The critical tree of a renormalization model as a growth-fragmentation process

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Abstract. We study a branching system which describes the evolution indexed by a continuous time parameter ranging in $[0, 1)$ of a population of cells; the size of each cell increases deterministically and linearly except when the cell splits into two daughter cells. The system appears as the scaling limit of the critical tree in the family of hierarchical renormalization models studied in [12], conditioned on survival; it is also a growth-fragmentation process in the sense of Bertoin [3]. We are interested in the empirical measure of the process representing the sizes of the cells that are alive at time $t \in [0, 1)$, and establish a general result, called the master formula, for exponential functionals of the empirical measure. The formula allows to determine the joint distribution of the sum of cell sizes and the number of cells at time t, which improves a previous result by Hu, Mallein and Pain [17] who proved joint weak convergence of these two quantities when $t \to 1^-$. The main result in our paper, established also relying on the master formula, is a law of large numbers for the empirical measure when $t \to 1^-$, the limiting distribution explicitly identified. Our system can be viewed as an exactly solvable example of a growth-fragmentation process.

Résumé. Nous étudions un système de branchement qui décrit l'évolutiond'une population de cellules ; le paramètre de temps est à valeurs dans $[0, 1)$; la taille de chaque cellule augmente de façon déterministe et linéaire sauf lorsque la cellule se divise en deux cellules filles. Le système apparaît comme la limite d'échelle de l'arbre critique conditionné par la survie d'une famille de modèles de renormalisation hiérarchiques étudiés dans [12] ; c'est aussi un processus de croissance-fragmentation au sens de Bertoin [3]. Nous nous intéressons à la mesure empirique du processus ponctuel représentant les tailles des cellules vivant au temps $t \in [0, 1)$ et nous établissons un résultat général, appelé formule maîtresse, pour les fonctionnelles exponentielles de la mesure empirique. La formule permet de déterminer la distribution conjointe de la somme des tailles des cellules et du nombre de cellules au temps t, ce qui améliore un résultat précédent de Hu, Mallein et Pain [17] ayant prouvé la convergence en loi jointe de ces deux quantités lorsque $t \to 1^-$. Le résultat principal de notre article, qui est établi via la formule maîtresse, est une loi des grands nombres pour la mesure empirique lorsque $t \to 1^-$, la loi limite étant explicitement identifiée. Notre système peut être considéré comme un exemple exactement soluble d'un processus de croissance-fragmentation.

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1. Introduction

We consider the evolution indexed by a continuous time parameter ranging in $[0, 1)$ of a population of cells. Each cell in the population has a size which grows linearly with time and independently splits into two daughter cells at a rate which depends both on the cell size and on the time. Initially there is a single cell of size $x \in [0, \infty)$. More precisely, the population evolves according to the following five rules:

$(1.1a)$ At time 0 there is only one cell whose size is x.

$(1.1b)$ A cell of size $\mathbf{y}$ at time $t \in [0, 1)$ that does not split between times t and $t' \in [t, 1)$ has size $\mathbf{y} + t' - t$ at time t'.

$(1.1c)$ A cell of size $\mathbf{y}$ at time $t \in [0, 1)$ undergoes a cell division at that time with rate $\frac{2\mathbf{y}}{(1-t)^2}$.

$(1.1d)$ When a cell division occurs, the parent cell of size $\mathbf{y}$ instantly dies and splits into two cells of respective random sizes $\mathbf{y}'$ and $\mathbf{y}''$ such that $\mathbf{y}=\mathbf{y}'+\mathbf{y}''$ and $\mathbf{y}'/\mathbf{y}$ is uniformly distributed on $[0,1]$.

$(1.1e)$ All branching times (i.e. times when a cell division occurs) and the random proportions of cell divisions are independent.

See Figure 1 below for an example.

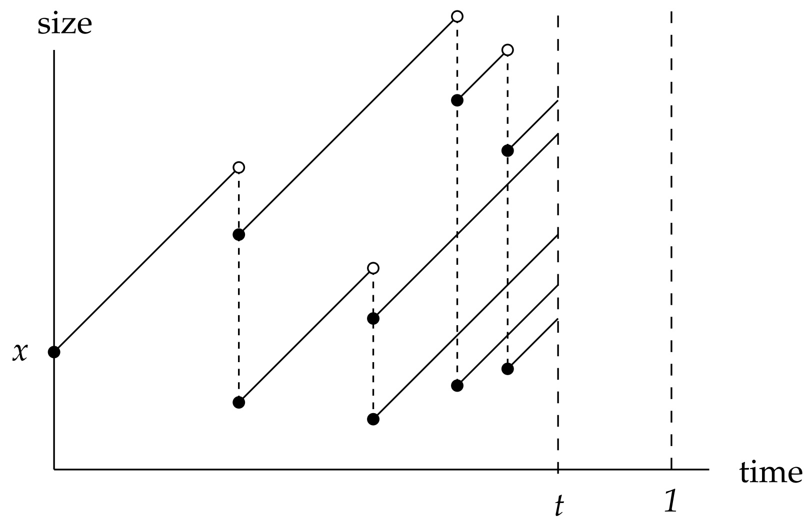

FIG 1. An example of the HR process (i.e. the branching system defined by $(1.1a)$-$(1.1e)$). Each cell increases linearly except at branching times. A white circle marks the end of a cell which splits into two cells, each marked by a black dot. In our example, five cells are alive at time t.

We call the branching system given by $(1.1a)$-$(1.1e)$ the *Hierarchical Renormalization process* (HR process for short) because even though no rigorous proof has been available yet, it is expected to be the scaling limit of the discrete-time renormalization model at criticality studied in [12], conditioned on survival; see [10] and [13]. Also, a continuous-time version of suitably rescaled hierarchical renormalization models has been proved to converge weakly to the branching system satisfying $(1.1a)$-$(1.1e)$; see Hu, Mallein and Pain [17]. In what follows, we write $\mathbf{P}_x$ for the law of the system with initial value x, and $\mathbf{E}_x$ for the corresponding expectation.

Although the existence of the HR process (i.e. the branching system satisfying $(1.1a)$-$(1.1e)$) is already part of the consequences of the aforementioned work by Hu, Mallein and Pain [17], we are going to give, in Section 2.1, an elementary construction of it, by means of a Poisson point process on $[0,1)\times[0,\infty)$. Let us first informally explain how such a population can be represented by paths in $[0,1)\times[0,\infty)$. At time t, the number of cells in the population is finite and we denote their respective sizes by $\mathbf{x}(t)=(\mathbf{x}_k(t))_{1\le k\le \mathbf{n}_t}$, the total numbers of cells being denoted by $\mathbf{n}_t$. We adopt the convention that the paths $t\mapsto \mathbf{x}_k(t)$ are right-continuous with left limits and since only finitely many cell divisions occur up to time $t<1$, cells can be linearly ordered according to the following convention: if at time t^-, the k-th cell undergoes a cell division, then the two resulting child-cells have respective ranks k and $k+1$ and the relative order of the other cells is unchanged. Namely, $\mathbf{x}_j(t^-)=\mathbf{x}_j(t)$ if $1\le j<k$, $\mathbf{x}_k(t^-)=\mathbf{x}_k(t)+\mathbf{x}_{k+1}(t)$, $\mathbf{x}_j(t)=\mathbf{x}_{j-1}(t^-)$ if $k+1<j\le \mathbf{n}_t$ and $\mathbf{n}_t=1+\mathbf{n}_{t^-}$. See Figure 2 below for an example.

To study the HR process $\mathbf{x}(t)=(\mathbf{x}_k(t))_{1\le k\le \mathbf{n}_t}$ when $t\to 1^-$, our main tool, referred to as the master formula, is equation (3.9) in Theorem 3.1: for all $x\in[0,\infty)$ and $t\in[0,1)$, and all measurable function $h:\ [0,\infty)^2\to\mathbb{R}$ satisfying certain regularity conditions, we have

$$(1.1)\qquad \mathbf{E}_x\Big[e^{\sum_{k=1}^{\mathbf{n}_t} h(t,\mathbf{x}_k(t))}\Big]=e^{h(0,x)}+\int_0^t \mathbf{E}_x\Big[e^{\sum_{k=1}^{\mathbf{n}_r} h(r,\mathbf{x}_k(r))}\sum_{j=1}^{\mathbf{n}_r}\mathcal{G}h(r,\mathbf{x}_j(r))\Big]\mathrm{d}r\,,$$

where $\mathcal{G}$ is an explicit *nonlinear* operator. To see why (1.1) is useful, let us observe that if a convenient function h satisfies $\mathcal{G}h=0$, then by (1.1), we have $\mathbf{E}_x[e^{\sum_{k=1}^{\mathbf{n}_t} h(t,\mathbf{x}_k(t))}]=e^{h(0,x)}$, which is a typical relation obtained by the method of characteristics.

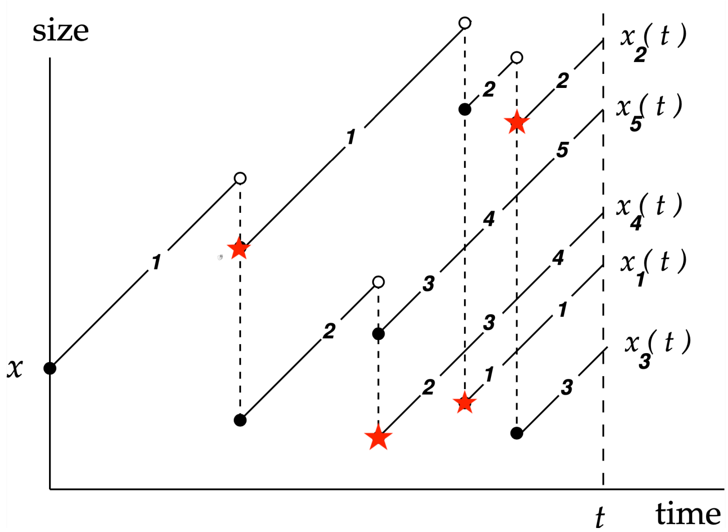

FIG 2. Same example as in Figure 1, with $\mathbf{n}_t = 5$. The cells at time t have sizes $(\mathbf{x}_j(r))_{j=1,2,\dots,5}$. At each time of cell division, two marks appear: a (red) star and a dot. The heights of these two marks represent the sizes of the two daughter cells: if the parent-cell has index k, then the star corresponds to the child-cell with index k and the dot corresponds to the child-cell with index $k+1$. Note that according to our way of labelling the cell numbers, cells may change their index without splitting.

In Section 3.1, we are going to give a precise statement of the master formula, and to develop several interesting consequences. Let us mention immediately two of the applications. The first, which is also the main result of the paper, is a law of large numbers (see Theorem 3.13) for the empirical measure of the HR process $\mathbf{x}(t) = (\mathbf{x}_k(t))_{1\le k\le \mathbf{n}_t}$ upon a suitable normalization: for all $x \in [0,\infty)$, $\mathbf{P}_x$-almost surely,

$$(1-t)^2 \sum_{k=1}^{\mathbf{n}_t} \delta_{\frac{1}{1-t}\mathbf{x}_k(t)} \to \tfrac{1}{3} R_\infty \boldsymbol{\gamma}_2, \qquad t \to 1^-, \tag{1.2}$$

weakly in the space of all finite Borel measures on $[0,\infty)$. Here, δ_y denotes the Dirac measure at y, $\boldsymbol{\gamma}_2$ is the gamma distribution on $[0,\infty)$: $\boldsymbol{\gamma}_2(\mathrm{d}y) = 4ye^{-2y}\mathrm{d}y$, and R_∞ is a certain positive random variable. The Laplace transform of R_∞ is given by

$$\mathbf{E}_x[\exp(-\lambda R_\infty)] = \exp(-\varphi(\sqrt{3\lambda}) - x\psi(\sqrt{3\lambda})), \qquad \lambda \ge 0, \tag{1.3}$$

where $\varphi(y) = 2\log(y^{-1}\sinh y)$ and $\psi(y) = y\varphi'(y) = 2(y\coth(y) - 1)$.

A consequence of (1.2) is the following convergence result for the normalized empirical measure of cell sizes: for all $x \in [0,\infty)$, $\mathbf{P}_x$-almost surely,

$$\frac{1}{\mathbf{n}_t} \sum_{k=1}^{\mathbf{n}_t} \delta_{\frac{1}{1-t}\mathbf{x}_k(t)} \xrightarrow[t\to 1^-]{\text{weakly}} \boldsymbol{\gamma}_2 \,. \tag{1.4}$$

Let us compare this result to the the evolution of cell sizes along a *genealogically typical* branch that behave as $(\mathbf{x}_1(t))_{t\in[0,1)}$ due to the symmetries of the model: Indeed, during a cell division event, a parent-cell of size x splits into two child-cells of sizes Ux and $(1-U)x$ where U has a uniform distribution on $[0,1]$ and so Ux and $(1-U)x$ are exchangeable in law. Combined with the Markovian character of the system, this leads to the following property: if one chooses in advance and in a deterministic way two ancestral lines of cells and follows their sizes over time, one obtains two processes which have the same law and thus in particular the same law as $(\mathbf{x}_1(t))_{t\in[0,1)}$. Then, (1.4) is in slight contrast with weak convergence of cell sizes along the genealogically typical branch $(\mathbf{x}_1(t))_{t\in[0,1)}$: we are going to see in Proposition 2.12 that

$$\frac{\mathbf{x}_1(t)}{1-t} \longrightarrow \nu, \qquad t \to 1^-, \tag{1.5}$$

weakly, where $\nu(\mathrm{d}y)=2y(y+1)e^{-2y}\,\mathbf{1}_{\{y>0\}}\mathrm{d}y$. Note that $\nu\neq\boldsymbol{\gamma}_2$ and that ν is stochastically dominated by $\boldsymbol{\gamma}_2$ since $\nu([y,\infty))=(1+y)^2e^{-2y}\geq\gamma_2([y,\infty))=(1+2y)e^{-2y}$ for all $y\in[0,\infty)$.

Loosely speaking, (1.5) says that typically a cell size at time t is of order of magnitude $1-t$ when $t\to1^-$. However, in Theorem 3.14 (Section 3.6), we prove that the maximal cell size at time t is much larger than $1-t$. Namely, for all $x\in[0,\infty)$, we prove that

$$\lim_{t\to1^-}\frac{\max_{1\leq k\leq \mathbf{n}_t}\mathbf{x}_k(t)}{(1-t)\log\frac{1}{1-t}}=1\qquad \mathbf{P}_x\text{-a.s.}\tag{1.6}$$

Let us go back to (1.2). Let $(\varrho_s)_{s\in[0,1]}$ be a 4-dimensional Bessel bridge with initial value $\varrho_0=2\sqrt{x}$ and terminal value $\varrho_1=0$. It is known (Pitman and Yor [21]) that $\mathbf{E}\big[\exp(-\frac{1}{2}\lambda\int_0^1\varrho_s^2\mathrm{d}s)\big]=\exp(-\varphi(\sqrt{\lambda})-x\psi(\sqrt{\lambda}))$. By comparison with (1.3), we see that $\frac{1}{3}R_\infty$ under $\mathbf{P}_x$ has the law of $\frac{1}{2}\int_0^1\varrho_s^2\mathrm{d}s$.

The relation between $\int_0^1\varrho_s^2\mathrm{d}s$ and the HR process was already known to Hu, Mallein and Pain [17]. In fact, it was proved in [17] that there exist constants c_1, $c_2\in(0,\infty)$ such that for all $x\geq0$ under $\mathbf{P}_x$,

$$\Big((1-t)\sum_{k=1}^{\mathbf{n}_t}\mathbf{x}_k(t)\,,\,(1-t)^2\mathbf{n}_t\Big)\longrightarrow\Big(c_1\int_0^1\varrho_s^2\mathrm{d}s\,,\,c_2\int_0^1\varrho_s^2\mathrm{d}s\Big),\qquad t\to1^-,\tag{1.7}$$

in distribution. It was one of our original motivations to understand this convergence, and to determine the constants c_1 and c_2. In Proposition 3.4 (that is a key step in the proof of the law of large numbers stated in (1.2) or in Thm 3.13) we are going to apply the master formula to prove that for all $x\geq0$, when $t\to1^-$,

$$(1-t)\sum_{k=1}^{\mathbf{n}_t}\mathbf{x}_k(t)\longrightarrow\tfrac{1}{3}R_\infty,\qquad(1-t)^2\mathbf{n}_t\longrightarrow\tfrac{1}{3}R_\infty,$$

$\mathbf{P}_x$-a.s. and in L^2. Since $\frac{1}{3}R_\infty$ is distributed as $\frac{1}{2}\int_0^1\varrho_s^2\mathrm{d}s$, it follows that $c_1=c_2=\frac{1}{2}$ in (1.7).

The rest of the paper is divided into two distinct parts. Preliminaries are in Section 2, and results in Section 3.

In Section 2, we give two elementary and complementary constructions of the Hierarchical Renormalization process (i.e. the branching system satisfying $(1.1a)$-$(1.1e)$). The first construction, described in Section 2.1, is based on a Poisson point process; the use of the Poisson point process makes it particularly easy to justify some independence results. The second construction, introduced in Section 2.3, relies on the first construction, and allows to exhibit a *time homogeneous* Markovian structure in the system via the following change of parameters

$$t=1-e^{-s},\tag{1.8}$$

for $t\in[0,1)$, or equivalently, for $s\in[0,\infty)$. Here the cell sizes have to be rescaled by e^s. Namely we consider

$$\big(e^s\mathbf{x}_k(1-e^{-s})\,;\,1\leq k\leq\mathbf{n}_{1-e^{-s}}\big)\,.\tag{1.9}$$

Both constructions are useful, though for different reasons. Consequently, we keep using the original time scale with $t\in[0,1)$ in Section 2.1 for the first construction, and starting from Section 2.3 (until the end of the paper), we switch to the new time scale with $s\in[0,\infty)$ in order to enjoy the homogeneous Markov property. In Section 2.3, we describe the evolution of a genealogically typical cell in the system (see right after (1.4) for an explanation of the words "genealogically typical"); this part is quite elementary, but useful for computations in the forthcoming Section 3. Section 2.4 is devoted to the presentation of our system as a time homogeneous Markovian growth-fragmentation process in the sense of Bertoin [3]. This is why we call the rescaled and time-changed process (1.9) the *HR growth-fragmentation* process. As such, it can be viewed as an exactly solvable example of growth-fragmentation process.

There is an extensive literature on growth-fragmentation processes, both from probabilistic point of view, and from differential equation point of view. Compared to existing results, our work focuses on a specific but remarkable model originating from statistical physics, for which we establish in Theorem 3.13 a law of large numbers for the rescaled empirical measure of the cell sizes. The literature provides usually more general but less precise results, often focusing on the *expectation* of the empirical measure. Namely, the measure $\boldsymbol{\mu}_s(\mathrm{d}y)$ such that

$$\int_{[0,\infty)}f(y)\,\boldsymbol{\mu}_s(\mathrm{d}y)=\mathbf{E}\Big[\sum_k f\big(e^s\mathbf{x}_k(1-e^{-s})\big)\Big]\,.$$

However, let us mention that in [15], Gonzalez, Horton and Kyprianou consider very general branching Markov processes: under a spectral assumption (H1) on the semigroup of the spatial motion and a moment assumption (H2) on the branching mechanism (see [15] page 810), the authors of this paper obtain a precise asymptotic of the integrated moments of the empirical measure of the particles of the branching Markov process.

We should also mention that in Bertoin and Watson [8], the authors have established a remarkable general law of large numbers for the normalized empirical measure itself. More precisely, for the general models investigated in [8], $\boldsymbol{\mu}_s$ is governed by the following dynamics $\frac{\mathrm{d}}{\mathrm{d}s}\int_{[0,\infty)} f(y)\boldsymbol{\mu}_s(\mathrm{d}y) = \int_{[0,\infty)}(\mathscr{A} f)(y)\boldsymbol{\mu}_s(\mathrm{d}y)$, where the infinitesimal generator $\mathscr{A}$ is of the form

$$\mathscr{A} f(x) = c(x)f'(x) + B(x)\int_{\mathcal{P}}\Big(\sum_{i=1}^{\infty} f(xp_i) - f(x)\Big)\,\kappa(x,\,\mathrm{d}\mathbf{p}), \tag{1.10}$$

for smooth functions in the domain of $\mathscr{A}$, where $\mathcal{P} := \{\mathbf{p} = (p_1, p_2, \ldots) : p_1 \geq p_2 \geq \cdots \geq 0,\, \sum_{i=1}^{\infty} p_i = 1\}$, with appropriate assumptions on the growth rate c and the fragmentation rate B that are continuous functions on $(0,\infty)$, and on the fragmentation (probability) kernel κ from $(0,\infty)$ to $\mathbf{p}$. In order to ensure the so-called Malthusian behavior of the system, [8] has adopted the stochastic approach developed previously by the authors ([4], [7]) and by Cavalli [9], by investigating an associated Markov process on $(0,\infty)$ under the assumption

$$\sup_{x>0}\frac{c(x)}{x} < \infty\,. \tag{1.11}$$

See (1.7) in [8], or (5) in [7].

Several specific cases that do not satisfy (1.11) have been studied for instance in Dadoun [11] and Bertoin, Budd, Curien and Kortchemski [5] where $c(x) = ax^{\alpha+1}$, $B(x) = bx^{\alpha}$ and κ is independent of x, and also in Shi [23] where $c(x) = x(a - \theta\log x)$, B is constant and κ does not depend on x. As explained in Remark 3.2, for our HR process as defined in $(1.1a)$-$(1.1e)$ and after the change of variables $t = 1 - e^{-s}$ in (1.8), it is seen that the generator also bears the form (1.10), with $c(x) = x+1$, $B(x) = 2x$ and $\kappa(x, \mathrm{d}\mathbf{p})$ is the law of $(x\max(U, 1-U)\,, x\min(U, 1-U), 0, 0\ldots)$ where U is uniform on $[0,1]$, so the assumption (1.11) fails. As such, we are not entitled to apply the result of [8] to our HR growth-fragmentation process. Also, the law of large numbers in [8] for the empirical measure, which is valid in a fairly general setting, holds in an appropriate sense of L^1 convergence, whereas our law of large numbers, built for the specific system satisfying $(1.1a)$-$(1.1e)$, holds almost surely (and in L^2). The main interest of our result is the identification of the limiting distribution of the empirical measure, for the specific model defined in $(1.1a)$-$(1.1e)$.

In the PDE literature, the authors mainly focus on the average number $u(s,x)$ of particles with size in $[x, x + \mathrm{d}x]$. Namely, $\boldsymbol{\mu}_s(\mathrm{d}y) = u(s,y)\mathrm{d}y$. In these papers, asymptotics on u proceed from the spectral analysis of $\mathscr{A}$: see Mischler and Scher [19] for a thorough overview on the PDE works and see Doumic and Escobedo [14] for a precise study of growth-fragmentations with binary splitting.

Let us mention that growth-fragmentation processes serve frequently as biological models (see the monographs by Perthame [20] and Rudnicki and Tyran-Kamińska [22]). They have recently appeared in various other contexts, such as excursions of Brownian motion in half-plane (Aïdékon and Da Silva [1]), growth-fragmentations with isolation on random recursive trees (Bansaye, Gu and Yuan [2]), random planar maps (Bertoin, Curien and Kortchemski [6]), Brownian motion indexed by the Brownian tree (Le Gall and Riera [18]), or ricocheted stable processes (Watson [24]).

Our main results are presented in Section 3. We have already mentioned the important tool of the master formula (1.1): it is presented in Section 3.1, and proved in Section 3.7 by means of the method of characteristics. As a simple application of it, we study a pair of martingales in Section 3.2, and prove almost sure and L^2 convergences of these martingales; In Section 3.3, we determine the joint law of $\sum_{k=1}^{\mathbf{n}_t}\mathbf{x}_k(t)$ and $\mathbf{n}_t$ for any given $t \in [0,\,1)$, in the new time scale $s \in [0,\,\infty)$ via (1.8); in particular, this yields the values of the constants $c_1 = c_2 = \frac{1}{2}$ in (1.7). The argument relies heavily on the strength of the master formula. Section 3.6 is devoted to the proof of (1.6). Section 3.4 contains exact computations of the *mean* of rescaled empirical measure of cell sizes. They yield some probability estimates (see Proposition 3.11) which are going to be used in Section 3.5, where the law of large numbers for the rescaled empirical measure is stated and proved.

We close the introduction by mentioning some possible extensions of the present work.

1. Overlaps. It seems more convenient to formulate this problem in the original time scale $t \in [0,\,1)$, without using the new time scale $s \in [0,\,\infty)$ via $t = 1 - e^{-s}$ as in (1.8). Let $t \in [0,\,1)$. Let $\mathbf{x}(t) = (\mathbf{x}_k(t))_{1\leq k\leq \mathbf{n}_t}$ denote the cell sizes at time t. For any integers $k,\, \ell \in [1,\, \mathbf{n}_t]$, let $T_{k,\ell}(t)$ be the largest $r \in [0,\,t]$ such that both $\mathbf{x}_k(t)$ and $\mathbf{x}_\ell(t)$ are descendants of $\mathbf{x}_i(r)$ for some $1 \leq i \leq \mathbf{n}_r$. In words, $T_{k,\ell}(t)$ is the generation/time of the most recent common ancestor of cells of sizes $\mathbf{x}_k(t)$ and $\mathbf{x}_\ell(t)$ at time t, whereas $t - T_{k,\ell}(t)$ represents the age at time t of $\mathbf{x}_k(t)$ and $\mathbf{x}_\ell(t)$ since their birth at time

$T_{k,\ell}(t)$. When k and ℓ are uniformly and independently chosen from $\{1,\dots,\mathbf{n}_t\}$, it is not too hard to prove that $T_{k,\,\ell}(t)$ converges weakly as $t\to 1^-$: it would be interesting to determine the limiting distribution of $T_{k,\ell}(t)$.

2. More general branching mechanism. The branching rate $\frac{2\mathbf{y}}{(1-t)^2}$ in $(1.1c)$ and the uniform fragmentation at branching times in $(1.1d)$ are specific features of our system; they originate from simplified hierarchical renormalization models at criticality studied in [12], conditioned on survival, if the initial distribution in the discrete-times satisfies a certain integrability condition; see [13]. When this integrability condition fails, the branching rates and the fragmentation rates become more complicated (See [10] and [13] for a precise description of these rates). Also one could forget the origin of our system, and consider it as a special case of a more general growth-fragmentation process. In all these cases, it would be interesting to see whether some of the results of the present paper could be extended.

3. Number of branchings along each branch. In the present work, we study the evolution of the number of cells and their sizes. Other properties of the cells could be considered relative to their ancestries, for example the number of branching events connecting a cell at time t to its ancestor cell at time $t=0$.

2. Two complementary constructions

The main purpose of this section is to provide two constructions of the system. The first construction is given in Section 2.1, based on a Poisson point process; it justifies, in particular, the use of "a genealogically typical branch" in the system. The second construction, presented in 2.3, reveals that each branch is a time homogeneous Markov process if we use the new time parameter $s\in[0,\infty)$ defined by

$$t=1-e^{-s}\,.$$

Time homogeneity simplifies our tasks in many aspects, so the new time parameter $s\in[0,\infty)$ is used from Section 2.3 on. We do some preliminary computations in Section 2.3 for the evolution of a single genealogically typical cell in the system; these computations will be used in Section 3 in deeper investigations on the system. In Section 2.4, we view our system as a Markovian growth-fragmentation process. For the sake of clarity, we postpone the proofs of some results until the end of the section: Sections 2.5, 2.6 and 2.7 are devoted to the proof of Lemmas 2.3, 2.7 and Proposition 2.12, respectively.

2.1. A construction by means of a Poisson point process

Notation and basic results on Poisson point processes. Let us now define more formally the cell population using a Poisson point process on $[0,1)\times[0,\infty)$. Let us briefly introduce some notation and recall some definitions on Poisson processes. We note by $\mathbb{N}$ the set of natural numbers $0,1,\dots$ and we use the notation $\mathbb{N}^*=\mathbb{N}\backslash\{0\}$. Unless the contrary is explicitly mentioned, all the random variables (r.v. for short) that we consider are defined on the same probability space $(\Omega,\mathscr{F},\mathbf{P})$. We agree on the following convention: a Poisson random variable has an infinite (resp. a zero) mean if it is constant to ∞ (resp. to 0).

Let μ be a sigma finite measure without atom on the Euclidian space $\mathbb{R}^d$ equipped with its Borel sigma field. A *Poisson point process* (PPP for short) on $\mathbb{R}^d$ with intensity measure μ is a countable random subset $\mathbf{\Pi}\subset\mathbb{R}^d$ that satisfies the following properties.

- (a) For all Borel subset $B\subset\mathbb{R}^d$, $\#(\mathbf{\Pi}\cap B)$ is a $\mathscr{F}$-measurable $\mathbb{N}\cup\{\infty\}$-valued random variable that has a Poisson distribution with mean $\mu(B)$.
- (b) If $B_1,\dots,B_n$ are pairwise disjoint Borel subsets of $\mathbb{R}^d$, then the r.v. $\#(\mathbf{\Pi}\cap B_1),\dots,$ $\#(\mathbf{\Pi}\cap B_n)$ are independent.

When it is convenient, we replace $\mathbf{\Pi}$ by its empirical measure $\sum_{X\in\mathbf{\Pi}}\delta_X$ and for all nonnegative measurable functions f on $\mathbb{R}^d$, we use the notation $\langle\mathbf{\Pi},f\rangle=\sum_{X\in\mathbf{\Pi}}f(X)$. We state here Mecke's formula that is used a couple of times: for any nonnegative mesurable function F,

$$\mathbf{E}\Big[\sum_{X\in\mathbf{\Pi}}F(X,\mathbf{\Pi}\backslash\{X\})\Big]=\int_{\mathbb{R}^d}\mu(\mathrm{d}x)\mathbf{E}[F(x,\mathbf{\Pi})]\,. \tag{2.1}$$

We construct the cell population using a PPP $\mathbf{\Pi}$ on the space $[0,1)\times[0,\infty)$ whose intensity measure is given by

$$\mu(\mathrm{d}t\,\mathrm{d}y)=\mathbf{1}_{[0,1)\times[0,\infty)}(t,y)\frac{2\,\mathrm{d}t\,\mathrm{d}y}{(1-t)^2}\,. \tag{2.2}$$

Note that for all $t_0 \in [0,1)$ and $x, y \in (0,\infty)$, $\#(\mathbf{\Pi} \cap ([0,t_0]\times[0,x]))$ is a Poisson r.v. with mean $2xt_0/(1-t_0)$ and that a.s. $\#(\mathbf{\Pi}\cap([t_0,1)\times[x,x+y]))=\infty$. We shall repeatedly use the following scaling property whose proof is elementary.

