EXCEPTIONALLY SMALL BALLS IN STABLE TREES

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Abstract. — The γ-stable trees are random measured compact metric spaces that appear as the scaling limit of Galton-Watson trees whose offspring distribution lies in a γ-stable domain, $\gamma \in (1,2]$. They form a specific class of Lévy trees (introduced by Le Gall and Le Jan in [24]) and the Brownian case $\gamma = 2$ corresponds to Aldous Continuum Random Tree (CRT). In this paper, we study fine properties of the mass measure, that is the natural measure on γ-stable trees. We first discuss the minimum of the mass measure of balls with radius r and we show that this quantity is of order $r^{\frac{\gamma}{\gamma-1}} (\log 1/r)^{-\frac{1}{\gamma-1}}$. We think that no similar result holds true for the maximum of the mass measure of balls with radius r, except in the Brownian case: when $\gamma = 2$, we prove that this quantity is of order $r^2 \log 1/r$. In addition, we compute the exact constant for the lower local density of the mass measure (and the upper one for the CRT), which continues previous results from [9, 10, 13].

Résumé (*Boules exceptionnellement petites dans les arbres stables*)

Les arbres γ-stables sont des espaces métriques compacts à mesure aléatoire qui apparaissent en tant que limite de mise à l'échelle d'arbres de Galton-Watson dont les distributions sont situées dans un domaine γ-stable, $\gamma \in]1,2]$. Ils forment une classe spécifique des arbres de Lévy (introduite par Le Gall et Le Jan dans [24]) et le cas brownien $\gamma = 2$ correspond aux arbres aléatoires du continuum d'Aldous (CRT).

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Dans cet article nous étudions les propriétés fines de la mesure de masse, qui est la mesure naturelle des arbres γ-stables. Nous discutons d'abord le minimum de la mesure de masse des boules de rayon r et nous montrons que cette quantité est de l'ordre de $r^{\frac{\gamma}{\gamma-1}} (\log 1/r)^{-\frac{1}{\gamma-1}}$. Nous pensons qu'aucun résultat similaire n'est vrai pour le maximum des mesures de masse de boule de rayon r, sauf dans le cas brownien : quand $\gamma = 2$ nous montrons que cette quantité est de l'ordre de $r^2 \log 1/r$. D'autre part, nous calculons la constante exacte de la densité local inférieure de la mesure de masse (et la supérieure pour le CRT), à la suite de résultats précédents de [9, 10, 13].

1. Introduction

Stable trees are particular instances of Lévy trees that form a class of random compact metric spaces introduced by Le Gall and Le Jan in [24] as the genealogy of Continuous State Branching Processes (CSBP for short). The class of stable trees contains Aldous's continuum random tree that corresponds to the Brownian case (see Aldous [2, 3]). Stable trees (and more generally Lévy trees) are the scaling limit of Galton-Watson trees (see [11] Chapter 2 and [8]). Various geometric and distributional properties of Lévy trees (and of stable trees, consequently) have been studied in [12] and in Weill [28]. Stable trees have been also studied in connection with fragmentation processes: see Miermont [25, 26], Haas and Miermont [20], Goldschmidt and Haas [18] for the stable cases and see Abraham and Delmas [1] for related models concerning more general Lévy trees. To study Brownian motion on stable trees, D. Croydon in [6] got partial results on balls with exceptionnally small mass measure.

Before stating the results, let us briefly explain the definition of stable trees before stating the main results of the paper. Let us fix the stable index $\gamma \in (1, 2]$ and let $X = (X_t)_{t \geq 0}$ be a spectrally positive γ-stable Lévy process that is defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$. More precisely, we suppose that $\mathbf{E}[\exp(-\lambda X_t)] = \exp(t\lambda^{\gamma})$, $t, \lambda \in [0, \infty)$. Note that X is a Brownian motion when $\gamma = 2$ and we shall refer to this case as to the *Brownian case*. As shown by Le Gall and Le Jan [24] (see also [11] Chapter 1), there exists a *continuous process* $H = (H_t)_{t \geq 0}$ such that for any $t \in [0, \infty)$, the following limit holds true in probability

$$(1) \qquad H_t := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_t^s < X_s < I_t^s + \varepsilon\}} \, ds.$$

Here I_t^s stands for $\inf_{s \leq r \leq t} X_r$. The process H is the γ-*stable height process*. Note that in the Brownian case, H is simply a reflected Brownian motion. Theorems 2.3.2 and 2.4.1 in [11] show that H is the scaling limit of the contour

function (or the depth-first exploration process) of an i.i.d. sequence of Galton-Watson trees whose offspring distribution is in the domain of attraction of a γ-stable law.

As in the discrete setting, the process H encodes a family of continuous trees: each excursion of H above 0 corresponds to the exploration process of a single continuous tree of the family. Let us make this statement more precise thanks to excursion theory. Recall that X has unbounded variation sample paths. We set $I_t = \inf_{s \in [0,t]} X_s$, that is the infimum process of X. Basic results on fluctuation theory (see Bertoin [4] VII.1) entail that $X - I$ is a strong Markov process in $[0, \infty)$ and that 0 is regular for $(0, \infty)$ and recurrent with respect to this Markov process. Moreover, $-I$ is a local time at 0 for $X - I$ (see Bertoin [4] Theorem VII.1). We denote by $\mathbf{N}$ the corresponding *excursion measure of $X - I$ above* 0. We denote by (a_j, b_j), $j \in \mathscr{I}$, the excursion intervals of $X - I$ above 0, and by $X^j = X_{(a_j + \cdot) \wedge b_j} - I_{a_j}$, $j \in \mathscr{I}$, the corresponding excursions. Next, observe that if $t \in (a_j, b_j)$, the value of H_t only depends on X^j. Moreover, one can show that $\bigcup_{j \in \mathscr{I}} (a_j, b_j) = \{t \geq 0 : H_t > 0\}$. This allows to define the height process under $\mathbf{N}$ as a certain measurable function $H(X)$ of X, that we simply denote by $(H_t)_{t \geq 0}$. For any $j \in \mathscr{I}$, we then set $H^j = H_{(a_j + \cdot) \wedge b_j}$ and the point measure

$$(2) \qquad \sum_{j \in \mathscr{I}} \delta_{(-I_{a_j}, H^j)}$$

is distributed as a Poisson point measure on $[0, \infty) \times C([0, \infty), \mathbb{R})$ with intensity $\ell \otimes \mathbf{N}$, where ℓ stands for the Lebesgue measure on $[0, \infty)$. Note that X and H under $\mathbf{N}$ are paths with the same lifetime given by

$$\zeta := \inf\{t \in (0, \infty) : H_t = 0\} \ .$$

Standard results in fluctuation theory imply that $0 < \zeta < \infty$, $\mathbf{N}$-a.e. and that

$$\mathbf{N}(1 - e^{-\lambda \zeta}) = \lambda^{1/\gamma} \ , \ \lambda \in [0, \infty) \ .$$

Thus, $\mathbf{N}(\zeta \in dr) = C \, r^{-\frac{1}{\gamma} - 1} \ell(dr)$, where $1/C = \gamma \Gamma(1 - \frac{1}{\gamma})$ (here, Γ stands for Euler's Gamma function). Note that $\mathbf{N}$-a.e. $H_t > 0$ iff $t \in (0, \zeta)$ and $H_0 = H_t = 0$, for any $t \in [\zeta, \infty)$. We refer to Chapter 1 in [11] for more details.

The excursion $(H_t)_{0 \leq t \leq \zeta}$ under $\mathbf{N}$ is the depth-first exploration process of a continuous tree that is defined as the following metric space: for any $s, t \in [0, \zeta]$, we set

$$(3) \qquad b(s,t) = \min_{s \wedge t \leq r \leq s \vee t} H_r \quad \text{and} \quad d(s,t) = H_t + H_s - 2b(s,t) \ .$$

The quantity $b(s,t)$ is the height of the branching point between the vertices visited at times s and t and $d(s,t)$ is therefore the distance in the tree of these vertices. We easily show that d is a pseudo-metric and we introduce the

equivalence relation $\sim$ on $[0, \zeta]$ by setting $s \sim t$ iff $d(s, t) = 0$. We then define the γ-*stable tree* as the quotient metric space

$$\big(\mathcal{T}, d\big) = \big([0, \zeta]/\!\sim, \, d\big) \, .$$

We denote by p $: [0, \zeta] \to \mathcal{T}$ the *canonical projection*. It is easy to see that p is continuous. Thus the γ-stable tree $(\mathcal{T}, d)$ is a (random) connected compact metric space. More precisely, Theorem 2.1 in [12] asserts that $(\mathcal{T}, d)$ is a $\mathbb{R}$-*tree*, namely a metric space such that the following holds true for any $\sigma, \sigma' \in \mathcal{T}$.

(a): There is a unique isometry $f_{\sigma, \sigma'}$ from $[0, d(\sigma, \sigma')]$ into $\mathcal{T}$ such that $f_{\sigma, \sigma'}(0) = \sigma$ and $f_{\sigma, \sigma'}(d(\sigma, \sigma')) = \sigma'$. We set $[\![\sigma, \sigma']\!] = f_{\sigma, \sigma'}([0, d(\sigma, \sigma')])$ that is the geodesic joining σ to σ'.

(b): If $g : [0, 1] \to \mathcal{T}$ is continuous injective, then $g([0, 1]) = [\![g(0), g(1)]\!]$.

We refer to Evans [16] or to Dress, Moulton and Terhalle [7] for a detailed account on $\mathbb{R}$-trees. An intrinsic approach of continuous trees has been developped by Evans Pitman and Winter in [17] (see also [12]): Theorem 1 in [17] asserts that the set of isometry classes of compact $\mathbb{R}$-trees equipped with the Gromov-Hausdorff distance (see Gromov [19]) is a Polish space. This geometric point of view has been used in [12] and [15] to study Lévy trees.

We need to introduce two additional features of γ-stable trees. First, we distinguish a special point $\rho := \mathrm{p}(0)$ in $\mathcal{T}$, that is called the *root*. We also equip $\mathcal{T}$ with the measure $\mathbf{m}$ that is induced by the Lebesgue measure ℓ on $[0, \zeta]$ via the canonical projection p. Namely, for any Borel subset A of $(\mathcal{T}, d)$,

$$\mathbf{m}(A) = \ell\big(\mathrm{p}^{-1}(A)\big) \, .$$

The measure $\mathbf{m}$ is called the *mass measure* of the γ-stable tree $\mathcal{T}$. Note that $\mathbf{m}(\mathcal{T}) = \zeta$. One can prove that $\mathbf{m}$ is diffuse and that its topological support is $\mathcal{T}$. Moreover, $\mathbf{m}$ is carried by the set of *leaves of $\mathcal{T}$* that is the set of the points σ such that $\mathcal{T} \backslash \{\sigma\}$ remains connected (see [12] for more details). The measured tree $(\mathcal{T}, d, \mathbf{m})$ is thus a *continuum tree*, as defined by Aldous in [3].

Let us discuss briefly the scaling property of $\mathcal{T}$. From the scaling property of X and from (1), we see that for any $r \in (0, \infty)$, under $\mathbf{P}$, $(r^{\frac{\gamma-1}{\gamma}} H_{t/r})_{t \geq 0}$ has the same distribution as H. Then, by (2), $(r^{\frac{\gamma-1}{\gamma}} H_{t/r})_{0 \leq t \leq r\zeta}$ under $r^{-\frac{1}{\gamma}} \mathbf{N}$ has the same "distribution" as $(H_t)_{0 \leq t \leq \zeta}$ under $\mathbf{N}$. Thus, the rescaled measured tree $(\mathcal{T}, r^{\frac{\gamma-1}{\gamma}} d, r\mathbf{m})$ under $r^{-\frac{1}{\gamma}} \mathbf{N}$ has the same "distribution" as $(\mathcal{T}, d, \mathbf{m})$ under $\mathbf{N}$. This allows to define for any $r \in (0, \infty)$, a probability distribution $\mathbf{N}(\cdot \,|\, \zeta = r)$ on $C([0, \infty), \mathbb{R})$ such that $r \mapsto \mathbf{N}(\cdot \,|\, \zeta = r)$ is weakly continuous and such that

$$\mathbf{N} = \int_0^\infty \mathbf{N}(\cdot \,|\, \zeta = r)\, \mathbf{N}(\zeta \in dr) \, .$$

Moreover, $(r^{\frac{\gamma-1}{\gamma}} H_{t/r})_{0 \le t \le r}$ under $\mathbf{N}(\cdot \,|\, \zeta = 1)$ has the same distribution as $(H_t)_{0 \le t \le r}$ under $\mathbf{N}(\cdot \,|\, \zeta = r)$. Since $\mathbf{m}(\mathscr{T}) = \zeta$, the tree $\mathscr{T}$ under $\mathbf{N}(\cdot \,|\, \zeta = 1)$ is interpreted as the γ-stable tree conditioned to have total mass equal to 1 and it is simply called the *normalised γ-stable tree*. When $\gamma = 2$, it corresponds (up to a scaling constant) to Aldous Continuum Random Tree as defined in [2] (see also Le Gall [22] for a definition via the normalised Brownian excursion). The normalised γ-stable tree is the weak limit when n goes to infinity of a rescaled Galton-Watson trees conditionned to have total size n and whose offspring distribution belongs to the domain of attraction of a γ-stable law: see Aldous [3] for the Brownian case and see [8] for the general case.

