\gamma_3(z))$ when $|z|$ is small, and on the other hand, the properties $\gamma_1'(0)=\gamma_3'(0)\neq 0$ and $R'(\gamma_1(0))=-R'(\gamma_3(0))\neq 0$ imply that $R(\gamma_1(z))\neq R(\gamma_3(z))$ when $|z|$ is small. Hence, for $z\neq 0$ and $|z|$ small enough, the numbers $R(\gamma_i(z))$, $i\in\{1,2,3\}$, are three distinct roots of $P(y,z)$ viewed as a polynomial of degree 3 in y. Since we know that $P(F_+(\lambda),\lambda)=0$, it follows that $F_+(\lambda)\in\{R(\gamma_1(\lambda)),R(\gamma_2(\lambda)),R(\gamma_3(\lambda))\}$ for $\lambda>0$ small. The case $F_+(\lambda)=R(\gamma_2(\lambda))$ is clearly excluded for λ small, and since $F_+(\lambda)$ is a monotone increasing function of λ, noting that $\gamma_1'(0)R'(\gamma_1(0))>0$ whereas $\gamma_3'(0)R'(\gamma_3(0))<0$, we get our claim $F_+(\lambda)=R(\gamma_1(\lambda))$ for $\lambda>0$ small.

In particular, we can extend F_+ to an analytic function in the neighborhood of 0, and we will then use the Lagrange inversion theorem to determine the coefficients of the Taylor expansion of F_+. To simplify notation, we set $\widetilde{F}_+(\lambda)=F_+(\lambda)-1/3$, $\widetilde{\gamma}(z)=\gamma_1(z)+36\sqrt{2}$ and for every $\lambda\geq 0$,

$$\widetilde{R}(\lambda)=R(\lambda-36\sqrt{2})-1/3.$$

Then, for $\lambda>0$ small, we have

$$\widetilde{F}_+(\lambda)=F_+(\lambda)-\frac{1}{3}=R(\gamma_1(\lambda))-\frac{1}{3}=\widetilde{R}(\widetilde{\gamma}(\lambda)).\tag{27}$$

On the other hand, the property $Q(\gamma_1(z))=z$ for $|z|<r_0$ shows that

$$\widetilde{\gamma}(\lambda)=\lambda\,\widetilde{\psi}(\widetilde{\gamma}(\lambda)),\tag{28}$$

with

$$\widetilde{\psi}(\lambda)=-\frac{124416}{(36\sqrt{2}-\lambda)(72\sqrt{2}-\lambda)}.$$

By (27), (28) and the Lagrange inversion theorem, we get for every $n\geq 1$,

$$[\lambda^n]F_+(\lambda)=[\lambda^n]\widetilde{F}_+(\lambda)=\frac{1}{n}[\lambda^{n-1}](\widetilde{R}'(\lambda)\widetilde{\psi}(\lambda)^n).$$

Note that

$$\widetilde{R}'(72\sqrt{2}\lambda)=R'(72\sqrt{2}(\lambda-1))=-\frac{\sqrt{2}}{48}(1-2\lambda)+\frac{1}{216\sqrt{2}}(1-2\lambda)^{-3}$$
$$\widetilde{\psi}(72\sqrt{2}\lambda)=-\frac{24}{(1-\lambda)(1-2\lambda)},$$

[1] The functions Q and ψ have been found using the Maple package *algcurve*

from which it follows that

$$[\lambda^{n-1}](\widetilde{R}'(72\sqrt{2}\lambda)\widetilde{\psi}(72\sqrt{2}\lambda)^n)$$
$$= (-24)^n \times \Big(\big(\frac{-\sqrt{2}}{48}[\lambda^{n-1}]\Big((1-2\lambda)^{-n+1}(1-\lambda)^{-n}\Big) + \frac{1}{216\sqrt{2}}[\lambda^{n-1}]\Big((1-2\lambda)^{-n-3}(1-\lambda)^{-n}\Big)\Big),$$

and finally

$$[\lambda^n]F_+(\lambda) = \frac{(-1)^n}{n}\,(3\sqrt{2})^{-n}\,\Big(-3[\lambda^{n-1}]\Big((1-2\lambda)^{-n+1}(1-\lambda)^{-n}\Big) + \frac{1}{3}[\lambda^{n-1}]\Big((1-2\lambda)^{-n-3}(1-\lambda)^{-n}\Big)\Big). \tag{29}$$