Lemma 2.1. *Let $\mathbf{\Pi}$ be a PPP on $[0,1)\times[0,\infty)$ with intensity measure μ given by (2.2). Let $f:[0,1)\to[0,\infty)$ be measurable and $t_0\in[0,1)$. We set*

$$\mathbf{\Pi}'=\Big\{\big(\tfrac{t-t_0}{1-t_0},\tfrac{y-f(t)}{1-t_0}\big)\,;\,(t,y)\in\mathbf{\Pi}: t\geq t_0 \text{ and } y>f(t)\Big\}$$

$$\text{and}\quad \mathbf{\Pi}''=\Big\{\big(\tfrac{t-t_0}{1-t_0},\tfrac{y}{1-t_0}\big)\,;\,(t,y)\in\mathbf{\Pi}: t\geq t_0 \text{ and } y\leq f(t)\Big\}\,.$$

Then, $\mathbf{\Pi}'$, $\mathbf{\Pi}''$ and $\mathbf{\Pi}\cap([0,t_0]\times[0,\infty))$ are independent, $\mathbf{\Pi}'$ has the same law as $\mathbf{\Pi}$, and $\mathbf{\Pi}''$ has the same law as $\big\{(t,y)\in\mathbf{\Pi}:(1-t_0)y\leq f(t_0+t(1-t_0))\big\}$.

Proof. Since $\mathbf{\Pi}\cap([0,t_0]\times[0,\infty))$, $\{(t,y)\in\mathbf{\Pi}: t\geq t_0 \text{ and } y>f(t)\}$ and $\{(t,y)\in\mathbf{\Pi}: t\geq t_0 \text{ and } y\leq f(t)\}$ are restrictions of $\mathbf{\Pi}$ to disjoint subsets of $[0,1)\times[0,\infty)$, there are independent PPP Next observe that $\mathbf{\Pi}'$ and $\mathbf{\Pi}''$ are images of the two previous PPP under two measurable bijective functions. It implies they are independent PPP. For all all nonnegative measurable g, an easy change of variable yields the following.

$$\begin{aligned}\mathbf{E}\big[\langle\mathbf{\Pi}',g\rangle\big] &= \int_{t_0}^{1}\!\mathrm{d}t\int_{f(t)}^{\infty}\!\mathrm{d}y\,\frac{2}{(1-t)^2}g\Big(\frac{t-t_0}{1-t_0},\frac{y-f(t)}{1-t_0}\Big)=(1-t_0)\int_{t_0}^{1}\!\mathrm{d}t\int_0^{\infty}\!\mathrm{d}y'\,\frac{2}{(1-t)^2}g\Big(\frac{t-t_0}{1-t_0},y'\Big)\\ &=\frac{2}{1-t_0}\int_0^{1-t_0}\!\mathrm{d}t'\int_0^{\infty}\!\mathrm{d}y\,\Big(1-\frac{t}{1-t_0}\Big)^{-2}g\Big(\frac{t'}{1-t_0},y\Big)=\int_0^1\!\mathrm{d}t\int_0^{\infty}\!\mathrm{d}y\,\frac{2}{(1-t)^2}g(t,y)\,.\end{aligned}$$

Therefore the intensity measure of $\mathbf{\Pi}'$ is μ, which implies it has the same law as $\mathbf{\Pi}$. The computation of the law of $\mathbf{\Pi}''$ is the consequence of a similar change of variable. ■

A deterministic integral equation. We now consider equations that are derived from the PPP $\mathbf{\Pi}$. Since we deal with possible rescaled versions of the PPP, it is convenient to proceed with a deterministic *countable* subset of points as defined below.

Definition 2.2. We denote by $\mathscr{P}$ the set of *countable* subsets $\Pi\subset[0,1)\times(0,\infty)$ satisfying the following conditions. For all $t_0\in[0,1)$ and for all $x,y\in(0,\infty)$, $\#\big(\Pi\cap(\{t_0\}\times(0,\infty))\big)\leq 1$ and

$$\#\big(\Pi\cap([0,t_0)\times[0,x])\big)<\infty,\quad \#\big(\Pi\cap([t_0,1)\times[x,x+y])\big)=\infty. \tag{2.3}$$

We equip $\mathscr{P}$ with the sigma field generated by the counting functions $\Pi\mapsto\#(\Pi\cap B)$, for all Borel subets B of $[0,1)\times[0,\infty)$. □

Let $\Pi\in\mathscr{P}$ and let $t_0\in[0,1)$. We say that $t\in[t_0,1)\mapsto(\mathtt{n}(t),\mathtt{m}(t))\in[0,\infty)^2$ is a solution to the equation $\mathtt{Eq}(t_0,\Pi)$ if for all $t\in[t_0,1)$

$$\underline{\mathtt{Eq}(t_0,\Pi)}:\quad \mathtt{m}(t)=\mathtt{m}(t_0)+\int_{t_0}^{t}\mathtt{n}(r)\,\mathrm{d}r \text{ and } \mathtt{n}(t)=\mathtt{n}(t_0)+\#\big\{(s,y)\in\Pi: s\in]t_0,t] \text{ and } y\leq\mathtt{m}(s)\big\}.$$

We easily check that *for any initial condition* $(\mathtt{n}(t_0),\mathtt{m}(t_0))\in[0,\infty)^2$ *there is a unique solution to* $\mathtt{Eq}(t_0,\Pi)$, and often write indifferently $\mathtt{n}_t$ and $\mathtt{n}(t)$, or $\mathtt{m}_t$ and $\mathtt{m}(t)$. This next implies a *flow property*: if $(\mathtt{n},\mathtt{m})$ is the solution of $\mathtt{Eq}(t_0,\Pi)$, then it is also a solution of $\mathtt{Eq}(t_0',\Pi)$, for all $t_0'\in[t_0,1)$. We finally observe that *solutions to the equation* $\mathtt{Eq}(t_0,\Pi)$ *increase with respect to their initial condition*. More precisely, let $\preceq$ be the partial order on $[0,\infty)^2$ given by $(a,b)\preceq(c,d)$ if and only if $a\leq c$ *and* $b\leq d$. Then, solutions $(\mathtt{n},\mathtt{m})$ of $\mathtt{Eq}(t_0,\Pi)$ preserve the order $\preceq$. The flow property combined with conservation of $\preceq$-order imply the following. Let $t_0,t_0'\in[0,1)$, such that $t_0\leq t_0'$. Let $(\mathtt{n},\mathtt{m})$ and $(\mathtt{n}',\mathtt{m}')$ be solutions of resp. $\mathtt{Eq}(t_0,\Pi)$ and $\mathtt{Eq}(t_0',\Pi)$. We check that

$$\text{If }\big(\mathtt{n}(t_0'),\mathtt{m}(t_0')\big)\preceq\big(\mathtt{n}'(t_0'),\mathtt{m}'(t_0')\big),\ \text{ then }\big(\mathtt{n}(t),\mathtt{m}(t)\big)\preceq\big(\mathtt{n}'(t),\mathtt{m}'(t)\big),\ \text{ for all } t\in[t_0',1). \tag{2.4}$$

The branching process associated with a point process. We then define the branching process with initial size $x\in[0,\infty)$ that satisfies $(1.1a)$-$(1.1e)$ and whose cell division events are represented by a fixed subset $\Pi\in\mathscr{P}$ as explained in the introduction.

To this end, let us first denote by $(\mathbf{n},\mathbf{m})$ the solution of $\mathtt{Eq}(0,\Pi)$ such that $(\mathbf{n}_0,\mathbf{m}_0)=(1,x)$. We introduce the following two sets that are derived from x and Π:

(2.5) $$\Pi_{\leq x} = \big\{(s,y)\in\Pi : y\leq \mathbf{m}_s\big\} \quad \text{and} \quad \Pi_{>x} = \big\{(s,y-\mathbf{m}_s)\,;\,(s,y)\in\Pi : y>\mathbf{m}_s\big\}.$$

The subset $\Pi_{\leq x}$ is actually the set of cell division events of the population with initial size x. More precisely, we denote by $(T_n,Y_n)_{n\in\mathbb{N}^*}$ the enumeration of $\Pi_{\leq x}$ such that $T_n<T_{n+1}$. Namely,

(2.6) $$\Pi_{\leq x} = \big\{(T_n,Y_n)\,;\,n\in\mathbb{N}^*\big\}.$$

By convention, we set $(T_0,Y_0)=(0,x)$. The subset $\Pi_{>x}$ is what remains of Π after "peeling" the points of $\Pi_{\leq x}$. The following lemma (whose proof is postponed to Section 2.5) and Lemma 2.1 are the key argument in the proof of the time-branching property of the model.

Lemma 2.3. *Let $\mathbf{\Pi}$ be a PPP on $[0,1)\times[0,\infty)$ whose intensity measure is given by (2.2). Let $x\in[0,\infty)$. Then, $\mathbf{\Pi}_{\leq x}$ and $\mathbf{\Pi}_{>x}$ are independent and $\mathbf{\Pi}_{>x}$ has the same law as $\mathbf{\Pi}$.*

Proof. See Section 2.5. ■

Thanks to (x,Π) we next define a sequence $\mathbf{y}_k(\cdot)$, $k\in\mathbb{N}$, of càdlàg functions from $[0,1)$ to $[0,\infty)$ that satisfy the following inequalities for all $t\in[0,1)$

(2.7) $$0=\mathbf{y}_0(t)<\mathbf{y}_1(t)<\ldots<\mathbf{y}_{\mathbf{n_t}}(t)=\mathbf{m}_t=\mathbf{y}_k(t)\,,\ \text{for all } k\geq\mathbf{n}_t.$$

In addition to these inequalities, a simple recursion shows that processes $\mathbf{y}_k$ are uniquely defined by the following conditions.

Def(1) *For all $t\in[0,1)$, $\mathbf{y}_0(t)=0$ and for all $k\in\mathbb{N}^*$, $\mathbf{y}_k(0)=x$.*

Def(2) *Recall the definition of $(T_n,Y_n)_{n\in\mathbb{N}}$ from (2.6). Let $n\in\mathbb{N}^*$ and $t\in[T_{n-1},T_n)$. Then $\mathbf{y}_k(t)=\mathbf{y}_k(T_{n-1})+(t-T_{n-1})\min\{k,n\}$, for all $k\in\mathbb{N}$. (Note that $\mathbf{n}_t=n$.)*

Def(3) *Let $n\in\mathbb{N}^*$. There is a unique $k_n\in\{1,\ldots,n\}$ such that $\mathbf{y}_{k_n-1}(T_n^-)<Y_n\leq\mathbf{y}_{k_n}(T_n^-)$. Then, for all $k\in\mathbb{N}$, we require the following:*

$$\mathbf{y}_k(T_n)=\begin{cases}\mathbf{y}_k(T_n^-) & \text{if } k<k_n\\ Y_n & \text{if } k=k_n\\ \mathbf{y}_{k-1}(T_n^-) & \text{if } k>k_n.\end{cases}$$

The cell sizes of the population *corresponding to x and Π are then given by*

(2.8) $$\mathbf{x}(t)=\big(\mathbf{x}_k(t)\big)_{1\leq k\leq\mathbf{n_t}}:=\big(\mathbf{y}_k(t)-\mathbf{y}_{k-1}(t)\big)_{1\leq k\leq\mathbf{n_t}}\,.$$

See Figure 3 below for an example.

Finite dimensional marginals of $\mathbf{x}$ are measurable functions of x and Π, so there exists a measurable function Φ from $[0,\infty)\times\mathscr{P}$ to the space of $(\bigcup_{n\in\mathbb{N}^*}[0,\infty)^n)$-valued càdlàg functions such that

(2.9) $$(\mathbf{x}(t))_{t\in[0,1)}=\Phi(x,\Pi)=\Phi(x,\Pi_{\leq x})\,.$$

Definition 2.4. Let $\mathbf{\Pi}$ be a PPP on $[0,1)\times[0,\infty)$ with intensity measure μ as specified in (2.2). A process $(\mathbf{x}(t))_{t\in[0,1)}$ that has the same law as $\Phi(x,\mathbf{\Pi})$ is referred to as a *Hierarchical Renormalization process* (HR process) with initial value x. We also keep notation $\mathbf{n}$, $\mathbf{m}$, $\mathbf{y}_k(\cdot)$ and (T_n,Y_n) as in Def(1), Def(2) and Def(3), and the subscript x in $\mathbf{P}_x$ and $\mathbf{E}_x$ to mean that we consider a system with initial value x; namely, $\mathbf{P}_x(\mathbf{m}_0=x)=1$. □

Claim 2.5. *We have constructed the process evolving according to the branching mechanism given by* (1.1*a*)-(1.1*e*) *in the introduction.*

Proof. First note that times $(T_n)_{n\in\mathbb{N}^*}$ correspond to cell divisions and observe that Def(2) implies that each cell grows linearly between to division events: namely, $\mathbf{x_k}(t)=\mathbf{x}_k(T_{n-1})+t-T_{n-1}$, for all $t\in[T_{n-1},T_n)$ and for all $k\leq n=\mathbf{n}_t$. At time T_n, we see by Def(3) that the k_n-th cell of size $\mathbf{x}_{k_n}(T_n^-)$ undergoes a division into two cells of sizes $\mathbf{x}_{k_n}(T_n)$ and $\mathbf{x}_{k_n+1}(T_n)$ such that $\mathbf{x}_{k_n}(T_n^-)=\mathbf{x}_{k_n}(T_n)+\mathbf{x}_{k_n+1}(T_n)$. Basic results on PPP imply that $\mathbf{x}_{k_n}(T_n)/\mathbf{x}_{k_n}(T_n^-)$ is uniformly distributed on $[0,1]$. ■

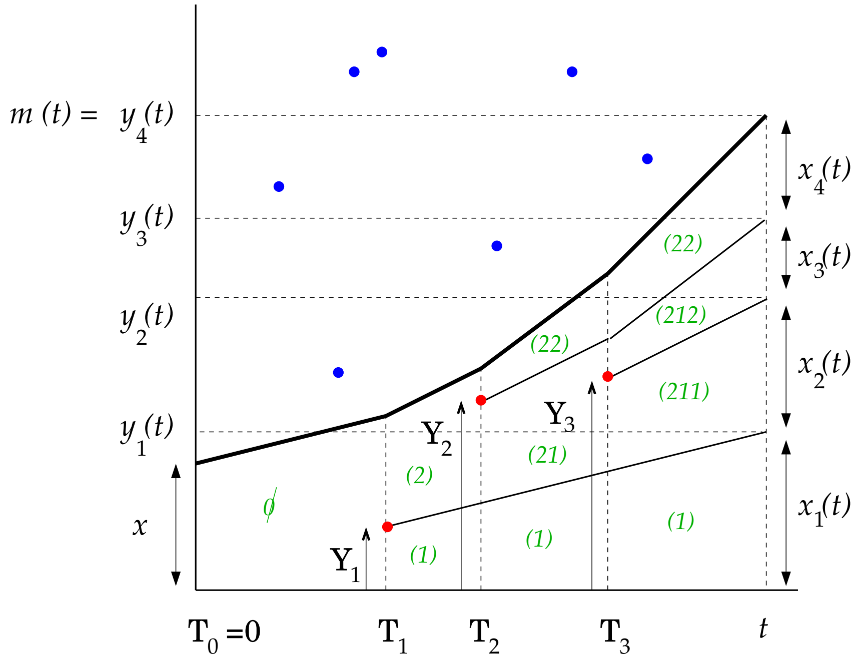

FIG 3. An example of the branching system associated with $(x, \mathbf{\Pi})$, with $\mathbf{n}_t = 4$. The atoms of $\mathbf{\Pi}$ are represented by dots. Only the dots lying under the broken line (in red) contribute to the evolution of our process: they are the atoms of $\mathbf{\Pi}_{\leq x}$. The cells are the areas between the broken lines. They are indexed by their genealogy which is coded by a finite word (in green) written with letters 1 and 2 as explained in Section 2.4: the ancestor cell corresponds to the empty word, its two daughter cells are indexed by (1) and (2), the two daughter cells of (2) are (21) and (22), the two daughter cells of (21) are (211) and (212), and so on.

2.2. *The time-branching renewal property*

We next state and prove the (inhomogeneous) *time-branching renewal property* enjoyed by our HR process. To this end, we first introduce the genealogy of the cells induced by our construction.

More precisely, let $\Pi \in \mathscr{P}$ as in Definition 2.2. Let $x \in [0, \infty)$ and let $t_0, t \in [0, 1)$ be such that $t_0 \leq t$. Recall that if $(\mathtt{m}, \mathtt{n})$ is a solution to $\mathtt{Eq}(t_0, \Pi)$, then $\dot{\mathtt{m}} = \mathtt{n}$. A quick inspection to the indexing rules `Def(3)` describing a cell division shows the following.

<u>`Def(4)`</u> *For all* $k \in \{0, \dots, \mathbf{n}_{t_0}\}$*, let* $(\dot{\mathbf{z}}_{k,t_0}, \mathbf{z}_{k,t_0})$ *be the solution of* $\mathtt{Eq}(t_0, \Pi)$ *such that*

$$\mathbf{z}_{k,t_0}(t_0) = \mathbf{y}_k(t_0) \quad \textit{and} \quad \dot{\mathbf{z}}_{k,t_0}(t_0) = k \, .$$

Then, the j*-th cell at time* t *is a descendent of the* k*-th cell at time* t_0 *if and only if*

$$\dot{\mathbf{z}}_{k-1,t_0}(t) < j \leq \dot{\mathbf{z}}_{k,t_0}(t) \, .$$

Then, for all $k \in \{1, \dots, \mathbf{n}_{t_0}\}$ and for all $t \in [t_0, 1)$ we define the following:

$$\mathbf{n}_t^{(k,t_0)} = \dot{\mathbf{z}}_{k,t_0}(t) - \dot{\mathbf{z}}_{k-1,t_0}(t) \, , \quad \mathbf{x}_j^{(k,t_0)}(t) = \mathbf{x}_{\dot{\mathbf{z}}_{k-1,t_0}(t)+j}(t) \, , \text{ for all } j \in \{1, \dots, \mathbf{n}_t^{(k,t_0)}\},$$

$$\text{and} \quad \mathbf{m}_t^{(k,t_0)} = \sum_{\dot{\mathbf{z}}_{k-1,t_0}(t) < j \leq \dot{\mathbf{z}}_{k,t_0}(t)} \mathbf{x}_j(t) \, . \tag{2.10}$$

Then, $\mathbf{n}_t^{(k,t_0)}$ (resp. $\mathbf{m}_t^{(k,t_0)}$) is *the number of descendents (resp. the total size of the descendents) at time* t *of the* k*-th cell at time* t_0 and $\mathbf{x}_j^{(k,t_0)}(t)$ is the size of the j*-th descendent at time* t *of the* k*-th cell at time* t_0.

For all $t_0, t \in [0,1)$ and all $k \in \{1, \ldots, \mathbf{n}_t\}$, we next introduce the following rescaled HR process of the subpopulation stemming from the k-th cell that is alive at time t.

$$(\theta_{k,t_0}\mathbf{x})(t) = \Big((1-t_0)^{-1} \mathbf{x}_j^{(k,t_0)}\big(t_0 + (1-t_0)t\big) \; ; \; 1 \leq j \leq \mathbf{n}^{(k,t_0)}_{t_0+(1-t_0)t} \Big) . \tag{2.11}$$

Next recall from (2.6) that $(T_n, Y_n)_{n \in \mathbb{N}^*}$ stands for the enumeration of $\Pi_{\leq x} = \{(s,y) \in \Pi : y \leq \mathbf{m}_s\}$ such that $T_n < T_{n+1}$. In particular T_1 is the first time a cell division occurs. Then,

$$\forall t \in [0, T_1), \quad \mathbf{n}_t = 1, \quad \mathbf{x}_1(t) = x + t, \quad \mathbf{x}_1(T_1) = Y_1 \quad \text{and} \quad \mathbf{x}_2(T_1) = x + T_1 - Y_1 \, .$$

Also note that $\mathbf{n}_{T_1} = 2$ and $\theta_{1,T_1}\mathbf{x}$ and $\theta_{2,T_1}\mathbf{x}$ are the rescaled HR process of the two subpopulations at the first time of cell division.

Proposition 2.6. *We keep the above notation when Π is equal to a PPP $\mathbf{\Pi}$ with intensity measure μ as in (2.2). Then following holds true.*

(i) *Conditionally on (T_1, Y_1), $\theta_{1,T_1}\mathbf{x}$ and $\theta_{2,T_1}\mathbf{x}$ are independent and distributed as HR processes with respective initial values $\mathbf{x}_1(T_1)/(1-T_1)$ and $\mathbf{x}_2(T_1)/(1-T_1)$. Moreover, for all measurable bounded $f : [0,1) \times [0,\infty)^2 \to \mathbb{R}$,*

$$\mathbf{E}_x\big[f(T_1, \mathbf{x}_1(T_1), \mathbf{x}_2(T_1))\big] = \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}u \, \frac{2(x+t)}{(1-t)^4} \, e^{-\frac{2(x+1)t}{1-t}} \, f\big(t, u(x+t), (1-u)(x+t)\big). \tag{2.12}$$

(ii) (Time-branching renewal property) *Let $t_0 \in [0,1)$. Conditionally on $\mathbf{\Pi} \cap ([0,t_0] \times [0,\infty))$, the processes $(\theta_{k,t_0}\mathbf{x})_{1 \leq k \leq \mathbf{n}_{t_0}}$ are independent and $\theta_{k,t_0}\mathbf{x}$ is distributed as the HR process with initial value $\mathbf{x}_k(t_0)/(1-t_0)$.*

Proof. For all $k \in \{1, \ldots, \mathbf{n}_{t_0}\}$ and all $t_0 \in [0,1)$ we set

$$\mathbf{\Pi}_{k,t_0} = \Big\{ \Big(\tfrac{t-t_0}{1-t_0}, \tfrac{y - \mathbf{z}_{k-1,t_0}(t)}{1-t_0} \Big) \, ; \, (t,y) \in \mathbf{\Pi} : t \geq t_0 \text{ and } \mathbf{z}_{k-1,t_0}(t) < y \leq \mathbf{z}_{k,t_0}(t) \Big\} \, .$$

Recall from (2.9) the definition of Φ. Then, a simple (deterministic) scaling argument implies that

$$\theta_{k,t_0}\mathbf{x} = \Phi\big(\mathbf{x}_k(t_0)/(1-t_0), \mathbf{\Pi}_{k,t_0}\big), \quad k \in \{1, \ldots, \mathbf{n}_{t_0}\} \, .$$

This also holds true when $t_0 = T_1$. Namely $\theta_{k,T_1}\mathbf{x} = \Phi(\mathbf{x}_k(T_1)/(1-T_1), \mathbf{\Pi}_{k,T_1})$ with $k \in \{1,2\}$. Therefore, the proposition is a consequence of the following two statements.

(i') *Conditionally on (T_1, Y_1), the point processes $\mathbf{\Pi}_{1,T_1}$ and $\mathbf{\Pi}_{2,T_1}$ are independent and for $k \in \{1,2\}$, $\mathbf{\Pi}_{k,T_1}$ has the same law as $\mathbf{\Pi}^*_{\leq \mathbf{x}_k(T_1)/(1-T_1)}$, where $\mathbf{\Pi}^*$ is an independent copy of $\mathbf{\Pi}$.*

(ii') *Conditionally on $\mathbf{\Pi} \cap ([0,t_0] \times [0,\infty))$, the point processes $(\mathbf{\Pi}_{k,t_0})_{1 \leq k \leq \mathbf{n}_{t_0}}$ are independent and $\mathbf{\Pi}_{k,t_0}$ has the same law as $\mathbf{\Pi}^*_{\leq \mathbf{x}_k(t_0)/(1-t_0)}$, where $\mathbf{\Pi}^*$ is an independent copy of $\mathbf{\Pi}$.*

Let us prove (i') and find the joint law of (T_1, Y_1). We first set $\mathbf{\Pi}^* = \big\{ \big(\frac{s-T_1}{1-T_1}, \frac{y}{1-T_1} \big) ; (s,y) \in \mathbf{\Pi} : s > T_1 \big\}$. For all $t \in [0,1]$, write $C(t) = \{(s,y) \in [0,t) \times [0,\infty) : y \leq x + s\}$. Then for all measurable functions $f : [0,1) \times [0,\infty) \to [0,\infty)$ and $F : \mathscr{P} \to [0,\infty)$, we have

$$\begin{aligned} &\mathbf{E}\big[f(T_1, Y_1) F(\mathbf{\Pi}^*)\big] \\ &= \mathbf{E}\Big[\sum_{(T,Y) \in \mathbf{\Pi}} \mathbf{1}_{\{Y \leq x+T\}} f(T,Y) \mathbf{1}_{\{\#(\mathbf{\Pi} \cap C(T)) = 0\}} \, F\Big(\big\{ \big(\tfrac{s-T}{1-T}, \tfrac{y}{1-T} \big) ; (s,y) \in \mathbf{\Pi} : s > T \big\} \Big) \Big]. \end{aligned}$$

By Mecke's formula, this leads to:

$$\begin{aligned} &\mathbf{E}\big[f(T_1, Y_1) F(\mathbf{\Pi}^*)\big] \\ &= \int_0^1 \int_0^\infty \mu(\mathrm{d}t \, \mathrm{d}y) \mathbf{1}_{\{y \leq x+t\}} \, f(t,y) \mathbf{E}\Big[\mathbf{1}_{\{\#(\mathbf{\Pi} \cap C(t)) = 0\}} \, F\Big(\big\{ \big(\tfrac{s-t}{1-t}, \tfrac{y}{1-t} \big) ; (s,y) \in \mathbf{\Pi} : s > t \big\} \Big) \Big] \\ &= \int_0^1 \int_0^\infty \frac{2 \, \mathrm{d}t \, \mathrm{d}y}{(1-t)^2} \mathbf{1}_{\{y \leq x+t\}} \, f(t,y) \, \mathbf{P}\big(\#(\mathbf{\Pi} \cap C(t)) = 0 \big) \, \mathbf{E}\big[F(\mathbf{\Pi})\big] \, , \end{aligned}$$

by Lemma 2.1 in the second equality. Since $\#(\mathbf{\Pi}\cap C(t))$ is a Poisson r.v. with mean

$$\mu(C(t)) = 2\log(1-t) + \frac{2(x+1)t}{1-t}\ , \tag{2.13}$$

we have $\mathbf{P}\big(\#(\mathbf{\Pi}\cap C(t))=0\big)=e^{-\mu(C(t))}$; thus

$$\mathbf{E}\big[f(T_1,Y_1)F(\mathbf{\Pi}^*)\big]=\int_0^1\!\int_0^\infty \frac{2\,\mathrm{d}t\,\mathrm{d}y}{(1-t)^2}\mathbf{1}_{\{y\le x+t\}}\,f(t,y)\,e^{-\mu(C(t))}\,\mathbf{E}\big[F(\mathbf{\Pi})\big].$$

In particular, by (2.13) and an easy computation, we get

$$\begin{aligned}\mathbf{E}\big[f(T_1,Y_1)\big] &= \int_0^1\!\int_0^\infty \frac{2\,\mathrm{d}t\,\mathrm{d}y}{(1-t)^2}\mathbf{1}_{\{y\le x+t\}}\,f(t,y)\,e^{-\mu(C(t))}\\ &= \int_0^1\!\mathrm{d}t\int_0^1\!\mathrm{d}u\,\frac{2(x+t)}{(1-t)^4}\,e^{-\frac{2(x+1)t}{1-t}}\,f\big(t,u(x+t)\big).\end{aligned}\tag{2.14}$$

This easily implies (2.12) and we also get

$$\mathbf{E}\big[f(T_1,Y_1)F(\mathbf{\Pi}^*)\big]=\mathbf{E}\big[f(T_1,Y_1)\big]\,\mathbf{E}\big[F(\mathbf{\Pi})\big].$$

In other words,

$$\mathbf{\Pi}^* \text{ and } (T_1,Y_1) \text{ are independent and } \mathbf{\Pi}^* \text{ has the same law as } \mathbf{\Pi}. \tag{2.15}$$

For all $\Pi\in\mathscr{P}$ and all $x\in[0,\infty)$, we recall from (2.5) the notation $\Pi_{\le x}$ and $\Pi_{>x}$. Then, observe that $\theta_{1,T_1}\mathbf{\Pi}=\mathbf{\Pi}^*_{\le \mathbf{x}_1(T_1)/(1-T_1)}$ and $\theta_{2,T_1}\mathbf{\Pi}=(\mathbf{\Pi}^*_{>\mathbf{x}_1(T_1)/(1-T_1)})_{\le \mathbf{x}_2(T_1)/(1-T_1)}$. This implies (i') by (2.15) and Lemma 2.3.