The mass measure $\mathbf{m}$ is in some sense the most spread out measure on $\mathscr{T}$ and it plays a crucial role in the study of stable trees. For instance Theorem 1.1 in [10] asserts that for any $\gamma \in (1,2]$, $\mathbf{N}$-a.e. the mass measure $\mathbf{m}$ is equal to a deterministic constant times the g_γ-packing measure where the gauge function is given by

$$(4) \qquad g_\gamma(r) = \frac{r^{\frac{\gamma}{\gamma-1}}}{(\log\log 1/r)^{\frac{1}{\gamma-1}}}\,, \qquad r \in (0, e^{-1})\,.$$

Actually, this result holds true for general Lévy trees (with a more involved gauge function). Here, the power exponent $\gamma/(\gamma - 1)$ reflects the scaling property. This value is also equal to the packing dimension of $\mathscr{T}$, and to its Hausdorff and its box counting dimensions (see [12]). The function g_γ is also the *lower density of $\mathbf{m}$* at typical points. More precisely, denote by $B(\sigma, r)$ the open ball in $\mathscr{T}$ with center $\sigma \in \mathscr{T}$ and radius $r \in (0, \infty)$. Then, Theorem 1.2 in [10] asserts that there exists a constant $C_\gamma \in (0, \infty)$ such that

$$(5) \qquad \mathbf{N}\text{-a.e. for } \mathbf{m}\text{-almost all } \sigma, \quad \liminf_{r \to 0} \frac{\mathbf{m}\big(B(\sigma, r)\big)}{g_\gamma(r)} = C_\gamma\,.$$

Theorem 1.2 in [10] also holds true for general Lévy trees and the constant is unknown. However, in the stable cases, we are able to compute explicitely C_γ, as shown by the following proposition.

PROPOSITION 1.1. — *For any $\gamma \in (1,2]$, $C_\gamma = \gamma - 1$.*

We also discuss the balls with exceptionally small mass measure. More precisely, we investigate the behaviour of $\inf_{\sigma \in \mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big)$ when r goes to 0. Our main result is the following.

THEOREM 1.2. — *For any $\gamma \in (1,2]$, we set*

$$(6) \qquad f_\gamma(r) = \frac{r^{\frac{\gamma}{\gamma-1}}}{(\log 1/r)^{\frac{1}{\gamma-1}}}\,, \qquad r \in (0,1)\,.$$

Then, there exist $k_\gamma, K_\gamma \in (0, \infty)$ such that **N**-*a.e.*

(7)
$$k_\gamma \;\leq\; \liminf_{r\to 0} \frac{1}{f_\gamma(r)} \inf_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big) \;\leq\; \limsup_{r\to 0} \frac{1}{f_\gamma(r)} \inf_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big) \;\leq\; K_\gamma \;.$$

To study Brownian motion on stable trees, D. Croydon in Proposition 5.1 [6] states a partial lower bound for $\inf_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big)$ that is sufficient to his purpose (but that does not provide the right scale function).

When $1 < \gamma < 2$, there is no exact upper density of $\mathbf{m}$ at typical points (see Proposition 1.9 in [9]). Moreover, Theorem 1.10 in [9] shows that $\mathscr{T}$ has no exact Hausdorff measure whose gauge function is regularly varying. We also strongly believe that there is no exact asymptotic function for $r \mapsto \sup_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big)$, when r goes to 0, but we will not consider this problem in this paper.

In the Brownian case, Theorem 1.1 in [13] asserts that **N**-a.e. the mass measure $\mathbf{m}$ is equal to a deterministic constant times the h-Hausdorff measure where the gauge function h is given by

$$h(r) = r^2 \log\log 1/r \;, \quad r \in (0, e^{-1}) \;.$$

In the proof of Theorem 1.1 in [13], it is proved that h is the upper density of $\mathbf{m}$ at typical points. In this paper we obtain the following specific constant.

PROPOSITION 1.3. — *Consider the Brownian case:* $\gamma = 2$. *Then*

(8)   **N**-*a.e. for* **m**-*almost all* σ, $\displaystyle \limsup_{r\to 0} \frac{\mathbf{m}\big(B(\sigma, r)\big)}{h(r)} = \frac{4}{\pi^2} \;.$

Moreover, in the Brownian case, the balls with exceptionally large mass have also an exact asymptotic function as shown by the following theorem.

THEOREM 1.4. — *Consider the Brownian case:* $\gamma = 2$. *Let us set*

$$f(r) = r^2 \log 1/r \;, \quad r \in (0, e^{-1}) \;.$$

Then, there exist $k, K \in (0, \infty)$ such that **N**-*a.e.*

(9)
$$k \;\leq\; \liminf_{r\to 0} \frac{1}{f(r)} \sup_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big) \;\leq\; \limsup_{r\to 0} \frac{1}{f(r)} \sup_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, r)\big) \;\leq\; K \;.$$

Observe that, in the Brownian case, Theorem 1.2 and Theorem 1.4 immediately imply the following result.

COROLLARY 1.5. — *Consider the Brownian case:* $\gamma = 2$. *Then, there are two constants $c, C \in (0, \infty)$ such that*

N-*a.e.* $\exists r_0 \in (0, \infty) \;:\;$ $\forall r \in (0, r_0), \forall \sigma \in \mathscr{T},$

$$\frac{c}{\log 1/r} \leq r^{-2}\mathbf{m}\big(B(\sigma, r)\big) \leq C \log 1/r.$$

Note that Proposition 1.1, Theorem 1.2, Proposition 1.3, Theorem 1.4 and Corollary 1.5 hold true under the normalised law $\mathbf{N}(\cdot \mid \zeta = 1)$.

The paper is organised as follows. Section 2 recalls useful technical results on the height process and basic geometric properties of stable trees. Section 3 is devoted to the tail estimates of the mass measure of specific subsets of stable trees. Section 4 is devoted to the proof of the results.

2. Preliminaries and basic results

2.1. Results on the stable height process

2.1.1. *Local times of the height process.* — Let H be the γ-stable height process under its excursion measure $\mathbf{N}$ as defined in the introduction. It is possible to define the local times of H under the excursion measure $\mathbf{N}$ as follows. For any $b \in (0, \infty)$, let us set $v(b) = \mathbf{N}(\sup_{t \in [0,\zeta]} H_t > b)$. The continuity of H under $\mathbf{P}$ and the Poisson decomposition (2) obviously imply that $v(b) < \infty$, for any $b > 0$. It is moreover clear that v is non-increasing and $\lim_{b \to \infty} v(b) = 0$. For every $a \in (0, \infty)$, we then define a continuous increasing process $(L_t^a)_{0 \leq t \leq \zeta}$, such that for every $b \in (0, \infty)$ and for any $t \in [0, \infty)$, one has

$$(10) \qquad \lim_{\varepsilon \to 0} \mathbf{N}\left(\mathbf{1}_{\{\sup H > b\}} \sup_{0 \leq s \leq t \wedge \zeta} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a - \varepsilon < H_r \leq a\}} - L_s^a \right| \right) = 0.$$

See [11] Section 1.3 for more details. The process $(L_t^a)_{0 \leq t \leq \zeta}$ is the *a-local time of the height process.* For any $a, \lambda, \mu \in [0, \infty)$, we set

$$(11) \qquad \kappa_a(\lambda, \mu) := \mathbf{N}\left(1 - e^{-\mu L_\zeta^a - \lambda \int_0^\zeta \mathbf{1}_{\{H_t < a\}} dt} \right).$$

The function κ is the Laplace exponent of a specific additive functional of a γ-stable Continuous States Branching Process (γ-stable CSBP, for short). An elementary result on CSBPs, whose proof can be found in Le Gall [23] Section II.3, entails that $a \mapsto \kappa_a(\lambda, \mu)$ is the unique solution of the following ordinary differential equation:

$$\frac{\partial \kappa_a}{\partial a}(\lambda, \mu) = \lambda - \kappa_a(\lambda, \mu)^\gamma \quad \text{and} \quad \kappa_0(\lambda, \mu) = \mu .$$

For more details, see [13] Section 4 page 405 or [9] Section 2.3 page 106. We note the following: if $\mu = \lambda^{1/\gamma}$, then $\kappa_a(\lambda, \mu) = \lambda^{1/\gamma}$. If $\mu < \lambda^{1/\gamma}$, (resp. if

$\mu > \lambda^{1/\gamma}$), then $a \mapsto \kappa_a(\lambda, \mu)$ is increasing (resp. decreasing). A simple change of variable implies that κ satisfies the following integral equation

$$(12) \qquad \int_{\mu}^{\kappa_a(\lambda,\mu)} \frac{du}{\lambda - u^{\gamma}} = a, \qquad \text{for any } a, \lambda, \mu \in [0, \infty) \text{ such that } \mu \neq \lambda^{\frac{1}{\gamma}}.$$

As a consequence, we get

$$(13) \qquad \kappa_{a+b}(\lambda, \mu) = \kappa_a\big(\lambda, \kappa_b(\lambda, \mu)\big), \quad a, b, \lambda, \mu \in [0, \infty) .$$

We also derive from (12) the following scaling property:

$$(14) \qquad c^{\frac{1}{\gamma-1}} \kappa_a\big(c^{-\frac{\gamma}{\gamma-1}}\lambda, c^{-\frac{1}{\gamma-1}}\mu\big) = \kappa_{a/c}(\lambda, \mu), \quad a, c, \lambda, \mu \in [0, \infty) .$$

When $\gamma \neq 2$ and $\lambda > 0$, it seems difficult to compute κ explicitly. However, when $\lambda = 0$, (12) implies that for any $a, \mu \in [0, \infty)$,

$$(15) \qquad \kappa_a(0, \mu) = \mathbf{N}\big(1 - e^{-\mu L_{\zeta}^a}\big) = \Big((\gamma-1)a + \frac{1}{\mu^{\gamma-1}}\Big)^{-\frac{1}{\gamma-1}} .$$

It is convenient to interpret these quantities in terms of the γ-tree $\mathscr{T}$. For any $a \in (0, \infty)$, first define the *a-local time measure* ℓ^a as the measure induced by $dL_\cdot^a$ via the canonical projection $p : [0, \zeta] \to \mathscr{T}$. Namely,

$$\langle \ell^a, f \rangle = \int_0^{\zeta} f(p(s)) \, dL_s^a ,$$

for any positive measurable application f on $\mathscr{T}$. Here $dL_\cdot^a$ stands for the Stieltjes measure associated with the non-decreasing function $s \mapsto L_s^a$. Note that the topological support of ℓ^a is included in the a-level set

$$(16) \qquad \mathscr{T}(a) = \{\sigma \in \mathscr{T} : d(\rho, \sigma) = a\} .$$

Next, we set

$$(17) \qquad \Gamma(\mathscr{T}) = \sup_{\sigma \in \mathscr{T}} d(\rho, \sigma)$$

that is the *total height of* $\mathscr{T}$. Then, observe that

$$(18) \qquad \langle \ell^a \rangle = L_{\zeta}^a, \quad \mathbf{m}\big(B(\rho, a)\big) = \int_0^{\zeta} \mathbf{1}_{\{H_t < a\}} dt \quad \text{and} \quad \Gamma(\mathscr{T}) = \sup_{t \in [0, \zeta]} H_t ,$$

where $\langle \ell^a \rangle$ stands for the total mass of ℓ^a. This implies

$$(19) \qquad \kappa_a(\lambda, \mu) = \mathbf{N}\Big(1 - e^{-\mu \langle \ell^a \rangle - \lambda \mathbf{m}(B(\rho, a))}\Big).$$

We recall from [11] Chapter 1 (proof of Corollary 1.4.2 page 41) that

$$(20) \qquad \mathbf{N}\text{-a.e.} \quad \mathbf{1}_{\{\sup H > a\}} = \mathbf{1}_{\{L_{\zeta}^a \neq 0\}} .$$

By letting μ go to ∞ in (15), we get

$$(21) \qquad v(a) = \mathbf{N}(\Gamma(\mathscr{T}) > a) = \mathbf{N}(\langle \ell^a \rangle \neq 0) = \big((\gamma-1)a\big)^{-\frac{1}{\gamma-1}} .$$

For any $a \in (0, \infty)$, we next define the probability measure $\mathbf{N}_a$ by setting

$$
(22) \qquad \mathbf{N}_a = \mathbf{N}\big(\ \cdot \ \big| \Gamma(\mathscr{T}) > a \big) = \mathbf{N}\big(\ \cdot \ \big| \langle \ell^a \rangle \neq 0 \big) \ .
$$

This combined with (15) implies that

$$
(23) \qquad \mathbf{N}_a\big(e^{-\mu \langle \ell^a \rangle}\big) = 1 - \Big(1 + \frac{1}{(\gamma-1)a\mu^{\gamma-1}} \Big)^{-\frac{1}{\gamma-1}} \ .
$$

By differentiating this equality at $\mu = 0$, one gets

$$
(24) \qquad \mathbf{N}_a\big(\langle \ell^a \rangle\big) = \big((\gamma-1)a \big)^{\frac{1}{\gamma-1}} = \frac{1}{v(a)} \ .
$$

2.1.2. *The branching property.* — We now describe the distribution of excursions of the height process above a given level (or equivalently of the corresponding stable subtrees above this level). We fix $a \in (0, \infty)$ and we denote by (l_j^a, r_j^a), $j \in \mathscr{J}_a$, the connected components of the open set $\{t \in (0, \zeta) : H_t > a\}$. For any $j \in \mathscr{J}_a$, denote by $H^{a,j}$ the corresponding excursion of H that is defined by $H_s^{a,j} = H_{(l_j^a + s) \wedge r_j^a} - a$, $s \in [0, \infty)$.

This decomposition is interpreted in terms of the tree as follows. Denote the closed ball in $\mathscr{T}$ with center ρ and radius a by $\bar{B}(\rho, a)$. Observe that the connected components of the open set $\mathscr{T} \backslash \bar{B}(\rho, a)$ are the subtrees $\mathscr{T}_j^{a,o} := \mathrm{p}((l_j^a, r_j^a))$, $j \in \mathscr{J}_a$. The closure $\mathscr{T}_j^a$ of $\mathscr{T}_j^{a,o}$ is simply $\{\sigma_j^a\} \cup \mathscr{T}_j^{a,o}$, where $\sigma_j^a = \mathrm{p}(l_j^a) = \mathrm{p}(r_j^a)$, that is the point in the a-level set $\mathscr{T}(a)$ at which $\mathscr{T}_j^{a,o}$ is grafted. Observe that the rooted measured tree $\big(\mathscr{T}_j^a, d, \sigma_j^a, \mathbf{m}(\ \cdot \ \cap \mathscr{T}_j^a) \big)$ is isometric to the tree coded by $H^{a,j}$.