To compute the right-hand side, we observe that, for every integers $m \geq 0$, $k \geq 1$ and $\ell \geq 1$, we have

$$[\lambda^m](1-2\lambda)^{-k}(1-\lambda)^{-\ell} = 2^m \binom{m+k-1}{m}\,{}_2F_1(-m,\ell;-m-k+1;\frac{1}{2}),$$

where ${}_2F_1$ stands for the Gauss hypergeometric function. This equality is easily checked by a direct calculation, noting that the hypergeometric series reduces to a finite sum in the case we are considering It follows that, for every $n \geq 2$,

$$[\lambda^{n-1}]\Big((1-2\lambda)^{-n+1}(1-\lambda)^{-n}\Big) = 2^{n-1}\binom{2n-3}{n-1}\,{}_2F_1(-n+1,n;-2n+3;\frac{1}{2}) \tag{30}$$
$$[\lambda^{n-1}]\Big((1-2\lambda)^{-n-3}(1-\lambda)^{-n}\Big) = 2^{n-1}\binom{2n+1}{n-1}\,{}_2F_1(-n+1,n;-2n-1;\frac{1}{2}). \tag{31}$$

Fortunately, Bailey's theorem (see [5, Theorem 3.5.4 (ii)]) gives an explicit formula for ${}_2F_1(a,1-a;b;\frac{1}{2})$ in terms of a ratio of products of values of the Gamma function, which we can apply here. Using also Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to eliminate the poles of the Gamma function, we arrive at

$$\begin{aligned}
{}_2F_1(-n+1,n;-2n+3;\frac{1}{2}) &= \frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n-\frac{1}{2})\,\Gamma(n-1)} = \frac{2^{2n-3}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(2n-2)}\\
{}_2F_1(-n+1,n;-2n-1;\frac{1}{2}) &= \frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(n+\frac{3}{2})\,\Gamma(n+1)} = \frac{2^{2n+1}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(2n+2)},
\end{aligned}$$

where we applied the duplication formula for the Gamma function, and we recall that we assume $n \geq 2$. Using (30) and (31), we get from (29) that

$$\begin{aligned}
[\lambda^n]F_+(\lambda) &= \frac{(-1)^{n+1}}{n!}\,(3\sqrt{2})^{-n}\,\frac{2^{3n}}{\sqrt{\pi}}\Big(\frac{3}{16}\,\frac{\Gamma(\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n-1)} - \frac{1}{3}\,\frac{\Gamma(\frac{n}{2}+\frac{3}{2})\,\Gamma(\frac{3n}{2}+1)}{\Gamma(n+3)}\Big)\\
&= \frac{(-1)^{n+1}}{n!}\,(3\sqrt{2})^{-n}\,\frac{2^{3n}}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n}{2}+\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n)}\Big(\frac{3}{8} - \frac{1}{3}\times\frac{(\frac{3n}{2}-1)(\frac{n}{2}+\frac{1}{2})\frac{3n}{2}}{(n+2)(n+1)n}\Big)\\
&= \frac{(-1)^{n+1}}{n!}\,(3\sqrt{2})^{-n}\,\frac{2^{3n}}{\sqrt{\pi}}\,\frac{1}{n+2}\,\frac{\Gamma(\frac{n}{2}+\frac{1}{2})\,\Gamma(\frac{3n}{2}-1)}{\Gamma(n)}\\
&= \frac{(-1)^{n+1}}{n!}\,(3\sqrt{2})^{-n}\,2^{2n+1}\,\frac{1}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}.
\end{aligned}$$

We have assumed $n \geq 2$, but a direct calculation from (29) shows that the last line of the preceding display also gives the correct value $[\lambda]F_+(\lambda) = 4\sqrt{2}/9$ for $n = 1$. Similarly as in the proof of Theorem 11, we conclude that, for every $n \geq 1$,

$$\mathbb{N}_0\Big((\mathcal{L}^0)^n\,e^{-\sigma_+/2}\Big) = \Big(\frac{2\sqrt{2}}{3}\Big)^n\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}.$$

On the other hand, the same scaling argument as in the proof of Theorem 11 (using now the fact that the density of σ_+ under $\mathbb{N}_0$ is $(3\sqrt{2\pi})^{-1}s^{-3/2}$) gives

$$\mathbb{N}_0((\mathcal{L}^0)^n e^{-\sigma_+/2}) = \int_0^\infty \frac{\mathrm{d}s}{3\sqrt{2\pi s^3}}\, e^{-s/2}\mathbb{N}_0\Big(s^{3n/4}(\mathcal{L}^0)^n\,\Big|\,\sigma_+=1\Big) = \frac{2^{\frac{3n}{4}-1}}{3\sqrt{\pi}}\,\Gamma(\frac{3n}{4}-\frac{1}{2})\,\mathbb{N}_0\Big((\mathcal{L}^0)^n\,\Big|\,\sigma_+=1\Big).$$

It follows that

$$\mathbb{N}_0\Big((\mathcal{L}^0)^n\,\Big|\,\sigma_+=1\Big) = 3\sqrt{\pi}\Big(\frac{2\sqrt{2}}{3}\Big)^n 2^{-\frac{3n}{4}+1}\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})\Gamma(\frac{3n}{4}-\frac{1}{2})} = \Big(\frac{2^{9/4}}{3}\Big)^n\,\frac{2}{n+2}\,\frac{\Gamma(\frac{3n}{4}+1)}{\Gamma(\frac{n}{2}+1)}.$$