To prove (ii'), we introduce the following notation. Let $\Pi\in\mathscr{P}$ satisfy (2.3) and let $(y_n)_{0\le n\le N}$ be a finite increasing sequence of real numbers such that $y_0=0$. For all $n\in\{0,\dots,N\}$, let us denote by $(\dot z_n,z_n)$ the solution of $\mathrm{Eq}(0,\Pi)$ such that $(\dot z_n(0),z_n(0))=(n,y_n)$ (note that $z_0(\cdot)$ is the null function). If $n\ge 1$, we set

$$\Pi_{(n)}=\big\{\big(t,y-z_{n-1}(t)\big)\,;\,(t,y)\in\Pi : z_{n-1}(t)<y\le z_n(t)\big\}\ .$$

Then, we easily check that $(\Pi_{(n)})_{1\le n\le N}=\Psi\big((y_n)_{0\le n\le N}\,,\,\Pi\big)$, where Ψ is measurable. Next, observe that $\Pi_{(1)}=\Pi_{\le y_1}$, that $\Pi_{>y_1}=\big\{\big(t,y-z_1(t)\big)\,;\,(t,y)\in\Pi : y>z_1(t)\big\}$ and therefore that

$$(\Pi_{(n+1)})_{1\le n<N}=\Psi\big((y_{n+1}-y_1)_{1\le n<N}\,,\,\Pi_{>y_1}\big)\ . \tag{2.16}$$

We then set $(\mathbf{\Pi}_{(n)})_{1\le n\le N}=\Psi\big((y_n)_{1\le n\le N},\mathbf{\Pi}\big)$. Since $\mathbf{\Pi}_{\le y_1}=\mathbf{\Pi}_{(1)}$, by (2.16) and by Lemma 2.3, we see that $\mathbf{\Pi}_{(1)}$ is independent of $(\mathbf{\Pi}_{(n+1)})_{1\le n<N}$, which has the same law as $\Psi\big((y_{n+1}-y_1)_{1\le n<N}\,,\,\mathbf{\Pi}\big)$. This easily implies the following.

$$\text{The } (\mathbf{\Pi}_{(n)})_{1\le n\le N} \text{ are independent and } \mathbf{\Pi}_{(n)} \text{ has the same law as } \mathbf{\Pi}_{\le y_n-y_{n-1}}. \tag{2.17}$$

We now set $\mathbf{\Pi}^*=\{(\frac{t-t_0}{1-t_0},\frac{y}{1-t_0});(t,y)\in\mathbf{\Pi}:t>t_0\}$. We first see that

$$(\mathbf{\Pi}_{k,t_0})_{1\le k\le \mathbf{n}_{t_0}}=\Psi\big((\mathbf{y}_k(t_0)/(1-t_0))_{1\le k\le\mathbf{n}_{t_0}},\mathbf{\Pi}^*\big)=\big(\mathbf{\Pi}^*_{(k)}\big)_{1\le k\le \mathbf{n}_{t_0}}\ . \tag{2.18}$$

Then, elementary results on PPP combined with Lemma 2.1 imply that $\mathbf{\Pi}\cap([0,t_0]\times[0,\infty))$ and $\mathbf{\Pi}^*$ are independent and that $\mathbf{\Pi}^*$ has the same law as $\mathbf{\Pi}$. It implies (ii') by (2.18) and (2.17). ■

The processes $\mathbf{n}$ and $\mathbf{m}$ both explode at 1^-: namely, for all $x\in[0,\infty)$, $\mathbf{P}_x$-a.s. $\lim_{t\to1^-}\mathbf{n}_t=\lim_{t\to1^-}\mathbf{m}_t=\infty$. However, the following lemma shows that for a fixed $t\in[0,1)$, the two r.v. $\mathbf{n}_t$ and $\mathbf{m}_t$ admit exponential moments. The proof is postponed to Section 2.6.

Lemma 2.7. *Let $x\in[0,\infty)$. For all $\lambda\in[0,1)$ and all $t\in[0,1]$,*

$$\mathbf{E}_x\big[\exp\big(\big(\tfrac12(1-t)\big)^{11}\lambda\,(\mathbf{n}_t+\mathbf{m}_t)\big)\big]\le\exp\big(\lambda(1+x)\big)\ .$$

Proof. See Section 2.6. ■

2.3. Evolution of a genealogically typical cell

Definition 2.4 of the HR process $(\mathbf{x}(t))_{t\in[0,1)}$ is a natural construction that is directly derived from the branching mechanism informally described in the introduction by $(1.1a)$-$(1.1e)$ and from a direct geometric coupling with a Poisson point process. However, this definition suffers from two drawbacks:

(1) *the process is not homogeneous in time*,

(2) *in the cell indexing system given by* $(\mathbf{x}_k(t))_{1\leq k\leq \mathbf{n}_t}$, *the numbering of the same cell can change while it does not undergo any cell division*.

To overcome these problems, we index the cells by their genealogy which is simply the infinite binary tree: this is the purpose of Section 2.4. We also perform a time change based on the time-branching property stated in Proposition 2.6 which makes the HR process time homogeneous. To this end, we first study in this section the evolution of the cell $t\in[0,1)\mapsto \mathbf{x}_1(t)$ that evolves as a genealogically typical cell. More precisely, we first give the change of time and space that transforms the evolution into $[0,\infty)$-valued Fellerian Markov process of which we give a simple description, as well as its infinitesimal generator and its invariant law.

We fix the following notation. Let $\mathbf{\Pi}$ be a PPP with intensity measure μ specified in (2.2) and let $x\in[0,\infty)$. We denote by $(\mathbf{n}^{(x)}_t, \mathbf{m}^{(x)}_t)_{t\in[0,1)}$ the solution of $\mathtt{Eq}(0,\mathbf{\Pi})$ such that $(\mathbf{n}^{(x)}_0, \mathbf{m}^{(x)}_0)=(1,x)$. We denote by $(\mathbf{x}^x(t)=(\mathbf{x}^x_k(t); 1\leq k\leq \mathbf{n}^{(x)}_t))_{t\in[0,1)}$ the HR process with initial size x that is derived from $\mathbf{\Pi}$ as explained in Section 2.1. We next denote by $(T^{(x)}_n, Y^{(x)}_n)_{n\geq 1}$ the points of $\mathbf{\Pi}_{\leq x}=\{(t,y)\in \mathbf{\Pi} : y\leq \mathbf{m}^{(x)}_t\}$ such that $T^{(x)}_n<T^{(x)}_{n+1}$.

Lemma 2.8. *We keep the above notation. Then, the following holds true.*

- (i) *We have* $\mathbf{x}^x_1(T^{(x)}_1-)=x+T^{(x)}_1$ *and* $\mathbf{x}^x_1(T^{(x)}_1)=Y^{(x)}_1$.
- (ii) *Let* $U=Y^{(x)}_1/(x+T^{(x)}_1)$. *Then,* $T^{(x)}_1$ *and* U *are independent,* U *is uniformly distributed on* $[0,1]$ *and* $\mathbf{P}(T^{(x)}_1>t)=(1-t)^{-2}\exp\big(-2(x+1)t/(1-t)\big)$, $t\in(0,1)$.
- (iii) *Let* $x'\in(x,\infty)$. *Then,* $0<T^{(x')}_1\leq T^{(x)}_1\leq T^{(0)}_1$ *and*

$$\forall t\in[0,T^{(x)}_1),\quad 0\leq \mathbf{x}^{x'}_1(t)-\mathbf{x}^x_1(t)\leq x'-x \quad \textit{and} \quad \forall t\in[T^{(x)}_1,1),\quad \mathbf{x}^{x'}_1(t)=\mathbf{x}^x_1(t)\,.$$

- (iv) *Almost surely, for all* $t\in(0,1)$, $x\in[0,\infty)\mapsto \mathbf{x}^x_1(t)$ *is nondecreasing,* $\lim_{x\to\infty}\mathbf{x}^x_1(t)=:\mathbf{x}^\infty_1(t)$ *exists in* $(0,\infty)$, *the process* $t\in(0,1)\mapsto \mathbf{x}^\infty_1(t)$ *is right-continuous with left limits and* $\lim_{t\to 0^+}\mathbf{x}^\infty_1(t)=\infty$.

Proof. Points (i) and (iii) are easy consequences of the construction. Point (ii) is an immediate consequence of (2.12) in Proposition 2.6.

Let us prove (iv). By (iii), $x\in[0,\infty)\mapsto \mathbf{x}^x_1(t)$ is nondecreasing. We denote by Ω_0 the event that $\mathbf{\Pi}\cap((0,\varepsilon)\times[0,\infty))\neq\emptyset$ for all $\varepsilon\in(0,1)$. Since $\mu((0,\varepsilon)\times[0,\infty))=\infty$, elementary results on PPP imply that $\mathbf{P}(\Omega_0)=1$ and we next argue deterministically on Ω_0. By (iii) observe that $x\mapsto T^{(x)}_1$ is strictly positive and nonincreasing. Moreover on Ω_0, for any $t\in(0,1)$, there is x_0 such that $T^{(x_0)}_1<t$ and (iii) implies that $\mathbf{x}^x_1(t)=\mathbf{x}^{x_0}_1(t)$ for all $x\geq x_0$. This shows that $\lim_{x\to\infty}\mathbf{x}^x_1(t)$ exists in $(0,\infty)$. The last two points of (iv) are readily checked on Ω_0. ∎

We view $[0,\infty]$ as the usual compactification of $[0,\infty)$. We denote by $C([0,\infty])$ the space of $\mathbb{R}$-valued continuous functions on $[0,\infty]$. For all $x\in[0,\infty]$, all $s\in[0,\infty)$ and all $f\in C([0,\infty])$, we then set

$$X^x_s=e^s\mathbf{x}^x_1(1-e^{-s}) \quad \text{and} \quad P_sf(x)=\mathbf{E}\big[f\big(X^x_s\big)\big]\,. \tag{2.19}$$

Informally, as specified in the very definition of the model, $\mathbf{x}_1$ evolves linearly between two jump times and the probability that it jumps during time interval $[t,t+\mathrm{d}t]$ under $\mathbf{P}(\,\cdot\,|\mathbf{x}_1(t)=y)$ is $\frac{2y\,\mathrm{d}t}{(1-t)^2}$. Therefore, the process X evolves exponentially between two jump times and the probability that it jumps during time interval $[s,s+\mathrm{d}s]$ under $\mathbf{P}(\,\cdot\,|X_s=y)$ is equal to $2y\,\mathrm{d}s$.

Lemma 2.9. *We keep the previous notation. The processes* X^x *are right-continuous with left limits* $[0,\infty]$*-valued Markov processes whose semi-group* $(P_s)_{s\in[0,\infty)}$ *is Feller. Moreover for all* $x\in[0,\infty]$ *and all* $s\in(0,\infty)$, X^x_{s-} *and* X^x_s *belong to* $(0,\infty)$.

Proof. Let $f\in C([0,\infty])$. Note that the X^x are right-continuous with left limits by definition (and by Lemma 2.8 (iv) for $x=\infty$). It follows from dominated convergence that $s\in[0,\infty)\mapsto P_sf(x)$ is right-continuous and that $\lim_{s\to 0^+}P_sf(x)=f(x)$ for all $x\in[0,\infty]$. Then, observe that $x\in[0,\infty]\mapsto X^x_s$ is continuous by Lemma 2.8 (iii) on $[0,\infty)$ and by Lemma 2.8

(iv) at ∞. This proves that $P_s f \in C([0,\infty])$. Next, fix $x, s_0, s_1 \in [0,\infty)$. The time-branching property stated in Proposition 2.6 immediately implies that $P_{s_0}(P_{s_1} f)(x) = P_{s_0+s_1} f(x)$. By letting x tend to ∞, we get $P_{s_0}(P_{s_1} f) = P_{s_0+s_1} f$ on $[0,\infty]$, which easily implies that the process is a Markov process with a Feller semi-group.

Observe that $\mathbf{x}_1^0(t) \le \mathbf{x}_1^x(t)$, that $t = \mathbf{x}_1^0(t)$ if $t \in [0, T_1^{(0)})$ and that $0 < \min\{Y_n^{(0)}; n \in \mathbb{N}^* : T_n^{(0)} \le t\} \le \mathbf{x}_1^0(t)$ for all $t \ge T_1^{(0)}$. This proves that the value 0 can only be taken at time 0 and that for all $t \in (0,1)$, the left limit $\mathbf{x}_1^x(t^-)$ is also strictly positive. We see from Lemma 2.8 (iv) that ∞ can only be taken at time 0. We have also proved for any $t \in (0,1)$ that there exists $x_0 \in [0,\infty)$ such that $\mathbf{x}_1^{x_0}(t') = \mathbf{x}_1^{\infty}(t')$ for all $t' \in (\frac{1}{2}t, 1)$; this implies that the left limit $\mathbf{x}_1^{\infty}(t^-)$ exists in $(0,\infty)$, which completes the proof of the lemma. ∎

Definition 2.10. A $[0,\infty]$-valued Markov process with Feller semi-group $(P_s)_{s\in[0,\infty)}$ is called the *genealogically typical (homogeneous) Markovian cell evolution*.

Convention. We denote generically such a process by $(X_s)_{s\in[0,\infty)}$ and we use the subscript $x \in [0,\infty]$ in $\mathbf{P}_x$ and $\mathbf{E}_x$ to mean that we consider a genealogically typical homogeneous Markovian cell evolution X with initial value x. Namely, $\mathbf{P}_x(X_0 = x) = 1$. □

The following lemma provides a simple description of the genealogically typical homogeneous Markovian cell evolution. Let us mention that the auxiliary variables $\mathcal{E}_n$ and $\mathcal{E}'_n$ are only used for practical reasons to facilitate the definition.

Lemma 2.11. *Let $x \in [0,\infty)$. Then, the following holds true.*

(i) *With $\mathbf{P}_x$-probability one, the jump times $(S_n)_{n\in\mathbb{N}^*}$ of X form an unbounded discrete subset of $[0,\infty)$. Namely, if we set $S_0 = 0$, then $S_{n+1} = \inf\{s \in (S_n,\infty) : X_s \ne X_{s-}\}$, $n \in \mathbb{N}$ and $\mathbf{P}_x$-a.s. $\lim_{n\to\infty} S_n = \infty$.*

(ii) *For all $n \in \mathbb{N}$ and all $s \in [S_n, S_{n+1})$, $X_s + 1 = e^{s-S_n}(X_{S_n} + 1)$.*

(iii) *For all $s \in [0,\infty)$, $X_s \le e^s(X_0+1) - 1$.*

(iv) *Let $U_n, \mathcal{E}_n, \mathcal{E}'_n$, $n \in \mathbb{N}^*$, be independent r.v. such that U_n is uniformly distributed on $[0,1]$ and $\mathbf{P}_x(\mathcal{E}_n > s) = e^{-2s}$ and $\mathbf{P}_x(\mathcal{E}'_n > s) = (s+1)^2 e^{-2s}$ for all $s \in [0,\infty)$. We define $(D_n, Z_n)_{n\in\mathbb{N}}$ by setting $(D_0, Z_0) = (0, x)$ and*

$$\forall n \in \mathbb{N}^*, \quad D_n = \log\Big(1 + \big(\tfrac{1}{Z_{n-1}}\mathcal{E}_n\big) \wedge \mathcal{E}'_n\Big) \quad \textit{and} \quad Z_n = U_n\big(e^{D_n}(Z_{n-1}+1) - 1\big).$$

Then, $(D_n, Z_n)_{n\in\mathbb{N}^*}$ and $(S_n - S_{n-1}, X_{S_n})_{n\in\mathbb{N}^*}$ have the same law under $\mathbf{P}_x$. In particular, for all measurable function $f : [0,\infty)^2 \to [0,\infty)$,

$$\mathbf{E}_x\big[f(S_1, X_{S_1})\big] = \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}u \; 2(e^s(x+1)-1) e^{2s - 2(x+1)(e^s-1)} f\big(s, u(e^s(x+1)-1)\big) \tag{2.20}$$

and $\mathbf{P}_x(S_1 > s) = \exp\big(2s - 2(e^s-1)(x+1)\big)$, for all $s \in [0,\infty)$.

Proof. We use notation X^x as defined in (2.19). First note that $\mathbf{x}_1^x$ (and therefore X^x) has infinitely many jumps because first $\mathbf{x}_1^x$ grows linearly at unit speed between two times of jumps because for any $t_0 \in [0,1)$, $\mathbf{\Pi}$ has infinitely many points (y,t) such that $y \le \mathbf{x}_1^x(t_0) + t - t_0$. Next observe that a jump time of X^x belongs to $\{-\log(1 - T_n^{(x)}); n \in \mathbb{N}\}$ and that $\lim_{n\to\infty} T_n^{(x)} = 1$, which easily completes the proof of (i). Since $\mathbf{x}_1^x$ grows linearly at unit speed between two jump times, we get (ii). We easily derive (iii) from (ii) by induction on n.

Let us prove (iv). To that end, we first relate $\mathcal{E}_1$ and $\mathcal{E}'_1$ to the Poissonian model as follows: for all $(x,t) \in [0,\infty) \times [0,1)$, we set $T_{x,t} = \{(r,y) : 0 \le s \le t, x \le y \le x+s\}$ and

$$Z_1 = \inf\{t > 0 : \#\Pi \cap ([0,t] \times [0,x]) \ne \emptyset\} \quad \text{and} \quad Z_2 = \inf\left\{t > 0 : \Pi \cap T_{x,t} \ne \emptyset\right\}.$$

First observe that $T_1^{(x)} = Z_1 \wedge Z_2$. Elementary computations on PPP imply $\mathbf{P}(Z_1 > t) = \exp\big(-\frac{2xt}{1-t}\big)$ and $\mathbf{P}(Z_2 > t) = (1-t)^{-2}\exp\big(-\frac{2t}{1-t}\big)$. Thus $(\mathcal{E}_1, \mathcal{E}'_1) = \big(\frac{Z_1}{x(Z_1+1)}, \frac{Z_2}{(Z_2+1)}\big)$ and since $S_1 = -\log(1 - T_1^{(x)})$, S_1 has the same law as D_1. Since $X_{S_1} = Y_1^{(x)}/(1 - T_1^{(x)})$, a simple computation based on Lemma 2.8 (ii) implies (2.20) and that (S_1, X_{S_1}) has the same law as (D_1, Z_1). Then (iv) is a consequence of (ii) and of recursive use of the strong Markov property at the stopping times $(S_n)_{n\in\mathbb{N}^*}$. Let us mention that all the r.v. $(\mathcal{E}_n, \mathcal{E}'_n)$, $n \ge 2$, can be also related to quantites in the Poissonian model. But we don't need to make explicit a full correspondence here. ∎

In the following proposition (whose proof is postponed to Section 2.7) we explicitly compute the law of X_s under $\mathbf{P}_x$ and we provide basic analytic results on the semi-group $(P_s)_{s\in[0,\infty)}$ in connection with the following operator defined for all functions f that are C^1 on $[0,\infty)$

$$\forall x\in[0,\infty), \quad Lf(x) = (x+1)f'(x) - 2xf(x) + 2\int_0^x f(y)\,\mathrm{d}y\,. \tag{2.21}$$

Proposition 2.12. *We keep the previous notation. Let $f\in C^2([0,\infty))$. Then, the following holds true.*

(i) *The function $(x,s)\in[0,\infty)^2\mapsto P_sf(x)$ is $C^{1,1}$ and*

$$\forall x,s\in[0,\infty), \quad \partial_s P_s f(x) = L(P_s f)(x) = P_s(Lf)(x)\,. \tag{2.22}$$

(ii) *For all $x,s\in[0,\infty)$, $\mathbf{P}_x(X_s\in\mathrm{d}y) = e^{2s-2(x+1)(e^s-1)}\,\delta_{(x+1)e^s-1} + \nu_1^{(s)} + \nu_2^{(s)}$, where*

$$\nu_1^{(s)}(\mathrm{d}y) := 2y(y+1)e^{-2y}\,\mathbf{1}_{[0,\,\mathrm{e}^s-1]}(y)\,\mathrm{d}y \quad \textit{and}$$
$$\nu_2^{(s)}(\mathrm{d}y) := 2(e^s-1)e^s\,e^{-2(y+1)(1-e^{-s})}\,\mathbf{1}_{[e^s-1,\,(x+1)e^s-1]}(y)\,\mathrm{d}y\,.$$

(iii) *The probability measure $\nu(\mathrm{d}y) = 2y(y+1)e^{-2y}\mathrm{d}y$ is the the unique law that is invariant under $(P_s)_{s\in[0,\infty)}$. Moreover, for all bounded and measurable $\phi\colon[0,\infty)\to\mathbb{R}$,*

$$\sup_{x\in[0,\infty)} \big|\langle \nu,\phi\rangle - \mathbf{E}_x[\phi(X_s)]\big| \leq 5\|\phi\|_\infty e^{-2(e^s-1-s)} \tag{2.23}$$

Proof. See Section 2.7. ■

2.4. The homogeneous Markovian HR growth-fragmentation process

In this section, we index the HR process $(\mathbf{x}(t))_{t\in[0,1)}$ in a more convenient way and make the necessary time and space changes in order to get a branching Markov process.

Indeed, as already mentioned, a disadvantage of the cell indexing $(\mathbf{x}_k(t))_{1\leq k\leq \mathbf{n}_t}$ is that the numbering of a cell changes even though it does not undergo any cell division. To overcome this problem, *we index the cells by their genealogy* . Note that in our description of the model, a cell that undergoes cell division *dies and gives birth to two new cells.* So the family tree of the cell population is simply the binary tree $\mathbb{T}_2$ that we see as the set of finite words written in the alphabet of two letters $\{1,2\}$. The ancestral cell is indexed by the empty word; the two children of the ancestral cell are indexed by the respective one-letter words (1) and (2), with the cell labeled (2) lying on top of the one labeled (1) in the graphical representation explained earlier. More generally, a cell labeled $u\in\mathbb{T}_2$ has two children labeled by the words $u*(1)$ and $u*(2)$ that are the two words with prefix u to which the letters 1 and 2 have been added respectively; here the cell indexed by $u*(2)$ is located on top of the one indexed by $u*(1)$ in the graphical representation. Note that the *lexicographical order* of the labels of the cell living at time t corresponds to the increasing indexation $k\in\{1,\dots,\mathbf{n}_t\}\mapsto\mathbf{x}_k(t)$. (See Figure 3 for a simple visual explanation)

Remark. In [3], Bertoin's way of indexing the cells to define their genealogy differs slightly from ours: In Bertoin's framework, cells are immortal and they are indexed by the words $v\in\bigcup_{n\in\mathbf{N}}(\mathbb{N}^*)^n$. More precisely, at a division-time, a parent-cell indexed by the word v does not die but goes from a size x to a size Ux (here, U is uniform on $[0,1]$) and gives birth to a child-cell of size $(1-U)x$ indexed by the word $v*(n)$ if it is the n-th child of the parent-cell v. Our framework and Bertoin's framework are equivalent ways to define a genealogy for the population of cells evolving according to the same dynamics (in particular it does not influence the set of sizes of the living cells at time s). The symmetry of the binary indexation suits better our purpose. □

The binary tree, notation. Before explaining the indexation more precisely, let us first introduce some notation about the *binary tree*

$$\mathbb{T}_2 = \bigcup_{n\in\mathbb{N}}\{1,2\}^n\,.$$

Here, we adopt the convention that $\{1,2\}^0 = \{\varnothing\}$ where $\varnothing$ is the empty word which is taken as the *root* of the binary tree $\mathbb{T}_2$. The vertices of $\mathbb{T}_2$ are generically denoted by the letters u, v, or w. Let $u=(a_1,\dots,a_n)\in\mathbb{T}_2$.

– We denote by $|u|=n$ its *length* or its *height* (in the genealogy) by adopting the conventions $|\varnothing|=0$.

– Let $v=(b_1,\dots b_m)\in\mathbb{T}_2$. Let us denote by $w=u*v=(a_1,\dots,a_n,b_1,\dots,b_m)\in\mathbb{T}_2$ the *concatenation* of u and v.

– We also define the operator θ_u consisting in removing u from words having u as prefix: $\theta_u w=v$ if $w=u*v$.

We observe that $u*\varnothing=\varnothing*u=u$ and that $|u*v|=|u|+|v|$.

– Let $k\in\mathbb{N}$, we note $u_{|k}=(a_1,\dots,a_{n\wedge k})$, with the convention $u_{|0}=\varnothing$. If $k\geq|u|=n$, then $u_{|k}=u$; if $k<|u|=n$, then $u_{|k}$ is the ancestor of u (in the broad sense) at generation k.

– The direct ancestor of u, also called the parent of u, is therefore $u_{|k}$ with $k=|u|-1$ and we denote it by $\overleftarrow{u}$ (we observe that it is well defined only when u is distinct from $\varnothing$).

– The lineage of u is noted $[\![\varnothing,u]\!]=\{u_{|k};0\leq k\leq|u|\}$ with the following additional notation: $]\!]\varnothing,u]\!]=[\![\varnothing,u]\!]\backslash\{\varnothing\}$, $[\![\varnothing,u[\![=[\![\varnothing,u]\!]\backslash\{u\}$ and $]\!]\varnothing,u[\![=[\![\varnothing,u]\!]\backslash\{\varnothing,u\}$.

– Let $u,v\in\mathbb{T}_2$. We note $u\wedge v\in\mathbb{T}_2$ the most recent common ancestor of u and v: $u\wedge v=u_{|m}=v_{|m}$ where $m=\sup\{k\in\mathbb{N}:u_{|k}=v_{|k}\}$.

The tree-indexed homogeneous Markovian cell population. Let $\Pi\in\mathscr{P}$ satisfy (2.3). Let $x\in[0,\infty)$ and let $(\mathbf{x}(t))_{t\in[0,1)}=\Phi(x,\Pi)$ the branching process generated by x and Π as defined in Section 2.1. In particular, $(\mathbf{n},\mathbf{m})$ is the solution of $\mathtt{Eq}(0,\Pi)$ with initial condition $(\mathbf{n}_0,\mathbf{m}_0)=(1,x)$ and recall from (2.6) that $(T_n,Y_n)_{n\geq1}$ is the indexation of $\Pi_{\leq x}=\{(t,y)\in\Pi:y\leq\mathbf{m}_t\}$ such that $T_n<T_{n+1}$ with the convenient convention that $(T_0,Y_0)=(0,x)$. Recall from `Def(3)` that for all $n\in\mathbb{N}^*$, the integer $k_n\in\{1,\dots,n\}$ is such that

$$\sum_{1\leq k<k_n}\mathbf{x}_k(T_n^-)<Y_n\leq\sum_{1\leq k\leq k_n}\mathbf{x}_k(T_n^-)\ .$$

Namely, the n-th cell division occurs at time T_n and at time T_n, the k_n-th cell is the sole cell undergoing a division.

We now work within the time scale $s=-\log(1-t)$ of the genealogically typical homogeneous Markovian cell evolution and we set

$$\forall n\in\mathbb{N},\quad \sigma_n=-\log(1-T_n)\ .$$

For any $s\in[0,\infty)$, we can now define G_s, the *generation at time* s as the random subset of $\mathbb{T}_2$ labelling the living cells at time s. It is formally defined as follows.

`Def(5)` *$G_0=\{\varnothing\}$ and for all $n\in\mathbb{N}$, $\#G_{\sigma_n}=n+1$ and $G_s=G_{\sigma_n}$ for all $s\in[\sigma_n,\sigma_{n+1})$.*

`Def(6)` *Let $n\in\mathbb{N}^*$. We denote by u_n the k_n-th largest word of $G_{\sigma_{n-1}}$ in the lexicographical order on $\mathbb{T}_2$. Then,*

$$G_{\sigma_n}=\big(G_{\sigma_{n-1}}\backslash\{u_n\}\big)\cup\big\{u_n*(1),u_n*(2)\big\}$$

The assumptions in (2.3) guarantee that every cell dies and undergoes a division, which implies that

$$\mathbb{T}_2=\bigcup_{s\in[0,\infty)}G_s\ .$$

Therefore the *time of death* of the cell labeled by $u\in\mathbb{T}_2$ in the homogeneous Markovian time scale is well-defined by

$$\zeta_u=\sup\big\{s\in[0,\infty):u\in G_s\big\}. \tag{2.24}$$

Namely, ζ_u is the time (in the homogeneous Markovian time scale) when the cell u divides. We therefore get $\{\zeta_u;u\in\mathbb{T}_2\}=\{\sigma_n;n\geq1\}$. The *time of birth* of the cell labeled by u (in the homogeneous Markovian time scale) is the time of death of its direct parent that is labeled by $\overleftarrow{u}$. Namely, the time of birth of u is $\zeta_{\overleftarrow{u}}$. Of course, note that $\zeta_{\overleftarrow{u}}<\zeta_u$ and that

$$\forall s\in[0,\infty),\quad G_s=\{u\in\mathbb{T}_2:\zeta_{\overleftarrow{u}}\leq s<\zeta_u\}\ . \tag{2.25}$$

The HR process $t\in[0,1)\mapsto(\mathbf{x}_k(t))_{1\leq k\leq\mathbf{n}_t}$ is rescaled and reindexed as follows. We first set

$$N_s=\mathbf{n}_{1-e^{-s}}\quad\text{and}\quad \mathcal{M}_s(\mathrm{d}y)=\sum_{1\leq k\leq N_s}\delta_{e^s\mathbf{x}_k(1-e^{-s})}(\mathrm{d}y)\ . \tag{2.26}$$

Note that $\#G_s=N_s$. Then, for all $u\in\mathbb{T}_2$, we define the processes $(X_u(s))_{s\in[0,\infty)}$ as follows.

`Def(7)` *Let $u \in \mathbb{T}_2$ and $s \in [0,\infty)$ such that $\zeta_{\overleftarrow{u}} \le s < \zeta_u$. Let k be the rank of u in G_s with respect to the lexicographical order on $\mathbb{T}_2$. Let us note that $1 \le k \le N_s$, which allows us to set*

$$X_u(s) = e^s \mathbf{x}_k(1 - e^{-s}) . \tag{2.27}$$

`Def(8)` *We extend the process X_u to $[0,\infty)$ by setting $X_u(s) = X_u(\zeta_u^-)$ for $s \in [\zeta_u, \infty)$ and*

$$\forall s \in [0, \zeta_{\overleftarrow{u}}), \quad X_u(s) = \sum_{v \in \rrbracket \varnothing, u \llbracket} X_v(s) \, \mathbf{1}_{\{\zeta_{\overleftarrow{v}} \le s < \zeta_v\}} .$$

Then, for all $u \in \mathbb{T}_2$, $X_u : [0,\infty) \to [0,\infty)$ is a càdlàg process. If $s < \zeta_u$, $X_u(s)$ is the size of the cell of the lineage of u that is alive at time s. If $s \ge \zeta_u$, $X_u(s)$ is the size of the cell labeled by u by the time of its death. The times ζ_v, $v \in \rrbracket \varnothing, u \llbracket$ are exactly the jumps times of the process $X_u(\cdot)$. Now recall that between two times of cell division, $t \mapsto \mathbf{x}_k(t)$ grows linearly. Therefore, by (2.27) in `Def(7)`, for all $v \in \rrbracket \varnothing, u \rrbracket$, the process $X_u(\cdot)$ grows exponentially on $[\zeta_{\overleftarrow{v}}, \zeta_v)$. Namely,

$$\forall s \in [\zeta_{\overleftarrow{v}}, \zeta_v), \quad X_u(s) + 1 = e^{s - \zeta_{\overleftarrow{v}}} \big(X_u(\zeta_{\overleftarrow{v}}) + 1 \big) . \tag{2.28}$$

In particular, we can deterministically reconstruct the $X_u(s)$, $u \in \mathbb{T}_2$, $s \in [0,\infty)$ from the r.v.

$$\big(\zeta_u - \zeta_{\overleftarrow{u}}, X_u(\zeta_{\overleftarrow{u}}) \big)_{u \in \mathbb{T}_2} . \tag{2.29}$$

If we see $(X_u(s); s \in [0,\infty), u \in \mathbb{T}_2)$ as a process indexed by $[0,\infty) \times \mathbb{T}_2$, then it is easy to verify that the finite dimensional marginals of this process are measurable functions of the process $\mathbf{x} : t \in [0,1) \mapsto (\mathbf{x}_k(t))_{1 \le k \le \mathbf{n}_t}$. So there exists a measurable function Ψ such that

$$\mathbf{X} := \big(X_u(s); s \in [0,\infty), u \in \mathbb{T}_2 \big) = \Psi(\mathbf{x}) . \tag{2.30}$$

Definition 2.13. Let $\mathbf{x}$ be a HR process with initial value $x \in [0,\infty)$ as in Definition 2.4. A process $\mathbf{X} = (X_u(s); s \in [0,\infty), u \in \mathbb{T}_2)$ that has the same law as $\Psi(\mathbf{x})$ is referred to as a *HR growth-fragmentation* process with initial size x.