We define the following point measure on $[0, \infty) \times C([0, \infty), \mathbb{R})$:

$$
(25) \qquad \mathscr{M}_a(dx \, dH) = \sum_{j \in \mathscr{J}_a} \delta_{(L_{l_j^a}^a, H^{a,j})}
$$

For any $s \in [0, \infty)$, we also set $\tilde{H}_s^a = H_{\tau_s^a}$, where the time-change τ_s^a is given by

$$
\tau_s^a = \inf \Big\{ t \geq 0 : \int_0^t dr \, \mathbf{1}_{\{H_r \leq a\}} > s \Big\} \ , \quad s \in (0, \infty) \ .
$$

The process $\tilde{H}^a = (\tilde{H}_s^a)_{s \geq 0}$ is the *height process below a* and the rooted compact $\mathbb{R}$-tree $(\bar{B}(\rho, a), d, \rho)$ is isometric to the tree coded by $\tilde{H}^a$. Let $\mathscr{G}_a$ be the sigma-field generated by $\tilde{H}^a$ augmented by the $\mathbf{N}$-negligible sets. From the approximation (10), it follows that L_ζ^a is measurable with respect to $\mathscr{G}_a$. Recall the notation $\mathbf{N}_a$ from (22).

The branching property at level a then asserts that under $\mathbf{N}_a$, conditionally given $\mathscr{G}_a$, $\mathscr{M}_a$ is distributed as a Poisson point measure with intensity $\mathbf{1}_{[0, L_\zeta^a]}(x) \ell(dx) \otimes \mathbf{N}(dH)$.

For more details, we refer to Proposition 1.3.1 in [11] or to the proof of Proposition 4.2.3 [11] (see also Theorem 4.2 [12]). We apply the branching property to prove the following lemma.

LEMMA 2.1. — *For any $a, b, \lambda \in (0, \infty)$, we set*

$$\Phi_{a,b}(\lambda) = \mathbf{N}_a\big(e^{-\lambda \mathbf{m}(B(\rho, a+b))}\big) \ .$$

Then,

$$\Phi_{a,b}(\lambda) = v(a)^{-1}\big(\kappa_a(\lambda, \infty) - \kappa_{a+b}(\lambda, 0)\big) \ ,$$

where $\kappa_a(\lambda, \infty)$ stands for $\lim_{\mu \to \infty} \kappa_a(\lambda, \mu)$, which is well-defined and finite.

Proof. — First note that

$$\mathbf{m}(B(\rho, a+b)) = \int_0^\zeta \mathbf{1}_{\{H_r < a+b\}} dr = \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr + \int_0^\zeta \mathbf{1}_{\{a \le H_r < a+b\}} dr$$

$$= \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr + \sum_{j \in \mathscr{I}_a} \int_0^{\zeta_j^a} \mathbf{1}_{\{H_r^{a,j} < b\}} dr \ ,$$

where $\zeta_j^a = r_j^a - l_j^a$ stands for the lifetime of $H^{a,j}$. The branching property then implies that

$$\mathbf{N}_a\big(e^{-\lambda \mathbf{m}(B(\rho, a+b))} \mid \mathcal{G}_a\big) = e^{-\lambda \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr} \exp\big(-L_\zeta^a \mathbf{N}\big(1 - e^{-\lambda \int_0^\zeta \mathbf{1}_{\{H_r < b\}} dr}\big)\big)$$

$$= e^{-\kappa_b(\lambda, 0) L_\zeta^a - \lambda \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr} \ .$$

Monotone convergence implies

$$\mathbf{N}_a\big(e^{-\kappa_b(\lambda, 0) L_\zeta^a - \lambda \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr}\big) = \lim_{\mu \to \infty} v(a)^{-1} \mathbf{N}\big((1 - e^{-\mu L_\zeta^a}) e^{-\kappa_b(\lambda, 0) L_\zeta^a - \lambda \int_0^\zeta \mathbf{1}_{\{H_r < a\}} dr}\big)$$

$$= \lim_{\mu \to \infty} v(a)^{-1}\big(\kappa_a\big(\lambda, \mu + \kappa_b(\lambda, 0)\big) - \kappa_a\big(\lambda, \kappa_b(\lambda, 0)\big)\big),$$

which easily implies the desired result thanks to (13).   □

From Lemma 2.1 and the scaling property (14), we get

$$\Phi_{a,b}(\lambda) = \Phi_{1,b/a}\big(a^{\frac{\gamma}{\gamma-1}} \lambda\big) \ ,$$

which implies that for any $a, b \in (0, \infty)$,

(26) $\mathbf{m}\big(B(\rho, a+b)\big)$ under $\mathbf{N}_a \overset{\text{(law)}}{=} a^{\frac{\gamma}{\gamma-1}} \mathbf{m}\big(B(\rho, 1 + \frac{a}{b})\big)$ under $\mathbf{N}_1$.

2.1.3. *Spinal decomposition.* — We recall another decomposition of the height process that is proved in [11] Chapter 2 (see [12] for a more specific statement and see [14] for further applications). This decomposition is used in the proof of Proposition 1.1 and Proposition 1.3.

Let $h : [0, \infty) \to [0, \infty)$ be a continuous function with compact support. Let us assume for clarity that $h(0) > 0$. We view h as the depth-first exploration process of a tree. Thus, the exploration starts at a vertex with height $h(0) > 0$ that we call the initial vertex. We obtain the subtrees grafted along the ancestral line of the initial vertex as follows: set $\underline{h}(s) = \inf_{[0,s]} h$ and denote by (l_i, r_i), $i \in \mathscr{I}(h)$, the excursion intervals of $h - \underline{h}$ away from 0 that are the connected components of the open set $\{s \in [0, \infty) : h(s) - \underline{h}(s) > 0\}$. For any $i \in \mathscr{I}(h)$, set

$$h^i(s) = \big((h - \underline{h})((l_i + s) \wedge r_i) \big)_{s \geq 0} .$$

Then, the subtrees along the ancestral line of the initial vertex are coded by the functions $(h^i \, ; \, i \in \mathscr{I}(h))$, and the tree coded by h^i is grafted at distance $h(0) - h(l_i)$ from the initial vertex. We next define $\mathscr{N}(h)$ as the point measure on $[0, \infty) \times C([0, \infty), \mathbb{R})$ given by

$$\mathscr{N}(h) = \sum_{i \in \mathscr{I}(h)} \delta_{(h(0) - h(l_i), \, h^i)} .$$

Recall that H stands for the γ-height process under its excursion measure $\mathbf{N}$. For any $t \in (0, \zeta)$, we set $\hat{H}^t := (H_{(t-s)_+})_{s \geq 0}$; here, $(\cdot)_+$ stands for the positive part function. We also set $\check{H}^t := (H_{(t+s) \wedge \zeta})_{s \geq 0}$, and we define the random point measure $\mathscr{N}_t$ on $[0, \infty) \times C([0, \infty), \mathbb{R})$ by

$$(27) \qquad \mathscr{N}_t = \mathscr{N}(\hat{H}^t) + \mathscr{N}(\check{H}^t) := \sum_{j \in \mathscr{J}_t} \delta_{(r_j^t, \, H^{(t),j})} .$$

This point measure records the subtrees grafted along the ancestral line of the vertex visited at time t in the coding of $\mathscr{T}$ by H. Namely, set $\sigma = \mathrm{p}(t) \in \mathscr{T}$. Then, the geodesic $[\![\rho, \sigma]\!]$ is the ancestral line of σ. Denote by $\mathscr{T}_j^\circ$, $j \in \mathscr{J}$, the connected components of the open set $\mathscr{T} \backslash [\![\rho, \sigma]\!]$ and denote by $\mathscr{T}_j$ the closure of $\mathscr{T}_j^\circ$. Then, there exists a point $\sigma_j \in [\![\rho, \sigma]\!]$ such that $\mathscr{T}_j = \{\sigma_j\} \cup \mathscr{T}_j^\circ$. The specific coding of $\mathscr{T}$ by H entails that for any $j \in \mathscr{J}$ there exists a unique $j' \in \mathscr{J}_t$ such that $d(\sigma, \sigma_j) = r_{j'}^t$ and such that the rooted compact $\mathbb{R}$-tree $(\mathscr{T}_j, d, \sigma_j)$ is isometric to the tree coded by $H^{(t),j'}$.

The law of $\mathscr{N}_t$ when t is chosen "at random" according to the Lebesgue measure is given as follows. To simplify the argument, we assume that the random variables we mention are defined on the same probability space $(\Omega, \mathscr{T}, \mathbf{P})$. Let $(U_t)_{t \geq 0}$ be a $(\gamma - 1)$-stable subordinator with initial value $U_0 = 0$, and

whose Laplace exponent is given by $-\log \mathbf{E}[\exp(-\lambda U_1)] = \gamma \lambda^{\gamma-1}$. Let

$$(28) \qquad \mathcal{N}^* = \sum_{j \in \mathcal{I}^*} \delta_{(r_j^*, H^{*j})}$$

be a random point measure on $[0, \infty) \times C([0, \infty), \mathbb{R})$ such that a regular version of the law of $\mathcal{N}^*$ conditionally given U is that of a Poisson point measure with intensity $dU_r \otimes \mathbf{N}(dH)$. Here, dU_r stands for the (random) Stieltjes measure associated with the non-decreasing path $r \mapsto U_r$. For any $a \in (0, \infty)$, we also set

$$(29) \qquad \mathcal{N}_a^* = \sum_{j \in \mathcal{I}^*} \mathbf{1}_{[0,a]}(r_j^*)\, \delta_{(r_j^*, H^{*j})}.$$

By Lemma 3.4 in [12], for any nonnegative measurable functional F,

$$(30) \qquad \mathbf{N}\left(\int_0^\zeta F(H_t, \mathcal{N}_t)\, dt \right) = \int_0^\infty \mathbf{E}\left[F(a, \mathcal{N}_a^*) \right] da .$$

We shall refer to this identity as to the *spinal decomposition of H at a random time*.

We use the spinal decomposition to compute the law of the mass measure of random balls in $\mathcal{T}$. To that end, we first fix $t \in (0, \zeta)$ and we express $\mathbf{m}(B(\mathrm{p}(t), r))$ in terms of $\mathcal{N}_t$ as follows. First, recall from (3) the definition of $b(s,t)$ and $d(s,t)$. Note that if $H_s = b(s,t)$ with $s \neq t$, then $\mathrm{p}(s)$ cannot be a leaf of $\mathcal{T}$. Let us fix a radius r in $[0, H_t]$. Since the leaves of $\mathcal{T}$ have full $\mathbf{m}$-measure, we get

$$\mathbf{m}(B(\mathrm{p}(t), r)) = \int_0^\zeta \mathbf{1}_{\{d(s,t)<r\}} ds = \int_0^\zeta \mathbf{1}_{\{0<H_s-b(s,t)<r-H_t+b(s,t)\}} ds.$$

The definition of $(\mathcal{N}(\hat{H}^t), \mathcal{N}(\check{H}^t))$ then entails

$$(31) \qquad \mathbf{m}(B(\mathrm{p}(t), r)) = \sum_{j \in \mathcal{I}_t} \mathbf{1}_{[0,r]}(r_j^t) \cdot \int_0^{\zeta_j^t} \mathbf{1}_{\{H_s^{(t),j}<r-r_j^t\}}\, ds,$$

where ζ_j^t stands for the lifetime of the path $H^{*\,t,j}$. For any $a \in (0, \infty)$ and for any $r \in [0, a]$, we next set

$$(32) \qquad M_r^*(a) = \sum_{j \in \mathcal{I}^*} \mathbf{1}_{[0,r \wedge a]}(r_j^*) \cdot \int_0^{\zeta_j^*} \mathbf{1}_{\{H_s^{*j} \leq r-r_j^*\}} ds ,$$

where ζ_j^* stands for the lifetime of the path H^{*j}. Then, $(M_r^*(a))_{0 \leq r \leq a}$ is a function of $\mathcal{N}_a^*$. It is a cadlag non-decreasing process and the spinal decomposition

(30) implies for any bounded measurable functional $F : \mathbb{D}([0, \infty), \mathbb{R}) \to \mathbb{R}$, we have

$$\mathbf{N}\left(\int_0^\zeta F\big((\mathbf{m}(B(\mathrm{p}(t), r)))_{r \geq 0}\big)\, dt\right) = \int_0^\infty \mathbf{E}\left[F((M_r^*(a))_{r \geq 0})\right]\, da\,,$$

which can be rephrased as follows in terms of the tree

$$(33) \qquad \mathbf{N}\left(\int_{\mathcal{T}} F\big((\mathbf{m}(B(\sigma, r)))_{r \geq 0}\big)\, \mathbf{m}(d\sigma)\right) = \int_0^\infty \mathbf{E}\left[F((M_r^*(a))_{r \geq 0})\right]\, da\,.$$

More generally, we need to consider more general quantities that are defined as follows. For any $\sigma, \sigma' \in \mathcal{T}$, recall that $[\![\sigma, \sigma']\!]$ stands for the geodesic joining σ to σ'. The branching point of σ and σ' is the unique point denoted by $\sigma \wedge \sigma'$ such that $[\![\rho, \sigma \wedge \sigma']\!] = [\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!]$. Then for any $\sigma \in \mathcal{T}$ and for any $r \geq r' \geq 0$, we set

$$C_{r',r}(\sigma) = \{\sigma' \in B(\sigma, r) : d(\sigma, \sigma' \wedge \sigma) \in (r', r]\}\,.$$

Note that if $a = d(\rho, \sigma)$, then $C_{r',r}(\sigma) = C_{r' \wedge a, r \wedge a}(\sigma)$. We next set for any $r \geq r' \geq 0$,

$$(34) \qquad M_{r',r}^*(a) = \sum_{j \in \mathscr{I}^*} \mathbf{1}_{(r' \wedge a, r \wedge a]}(r_j^*) \int_0^{\zeta_j^*} \mathbf{1}_{\{H_s^{*j} < r - r_j^*\}}$$

Then, the arguments used to prove (33) entail the following:

$$\mathbf{N}\left(\int_{\mathcal{T}} F\big((\mathbf{m}(C_{r',r}(\sigma)))_{r \geq r' \geq 0}\big)\, \mathbf{m}(d\sigma)\right) = \int_0^\infty \mathbf{E}\left[F((M_{r',r}^*(a))_{r \geq r' \geq 0})\right]\, da\,.$$

Note that $M_r^*(a) = M_{0,r}^*(a)$. The random variables $M_{r',r}^*(a)$ play an important role in the proof of Proposition 1.1 and Proposition 1.3. We gather in the following lemma their basic properties that are easy consequences of the definition (we refer to Lemma 2.11 and to Remark 2.12 in [9] page 115 for more details).