The right-hand side is the n-th moment of $(2^{9/4}/3)\,D\,T^{-1/2}$, where the pair (D,T) is as in the theorem. This completes the proof. $\square$

Interpretation in random geometry. We now explain briefly how both theorems of this section can be interpreted in the setting of continuous models of random geometry. It is best to start with the discrete picture of planar quadrangulations. For every integer $n\geq 1$, let Q_n be a uniformly distributed rooted and pointed quadrangulation with n faces and write d_{gr} for the graph distance on the vertex set $V(Q_n)$ of Q_n. The Schaeffer bijection (see e.g. [24, Section 5]) allows us to code Q_n by a uniformly distributed labeled tree with n edges, which we denote by T_n (a labeled tree is a rooted plane tree whose vertices are assigned integer labels ℓ_v, in such a way that the label of the root vertex ρ is $\ell_\rho=0$ and the labels of two adjacent vertices differ by at most 1 in absolute value). Furthermore the vertex set $V(Q_n)$ is canonically identified with $V(T_n)\cup\{\partial\}$, where $V(T_n)$ denotes the vertex set of T_n and ∂ is an extra vertex corresponding to the distinguished vertex of Q_n. Through this identification, the graph distance $\mathrm{d}_{gr}(\partial,v)$ between ∂ and another vertex v of Q_n can be expressed as $\ell_v-\min\{\ell_w : w\in V(T_n)\}+1$. Now consider the set $S_n=\{v\in V(Q_n) : \mathrm{d}_{gr}(\partial,v)=\mathrm{d}_{gr}(\partial,\rho)\}$ of all vertices v of Q_n that are at the same distance as ρ from the distinguished vertex ∂. From the previous observations, S_n is identified to $\{v\in V(T_n) : \ell_v=0\}$. It then follows from [9, Theorem 3.6] that the distribution of $n^{-3/4}\#S_n$ converges as $n\to\infty$ to the distribution of $2^{-1/4}3^{-1/2}\mathcal{L}^0$ under $\mathbb{N}_0(\cdot\,|\,\sigma=1)$, which is given in Theorem 11.

Consider then the (standard) Brownian map $(\mathbf{m}_\infty,D)$. This is a random compact metric space that can be constructed from Brownian motion indexed by the Brownian tree, which we denote here by $(V_a)_{a\in\mathcal{T}_\zeta}$ as in Section 1 above, under the probability measure $\mathbb{N}_0(\cdot\,|\,\sigma=1)$ — see e.g. the introduction of [21] for details. In this construction, the space $\mathbf{m}_\infty$ is obtained as a quotient space of $\mathcal{T}_\zeta$, and comes with two distinguished points, namely the point ρ corresponding to the root of $\mathcal{T}_\zeta$, and another point denoted by x_* in [21], which corresponds to the point of $\mathcal{T}_\zeta$ where V_a achieves its minimum. Note that ρ and x_* can be viewed as independently and uniformly distributed on $\mathbf{m}_\infty$. The “sphere” $\{x\in\mathbf{m}_\infty : D(x_*,x)=D(x_*,\rho)\}$ then corresponds to $\{a\in\mathcal{T}_\zeta : V_a=0\}$, and so the local time $\mathcal{L}^0$ is naturally interpreted as the “measure” of this sphere (here the word measure should refer to a suitable Hausdorff measure, although this has not been justified rigorously). This interpretation is made very plausible by the discrete result for quadrangulations described above.

To get a similar interpretation for Theorem 12, we consider the free Brownian map (M,Δ), which is the scaling limit of quadrangulations distributed according to Boltzmann weights and can again be constructed from Brownian motion indexed by the Brownian tree, but now under the σ-finite measure $\mathbb{N}_0$ (see e.g. [23, Section 3]). As in the case of the standard Brownian map, the space M is defined as a quotient space of $\mathcal{T}_\zeta$ and comes with two distinguished points denoted by ρ and x_*. Furthermore, the sphere $\{x\in M : \Delta(x_*,x)=\Delta(x_*,\rho)\}$ corresponds to $\{a\in\mathcal{T}_\zeta : V_a=0\}$, and the ball $\{x\in M : \Delta(x_*,x)\leq\Delta(x_*,\rho)\}$ corresponds to $\{a\in\mathcal{T}_\zeta : V_a\leq 0\}$. So Theorem 12 can be viewed as providing the conditional distribution of the measure of the sphere $\{x\in M : \Delta(x_*,x)=\Delta(x_*,\rho)\}$ given the volume of the ball it encloses.

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