Convention. We use the subscript x in $\mathbf{P}_x$ and $\mathbf{E}_x$ to mean that the HR growth-fragmentation process has an initial value x. Namely, $\mathbf{P}_x(X_\varnothing(0) = x) = 1$. □

The time-branching renewal property can be rewritten as follows.

Proposition 2.14. *Let $\mathbf{X} = (X_u)_{u \in \mathbb{T}_2}$ be a HR growth-fragmentation process with initial size $x \in [0,\infty)$. For all $s \in [0,\infty)$, we denote by $\mathscr{F}_s$ the sigma field generated by the r.v. $X_u(r)$, $r \in [0,s]$, $u \in \mathbb{T}_2$. We also set,*

$$\forall u \in \mathbb{T}_2, \quad \theta_{u,s} \mathbf{X} = (X_{u*v}(s + \cdot))_{v \in \mathbb{T}_2} . \tag{2.31}$$

For all $s \in [0,\infty)$, recall from (2.25) the definition of G_s and recall from (2.26) the definition of $\mathcal{M}_s$. Then, the following holds true.

- (i) *$\zeta_\varnothing$ is an $(\mathscr{F}_s)_{s \in [0,\infty)}$-stopping time. Moreover, conditionally on $\mathscr{F}_{\zeta_\varnothing}$, $\theta_{(1),\zeta_\varnothing} \mathbf{X}$ and $\theta_{(2),\zeta_\varnothing} \mathbf{X}$ are independent and they are distributed as HR growth-fragmentation processes with respective initial values $X_{(1)}(\zeta_\varnothing)$ and $X_{(2)}(\zeta_\varnothing)$. Moreover, for all bounded measurable $f : [0,\infty)^3 \to \mathbb{R}$,*

$$\begin{aligned} &\mathbf{E}_x\big[f\big(\zeta_\varnothing, X_{(1)}(\zeta_\varnothing), X_{(2)}(\zeta_\varnothing)\big)\big] = \\ &\int_0^\infty \! \mathrm{d}s \int_0^1 \! \mathrm{d}u \;\, 2(e^s(x+1)-1) e^{2s - 2(x+1)(e^s - 1)} f\big(s, u(e^s(x+1)-1), (1-u)(e^s(x+1)-1)\big). \end{aligned} \tag{2.32}$$

- (ii) (Time-branching renewal property) *Let $s_0 \in [0,\infty)$. Then, G_{s_0} is $\mathscr{F}_{s_0}$-mesurable and conditionally on $\mathscr{F}_{s_0}$, the processes $\theta_{u,s_0} \mathbf{X}$, $u \in G_{s_0}$, are independent and $\theta_{u,s_0} \mathbf{X}$ is a HR growth-fragmentation process with initial size $X_u(s_0)$.*

- (iii) *The process $(\mathcal{M}_s)_{s \in [0,\infty)}$ is a measure-valued Markov process in the following sense. For all $s_0, s \in [0,\infty)$ and for all $u \in G_{s_0}$, set*

$$G_s^u = \{ v \in \mathbb{T}_2 : u * v \in G_{s_0 + s} \} \quad \textit{and} \quad \mathcal{M}_s^u(\mathrm{d}y) = \sum_{v \in G_s^u} \delta_{X_{u*v}(s_0+s)}(\mathrm{d}y). \tag{2.33}$$

Then, $\mathcal{M}_{s_0+s} = \sum_{u \in G_{s_0}} \mathcal{M}^u_s$ *where conditionally on* $\mathscr{F}_{s_0}$*, the processes* $(\mathcal{M}^u_s)_{s\in[0,\infty)}$*,* $u \in G_{s_0}$ *are independent and* $(\mathcal{M}^u_s)_{s\in[0,\infty)}$ *has the same law as* $(\mathcal{M}_s)_{s\in[0,\infty)}$ *under* $\mathbf{P}_{X_u(s_0)}$*.*

Proof. We set $t_0 = 1 - e^{-s_0}$. By (2.26), $\mathbf{n}_{t_0} = N_{s_0} = \# G_{s_0}$. Let $u_1, \ldots, u_{N_{s_0}}$ be the elements of G_{s_0} listed in increasing lexicographical order. Let $k \in \{1, \ldots, N_{s_0}\}$. Then, by `Def(7)`, we get $X_{u_k}(s_0) = \mathbf{x}_k(t_0)$. Recall from (2.11) the definition of the rescaled HR process $\theta_{k,t_0}\mathbf{x}$. Then, we easily observe that $\Psi(\theta_{k,t_0}\mathbf{x}) = \theta_{s,u_k}\mathbf{X}$ and Proposition 2.6 completes the proof of (i) and (ii) (and (2.32) is an immediate consequence of (2.12)). Then the Markov property is an immediate consequence of the time-branching property. ■

A direct construction. Let $\mathbf{X} = (X_u(s); s \in [0,\infty), u \in \mathbb{T}_2)$ be a HR growth-fragmentation process with initial size $x \in [0,\infty)$. It is clear that the size of the first cell $X_\varnothing$ follows the genealogically typical homogeneous Markovian cell evolution up to its first jump time (as described in Lemma 2.11) and at time $\zeta_\varnothing$ the first cell uniformly splits into two cells of respective sizes $X_{(1)}(\zeta_\varnothing)$ and $X_{(2)}(\zeta_\varnothing)$. Then Proposition 2.14 (i) asserts that the resulting two subpopulations evolve independently as two populations with respective initial sizes $X_{(1)}(\zeta_\varnothing)$ and $X_{(2)}(\zeta_\varnothing)$. This implies the following direct construction of the HR growth-fragmentation process.

Let $(U_u, \mathcal{E}_u, \mathcal{E}'_u)_{u\in\mathbb{T}_2}$ be independent r.v. that are defined on an auxiliary probability space $(\Omega', \mathscr{F}', \mathbf{P}')$ and that are such that

$$U_u \text{ is uniform on } [0,1], \quad \mathbf{P}'(\mathcal{E}_u > s) = e^{-2s} \quad \text{and} \quad \mathbf{P}'(\mathcal{E}'_u > s) = (s+1)^2 e^{-2s}, \quad s \in [0,\infty)\,.$$

We define $(D^{(x)}_u, Z^{(x)}_u)_{u\in\mathbb{T}_2}$ by setting $Z^{(x)}_\varnothing = x$ and for all $u \in \mathbb{T}_2$, $D^{(x)}_u = \log\Big(1 + \big(\frac{1}{Z^{(x)}_u}\mathcal{E}_u\big) \wedge \mathcal{E}'_u\Big)$,

$$Z^{(x)}_{u*(1)} = U_u\big(e^{D^{(x)}_u}(Z^{(x)}_u + 1) - 1\big) \quad \text{and} \quad Z^{(x)}_{u*(2)} = (1-U_u)\big(e^{D^{(x)}_u}(Z^{(x)}_u + 1) - 1\big).$$

If $x = 0$, then we set $D^{(0)}_\varnothing = \mathcal{E}'_\varnothing$. Then,

$$\big(D^{(x)}_u, Z^{(x)}_u\big)_{u\in\mathbb{T}_2} \text{ under } \mathbf{P}' \overset{\text{(law)}}{=} \big(\zeta_u - \zeta_{\overleftarrow{u}}, X_u(\zeta_{\overleftarrow{u}})\big)_{u\in\mathbb{T}_2} \text{ under } \mathbf{P}_x. \tag{2.34}$$

For all $(x,s) \in [0,\infty)^2$ and for all $u \in \mathbb{T}_2$, we set $\zeta^{(x)}_\varnothing = 0$, $\zeta^{(x)}_u = \sum_{v \in \rrbracket \varnothing, u \rrbracket} D^{(x)}_v$ and

$$X^{(x)}_u(s) = \begin{cases} \sum_{v\in\rrbracket\varnothing,u\rrbracket} \big(e^{s-\zeta^{(x)}_{\overleftarrow{v}}}(Z^{(x)}_v + 1) - 1\big)\mathbf{1}_{\{\zeta^{(x)}_{\overleftarrow{v}} \le s < \zeta^{(x)}_v\}} & \text{if } s \in [0, \zeta^{(x)}_u) \\ X^{(x)}_u(\zeta^{(x)}_u -) & \text{if } s \in [\zeta^{(x)}_u, \infty). \end{cases}$$

Proposition 2.15. *Let* $\mathbf{X} = (X_u(s); s \in [0,\infty), u \in \mathbb{T}_2)$ *be a HR growth-fragmentation process with initial size* $x \in [0,\infty)$ *and let* $(X^{(x)}_u)_{u\in\mathbb{T}_2}$ *be defined as above on the auxiliary probability space* $(\Omega', \mathscr{F}', \mathbf{P}')$*. Then, the following holds true.*

(i) *The system* $\big(X^{(x)}_u(s); s\in[0,\infty), u\in\mathbb{T}_2\big)$ *under* $\mathbf{P}'$ *has the same law as* $\mathbf{X}$ *under* $\mathbf{P}_x$*.*

(ii) *For all* $s, x \in [0,\infty)$*, we set,*

$$G^{(x)}_s = \{u \in \mathbb{T}_2 : \zeta^{(x)}_{\overleftarrow{u}} \le s < \zeta^{(x)}_u\} \quad \textit{and} \quad \mathcal{M}^{(x)}_s(\mathrm{d}y) = \sum_{u \in G^{(x)}_s} \delta_{X^{(x)}_u(s)}(\mathrm{d}y).$$

Then, for all $(s_0, x_0) \in [0,\infty)^2$*,* $\mathbf{P}'$-a.s. $G^{(x)}_s = G^{(x_0)}_{s_0}$ *for all* (s,x) *sufficiently close to* (s_0,x_0) *and for all* $u \in \mathbb{T}_2$*,* $(s,x) \mapsto X^{(x)}_u(s)$ *is continuous at* (s_0, x_0)*.*

Proof. Note that (i) is a consequence of (2.34), of the remark concerning (2.29), of (2.28) and of `Def(8)`. Let us prove (ii). To this end, first observe that for all $u \in \mathbb{T}_2$, the law of ζ_u is diffuse. Therefore

$$\mathbf{P}'\text{-a.s. for all } u \in \mathbb{T}_2, \quad \mathbf{1}_{\{\zeta^{(x_0)}_{\overleftarrow{u}} \le s < \zeta^{(x_0)}_u\}} = \mathbf{1}_{\{\zeta^{(x_0)}_{\overleftarrow{u}} < s < \zeta^{(x_0)}_u\}}.$$

Next, observe that $x \mapsto (D^{(x)}_u, Z^{(x)}_u)$ is continuous for all $u \in \mathbb{T}_2$, which readily completes the proof of (ii). ■

2.5. Proof of Lemma 2.3

We prove this result by approximation. We fix $n\in\mathbb{N}^*$, $\varepsilon, t_0\in(0,1)$ and $s_0=0<s_1<\ldots<s_n<t_0=s_{n+1}<1$. We assume that $\varepsilon<\frac{1}{n+1}\min_{0\leq k\leq n}(s_{k+1}-s_k)$. Recall from (2.5) the definitions of $\mathbf{\Pi}_{\leq x}$ and of $\mathbf{\Pi}_{>x}$. Recall from (2.6) that $(T_n, Y_n)_{n\in\mathbb{N}^*}$ stands for the enumeration of $\mathbf{\Pi}_{\leq x}$ such that $T_n<T_{n+1}$. Then, on the event

$$A_n=\big\{\mathbf{n}_{t_0}=n\,;\,T_1\in[s_1,s_1+\varepsilon]\,;\,\ldots\,;\,T_n\in[s_n,s_n+\varepsilon]\big\},$$

for all $t\in[0,t_0)$, we get

$$g(t):=x+\sum_{1\leq k\leq n+1}(t-s_{k-1}-\varepsilon)\mathbf{1}_{[s_{k-1}+\varepsilon,1)}(t)\;\leq\;\mathbf{m}_t\;\leq\;x+\sum_{1\leq k\leq n+1}(t-s_{k-1})\mathbf{1}_{[s_{k-1},1)}(t)=:f(t).$$

We next introduce the subsets $\mathbf{\Pi}_{>f}=\big\{(t,y-f(t))\,;(t,y)\in\mathbf{\Pi}:y>f(t)\big\}$ and $\mathbf{\Pi}_{\leq f}=\big\{(t,y)\in\mathbf{\Pi}:y\leq f(t)\big\}$. Then on the event $B_n:=A_n\cap\{\#\mathbf{\Pi}_{\leq f}=n\}$, $\mathbf{\Pi}_{>f}$ and $\mathbf{\Pi}_{>x}$ are close in the following sense: let $z\in(x,\infty)$ and let $h:[0,1)\times[0,\infty)\to[0,\infty)$ be continuous with compact support in $[0,t_0]\times[0,z]$; observe that $f(t_0)\leq x+(n+1)t_0$ and that for all $(t,y)\in[0,t_0)\times[0,\infty)$, $0\leq(y-\mathbf{m}_t)_+-(y-f(t))_+\leq f(t)-\mathbf{m}_t\leq(n+1)\varepsilon$. Then, on B_n, we get

$$|\langle\mathbf{\Pi}_{>f},h\rangle-\langle\mathbf{\Pi}_{>x},h\rangle|\leq w_h\big((n+1)\varepsilon\big)\mathbf{\Pi}(K)\;,$$

where for all $\delta\in(0,\infty)$, we have set

$$w_h(\delta)=\max\big\{\,|h(t,y)-h(t,y')|\,;\,t\in[0,t_0]\text{ and }y,y'\in[0,\infty):|y-y'|\leq\delta\big\}$$

that tends to 0 with δ and where $K=[0,t_0]\times[0,z+x+(n+1)t_0]$. Let $F:\mathbb{R}^{2n}\to[0,\infty)$ be bounded and measurable. Observe that $F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{B_n}$ only depends on $\mathbf{\Pi}_{\leq f}$. Thus, by Lemma 2.1 (with $t_0=0$), we get

$$\mathbf{E}\big[e^{-\langle\mathbf{\Pi}_{>f},h\rangle}F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{B_n}\big]=\mathbf{E}\big[e^{-\langle\mathbf{\Pi},h\rangle}\big]\mathbf{E}\big[F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{B_n}\big]$$

and the previous arguments imply

$$\begin{aligned}&\Big|\mathbf{E}\big[e^{-\langle\mathbf{\Pi}_{>x},h\rangle}F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{B_n}\big]-\mathbf{E}\big[e^{-\langle\mathbf{\Pi},h\rangle}\big]\,\mathbf{E}\big[F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{B_n}\big]\Big|\\ &\qquad\leq\|F\|_\infty\mathbf{E}\big[\big(e^{w_h((n+1)\varepsilon)\mathbf{\Pi}(K)}-1\big)\mathbf{1}_{B_n}\big]\;.\end{aligned}\tag{2.35}$$

For all $p\in\mathbb{N}^*$ and all $k\in\{1,\ldots,n\}$, we next set $T_k^{(p)}=2^{-p}\lfloor 2^pT_k\rfloor$ and for all $t\in[0,1)$, we also set $f_p(t)=x+\sum_{1\leq k\leq n+1}(t-T_{k-1}^{(p)})\mathbf{1}_{[T_{k-1}^{(p)},1)}(t)$. Then, we introduce the event

$$C_{n,p}=\Big\{\#\big\{(t,y)\in\mathbf{\Pi}:t\leq t_0\text{ and }y\leq f_p(t)\big\}=\mathbf{n}_{t_0}=n\text{ and }\min_{0\leq k\leq n}(t_0\wedge T_{k+1}-T_k)>(n+1)2^{-p}\Big\}\;.$$

Then, by (2.35) and summation over suitable dyadic numbers $s_1,\ldots,s_n$ with $\varepsilon=2^{-p}$, we get

$$\begin{aligned}&\Big|\mathbf{E}\big[e^{-\langle\mathbf{\Pi}_{>x},h\rangle}F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{C_{n,p}}\big]-\mathbf{E}\big[e^{-\langle\mathbf{\Pi},h\rangle}\big]\,\mathbf{E}\big[F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{C_{n,p}}\big]\Big|\\ &\qquad\leq\|F\|_\infty\mathbf{E}\big[e^{w_h((n+1)2^{-p})\mathbf{\Pi}(K)}-1\big]\;.\end{aligned}$$

Now observe that $\mathbf{P}$-a.s. for all sufficiently large p, $\mathbf{1}_{C_{n,p}}=\mathbf{1}_{\{\mathbf{n}_{t_0}=n\}}$. Therefore, by letting p go to ∞ in the previous inequality, we get

$$\mathbf{E}\big[e^{-\langle\mathbf{\Pi}_{>x},h\rangle}F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{\{\mathbf{n}_{t_0}=n\}}\big]=\mathbf{E}\big[e^{-\langle\mathbf{\Pi},h\rangle}\big]\mathbf{E}\big[F\big((T_k,Y_k)_{1\leq k\leq n}\big)\mathbf{1}_{\{\mathbf{n}_{t_0}=n\}}\big],$$

which implies the desired result since it holds for all n, $t_0\in[0,1)$, all nonnegative bounded measurable F and all continuous h with compact support. ∎

2.6. *Proof of Lemma 2.7*

Before proving Lemma 2.7, we briefly show the following deviation inequality for a Poisson r.v. Z with mean θ: namely,

$$\forall y\in[0,\infty), \quad \mathbf{P}\big(Z \geq y+\theta\big) \leq \exp\Big(-\frac{y^2}{2(y+\theta)}\Big). \tag{2.36}$$

Indeed, by Markov inequality $\mathbf{P}(Z\geq y+\theta)\leq\exp(\theta(e^{\lambda}-1)-\lambda(\theta+y))$, for all $\lambda\in[0,\infty)$. By taking $\lambda=\log(1+\frac{y}{\theta})$, we get $\mathbf{P}(Z\geq y+\theta)\leq\exp(-\theta h(y/\theta))$ where $h(r)=(1+r)\log(1+r)-r$. Then, note that $h(r)=\int_0^r \mathrm{d}s\int_0^s \frac{\mathrm{d}u}{1+u}\geq \frac{1}{1+r}\int_0^r \mathrm{d}s\int_0^s \mathrm{d}u=\frac{r^2}{2(1+r)}$, which implies the desired result. □

Recall that $\mathbf{\Pi}$ is a PPP with intensity measure μ given by (2.2). We first fix $t\in[0,1)$, $x\in[0,\infty)$, $k\in\mathbb{N}$ and we set $C_{k,t}=\{(s,y)\in[0,t]\times[0,\infty) : y\leq x+ks\}$. Observe that for all $r\in[0,\infty)$,

$$\mathbf{P}_x\big(\mathbf{n}_t>r\big)\leq\mathbf{P}_x\big(\mathbf{n}_t\geq\lfloor r\rfloor+1\big)\leq\mathbf{P}_x\big(\#(\mathbf{\Pi}\cap C_{\lfloor r\rfloor,t})\geq\lfloor r\rfloor\big)\,.$$

Observe that $\#(\mathbf{\Pi}\cap C_{k,t})$ is Poisson r.v. with mean

$$\mu(C_{k,t})=\int_0^t\frac{2(x+ks)ds}{(1-s)^2}=a(t)k+xb(t)\,,$$

where $b(t)=2t/(1-t)$ and $a(t)=b(t)+2\log(1-t)$. We first assume that $t\in[0,1/2]$. Note that a and b increase and that $0\leq b(t)\leq b(1/2)=2$ and $0\leq a(t)\leq a(1/2)<2-2(2^{-1}+2^{-3})=3/4$. Thus, for all $t\in[0,1/2]$, $\mu(C_{k,t})\leq\frac{3}{4}k+2x$. Let $r\geq1+16x$. Write $\lfloor r\rfloor=\frac{3}{4}\lfloor r\rfloor+2x+\frac{1}{4}\lfloor r\rfloor-2x$ and note that $\frac{1}{4}\lfloor r\rfloor-2x>0$. Therefore the deviation inequality (2.36) for Poisson r.v. implies that

$$\mathbf{P}_x\big(\#(\mathbf{\Pi}\cap C_{\lfloor r\rfloor,t})\geq\lfloor r\rfloor\big)\leq\exp\Big(-\frac{(\frac{1}{4}\lfloor r\rfloor-2x)^2}{2\lfloor r\rfloor}\Big)\leq\exp\big(-\tfrac{1}{32}\lfloor r\rfloor\big(1-\tfrac{1}{2}\big)^2\big)\leq\exp(-\tfrac{1}{256}r)\,.$$

Thus for all $t\in[0,1/2]$, and all $\lambda\in[0,1)$,

$$\begin{aligned}\mathbf{E}_x\big[e^{2^{-9}\lambda\mathbf{n}_t}\big]&=1+\int_0^\infty 2^{-9}\lambda e^{2^{-9}\lambda r}\mathbf{P}_x(\mathbf{n}_t>r)\,\mathrm{d}r\leq1+\int_0^{1+16x}2^{-9}\lambda e^{2^{-9}\lambda r}\,\mathrm{d}r+\int_{1+16x}^\infty 2^{-9}\lambda e^{2^{-9}\lambda r}e^{-2^{-8}r}\,\mathrm{d}r\\&\leq e^{2^{-9}\lambda(1+16x)}+\frac{\frac{1}{2}\lambda}{1-\frac{1}{2}\lambda}\leq\frac{1}{1-\frac{1}{2}\lambda}e^{2^{-9}\lambda(1+16x)}\leq e^{-\log(1-\frac{1}{2}\lambda)+2^{-9}\lambda+2^{-5}\lambda x}\leq e^{2\lambda+2^{-5}\lambda x}\end{aligned}$$

since $-\log(1-\frac{1}{2}\lambda)\leq\lambda$ for all $\lambda\in[0,1)$. We then observe that $\mathbf{m}_t\leq x+t\mathbf{n}_t\leq x+\frac{1}{2}\mathbf{n}_t$. Thus,

$$\mathbf{E}_x\big[e^{2^{-10}\lambda(\mathbf{n}_t+\mathbf{m}_t)}\big]\leq e^{2^{-10}\lambda x}\,\mathbf{E}_x\big[e^{2^{-9}\lambda\mathbf{n}_t}\big]\leq e^{2\lambda+2^{-4}\lambda x}\,.$$

To simplify notation we denote by $(\mathbf{n}',\mathbf{m}')$ the process derived from an independent copy $\mathbf{\Pi}'$ of $\mathbf{\Pi}$. By the time-branching property stated in Proposition 2.6, for all $\lambda,t_0\in[0,1)$ and all $t\in[0,1/2]$, we get $\mathbf{P}_x$-a.s. ($\mathscr{G}_{t_0}$ denoting the sigma field generated by all the atoms of $\mathbf{\Pi}$ in $[0,t_0]\times[0,\infty)$)

$$\begin{aligned}\mathbf{E}_x\big[e^{2^{-11}\lambda(\mathbf{n}_{t_0+t(1-t_0)}+\mathbf{m}_{t_0+t(1-t_0)})}\,\big|\,\mathscr{G}_{t_0}\big]&=\prod_{1\leq k\leq\mathbf{n}_{t_0}}\mathbf{E}'_{\mathbf{x}_k(t_0)}\big[e^{2^{-11}\lambda(\mathbf{n}'_t+(1-t_0)\mathbf{m}'_t)}\big]\\&\leq\prod_{1\leq k\leq\mathbf{n}_{t_0}}\mathbf{E}'_{\mathbf{x}_k(t_0)}\big[e^{2^{-11}\lambda(\mathbf{n}'_t+\mathbf{m}'_t)}\big]\\&\leq e^{\lambda\mathbf{n}_{t_0}+2^{-5}\lambda\mathbf{m}_{t_0}}\leq e^{\lambda(\mathbf{n}_{t_0}+\mathbf{m}_{t_0})},\end{aligned}\tag{2.37}$$

since $\mathbf{m}_{t_0}=\sum_{1\leq k\leq\mathbf{n}_{t_0}}\mathbf{x}_k(t_0)$. Then we set $t_n=1-2^{-n}$ and a simple recurrence based on (2.37) implies that $\mathbf{E}_x[\exp(2^{-11n}\lambda(\mathbf{n}_{t_n}+\mathbf{m}_{t_n}))]\leq\exp(\lambda(1+x))$ for all $\lambda\in[0,1)$. For all $t\in[0,1)$, there exists $n\in\mathbb{N}$ such that $t_n\leq t<t_{n+1}$, which implies that $(\frac{1}{2}(1-t))^{11}(\mathbf{n}_t+\mathbf{m}_t)\leq2^{-11(n+1)}(\mathbf{n}_{t_{n+1}}+\mathbf{m}_{t_{n+1}})$, which proves the lemma. ■

2.7. Proof of Proposition 2.12

We first prove the following lemma.

Lemma 2.16. *Let $g\in C([0,\infty])$ and let $f\in C^2([0,\infty))$. For all $x\in[0,\infty)$, we set*

$$C(f,x)=\max_{y\in[0,3(x+1)]}\big(|f(y)|+|f'(y)|+|f''(y)|\big) \tag{2.38}$$

Then, the following holds true.

(i) *The function $(x,s)\in[0,\infty]\times[0,\infty)\mapsto P_sg(x)$ is continuous.*

(ii) *For all $s\in[0,\infty)$, the function $x\in[0,\infty)\mapsto P_sf(x)$ is C^2 and*

$$\partial_x(P_sf)(x)=e^s(\partial_xf)(e^s(x+1)-1)\mathbf{P}_x(S_1>s)\ . \tag{2.39}$$

(iii) *There exists $c_0\in(0,\infty)$ such that for all $x,s_0\in[0,\infty)$ and all $s\in(0,1)$,*

$$\big|\tfrac{1}{s}\big(P_{s_0+s}f(x)-P_{s_0}f(x)\big)-P_{s_0}(Lf)(x)\big|\leq c_0\,s\,e^{2s_0}(x+1)^2C\big(f,3e^{s_0}(x+1)\big)\ . \tag{2.40}$$

Proof. Let us prove (i). Observe that for all $s\in[0,\infty)$, a.s. $\mathbf{\Pi}\cap(\{1-e^{-s}\}\times[0,\infty))=\emptyset$ and therefore a.s. for all $x\in[0,\infty]$, $s\notin\{-\log(1-T_n^{(x)});n\in\mathbb{N}^*\}$ which implies that s is a.s. not a jump time of any of the X^x, $x\in[0,\infty]$. By Lemma 2.8 we therefore see that a.s. $X_r^y\to X_s^x$ as $(r,y)\mapsto(s,x)$, which implies the desired result by dominated convergence.

Let us prove (ii). Let us first mention that $X_s^x=X_s^{x+\varepsilon}$ for $s>S_1^{(x)}$. We then set $S_1^{(x)}=-\log(1-T_1^{(x)})$ that is the first jump of X^x. Let $\varepsilon\in(0,1)$. By Lemma 2.8, $0\leq X_s^{x+\varepsilon}-X_s^x\leq e^s\varepsilon$ if $s<S_1^{(x)}$ and $X_s^{x+\varepsilon}=X_s^x$ if $s\geq S_1^{(x)}$. Since $X_s^x\leq e^s(x+1)-1$, the mean-value theorem implies

$$|f(X_s^{x+\varepsilon})-f(X_s^x)|\leq\varepsilon e^sC(f,e^s(x+1))\ . \tag{2.41}$$

Next note that a.s. for all sufficiently small ε, we get $S_1^{(x+\varepsilon)}=S_1^{(x)}$ and if $s<S_1^{(x)}$, then $X_s^{x+\varepsilon}=e^s\varepsilon+X_s^x=e^s(x+\varepsilon+1)-1$. Thus

$$\lim_{\varepsilon\to0^+}\varepsilon^{-1}(f(X_s^{x+\varepsilon})-f(X_s^x))=e^s(\partial_xf)(e^s(x+1)-1)\mathbf{1}_{\{s<S_1^{(x)}\}}\ .$$

Dominated convergence entails that P_sf has everywhere a right derivative h given by (2.39). Thanks to (2.41) and by dominated convergence, we get that $P_sf(x)=P_sf(0)+\int_0^x h(y)\,\mathrm{d}y$. Since $x\mapsto\mathbf{P}_x(S_1>s)$ is C^1 (by Lemma 2.11 (iv)), we get that h is C^1, which implies that $x\mapsto P_sf(x)$ is C^2.