LEMMA 2.2. — *Let us fix* $a \in (0, \infty)$. *The following holds true.*

(i) *Let* $(r_n)_{n \geq 0}$ *be a sequence such that* $0 < r_{n+1} \leq r_n \leq a$ *and* $\lim_{n \to \infty} r_n = 0$. *Then, the random variables* $(M_{r_{n+1}, r_n}^*(a))_{n \geq 0}$ *are independent.*

(ii) *The increments of* $r \in [0, a] \mapsto M_r^*(a)$ *are not independent. However, for any* $0 \leq r' \leq r \leq a$, *we have*

$$M_r^*(a) - M_{r'}^*(a) = M_{r',r}^*(a) + \sum_{j \in \mathscr{I}^*} \mathbf{1}_{[0, r']}(r_j^*) \int_0^{\zeta_j^*} \mathbf{1}_{\{r' - r_j^* \leq H_s^{*j} < r - r_j^*\}}\,.$$

It implies that $M_r^*(a) \geq M_r^*(a) - M_{r'}^*(a) \geq M_{r',r}^*(a)$. *Note that* $M_{r',r}^*(a)$ *is independent of* $M_{r'}^*(a)$.

The law of $M^*_{r',r}(a)$ is characterised by its Laplace transform:

(35) $\qquad \mathbf{E}\big[\exp(-\lambda M^*_{r',r}(a))\big] = 1 - \dfrac{\kappa_{r-r'}(\lambda,0)^\gamma}{\lambda}, \quad \lambda \in [0,\infty)\ .$

Proof of (35). — Recall that conditionally given U, $\mathscr{N}^*$ is a Poisson point process with intensity $dU_r \otimes \mathbf{N}$. Thus,

$$\mathbf{E}\big[\exp(-\lambda M^*_{r',r}(a))\,|\,U\big] = \exp\Big(-\int_{(r',r]} dU_s\,\kappa_{r-s}(\lambda,0)\Big).$$

We get $\mathbf{E}[\exp(-\lambda M^*_{r',r}(a))] = \exp(-\gamma \int_0^{r-r'} \kappa_s(\lambda,0)^{\gamma-1} ds)$. By a change of variable based on (12), we obtain $\gamma \int_0^{r-r'} \kappa_s(\lambda,0)^{\gamma-1} ds = \log\lambda - \log\big(\lambda - \kappa_{r-r'}(\lambda,0)\big)$, which entails (35). $\qquad\square$

We also introduce the following notation:

(36) $\qquad\qquad\qquad M_* := M^*_{0,1}(1) = M^*_1(1)\ .$

Then, (35) and the scaling property (14) imply that for any $0 \le r' \le r \le a$,

(37) $\qquad\qquad (r-r')^{-\frac{\gamma}{\gamma-1}} M^*_{r',r}(a) \overset{(\text{law})}{=} M_*\ .$

We see in particular that $r^{-\frac{\gamma}{\gamma-1}} M^*_r(a)$ has the same law as M_*. The tail at $0+$ of the distribution of M_* is studied in Section 3.

2.2. Balls and truncated subtrees.

— Recall that $(H_t)_{0 \le t \le \zeta}$ stands for the excursion of the γ-stable process under $\mathbf{N}$ and that $(\mathscr{T},d)$ is the γ-stable tree coded by H. Recall that for any $\sigma,\sigma' \in \mathscr{T}$, $[\![\sigma,\sigma']\!]$ stands for the (unique) geodesic joining σ to σ'. For any $\sigma \in \mathscr{T}$, we set

$$\mathscr{T}_\sigma = \big\{\sigma' \in \mathscr{T}\,:\,\sigma \in [\![\rho,\sigma']\!]\big\}\ ,$$

that is the subtree stemming from σ. We then set $\Gamma(\mathscr{T}_\sigma) = \sup_{\sigma' \in \mathscr{T}_\sigma} d(\sigma,\sigma')$ that is the total height of $\mathscr{T}_\sigma$. Next, for any $a,\varepsilon \in (0,\infty)$, we set

$$\mathscr{T}(a) = \big\{\sigma \in \mathscr{T} : d(\rho,\sigma) = a\big\} \quad\text{and}\quad \mathscr{T}(a,\varepsilon) = \big\{\sigma \in \mathscr{T}(a) : \Gamma(\mathscr{T}_\sigma) > \varepsilon\big\}\ .$$

Since $\mathscr{T}$ is a compact metric space $\mathscr{T}(a,\varepsilon)$ is a finite subset and we set $Z_a(\varepsilon) = \#\mathscr{T}(a,\varepsilon)$. Then,

$$\mathscr{T}(a,\varepsilon) = \big\{\sigma_1,\ldots,\sigma_{Z_a(\varepsilon)}\big\}\ ,$$

where, the σ_i is the i-th point to be visited by H. For any $\eta \in (0,\infty)$, we set

$$\mathscr{D}_{a,\varepsilon,\eta} = \big\{T_i\,;\,1 \le i \le Z_a(\varepsilon)\big\} \quad\text{where}\quad T_i = \mathscr{T}_{\sigma_i} \cap B(\sigma_i,\eta).$$

The T_is are the subtrees above level a that are higher than ε and that are truncated at height η. We simply call them the (a,ε)-*subtrees truncated at height* η.

Recall from (21) that $v(\varepsilon) = \mathbf{N}(\Gamma(\mathcal{T}) > \varepsilon) = ((\gamma - 1)\varepsilon)^{-\frac{1}{\gamma-1}}$. Recall from the subsection stating the branching property, that $\mathcal{G}_a$ stands for the sigma-field generated by the height process below a. Recall from (22) that $\mathbf{N}_a = \mathbf{N}(\cdot \mid \sup H > a)$. By the branching property at level a,

(i) under $\mathbf{N}_a$ and conditionally given $\mathcal{G}_a$, $Z_a(\varepsilon)$ is a Poisson random variable with parameter $v(\varepsilon)\langle \ell^a \rangle$.

Moreover, each $\mathcal{T}_{\sigma_i}$ is coded by an excursion above level a that is higher than ε. Therefore,

(ii) conditionally given $Z_a(\varepsilon)$, the truncated subtrees T_i are independent and distributed as $B(\rho, \eta)$ under $\mathbf{N}_\varepsilon$.

Consequently, for any integer $k \geq 1$, for any measurable functions $F_1, \dots, F_k : [0, \infty) \to [0, \infty)$, we have
(38)
$$\mathbf{N}_a\Big(\mathbf{1}_{\{Z_a(\varepsilon)=k\}} \prod_{1 \leq i \leq k} F_i\big(\mathbf{m}(T_i)\big) \Big| \mathcal{G}_a\Big) = \frac{(v(\varepsilon)\langle\ell^a\rangle)^k}{k!} e^{-v(\varepsilon)\langle\ell^a\rangle} \prod_{1 \leq i \leq k} \mathbf{N}_\varepsilon\big(F_i\big(\mathbf{m}(B(\rho, \eta))\big)\big).$$

The distribution of $\mathbf{m}(B(\rho, \eta))$ under $\mathbf{N}_\varepsilon$ plays an important role in the proofs and it is studied in Section 3.

The two following lemmas are used in the proofs of Theorem 1.2 and Theorem 1.4. We first show that any ball contains a reasonably high truncated subtree.

LEMMA 2.3. — *Let $r \in (0, 1/2)$. Let n_r be the positive integer given by $2^{-n_r} < r \leq 2^{-n_r+1}$. Let $\sigma \in \mathcal{T}$ be such that $d(\rho, \sigma) \geq 2.2^{-n_r-2}$ and let $k \geq 1$ be the integer such that $(k+1)2^{-n_r-2} \leq d(\rho, \sigma) < (k+2)2^{-n_r-2}$.*

Then, there exists a unique truncated subtree $T \in \mathcal{D}_{k2^{-n_r-2}, 2^{-n_r-2}, 2^{-n_r-2}}$ such that $T \subset B(\sigma, r)$.

Proof. — To simplify the notation, we set $a = k2^{-n_r-2}$ and $\varepsilon = \eta = 2^{-n_r-2}$. There is a unique $i \in \{1, \dots, Z_a(\varepsilon)\}$ such that $\sigma_i \in [\![\rho, \sigma]\!]$ (and note that $d(\rho, \sigma_i) = a = k2^{-n_r-2}$). Then, $T_i = \mathcal{T}_{\sigma_i} \cap B(\sigma_i, \eta) \subset B(\sigma, r)$. Indeed, for any $\sigma' \in T_i$, $d(\sigma, \sigma') \leq d(\sigma, \sigma_i) + d(\sigma', \sigma_i)$. Since $\sigma' \in B(\sigma_i, \eta)$, $d(\sigma', \sigma_i) \leq \eta = 2^{-n_r-2}$, and since $\sigma_i \in [\![\rho, \sigma]\!]$, we get

$$d(\sigma, \sigma_i) = d(\rho, \sigma) - d(\rho, \sigma_i) \leq (k+2)2^{-n_r-2} - a = 2.2^{-n_r-2}.$$

Thus, $d(\sigma, \sigma') \leq 3.2^{-n_r-2} < r$, which completes the proof. $\qquad\square$

Conversely, one proves that any ball is contained in a reasonably small truncated subtree.

LEMMA 2.4. — *Let $r \in (0, 1/2)$. Let n_r be the positive integer given by $2^{-n_r} < r \leq 2^{-n_r+1}$. Let $\sigma \in \mathscr{T}$ be such that $d(\rho, \sigma) \geq 2.2^{-n_r+1}$ and let $l \geq 1$, be the integer such that $(l+1)2^{-n_r+1} \leq d(\rho, \sigma) < (l+2)2^{-n_r+1}$.*

Then, there exists a unique truncated subtree $T \in \mathscr{D}_{l2^{-n_r+1}, 2^{-n_r+1}, 3.2^{-n_r+1}}$ such that $B(\sigma, r) \subset T$.

Proof. — To simplify the notation, we set $a = l2^{-n_r+1}$, $\varepsilon = 2^{-n_r+1}$ and $\eta = 3\varepsilon$. There is a unique $i \in \{1, \ldots, Z_a(\varepsilon)\}$ such that $\sigma_i \in [\![\rho, \sigma]\!]$ (and note that $d(\rho, \sigma_i) = a = l2^{-n_r+1}$). We check that $B(\sigma, r) \subset T_i = \mathscr{T}_{\sigma_i} \cap B(\sigma_i, \eta)$. Indeed, let $\sigma' \in B(\sigma, r)$ and let $\sigma \wedge \sigma'$ be the branching point of σ and σ'. Namely, $[\![\rho, \sigma \wedge \sigma']\!] = [\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!]$. We get

$$d(\rho, \sigma \wedge \sigma') = \min\big(d(\rho, \sigma), d(\rho, \sigma')\big)$$
$$> d(\rho, \sigma) - r \geq (l+1)2^{-n_r+1} - 2^{-n_r+1} = l2^{-n_r+1} \ .$$

Thus, $d(\rho, \sigma_i) \leq d(\rho, \sigma \wedge \sigma')$ and since σ_i and $\sigma \wedge \sigma'$ belongs to $[\![\rho, \sigma]\!]$, it implies that $\sigma' \in \mathscr{T}_{\sigma_i}$. Moreover, $d(\sigma', \sigma_i) \leq d(\sigma', \sigma) + d(\sigma, \sigma_i) < r + 2.2^{-n_r+1} \leq 3.2^{-n_r+1} = \eta$, which completes the proof. $\square$

3. Tail estimates

3.1. Tail of the distribution of $\mathbf{m}(B(\rho, 1+c))$ under $\mathbf{N}_1$. — Recall that $v(1) = \mathbf{N}(\sup H > 1)$ and that $\mathbf{N}_1 = \mathbf{N}(\cdot \,|\, \sup H > 1)$. Recall from (11) the definition of $\kappa_a(\lambda, \mu)$ and Lemma 2.1 that asserts that for any $c \in [0, \infty)$,

$$(39) \qquad \mathbf{N}_1\big(e^{-\lambda \mathbf{m}(B(\rho, 1+c))}\big) = \Phi_{1,c}(\lambda) = v(1)^{-1}\big(\kappa_1(\lambda, \infty) - \kappa_{1+c}(\lambda, 0)\big) \ .$$

LEMMA 3.1. — *For any $\gamma \in (1, 2]$ and any $c \in [0, \infty)$, we get*

$$-\log \mathbf{N}_1\big(\mathbf{m}(B(\rho, 1+c)) \leq y\big) \underset{y \to 0+}{\sim} \left(\frac{\gamma - 1}{y}\right)^{\gamma - 1} \ .$$

Proof. — By De Bruijn's Tauberian theorem, we only need to get an equivalent to $-\log \Phi_{1,c}(\lambda)$ when λ goes to infinity. To that end, we use (39) and we first get an estimate of $\kappa_1(\lambda, \infty)$: we take $a = 1$ in (12) and we let μ go to infinity to obtain

$$\int_{\kappa_1(\lambda, \infty)}^{\infty} \frac{du}{u^\gamma - \lambda} = 1 \ .$$