Let us prove (iii). Simple bounds combined with Lemma 2.11 (iii) imply that there exists $c_1\in(0,\infty)$ such that for all $x\in[0,\infty)$ and all $s\in(0,1)$,

$$\tfrac{1}{s}\mathbf{P}_x(S_1\leq s)\leq c_1(x+1)\quad\text{and}\quad\big|2x-\tfrac{1}{s}\mathbf{P}_x(S_1\leq s)\big|\leq c_1s(x+1)^2. \tag{2.42}$$

To simplify notation, we next set $A(s,x)=s^{-1}(f(e^s(x+1)-1)-f(x))$. Note that if $s\in(0,1)$, then $e^s(x+1)-1\leq 3(x+1)$. Then, the Taylor Lagrange Formula combined with simple bounds imply that there exists $c_2\in(0,\infty)$ such that for all $x\in[0,\infty)$ and all $s\in(0,1)$ that

$$|A(s,x)|\leq c_2(x+1)C(f,x)\quad\text{and}\quad\big|A(s,x)-(x+1)\partial_xf(x)\big|\leq c_2s(x+1)^2C(f,x) \tag{2.43}$$

Observe that under $\mathbf{P}_x$, that $X_s\leq e^s(x+1)-1$ with an equality if $S_1>s$. Then,

$$s^{-1}\big(P_sf(x)-f(x)\big)=A(s,x)\mathbf{P}_x(S_1>s)+s^{-1}\mathbf{E}_x\big[\big(f(X_s)-f(x)\big)\mathbf{1}_{\{S_1\leq s\}}\big]\ .$$

If $s\in(0,1)$, then $e^s(x+1)-1\leq3(x+1)$. To simplify notation we set

$$A^*(s,x)=\tfrac{1}{s}\big(P_sf(x)-f(x)\big).$$

Thus by the first inequalities in (2.42) and (2.43), there exists $c_3\in(0,\infty)$ such that for all $x\in[0,\infty)$ and all $s\in(0,1)$

$$|A^*(s,x)|\leq c_3(x+1)C(f,x)\ . \tag{2.44}$$

By Lemma 2.11, $\mathbf{P}_x$-a.s. $X_{S_1}=U(e^{S_1}(x+1)-1)$, where U is uniform on $[0,1]$ and independent of S_1. Then, by the strong Markov property at S_1, we get

$$A^*(s,x)=\mathbf{E}_x\big[\tfrac{1}{s}(f(X_s)-f(x))\big]=A(s,x)\mathbf{P}_x(S_1>s)+\mathbf{E}_x\big[\tfrac{1}{s}\big((P_{s-S_1}f)\big(X_{S_1}\big)-f(x)\big)\mathbf{1}_{\{S_1\leq s\}}\big].$$

Then, we get $\frac{1}{s}(P_sf(x)-f(x))-Lf(x)=B_1(s)+B_2(s)+B_3(s)$, where:

(a) $B_1(s)=(A(s,x)-(x+1)\partial_x f(x))\mathbf{P}_x(S_1>s)-(x+1)\partial_x f(x)\mathbf{P}_x(S_1\leq s)$,

(b) $B_2(s)=\frac{1}{x}\big(\frac{1}{s}\mathbf{P}_x(S_1\leq s)-2x\big)\int_0^x(f(y)-f(x))\,\mathrm{d}y$,

(c) $B_3(s)=\mathbf{E}_x\big[\frac{1}{s}\big((P_{s-S_1}f)\big(X_{S_1}\big)-f(Ux)\big)\mathbf{1}_{\{S_1\leq s\}}\big]$.

By (2.42) and (2.43), there exists $c_4\in(0,\infty)$ such that $|B_1(s)|+|B_2(s)|\leq c_4 s(x+1)^2C(f,x)$. Then note that

$$B_3(s)=\tfrac{1}{s}\mathbf{E}_x\big[\big((s-S_1)A^*(s-S_1,X_{S_1})+f(X_{S_1})-f(Ux)\big)\mathbf{1}_{\{S_1\leq s\}}\big].$$

Note that under $\mathbf{P}_x$ on $\{S_1\leq s\}$, $X_{S_1}\leq e^s(x+1)-1\leq 3(x+1)$. Thus $|A^*(s-S_1,X_{S_1})|\leq 4c_3(x+1)C(f,3(x+1))$ on $\{S_1\leq s\}$. The mean-value theorem implies that $|f(X_{S_1})-f(Ux)|\leq(e^s-1)(x+1)C(f,x)$ on $\{S_1\leq s\}$. By (2.42), (2.43) and (2.44), there is $c_5\in(0,\infty)$ such that $|B_3(s)|\leq c_5 s(x+1)C(f,3(x+1))$. This, combined with the previous bounds, implies there exists $c_0\in(0,\infty)$ such that (2.40) holds with $s_0=0$.

Next note that $\mathbf{P}_x$-a.s. $X_{s_0}+1\leq e^{s_0}(x+1)$ and by the previous inequality, we $\mathbf{P}_x$-a.s. get $|\frac{1}{s}P_sf(X_{s_0})-f(X_{s_0}))-Lf(X_{s_0})|\leq c_0 se^{2s_0}(x+1)^2C(f,3e^{s_0}(x+1))$. The Markov property at time s_0 implies (2.40). ■

Proof of Proposition 2.12 (i)**.**

We deduce Proposition 2.12 (i) from Lemma 2.16. We first assume that $f\in C^2([0,\infty))$ as in the previous lemma. By (2.40) and the semi-group property we get

$$\lim_{s\to 0^+}\big|\tfrac{1}{s}\big(P_{s_0+s}f(x)-P_{s_0}f(x)\big)-P_{s_0}(Lf)(x)\big|=\lim_{s\to 0^+}\big|\tfrac{1}{s}\big(P_{s_0}f(x)-P_{s_0-s}f(x)\big)-P_{s_0-s}(Lf)(x)\big|=0.$$

Since $\lim_{s\to 0^+}P_{s_0-s}(Lf)(x)=P_{s_0}(Lf)(x)$ by Lemma 2.16 (i), we have proved that $s_0\mapsto P_{s_0}f(x)$ is C^1 and that $\partial_{s_0}P_{s_0}f(x)=P_{s_0}(Lf)(x)$. Next, since $s\mapsto P_sf$ is C^1 and by the semi-group property, we get

$$\lim_{s\to 0^+}\tfrac{1}{s}(P_{s_0+s}f(x)-P_{s_0}f(x))=\lim_{s\to 0^+}\tfrac{1}{s}(P_s(P_{s_0}f)(x)-P_{s_0}f(x))=L(P_{s_0}f)(x)\ ,$$

which completes the proof of Proposition 2.12 (i). ■

We next fix $x\in[0,\infty)$ and for all $s,\lambda\in[0,\infty)$, we set $\mathbf{a}(s,\lambda)=\mathbf{E}_x\big[e^{-\lambda X_s}\big]$. We denote by $\dot{\mathbf{a}}$ the time derivative of $\mathbf{a}$ and by $\mathbf{a}'$ the derivative with respect to λ.

Lemma 2.17. *We keep the previous notation. Then, the following holds true.*

(i) *The function* $\mathbf{a}$ *satisfies the following first order partial differential equation:*

$$\forall(s,\lambda)\in[0,\infty)\times(0,\infty),\quad \dot{\mathbf{a}}(s,\lambda)=(\lambda+2)\mathbf{a}'(s,\lambda)-\Big(\lambda+\frac{2}{\lambda}\Big)\mathbf{a}(s,\lambda)+\frac{2}{\lambda}\ , \tag{2.45}$$

with the initial condition $\mathbf{a}(0,\lambda)=\exp(-\lambda x)$.

(ii) *The unique solution of (2.45) is given for* $(s,\lambda)\in[0,\infty)\times(0,\infty)$ *by*

$$\mathbf{a}(s,\lambda)=\frac{\lambda e^{3s-x(\lambda+q)-q}}{\lambda+q}+\frac{2}{(\lambda+2)^2}+\frac{4}{(\lambda+2)^3}-\Big(\frac{2}{(\lambda+2)^2}-\frac{4}{(\lambda+2)^3}\Big)e^{-q}-\frac{8\lambda e^{-q}}{(\lambda+2)^3(\lambda+q)},$$

where $q=q(s,\lambda)=(\lambda+2)(e^s-1)$.

Proof. We first prove (i) by applying Proposition (2.12) (i) to the function $\mathbf{e}_\lambda(y)=e^{-\lambda y}$. Note that

$$L\mathbf{e}_\lambda(y)=-(\lambda+2)y\mathbf{e}_\lambda(y)-(\lambda+2\lambda^{-1})\mathbf{e}_\lambda(y)+2\lambda^{-1}=(\lambda+2)\partial_\lambda\mathbf{e}_\lambda(y)-(\lambda+2\lambda^{-1})\mathbf{e}_\lambda(y)+2\lambda^{-1}.$$

Then, observe that $\mathbf{a}(s,\lambda)=(P_s\mathbf{e}_\lambda)(x)$ that is C^1 in s by Proposition (2.12) (i). By dominated convergence, for $\lambda\in(0,\infty)$, we have $\partial_\lambda(P_s\mathbf{e}_\lambda)(x)=(P_s(\partial_\lambda\mathbf{e}_\lambda))(x)$. Then (2.22) immediately implies (2.45).

To solve (2.45), we use the method of characteristics. Namely we want to find a C^1 function $\ell:[0,\infty)\to[0,\infty)$ such that $\mathbf{a}(s,\ell_s)$ satisfies a first order ordinary differential equation. A quick computation shows the following. We set

$$\forall y,s\in\mathbb{R},\quad \ell_s(y)=(y+2)e^{-s}-2\,. \tag{2.46}$$

Note that $\ell_{s+r}(y)=\ell_s(\ell_r(y))$, for all $s,r\in\mathbb{R}$. We fix $y\in(0,\infty)$. Then $b_s=\mathbf{a}(s,\ell_s(y))$ satisfies

$$\forall s\in\big[0,\log(1+\tfrac{1}{2}y)\big),\quad \dot{b}_s=-\Big(\ell_s(y)+\frac{2}{\ell_s(y)}\Big)b_s+\frac{2}{\ell_s(y)}\quad\text{and}\quad b_0=e^{-xy}. \tag{2.47}$$

Then, we explicitly solve (2.47). To this end, we set $v_s(y)=\ell_s(y)+3s+\log\ell_s(y)$. Observe that $-\dot{v}_s(y)=\ell_s(y)+\frac{2}{\ell_s(y)}$. Thus, the unique solution of (2.47) is given by

$$\mathbf{a}\big(s,\ell_s(y)\big)=b_s=e^{v_s(y)-v_0(y)-yx}+\int_0^s\frac{2e^{v_s(y)-v_r(y)}}{\ell_r(y)}\mathrm{d}r\,,$$

that holds true for all $y\in(0,\infty)$ and $s\in[0,\infty)$ such that $\ell_s(y)>0$. In particular, it holds true for $y=\ell_{-s}(\lambda)$ since $\ell_s(y)=\ell_0(\lambda)=\lambda>0$. In this case, observe that $v_s(\ell_{-s}(\lambda))-v_r(\ell_{-s}(\lambda))=\lambda-\ell_{r-s}(\lambda)+3(s-r)+\log\lambda-\log\ell_{r-s}(\lambda)$. Next, we observe that $q(s,\lambda)+\lambda=\ell_{-s}(\lambda)$. Then, for all $(s,\lambda)\in[0,\infty)\times(0,\infty)$, we get

$$\begin{aligned}\mathbf{a}(s,\lambda)&=\frac{\lambda e^{3s-q-x(\lambda+q)}}{\lambda+q}+2\int_0^s\frac{\lambda e^{3r-q(r,\lambda)}}{(\lambda+q(r,\lambda))^2}\,\mathrm{d}r\\&=\frac{\lambda e^{3s-q-x(\lambda+q)}}{\lambda+q}+\frac{2\lambda}{(\lambda+2)^3}\int_0^q\frac{(b+\lambda+2)^2e^{-b}}{(\lambda+b)^2}\,\mathrm{d}b,\end{aligned}$$

with a change of variables $b=q(r,\lambda)$. Note that $\int_0^q(b+\lambda+2)^2(b+\lambda)^{-2}e^{-b}\,\mathrm{d}b=1-e^{-q}+4\lambda^{-1}-4(\lambda+q)^{-1}e^{-q}$, which implies the desired result. ■

Proof of Proposition 2.12 (ii)**.**

We explicitly invert the Laplace transform $\mathbf{a}(s,\lambda)=\mathbf{E}_x[e^{-\lambda X_s}]$ obtained in Lemma 2.17 (ii). To this end we introduce the following notation. Let π be a signed measure on the Borel subsets of $[0,\infty)$. Namely, there are two finite nonnegative measures π_+ and π_- on $[0,\infty)$ such that $\pi=\pi_+-\pi_-$. For all bounded measurable function f from $[0,\infty)$ to $\mathbb{R}$, we use the notation $\langle\pi,f\rangle=\langle\pi_+,f\rangle-\langle\pi_-,f\rangle$ and we also denote by

$$\forall\lambda\in[0,\infty),\quad \overline{\pi}(\lambda)=\int_{[0,\infty)}e^{-\lambda y}\pi_+(\mathrm{d}y)\;-\int_{[0,\infty)}e^{-\lambda y}\pi_-(\mathrm{d}y) \tag{2.48}$$

that is the Laplace transform of π that characterizes π. We denote by $*$ the convolution product of finite nonnegative measures on $[0,\infty)$. Let μ be a finite positive measure. Then, it makes sense to define the signed measure $\mu*\pi=\pi*\mu$ as $\mu*\pi_+-\mu*\pi_-$.

For all $n\in\mathbb{N}^*$ and all $\beta\in(0,\infty)$, we denote by $\pi_{n,\beta}$ the (n,β)-gamma distribution, namely the probability law on $[0,\infty)$ with density

$$\pi_{n,\beta}(\mathrm{d}y)=\frac{1}{(n-1)!}\beta^ny^{n-1}e^{-\beta y}\mathrm{d}y\,. \tag{2.49}$$

We also set,

$$\boldsymbol{\gamma}_n=\pi_{n,2},\qquad\text{and}\quad \boldsymbol{\pi}_1^{(s)}=\delta_0-\pi_{1,2(1-e^{-s})},\quad s\in(0,\infty)\,. \tag{2.50}$$

Then for all $y_0\in[0,\infty)$, we observe that

$$\overline{\delta_{y_0}*\pi_{n,\beta}}\,(\lambda)=\big(\tfrac{\beta}{\lambda+\beta}\big)^ne^{-\lambda y_0}\,, \tag{2.51}$$

Recall from (2.50) the notation $\boldsymbol{\gamma}_n$ and $\boldsymbol{\pi}_1^{(s)}$ and observe that (2.51) implies that

$$e^{-q(s,\lambda)}=e^{-2(e^s-1)}\overline{\delta_{e^s-1}}(\lambda), \qquad e^{-x(\lambda+q(s,\lambda))-q(s,\lambda)}=e^{-2(e^s-1)(x+1)}\overline{\delta_{(x+1)e^s-1}}(\lambda)$$
$$\text{and} \quad \frac{\lambda}{\lambda+q(s,\lambda)}=e^{-s}\,\overline{\boldsymbol{\pi}_1^{(s)}}(\lambda).$$

By Lemma 2.17 (ii), this readily implies the following.

$$\mathbf{P}_x(X_s\in \mathrm{d}y)=\tfrac{1}{2}(\boldsymbol{\gamma_2}+\boldsymbol{\gamma_3})+e^{-2(e^s-1)}\,\delta_{e^s-1}*\boldsymbol{\varpi}_s+e^{2s-2(x+1)(e^s-1)}\,\delta_{(x+1)e^s-1}*\boldsymbol{\pi}_1^{(s)}, \tag{2.52}$$

where $\boldsymbol{\varpi}_s=\frac{1}{2}(\boldsymbol{\gamma_3}-\boldsymbol{\gamma_2})-e^{-s}\boldsymbol{\gamma}_3*\boldsymbol{\pi}_1^{(s)}$. A long but straightforward computation then entails Proposition 2.12 (ii).

Proof of Proposition 2.12 (iii)**.**

Let us first prove (2.23). To that end, we use (2.52) and we first observe that $\nu=\frac{1}{2}(\boldsymbol{\gamma_2}+\boldsymbol{\gamma_3})$. If π is a signed measure, then we denote by $|\pi|$ its total variation and we recall that $\mathbf{1}$ stands for the function that is constant to 1. Let $\phi\colon[0,\infty)\to\mathbb{R}$ be bounded and measurable. Then, by (2.52) we get for all $x\in[0,\infty)$

$$\begin{aligned}\big|\langle\nu,\phi\rangle-\mathbf{E}_x[\phi(X_s)]\big| &\leq \|\phi\|_\infty\Big(e^{-2(e^s-1)}\langle|\boldsymbol{\varpi}_s|,\mathbf{1}\rangle+e^{2s-2(x+1)(e^s-1)}\langle|\boldsymbol{\pi}_1^{(s)}|,\mathbf{1}\rangle\Big)\\ &\leq e^{-2(e^s-1-s)}\|\phi\|_\infty\Big(\langle|\boldsymbol{\varpi}_s|,\mathbf{1}\rangle+\langle|\boldsymbol{\pi}_1^{(s)}|,\mathbf{1}\rangle\Big),\end{aligned}$$

which immediately entails (2.23) since $\langle|\boldsymbol{\varpi}_s|,\mathbf{1}\rangle\leq 3$ and $\langle|\boldsymbol{\pi}_1^{(s)}|,\mathbf{1}\rangle\leq 2$.

By (2.23) and the semi-group property, for all $g\in C([0,\infty])$ and all $s_0\in[0,\infty)$ we get for all $x\in[0,\infty)$,

$$\langle\nu,g\rangle=\lim_{s\to\infty}P_sg(x)=\lim_{s\to\infty}P_{s+s_0}g(x)=\lim_{s\to\infty}P_s(P_{s_0}g)(x)=\langle\nu,P_{s_0}g\rangle,$$

which prove that ν is invariant under $(P_s)_{s\in[0,\infty)}$. Uniqueness follows immediately from (2.23). This completes the proof of Proposition 2.12 (iii). ∎

3. Law of large numbers for the empirical measure

We reach the main part of the paper. In Section 3.1, we present the master formula which is our main tool. The rest of this part is devoted to several applications of the master formula, with the only exception of Section 3.7 where the master formula is proved. A first application of the master formula — though we do not need its full strength — appears in Section 3.2, where we prove almost sure and L^2 convergences of $e^{-2s}M_s$ and $e^{-2s}N_s$, and of the new martingale $R_s=e^{-2s}(2M_s+N_s)$, when $s\to\infty$. In Section 3.3, we show another, more substantial, application of the master formula, and determine the exact law of (M_s,N_s) as well as that of R_∞ (the limit of R_s when $s\to\infty$). Section 3.4 provides explicit computations of the mean rescaled empirical measure of cell sizes, whereas Section 3.5 contains the most important application of the master formula in the paper: a law of large numbers for the empirical measure of cell sizes. In Section 3.6, we study the limit, when $s\to\infty$, of the maximal cell size at time s.

Throughout the section, we work under the new time scale $s\in[0,\infty)$ which satisfies

$$t=1-e^{-s}\in[0,1).$$

3.1. The master formula

Let $f\in C^{1,1}([0,\infty)^2,\mathbb{R})$. Namely, f is continuously differentiable in each coordinate: the derivative of f with respect to the first coordinate − that we consider as a time parameter − is noted by $\dot f$ and the derivative with respect to the second coordinate − that we consider as a space parameter − is noted by f'. We then introduce the following notation for all $s,x\in[0,\infty)$,

$$\mathcal{L}f(s,x) \;=\; (x+1)\big(f'(s,x)-2f(s,x)\big)+4\int_0^x f(s,y)\mathrm{d}y \tag{3.1}$$
$$\text{and}\quad Q_f(s,x)=2e^{2s}\int_0^x K\big(e^{-2s}Df(s,x,y)\big)\mathrm{d}y$$

where we have set

(3.2) $$Df(s,x,y)=f(s,x-y)+f(s,y)-f(s,x),\ y\in[0,x] \quad \text{and} \quad K(z)=e^z-1-z,\ z\in\mathbb{C}\ .$$

Let $\mathbf{X}=(X_u(s))_{u\in\mathbb{T}_2,s\in[0,\infty)}$ be a HR growth-fragmentation process as defined in Section 2.4 (Definition 2.13). Recall that $G_s=\{u\in\mathbb{T}_2 : \zeta_{\overleftarrow{u}}\leq s<\zeta_u\}$ is the set of cells alive at time s. Let

(3.3) $$N_s=\#G_s, \quad M_s=\sum_{u\in G_s}X_u(s), \quad \mathcal{M}_s(\mathrm{d}y)=\sum_{u\in G_s}\delta_{X_u(s)}(\mathrm{d}y) \quad \text{and} \quad \overline{\mathcal{M}}_s=e^{-2s}\mathcal{M}_s\ .$$

For any continuous $g:[0,\infty)^2\to\mathbb{R}$, we also use the notation $\langle\overline{\mathcal{M}}_s,g\rangle=\sum_{u\in G_s}e^{-2s}g(s,X_u(s))$. We recall from Lemma 2.7 applied to $t=1-e^{-s}$ that N_s and M_s have exponential moments. More precisley, for all $x,s\in[0,\infty)$ and all $\lambda\in[0,1]$,

(3.4) $$\mathbf{E}_x\big[\exp\big(2^{-11}e^{-11s}\lambda\,(N_s+M_s)\big)\big]\leq\exp\big(\lambda(1+x)\big)\ .$$

The following theorem is the key technical tool to derive the equations satisfied by $\mathcal{M}_s$ and to prove the law of large numbers as well as other related results.

Theorem 3.1. *We keep the previous notations. Let $f\in C^{1,1}([0,\infty)^2,\mathbb{R})$ and $s_0\in(0,\infty)$. To simplify notation we set $a=(\frac{1}{2}e^{-s_0})^{11}/10$ and we assume that*

(3.5) $$b_f=1+\sup\Big\{e^{-a(x+1)}\big(|f(s,x)|+|\dot{f}(s,x)|+|f'(s,x)|\big)\ ;\ (s,x)\in[0,s_0+1]\times[0,\infty)\Big\}<\infty.$$

(i) *The r.v. $\langle\overline{\mathcal{M}}_s,f\rangle$ is square integrable under $\mathbf{P}_x$ and*

(3.6) $$\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,f\rangle\big]=f(0,x)+\int_0^s\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\dot{f}+\mathcal{L}f\rangle\big]\mathrm{d}r \quad \textit{and}$$

(3.7) $$\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,f\rangle^2\big]=f(0,x)^2+\int_0^s\mathbf{E}_x\Big[2\langle\overline{\mathcal{M}}_r,f\rangle\langle\overline{\mathcal{M}}_r,\dot{f}+\mathcal{L}f\rangle+\langle\overline{\mathcal{M}}_r,R_f\rangle\Big]\,\mathrm{d}r$$

where we have set $R_f(s,x)=2e^{-2s}\int_0^x(Df(s,x,y))^2\mathrm{d}y$, for all $s,x\in[0,\infty)$.

(ii) **(Master formula)** *Assume furthermore that for all $(s,x)\in[0,s_0+1]\times[0,\infty)$,*

(3.8) $$f(s,x)\leq a(x+1) \quad \textit{and} \quad \forall y\in[0,x],\ Df(s,x,y)\leq a(x+1).$$

Then, for all $(s,x)\in[0,s_0]\times[0,\infty)$, the r.v. $e^{\langle\overline{\mathcal{M}}_s,f\rangle}$ and $e^{\langle\overline{\mathcal{M}}_s,f\rangle}\langle\overline{\mathcal{M}}_s,\dot{f}+\mathcal{L}f+Q_f\rangle$ are $\mathbf{P}_x$-integrable, and

(3.9) $$\mathbf{E}_x\Big[e^{\langle\overline{\mathcal{M}}_s,f\rangle}\Big]=e^{f(0,x)}+\int_0^s\mathbf{E}_x\Big[e^{\langle\overline{\mathcal{M}}_r,f\rangle}\langle\overline{\mathcal{M}}_r,\dot{f}+\mathcal{L}f+Q_f\rangle\Big]\mathrm{d}r\ .$$

Proof. See Section 3.7. ■

Remark 3.2. Let $f:[0,\infty)\to\mathbb{R}$ be C^1 and satisfy (3.5) (it does not depend on the time parameter). By differentiating (3.6) in s we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_x[\langle\overline{\mathcal{M}}_s,f\rangle]=e^{-2s}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_x[\langle\mathcal{M}_s,f\rangle]-2e^{-2s}\mathbf{E}_x[\langle\mathcal{M}_s,f\rangle]=e^{-2s}\mathbf{E}_x[\langle\mathcal{M}_s,\mathcal{L}\rangle]$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}_x\big[\langle\mathcal{M}_s,f\rangle\big]=\mathbf{E}_x\big[\langle\mathcal{M}_s,\mathscr{A}f\rangle\big] \quad \text{where}$$

$$\mathscr{A}f(x)=(x+1)f'(s,x)+2x\int_0^1\big(f(ux)+f((1-u)x)-f(x)\big)\mathrm{d}u\ .$$

In the framework of [8] as explained in (1.10) in the introduction, it corresponds to $c(x)=x+1$, $B(x)=2x$ and $\kappa(x,\mathrm{d}\mathbf{p})$ is the law of $(x\max(U,1-U)\,,x\min(U,1-U),0,0\ldots)$ where U is uniform on $[0,1]$. □

3.2. Convergence of $e^{-2s}M_s$, $e^{-2s}N_s$ and of the martingale $R_s = e^{-2s}(2M_s + N_s)$

Recall that $\mathbf{X} = (X_u(s))_{u\in\mathbb{T}_2, s\in[0,\infty)}$ is a HR growth-fragmentation process with initial size $x\in[0,\infty)$ and that for all $s\in[0,\infty)$, $\mathscr{F}_s$ stands for the sigma field generated by the r.v. $X_u(r)$, $r\in[0,s]$, $u\in\mathbb{T}_2$.

Lemma 3.3. *Let $f\in C^{1,1}([0,\infty)^2, \mathbb{R})$ satisfy (3.5), and let $c, s_0\in(0,\infty)$. Assume that*

$$\forall (s,x)\in[0,s_0+1]\times[0,\infty), \quad |f(s,x)|\leq c(1+x) \quad \textit{and} \quad \dot{f}(s,x) + \mathcal{L}f(s,x) = 0.$$

Then, for all $x\in[0,\infty)$, under $\mathbf{P}_x$, $\big(\langle\overline{\mathcal{M}}_s, f\rangle\big)_{s\in[0,s_0]}$ is a càdlàg $(\mathscr{F}_s)_{s\in[0,s_0]}$-martingale.

Proof. By Theorem 3.1, $\langle\overline{\mathcal{M}}_s, f\rangle$ is integrable. Let $s, s'\in[0,s_0]$ be such that $s+s'\leq s_0$. We set $g(r,y) = f(r+s', y)$ for $r\in[0, s_0 - s']$, and observe that $\langle\overline{\mathcal{M}}_{s+s'}, f\rangle = \sum_{u\in G_{s'}} e^{-2s'}\langle\mathcal{M}^u_s, g\rangle$ where $\mathcal{M}^u_s$ is as in (2.33). By Proposition 2.14 (iii), $\mathbf{E}_x[\langle\overline{\mathcal{M}}_{s+s'}, f\rangle\,|\,\mathscr{F}_{s'}] = \sum_{u\in G_{s'}} e^{-2s'}\mathbf{E}_{X_u(s')}[\langle\mathcal{M}_s, g\rangle]$. Since $\dot{g} + \mathcal{L}g = 0$, it follows from (3.6) that $\mathbf{E}_y[\langle\mathcal{M}_s, g\rangle] = g(0,y) = f(s',y)$. Therefore, $\mathbf{E}_x[\langle\overline{\mathcal{M}}_{s+s'}, f\rangle\,|\,\mathscr{F}_{s'}] = \sum_{u\in G_{s'}} e^{-2s'} f(s', X_u(s')) = \langle\overline{\mathcal{M}}_{s'}, f\rangle$. ■

Proposition 3.4. *We keep the previous notation, and set*

$$R_s = e^{-2s}(2M_s + N_s), \qquad D_s = e^s(M_s - N_s), \qquad s\in[0,\infty).$$

Let $x\in[0,\infty)$. Then, under $\mathbf{P}_x$, the following holds true.

(i) *The processes $(R_s)_{s\in[0,\infty)}$ and $(D_s)_{s\in[0,\infty)}$ are $(\mathscr{F}_s)_{s\in[0,\infty)}$ càdlàg martingales, and*

$$\mathbf{E}_x[M_s] = \tfrac{1}{3}(2x+1)e^{2s} + \tfrac{1}{3}(x-1)e^{-s} \quad \textit{and} \quad \mathbf{E}_x[N_s] = \tfrac{1}{3}(2x+1)e^{2s} - \tfrac{2}{3}(x-1)e^{-s}. \tag{3.10}$$

(ii) *For all $s\in[0,\infty)$,*

$$\mathbf{E}_x\big[R_s^2\big] = (2x+1)^2 + \tfrac{1}{5}(4x+1) - \tfrac{1}{3}(2x+1)e^{-2s} - \tfrac{2}{15}(x-1)e^{-5s}$$

$$\textit{and} \quad \mathbf{E}_x\big[D_s^2\big] = \tfrac{1}{6}(2x+1)e^{4s} + \tfrac{2}{3}(x-1)e^{s} + (x-1)^2 - x + \tfrac{1}{2}.$$

Moreover, $\lim_{s\to\infty} R_s = R_\infty$ in L^2 and $\mathbf{P}_x$-a.s.

(iii) $\lim_{s\to\infty} e^{-2s}M_s = \lim_{s\to\infty} e^{-2s}N_s = \frac{1}{3}R_\infty$ *in L^2 and $\mathbf{P}_x$*-a.s.

Proof. We set $f_1(s,x) = 2x+1$ and $f_2(s,x) = e^{3s}(x-1)$, for all $s,x\in[0,\infty)$. These two functions are space-affine solutions of $\dot{f} + \mathcal{L}f = 0$. Note that $\langle\overline{\mathcal{M}}_s, f_1\rangle = R_s$ and that $\langle\overline{\mathcal{M}}_s, f_2\rangle = D_s$, which proves that $(R_s)_{s\in[0,\infty)}$ and $(D_s)_{s\in[0,\infty)}$ are càdlàg $(\mathscr{F}_s)_{s\in[0,\infty)}$-martingales by Lemma 3.3. In particular, $\mathbf{E}_x(R_s) = 2x+1$ and $\mathbf{E}_x(D_s) = x-1$, which implies (3.10).

Recall notation $Df(s,x,y)$ from (3.2). Observe that $Df_1(r,x,y) = 1$ and $Df_2(s,x,y) = -e^{3s}$. Recall from Theorem 3.1 (i), the definition of R_f and observe that $R_{f_1}(s,x) = 2e^{-2s}x$ and $R_{f_2}(s,x) = 2e^{4s}x$. Thus $\langle\overline{\mathcal{M}}_s, R_{f_1}\rangle = 2e^{-4s}M_s$ and $\langle\overline{\mathcal{M}}_s, R_{f_2}\rangle = 2e^{2s}M_s$. We then get (ii) by (3.7) in Theorem 3.1 (i) and by the explicit computation of $\mathbf{E}_x[M_s]$ from (3.10).

Let us prove (iii). Let $n\in\mathbb{N}$. We observe that

$$\begin{aligned}\mathbf{E}_x\Big[\sup_{s\in[n,n+1]}(e^{-3s}D_s)^2\Big] &\leq e^{-6n}\mathbf{E}_x\Big[\sup_{s\in[n,n+1]}D_s^2\Big] \leq 4e^{-6n}\mathbf{E}_x[D_{n+1}^2] \\ &\leq 4e^{-6n}e^{4(n+1)}(x^2+x+2) = 4e^4e^{-2n}(x^2+x+2).\end{aligned}$$

Here, we use Doob's L^2 inequality for martingales in the second inequality and the explicit computation of $\mathbf{E}_x[D_s^2]$ from (ii). Therefore, $\sum_{n\in\mathbb{N}}\mathbf{E}_x[\sup_{s\in[n,n+1]}(e^{-3s}D_s)^2] < \infty$. This implies that $\lim_{s\to\infty} e^{-3s}|D_s| = \lim_{s\to\infty}|e^{-2s}M_s - e^{-2s}N_s| = 0$ in L^2 and $\mathbf{P}_x$-a.s. which readily completes the proof of (iii). ■

3.3. The laws of (M_s, N_s) and of R_∞

We investigate the joint law of (M_s, N_s) and the law of R_∞; they are closely related (see below) to the following second order ordinary differential equation

$$\ddot{z} = \dot{z} + 2(1 - e^{-z}) \tag{3.11}$$

that is briefly discussed here.

For all $\lambda, \mu \in \mathbb{R}$, we set $F(\lambda,\mu) = \big(\mu\,,\, \mu + 2(1-e^{-\lambda})\big)$. Then, $\dot{Z} = F(Z)$ where $Z = (z, \dot z)$ and where z is a solution of (3.11). We see that $(0,0)$ is the only critical point (or: equilibrium point) of F. Moreover

$$dF_{(\lambda,\mu)} = \begin{pmatrix} 0 & 1 \\ 2e^{-\lambda} & 1 \end{pmatrix} \quad \text{and} \quad dF_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} =: A\,.$$

Therefore F is locally Lipschitz, and for all $(\lambda,\mu) \in \mathbb{R}^2$, there exists a maximal solution $z : I_{\lambda,\mu} \to \mathbb{R}$ to (3.11) such that $(z(0), \dot z(0)) = (\lambda, \mu)$ (here $I_{\lambda,\mu}$ is therefore an open interval containing 0), and we denote by $s \in I_{\lambda,\mu} \mapsto \phi_s(\lambda,\mu)$ this maximal solution, and by $\Phi_s(\lambda,\mu) = (\phi_s(\lambda,\mu), \dot\phi_s(\lambda,\mu))$ the flow associated with (3.11).

The eigenvalues of A are 2 et -1 with respective eigenvectors $\mathbf{v}_2 = (1,2)$ et $\mathbf{v}_{-1} = (1,-1)$ (since the eigenvalues of A are not purely imaginary numbers, the critical point is said to be hyperbolic). In the case of a 2-dimensional hyperbolic critical point and since F is C^∞, the refined version of Hartman–Grobman Theorem in [16] asserts that there are two open neighborhoods U and V of $(0,0)$ and an homeomorphism $H : U \to V$ that is differentiable at $(0,0)$ with $\mathrm{d}_{(0,0)} H = \mathrm{Id}$, such that

$$\Phi_s\big(H(\lambda,\mu)\big) = H\big(e^{sA}\tbinom{\lambda}{\mu}\big) = H\Big(\tfrac{\lambda+\mu}{3}e^{2s} + \tfrac{2\lambda-\mu}{3}e^{-s}\,,\; \tfrac{2(\lambda+\mu)}{3}e^{2s} - \tfrac{2\lambda-\mu}{3}e^{-s}\Big)\,, \tag{3.12}$$

for all $(\lambda,\mu) \in U$ and $s \in \mathbb{R}$ such that $e^{sA}\binom{\lambda}{\mu} \in U$.

Lemma 3.5. *Equation (3.11) admits a solution defined on $[0,\infty)$ for all initial conditions $z(0)$ and $\dot z(0)$ in $[0,\infty)$. Moreover, this solution is strictly increasing and strictly convex if $(z(0), \dot z(0))$ is distinct from $(0,0)$.*

Proof. Let $(\lambda,\mu) \in [0,\infty)^2 \backslash \{(0,0)\}$ and let $z : I_{\lambda,\mu} \to \mathbb{R}$ be the maximal solution of (3.11) such that $(z(0), \dot z(0)) = (\lambda,\mu)$. We denote by $s_* > 0$ the right end of $I_{\lambda,\mu}$: we want to prove that $s_* = \infty$. We first prove that z and $\dot z$ are strictly positive on $(0, s_*)$. Indeed, suppose that $\dot z$ takes negative values on $[0, s_*)$ and let $s_1 = \inf\{s \in (0,s_*) : \dot z(s) \le 0\}$. If $s_1 > 0$, then z is strictly increasing on $[0, s_1)$ and thus $z(s_1) > 0$ and $\dot z(s) = 2(s - s_1)(1 - e^{-z(s_1)} + o(1))$. Therefore, $\dot z$ would be strictly negative in a left neighborhood of s_1, which contradicts the definition of s_1. If $s_1 = 0$, then $\mu = 0$, which implies that $\lambda > 0$, since we assume that $(\lambda,\mu) \neq (0,0)$. So $\dot z(s) = 2s(1 - e^{-\lambda} + o(1))$ and $\dot z$ is strictly positive in a right neighborhood of $s_1 = 0$, which also contradicts the definition of s_1. So it proves that $\dot z$ is strictly positive on $(0, s_*)$, and so is z. This shows that z is strictly increasing and strictly convex on $[0, s_*)$.

Since we proved that z stays nonnegative, Equation (3.11) implies that $\ddot z \le \dot z + 2$. Therefore, $\dot z(s) \le (\dot z(0) + 2)e^s$ and thus $z(s) \le z(0) - \dot z(0) - 2 + (\dot z(0) + 2)e^s$. Since the domain of definition of F is the whole plane, if s_* were finite, then $(z, \dot z)$ would have to explode in finite time which would contradict the previous upper bounds. Thus $s_* = \infty$, which completes the proof of the lemma. ∎

Proposition 3.6. *For all $\lambda, \mu \in [0,\infty)$, we denote by $s \in [0,\infty) \mapsto \phi_s(\lambda,\mu)$ the solution of (3.11) on $[0,\infty)$ with inital conditions $\phi_0(\lambda,\mu) = \lambda$ and $\dot\phi_0(\lambda,\mu) = \mu$. Then, for all $s, x \in [0,\infty)$,*

$$\mathbf{E}_x\big[\exp\big(-\lambda N_s - \mu M_s\big)\big] = \exp\big(-\phi_s(\lambda,\mu) - x\dot\phi_s(\lambda,\mu)\big)\,. \tag{3.13}$$

Proof. We fix $x, s, \lambda, \mu \in [0,\infty)$ and for all $r \in [0,s]$, we set $\beta(r) = \phi_{s-r}(\lambda,\mu)$. Note that $\ddot\beta = -\dot\beta + 2(1 - e^{-\beta})$ on $[0,s]$. We also set $f(r,y) = e^{2r}(\dot\beta(r) y - \beta(r))$. Note that $\dot\beta(r) = -\dot\phi_{s-r}(\lambda,\mu)$. Therefore,

$$\langle \overline{\mathcal{M}}_s, f\rangle = -\lambda N_s - \mu M_s \quad \text{and} \quad \langle \overline{\mathcal{M}}_0, f\rangle = -\phi_s(\lambda,\mu) - M_0\dot\phi_s(\lambda,\mu)\,.$$

We want to apply Theorem 3.1 (ii) to f. To this end, first note that f obviously satisfies condition (3.5). Since $\dot\beta(r) \le 0 \le \beta(r)$, we get $f(s,r) \le 0$. Then observe that $Df(r,x,y) = -e^{2r}\beta(r) \le 0$. Therefore f also satisfies (3.8). Thus, (3.9) applies and we get

$$\mathbf{E}_x\big[e^{-\lambda N_s - \mu M_s}\big] = \mathbf{E}_x\big[e^{\langle \overline{\mathcal{M}}_s, f\rangle}\big] = e^{-\phi_s(\lambda,\mu) - x\dot\phi_s(\lambda,\mu)} + \int_0^s \mathbf{E}_x\Big[e^{\langle \overline{\mathcal{M}}_r, f\rangle}\langle \overline{\mathcal{M}}_r, \dot f + \mathcal{L}f + Q_f\rangle\Big]\mathrm{d}r.$$

A straightforward computation then implies that $\dot f + \mathcal{L}f + Q_f = 0$ which implies (3.13). ∎

Thanks to the previous proposition we are going to find the law of the common limit of $e^{-2s}M_s$ and of $e^{-2s}N_s$ as s tends to ∞ that is $\frac{1}{3}R_\infty$ (see Proposition 3.4 (iii)). Before stating the result, let us briefly discuss how the result has been found. The law of R_s is related to the solutions of (3.11) whose initial values lie on the line directed by the eigenvector

$\mathbf{v}_2=(1,2)$ of A. By (3.12), for all sufficiently small $\lambda\in[0,\infty)$ and for all $s\in[0,\infty)$, we get $\Phi_s(H(\lambda e^{-2s},2\lambda e^{-2s}))=H(\lambda,2\lambda)$. As already mentioned, H is differentiable at $(0,0)$ and $d_{(0,0)}H=\mathrm{Id}$. Then, we get $H_1(\lambda e^{-2s},2\lambda e^{-2s})=\lambda e^{-2s}(1+o(1))$ and $H_2(\lambda e^{-2s},2\lambda e^{-2s})=2\lambda e^{-2s}(1+o(1))$ as s tends to ∞. Therefore

$$\mathbf{E}_x\big[e^{-\lambda R_s}\big]\sim_{s\to\infty}\mathbf{E}_x\big[e^{-H_1(\lambda e^{-2s},2\lambda e^{-2s})N_s-H_2(\lambda e^{-2s},2\lambda e^{-2s})M_s}\big]=e^{-H_1(\lambda,2\lambda)-xH_2(\lambda,2\lambda)}$$

which implies that $\mathbf{E}_x[\exp(-\lambda R_\infty)]=\exp(-H_1(\lambda,2\lambda)-xH_2(\lambda,2\lambda))$. Previous works on related models suggest that R_∞ has the same law as $\frac12\int_0^1\varrho_s^2\mathrm{d}s$, where $(\varrho_s)_{s\in[0,1]}$ is a 4-dimensional Bessel bridge with initial value $\varrho_0=2\sqrt{x}$ and terminal value $\varrho_1=0$. The Laplace transform of $\frac12\int_0^1\varrho_s^2\mathrm{d}s$ is given in terms of the following functions

$$\forall y\in\mathbb{R},\quad \varphi(y)=2\log\big(y^{-1}\sinh y\big)\quad\text{and}\quad \psi(y)=y\varphi'(y)=2\big(y\coth(y)-1\big)\,. \tag{3.14}$$

Namely, $\mathbf{E}\big[\exp(-\frac12\lambda\int_0^1\varrho_s^2\mathrm{d}s)\big]=\exp(-\varphi(\sqrt{\lambda})\,-\,x\psi(\sqrt{\lambda}))$. It therefore suggests that $H_1(\lambda,2\lambda)=\varphi(\sqrt{3\lambda})$. and $H_2(\lambda,2\lambda)=\psi(\sqrt{3\lambda})$. This is indeed true, as this proposition shows.

Proposition 3.7. *Let φ and ψ be defined by (3.14). For all $x,s,\lambda\in[0,\infty)$, we get*

$$\mathbf{E}_x\big[e^{-\varphi(\sqrt{3\lambda})N_s-\psi(\sqrt{3\lambda})M_s}\big]=e^{-\varphi(e^s\sqrt{3\lambda})-x\psi(e^s\sqrt{3\lambda})}\quad\textit{and}\quad\mathbf{E}_x\big[e^{-\lambda R_\infty}\big]=e^{-\varphi(\sqrt{3\lambda})-x\psi(\sqrt{3\lambda})}. \tag{3.15}$$

Proof. If we set $x=\varphi(\sqrt{3\lambda})$ and $y=\psi(\sqrt{3\lambda})$, then we easily check that $s\in[0,\infty)\mapsto\varphi(e^s\sqrt{3\lambda})$ is a solution of (3.11). Since $\partial_s\varphi(e^s\sqrt{3\lambda})=\psi(e^s\sqrt{3\lambda})$, Proposition 3.6 implies the first equality of (3.15). This equality implies that

$$\mathbf{E}_x\big[e^{-\varphi(e^{-s}\sqrt{3\lambda})N_s-\psi(e^{-s}\sqrt{3\lambda})M_s}\big]=e^{-\varphi(\sqrt{3\lambda})-x\psi(\sqrt{3\lambda})}.$$

As y tends to 0, $\varphi(y)=\frac13y^2(1+O(y^2))$ and $\psi(y)=\frac23y^2(1+O(y^2))$. Thus, $\varphi(e^{-s}\sqrt{3\lambda})N_s+\psi(e^{-s}\sqrt{3\lambda})M_s=\lambda R_s(1+O(e^{-2s}\lambda))$ $\mathbf{P}_x$-a.s. as $s\to\infty$, which implies the second equality of (3.15). ■

3.4. *Explicit computation of the mean rescaled empirical measure*

Recall from (3.3) the definition of the (rescaled) empirical measure $\overline{\mathcal{M}}_s$ of the sizes of the HR growth-fragmentation processus. We now study the convergence of the mean $\boldsymbol{\mu}_s^{(x)}$ of $\overline{\mathcal{M}}_s$ under $\mathbf{P}_x$ that is a finite Borel measure on $[0,\infty)$ given by

$$\int_{[0,\infty)}f(y)\,\boldsymbol{\mu}_s^{(x)}(dy)=\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,f\rangle\big]=\mathbf{E}_x\Big[\sum_{u\in G_s}e^{-2s}f(X_s(u))\Big]. \tag{3.16}$$

for all measurable $f:[0,\infty)\to[0,\infty)$. We fix $x\in[0,\infty)$ and for all $s,\lambda\in[0,\infty)$, we denote by $\Lambda_s(\lambda)$ the Laplace transform of $\boldsymbol{\mu}_s^{(x)}$:

$$\Lambda_s(\lambda)=\int_{[0,\infty)}e^{-\lambda y}\,\boldsymbol{\mu}_s^{(x)}(dy)\,.$$

In what follows, the time derivative is denoted by $\dot\Lambda_s(\lambda)$ and the derivative with respect to λ by $\Lambda_s'(\lambda)$. We systematically omit the dependence in the variable x to simplify the notation.

Lemma 3.8. *For all $x,s\in[0,\infty)$, we set $\mathtt{U}(s)=\frac13(2x+1)-\frac23(x-1)e^{-3s}$. Then Λ satisfies the following first order partial differential equation.*

$$\dot\Lambda_s(\lambda)=(\lambda+2)\Lambda_s'(\lambda)-(\lambda+2+4\lambda^{-1})\Lambda_s(\lambda)+4\lambda^{-1}\mathtt{U}(s),\qquad\lambda\in(0,\infty), \tag{3.17}$$

with the initial condition $\Lambda_0(\lambda)=e^{-\lambda x}$.

Proof. Let us set $\mathbf{e}_\lambda(y)=e^{-\lambda y}$. Recall from (3.1) the definition of $\mathcal{L}$ and note that

$$\mathcal{L}\mathbf{e}_\lambda(y)=-(\lambda+2)y\mathbf{e}_\lambda(y)-(\lambda+2+4\lambda^{-1})\mathbf{e}_\lambda(y)+4\lambda^{-1}=(\lambda+2)\partial_\lambda\mathbf{e}_\lambda(y)-(\lambda+2+4\lambda^{-1})\mathbf{e}_\lambda(y)+4\lambda^{-1}.$$

Let us denote by $\mathbf{1}$ the function that is constant to 1. By (3.6) in Theorem 3.1 (ii), we see that

$$\Lambda_s(\lambda)=e^{-\lambda x}+\int_0^s\Big((\lambda+2)\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\partial_\lambda\mathbf{e}_\lambda\rangle\big]-(\lambda+2+4\lambda^{-1})\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\mathbf{e}_\lambda\rangle\big]+4\lambda^{-1}\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\mathbf{1}\rangle\big]\Big)\mathrm{d}r,$$

which implies (3.17) since $\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\mathbf{1}\rangle\big]=\mathtt{U}(r)$ by (3.10) and since $\Lambda_s'(\lambda)=\partial_\lambda\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,\mathbf{e}_\lambda\rangle\big]=\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,\partial_\lambda\mathbf{e}_\lambda\rangle\big]$ by dominated convergence and the fact that $\mathbf{E}_x[M_s]<\infty$. ■

Lemma 3.9. *Write $q(s,\lambda)=(\lambda+2)(e^s-1)$ for $(s,\lambda)\in[0,\infty)\times(0,\infty)$. Then, Equation (3.17) has a unique solution $\Lambda_s(\lambda)=\sum_{0\leq j\leq 6}T_j$ where*

$$T_0=\frac{4(2x+1)}{3(\lambda+2)^2},\quad T_1=-e^{-3s}\frac{8(x-1)(\lambda^3+9\lambda^2+24\lambda+8)}{3(\lambda+2)^5},\quad T_2=-\frac{4(2x+1)\lambda^2e^{-q}}{3(\lambda+2)^2(\lambda+q)^2},$$

$$T_3=e^{-3s}\frac{8(x-1)\lambda^2(\lambda+q+9)e^{-q}}{3(\lambda+2)^5},\quad T_4=e^{-3s}\frac{64(x-1)\lambda^2e^{-q}}{(\lambda+2)^5(\lambda+q)},$$

$$T_5=e^{-3s}\frac{64(x-1)\lambda^2e^{-q}}{3(\lambda+2)^5(\lambda+q)^2}\quad\textit{and}\quad T_6=e^{2s}e^{-x(\lambda+q)}\frac{\lambda^2e^{-q}}{(\lambda+q)^2}\,.$$

Proof. We fix $x\in[0,\infty)$ and $\lambda\in(0,\infty)$. To solve (3.17) we use the method of characteristics as in the proof of Proposition 2.12 (ii). Recall from (2.46) the definition of $\ell_s(y)=(y+2)e^{-s}-2$, $s,y\in\mathbb{R}$ and recall that $\ell_{s+r}(y)=\ell_s(\ell_r(y))$, $s,r\in\mathbb{R}$. We fix $y\in(0,\infty)$. Then by Lemma 3.8, $L_s=\Lambda_s(\ell_s(y))$ satisfies

$$\forall s\in\big[0,\log(1+\tfrac{1}{2}y)\big),\quad \dot{L}_s=-\Big(2+\ell_s(y)+\frac{4}{\ell_s(y)}\Big)L_s+\frac{4\mathtt{U}(s)}{\ell_s(y)}\quad\text{and}\quad L_0=e^{-xy}. \tag{3.18}$$

We set $h_s(y)=2+\ell_s(y)+2s+2\log\ell_s(y)$. Observe that $-\dot{h}_s(y)=2+\ell_s(y)+\frac{4}{\ell_s(y)}$. Thus, the unique solution of (3.18) is given by

$$\Lambda_s\big(\ell_s(y)\big)=L_s=e^{h_s(y)-h_0(y)-yx}+\int_0^s\frac{4\mathtt{U}(r)e^{h_s(y)-h_r(y)}}{\ell_r(y)}\mathrm{d}r\,,$$

that holds true for all $y\in(0,\infty)$ and $s\in[0,\infty)$ such that $\ell_s(y)>0$. In particular, it holds true for $y=\ell_{-s}(\lambda)$ since $\ell_s(y)=\ell_0(\lambda)=\lambda>0$. In this case, observe that $h_s(\ell_{-s}(\lambda))-h_r(\ell_{-s}(\lambda))=\lambda+2(s-r)+2\log\lambda-\ell_{r-s}(\lambda)-2\log\ell_{r-s}(\lambda)$. Thus, for all $s,\lambda\in[0,\infty)$, we get

$$\Lambda_s(\lambda)=\frac{\lambda^2e^{2s}}{\ell_{-s}(\lambda)^2}e^{\lambda-(x+1)\ell_{-s}(\lambda)}+\int_0^s\frac{4\lambda^2\mathtt{U}(s-r)e^{2r+\lambda-\ell_{-r}(\lambda)}}{\ell_{-r}(\lambda)^3}\mathrm{d}r$$

Now we observe that $q(s,\lambda)+\lambda=\ell_{-s}(\lambda)$. Then, we get

$$\begin{aligned}\Lambda_s(\lambda)&=T_6+\tfrac{4}{3}\lambda^2(2x+1)\int_0^s\frac{\mathrm{e}^{2r}\mathrm{e}^{-q(r,\lambda)}}{(q(r,\lambda)+\lambda)^3}\,\mathrm{d}r-\tfrac{8}{3}\lambda^2(x-1)\mathrm{e}^{-3s}\int_0^s\frac{\mathrm{e}^{5r}\mathrm{e}^{-q(r,\lambda)}}{(q(r,\lambda)+\lambda)^3}\,\mathrm{d}r\\&=T_6+\frac{4(2x+1)\lambda^2}{3(\lambda+2)^2}\int_0^q\frac{(b+\lambda+2)e^{-b}}{(b+\lambda)^3}\,\mathrm{d}b-\frac{8(x-1)\mathrm{e}^{-3s}\lambda^2}{3(\lambda+2)^5}\int_0^q\frac{(b+\lambda+2)^4e^{-b}}{(b+\lambda)^3}\,\mathrm{d}b.\end{aligned}$$

Note that $\int_0^q(b+\lambda+2)(b+\lambda)^{-3}e^{-b}\,\mathrm{d}b=\lambda^{-2}-(\lambda+q)^{-2}e^{-q}$ and that

$$\int_0^q\frac{(b+\lambda+2)^4e^{-b}}{(b+\lambda)^3}\,\mathrm{d}b=\frac{\lambda^3+9\lambda^2+24\lambda+8}{\lambda^2}-(\lambda+9+q)e^{-q}-\frac{24e^{-q}}{\lambda+q}-\frac{8e^{-q}}{(\lambda+q)^2},$$

which implies the desired result. ■

We now explain how to explicitly invert the Laplace transform Λ_s. Let π be a signed measure on the Borel subsets of $[0,\infty)$. Namely, there are two finite nonnegative measures π_+ and π_- on $[0,\infty)$ such that $\pi=\pi_+-\pi_-$. Recall from (2.48) that for any signed measure $\pi=\pi_+-\pi_-$ on $[0,\infty)$ we use the notation $\overline{\pi}(\lambda)=\int_{[0,\infty)}e^{-\lambda y}\pi_+(\mathrm{d}y)-\int_{[0,\infty)}e^{-\lambda y}\pi_-(\mathrm{d}y)$, $\lambda\in[0,\infty)$, for the Laplace transform of π. We recall that $*$ stands for the convolution product of a nonnegative finite measure with a signed measure. Recall from (2.49) that $\pi_{n,\beta}$ stands for the (n,β)-gamma distribution (here, $n\in\mathbb{N}^*$ and $\beta\in(0,\infty)$). Let $y_0\in[0,\infty)$. Then, recall from (2.51) that

$$\overline{\delta_{y_0}*\pi_{n,\beta}}\,(\lambda)=\big(\tfrac{\beta}{\lambda+\beta}\big)^ne^{-\lambda y_0}\,.$$

Also recall from (2.50) that $\boldsymbol{\gamma}_n = \pi_{n,2}$ and that $\boldsymbol{\pi}_1^{(s)} = \delta_0 - \pi_{1,2(1-e^{-s})}$. We also set

$$\boldsymbol{\pi}_2^{(s)} = \delta_0 - 2\pi_{1,2(1-e^{-s})} + \pi_{2,2(1-e^{-s})}, \quad \boldsymbol{\rho} = 2\boldsymbol{\gamma}_2 + 3\boldsymbol{\gamma}_3 - 3\boldsymbol{\gamma}_5, \tag{3.19}$$
$$\boldsymbol{\mu} = \boldsymbol{\gamma}_2 - 2\boldsymbol{\gamma}_3 + \boldsymbol{\gamma}_4 \quad \text{and} \quad \boldsymbol{\nu} = \boldsymbol{\gamma}_3 - 2\boldsymbol{\gamma}_4 + \boldsymbol{\gamma}_5 \, .$$

Recall that $q(s,\lambda) = (\lambda+2)(e^s-1)$. Then, we easily check that $\overline{\boldsymbol{\mu}}(\lambda) = \frac{4\lambda^2}{(\lambda+2)^4}$, $\overline{\boldsymbol{\nu}}(\lambda) = \frac{8\lambda^2}{(\lambda+2)^5}$ and

$$e^{-s-2(e^s-1)} . \, \overline{\delta_{e^s-1} * \boldsymbol{\pi}_1^{(s)}} \, (\lambda) = \frac{\lambda e^{-q}}{\lambda+q}, \quad e^{-2s-2(e^s-1)} . \, \overline{\delta_{e^s-1} * \boldsymbol{\pi}_2^{(s)}} \, (\lambda) = \frac{\lambda^2 e^{-q}}{(\lambda+q)^2} \, ,$$

$$\text{and} \qquad \tfrac{1}{8} e^{-2(e^s-1)} . \, \overline{\delta_{e^s-1} * (2e^s \boldsymbol{\mu} + 7\boldsymbol{\nu})} \, (\lambda) = \frac{\lambda^2 (\lambda+q+9) e^{-q}}{(\lambda+2)^5} .$$

This, combined with Lemma 3.9, implies the following proposition.

Proposition 3.10. *Recall that $\boldsymbol{\mu}_s^{(x)}$ is the mean of $\overline{\mathcal{M}}_s$ under $\mathbf{P}_x$. Recall from (3.19) the notation $\boldsymbol{\gamma}_2$, $\boldsymbol{\rho}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\boldsymbol{\pi}_1^{(s)}$, $\boldsymbol{\pi}_2^{(s)}$. Then, for all $x \in [0,\infty)$ and $s \in (0,\infty)$,*

$$\boldsymbol{\mu}_s^{(x)} = \tfrac{1}{3}(2x+1)\boldsymbol{\gamma}_2 - \tfrac{1}{3}(x-1)e^{-3s}\boldsymbol{\rho} + e^{-2s-2(e^s-1)}\delta_{e^s-1} * \boldsymbol{\kappa}_s + e^{-2(e^s-1)(x+1)}\delta_{e^s(x+1)-1} * \boldsymbol{\pi}_2^{(s)},$$

where $\boldsymbol{\kappa}_s$ is the signed measure given by

$$\boldsymbol{\kappa}_s = -\tfrac{1}{3}(2x+1)\,\boldsymbol{\gamma}_2 * \boldsymbol{\pi}_2^{(s)} + \tfrac{1}{3}(x-1)\big(2\boldsymbol{\mu} + 7e^{-s}\boldsymbol{\nu} + 12e^{-2s}\boldsymbol{\gamma}_4 * \boldsymbol{\pi}_1^{(s)} - 12e^{-2s}\boldsymbol{\gamma}_5 * \boldsymbol{\pi}_1^{(s)} + 2e^{-3s}\boldsymbol{\gamma}_5 * \boldsymbol{\pi}_2^{(s)}\big).$$

Proof. Follows from Lemma 3.9 by means of long but straightforward computations. ■

We next use the previous proposition to get the following estimates.

Proposition 3.11. *Let $x \in [0,\infty)$ and let $s \in (0,\infty)$. Let $\phi : [0,\infty) \to \mathbb{R}$ be bounded and measurable. We set*

$$u_\phi(s,x) = \langle \boldsymbol{\mu}_s^{(x)}, \phi \rangle = \mathbf{E}_x\big[\langle \overline{\mathcal{M}}_s, \phi \rangle\big] \, . \tag{3.20}$$

Then the following holds true.

- (i) $\big|u_\phi(s,x) - \frac{1}{3}(2x+1)\langle \boldsymbol{\gamma}_2, \phi\rangle\big| \leq 39e^{-3s}\|\phi\|_\infty(2x+1)$.
- (ii) *For $y \in [0,x]$, we have $|Du_\phi(s,x,y)| = |u_\phi(s,x-y) + u_\phi(s,y) - u_\phi(s,x)| \leq 17\|\phi\|_\infty$.*
- (iii) *Suppose that ϕ is C^1. Then, the function $x \in [0,\infty) \mapsto u_\phi(s,x)$ is C^1. If furthermore ϕ and ϕ' are bounded, then for all $a \in (0,\infty)$,*

$$\sup\big\{e^{-ax}|u_\phi(s,x)| + e^{-ax}|\partial_x u_\phi(s,x)| \, ; \, x \in [0,\infty)\big\} < \infty \, . \tag{3.21}$$

Proof. If π is a signed measure, then we denote by $|\pi|$ its total variation. Recall that $\mathbf{1}$ stands for the function that is constant to 1. By Proposition 3.10 and since $2(e^s-1) > s$, we get

$$e^{3s}\big|u_\phi(s,x) - \tfrac{1}{3}(2x+1)\langle \boldsymbol{\gamma}_2, \phi\rangle\big| \leq \|\phi\|_\infty \Big(\tfrac{1}{3}(x+1)\langle|\boldsymbol{\rho}|, \mathbf{1}\rangle + \langle|\boldsymbol{\kappa}_s|, \mathbf{1}\rangle + \langle|\boldsymbol{\pi}_2^{(s)}|, \mathbf{1}\rangle\Big).$$

Now observe that $\langle|\boldsymbol{\rho}|, \mathbf{1}\rangle \leq 8$, $\langle|\boldsymbol{\mu}|, \mathbf{1}\rangle \leq 4$, $\langle|\boldsymbol{\nu}|, \mathbf{1}\rangle \leq 4$, $\langle|\boldsymbol{\pi}_1^{(s)}|, \mathbf{1}\rangle \leq 2$ and $\langle|\boldsymbol{\pi}_2^{(s)}|, \mathbf{1}\rangle \leq 4$. Thus, $\langle|\boldsymbol{\kappa}_s|, \mathbf{1}\rangle \leq \frac{1}{3}(100x+96)$, which easily implies (i).