We set $a(\lambda) = \log\big(\frac{\kappa_1(\lambda, \infty)^\gamma}{\lambda} - 1\big)$ and we use the change of variable $y = \log(\lambda^{-1}u^\gamma - 1)$ in the previous integral equation to get

$$\gamma \lambda^{1 - \frac{1}{\gamma}} = \int_{a(\lambda)}^{\infty} \frac{dy}{(1 + e^y)^{1 - \frac{1}{\gamma}}} \ .$$

Note that $a(\lambda)$ decreases to $-\infty$ when λ goes to ∞. Thus, there exists $\lambda_0 \in (0, \infty)$ such that $a(\lambda) < 0$, for any $\lambda > \lambda_0$. Next, observe that

$$(40) \quad Q_0(\lambda) := \int_0^\infty \frac{dy}{(1 + e^y)^{1 - \frac{1}{\gamma}}} - \int_{a(\lambda)}^0 \left(1 - \frac{1}{(1 + e^y)^{1 - \frac{1}{\gamma}}}\right) dy \xrightarrow[\lambda \to \infty]{} Q_0(\infty) \in \mathbb{R},$$

and that $\gamma \lambda^{1 - \frac{1}{\gamma}} = Q_0(\lambda) - a(\lambda)$, for any $\lambda > \lambda_0$. Namely,

$$(41) \quad \kappa_1(\lambda, \infty) = \lambda^{\frac{1}{\gamma}} \left(1 + \exp\left(Q_0(\lambda) - \gamma \lambda^{\frac{\gamma - 1}{\gamma}}\right)\right)^{\frac{1}{\gamma}}.$$

Similarly, we get an estimate for $\kappa_{1+c}(\lambda, 0)$: we take $a = 1 + c$ and $\mu = 0$ in (12):

$$\int_0^{\kappa_{1+c}(\lambda, 0)} \frac{du}{\lambda - u^\gamma} = 1 + c.$$

We set $b(\lambda) = -\log\left(1 - \frac{\kappa_{1+c}(\lambda, 0)^\gamma}{\lambda}\right)$ and we take $y = -\log(1 - \lambda^{-1} u^\gamma)$ to get

$$(1 + c)\gamma \lambda^{1 - \frac{1}{\gamma}} = \int_0^{b(\lambda)} \frac{dy}{(1 - e^{-y})^{1 - \frac{1}{\gamma}}} = b(\lambda) + Q_1(\lambda),$$

where

$$(42) \quad Q_1(\lambda) := \int_0^{b(\lambda)} \left(\frac{1}{(1 - e^{-y})^{1 - \frac{1}{\gamma}}} - 1\right) dy \xrightarrow[\lambda \to \infty]{} Q_1(\infty) \in [0, \infty).$$

Thus,

$$(43) \quad \kappa_{1+c}(\lambda, 0) = \lambda^{\frac{1}{\gamma}} \left(1 - \exp\left(Q_1(\lambda) - (1 + c)\gamma \lambda^{\frac{\gamma - 1}{\gamma}}\right)\right)^{\frac{1}{\gamma}}.$$

By (39), (40), (41), (42) and (43), we get

$$-\log \mathbf{N}_1\left(e^{-\lambda \mathbf{m}(B(\rho, 1+c))}\right) \underset{\lambda \to \infty}{\sim} \gamma \lambda^{\frac{\gamma - 1}{\gamma}},$$

and De Bruijn's Tauberian theorem entails the desired result (see Theorem 4.12.9 page 254 in [5]). $\qquad \square$

In the Brownian case, computations are explicit: we easily derive from (12) that for any $a, \lambda, \mu \in [0, \infty)$ such that $\sqrt{\lambda} \neq \mu$,

$$(44) \quad \kappa_a(\lambda, \mu) = \sqrt{\lambda} \cdot \frac{e^{a\sqrt{\lambda}}(\sqrt{\lambda} + \mu) - e^{-a\sqrt{\lambda}}(\sqrt{\lambda} - \mu)}{e^{a\sqrt{\lambda}}(\sqrt{\lambda} + \mu) + e^{-a\sqrt{\lambda}}(\sqrt{\lambda} - \mu)}.$$

Recall that $\coth(x) = (e^x + e^{-x})/(e^x - e^{-x}) = 1/\tanh(x)$, and note that $v(1) = 1$. Thus, (39) implies that

$$\mathbf{N}_1\left(e^{-\lambda \mathbf{m}(B(\rho, 1+c))}\right) = \sqrt{\lambda}\left(\coth(\sqrt{\lambda}) - \tanh((1 + c)\sqrt{\lambda})\right).$$

We next use the well-known formulas
(45)
$$x \tanh(x) = \sum_{n \geq 0} \frac{2x^2}{x^2 + \frac{\pi^2}{4}(2n+1)^2} \quad \text{and} \quad x \coth(x) = 1 + \sum_{n \geq 1} \frac{2x^2}{x^2 + \pi^2 n^2} \, .$$

Therefore,
$$\int_0^\infty \mathbf{N}_1\big(\mathbf{m}(B(\rho, 1+c)) \geq y\big) e^{-\lambda y} \, dy = \lambda^{-1}\big(1 - \mathbf{N}_1\big(e^{-\lambda \mathbf{m}(B(\rho, 1+c))}\big)\big)$$

$$= \frac{2}{1+c} \sum_{n \geq 0} \frac{1}{\lambda + \left(\frac{\pi(2n+1)}{2(1+c)}\right)^2} - 2 \sum_{n \geq 1} \frac{1}{\lambda + \pi^2 n^2}.$$

This easily implies the following.

LEMMA 3.2. — *Consider the Brownian case: $\gamma = 2$. Then, for any $y \in [0, \infty)$,*

$$\mathbf{N}_1\big(\mathbf{m}(B(\rho, 1+c)) \geq y\big) = \frac{2}{1+c} \sum_{n \geq 0} \exp\big(-\tfrac{\pi^2 (2n+1)^2}{4(1+c)^2} \, y\big) - 2 \sum_{n \geq 1} \exp(-\pi^2 n^2 \, y)$$

$$\underset{y \to \infty}{\sim} \frac{2}{1+c} \, \exp\big(-\tfrac{\pi^2}{4(1+c)^2} \, y\big) \, .$$

This result shall be used in the proof of Theorem 1.4.

3.2. Tail of the distribution of M_*. — Recall from (36) that $M_* = M_1^*(1)$. We set $\mathcal{L}(\lambda) = \mathbf{E}\big[\exp(-\lambda M_*)\big]$ and from (35), we get

(46) $$\mathcal{L}(\lambda) = \mathbf{E}\big[\exp(-\lambda M_*)\big] = 1 - \frac{\kappa_1(\lambda, 0)^\gamma}{\lambda}, \quad \lambda \in [0, \infty) \, .$$

The following lemma provides an equivalent of the tail at $0+$ of the distribution of M_* that is used in the proof of Proposition 1.1.

LEMMA 3.3. — *For any $\gamma \in (1, 2]$ we have the following estimate.*

$$\lim_{y \to 0+} y^{-\frac{\gamma-1}{2}} \exp\big(\tfrac{1}{y^{\gamma-1}}\big) \, \mathbf{P}\big(M_* \leq (\gamma-1) y\big) = e^{C_\gamma} \sqrt{\frac{\gamma(\gamma-1)}{2\pi}} \, .$$

where C_γ is a constant given by

(47) $$C_\gamma = \int_0^1 u^{-1}\big((1-u)^{-\frac{\gamma-1}{\gamma}} - 1\big) du = \sum_{n \geq 1} \frac{1}{n} \left| \binom{\frac{1-\gamma}{\gamma}}{n} \right| \, ,$$

Proof. — First observe that for any $u \in [0, 1)$, we have

(48) $$(1-u)^{-\frac{\gamma-1}{\gamma}} = \sum_{n \geq 0} (-1)^n \binom{\frac{1-\gamma}{\gamma}}{n} u^n = 1 + \sum_{n \geq 1} \left| \binom{\frac{1-\gamma}{\gamma}}{n} \right| u^n \, .$$

This easily entails the second equality in (47). For any $y \in [0, 1)$, we set

$$F(y) = \int_y^1 \frac{du}{u(1-u)^{\frac{\gamma-1}{\gamma}}}$$

By (12) and by a simple change of variable, $F(\mathcal{L}(\lambda)) = \gamma \lambda^{\frac{\gamma-1}{\gamma}}$. Note that

$$F(y) = \int_y^1 u^{-1} du + \int_0^1 u^{-1} \big((1-u)^{-\frac{\gamma-1}{\gamma}} - 1\big) du - \int_0^y u^{-1} \big((1-u)^{-\frac{\gamma-1}{\gamma}} - 1\big) du$$

$$= -\log y + C_\gamma - h(y) \ .$$

Here $h : [0, 1] \to [0, \infty)$ is increasing, $h(0) = 0$, $h(1) = C_\gamma$, and $h(y) = \sum_{n \geq 1} a_n y^n$, where for any $n \geq 1$,

$$(49) \qquad \frac{h^{(n)}(0)}{n!} = \frac{1}{n} \left| \binom{\frac{1-\gamma}{\gamma}}{n} \right| = \frac{1}{n} \prod_{k=1}^n \left(1 - \frac{1}{\gamma k} \right) \in (0, 1) \ .$$

Thus,

$$(50) \qquad \mathcal{L}(\lambda) = e^{C_\gamma} \exp\big(-\gamma \lambda^{\frac{\gamma-1}{\gamma}}\big) \exp\big(-h(\mathcal{L}(\lambda))\big) \ .$$

We next use Fubini for sums of nonnegative real numbers to prove that for any $y \in [0, 1]$, and any integer $m \geq 1$,
(51)

$$h(y)^m = \sum_{\substack{n \geq m}} y^n \cdot \sum_{\substack{q_1,\ldots,q_m \geq 1 \\ q_1+\ldots+q_m=n}} a_{q_1} \ldots a_{q_m} = \sum_{n \geq m} y^n \cdot \sum_{\substack{p_1+\ldots+p_n=m \\ p_1+2p_2+\ldots+np_n=n}} \frac{m!}{p_1! \ldots p_n!} a_1^{p_1} \ldots a_n^{p_n}.$$

Thus, for any $y \in [0, 1]$, $\exp(h(y)) = 1 + \sum_{n \geq 1} d_n y^n$, $d_n = \sum a_1^{p_1} \ldots a_n^{p_n}/(p_1! \ldots p_n!)$, where the sum is over all the $p_1, \ldots, p_n \geq 0$ such that $p_1 + 2p_2 + \ldots + np_n = n$. Standard arguments on analytic functions imply that there exists $r_1 > 0$ such that

$$\exp(-h(y)) = 1 + \sum_{n \geq 1} c_n y^n \ , \quad y \in [0, r_1)$$

and (51) easily entails that for any $n \geq 1$,

$$c_n = \sum_{\substack{p_1,\ldots,p_n \geq 0 \\ p_1+2p_2+\ldots+np_n=n}} \frac{(-1)^{p_1+\ldots+p_n}}{p_1! \ldots p_n!} a_1^{p_1} \ldots a_n^{p_n} \ .$$

Consequently, $|c_n| \leq d_n$, and $r_1 \geq 1$, which implies that

$$(52) \qquad \sum_{n \geq 0} |c_n| \leq \exp(h(1)) = e^{C_\gamma} \ .$$

The previous arguments and (50) imply

$$(53) \quad \mathcal{L}(\lambda) = e^{C_\gamma} \exp\big(-\gamma \lambda^{\frac{\gamma-1}{\gamma}}\big) + \sum_{n \geq 1} e^{C_\gamma} c_n \exp\big(-\gamma \lambda^{\frac{\gamma-1}{\gamma}}\big) \mathcal{L}(\lambda)^n \ , \quad \lambda \in [0, \infty) \ .$$

We next introduce a non-negative random variable S defined on $(\Omega, \mathscr{F}, \mathbf{P})$, that has a stable distribution whose Laplace transform is given by

$$\mathbf{E}\big[\exp\left(-\lambda S\right)\big] = \exp\left(-\gamma \lambda^{\frac{\gamma-1}{\gamma}}\right).$$

We use the following standard tail estimate due to Skorohod [27] (see also Example 4.1 in [21]).

$$(54) \qquad \lim_{y \to 0+} y^{-\frac{\gamma-1}{2}} \exp\left(\frac{1}{y^{\gamma-1}}\right) \mathbf{P}\big(S \le (\gamma-1)\,y\big) = \sqrt{\frac{\gamma(\gamma-1)}{2\pi}}.$$

We denote by q the density of S and by μ the distribution of M_*. The bound (52) implies that

$$R(dx) := e^{C_\gamma} q(x)dx + \sum_{n \ge 1} e^{C_\gamma} c_n (q * \mu^{*n})(dx)$$

is a Borel signed measure on $[0, \infty)$ whose total variation is bounded by $2\exp(2C_\gamma)$. Moreover (53), implies that $\int_{[0,\infty)} e^{-\lambda x} R(dx) = \mathcal{L}(\lambda)$, $\lambda \in [0, \infty)$. Standard arguments on Laplace transform imply that $R = \mu$. Denote by $(Y_n)_{n \ge 1}$, a sequence of i.i.d. copies of M_* that are also independent of S. Since $R = \mu$, for any $y \in [0, \infty)$,

$$\mathbf{P}(M_* \le (\gamma-1)y) = e^{C_\gamma}\mathbf{P}(S \le (\gamma-1)y) + \sum_{n \ge 1} e^{C_\gamma} c_n \mathbf{P}(S + Y_1 + \ldots + Y_n \le (\gamma-1)y).$$

The obvious bound

$$\mathbf{P}\big(S + Y_1 + \ldots + Y_n \le (\gamma-1)y\big) \le \mathbf{P}\big(S \le (\gamma-1)y\big) \mathbf{P}\big(M_* \le (\gamma-1)y\big)^n$$

entails

$$\left| \frac{\mathbf{P}(M_* \le (\gamma-1)y)}{e^{C_\gamma}\mathbf{P}(S \le (\gamma-1)y)} - 1 \right| \le \sum_{n \ge 1} |c_n|\,\mathbf{P}\big(M_* \le (\gamma-1)y\big)^n \xrightarrow[y \to 0+]{} 0,$$

which entails the desired result thanks to (54). $\qquad\qquad\square$

In the Brownian case, the computations are explicit. By (44), we get

$$\int_0^\infty e^{-\lambda y} \mathbf{P}(M_* \ge y) = \lambda^{-1}\big(1 - \mathcal{L}(\lambda)\big) = \lambda^{-1}\tanh^2(\sqrt{\lambda}).$$

Observe that $\tanh^2(x) = \tanh'(0) - \tanh'(x)$. If we set $a_n = \pi^2(2n+1)^2/4$, then (45) implies

$$\tanh^2(x) = \sum_{n \ge 0} \frac{2x^2}{a_n(x^2 + a_n)} + \frac{4x^2}{(x^2 + a_n)^2}.$$

Thus

$$\int_0^\infty e^{-\lambda y} \mathbf{P}(M_* \geq y) = \sum_{n \geq 0} \frac{2}{a_n(\lambda + a_n)} + \frac{4}{(\lambda + a_n)^2} \,,$$

which easily implies the following lemma.