Let us prove (ii). By Proposition 3.10, $\boldsymbol{\mu}_s^{(x)}$ can be written as follows.

$$\boldsymbol{\mu}_s^{(x)}(\mathrm{d}z) = x p_1(\mathrm{d}z) + p_2(\mathrm{d}z) + m(x, \mathrm{d}z) \tag{3.22}$$

where $m(x,\mathrm{d}z) = e^{-2(e^s-1)(x+1)}(\delta_{e^s(x+1)-1} * \boldsymbol{\pi}_2^{(s)})(\mathrm{d}z)$ and the measures p_1 and p_2 do not depend on x. Therefore,

$$\boldsymbol{\mu}_s^{(y)}(\mathrm{d}z) + \boldsymbol{\mu}_s^{(x-y)}(\mathrm{d}z) - \boldsymbol{\mu}_s^{(x)}(\mathrm{d}z) = \boldsymbol{\mu}_s^{(0)}(\mathrm{d}z) + m(x-y, \mathrm{d}z) + m(y, \mathrm{d}z) - m(x, \mathrm{d}z) - m(0, \mathrm{d}z).$$

Note that $|\int_{[0,\infty)} m(y,\mathrm{d}z)\phi(z)| \leq e^{-2(e^s-1)(y+1)} 4\|\phi\|_\infty \leq 4\|\phi\|_\infty$. Thus $|Du_\phi(s,x,y)| \leq |u_\phi(s,0)| + 16\|\phi\|_\infty$. Then observe that

$$|u_\phi(s,0)| \leq \mathbf{E}_0\big[\langle \overline{\mathcal{M}}_s, |\phi|\rangle\big] \leq \|\phi\|_\infty e^{-2s}\mathbf{E}_0[N_s] \leq \|\phi\|_\infty$$

by (3.10) in the last inequality. This immediately implies (ii).

Let us prove (iii). We assume that ϕ is C^1 and we fix $s\in[0,\infty)$. By (3.22), we get

$$u_\phi(s,x)=x\langle p_1,\phi\rangle+\langle p_2,\phi\rangle+e^{-2(e^s-1)(x+1)}\int_{[0,\infty)}\boldsymbol{\pi}_2^{(s)}(\mathrm{d}z)\,\phi\big(z+e^s(x+1)-1\big)$$

that is clearly a C^1 function in the variable x and

$$\partial_x u_\phi(s,x)=\langle p_1,\phi\rangle-2(e^s-1)e^{-2(e^s-1)(x+1)}\int_{[0,\infty)}\boldsymbol{\pi}_2^{(s)}(\mathrm{d}z)\,\phi\big(z+e^s(x+1)-1\big)$$
$$+\,e^{s-2(e^s-1)(x+1)}\int_{[0,\infty)}\boldsymbol{\pi}_2^{(s)}(\mathrm{d}z)\,\phi'\big(z+e^s(x+1)-1\big)\,.$$

This readily implies (3.21) since ϕ and ϕ' are bounded. ∎

3.5. Law of large numbers for the empirical measure

Recall that for all $s\in[0,\infty)$, $\mathscr{F}_s$ stands for the sigma field generated by the r.v. $X_u(r)$, $r\in[0,s]$, $u\in\mathbb{T}_2$.

Lemma 3.12. *Let $\phi\in C^1([0,\infty),\mathbb{R})$. We assume that ϕ and ϕ' are bounded. Recall from (3.20) the definition of u_ϕ. Then, the following holds true.*

- (i) *The function u_ϕ satisfies the equation $\dot u_\phi-\mathcal{L}u_\phi=0$.*
- (ii) *For all $s_1\in(0,\infty)$, $\big(\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s\,,\cdot)\rangle\big)_{s\in[0,s_1]}$ is a square integrable $(\mathscr{F}_s)_{s\in[0,s_1]}$-martingale with respect to $\mathbf{P}_x$.*

Proof. Let $s,s_0\in[0,\infty)$ and observe that

$$\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_{s+s_0},\phi\rangle\,\big|\,\mathscr{F}_s\big]=\mathbf{E}_x\Big[\sum_{s\in G_s}e^{-2s}\langle\overline{\mathcal{M}}^u_{s_0},\phi\rangle\Big|\mathscr{F}_s\Big]=\sum_{s\in G_s}e^{-2s}u_\phi(s_0,X_u(s))=\langle\overline{\mathcal{M}}_s,u_\phi(s_0\,,\cdot)\rangle$$

by the time-branching property (Proposition 2.14 (iii)). By Proposition 3.11 (iii), the function $x\in[0,\infty)\mapsto u_\phi(s_0,x)$ is C^1 and we can apply (3.6) in Theorem 3.1 (i) to get that

$$u_\phi(s+s_0,x)=\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,u_\phi(s_0,\cdot)\rangle\big]=u_\phi(s_0,x)+\int_0^s\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,\mathcal{L}u_\phi(s_0,\cdot)\rangle\big]\mathrm{d}r.$$

This shows that u_ϕ is C^1 in time, and we get (i) by taking the right-derivative with respect to s in the last equation.

To prove (ii), we set $f(s,x)=u_\phi(s_1-s,x)$ for $s\in[0,s_1]$. Then (i) implies that $\dot f+\mathcal{L}f=0$, and Proposition 3.11 (i) implies that there exists a constant $c\in(0,\infty)$ such that $|f(s,x)|\leq c(1+x)$ for all $(s,x)\in[0,s_1]\times[0,\infty)$. Then, we can apply Lemma 3.3 which proves that $\big(\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s\,,\cdot)\rangle\big)_{s\in[0,s_1]}$ is a martingale with respect to $(\mathscr{F}_s)_{s\in[0,s_1]}$. By Proposition 3.11 (i), we get $|\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s\,,\cdot)\rangle|\leq 39\|\phi\|_\infty R_s$, which completes the proof of (ii) since R_s is square integrable with respect to $\mathbf{P}_x$. ∎

We next prove the following law of large numbers for the measure $\overline{\mathcal{M}}_s$ as $s\to\infty$. Recall from (2.50) the definition of the gamma distribution $\boldsymbol{\gamma}_2$.

Theorem 3.13. ***(Law of large numbers)*** *We fix $x\in[0,\infty)$, and keep the previous notation.*

- (i) *Let $\phi\in C^1([0,\infty),\mathbb{R})$ be such that ϕ and ϕ' are bounded. Then, for all $s\in(0,\infty)$,*

$$\sqrt{\mathbf{E}_x\Big[\big(\,\langle\overline{\mathcal{M}}_s,\phi\rangle-\tfrac{1}{3}R_\infty\langle\boldsymbol{\gamma}_2,\phi\rangle\,\big)^2\Big]}\leq 135\,e^{-\frac{3}{4}s}\|\phi\|_\infty(x+1). \tag{3.23}$$

- (ii) $\mathbf{P}_x$-a.s. $\lim_{s\to\infty}\overline{\mathcal{M}}_s=\frac{1}{3}R_\infty\boldsymbol{\gamma}_2$ *with respect to the weak convergence of finite Borel measures on* $[0,\infty)$.

Proof. Let us first prove (3.23). We fix $s_1\in(0,\infty)$. Recall from (3.20) the definition of u_ϕ and observe that $u_\phi(0,y)=\phi(y)$. Then, by Lemma 3.12 (ii) for all $s\in[0,s_1]$ and all $x\in[0,\infty)$, we get

$$\mathbf{E}_x\big[\big(\langle\overline{\mathcal{M}}_{s_1},\phi\rangle-\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle\big)^2\big]=\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_{s_1},\phi\rangle^2\big]-\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle^2\big]\ .$$

By Proposition 3.11 (iii), we are entitled to apply (3.7) in Theorem 3.1 (i) to get that

$$\mathbf{E}_x\big[\big(\langle\overline{\mathcal{M}}_{s_1},\phi\rangle-\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle\big)^2\big]=\int_s^{s_1}\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,R_f\rangle\big]\,\mathrm{d}r\ ,$$

where we have set $f(s,\cdot)=u_\phi(s_1-s,\cdot)$ to simplify the notation (so that f satisfies (3.5) and (3.8), with $\dot f+\mathcal{L}f=0$). Observe that, $Df(r,x,y)=u_\phi(s_1-r,x-y)+u_\phi(s_1-r,y)-u_\phi(s_1-r,x)$. Then, by Proposition 3.11 (ii),

$$R_f(r,x)=2e^{-2r}\int_0^x(Df(r,x,y))^2\mathrm{d}y\le 2\times 17^2e^{-2r}\|\phi\|_\infty^2\,x.$$

Thus, $\mathbf{E}_x\big[\langle\overline{\mathcal{M}}_r,R_f\rangle\big]\le 2\times 17^2\|\phi\|_\infty^2e^{-4r}\mathbf{E}_x[M_r]$. By (3.10), $e^{-2r}\mathbf{E}_x[M_r]\le x+1$. Thus,

$$\|\langle\overline{\mathcal{M}}_{s_1},\phi\rangle-\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle\|_2\le 17e^{-s}\|\phi\|_\infty\sqrt{x+1}\le 17e^{-s}\|\phi\|_\infty(x+1)$$

where $\|\cdot\|_2$ stands for the L^2 norm with respect to $\mathbf{P}_x$.

We next compare $\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle$ to $\frac13\langle\boldsymbol{\gamma}_2,\phi\rangle R_\infty$. To this end, we set $c=\frac13\langle\boldsymbol{\gamma}_2,\phi\rangle$ and $g(x)=2x+1$. Note that $\frac13\langle\boldsymbol{\gamma}_2,\phi\rangle R_s=\langle\overline{\mathcal{M}}_s,cg\rangle$. Then, Proposition 3.11 (i) asserts that $|u_\phi(s_1-s,y)-cg(y)|\le 39e^{-3(s_1-s)}\|\phi\|_\infty g(y)$. Thus,

$$\big|\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle-cR_s\big|\le 39e^{-3(s_1-s)}\|\phi\|_\infty R_s.$$

By Proposition 3.4 (ii), $\mathbf{E}_x[R_s^2]\le(2x+1)^2+\frac15(4x+1)\le 9(x+1)^2$. Thus,

$$\|\langle\overline{\mathcal{M}}_s,u_\phi(s_1-s,\cdot)\rangle-cR_s\|_2\le 117e^{-3(s_1-s)}\|\phi\|_\infty(x+1).$$

The same Proposition 3.4 (ii) implies that

$$\mathbf{E}_x\big[(R_\infty-R_s)^2\big]=\mathbf{E}_x\big[R_\infty^2\big]-\mathbf{E}_x\big[R_s^2\big]=\tfrac13(2x+1)e^{-2s}+\tfrac{2}{15}(x-1)e^{-5s}\le(2x+1)e^{-2s}\ .$$

Thus,

$$\|cR_\infty-cR_s\|_2\le ce^{-s}\sqrt{2x+1}\le\tfrac13e^{-s}\|\phi\|_\infty\sqrt{2x+1}\le e^{-s}\|\phi\|_\infty(x+1).$$

For all $s,s_1\in[0,\infty)$ such that $s\le s_1$, we get

$$\|\langle\overline{\mathcal{M}}_{s_1},\phi\rangle-cR_\infty\|_2\le(18e^{-s}+117e^{-3(s_1-s)})\|\phi\|_\infty(x+1).$$

By choosing $s=\frac34s_1$, we get for all $s_1\in(0,\infty)$

$$\sqrt{\mathbf{E}_x\big[\big(\langle\overline{\mathcal{M}}_{s_1},\phi\rangle-\tfrac13R_\infty\langle\boldsymbol{\gamma}_2,\phi\rangle\big)^2\big]}\le 135\,e^{-\frac34s_1}\|\phi\|_\infty(x+1),$$

which is the desired result.

Let us prove (ii). To this end, we fix ϕ to be C^1 and we assume that ϕ *and* ϕ' *are bounded, that* ϕ *is nonnegative and non decreasing*, and we first want to prove that

$$\text{(3.24)}\qquad \mathbf{P}_x\text{-a.s.}\quad \lim_{s\to\infty}\langle\overline{\mathcal{M}}_s,\phi\rangle=\tfrac13R_\infty\langle\boldsymbol{\gamma}_2,\phi\rangle\ .$$

For all $n\in\mathbb{N}^*$, we set $s_n=\log n$ and by (3.23), we get $\mathbf{E}_x\big[\sum_{n\ge1}\big(\langle\overline{\mathcal{M}}_{s_n},\phi\rangle-\frac13R_\infty\langle\boldsymbol{\gamma}_2,\phi\rangle\big)^2\big]<\infty$. Therefore

$$\text{(3.25)}\qquad \mathbf{P}_x\text{-a.s.}\quad \lim_{n\to\infty}\langle\overline{\mathcal{M}}_{s_n},\phi\rangle=\tfrac13R_\infty\langle\boldsymbol{\gamma}_2,\phi\rangle\ .$$

We next show that $s\mapsto\langle\overline{\mathcal{M}}_s,\phi\rangle$ does not fluctuate much over $[s_n,s_{n+1}]$. To this end we fix $n\in\mathbb{N}^*$ and $s,s'\in[s_n,s_{n+1}]$ such that $s\le s'$. We also set

$$\varepsilon_n=2(s_{n+1}-s_n)\quad\text{and}\quad\Delta_n=\sup_{s\in[s_n,\infty)}\left|e^{-2s}N_s-\tfrac{1}{3}R_\infty\right|.$$

Recall from Proposition 3.4 that $\mathbf{P}_x$-a.s. $\lim_{n\to\infty}\Delta_n=0$. Next, recall that G_s (resp. $G_{s'}$) stands for the set of cells that are alive at time s (resp. at time s'). Then note that if $u\in G_s\cap G_{s'}$, then $\zeta_{\overleftarrow{u}}\le s\le s'<\zeta_u$ and (2.28) implies that $X_u(s')=e^{s'-s}(X_u(s)+1)-1\ge X_u(s)$. Since ϕ is non decreasing, $\phi(X_u(s'))\ge\phi(X_u(s))$ and

$$\begin{aligned}
\langle\overline{\mathcal{M}}_s,\phi\rangle &= \sum_{u\in G_s\cap G_{s'}}e^{-2s}\phi(X_u(s)) \;+\; \sum_{u\in G_s\backslash(G_s\cap G_{s'})}e^{-2s}\phi(X_u(s))\\
&\le e^{2(s'-s)}\sum_{u\in G_s\cap G_{s'}}e^{-2s'}\phi(X_u(s')) \;+\; \sum_{u\in G_s\backslash(G_s\cap G_{s'})}e^{-2s}\phi(X_u(s))\\
&= e^{2(s'-s)}\langle\overline{\mathcal{M}}_{s'},\phi\rangle \;-\; \sum_{u\in G_{s'}\backslash(G_s\cap G_{s'})}e^{-2s}\phi(X_u(s')) \;+\; \sum_{u\in G_s\backslash(G_s\cap G_{s'})}e^{-2s}\phi(X_u(s))\\
&\le e^{2(s'-s)}\langle\overline{\mathcal{M}}_{s'},\phi\rangle+e^{-2s}\|\phi\|_\infty\#(G_s\backslash(G_s\cap G_{s'}))\\
&\le e^{2(s'-s)}\langle\overline{\mathcal{M}}_{s'},\phi\rangle+e^{-2s}\|\phi\|_\infty(N_{s'}-N_s)\,,
\end{aligned}$$

since ϕ is nonnegative and bounded, and since $\#(G_s\backslash(G_s\cap G_{s'}))\le N_{s'}-N_s$. Thus,

$$\begin{aligned}
\langle\overline{\mathcal{M}}_s,\phi\rangle &\le e^{\varepsilon_n}\langle\overline{\mathcal{M}}_{s'},\phi\rangle+\|\phi\|_\infty(1+e^{\varepsilon_n})\Delta_n+\tfrac{1}{3}R_\infty\|\phi\|_\infty(e^{\varepsilon_n}-1)\\
&\le e^{\varepsilon_n}\langle\overline{\mathcal{M}}_{s_{n+1}},\phi\rangle+\|\phi\|_\infty(1+e^{\varepsilon_n})\Delta_n+\tfrac{1}{3}R_\infty\|\phi\|_\infty(e^{\varepsilon_n}-1).
\end{aligned}\tag{3.26}$$

On the other hand, replacing (s,s') in (3.26) by (s_n,s) gives that

$$\langle\overline{\mathcal{M}}_s,\phi\rangle\ge e^{-\varepsilon_n}\langle\overline{\mathcal{M}}_{s_n},\phi\rangle-\|\phi\|_\infty(1+e^{-\varepsilon_n})\Delta_n-\tfrac{1}{3}R_\infty\|\phi\|_\infty(1-e^{-\varepsilon_n}).$$

Combining the last two inequalities with (3.25) implies (3.24).

We now complete the proof of (ii). For all $r,r'\in\mathbb{Q}_+$ such that $r<r'$ there exists a C^1 function $\phi_{r,r'}$ on $[0,\infty)$ that is non decreasing and such that $\phi'_{r,r'}$ is bounded and $\mathbf{1}_{(r',\infty)}(y)\le\phi_{r,r'}(y)\le\mathbf{1}_{(r,\infty)}(y)$, for all $y\in[0,\infty)$. We denote by

$$\Omega_0=\bigcap_{r,r'\in\mathbb{Q}_+:\,r<r'}\left\{\lim_{s\to\infty}\langle\overline{\mathcal{M}}_s,\phi_{r,r'}\rangle=\tfrac{1}{3}R_\infty\langle\boldsymbol{\gamma}_2,\phi_{r,r'}\rangle\right\}.$$

By (3.24), $\mathbf{P}_x(\Omega_0)=1$, for all $x\in[0,\infty)$. We then fix $y\in(0,\infty)$ and we choose $(r_n)_{n\in\mathbb{N}}$ and $(r'_n)_{n\in\mathbb{N}}$ that are two sequences of nonnegative rational numbers converging to y such that $r_n\le r_{n+1}\le y<r'_{n+1}<r'_n$ for all $n\in\mathbb{N}$. Then $\phi_{r'_n,r'_n+2^{-n}}(z)\le\mathbf{1}_{(y,\infty)}(z)\le\phi_{(r_n-2^{-n})_+,r_n}(z)$ for all $z\in[0,\infty)$. Thus, on Ω_0, for all $n\in\mathbb{N}$,

$$\tfrac{1}{3}R_\infty\langle\boldsymbol{\gamma}_2,\phi_{r'_n,r'_n+2^{-n}}\rangle\le\liminf_{s\to\infty}\overline{\mathcal{M}}_s((y,\infty))\le\limsup_{s\to\infty}\overline{\mathcal{M}}_s((y,\infty))\le\tfrac{1}{3}R_\infty\langle\boldsymbol{\gamma}_2,\phi_{(r_n-2^{-n})_+,r_n}\rangle$$

Since $\boldsymbol{\gamma}_2$ is a diffuse probability measure, we get

$$\lim_{n\to\infty}\langle\boldsymbol{\gamma}_2,\phi_{r'_n,r'_n+2^{-n}}\rangle=\lim_{n\to\infty}\langle\boldsymbol{\gamma}_2,\phi_{(r_n-2^{-n})_+,r_n}\rangle=\boldsymbol{\gamma}_2((y,\infty)).$$

Therefore,

$$\text{on }\Omega_0\text{, for all }y\in(0,\infty),\quad\lim_{s\to\infty}\overline{\mathcal{M}}_s((y,\infty))=\tfrac{1}{3}R_\infty\boldsymbol{\gamma}_2((y,\infty)),$$

which implies the desired result. ■

3.6. The maximal cell size at a given time

Let $\mathbf{X}=(X_u(s))_{u\in\mathbb{T}_2,s\in[0,\infty)}$ be a HR growth-fragmentation process as in Definition 2.13. Recall that $G_s=\{u\in\mathbb{T}_2:\zeta_{\overleftarrow{u}}\le s<\zeta_u\}$ is the set of cells alive at time s. We prove the following almost sure convergence (when $s\to\infty$) for the maximal cell size at time s.

Theorem 3.14. *For all $s\in[0,\infty)$, we set $\overline{X}_s=\max_{u\in G_s} X_u(s)$ that is the size of the largest cell alive at time s. Then, for all $x\in[0,\infty)$,*

$$\lim_{s\to\infty}\frac{\overline{X}_s}{s}=1 \qquad \mathbf{P}_x\text{-a.s.} \tag{3.27}$$

Proof. Let $s\in[0,\infty)$. Recall from (3.3) the definition of $\overline{\mathcal{M}}_s$ and from (3.16) the definition of $\boldsymbol{\mu}_s^{(x)}$. For all $z\in[0,\infty)$, we get

$$\begin{aligned}\mathbf{P}_x\big(\overline{X}_s\geq z\big) &\leq \mathbf{E}_x\Big[\sum_{u\in G_s}\mathbf{1}_{\{X_u(s)\geq z\}}\Big]=\mathbf{E}_x\big[e^{2s}\langle\overline{\mathcal{M}}_s,\mathbf{1}_{[z,\infty)}\rangle\big]=e^{2s}\boldsymbol{\mu}_s^{(x)}([z,\infty))\\ &\leq \tfrac{1}{3}(2x+1)(1+2z)e^{-2z+2s}+39(2x+1)e^{-s}\end{aligned} \tag{3.28}$$

by Proposition 3.11 (i) applied to $\phi=\mathbf{1}_{[z,\infty)}$ (and noting that $\boldsymbol{\gamma}_2([z,\infty))=\int_z^\infty 4ye^{-2y}\mathrm{d}y=(1+2z)e^{-2z}$).

We fix $\varepsilon\in(0,1)$. For all $n\in\mathbb{N}^*$, we set $s_n=n^{\frac{1}{2+\varepsilon}}$ and $z_n=s_n+(2+\varepsilon)\log s_n$. By (3.28) we get

$$\mathbf{P}_x\big(\overline{X}_{s_n}\geq z_n\big)\leq \tfrac{1}{3}(2x+1)s_n^{-2(2+\varepsilon)}(1+2z_n)+39(2x+1)e^{-s_n}\sim \tfrac{2}{3}(2x+1)n^{-\frac{3+2\varepsilon}{2+\varepsilon}}.$$

Therefore, $\sum_{n\in\mathbb{N}^*}\mathbf{P}_x\big(\overline{X}_{s_n}\geq z_n\big)<\infty$. So by the Borel–Cantelli Lemma,

$$\mathbf{P}_x\text{-a.s. for all sufficiently large } n\in\mathbb{N}^*, \quad \overline{X}_{s_n}\leq s_n+(2+\varepsilon)\log s_n. \tag{3.29}$$

On the other hand, by (2.28), we have, for all $u\in\mathbb{T}_2$ and for all real numbers $s'\geq s\geq 0$,

$$X_u(s')\leq e^{s'-s}(X_u(s)+1)-1\,. \tag{3.30}$$

Let $u'\in G_{s'}$ be such that $X_{u'}(s')=\overline{X}_{s'}$. Then, there exists $u\in G_s$ such that $X_{u'}(s)=X_u(s)$ and thus $X_{u'}(s)\leq\overline{X}_s$. Therefore, (3.30) implies that for all real numbers $s'\geq s\geq 0$,

$$\overline{X}_{s'}\leq e^{s'-s}(\overline{X}_s+1)-1\,. \tag{3.31}$$

To simplify notation, we set $\Delta_n=s_{n+1}-s_n$. By (3.31), we get that for $s\in[s_n,s_{n+1}]$,

$$\overline{X}_s-s\leq e^{\Delta_n}\big(\overline{X}_{s_n}-s_n\big)+(1+s_n)\big(e^{\Delta_n}-1\big).$$

By the mean-value theorem, $\Delta_n\leq(2+\varepsilon)^{-1}n^{-\frac{1+\varepsilon}{2+\varepsilon}}$ (in particular, $\lim_{n\to\infty}\Delta_n=0$), hence $(1+s_n)\big(e^{\Delta_n}-1\big)\leq 2s_n\Delta_n\leq 2(2+\varepsilon)^{-1}n^{-\frac{\varepsilon}{2+\varepsilon}}\to 0$. Thus, by (3.29), $\mathbf{P}_x$-a.s. for all n large enough and for all $s\in[s_n,s_{n+1}]$,

$$\overline{X}_s-s\leq e^{\Delta_n}(2+\varepsilon)\log s+\tfrac{2}{2+\varepsilon}n^{-\frac{\varepsilon}{2+\varepsilon}}$$

which implies that $\mathbf{P}_x$-a.s. $\limsup_{s\to\infty}\big(\overline{X}_s-s\big)/\log s\leq 2+\varepsilon$. It entails the lower bound in (3.27).

To prove the first bound in (3.27), we rely on the time-branching property and on (2.23) that shows that the genealogically typical evolution of a particle is rapidly stationary. More precisely, let us recall how the genealogically typical evolution appears in the HR growth-fragmentation process $(X_u(s))_{s\in[0,\infty),u\in\mathbb{T}_2}$. To this end, let $s\in[0,\infty)$ be fixed and let us denote by u^* the smallest $u\in G_s$ with respect to the lexicographical order (u^* is thus a word written with $|u^*|$ successive "1"s). Then,

$$\text{under } \mathbf{P}_x \quad X_{u^*}(s)\overset{\text{(law)}}{=}X_s\,, \tag{3.32}$$

where X_s stands for the genealogically typical evolution of a cell as in Definition 2.10.

We next fix $s_0,s\in[0,\infty)$ such that $s_0<s$ and we set $\delta=s-s_0$. Recall that $N_{s_0}=\#G_{s_0}$ is the number of cells that are alive at time s_0 and denote by $u_1,\dots,u_{N_{s_0}}$ the words $u\in G_{s_0}$ listed in the lexicographical order. For all $j\in\{1,\dots,N_{s_0}\}$, we denote by u_j^* the smallest $u\in G_s$ (with respect to the lexicographical order) that is a descendent of u_j, i.e. such that $u_j\in[\![\varnothing,u_j^*]\!]$. Then, the time-branching property in Proposition 2.14 (ii) combined with (3.32) imply the following: let $\mathscr{F}_{s_0}$ be the sigma field generated by the r.v. $X_u(r)$, $r\in[0,s_0]$, $u\in\mathbb{T}_2$; conditionally on $\mathscr{F}_{s_0}$, the r.v. $(X_{u_j^*}(s))_{1\leq j\leq N_{s_0}}$

are independent and the conditional law of $X_{u_j^*}(s)$ is that of X_δ under $\mathbf{P}_{X_{u_j}(s_0)}$. Since $\overline{X}_s \geq \max_{1\leq j\leq N_{s_0}} X_{u_j^*}(s)$, we obtain, for all $z\in[0,\infty)$,

$$\mathbf{P}_x\text{-a.s.}\quad \mathbf{E}_x\big[\mathbf{1}_{\{\overline{X}_s\leq z\}}\big|\mathscr{F}_{s_0}\big] \leq \prod_{1\leq j\leq N_{s_0}} \mathbf{P}_{X_{u_j}(s_0)}\big(X_\delta \leq z\big)\,. \tag{3.33}$$

Recall that $\nu(\mathrm{d}y)=2y(y+1)e^{-2y}\mathrm{d}y$ is the the unique stationary law of the process $(X_r)_{r\in[0,\infty)}$ and recall from (2.23) in Proposition 2.12 (ii) that for all $x'\in[0,\infty)$,

$$\big|\mathbf{P}_{x'}(X_\delta\leq z)-1+(1+z)^2e^{-2z}\big|\leq 5e^{-2(e^\delta-1-\delta)}$$

since $\nu([z,\infty))=(1+z)^2e^{-2z}$. We next assume that $\lambda:=(1+z)^2e^{-2z}-5e^{-2(e^\delta-1-\delta)}\in(0,1)$ and we derive from (3.33) and the previous inequality that

$$\mathbf{P}_x\big(\overline{X}_s\leq z\big)\leq \mathbf{E}_x\big[(1-\lambda)^{N_{s_0}}\big]\leq \mathbf{E}_x\big[e^{-\lambda N_{s_0}}\big]. \tag{3.34}$$

The following lemma allows to control $\mathbf{E}_x\big[e^{-\lambda N_{s_0}}\big]$ when λ is small but $e^{2s_0}\lambda$ is large.

Lemma 3.15. *Let $x,s_0\in[0,\infty)$ and let $\varepsilon\in(0,1)$. There exists $\lambda_\varepsilon\in(0,\infty)$ that only depends on ε such that for all $\lambda\in[0,\lambda_\varepsilon]$, we get*

$$\mathbf{E}_x\big[e^{-\lambda N_{s_0}}\big]\leq \exp\Big(2\lambda e^{3s_0/2}-\varphi\Big(\sqrt{\tfrac{\lambda e^{2s_0}}{1+\varepsilon}}\Big)\Big)+(x^2+x+2)e^{-s_0} \tag{3.35}$$

where we recall that $\varphi(y)=2\log(y^{-1}\sinh y)$, $y\in[0,\infty)$.