LEMMA 3.4. — *Consider the Brownian case:* $\gamma = 2$. *Then, for any* $y \in [0, \infty)$,

$$\mathbf{P}(M_* \geq y) = \sum_{n \geq 0} 4\big(\tfrac{2}{\pi^2(2n+1)^2} + y\big) \exp\big(-\tfrac{\pi^2}{4}(2n+1)^2 y\big)$$

$$\underset{y \to \infty}{\sim} 4y \exp\big(-\tfrac{\pi^2}{4} y\big) \,.$$

This lemma shall be used in the proof of Proposition 1.3.

4. Proofs

4.1. Proof of Theorem 1.2. — We fix $\gamma \in (1, 2]$ and we consider the γ-stable tree $(\mathscr{T}, d)$ with root ρ coded by the γ-height process H under its excursion measure $\mathbf{N}$, as defined in the introduction. Recall from Section 2.2 the definition of the (a, ε)-subtrees truncated at height η, whose set is denoted by $\mathscr{D}_{a,\varepsilon,\eta} = \{T_i \, ; \, 1 \leq i \leq Z_a(\varepsilon)\}$. Recall that

$$f_\gamma(r) = \frac{r^{\frac{\gamma}{\gamma-1}}}{(\log 1/r)^{\frac{1}{\gamma-1}}} \,, \quad r \in (0, 1) \,.$$

Lower Bound. — Let fix a positive integer R_0 and a real number $\alpha \in (0, \infty)$, that will be specified later. For any integer $n \geq 4$, we set

$$V(n) = \mathbf{1}_{\{\mathbf{m}(B(\rho, 2^{-n}) \leq \alpha f_\gamma(2^{-n})\}}$$
$$+ \sum_{1 \leq k < R_0 2^n} \#\{T_i \in \mathscr{D}_{k2^{-n}, 2^{-n}, 2^{-n}} \, : \, \mathbf{m}(T_i) \leq \alpha f_\gamma(2^{-n})\}.$$

We first prove that

$$(55) \qquad V(n) = 0 \implies \inf_{\sigma \in B(\rho, R_0)} \mathbf{m}\big(B(\sigma, 2^{-n+3})\big) > \alpha \, f_\gamma(2^{-n}) \,.$$

Indeed, we apply Lemma 2.3 with $r = 2^{-n+3}$. Thus, $n_r = n - 2$ and we have $2^{-n_r} < r = 2^{-n_r+1}$. Let $\sigma \in \mathscr{T}$ be such that $d(\rho, \sigma) \leq R_0$. We first consider the case where $d(\rho, \sigma) \geq 2.2^{-n_r-2} = 2^{-n+1} = r/4$. Let $k \in \mathbb{N}$ be such that $(k+1)2^{-n_r-2} \leq d(\rho, \sigma) < (k+2)2^{-n_r-2}$. Observe that $1 \leq k < R_0 2^n$. Lemma 2.3 implies that there exists a truncated subtree $T \in \mathscr{D}_{k2^{-n}, 2^{-n}, 2^{-n}}$ such that $T \subset B(\sigma, 2^{-n+3})$. Consequently, if $V(n) = 0$, then $\alpha f_\gamma(2^{-n}) < \mathbf{m}(B(\sigma, 2^{-n+3}))$.

We next consider the case where $d(\rho, \sigma) < 2^{-n+1}$. Then it is easy to see that $B(\rho, 2^{-n}) \subset B(\sigma, 3.2^{-n}) \subset B(\sigma, 2^{-n+3})$. Thus, if $V(n) = 0$, $\mathbf{m}(B(\sigma, 2^{-n+3})) \geq \mathbf{m}(B(\rho, 2^{-n}) > \alpha f_\gamma(2^{-n})$, which completes the proof of (55).

We next claim that it is possible to find α such that

$$(56) \qquad \sum_{n \geq 4} \mathbf{N}\big(V(n)\mathbf{1}_{\{\Gamma(\mathcal{T}) > 2^{-n}\}}\big) < \infty \, .$$

We first set $x_n = \mathbf{N}\big(V(n)\mathbf{1}_{\{\Gamma(\mathcal{T}) > 2^{-n}\}}\big)$ and

$$\begin{aligned} y_n &= \mathbf{N}\big(\, \mathbf{m}(B(\rho, 2^{-n})) \leq \alpha f_\gamma(2^{-n}) \quad \text{and} \quad \Gamma(\mathcal{T}) > 2^{-n} \,\big) \\ &= v(2^{-n})\mathbf{N}_{2^{-n}}\big(\, \mathbf{m}(B(\rho, 2^{-n})) \leq \alpha f_\gamma(2^{-n}) \,\big) \, . \end{aligned}$$

To simplify the notation, we also set $Z(k, n) = Z_{k2^{-n}}(2^{-n})$. We get the following.

$$\begin{aligned} x_n &\leq y_n + \sum_{1 \leq k < R_0 2^n} \mathbf{N}\Big(\sum_{1 \leq i \leq Z(k,n)} \mathbf{1}_{\{\mathbf{m}(T_i) \leq \alpha f_\gamma(2^{-n})\}} \Big) \\ &\leq y_n + \sum_{1 \leq k < R_0 2^n} v(k2^{-n})\mathbf{N}_{k2^{-n}}\Big(\sum_{1 \leq i \leq Z(k,n)} \mathbf{1}_{\{\mathbf{m}(T_i) \leq \alpha f_\gamma(2^{-n})\}} \Big) \end{aligned}$$

Recall from the definition of the branching property that $\mathscr{G}_{k2^{-n}}$ stands for the sigma-field generated by the tree below $k2^{-n}$. Recall from Section 2.2 that conditionnally given $\mathscr{G}_{k2^{-n}}$, $Z(k, n)$ is a Poisson random variable with parameter $v(2^{-n})\langle \ell^{k2^{-n}} \rangle$ and by (38), we get

$$\begin{aligned} \mathbf{N}_{k2^{-n}}\Big(&\sum_{1 \leq i \leq Z(k,n)} \mathbf{1}_{\{\mathbf{m}(T_i) \leq \alpha f_\gamma(2^{-n})\}} \Big) \\ &= \mathbf{N}_{k2^{-n}}\big(Z(k, n)\big)\, \mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho, 2^{-n})) \leq \alpha f_\gamma(2^{-n})\big) \\ &= \mathbf{N}_{k2^{-n}}\big(\langle \ell^{k2^{-n}} \rangle\big)\, y_n \, . \end{aligned}$$

Consequently, $x_n \leq y_n + \sum_{1 \leq k < R_0 2^n} v(k2^{-n})\mathbf{N}_{k2^{-n}}\big(\langle \ell^{k2^{-n}} \rangle\big)\, y_n$. We next use (24) that implies $v(k2^{-n})\mathbf{N}_{k2^{-n}}\big(\langle \ell^{k2^{-n}} \rangle\big) = 1$. Thus, we get $x_n \leq R_0 2^n y_n$. Namely,

$$(57)$$
$$\mathbf{N}\big(V(n)\mathbf{1}_{\{\Gamma(\mathcal{T}) > 2^{-n}\}}\big) \leq R_0\, 2^n v(2^{-n})\, \mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho, 2^{-n})) \leq \alpha f_\gamma(2^{-n})\big) \, .$$

By (21), $2^n v(2^{-n}) = (\gamma - 1)^{-\frac{1}{\gamma-1}} 2^{\frac{\gamma n}{\gamma-1}}$ and the scaling property (26) implies

$$\mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho, 2^{-n})) \leq \alpha f_\gamma(2^{-n})\big) = \mathbf{N}_1\big(\mathbf{m}(B(\rho, 1)) \leq \alpha(\log 2^n)^{-\frac{1}{\gamma-1}}\big) \, .$$

By Lemma 3.1, there is a constant $q \in (0, \infty)$ that only depends on γ, such that for any $n \geq 4$,

$$\mathbf{N}_1\big(\mathbf{m}(B(\rho, 1)) \leq \alpha(\log 2^n)^{-\frac{1}{\gamma-1}}\big) \leq \exp\big(-\alpha^{-(\gamma-1)}q \log 2^n\big) \, .$$

This inequality combined with (57) entails that for any $n \geq 4$,

$$\mathbf{N}\big(V(n)\mathbf{1}_{\{\Gamma(\mathscr{T})>2^{-n}\}}\big) \leq R_0(\gamma-1)^{-\frac{1}{\gamma-1}} \exp\big(-\big(\alpha^{-(\gamma-1)}q\log 2 - \tfrac{\gamma}{\gamma-1}\log 2\big)n\big),$$

which implies (56) if $\alpha < \big(\frac{(\gamma-1)q}{\gamma}\big)^{\frac{1}{\gamma-1}}$.

Since $V(n) \in \mathbb{N}$, (56) implies that $\mathbf{N}$-a.e. for any sufficiently large n, we have $V(n)\mathbf{1}_{\{\Gamma(\mathscr{T})>2^{-n}\}} = 0$. Since $\mathbf{N}(\Gamma(\mathscr{T}) = 0) = 0$, it implies that $\mathbf{N}$-a.e. for any sufficiently large n, $V(n) = 0$. We next use (55), to get the following: there exists $\alpha_\gamma \in (0,\infty)$, that only depends on γ, such that for any positive integer R_0,

$$\mathbf{N}\text{-a.e.} \quad \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0, \quad \inf_{\sigma \in B(\rho, R_0)} \mathbf{m}\big(B(\sigma, 2^{-n+3})\big) \geq \alpha_\gamma\, f_\gamma(2^{-n}).$$

Note that $\mathbf{N}$-a.e. there exists R_0 such that $B(\rho, R_0) = \mathscr{T}$. Since α_γ does not depend on R_0, it entails

$$(58) \qquad \mathbf{N}\text{-a.e.} \qquad \liminf_{n\to\infty} \frac{1}{f_\gamma(2^{-n})} \inf_{\sigma \in \mathscr{T}} \mathbf{m}\big(B(\sigma, 2^{-n+3})\big) \geq \alpha_\gamma.$$

Upper Bound. — Let R_0 be a positive integer and let $h_0 \in (0,\infty)$. We also fix $\beta \in (0,\infty)$, that will be specified later. We introduce the following event

$$A_n = \big\{\Gamma(\mathscr{T}) > h_0\big\} \cap \big\{\inf_{\sigma \in B(\rho, R_0)} \mathbf{m}(B(\rho, 2^{-n})) > \beta f_\gamma(2^{-n})\big\}.$$

We assume that $n \geq 4$, and that $h_0 > 2^{-n+1}$. Let $l \geq 1$ be such that $(l+1)2^{-n} \leq h_0 < (l+2)2^{-n}$. We argue on the event A_n: let $\sigma \in \mathscr{T}$ be such that $d(\rho,\sigma) = h_0$; we apply Lemma 2.4 with $r = 2^{-n}$; thus $n_r = n+1$ (namely, $2^{-n-1} = 2^{-n_r} < r = 2^{-n_r+1} = 2^{-n}$); it implies that there exists a truncated subtree $T \in \mathscr{D}_{l2^{-n},2^{-n},3.2^{-n}}$ such that $B(\sigma, 2^{-n}) \subset T$. This proves
$$(59)$$
$$A_n \subset B_n := \big\{\Gamma(\mathscr{T}) > l2^{-n}\big\} \cap \big\{\forall T_i \in \mathscr{D}_{l2^{-n},2^{-n},3.2^{-n}} : \mathbf{m}(T_i) > \beta f_\gamma(2^{-n})\big\}.$$

To simplify the notation we set $Z(l,n) = Z_{l2^{-n}}(2^{-n})$. We then use (38) to get

$$\mathbf{N}(B_n) = \mathbf{N}(\Gamma(\mathscr{T}) > l2^{-n})\, \mathbf{N}_{l2^{-n}}\Big(\prod_{1 \leq i \leq Z(l,n)} \mathbf{1}_{\{\mathbf{m}(T_i)>\beta f_\gamma(2^{-n})\}}\Big)$$

$$= v(l2^{-n})\, \mathbf{N}_{l2^{-n}}\Big(\mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho, 3.2^{-n})) > \beta f_\gamma(2^{-n})\big)^{Z(l,n)}\Big).$$

Recall that under $\mathbf{N}_{l2^{-n}}$, conditionally given $\mathscr{G}_{l2^{-n}}$, the random variable $Z(l,n)$ has a Poisson distribution with mean $v(2^{-n})\langle \ell^{l2^{-n}} \rangle$. We then get

$$(60) \qquad \mathbf{N}(B_n) = v(l2^{-n})\, \mathbf{N}_{l2^{-n}}\big(\exp\big(-z_n \langle \ell^{l2^{-n}} \rangle\big)\big),$$

where we have set

$$z_n = v(2^{-n})\, \mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho, 3.2^{-n})) \leq \beta f_\gamma(2^{-n})\big).$$

By (21) and the scaling property (14), we get

$$z_n = (\gamma - 1)^{-\frac{1}{\gamma-1}} 2^{\frac{n}{\gamma-1}} \mathbf{N}_1\big(\mathbf{m}(B(\rho, 3) \leq \beta(\log 2^n)^{-\frac{1}{\gamma-1}}\big) .$$

By Lemma 3.1, there exists $q, q' \in (0, \infty)$ that only depend on γ such that for any $n \geq 4$,

$$z_n \geq w_n := q' \exp\left(\left(\frac{1}{\gamma-1}\log 2 - \beta^{-(\gamma-1)}q\log 2\right)n\right) .$$

We fix $\beta > \big((\gamma-1)q\big)^{\frac{1}{\gamma-1}}$ so that $\theta_\gamma := \frac{1}{\gamma-1}\log 2 - \beta^{-(\gamma-1)}q\log 2 > 0$. Thus, $w_n = q'\exp(\theta_\gamma n) \to \infty$, when $n \to \infty$. By (60), we get

$$\mathbf{N}(B_n) \leq v(l2^{-n}) \mathbf{N}_{l2^{-n}}\big(\exp\big(-w_n\langle \ell^{l2^{-n}}\rangle\big)\big) .$$

Recall that $h_0 > 2^{-n+1}$, which implies that $l \geq 1$. Thus $l2^{-n} \geq h_0/3$. Since v decreases, $v(l2^{-n}) \leq v(h_0/3)$. Next, recall (23) and observe that $a \mapsto \mathbf{N}_a(\exp(-\mu\langle\ell^a\rangle))$ is decreasing. Thus,

$$(61) \qquad \mathbf{N}(B_n) \leq v(h_0/3) \mathbf{N}_{h_0/3}\big(\exp\big(-w_n\langle\ell^{h_0/3}\rangle\big)\big) .$$

Since $\lim_{n\to\infty} w_n = \infty$, we easily derive from (23) with $a = h_0/3$ that

$$\mathbf{N}_{h_0/3}\big(\exp\big(-w_n\langle\ell^{h_0/3}\rangle\big)\big) \underset{n\to\infty}{\sim} \frac{3}{(\gamma-1)^2 (q')^{\gamma-1}h_0} \exp\big(-(\gamma-1)\theta_\gamma n\big) .$$

Thus, and (59) and (61) immediately entail $\sum_{n\geq 4} \mathbf{N}(A_n) < \infty$. By the Borel-Cantelli Lemma, there exists $\beta_\gamma \in (0, \infty)$, that only depends on γ, such that for any $h_0, R_0 \in (0, \infty)$, $\mathbf{N}$-a.e. for any sufficiently large n, $\mathbf{1}_{A_n} = 0$. Since $\mathbf{N}$-a.e. $0 < \Gamma(\mathscr{T}) < \infty$, one gets

$$(62) \qquad \mathbf{N}\text{-a.e.} \qquad \limsup_{n\to\infty} \frac{1}{f_\gamma(2^{-n})} \inf_{\sigma\in\mathscr{T}} \mathbf{m}\big(B(\sigma, 2^{-n})\big) \leq \beta_\gamma .$$

Since f_γ is regularly varying at 0, (58) and (62) entail Theorem 1.2. $\qquad\square$

4.2. Proof of Theorem 1.4.

— The proof is close to that of Theorem 1.2. Here, we fix $\gamma = 2$ and we recall that $f(r) = r^2 \log 1/r$, $r \in (0, 1)$.

Upper Bound. — We fix a positive integer R_0 and a real number $\alpha \in (0, \infty)$ that is specified later. For any integer $n \geq 4$, we set

$$W(n) = \mathbf{1}_{\{\mathbf{m}(B(\rho, 3.2^{-n})\geq \alpha f(2^{-n})\}}$$

$$+ \sum_{1\leq l < R_0 2^n} \#\{T_i \in \mathscr{D}_{l2^{-n}, 2^{-n}, 3.2^{-n}} \ : \ \mathbf{m}(T_i) \geq \alpha f(2^{-n})\}.$$

Arguing as previously, we apply Lemma 2.4 with $r = 2^{-n}$ to prove that

$$(63) \qquad W(n) = 0 \quad \Longrightarrow \quad \sup_{\sigma\in B(\rho, R_0)} \mathbf{m}\big(B(\sigma, 2^{-n})\big) \leq \alpha f(2^{-n}) .$$

We next claim that it is possible to find α such that

$$(64) \qquad \sum_{n \geq 4} \mathbf{N}\big(W(n)\mathbf{1}_{\{\Gamma(\mathcal{T})>2^{-n}\}}\big) < \infty \ .$$

To simplify the notation we first set $x_n = \mathbf{N}\big(W(n)\mathbf{1}_{\{\Gamma(\mathcal{T})>2^{-n}\}}\big)$ and

$$\begin{aligned}
y_n &= \mathbf{N}\big(\, \mathbf{m}(B(\rho,3.2^{-n})) \geq \alpha f(2^{-n}) \quad \text{and} \quad \Gamma(\mathcal{T})>2^{-n}\,\big) \\
&= v(2^{-n})\mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho,3.2^{-n})) \geq \alpha f(2^{-n})\big) \ .
\end{aligned}$$

We also set $Z(l,n) = Z_{l2^{-n}}(2^{-n})$. Then, we get the following.

$$\begin{aligned}
x_n &\leq y_n + \sum_{1 \leq l < R_0 2^n} \mathbf{N}\Big(\sum_{1 \leq i \leq Z(l,n)} \mathbf{1}_{\{\mathbf{m}(T_i) \geq \alpha f(2^{-n})\}} \Big) \\
&\leq y_n + \sum_{1 \leq l < R_0 2^n} v(l2^{-n})\mathbf{N}_{l2^{-n}}\Big(\sum_{1 \leq i \leq Z(l,n)} \mathbf{1}_{\{\mathbf{m}(T_i) \geq \alpha f(2^{-n})\}} \Big) \\
&\leq y_n + \sum_{1 \leq l < R_0 2^n} v(l2^{-n})\, \mathbf{N}_{l2^{-n}}\big(Z(l,n)\big)\, \mathbf{N}_{2^{-n}}\Big(\mathbf{m}(B(\rho,3.2^{-n})) \geq \alpha f(2^{-n})\Big) \\
&\leq y_n + \sum_{1 \leq l < R_0 2^n} v(l2^{-n})\mathbf{N}_{l2^{-n}}\big(\langle \ell^{l2^{-n}} \rangle\big)\, y_n \ .
\end{aligned}$$

Here, we used (38) in the third line.

Recall from (24) that $v(l2^{-n})\mathbf{N}_{l2^{-n}}\big(\langle \ell^{l2^{-n}} \rangle\big) = 1$. Thus, $x_n \leq R_0 2^n y_n$. We next get an equivalent of y_n: by (21) with $\gamma = 2$, we have $v(2^{-n}) = 2^n$; the scaling property (26) and Lemma 3.2 with $c = 2$ imply

$$\begin{aligned}
y_n &= 2^n \mathbf{N}_{2^{-n}}\big(\mathbf{m}(B(\rho,3.2^{-n})) \geq \alpha f(2^{-n})\big) \\
&= 2^n \mathbf{N}_1\big(\mathbf{m}(B(\rho,3)) \geq \alpha \log 2^n\big) \\
&\underset{n \to \infty}{\sim} \frac{2}{3} \exp\big(-\big(\tfrac{\pi^2 \log 2}{36}\alpha - \log 2\big) n \big) \ .
\end{aligned}$$

Thus,

$$x_n \leq R_0 2^n y_n \underset{n \to \infty}{\sim} \frac{2}{3} R_0 \exp\big(-\big(\tfrac{\pi^2 \log 2}{36}\alpha - 2\log 2\big) n \big)$$

which implies (64) if $\alpha > 72/\pi^2$. We argue as in the proof of Theorem 1.2 to prove that (64) implies

$$(65) \qquad \mathbf{N}\text{-a.e.} \qquad \limsup_{n \to \infty} \frac{1}{f(2^{-n})} \sup_{\sigma \in \mathcal{T}} \mathbf{m}\big(B(\sigma,2^{-n})\big) \leq \frac{73}{\pi^2} \ .$$

Lower Bound. — Let R_0 be a positive integer and let $h_0 \in (0,\infty)$. We also fix $\beta \in (0,\infty)$, that is specified later. We introduce the following event

$$C_n = \big\{\Gamma(\mathcal{T}) > h_0\big\} \cap \big\{ \sup_{\sigma \in B(\rho,R_0)} \mathbf{m}(B(\rho,2^{-n+3})) < \beta f(2^{-n}) \big\}$$

We assume that $n \geq 4$, and that $h_0 \geq 2^{-n+1}$. Let $k \geq 1$ be such that $(k+1)2^{-n} \leq h_0 < (k+2)2^{-n}$. We argue on the event C_n: let $\sigma \in \mathcal{T}$ be such that

$d(\rho,\sigma) = h_0$; we apply Lemma 2.3 with $r = 2^{-n+3}$ (and thus $n_r = n - 2$) to prove that there exists $T \in \mathscr{D}_{k2^{-n},2^{-n},2^{-n}}$ such that $T \subset B(\sigma, 2^{-n+3})$. Thus,

$$(66) \qquad C_n \subset D_n := \left\{ \Gamma(\mathscr{T}) > k2^{-n} \right\} \cap \left\{ \forall T_i \in \mathscr{D}_{k2^{-n},2^{-n},2^{-n}} : \mathbf{m}(T_i) < \beta f(2^{-n}) \right\}.$$

To simplify the notation we set $Z(k,n) = Z_{k2^{-n}}(2^{-n})$. We then use (38) to get

$$\mathbf{N}(D_n) = \mathbf{N}(\Gamma(\mathscr{T}) > k2^{-n}) \, \mathbf{N}_{k2^{-n}} \left(\prod_{1 \le i \le Z(k,n)} \mathbf{1}_{\{\mathbf{m}(T_i) < \beta f(2^{-n})\}} \right)$$

$$= v(k2^{-n}) \, \mathbf{N}_{k2^{-n}} \left(\mathbf{N}_{2^{-n}} \big(\mathbf{m}(B(\rho, 2^{-n})) < \beta f(2^{-n}) \big)^{Z(k,n)} \right).$$

Recall that under $\mathbf{N}_{k2^{-n}}$, conditionally given $\mathscr{G}_{k2^{-n}}$, the random variable $Z(k,n)$ has a Poisson distribution with mean $v(2^{-n})\langle \ell^{k2^{-n}} \rangle$. We then get

$$\mathbf{N}(D_n) = v(k2^{-n}) \, \mathbf{N}_{k2^{-n}} \left(\exp \left(-z_n \langle \ell^{k2^{-n}} \rangle \right) \right),$$

where we have set

$$z_n = v(2^{-n}) \, \mathbf{N}_{2^{-n}} \big(\mathbf{m}(B(\rho, 2^{-n})) \ge \beta f(2^{-n}) \big).$$

We next apply (21) and (23) with $\gamma = 2$ to get $v(a) = 1/a$ and

$$\mathbf{N}_a(\exp(-\mu \langle \ell^a \rangle)) = (1 + a\mu)^{-1}.$$

Since $h_0 \ge 2^{-n+1}$, we have $k2^{-n} \ge h_0/3$ and we get

$$(67) \qquad \mathbf{N}(D_n) = \frac{1}{k2^{-n}(1 + k2^{-n}z_n)} \le \frac{1}{\frac{1}{3}h_0(1 + \frac{1}{3}h_0 z_n)},$$

We next apply (14) and Lemma 3.2 with $c = 0$, to obtain

$$z_n = 2^n \mathbf{N}_1 \big(\mathbf{m}(B(\rho, 1)) \ge \beta \log 2^n \big) \underset{n \to \infty}{\sim} 2 \exp \left(\frac{\log 2}{4}(4 - \pi^2 \beta)n \right).$$

Then, for any $\beta < 4/\pi^2$, (66) and (67) entail that $\sum_{n \ge 4} \mathbf{N}(C_n) < \infty$. Thus, for any $h_0, R_0 \in (0, \infty)$, $\mathbf{N}$-a.e. for any sufficiently large n, $\mathbf{1}_{C_n} = 0$. Since $\mathbf{N}$-a.e. $0 < \Gamma(\mathscr{T}) < \infty$, one easily gets

$$(68) \qquad \mathbf{N}\text{-a.e.} \qquad \liminf_{n \to \infty} \frac{1}{f(2^{-n})} \sup_{\sigma \in \mathscr{T}} \mathbf{m}\big(B(\sigma, 2^{-n+3}) \big) \ge \frac{3}{\pi^2}.$$

Since f is regularly varying at 0, (65) and (68) entail Theorem 1.4. $\qquad\square$

4.3. Proof of Proposition 1.1. — Let us fix $\gamma \in (1, 2]$. Recall that

$$g_\gamma(r) = \frac{r^{\frac{\gamma}{\gamma-1}}}{(\log \log 1/r)^{\frac{1}{\gamma-1}}}, \quad r \in (0, e^{-1}).$$

Recall from (32) the definition of $M_r^*(a)$. We only need to prove the following: for any $a \in (0, \infty)$,

$$(69) \qquad \textbf{P}\text{-a.s.} \quad \liminf_{r \to 0} \frac{M_r^*(a)}{g_\gamma(r)} = \gamma - 1 \, .$$

Indeed, by (33), (69) we get

$$\mathbf{N}\left(\int_{\mathscr{T}} \mathbf{1}_{\{\liminf_{r \to 0} \mathbf{m}(B(\sigma, r))/g_\gamma(r) \neq \gamma - 1\}} \mathbf{m}(d\sigma) \right) = 0 \, ,$$

that immediately entails Proposition 1.1.