Proof. To prove (3.35), we rely on the exact computation (3.15) in Proposition 3.7 and on the second moments of M_{s_0} and N_{s_0} given in Proposition 3.4. Namely, recall from this proposition the notation $D_{s_0}=e^{s_0}(M_{s_0}-N_{s_0})$. By Proposition 3.4 (ii), we get $\mathbf{E}_x[D_{s_0}^2]\leq(x^2+x+2)e^{4s_0}$. Therefore, for all $\lambda\in[0,\infty)$,

$$\begin{aligned}
\mathbf{E}_x\big[e^{-\lambda N_{s_0}}\big] &= \mathbf{E}_x\big[e^{-\lambda N_{s_0}}\mathbf{1}_{\{|M_{s_0}-N_{s_0}|\leq e^{3s_0/2}\}}\big]+\mathbf{E}_x\big[e^{-\lambda N_{s_0}}\mathbf{1}_{\{|M_{s_0}-N_{s_0}|> e^{3s_0/2}\}}\big]\\
&\leq \mathbf{E}_x\big[e^{-\frac13\lambda N_{s_0}-\frac23\lambda(M_{s_0}-e^{3s_0/2})}\mathbf{1}_{\{|M_{s_0}-N_{s_0}|\leq e^{3s_0/2}\}}\big]+\mathbf{P}_x\big(|M_{s_0}-N_{s_0}|>e^{3s_0/2}\big)\\
&\leq e^{\frac23\lambda e^{3s_0/2}}\mathbf{E}_x\big[e^{-\frac13\lambda N_{s_0}-\frac23\lambda M_{s_0}}\big]+\mathbf{E}_x\big[D_{s_0}^2\big]e^{-5s_0}\\
&\leq e^{\frac23\lambda e^{3s_0/2}}\mathbf{E}_x\big[e^{-\frac13\lambda N_{s_0}-\frac23\lambda M_{s_0}}\big]+(x^2+x+2)e^{-s_0}\,.
\end{aligned} \tag{3.36}$$

Recall from (3.14) that $\psi(y)=2(y\coth(y)-1)$ and observe that $\varphi(\sqrt{\lambda})\sim\frac13\lambda$ and $\psi(\sqrt{\lambda})\sim\frac23\lambda$ as λ goes to 0. Thus there is $\lambda_\varepsilon\in(0,\infty)$ that only depends on ε such that for all $\lambda\in[0,\lambda_\varepsilon]$, we get $\varphi\big(\sqrt{\frac{\lambda}{1+\varepsilon}}\big)\leq\frac13\lambda$ and $\psi\big(\sqrt{\frac{\lambda}{1+\varepsilon}}\big)\leq\frac23\lambda$, which implies that

$$\begin{aligned}
\mathbf{E}_x\big[e^{-\frac13\lambda N_{s_0}-\frac23\lambda M_{s_0}}\big] &\leq \mathbf{E}_x\Big[\exp\Big(-\varphi\Big(\sqrt{\tfrac{\lambda}{1+\varepsilon}}\Big)N_{s_0}-\psi\Big(\sqrt{\tfrac{\lambda}{1+\varepsilon}}\Big)M_{s_0}\Big)\Big]\\
&= \exp\Big(-\varphi\Big(\sqrt{\tfrac{\lambda e^{2s_0}}{1+\varepsilon}}\Big)-x\psi\Big(\sqrt{\tfrac{\lambda e^{2s_0}}{1+\varepsilon}}\Big)\Big)
\end{aligned}$$

by the exact computation (3.15) in Proposition 3.7. This inequality combined with (3.36) immediately entail (3.35). ■

End of the proof of Theorem 3.14. Recall that $\lambda=(1+z)^2e^{-2z}-5e^{-2(e^\delta-1-\delta)}$ and that $\delta=s-s_0$. We now choose

$$\delta=\log s+\log 4\quad\text{and}\quad z=s-\varepsilon\log s-\log 4\,.$$

As s goes to ∞, we have

$$e^{-s_0}\sim 4se^{-s},\quad \lambda\sim 16s^{2+2\varepsilon}e^{-2s},\quad \lambda e^{2s_0}\sim s^{2\varepsilon},\quad \lambda e^{3s_0/2}\sim 2s^{\frac12+2\varepsilon}e^{-s/2},$$

and therefore

$$a(s):=\varphi\Big(\sqrt{\tfrac{\lambda e^{2s_0}}{1+\varepsilon}}\Big)-2\lambda e^{3s_0/2}\sim\tfrac{2}{\sqrt{1+\varepsilon}}s^\varepsilon.$$

It follows that there exists $s_\varepsilon \in (0,\infty)$ that only depends on ε such that $a(s) \geq s^\varepsilon$ for all $s\in[s_\varepsilon,\infty)$. By (3.34), we thus have proved that for all $x\in[0,\infty)$ and all $s\in[s_\varepsilon,\infty)$,

$$\mathbf{P}_x\big(\overline{X}_s \leq s-\varepsilon\log s-\log 4\big) \leq e^{-s^\varepsilon} + 4(x^2+x+2)se^{-s}.$$

For all $n\in\mathbb{N}^*$, we set $s'_n=(2\log n)^{\frac{1}{\varepsilon}}$. The previous inequality and the Borel–Cantelli lemma imply that

$$\mathbf{P}_x\text{-a.s. for all sufficiently large } n\in\mathbb{N}^*, \quad \overline{X}_{s'_n} \geq s'_n-\varepsilon\log s'_n-\log 4\,. \tag{3.37}$$

Write $\Delta'_n=s'_n-s'_{n-1}$. By (3.31), for $s\in[s'_{n-1},s'_n]$, we get

$$\overline{X}_s - s \geq e^{-\Delta'_n}\big(\overline{X}_{s'_n}-s'_n\big) - (1+s'_n)\big(1-e^{-\Delta'_n}\big)$$

By the mean-value theorem and elementary inequalities, there are $c_{1,\varepsilon}$ and $c_{2,\varepsilon}\in(0,\infty)$ that only depend on ε such that $\Delta'_n\leq c_\varepsilon n^{-1}(\log n)^{\frac{1}{\varepsilon}-1}$ and $(1+s'_n)\Delta'_n\leq 2s'_n\Delta'_n\leq c_{2,\varepsilon}n^{-1}(\log n)^{\frac{2}{\varepsilon}-1}$. Thus by (3.37), $\mathbf{P}_x$-a.s. for all n large enough and for all $s\in[s'_{n-1},s'_n]$, we get

$$\overline{X}_s - s \geq -e^{-\Delta'_n}(\varepsilon\log s'_n+\log 4) - c_{2,\varepsilon}n^{-1}(\log n)^{\frac{2}{\varepsilon}-1}$$

which implies that $\mathbf{P}_x$-a.s. $\liminf_{s\to\infty}\big(\overline{X}_s-s\big)/\log s \geq -\varepsilon$. It entails the lower bound in (3.27). This completes the proof of Theorem 3.14. ■

3.7. Proof of Theorem 3.1

Recall that $a=(\frac{1}{2}e^{-s_0})^{11}/10$ and note that $a\in(0,1)$. To simplify notation we set $Z=e^{a(N_{s_0}+M_{s_0})}$. Then, by (3.4), for all $\lambda\in[0,1]$ and for all $x\in[0,\infty)$, we get

$$Z\geq 1 \quad\text{and}\quad \mathbf{E}_x\big[Z^{10\lambda}\big]\leq e^{\lambda(1+x)}\,.$$

We first fix $x\in[0,\infty)$ and f that satisfies (3.5) and (3.8). Let us prove that the r.v. $e^{\langle\overline{\mathcal{M}}_s,f\rangle}$ and $e^{\langle\overline{\mathcal{M}}_s,f\rangle}\langle\overline{\mathcal{M}}_s,\dot{f}+\mathcal{L}f+Q_f\rangle$ are $\mathbf{P}_x$-integrable. To this end we shall use repeatedly the following elementary inequalites for all $c,d\in(0,\infty)$ and all $k\in\{0,1,2\}$

$$\forall x\in[0,\infty),\quad (x+1)^c\leq c^c d^{-c}e^{d(x+1)} \quad\text{and}\quad \forall z\in\mathbb{C},\quad |K^{(k)}(z)|\leq|z|^{2-k}e^{(\mathrm{Re}\,z)_+}. \tag{3.38}$$

For all $z\in\mathbb{C}$ such that $\mathrm{Re}(z)\in[0,1]$, we set

$$Y_s(zf) = e^{z\langle\overline{\mathcal{M}}_s,f\rangle} \quad\text{and}\quad W_s(zf)=e^{z\langle\overline{\mathcal{M}}_s,f\rangle}\langle\overline{\mathcal{M}}_s\,,z\dot{f}+z\mathcal{L}f+Q_{zf}\rangle\,.$$

We first prove that $Y_s(zf)$, $\partial_zY_s(zf)$, $\partial_z^2Y_s(zf)$, $W_s(zf)$, $\partial_zW_s(zf)$ and $\partial_z^2W_s(zf)$ are $\mathbf{P}_x$-integrable. To this end, on $[0,s_0]\times[0,\infty)$, observe that

$$|\dot{f}+\mathcal{L}f|\leq 8b_f(1+x)e^{a(x+1)} \quad\text{and}\quad |Df|\leq 3b_fe^{a(x+1)}\,. \tag{3.39}$$

Recall that $\mathrm{Re}(z)\in[0,1]$. Thus by (3.8) and the second inequality in (3.38), for $k\in\{0,1,2\}$,

$$\begin{aligned}|\partial_z^{(k)}Q_{zf}(s,x)| &\leq 2e^{-2s}|z|^{2-k}\int_0^x(Df(s,x,y))^2e^{e^{-2s}\mathrm{Re}(z)(Df(s,x,y))_+}\,\mathrm{d}y\\ &\leq 18b_f^2|z|^{2-k}(x+1)e^{3a(x+1)}.\end{aligned} \tag{3.40}$$

We next observe that

$$|\langle\overline{\mathcal{M}}_s,f\rangle|\leq e^{-2s}b_f\sum_{s\in G_s}e^{a(1+X_u(s)}\leq b_fN_se^{a(N_s+M_s)}\leq b_fa^{-1}e^{2a(N_s+M_s)}\leq b_fa^{-1}Z^2,$$

since $\#G_s = N_s$ and by the first inequality in (3.38). Again, by the first inequality in (3.38) we get for all $d\in(0,10]$ and all $c\in(0,\infty)$,

$$\begin{aligned}\langle \overline{\mathcal{M}}_s, x\mapsto (1+x)^c e^{da(x+1)}\rangle &= e^{-2s}\sum_{u\in G_s}(1+X_u(s))^c e^{da(1+X_u(s))} \le \sum_{u\in G_s} c^c a^{-c} e^{(d+1)a(1+X_u(s))}\\ &\le \sum_{u\in G_s} c^c a^{-c} e^{(d+1)a(N_s+M_s)} = c^c a^{-c} N_s e^{(d+1)a(N_s+M_s)}\\ &\le c^c a^{-c-1} e^{(d+2)a(N_s+M_s)} \le c^c a^{-c-1} Z^{d+2}. \end{aligned} \tag{3.41}$$

By (3.8), we also get $|Y_s(zf)| = e^{\mathrm{Re}(z)\langle\overline{\mathcal{M}}_s, f\rangle} \le e^{a(M_s+N_s)} \le Z$.

Let us next bound $|W_s(zf)|$. By (3.40) and (3.41), we get $|\langle\overline{\mathcal{M}}_s, Q_{zf}\rangle| \le 18|z^2|b_f^2 a^{-2} Z^5$; by (3.39) and (3.41), we also get $|\langle\overline{\mathcal{M}}_s, z(\dot f + \mathcal{L}f)\rangle| \le 8|z|b_f a^{-2} Z^3$. Since $b_f \ge 1$, we finally get $|W_s(zf)| \le 26(b_f/a)^2(1\vee|z|^2)Z^6$. Similarly $|\partial_z W_s(zf)| \le 52(b_f/a)^3(1\vee|z|^2)Z^8$ and $|\partial_z W_s(zf)| \le 96(b_f/a)^4(1\vee|z|^2)Z^{10}$. Similar computations for $Y_s(zf)$ imply that for all $(s,x)\in[0,s_0]\times[0,\infty)$ and all $z\in\mathbb{C}$ such that $\mathrm{Re}(z)\in[0,1]$,

$$\begin{aligned}|W_s(zf)| + |\partial_z W_s(zf)| + |\partial_z^2 W_s(zf)| &\le 174\,(b_f/a)^4(1\vee|z|)^2 Z^{10}\\ \text{and}\quad |Y_s(zf)| + |\partial_z Y_s(zf)| + |\partial_z^2 Y_s(zf)| &\le 3(b_f/a)^2 Z^5.\end{aligned} \tag{3.42}$$

These bounds with $z=1$ imply that the r.v. $e^{\langle\overline{\mathcal{M}}_s, f\rangle}$ and $e^{\langle\overline{\mathcal{M}}_s, f\rangle}\langle\overline{\mathcal{M}}_s, \dot f + \mathcal{L}f + Q_f\rangle$ are $\mathbf{P}_x$-integrable.

Next let us assume that (3.9) holds true under a more restrictive assumption. More precisely, let us suppose that the following lemma holds true.

Lemma 3.16. *Let $f\in C^{1,1}([0,\infty)^2,\mathbb{R})$ be such that for all $(s,x)\in[0,s_0+1]\times[0,\infty)$,*

$$\frac{|f(s,x)| + |\dot f(s,x)|}{x+1} + |f'(s,x)| < \tfrac{1}{3}a^2 \tag{3.43}$$

Then, f satisfies (3.5) and (3.8), and

$$\forall (s,x)\in[0,s_0]\times[0,\infty),\quad \mathbf{E}_x[Y_s(f)] = e^{f(0,x)} + \int_0^s \mathbf{E}_x\big[W_r(f)\big]\,\mathrm{d}r. \tag{3.44}$$

Proof. *We prove this lemma later on.* ■

Let us assume that Lemma 3.16 holds true and let us first prove that it entails (3.6) and (3.7). Let $f\in C^{1,1}([0,\infty)^2,\mathbb{R})$ satisfy (3.5). We truncate f and multiply it by a small constant so that the resulting function satisfies (3.43). More precisely, let $\rho\in(0,\infty)$ and let $\phi_\rho:[0,\infty)\to\mathbb{R}$ be C^2, nonincreasing and such that $|\phi_\rho'|\le 1$ and $\mathbf{1}_{[0,\rho]} \le \phi_\rho \le \mathbf{1}_{[0,\rho+2]}$. To simplify the notation, we set $f_\rho(s,x) = f(s,x)\phi_\rho(x)$. Then, there exists $\lambda_0\in(0,1)$ such that for all $\lambda\in[0,\lambda_0]$, λf_ρ satisfies the assumption of Lemma 3.16 and thus $\mathbf{E}_x[Y_s(\lambda f_\rho)] = e^{\lambda f_\rho(0,x)} + \int_0^s \mathbf{E}_x\big[W_r(\lambda f_\rho)\big]\,\mathrm{d}r$. By (3.42), we can differentiate in λ the previous equation to get

$$\mathbf{E}_x[\partial_\lambda^{(k)} Y_s(\lambda f_\rho)] = (f_\rho(0,x))^k e^{\lambda f_\rho(0,x)} + \int_0^s \mathbf{E}_x\big[\partial_\lambda^{(k)} W_r(\lambda f_\rho)\big]\,\mathrm{d}r$$

for $k\in\{1,2\}$. Taking $\lambda=0$ in the previous equation implies (3.6) et (3.7) for f_ρ.

Next observe that $\langle\overline{\mathcal{M}}_s, f_\rho\rangle$, $\langle\overline{\mathcal{M}}_s, \dot f_\rho + \mathcal{L}f_\rho\rangle$ and $\langle\overline{\mathcal{M}}_s, R_{f_\rho}\rangle$ converge to respectively $\langle\overline{\mathcal{M}}_s, f\rangle$, $\langle\overline{\mathcal{M}}_s, \dot f + \mathcal{L}f\rangle$ and $\langle\overline{\mathcal{M}}_s, R_f\rangle$ as $\rho\to\infty$. Then note that $b_{f_\rho}\le 2b_f$. Therefore, by (3.42) we get $|\langle\overline{\mathcal{M}}_s, f_\rho\rangle| \le 2b_f a^{-1}Z^2$, $|\langle\overline{\mathcal{M}}_s, \dot f_\rho + \mathcal{L}f_\rho\rangle| \le 12 b_f a^{-2} Z^3$ and $|\langle\overline{\mathcal{M}}_s, R_{f_\rho}\rangle| \le 72 b_f^2 a^{-2} Z^4$ for all $s\in[0,s_0]$ and $\rho\in[0,\infty)$. Thus, dominated convergence applies in (3.6) and (3.7) to f_ρ when $\rho\to\infty$, which completes the proof of (ii).

We next prove that Lemma 3.16 implies (3.9) in the general case. So in addition to (3.5), we assume that f satisfies (3.8) and we keep the previous notation for f_ρ and λ_0.

We first observe that for all $s\in[0,s_0]$, $z\in\mathbb{C}\mapsto Y_s(zf_\rho)$ and $z\in\mathbb{C}\mapsto W_s(zf_\rho)$ are analytical. We then introduce the open half-disk $U=\{z\in\mathbb{C} : |z|<1 \text{ and } 0<\mathrm{Re}(z)<1\}$. By (3.42), the function $g(z) = \mathbf{E}_x[Y_s(zf_\rho)] - e^{zf_\rho(0,x)} - \int_0^s \mathbf{E}_x[W_s(zf_\rho)]\mathrm{d}r$ is well-defined on $\overline{U}$ and it is analytical on U. As mentioned previously, for all $\lambda\in[0,\lambda_0]$, λf_ρ satisfies the assumption of Lemma 3.16. Therefore, for all $\lambda\in(0,\lambda_0]\subset U$, we get $g(\lambda)=0$. It implies that g vanishes on $\overline{U}$ and in particular at $z=1$. Namely, $\mathbf{E}_x[Y_s(f_\rho)] = e^{f_\rho(0,x)} + \int_0^s \mathbf{E}_x[W_s(f_\rho)]\,\mathrm{d}r$. We get (3.9) by dominated convergence as $\rho\to\infty$ by the upper bounds (3.42). This completes the proof of Theorem 3.1 under the assumption that Lemma 3.16 holds true.

Proof of Lemma 3.16. We first prove the following lemma that provide a control of $\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_s,f\rangle}\big]$ in a right-neighbourhood of $s=0$ and that is uniform in the space variable x.

Lemma 3.17. *Let f be as in Lemma 3.16. Then, for all $x\in[0,\infty)$,*

$$\Big|e^{-f(0,x)}\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_s,f\rangle}\big]-1\Big|\le 14a^{-1}s\,e^{6a(x+1)},\ s\in[0,a],\quad \textit{and} \tag{3.45}$$

$$\lim_{s\to0^+}\tfrac{1}{s}\big(e^{-f(0,x)}\mathbf{E}_x[e^{\langle\overline{\mathcal{M}}_s,f\rangle}]-1\big)=\dot f(0,x)+\mathcal{L}f(0,x)+Q_f(0,x). \tag{3.46}$$

Proof. By the time-branching property at the first time of cell division $\zeta_\varnothing$ (Proposition 2.14 (i)) we get $\mathbf{E}_x[e^{\langle\overline{\mathcal{M}}_s,f\rangle}]=T_1(s,x)+T_2(s,x)$ where

$$T_1(s,x)=\mathbf{P}_x(\zeta_\varnothing>s)\exp\big(e^{-2s}f(s,e^s(x+1)-1)\big),$$
$$T_2(s,x)=\int_0^s\mathbf{P}_x(\zeta_\varnothing\in\mathrm{d}r)\int_0^1\mathrm{d}u\ \mathbf{E}_{u(e^r(x+1)-1)}\big[e^{e^{-2r}\langle\overline{\mathcal{M}}_{s-r},f\rangle}\big]\,\mathbf{E}_{(1-u)(e^r(x+1)-1)}\big[e^{e^{-2r}\langle\overline{\mathcal{M}}_{s-r},f\rangle}\big],$$

and where $\mathbf{P}_x(\zeta_\varnothing\in\mathrm{d}s)=2(e^s(x+1)-1)e^{2s-2(e^s-1)(x+1)}\mathrm{d}s$. See Lemma 2 p. 825 in Gonzalez, Horton and Kyprianou [15] for similar decompositions at the first branching point for more general models.

First note that $\dot T_1/T_1=e^{-2s}(\dot f-2f)+e^{-s}(x+1)f'-2(e^s(x+1)-1)$. So we see that

$$\lim_{s\to0^+}\tfrac{1}{s}\big(e^{-f(0,x)}T_1(s,x)-1\big)=\frac{\dot T_1(0,x)}{T_1(0,x)}=\dot f(0,x)-2f(0,x)+(x+1)f'(0,x)-2x. \tag{3.47}$$

We use (3.43) to observe that $|\dot T_1/T_1|\le2(e^s+a^2)(x+1)$ and that $T_1\le e^{a^2(x+1)}$. Since $s\le a\le1$, we get $|\dot T_1|\le8(x+1)e^{a^2(x+1)}\le8a^{-1}e^{(a+a^2)(x+1)}\le8a^{-1}e^{2a(x+1)}$. Thus,

$$\forall(s,x)\in[0,\infty)^2,\quad \big|e^{-f(0,x)}T_1(s,x)-1\big|=e^{-f(0,x)}\Big|\int_0^s\dot T_1(r,x)\,\mathrm{d}r\Big|\le8a^{-1}s\,e^{2a(x+1)}\,. \tag{3.48}$$

We next bound T_2. To this end, observe that by (3.43), $|e^{-2r}\langle\overline{\mathcal{M}}_{s-r},f\rangle|\le a^2(N_{s-r}+M_{s-r})$, for all $r\in[0,s]$. Thus by (3.4) applied to $\lambda=a$, we get $\mathbf{E}_y\big[e^{e^{-2r}\langle\overline{\mathcal{M}}_{s-r},f\rangle}\big]\le e^{a(1+y)}$ for all $y\in[0,\infty)$. Thus,

$$T_2(s,x)\le2s(e^s(x+1)-1)e^{a(1+e^s(x+1))}\le6s(x+1)e^{4a(x+1)}\le6a^{-1}s\,e^{5a(x+1)},$$

Thus by (3.43), $e^{-f(0,x)}T_2(s,x)\le6a^{-1}s\,e^{6a(x+1)}$ and we get (3.45) by (3.48).

By the continuity property in Proposition 2.15 and dominated convergence (using (3.4)), we get $\mathbf{E}_{u(e^r(x+1)-1)}\big[e^{e^{-2r}\langle\overline{\mathcal{M}}_{s-r},f\rangle}\big]\to e^{f(0,ux)}$ as r and s go to 0 with $r\le s$. We get a similar limit when u is replaced by $1-u$, which implies, by dominated convergence again, that

$$\begin{aligned}\lim_{s\to0^+}\tfrac{1}{s}e^{-f(0,x)}T_2(s,x)&=2x\int_0^1\mathrm{d}u\,e^{f(0,ux)+f(0,(1-u)x)-f(0,x)}=2\int_0^x\mathrm{d}y\,e^{Df(0,x,y)}\\&=2\int_0^x\mathrm{d}y\,K\big(Df(0,x,y)\big)+2x-2xf(0,x)+4\int_0^x\mathrm{d}y\,f(0,y).\end{aligned}$$

This implies (3.46) by (3.47). ∎

Lemma 3.18. *Let f be as in Lemma 3.16. For all $x\in[0,\infty)$ and $s_1\in[0,s_0]$, we* $\mathbf{P}_x$-a.s. *get*

$$\Big|\tfrac{1}{s}\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_{s_1+s},f\rangle}-e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\big|\mathscr{F}_{s_1}\big]\Big|\le14a^{-2}e^{10a(N_{s_0}+M_{s_0})},\ \ s\in(0,\min\{a,s_0\}], \tag{3.49}$$

$$\lim_{s\to0^+}\tfrac{1}{s}\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_{s_1+s},f\rangle}-e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\big|\mathscr{F}_{s_1}\big]=e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\langle\overline{\mathcal{M}}_{s_1},\dot f+\mathcal{L}f+Q_f\rangle\,. \tag{3.50}$$

Proof. We recall the notation G_{s_1} introduced in Proposition 2.14 (iii). We denote the vertices of G_{s_1} listed in the lexicographical order by $\{u_1,\dots,u_{N_{s_1}}\}$. We also recall the notation of $\mathcal{M}_s^{(u_k)}$ introduced in (2.33) and we set $\overline{\mathcal{M}}_s^{(u_k)}=e^{-2s}\mathcal{M}_s^{(u_k)}$. By Proposition 2.14 (iii), $\overline{\mathcal{M}}_{s_1+s}=e^{-2s_1}\sum_{1\le k\le N_{s_1}}\overline{\mathcal{M}}_s^{(u_k)}$ and conditionally on $\mathscr{F}_{s_1}$, the random measures $(\overline{\mathcal{M}}_s^{(u_k)})_{1\le k\le N_{s_1}}$ are independent; moreover $\overline{\mathcal{M}}_s^{(u_k)}$ has the same law as $\overline{\mathcal{M}}_s$ under $\mathbf{P}_{X_{u_k}(s_1)}$. To simplify notation, we set $g(s,x)=e^{-2s_1}f(s_1+s,x)$. Thus, $\langle\overline{\mathcal{M}}_{s_1+s},f\rangle=\sum_{1\le k\le N_{s_1}}\langle\overline{\mathcal{M}}_s^{(u_k)},g\rangle$ and

$$\mathbf{P}_x\text{-a.s.}\quad \mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_{s_1+s},f\rangle}\big|\mathscr{F}_{s_1}\big]=\prod_{1\le k\le N_{s_1}}\mathbf{E}_{X_{u_k}(s_1)}\big[e^{\langle\overline{\mathcal{M}}_s,g\rangle}\big]=e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\prod_{1\le k\le N_{s_1}}\big(1+\lambda_k(s)\big) \tag{3.51}$$

where for all $k\in\{1,\dots,N_{s_1}\}$, we have set $\lambda_k(s)=e^{-g(0,X_{u_k}(s_1))}\mathbf{E}_{X_{u_k}(s_1)}\big[e^{\langle\overline{\mathcal{M}}_s,g\rangle}\big]-1$. Note that by (3.45) in Lemma 3.17, we first get for all $1\le k\le N_{s_1}$,

$$|\lambda_k(s)|\le 14a^{-1}s\,e^{6a(1+X_{u_k}(s_1))}. \tag{3.52}$$

By (3.46) in Lemma 3.17, we get that $\mathbf{P}_x$-a.s.,

$$\begin{aligned}\lim_{s\to0^+}\tfrac{1}{s}\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_{s_1+s},f\rangle}-e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\big|\mathscr{F}_{s_1}\big]&=e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\lim_{s\to0^+}\sum_{1\le k\le N_{s_1}}\tfrac{1}{s}\lambda_k(s)\\&=e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\sum_{1\le k\le N_{s_1}}(\dot g+\mathcal{L}g+Q_g)(0,X_{u_k}(s_1))\end{aligned}$$

which implies (3.50).

We next prove (3.49). To this end first observe that by (3.43), $|g(s,x)|\le a^2(x+1)$. Thus, by (3.4) applied to $\lambda=a$ (recalling that $s\le s_0$), we get

$$0\le1+\lambda_k(s)\le e^{-g(0,X_{u_k}(s_1))}\mathbf{E}_{X_{u_k}(s_1)}\big[e^{a^2(N_s+M_s)}\big]\le e^{-g(0,X_{u_k}(s_1))}e^{a(1+X_{u_k}(s_1))}\le e^{2a(1+X_{u_k}(s_1))}.$$

For all $k\in\{1,\dots,N_{s_1}\}$, we set $x_k=\prod_{1\le m\le k}(1+\lambda_m(s))$ and $x_0=1$. Then, the previous inequality implies that $0\le x_k\le e^{2a(N_{s_1}+M_{s_1})}$. Combined with (3.52), it yields

$$|x_k-x_{k-1}|=|\lambda_k(s)|x_{k-1}\le|\lambda_k(s)|e^{2a(N_{s_1}+M_{s_1})}\le 14a^{-1}s\,e^{6a(1+X_{u_k}(s_1))}e^{2a(N_{s_1}+M_{s_1})},$$

by (3.52). Thus, we obtain that

$$\begin{aligned}\Big|\prod_{1\le k\le N_{s_1}}\big(1+\lambda_k(s)\big)\ -1\Big|=\Big|\sum_{1\le k\le N_{s_1}}x_k-x_{k-1}\Big|&\le e^{2a(N_{s_1}+M_{s_1})}\sum_{1\le k\le N_{s_1}}14a^{-1}s\,e^{6a(1+X_{u_k}(s_1))}\\&\le 14a^{-1}sN_{s_1}e^{8a(N_{s_1}+M_{s_1})}\le 14a^{-2}se^{9a(N_{s_1}+M_{s_1})}.\end{aligned}$$

Thus by (3.51) we get

$$\Big|\tfrac{1}{s}\mathbf{E}_x\big[e^{\langle\overline{\mathcal{M}}_{s_1+s},f\rangle}-e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\big|\mathscr{F}_{s_1}\big]\Big|\le\tfrac{1}{s}e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\Big|\prod_{1\le k\le N_{s_1}}\big(1+\lambda_k(s)\big)\ -1\Big|\le 14a^{-2}e^{10a(N_{s_1}+M_{s_1})}$$

since $e^{\langle\overline{\mathcal{M}}_{s_1},f\rangle}\le e^{a(N_{s_1}+M_{s_1})}$, which implies (3.49) since $N_{s_1}+M_{s_1}\le N_{s_0}+M_{s_0}$. ■

End of the proof of Lemma 3.16. Recall the following notation

$$Y_s(f)=e^{\langle\overline{\mathcal{M}}_s,f\rangle}\quad\text{and}\quad W_s(f)=Y_s(f)\langle\overline{\mathcal{M}}_s,\dot f+\mathcal{L}f+Q_f\rangle\,.$$

Clearly $s\mapsto Y_s(f)$ and $s\mapsto W_s(f)$ are càdlàg. To simplify notation we set $S=14a^{-2}e^{10a(N_{s_0}+M_{s_0})}$, so $\mathbf{E}_x[S]<14a^{-2}e^{(x+1)/2}$ by (3.4). By Lemma 3.18, for $s_1\in[0,s_0]$ and $s\in(0,\min\{a,s_0\}]$, $\mathbf{P}_x$-a.s.,

$$\tfrac{1}{s}\big|\mathbf{E}_x\big[Y_{s_1+s}(f)-Y_{s_1}(f)\big|\mathscr{F}_{s_1}\big]\big|\le S\quad\text{and}\quad\lim_{s\to0^+}\tfrac{1}{s}\mathbf{E}_x\big[Y_{s_1+s}(f)-Y_{s_1}(f)\big|\mathscr{F}_{s_1}\big]=W_{s_1}(f)\,.$$

By dominated convergence and (3.49), $\lim_{s\to 0^+} \mathbf{E}_x\left[\left|W_{s_1}(f) - \frac{1}{s}\mathbf{E}_x\left[Y_{s_1+s}(f) - Y_{s_1}(f)\middle|\mathscr{F}_{s_1}\right]\right|\right] = 0$. By dominated convergence and (3.49) again, we get

$$\lim_{s\to 0^+}\int_0^{s_0}\mathbf{E}_x\left[\tfrac{1}{s}\mathbf{E}_x\left[Y_{s_1+s}(f) - Y_{s_1}(f)\middle|\mathscr{F}_{s_1}\right]\right]\mathrm{d}s_1 = \int_0^{s_0}\mathbf{E}_x[W_{s_1}(f)]\,\mathrm{d}s_1.$$

Now observe that

$$\int_0^{s_0}\mathbf{E}_x\left[\tfrac{1}{s}\,\mathbf{E}_x\left[Y_{s_1+s}(f) - Y_{s_1}(f)\middle|\mathscr{F}_{s_1}\right]\right]\mathrm{d}s_1 = \frac{1}{s}\int_0^{s_0}\left(\mathbf{E}_x\left[Y_{s_1+s}(f)\right] - \mathbf{E}_x[Y_{s_1}(f)]\right)\mathrm{d}s_1$$
$$= \frac{1}{s}\int_0^{s}\mathbf{E}_x[Y_{s_0+s_1}(f)]\,\mathrm{d}s_1 - \frac{1}{s}\int_0^{s}\mathbf{E}_x[Y_{s_1}(f)]\,\mathrm{d}s_1 \underset{s\to 0^+}{\longrightarrow} \mathbf{E}_x[Y_{s_0}(f)] - \mathbf{E}_x[Y_0(f)].$$

by dominated convergence. Thus $\mathbf{E}_x[Y_{s_0}(f)] = \mathbf{E}_x[Y_0(f)] + \int_0^{s_0}\mathbf{E}_x[W_s(f)]\,\mathrm{d}s$, which immediately implies (3.44). It completes the proof of Lemma 3.16 and therefore also the proof of Theorem 3.1. ■

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