Lower bound in (69). — We fix $a \in (0, \infty)$. Recall from (36) and (37) that for any $r \in (0, a]$,

$$(70) \qquad r^{-\frac{\gamma}{\gamma - 1}} M_r^*(a) = M_1^*(1) := M_* $$

and recall Lemma 3.3 that gives the tails of M_* at $0+$. We set

$$Q = e^{C_\gamma} \sqrt{\frac{\gamma(\gamma - 1)}{2\pi}} \, ,$$

that is the right limit in Lemma 3.3. Fix $\varrho \in (0, 1)$ and $c \in (0, \infty)$. Then, (70) and Lemma 3.3 imply that

$$\mathbf{P}\left(M_{\varrho^n}^*(a) \leq (\gamma - 1)c g_\gamma(\varrho^n) \right) = \mathbf{P}\left(M_* \leq (\gamma - 1)c(\log\log(\varrho^{-n}))^{-\frac{1}{\gamma - 1}} \right)$$
$$\underset{n \to \infty}{\sim} Q c^{\frac{\gamma - 1}{2}} (\log 1/\varrho)^{-c^{-(\gamma - 1)}} (\log n)^{-1/2} n^{-c^{-(\gamma - 1)}} \, .$$

By Borel-Cantelli, for any $c < 1$, $\textbf{P}$-a.s. $\liminf_{n \to \infty} M_{\varrho^n}^*(a)/g_\gamma(\varrho^n) \geq (\gamma - 1)c$. An easy argument, entails that $\textbf{P}$-a.s.

$$\liminf_{n \to \infty} \frac{M_{\varrho^n}^*(a)}{g_\gamma(\varrho^n)} \geq \gamma - 1 \, .$$

For any $r \in (0, 1/\varrho)$, let $n(r) \in \mathbb{N}$ be such that $\varrho^{n(r)} < r \leq \varrho^{n(r) - 1}$. Thus,

$$\frac{M_r^*(a)}{g_\gamma(r)} \geq \varrho^{\frac{\gamma}{\gamma - 1}} \left(\frac{\log(\log(\varrho^{1 - n(r)}))}{\log(\log(\varrho^{-n(r)}))} \right)^{\frac{1}{\gamma - 1}} \frac{M_{\varrho^{n(r)}}^*(a)}{g_\gamma(\varrho^{n(r)})} \, .$$

Thus, for any $\varrho \in (0, 1)$, $\textbf{P}$-a.s. $\liminf_{r \to 0} M_r^*(a)/g_\gamma(r) \geq \varrho^{\frac{\gamma}{\gamma - 1}}(\gamma - 1)$, and by letting ϱ go to 1, we get

$$(71) \qquad \textbf{P}\text{-a.s.} \qquad \liminf_{r \to 0} \frac{M_r^*(a)}{g_\gamma(r)} \geq \gamma - 1 \, .$$

Upper bound in (69). — For any $n \geq 2$, we set $r_n = (\log n)^{-n}$ and

$$\varepsilon_n = \mathbf{1}_{\{M^*_{r_n}(a) \leq (\gamma - 1)g_\gamma(r_n)\}} \,,$$

and $S_n = \varepsilon_2 + \ldots + \varepsilon_n$. Then, (70) and Lemma 3.3 imply that

$$(72) \qquad \mathbf{E}\left[\varepsilon_n\right] \underset{n \to \infty}{\sim} Q\left(\log \log n\right)^{-1}(\log n)^{-1/2}n^{-1} \,.$$

Therefore, $\lim_{n \to \infty} \mathbf{E}[S_n] = \infty$. Next observe that

$$(73) \qquad \mathbf{E}[S_n^2] = \mathbf{E}[S_n] + 2 \sum_{2 \leq k < l \leq n} \mathbf{E}[\varepsilon_k \varepsilon_l] \,.$$

We then use the following lemma.

LEMMA 4.1. — *There exists a constant $q \in (0, \infty)$ that only depend on γ such that for any $2 \leq k < l$, $\mathbf{E}[\varepsilon_k \varepsilon_l] \leq q\mathbf{E}[\varepsilon_k]\mathbf{E}[\varepsilon_l]$.*

Proof. — First recall that $(U_t, t \geq 0)$ is a subordinator defined on $(\Omega, \mathscr{F}, \mathbf{P})$ with Laplace exponent $\lambda \mapsto \gamma\lambda^{\gamma-1}$. Then, recall that $\mathscr{N}^* = \sum_{j \in \mathscr{I}^*} \delta_{(r_j^*, H^{*j})}$ is a random point measure on $[0, \infty) \times C([0, \infty), \mathbb{R})$ defined on $(\Omega, \mathscr{F}, \mathbf{P})$ such that conditionally given U, $\mathscr{N}^*$ is distributed as a Poisson point measure with intensity $dU_r \otimes \mathbf{N}(dH)$. Next recall for any $0 \leq r' \leq r \leq a$, the notation

$$M^*_{r',r}(a) = \sum_{j \in \mathscr{I}^*} \mathbf{1}_{(r',r]}(r_j^*) \int_0^{\zeta_j^*} \mathbf{1}_{\{H_s^{*j} \leq r - r_j^*\}} ds \,,$$

where ζ_j^* stands for the lifetime of H^{*j}, for any $j \in \mathscr{I}^*$. Recall from Lemma 2.2 that $M^*_{r',r}(a) \leq M^*_r(a)$ and that $M^*_{r',r}(a)$ is independent from $M^*_{r'}(a)$. Thus, observe that for any $2 \leq k < l$,

$$\{ M^*_{r_k}(a) \leq (\gamma - 1)g_\gamma(r_k) \} \cap \{ M^*_{r_l}(a) \leq (\gamma - 1)g_\gamma(r_l) \}$$
$$\subset \{ M^*_{r_l, r_k}(a) \leq (\gamma - 1)g_\gamma(r_k) \} \cap \{M^*_{r_l}(a) \leq (\gamma - 1)g_\gamma(r_l) \} \,.$$

Thus,

$$\mathbf{E}[\varepsilon_k \varepsilon_l] \leq \mathbf{P}\left(M^*_{r_l, r_k}(a) \leq (\gamma - 1)g_\gamma(r_k) \right) \mathbf{E}[\varepsilon_l] \,.$$

Next recall from (37) that $(r - r')^{-\frac{\gamma}{\gamma-1}} M^*_{r',r}(a)$ has the same law as M_*. Consequently,

$$\mathbf{E}[\varepsilon_k \varepsilon_l] \leq \mathbf{P}\left((1 - (r_l/r_k))^{\frac{\gamma}{\gamma-1}} M_* \leq (\gamma - 1)(\log \log 1/r_k)^{-\frac{1}{\gamma-1}} \right) \mathbf{E}[\varepsilon_l]$$
$$\leq \mathbf{P}\left((1 - (r_{k+1}/r_k))^{\frac{\gamma}{\gamma-1}} M_* \leq (\gamma - 1)(\log \log 1/r_k)^{-\frac{1}{\gamma-1}} \right) \mathbf{E}[\varepsilon_l] \,.$$

Now observe that

$$r_{k+1}/r_k = (\log k)^{-1} + O((\log k)^{-2}) \quad \text{and} \quad \log \log 1/r_k = \log k + \log \log \log k \,.$$

By Lemma 3.3, we get

$$\mathbf{P}\left(M_* \le (\gamma - 1)(1 - (r_{k+1}/r_k))^{-\frac{\gamma}{\gamma-1}} (\log\log 1/r_k)^{-\frac{1}{\gamma-1}} \right)$$
$$\underset{k\to\infty}{\sim} Q\left(1 - \tfrac{r_{k+1}}{r_k}\right)^{-\frac{\gamma}{2}} (\log\log 1/r_k)^{-1/2} \exp\left(-\left(1 - \tfrac{r_{k+1}}{r_k}\right)^{\gamma} \log\log 1/r_k \right),$$

and since

$$\left(1 - \tfrac{r_{k+1}}{r_k}\right)^{\gamma} \log\log 1/r_k = \log k + \log\log\log k - \gamma + O\left(\frac{\log\log\log k}{\log k}\right),$$

we get,

$$\mathbf{P}\left(M_* \le (\gamma - 1)(1 - (r_{k+1}/r_k))^{-\frac{\gamma}{\gamma-1}} (\log\log 1/r_k)^{-\frac{1}{\gamma-1}} \right)$$
$$\underset{k\to\infty}{\sim} e^{\gamma} Q (\log\log k)^{-1} (\log k)^{-1/2} k^{-1},$$

which easily completes the proof of the lemma by (72). $\qquad\square$

The previous lemma and (73) imply that $\limsup_{n\to\infty} \mathbf{E}[S_n^2]/(\mathbf{E}[S_n])^2 \le q$. By the Kochen-Stone Lemma, we get $\mathbf{P}(\sum_{n\ge2} \varepsilon_n = \infty) \ge 1/q > 0$. Namely, with the lower bound (71), it entails that

$$(74) \qquad \mathbf{P}\left(\liminf_{r\to0} M_r^*(a)/g_\gamma(r) = \gamma - 1 \right) \ge 1/q > 0.$$

Standard arguments on Poisson point processes imply that $\liminf_{r\to0} M_r^*(a)/g_\gamma(r)$ is measurable with respect to the tail sigma-field of U at $0+$. By Blumenthal zero-one law, (74) entails (69), which completes the proof of Proposition 1.1. $\qquad\square$

4.4. Proof of Proposition 1.3. — Let us fix $\gamma = 2$ and let us recall that $h(r) = r^2 \log\log 1/r$, $(0, e^{-1})$. Recall from (32) the definition of $M_r^*(a)$. We only need to prove the following: for any $a \in (0, \infty)$,

$$(75) \qquad \mathbf{P}\text{-a.s.} \quad \limsup_{r\to0} \frac{M_r^*(a)}{h(r)} = \frac{4}{\pi^2}.$$

Indeed by the formula (33), we get

$$\mathbf{N}\left(\int_{\mathscr{T}} \mathbf{1}_{\{\limsup_{r\to0} \mathbf{m}(B(\sigma,r))/h(r) \ne 4/\pi^2\}} \mathbf{m}(d\sigma) \right) = 0,$$

that immediately entails Proposition 1.3.

Upper bound in (75). — Recall from (37) that $r^{-2}M_r^*(a)$ has the law as M_*. We fix $\varrho \in (0,1)$ and $c \in (0,\infty)$. By Lemma 3.4, we get

$$\mathbf{P}(M_{\varrho^n}^*(a) \geq ch(\varrho^n)) = \mathbf{P}(M_* \geq c\log\log\varrho^{-n}) \underset{n\to\infty}{\sim} 4c\,(\log 1/\varrho)^{-\frac{\pi^2}{4}c}\,n^{-\frac{\pi^2}{4}c}\log n\,.$$

The Borel-Cantelli Lemma and an easy argument imply that **P**-a.s.

$$\limsup_{n\to\infty} \frac{M_{\varrho^n}^*(a)}{h(\varrho^n)} \leq \frac{4}{\pi^2}\,.$$

Let $r \in (0,1)$. There exists $n(r) \in \mathbb{N}$ such that $\varrho^{n(r)+1} < r \leq \varrho^{n(r)}$. Thus,

$$\frac{M_r^*(a)}{h(r)} \leq \frac{1}{\varrho^2}\frac{M_{\varrho^{n(r)}}^*(a)}{h(\varrho^{n(r)})}\,.$$

Consequently, for any $\varrho \in (0,1)$, we have **P**-a.s. $\limsup_{r\to 0} M_r^*(a)/h(r) \leq 4(\pi\varrho)^{-2}$. By letting ϱ go to 1, we get for any $a \in (0,\infty)$,

$$(76) \qquad\qquad \mathbf{P}\text{-a.s.} \quad \limsup_{r\to 0} \frac{M_r^*(a)}{h(r)} \leq \frac{4}{\pi^2}\,.$$

Lower bound in (75). — For any $0 \leq r' \leq r \leq a$, recall from (34) the definition of $M_{r',r}^*(a)$ and recall from Lemma 2.2 that for any sequence $(r_n)_{n\geq 0}$ such that $0 \leq r_{n+1} \leq r_n \leq a$, and $\lim_{n\to\infty} r_n = 0$, the random variables $M_{r_{n+1},r_n}^*(a)$, $n \geq 0$, are independent. Here, we take $r_n = \varrho^n$, with $\varrho \in (0,1)$. We fix $c \in (0,\infty)$ and for any $n \geq 1$, we set

$$\varepsilon_n = \mathbf{1}_{\left\{M_{\varrho^{n+1},\varrho^n}^*(a) \geq \mathrm{ch}(\varrho^n)\right\}}\,.$$

The scaling property (37) and Lemma 3.4 entail

$$\mathbf{E}[\varepsilon_n] = \mathbf{P}\big(\varrho^{2n}(1-\varrho)^2 M_* \geq ch(\varrho^n)\big)$$

$$\underset{n\to\infty}{\sim} \frac{4c}{(1-\varrho)^2}(\log 1/\varrho)^{-\frac{\pi^2}{4(1-\varrho)^2}c}\,n^{-\frac{\pi^2}{4(1-\varrho)^2}c}\log n.$$

If $c \leq 4(1-\varrho)^2\pi^{-2}$, then $\sum_{n\geq 1}\mathbf{E}[\varepsilon_n] = \infty$. Since the ε_n's are independent, the usual converse of the Borel-Cantelli Lemma entails that **P**-a.s. $\sum_{n\geq 1}\varepsilon_n = \infty$. Now observe that for any $n \geq 1$,

$$\varepsilon_n \leq \mathbf{1}_{\left\{M_{\varrho^n}^*(a)\geq ch(\varrho^n)\right\}}\,.$$

Therefore, for any $\varrho \in (0,1)$ and any $c \in (0,\infty)$ such that $c \leq 4(1-\varrho)^2\pi^{-2}$ we **P**-a.s. have

$$\limsup_{r\to\infty} \frac{M_r^*(a)}{h(r)} \geq \limsup_{n\to\infty} \frac{M_{\varrho^n}^*(a)}{h(\varrho^n)} \geq c\,.$$

It easily entails the desired lower bound. It completes the proof of (75) and that of Proposition 1.3. $\qquad\square$

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